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12

MASTERING HSC MATHEMATICS

YEAR 12 EXTENSION 1 MATHEMATICS

NEW STAGE 6 HSC SYLLABUS

FOR STUDENTS AND TEACHERS

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Features of this book

This book is suitable for all students studying the HSC Mathematics Advanced and HSC Mathematics Extension 1 course. It has been designed in a thoroughly organised manner to help students master each syllabus topic in the new Stage 6 HSC Mathematics Advanced course. This book will teach, consolidate, test and challenge students. It is an essential resource for all students and teachers.

In flavour with the new course, this book has the following features:

- Technology-based questions.
- Interpretation questions.
- Modelling and application problems.
- Verification questions.

Within each chapter, there are subsections divided as follows.

Fundamentals

The carefully constructed *fundamentals* section appears before the main body of questions. The purpose of this section is to

- test all key formulae, definitions, concepts and theory.
- test essential mathematical terms and language through cloze-passages.
- ensure that the student has knowledge of the essential prerequisites.
- provide a summary of basic requirements for the topic.

Questions

This is the main body of questions with the following features.

- Step-by-step questions to assist the student with more difficult problems.
- Carefully graded exercises.
- “Show”-type questions, both guides the student, and offers good exam preparation.
- Proofs and explanations to strengthen understanding and develop problem-solving skills.
- Application questions to demonstrate future uses of learned theory.
- Technology-based questions to teach and reinforce concepts.

Challenge

These are more difficult questions that provide

- a challenge for students wishing to test their mastery of the topic.
- rigour and higher-order thinking skills.
- extension and more in-depth treatment of the unit of work.

Chapter Review

This section appears at the end of every chapter, and offers the following.

- Revision and consolidation of the previous exercises.
- Questions that require a combination of ideas from previous exercises.

Investigations

These tasks are potential assignments and research projects. Teachers may use and adapt these to cover the new NESAs requirements on investigative assessment tasks. This section provides for the student

- application and modelling scenarios.
- research tasks involving data collection and analysis.
- scaffolding of learning tasks.
- open-ended style problems for discussions.
- opportunity to use appropriate technology effectively in a range of contexts.
- opportunity for students to demonstrate critical thinking.

Answers

- Quick answers to questions.
- “Show” and “prove” answers can be found in the full worked solutions.

Full worked Solutions

- Can be found online for free, or a full-colour hard copy purchased for convenience.
- Provide complete worked solutions to all questions, except investigative tasks to maintain the open-ended nature of the tasks.
- Includes several alternative solutions to problems, where possible.

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1

MATHEMATICAL INDUCTION

- Induction involving sums
- Induction involving divisibility

Exercise 1A

Induction involving sums



Fundamentals

Fundamentals 1

Every mathematical induction proof must have four main steps:

Let $P(n)$ be a proposition or hypothesis defined for $n = 1, 2, 3, \dots$

- Step 1** Test the rule for the f_____ allowable value of n . Usually it is $n = \underline{\quad}$, but check carefully because it may be some other value of n .
- Step 2** Let $n = k$ be any integer value for which $P(k)$ is true, so we can a_____ $P(k)$ is true. Write down the statement that you are required to prove for $n = \underline{\quad}$.
- Step 3** P_____ the above statement by starting from the LHS. At some point to help complete the proof, you *must* use the a_____ from the previous step.
- Step 4** In **Step 3**, we proved that if $P(k)$ is true for any value of k , then _____ is also true. However from **Step 1**, $P(1)$ is true so inductively _____ is true, and _____ is true etc. Hence, by mathematical induction, $P(n)$ is true for all positive integers.

Question 1 [Drills for the base case]

Prove that the following statements are true for $n = 1$ and $n = 2$

- (a) $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$
- (b) $1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1}n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$
- (c) $1^2 + 4^2 + 7^2 + \dots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$
- (d) $1 + 2 \times 2 + 3 \times 2^2 + \dots + n \times 2^{n-1} = 1 + (n-1)2^n$
- (e) $\frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \dots + \frac{1}{(3n-2)(3n-1)} = \frac{n}{3n+1}$

Question 2 [Drills for producing the RTP]

Consider the following expressions for $T(k)$. Find and simplify $T(k+1)$.

- (a) $T(k) = 2k$ (b) $T(k) = 2k - 1$
- (c) $T(k) = 2k + 1$ (d) $T(k) = k2^{k-1}$
- (e) $T(k) = \frac{1}{(3k-2)(3k+1)}$ (f) $T(k) = \frac{1}{(2k-1)(2k+1)}$

Question 3 [Drills for proving the RTP]

Prove the following identities by starting from the left-hand side.

$$(a) \quad \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$(b) \quad 1 - \frac{1}{2k+1} + \frac{2}{(2k+1)(2k+3)} = 1 - \frac{1}{2k+3}$$

$$(c) \quad \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$$

$$(d) \quad \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{1}{4} - \frac{1}{2(k+2)(k+3)}$$

$$(e) \quad \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$(f) \quad \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

$$(g) \quad \left(\frac{k+1}{2k}\right) \times \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$$

$$(h) \quad \frac{1}{x^k(x-1)} - \frac{1}{x^{k+1}} = \frac{1}{x^{k+1}(x-1)}$$



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Question 4 Prove that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}n(n+1)(2n+1),$$

for all integers $n \geq 1$ using the principle of mathematical induction. Complete the steps below to prove the result.

(a) **Step 1:** Prove the statement is true for $n = 1$.

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1),$$

for $k \geq 1$. Write down the statement required to prove true for $n = k + 1$.

RTP $1^2 + 2^2 + 3^2 + \cdots + k^2 + \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\text{LHS} = \underline{\hspace{4cm}}$$

$$= \underline{\hspace{4cm}}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \text{RHS}$$

(d) Find the value of

$$1^2 + 2^2 + 3^2 + \cdots + 60^2$$

(e) Find the value of

$$10^2 + 11^2 + 12^2 + \cdots + 30^2$$

Question 5 Prove each of the following using mathematical induction for all integers $n \geq 1$.

(a) $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$

(b) $1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \frac{n}{2}(6n^2 - 3n - 1)$

(c) $1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + n \times 2^{n-1} = 1 + (n-1)2^n$

(d) $2 + 10 + 24 + \cdots + n(3n-1) = n^2(n+1)$

(e) $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$

$$(f) \quad 1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$$

$$(g) \quad 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

$$(h) \quad 1 + x + x^2 + x^3 + x^4 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Question 6 Prove each of the following using mathematical induction for all integers $n \geq 1$.

$$(a) \quad 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

$$(b) \quad 2 - 6 + 18 - 54 + \cdots + 2(-3)^{n-1} = \frac{1}{2}(1 - (-3)^n)$$

Question 7 Prove each of the following using mathematical induction for all integers $n \geq 1$.

$$(a) \quad \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$(b) \quad \frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \frac{1}{7 \times 10} + \cdots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

$$(c) \quad \frac{2}{1 \times 3} + \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \cdots + \frac{2}{(2n-1)(2n+1)} = 1 - \frac{1}{2n+1}$$

$$(d) \quad \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

Question 8

(a) Simplify the following

$$(i) \quad \frac{1}{1 \times 2} + \frac{1}{2 \times 3}$$

$$(ii) \quad \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4}$$

$$(iii) \quad \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5}$$

(b) Hypothesise a simpler expression for $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n \times (n+1)}$

(c) Prove your hypothesis from (b) for all integers $n \geq 1$ using mathematical induction.

(d) Use the above result to evaluate

$$(i) \quad \frac{1}{10 \times 11} + \frac{1}{11 \times 12} + \frac{1}{12 \times 13} + \cdots + \frac{1}{49 \times 50}$$

$$(ii) \quad \frac{1}{20 \times 21} + \frac{1}{21 \times 22} + \frac{1}{22 \times 23} + \cdots + \frac{1}{99 \times 100}$$

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Question 9

- (a) Show that $(k+1)! + (k+1) \cdot (k+1)! = (k+2)!$
(b) Hence, use mathematical induction to prove that for all integers $n \geq 1$

$$1 + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

Question 10

Prove each of the following using mathematical induction.

- (a) $(1^2 + 1)1! + (2^2 + 1)2! + (3^2 + 1)3! + \cdots + (n^2 + 1)n! = n(n+1)!$ for all integers $n \geq 1$.
(b) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n-1}{n!} = 1 - \frac{1}{n!}$ for all integers $n \geq 2$.

Question 11

Use mathematical induction to prove that

- (a) $\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1$, for $n \geq 1$.
(b) $\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{n}\right) = \frac{1}{n}$, for $n \geq 2$.
(c) $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$, for $n \geq 2$.
(d) $\left(1 - \frac{1}{2}\right) \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{4}\right) \left(1 - \frac{2}{5}\right) \cdots \left(1 - \frac{2}{n}\right) = \frac{1}{n(n-1)}$, for $n \geq 3$.

Question 12

Use mathematical induction to prove that

- (a) $\log 1 + \log 2 + \log 3 + \cdots + \log n = \log(n!)$, for $n \geq 1$
(b) $\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{n}{n-1}\right) = \log(n)$, for $n \geq 2$

Question 13

- (a) Use mathematical induction to prove that for all integers $n \geq 2$

$$\frac{1}{2^2 - 1} + \frac{1}{3^2 - 1} + \frac{1}{4^2 - 1} + \cdots + \frac{1}{n^2 - 1} = \frac{(n-1)(3n+2)}{4n(n+1)}$$

- (b) Hence, find the limiting value of

$$\frac{1}{2^2 - 1} + \frac{1}{3^2 - 1} + \frac{1}{4^2 - 1} + \frac{1}{5^2 - 1} + \cdots$$

Question 14 [Guide for the harder trigonometric series problems]

Prove that

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos(2n-1)x = \frac{\sin 2nx}{2 \sin x},$$

for all integers $n \geq 1$ using the principle of mathematical induction. Complete the steps below to prove the result.

- (a) Use the product-to-sum formulae to complete the following identity.

$$2 \cos A \sin B = \sin(\text{---}) - \sin(\text{---})$$

- (b) **Step 1:** Prove the statement is true for $n = 1$.
 (c) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos(2k-1)x = \frac{\sin 2kx}{2 \sin x},$$

for $k \geq 1$. Write down the statement required to prove true for $n = k + 1$.

$$\text{RTP} \quad \cos x + \cos 3x + \cos 5x + \cdots + \cos(2k-1)x + \text{---} = \text{---}$$

- (d) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\begin{aligned} \text{LHS} &= \frac{\sin 2kx}{2 \sin x} + \cos(2k+1)x \\ &= \frac{\sin 2kx + 2 \cos(2k+1)x \sin x}{2 \sin x} \\ &= \frac{\sin 2kx + \sin(\text{---})x - \sin(\text{---})x}{2 \sin x}, \quad \text{using (a)} \\ &= \frac{\sin(2k+2)x}{2 \sin x} \\ &= \text{RHS} \end{aligned}$$

Question 15 Use similar technique to the previous question to prove that

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{1 - \cos 2nx}{2 \sin x}$$

for all integers $n \geq 1$ using mathematical induction.



Challenge Problems

Problem 1 Use mathematical induction to prove that for if $x \neq 0, 1$ and $n \geq 1$, then

$$\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \cdots - \frac{1}{x^n} = \frac{1}{x^n(x-1)}$$

Problem 2 Use mathematical induction to prove that for all integers $n \geq 1$

$$(n+1)(n+2)(n+3) \cdots (2n-1)(2n) = 2^n (1 \times 3 \times 5 \times \cdots \times (2n-1))$$

Problem 3

(a) Use mathematical induction to prove that for all integers $n \geq 1$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

(b) Use mathematical induction to prove that for all integers $n \geq 1$

$$(1+2+\cdots+n)^2 = 1^3 + 2^3 + \cdots + n^3$$

(c) Deduce that

$$(1+2+\cdots+n)^2 = \frac{1}{4}n^2(n+1)^2$$

(d) Find another way of proving this result.

Hint: Think of arithmetic progressions

Problem 4

(a) Factorise $4n^3 + 18n^2 + 23n + 9$, given that it has an integer zero.

(b) Hence, use mathematical induction to prove that for all integers $n \geq 1$

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n-1)(2n+1) = \frac{n}{3}(4n^2 + 6n - 1)$$

Problem 5 Use mathematical induction to prove the following formulae for all integers $n \geq 1$.

(a) $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$

(b) $a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) = \frac{n}{2}(2a + (n-1)d)$

Exercise 1B

Induction involving divisibility

Fundamentals

Fundamentals 1

Recall the main steps involved in any induction proof for a statement $P(n)$ for $n = 1, 2, 3, \dots$

Base Case: Verify $P(\text{---})$.

Inductive Hypothesis: Assume the statement is true for $n = k$ i.e. $P(\text{---})$ is true.

Inductive Step: Show $P(k)$ implies $P(\text{---})$.

Fundamentals 2

To show that a number is divisible by some integer p , we must prove that it can be expressed in the form --- where M is some integer.

Fundamentals 3

Be mindful of statements where $P(n)$ is only true for odd or even integers.

- (a) If $P(n)$ is true for odd integers only, start from a base case of $n = \text{---}$. Assume $P(k)$ is true for odd k and instead of proving $n = k + 1$, prove $n = \text{---}$.
- (b) If $P(n)$ is true for even integers only, start from a base case of $n = \text{---}$. Assume $P(k)$ is true for even k and instead of proving $n = k + 1$, prove $n = \text{---}$.

Question 1 [Drill question]

Write down what we are required to prove, in terms of some positive integer N .

- (a) $n^3 + 2n$ is divisible by 3
- (b) $4^n + 2$ is divisible by 6
- (c) $n^2 + n$ is even
- (d) $9^n + 7$ is divisible by 8
- (e) $5^n + 3$ is divisible by 4
- (f) $3^{2n} - 1$ is divisible by 8

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Question 2 [Guide for the basic types with one power term]

Prove that $5^n - 1$ is divisible by 4 for all integers $n \geq 1$ using the principle of mathematical induction. Complete the steps below to prove the result.

(a) **Step 1:** Prove the statement is true for $n = 1$.

$$5^1 - 1 = \underline{\quad}, \text{ which is divisible by 4.}$$

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$5^k - 1 = 4M,$$

where $k \geq 1$ and M is an integer. Write down the statement required to prove true for $n = k + 1$.

$$5^{k+1} - 1 = \underline{\hspace{2cm}}$$

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$5^{k+1} - 1 = 5 \times 5^k - 1$$

$$= 5(\underline{\hspace{2cm}}) - 1, \quad \text{since } 5^k = \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}}$$

$$= 4(\underbrace{\underline{\hspace{2cm}}}_{\text{Integer}})$$

Hence, $5^{n+1} - 1$ is divisible by 4 for integer $n \geq 1$.

Question 3 Prove each of the following by mathematical induction when n is a positive integer

(a) $9^n + 7$ is divisible by 8

(b) $3^n - 1$ is divisible by 2

(c) $4^n + 14$ is divisible by 6

(d) $13 \times 6^n + 2$ is divisible by 10

(e) $3^{2n} - 1$ is divisible by 8

(f) $7^{2n-1} + 5$ is divisible by 12

Question 4 [Guide for the harder types with two power terms]

Prove that $8^n - 3^n$ is divisible by 5 for all integers $n \geq 1$ using the principle of mathematical induction. Complete the steps below to prove the result.

(a) **Step 1:** Prove the statement is true for $n = 1$.

$$8^1 - 3^1 = \underline{\quad}, \text{ which is divisible by 5.}$$

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$8^k - 3^k = \underline{\quad},$$

where $k \geq 1$ and M is an integer. Write down the statement required to prove true for $n = k + 1$.

$$8^{k+1} - 3^{k+1} = \underline{\quad}$$

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\begin{aligned} 8^{k+1} - 3^{k+1} &= \underline{\quad} (8^k) - \underline{\quad} (3^k) \\ &= \underline{\quad} (5M + 3^k) - \underline{\quad} (3^k), \quad \text{since } 8^k = \underline{\quad} \\ &= \underline{\quad} \\ &= 5 \left(\underbrace{\underline{\quad}}_{\text{Integer}} \right) \end{aligned}$$

Hence, $8^n - 3^n$ is divisible by 5 for integer $n \geq 1$.

(d) Prove the same result by using $3^k = 8^k - 5M$ instead.

Question 5 Prove each of the following by mathematical induction where n is a positive integer

- | | |
|---|---|
| (a) $8^n - 5^n$ is divisible by 3 | (b) $2 \times 11^n + 5^n$ is divisible by 3 |
| (c) $7^{2n} + 7^n + 4$ is divisible by 6 | (d) $5^{2n} + 5^n + 2$ is divisible by 4 |
| (e) $5^{3n} + 5^{2n} + 5^n + 1$ is divisible by 4 | (f) $9^{n+2} - 4^n$ is divisible by 5 |
| (g) $3^{3n} + 2^{n+2}$ is divisible by 5 | (h) $5^{2n} - 4^{2n}$ is divisible by 9 |

12 Chapter 1: Mathematical Induction

Question 6 [The statement is only true every other term]

Let $n \geq 1$ be an integer. Prove that $3^n + 5^n$ is divisible by 8 for *odd* values of n using the principle of mathematical induction. Complete the steps below to prove the result.

(a) **Step 1:** Prove the statement is true for $n = 1$.

$$3^1 + 5^1 = \underline{\quad}, \text{ which is divisible by 8.}$$

(b) **Step 2:** Let $n = k$ be any integer value where the statement is true, so

$$3^k + 5^k = \underline{\quad},$$

where $k \geq 1$ and M is an integer. Write down the statement required to prove true for $n = k + 2$.

$$3^{k+2} + 5^{k+2} = \underline{\quad}$$

and briefly explain why we use $n = k + 2$ here instead of the usual $n = k + 1$.

(c) **Step 3:** Prove that $n = k$ implies $n = k + 1$ is true.

$$\begin{aligned} 3^{k+2} + 5^{k+2} &= 9(3^k) + 25(5^k) \\ &= \underline{\quad} (\underline{\quad}) + \underline{\quad} (5^k) \\ &= \underline{\quad} \\ &= 8 (\underbrace{\underline{\quad}}_{\text{Integer}}) \end{aligned}$$

Hence, $3^n + 5^n$ is divisible by 8 for odd values of n .

Question 7 Use mathematical induction to prove that if n is a positive integer, then

- (a) $3^n + 7^n$ is divisible by 10 for odd values of n . (b) $4^n + 5^n$ is divisible by 9 for odd values of n .
(c) $n^2 + 2n$ is divisible by 8 for even values of n . (d) $5^n - 4$ is divisible by 3 for even values of n .

Question 8 Prove by induction that for all integers $n \geq 1$

- (a) $n^2 + n$ is divisible by 2 (b) $n^3 + 2n$ is divisible by 3 (c) $n^3 - n$ is divisible by 6

Question 9

- (a) Complete the table of values below

n	1	2	3	4	5	6	7
$2^n - (-1)^n$	3						

- (b) Make a hypothesis about the value of $2^n - (-1)^n$.
 (c) Prove your hypothesis using mathematical induction.

Challenge Problems

Problem 1 [Harder standard-type problems]

Use mathematical induction to prove that for integers $n \geq 1$.

- (a) $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. (b) $49^n + 16n - 1$ is divisible by 64.
 (c) $8^{2n+1} + 6^{2n-1}$ is divisible by 7. (d) $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Problem 2 [General divisibility result]

Define the polynomial $P_n(x) = x^n - 1$. Use mathematical induction to prove that in general for all integers $n \geq 1$, $P_n(x)$ is divisible by $x - 1$.

Problem 3 Use mathematical induction to prove that for integers $n \geq 1$, the expression

$$x^n - y^n$$

is divisible by $x - y$.

Hint: For your assumption, let $x^k - y^k = (x - y)Q(x, y)$, where $Q(x, y)$ is some polynomial in terms of x and y .

Problem 4 Let a and b be any positive integers. Use mathematical induction to prove that

$$a^n + b(2a + b)^{n-1}$$

is divisible by $a + b$ for all integers $n \geq 1$.

Chapter 1 Review

Mathematical Induction

Review

Question 1 Prove each of the following using mathematical induction for all integers $n \geq 1$.

(a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

(b) $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$

(c) $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n}{3}(4n^2 - 1)$

(d) $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$

(e) $1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

(f) $3^2 + 6^2 + 9^2 + \cdots + (3n)^2 = \frac{3}{2}n(n + 1)(2n + 1)$

(g) $3 - 6 + 12 - 24 + \cdots + 3(-2)^{n-1} = 1 - (-2)^n$

(h) $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n + 1)^2 = \frac{1}{12}n(n + 1)(n + 2)(3n + 5)$

Question 2

(a) Prove that

$$3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1),$$

for all integers $n \geq 1$ using the principle of mathematical induction.

(b) Find the value of n that makes the series end with 2019, and hence find the value of

$$3 + 7 + 11 + \cdots + 2019$$

(c) Find the value of

$$99 + 103 + 107 + \cdots + 499$$

Question 3 Use mathematical induction to prove that

$$(2^2 - 1)(3^2 - 1) \cdots (n^2 - 1) = \frac{(n!)^2(n + 1)}{2n}$$

for integer $n \geq 2$.

Question 4 Use mathematical induction to prove the following for all integers $n \geq 1$.

$$(a) \quad \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$$

$$(b) \quad \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \cdots + \frac{2^n - 1}{2^n} = n + \frac{1}{2^n} - 1$$

$$(c) \quad \frac{1^2}{1 \times 3} + \frac{2^2}{3 \times 5} + \frac{3^2}{5 \times 7} + \cdots + \frac{n^2}{(2n-1)(2n+1)} = \frac{n(n+1)}{2(2n+1)}$$

Question 5

(a) Use mathematical induction to prove that

$$\left(\frac{1 \times 5}{3^2}\right) \left(\frac{3 \times 7}{5^2}\right) \left(\frac{5 \times 9}{7^2}\right) \cdots \left(\frac{(2n-1)(2n+3)}{(2n+1)^2}\right) = \frac{2n+3}{6n+3}$$

for integer $n \geq 1$.

(b) Hence, find the limiting value of

$$\left(\frac{1 \times 5}{3^2}\right) \left(\frac{3 \times 7}{5^2}\right) \left(\frac{5 \times 9}{7^2}\right) \cdots$$

Question 6

(a) Use mathematical induction to prove that for all integers $n \geq 1$,

$$\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

(b) Hence, find the limiting value of $\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \cdots$

Question 7

(a) Use mathematical induction to prove that for all integers $n \geq 1$,

$$\frac{1}{4 \times 1^2 - 1} + \frac{1}{4 \times 2^2 - 1} + \frac{1}{4 \times 3^2 - 1} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n+1}$$

(b) Hence, find the limiting value of

$$\frac{1}{4 \times 1^2 - 1} + \frac{1}{4 \times 2^2 - 1} + \frac{1}{4 \times 3^2 - 1} + \cdots$$

 Investigation Task

Other types of induction

In this course, we typically see two types of induction problems. One is a ‘series’-type problem and the other is a ‘divisibility’-type problem.

In fact, mathematical induction is useful in far more applications than the two mentioned above. This investigation task will allow students to explore those other different types to really see the versatility of this proof technique. Use the method of mathematical induction to prove each of the statements below.

Question 1 [Differentiation]

Prove that if $y = (x + 2) \ln(x + 2)$ and $x + 2 > 0$, then

$$\frac{d^n y}{dx^n} = (-1)^n \frac{(n-2)!}{(x+2)^{n-1}}$$

for integers $n \geq 2$.

Question 2 [Inequalities]

Prove that $2^n \geq n^2$ for integers $n \geq 4$.

Question 3 [Geometry]

A plane has n distinct non-parallel lines such that no three are concurrent.

Prove that in total there are $\frac{n}{2}(n-1)$ points of intersection for integers $n \geq 2$.

Question 4 [Function composition]

Let $f(x) = \frac{x}{\sqrt{1+x^2}}$. Let $f \circ f(x) = f_2(x)$ and similarly $f \circ f \circ f(x) = f_3(x)$. Prove that

$$f_n(x) = \frac{x}{\sqrt{1+nx^2}}$$

for integers $n \geq 2$.

Question 5 [Counting scenarios]

A room contains n people. Prove that if everybody shakes hands with each other exactly once, there are $\frac{n}{2}(n-1)$ handshakes in total for integers $n \geq 2$. Bonus points for explaining the connection between this problem and the *Geometry* induction problem above.

 Investigation Task

Tower of Hanoi

The Tower of Hanoi is a well-known mathematical puzzle/game that has applications in computer science. The task is to produce a two page article about the Tower of Hanoi. Your article should contain

- A brief history of it.
- A description of it and an example of a solution for 3 discs.
- The minimal number of moves for n discs.
- The motivation behind that formula.
- A proof of the formula using induction.

 Investigation Task

Strong Induction

In the past, mathematical induction proofs have been somewhat similar. First you prove the statement for some base case, then assume the statement is true for some $n = k$, then prove that if $n = k$ is true then $n = k + 1$ is true. However, this technique does not always work for some problems and for some of these scenarios, a stronger form of induction named (rather intuitively) *strong induction* is needed. This investigation task will allow students to explore this new form of induction, and understand why and when it is needed.

Question 1 Research and write a half page article explaining what *strong induction* is, and why it is ‘strong’.

Question 2 Consider the recurrence

$$T_n = 5T_{n-1} - 6T_{n-2}$$

for integer values of $n \geq 3$. The recurrence has seed values (starting points) $T_1 = 5$ and $T_2 = 13$.

- (a) Calculate by hand T_3 , T_4 and T_5 .
- (b) Obviously, calculating something like T_{100} recursively would be impractical. Try the closed form $T_n = 2^n + 3^n$. Does it seem to ‘generate’ the correct values?
- (c) Try to prove that the statement is true using Extension 1 mathematical induction. At some point you should be stuck in your proof. Identify the precise step where you get stuck and explain what you would need to proceed with your proof.
- (d) Hence, modify your proof and prove that the closed form works using *strong induction* for integer $n \geq 1$.

Question 3 [Binet’s Formula]

- (a) Research and write a short paragraph explaining what Binet’s Formula is.
- (b) Prove Binet’s Formula using strong induction.

Question 4 [Fundamental Theorem of Arithmetic]

- (a) Research and write a short paragraph about the Fundamental Theorem of Arithmetic.
- (b) Prove the Fundamental Theorem of Arithmetic using strong induction.

Question 5 Your task is to write the section on Strong Induction for a mathematics textbook. Your submission should be as detailed as you would expect a textbook to be for a final-year high school student. It should contain theory, worked examples, and a small problem set featuring a wide variety of problems.

 Investigation Task

Logical fallacies

Mathematical induction is a powerful proof technique, but only when done properly. A poorly done proof or just a plain mis-application of the technique can be misleading. This investigation task will show students that they do need to be careful when applying the technique, and that it is not just a silver bullet that can be used to prove anything related to integers, as tempting as it may seem.

Question 1 Try to use mathematical induction to prove that $n^2 - n + 41$ is a prime number for all integer values of $n \geq 1$.

Question 2 Find the flaw in the following proof by induction that $4^n + 6n$ is divisible by 6 for integer $n \geq 1$.

Assume true for $n = k$:

$$4^k + 6k = 6M$$

where M is an integer.

Prove true for $n = k + 1$:

$$\begin{aligned} 4^{k+1} + 6(k+1) &= 4 \times 4^k + 6k + 6 \\ &= 4(6M - 6k) + 6k + 6 \\ &= 24M - 24k + 6k + 6 \\ &= 6(4M - 3k + 1) \end{aligned}$$

Hence, the statement is true by induction.

Question 3 If you try to prove a statement by induction, and the proof fails, does it imply that the statement was false in the first place? Support your answer with examples and/or counter examples.

Question 4 Come up with three 'fake' proofs by induction that may seem true to a reader initially. In each one, explain in detail why it does not work and if possible, correct the statement or proof so that it *does* work.

2

VECTORS

- Introduction to vectors
- Addition, subtraction and scalar multiples of vectors
- Magnitude of a vector
- The dot product and applications
- Vector projections
- Proofs using vectors
- Displacement and velocity
- Resolving forces and equilibrium
- Projectile motion

Exercise 2A

Introduction to vectors



Fundamentals

Fundamentals 1

- A vector is a quantity that has a m _____ and d _____.
- If two vectors have the same magnitude and same direction, then they are said to be e _____.
- A vector that has its tail at the origin is called a p _____ vector.
- The same vector, but originating from a point other than the origin, is called the d _____ vector.
- All d _____ vectors with the same direction are p _____.

Fundamentals 2

- A vector in 2-dimensional space has two components being the h _____ component and the v _____ component.
- One representation of a vector is $x\hat{i} + y\hat{j}$, where x is the horizontal/vertical (circle one) component and y is the horizontal/vertical (circle one) component.
- Another way of representing this same vector is using column notation $\begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$.
- The vector may also be represented as the ordered pair $\langle \text{---}, \text{---} \rangle$.
- A vector may be expressed as \overrightarrow{AB} , \mathbf{a} , or $\underline{\quad}$.

Fundamentals 3

If $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then

$$\begin{aligned} \overrightarrow{AB} &= \langle \text{---}, \text{---} \rangle \\ &= \text{---}\hat{i} + \text{---}\hat{j} \\ &= \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix} \end{aligned}$$

Challenge Problems**Problem 1** [Parallel vectors]

Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ be a vector. Explain why $k\mathbf{a} = kx\mathbf{i} + ky\mathbf{j}$ is a vector that is parallel to the original vector \mathbf{a} .

Problem 2 [Unit vector]

- (a) Show that the magnitude of the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j}$ is $\sqrt{x^2 + y^2}$.
- (b) Hence, use the result from **Problem 1** to explain why the vector parallel to \mathbf{a} but with unit length is $\frac{1}{\sqrt{x^2 + y^2}}\mathbf{a}$.

Exercise 2B

Addition, subtraction and scalar multiples of vectors

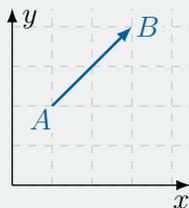
Fundamentals

Fundamentals 1

- (a) When a vector \overrightarrow{AB} is multiplied by a constant k to give $k\overrightarrow{AB}$, then k is called the s_____.
- (b) If $k > 1$, then $k\overrightarrow{AB}$ is longer/shorter (circle one) than the original vector \overrightarrow{AB} .
- (c) If $k < 1$, then $k\overrightarrow{AB}$ is longer/shorter (circle one) than the original vector \overrightarrow{AB} .
- (d) If $k = -1$, then $k\overrightarrow{AB}$ is the same as \overrightarrow{AB} , but f_____ the other way around.
- (e) Vectors that are scalar multiples of each other are p_____ to each other on the plane.
- (f) In other words if $\mathbf{u} = k\mathbf{v}$ for $k \neq 0$, then \mathbf{u} and \mathbf{v} are p_____.
- (g) Conversely, if \mathbf{u} and \mathbf{v} are p_____, then $\mathbf{u} = k\mathbf{v}$ for some $k \neq 0$.

Fundamentals 2

The diagram below shows the vector \overrightarrow{AB} .



- (a) Draw the equivalent position vector. (b) Draw $2\overrightarrow{AB}$.
- (c) Draw $\frac{1}{2}\overrightarrow{AB}$. (d) Draw $-\overrightarrow{AB}$.

Fundamentals 3

(a) $\overrightarrow{AB} + \overrightarrow{BC} = \underline{\hspace{2cm}}$

(b) In general,

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \overrightarrow{A_3A_4} + \cdots + \overrightarrow{A_{n-1}A_n} =$$

- (c) This is sometimes referred to as the p_____ rule.

Question 1 Let $\mathbf{a} = \underline{i} + 2\underline{j}$ and $\mathbf{b} = -2\underline{i} + 3\underline{j}$. Calculate the following.

- (a) $-\mathbf{a}$ (b) $2\mathbf{b}$ (c) $\mathbf{a} + \mathbf{b}$
 (d) $\mathbf{a} - \mathbf{b}$ (e) $2\mathbf{a} + 3\mathbf{b}$ (f) $3\mathbf{a} - 2\mathbf{b}$

Question 2 Let $\mathbf{a} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Calculate the following.

- (a) $-\mathbf{b}$ (b) $\frac{1}{2}\mathbf{b}$ (c) $\mathbf{a} + \mathbf{b}$
 (d) $\mathbf{a} - \mathbf{b}$ (e) $\mathbf{a} - 2\mathbf{b}$ (f) $2\mathbf{a} - 3\mathbf{b}$

Question 3 Let $\mathbf{a} = \langle -2, 4 \rangle$ and $\mathbf{b} = \langle 5, 6 \rangle$. Calculate the following.

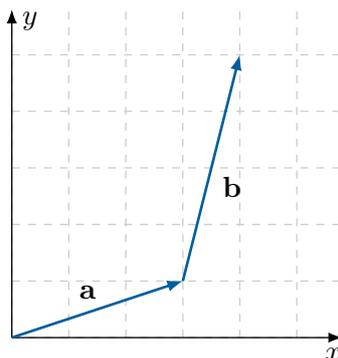
- (a) $3\mathbf{a} + 5\mathbf{b}$ (b) $4\mathbf{b} - 2\mathbf{a}$ (c) $-\frac{1}{2}\mathbf{a} - \mathbf{b}$

Question 4 Find the value(s) of k such that the following vectors are parallel.

- (a) $\langle 2, k \rangle, \langle 6, 3 \rangle$ (b) $\langle -2k, 6 \rangle, \langle 4, 12 \rangle$ (c) $\langle k, k^2 \rangle, \langle -4, 6 \rangle$

Question 5 [Confirming tip-to-tail addition numerically]

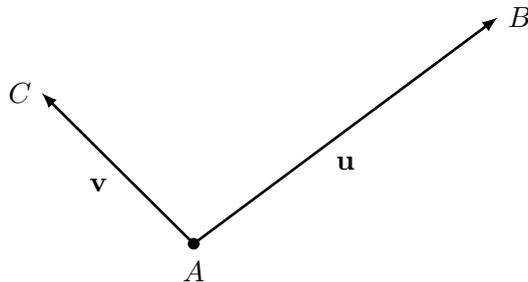
The diagram below shows $\mathbf{a} = 3\underline{i} + \underline{j}$ and $\mathbf{b} = \underline{i} + 4\underline{j}$ on the plane.



- (a) Draw $\mathbf{a} + \mathbf{b}$.
 (b) Read off from your diagram an expression for $\mathbf{a} + \mathbf{b}$.
 (c) Calculate $\mathbf{a} + \mathbf{b}$ by adding their components.

Question 6 [Adding vectors that are connected tail-to-tail]

The diagram below shows two vectors \mathbf{u} and \mathbf{v} .

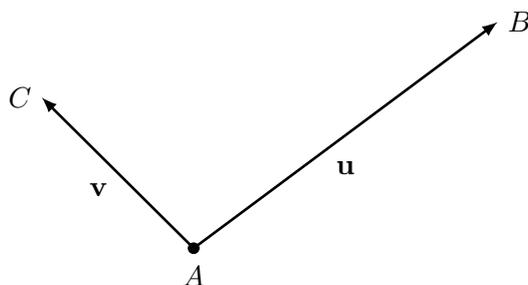


- 'Complete' the parallelogram.
- Hence, draw $\mathbf{u} + \mathbf{v}$.
- Complete the following statement.

To add two vectors \mathbf{u} and \mathbf{v} that are connected tail-to-tail, first complete the p_____ that they outline. Connect the tail to the o_____ vertex, and that is vector $\mathbf{u} + \mathbf{v}$.

Question 7 [Subtracting vectors when connected tail-to-tail]

The diagram below shows two vectors \mathbf{u} and \mathbf{v} .

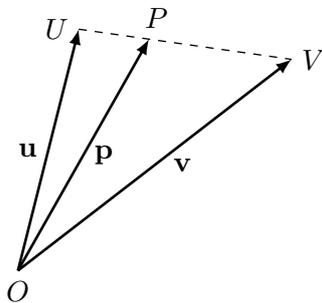


- Find \overrightarrow{BC} in terms of \mathbf{u} and \mathbf{v} .
- Write down \overrightarrow{CB} in terms of \mathbf{u} and \mathbf{v} .
- Complete the following statement.

Given two vectors \mathbf{u} and \mathbf{v} that are connected tail-to-tail, the vector $\mathbf{u} - \mathbf{v}$ is just the line connecting the tip of \mathbf{v} to the tip of \mathbf{u} . Conversely the vector $\mathbf{v} - \mathbf{u}$ is just the line connecting the tip of _____ to the tip of _____.

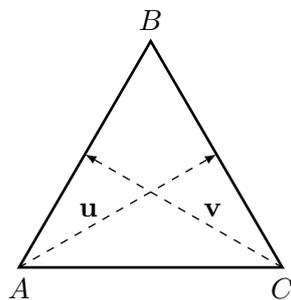
Challenge Problems

Problem 1 The diagram below shows a point P on an interval UV , which is twice as far from V as it is from U . Let $\mathbf{u} = \overrightarrow{OU}$, $\mathbf{v} = \overrightarrow{OV}$ and $\mathbf{p} = \overrightarrow{OP}$.



- Write down the vector that represents $\mathbf{u} + \overrightarrow{UV}$.
- Write down the vector that represents $\mathbf{u} + \overrightarrow{UP}$.
- Deduce that $\mathbf{p} = \frac{2}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}$.

Problem 2 The diagram below shows an equilateral triangle ABC and two vectors \mathbf{u} and \mathbf{v} that connect each vertex to the midpoint to the opposite side.



- Show that $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \mathbf{u}$
- Find a similar expression for \mathbf{v} in terms of \overrightarrow{AB} and \overrightarrow{BC} .
- Hence, show that $\overrightarrow{AB} = \frac{2}{3}(2\mathbf{u} + \mathbf{v})$.
- Find a similar expression for \overrightarrow{BC} in terms of \mathbf{u} and \mathbf{v} .

Exercise 2C

Magnitude of a vector

Fundamentals

Fundamentals 1

Let $\mathbf{u} = x\hat{i} + y\hat{j}$.

- (a) The magnitude of a vector \mathbf{u} is denoted by ____.
- (b) The magnitude of a displacement vector \overrightarrow{AB} represents the d_____ between the two points A and B .
- (c) $|\mathbf{u}| = |x\hat{i} + y\hat{j}| = \underline{\hspace{2cm}}$

Fundamentals 2

Let \mathbf{u} be a non-zero vector.

- (a) The vector that points in the direction of \mathbf{u} but with length 1 is called the \mathbf{u} ____ vector.
- (b) The \mathbf{u} ____ vector is denoted by ____.
- (c) The expression for ____ is obtained by dividing the the vector \mathbf{u} by its length, so

$$\underline{\hspace{1cm}} = \underline{\hspace{1cm}} \mathbf{u}$$

Question 1 Let $\mathbf{a} = \hat{i} + 5\hat{j}$, $\mathbf{b} = 3\hat{i} - 2\hat{j}$ and $\mathbf{c} = -2\hat{i} - 4\hat{j}$. Calculate the following.

- (a) $|\mathbf{a}|$ (b) $|\mathbf{-a}|$ (c) $|\mathbf{b}|$ (d) $|\mathbf{c}|$
 (e) $|\mathbf{a} - \mathbf{b}|$ (f) $|\mathbf{b} - \mathbf{a}|$ (g) $|\hat{\mathbf{a}}|$ (h) $|\hat{\mathbf{b}}|$

Question 2 Find the unit vector pointing in the direction of each of the following vectors.

- (a) $\mathbf{a} = 3\hat{i} - 4\hat{j}$ (b) $\mathbf{b} = \hat{i} + \hat{j}$ (c) $\mathbf{c} = -5\hat{i} + 12\hat{j}$
 (d) $\mathbf{d} = 2\hat{i} + 4\hat{j}$ (e) $\mathbf{e} = 3\hat{i}$ (f) $\mathbf{f} = -2\hat{j}$

Question 3 [Proofs]

Prove the following results about magnitudes.

- (a) $|\mathbf{v}| = |-\mathbf{v}|$ (b) $|\mathbf{v} - \mathbf{w}| = |\mathbf{w} - \mathbf{v}|$

Question 4 Consider the points $A(3, 0)$, $B(4, 5)$, $C(-1, 4)$ and $D(-2, -1)$ on the plane.

- Show that $\overrightarrow{AB} = \overrightarrow{DC}$.
- Show that $\overrightarrow{BC} = \overrightarrow{AD}$.
- From this alone, what kind of quadrilateral does $ABCD$ form?
- Calculate $|\overrightarrow{AB}|$ and $|\overrightarrow{AD}|$. Explain why this implies that $ABCD$ is now at least a rhombus.
- Prove that $ABCD$ is a rhombus, but not a square.

Hint: What other magnitudes can we calculate that disprove it being a square?

Question 5 Show that the points $A(-3, -2)$, $B(2, -1)$, $C(3, 4)$ and $D(-2, 3)$ form a rhombus in the plane, but not a square.

Question 6 [Important result for magnitudes]

Consider $\mathbf{a} = k\mathbf{v}$, where $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$.

- Write down an expression for $k\mathbf{v}$.
- Calculate $|\mathbf{a}|$.
- Hence, show that $|k\mathbf{v}| = k|\mathbf{v}|$.
- Verify this result with $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $k = 2$.

Challenge Problems

Problem 1 Let

$$\mathbf{u} = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$$

- Find $|\mathbf{u}|$.
- Explain how your answer was to be expected.

Problem 2 [Cauchy Schwarz inequality from the triangle inequality]

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors.

- Draw two random vectors \mathbf{u} and \mathbf{v} connected tip-to-tail, then draw $\mathbf{u} + \mathbf{v}$. Explain why

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

- Hence, show that

$$(u_1v_1 + u_2v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

Question 4 Define $\mathbf{a} = \langle k, 4 \rangle$ and $\mathbf{b} = \langle 2, -1 \rangle$. Suppose that vectors \mathbf{a} and \mathbf{b} are perpendicular.

- (a) Write down the value of $\mathbf{a} \cdot \mathbf{b}$.
 (b) Hence, find the value of k .

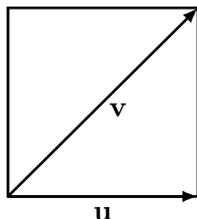
Question 5 Find the value of k such that the following vectors are perpendicular.

- (a) $\langle 4, k \rangle, \langle -2, 4 \rangle$ (b) $\langle 2k, 6 \rangle, \langle 1, k \rangle$ (c) $\langle k - 1, k + 1 \rangle, \langle 6, -8 \rangle$

Question 6 Define $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle b, a \rangle$. Let the angle between the two vectors be θ . Show that

$$\sec \theta = \frac{1}{2} \left(\frac{a}{b} + \frac{b}{a} \right)$$

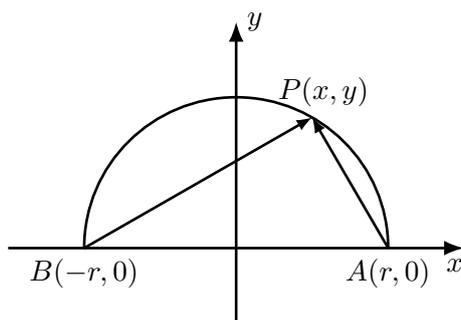
Question 7 The diagram below shows a unit vector \mathbf{u} being one of the sides of a square, and \mathbf{v} being the diagonal of the square.



Calculate $\mathbf{u} \cdot \mathbf{v}$.

Question 8 [Thales' Theorem]

Let $P(x, y)$ be any point on a semi-circle of radius r .



Define coordinates $A(r, 0)$ and $B(-r, 0)$ to be the end-points of the circle.

- (a) Write down vectors \overrightarrow{AP} and \overrightarrow{BP} in column vector form.
 (b) Find $\overrightarrow{AP} \cdot \overrightarrow{BP}$.
 (c) What is the geometric significance of this result?

Question 9 [Proofs of standard results]

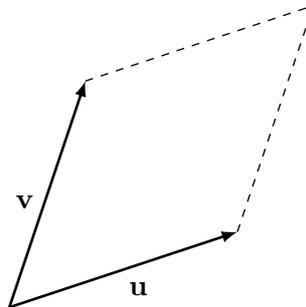
Let \mathbf{a} and \mathbf{b} be non-zero vectors. Prove each of the following results by using the relationship $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the two vectors.

- (a) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- (b) If two vectors are parallel and pointing in the same direction, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$
Hint: If they are pointing in the same direction, what is the angle between the two vectors?
- (c) If two vectors are parallel and pointing in the opposite direction, then $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$
Hint: If they are pointing in opposite directions, what is the angle between the two vectors?
- (d) If two vectors are perpendicular, then $\mathbf{a} \cdot \mathbf{b} = 0$
- (e) If $\mathbf{a} \cdot \mathbf{b} = 0$, then vectors \mathbf{a} and \mathbf{b} are perpendicular.

Question 10 [Commutativity of the dot product]

Let $\mathbf{u} = u_1\hat{i} + u_2\hat{j}$ and $\mathbf{v} = v_1\hat{i} + v_2\hat{j}$. Prove that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

Question 11 Let $\mathbf{u} = u_1\hat{i} + u_2\hat{j}$ and $\mathbf{v} = v_1\hat{i} + v_2\hat{j}$ be two vectors on the plane, as shown in the diagram below.



- (a) Copy the diagram above, and draw $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. What is the significance of these two vectors in relation to the parallelogram that gets traced out by \mathbf{u} and \mathbf{v} ?
- (b) Suppose $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are perpendicular. Write down the value of $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$.
- (c) Explicitly calculate $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$ and show that

$$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = u_1^2 - v_1^2 + u_2^2 - v_2^2$$

- (d) Deduce that $|\mathbf{u}| = |\mathbf{v}|$.
- (e) Interpret the result above geometrically.

Question 12 [Proof of the compound-angle formula expansion]

Let \mathbf{u} and \mathbf{v} be vectors on the unit circle with angles α and β respectively from the positive x -axis.

- Show that $\mathbf{u} \cdot \mathbf{v} = \cos(\alpha - \beta)$.
- Explain why $\mathbf{u} = \langle \cos \alpha, \sin \alpha \rangle$, and write down a similar expression for \mathbf{v} .
- Deduce that $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$.

Challenge Problems**Problem 1** [Constructing an angle bisector from two vectors]

Let \mathbf{u} and \mathbf{v} be non-zero vectors. Define the vector

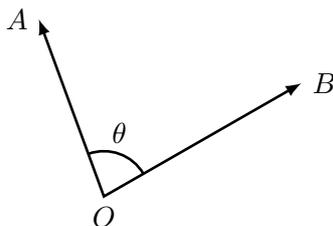
$$\mathbf{w} = |\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}$$

Let α be the angle between \mathbf{u} and \mathbf{w} , and let β be the angle between \mathbf{v} and \mathbf{w} .

- Show that $\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}|(\mathbf{u} \cdot \mathbf{v} + |\mathbf{u}||\mathbf{v}|)$
- Hence, show that $|\mathbf{w}| \cos \alpha = \mathbf{u} \cdot \mathbf{v} + |\mathbf{u}||\mathbf{v}|$
- What result would you expect if you instead started from $\mathbf{v} \cdot \mathbf{w}$?
- Deduce that $\alpha = \beta$.
- Describe the geometric significance of this result.

Problem 2 [Proof of the cosine formula for the dot product]

The diagram below shows two vectors \overrightarrow{OA} and \overrightarrow{OB} with vector representations $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ respectively.



Let the angle between the two vectors be θ .

- Write down the representation of \overrightarrow{BA} in terms of \mathbf{a} and \mathbf{b} .
- Explain briefly why

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta = |\mathbf{a} - \mathbf{b}|^2$$

- Express $|\mathbf{a} - \mathbf{b}|^2$ as a dot product and expand it.
- Hence, prove that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$.

Problem 3 [Cauchy-Schwarz inequality]

Let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ be vectors, where a, b, c and d are any real numbers.

- (a) Write down the cosine formula for $\mathbf{u} \cdot \mathbf{v}$.
- (b) Hence, show that $(\mathbf{u} \cdot \mathbf{v})^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2$, and state when equality occurs.
- (c) Deduce that

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$$

and state when equality occurs.

Problem 4 [Parallelogram Law]

The *Parallelogram Law* states that if \mathbf{a} and \mathbf{b} are vectors, then

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

- (a) Describe the geometric meaning of this result.
- (b) Prove the Parallelogram Law.



Exercise 2E

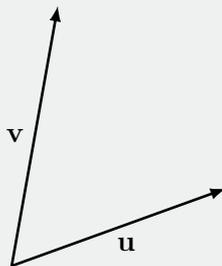
Vector projections



Fundamentals

Fundamentals 1

The diagram below shows vectors \mathbf{u} and \mathbf{v} .



Copy the diagram into your book, and draw the

- (a) vector projection of \mathbf{v} onto \mathbf{u} . (b) vector projection of \mathbf{u} onto \mathbf{v} .

Fundamentals 2

Let \mathbf{u} and \mathbf{v} be non-zero vectors. Write down the formula for the

- (a) scalar projection of \mathbf{u} onto \mathbf{v} . (b) vector projection of \mathbf{u} onto \mathbf{v} .

Question 1 Define $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle 3, 1 \rangle$. Find the

- (a) scalar projection of \mathbf{u} onto \mathbf{v} . (b) vector projection of \mathbf{u} onto \mathbf{v} .
 (c) scalar projection of \mathbf{v} onto \mathbf{u} . (d) vector projection of \mathbf{v} onto \mathbf{u} .

Question 2 Define $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 2, 1 \rangle$. Find the

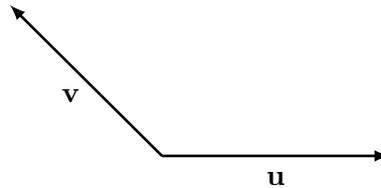
- (a) scalar projection of \mathbf{u} onto \mathbf{v} . (b) vector projection of \mathbf{u} onto \mathbf{v} .
 (c) scalar projection of \mathbf{v} onto \mathbf{u} . (d) vector projection of \mathbf{v} onto \mathbf{u} .

Question 3 For the following pairs of vectors \mathbf{u} and \mathbf{v} , find the vector projection of \mathbf{u} onto \mathbf{v} .

- (a) $\mathbf{u} = \underline{i} + 4\underline{j}$, $\mathbf{v} = 3\underline{i} + 6\underline{j}$ (b) $\mathbf{u} = -3\underline{i} + \underline{j}$, $\mathbf{v} = 2\underline{i} + 4\underline{j}$
 (c) $\mathbf{u} = -4\underline{i} + 3\underline{j}$, $\mathbf{v} = -2\underline{i} - \underline{j}$ (d) $\mathbf{u} = 2\underline{i} - 5\underline{j}$, $\mathbf{v} = 4\underline{i} - 3\underline{j}$

Question 4 Repeat the above question, but for the vector projection of \mathbf{v} onto \mathbf{u} .

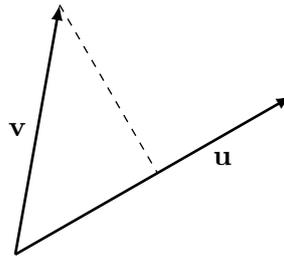
Question 5 The diagram below shows two vectors \mathbf{u} and \mathbf{v} . Draw the



- (a) vector projection of \mathbf{u} onto \mathbf{v} . (b) vector projection of \mathbf{v} onto \mathbf{u} .

Question 6 [Proof of the vector projection]

Let \mathbf{u} and \mathbf{v} be non-zero vectors with angle θ between them, as shown in the diagram below.



- (a) Complete the following formula.

$$\mathbf{u} \cdot \mathbf{v} = \text{---} \text{---} \cos \theta$$

- (b) State the significance of $|\mathbf{v}| \cos \theta$.

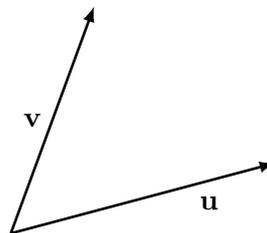
- (c) Deduce that the scalar projection of \mathbf{v} onto \mathbf{u} is $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$.

- (d) Write down an expression for the unit vector pointing in the direction of \mathbf{u} .

- (e) Deduce that the vector projection of \mathbf{v} onto \mathbf{u} is $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$.

Question 7 [Orthogonal projection]

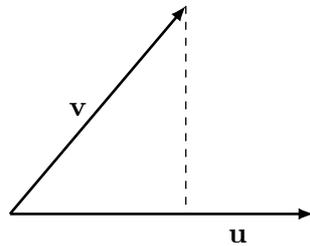
Let \mathbf{u} and \mathbf{v} be non-zero vectors, as shown in the diagram below.



- (a) Describe the significance of the vector $\mathbf{w} = \mathbf{v} - \text{proj}_{\mathbf{u}} \mathbf{v}$.
- (b) Calculate $\mathbf{u} \cdot \mathbf{w}$. Was this result a surprise? Explain.
- (c) Look up the word *orthogonal* and explain why \mathbf{w} is called the *orthogonal projection* of \mathbf{v} onto \mathbf{u} .

Question 8 [Area of a triangle]

Let \mathbf{u} and \mathbf{v} be non-zero vectors, as shown in the diagram below.



- (a) Show that the area of the triangle enclosed by \mathbf{u} , \mathbf{v} , and the perpendicular line from the tip of \mathbf{v} onto \mathbf{u} , is given by

$$A = \frac{1}{2} |\text{scal}_{\mathbf{u}} \mathbf{v}| |\mathbf{v} - \text{proj}_{\mathbf{u}} \mathbf{v}|$$

Hint: Use the result from the previous question.

- (b) Suppose $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Calculate the area of the triangle formed by \mathbf{u} , \mathbf{v} , and the orthogonal projection of \mathbf{v} onto \mathbf{u} .
- (c) Consider instead the area of the triangle formed by \mathbf{u} , \mathbf{v} , and the orthogonal projection of \mathbf{u} onto \mathbf{v} (other way around). Calculate this area. Did you get the same result? Explain.

Question 9 Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ a \end{bmatrix}$.

- (a) Show that the orthogonal projection of \mathbf{v} onto \mathbf{u} is

$$\left\langle b \left(\frac{b^2 - a^2}{a^2 + b^2} \right), a \left(\frac{a^2 - b^2}{a^2 + b^2} \right) \right\rangle$$

- (b) Hence, show that the area of the triangle formed by \mathbf{u} , \mathbf{v} , and the orthogonal projection of \mathbf{v} onto \mathbf{u} is

$$A = ab \left| \frac{a^2 - b^2}{a^2 + b^2} \right|$$

- (c) By looking at your previous result algebraically, explain why if we instead use the orthogonal projection of \mathbf{u} onto \mathbf{v} , we would expect to get the same thing.
- (d) By thinking about the bigger picture and how these vectors look with respect to each other, explain how we can expect the areas to be the same.

⚙️ Challenge Problems

Problem 1 Let \mathbf{u} and \mathbf{v} be non-zero vectors. Find the conditions for which the following are true, and prove them.

(a) $\text{scal}_{\mathbf{u}} \mathbf{v} = \text{scal}_{\mathbf{v}} \mathbf{u}$

(b) $\text{proj}_{\mathbf{u}} \mathbf{v} = \text{proj}_{\mathbf{v}} \mathbf{u}$

Problem 2 Let $\mathbf{u} = \langle 3, 4 \rangle$. Find a possible vector \mathbf{v} such that $\text{scal}_{\mathbf{u}} \mathbf{v} = 2$.

Problem 3 [Equivalent expressions of the vector projection]

Let $\text{proj}_{\mathbf{u}} \mathbf{v}$ be the vector projection of \mathbf{v} onto \mathbf{u} . By starting with the standard formula for $\text{proj}_{\mathbf{u}} \mathbf{v}$, show that it can also be expressed as $(\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}$.



Exercise 2F

Proofs using vectors



Fundamentals

Fundamentals 1

If vectors \vec{AB} and \vec{CD} are

- parallel, then $\vec{AB} = k\vec{CD}$, where k is a constant.
- perpendicular, then $\underline{\quad} \cdot \underline{\quad} = 0$.
- equal, then \vec{AB} is parallel to \vec{CD} , and $|\vec{AB}| = |\vec{CD}|$.

Fundamentals 2

Describe in general the approach to prove the following statements.

- Two vectors \vec{AB} and \vec{CD} are perpendicular.
- Two vectors \vec{AB} and \vec{CD} are parallel.
- Three points A , B and C are collinear.
- A point M is the midpoint of AB .
- Two vectors \vec{AB} and \vec{CD} are equal in length.
- Two intervals AB and CD bisect each other.

Fundamentals 3

Complete the following properties of the dot product.

- $\mathbf{a} \cdot \mathbf{a} = \underline{\quad}$
- $\mathbf{a} \cdot \mathbf{b} = \underline{\quad} \cdot \underline{\quad}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \underline{\quad}$

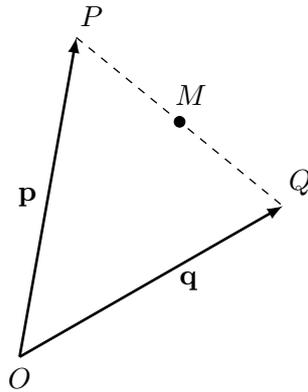
Fundamentals 4

- If $\vec{AB} = k\vec{BC}$ for some constant k , then the points A , B and C are collinear.
- A vector \vec{AB} can be expressed in terms of the parallel vectors \mathbf{a} and \mathbf{b} by using

$$\vec{AB} = \mathbf{b} - \mathbf{a}$$

Note: For the sake of brevity, the remainder of this exercise will assume that unless otherwise stated, any point P will have position vector $\mathbf{p} = \vec{OP}$, and similarly for other letters.

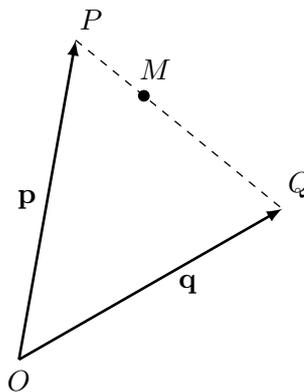
Question 1 The diagram below shows vectors $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$ on the xy -plane. Let M be the midpoint of the interval PQ .



- (a) Complete the following proof of the fact that $\overrightarrow{OM} = \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$.

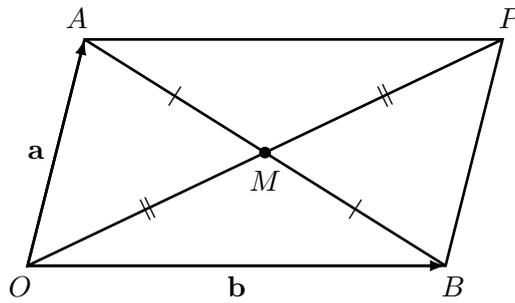
$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP} + \text{---} \\ &= \overrightarrow{OP} + \frac{1}{2}\text{---} \\ &= \mathbf{p} + \frac{1}{2}(\text{---}) \\ &= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}\end{aligned}$$

- (b) Now, suppose that M instead divided the interval PQ in the ratio $1 : 2$, as shown in the diagram below.



Use a similar proof to (a) to show that $\overrightarrow{OM} = \frac{2}{3}\mathbf{p} + \frac{1}{3}\mathbf{q}$

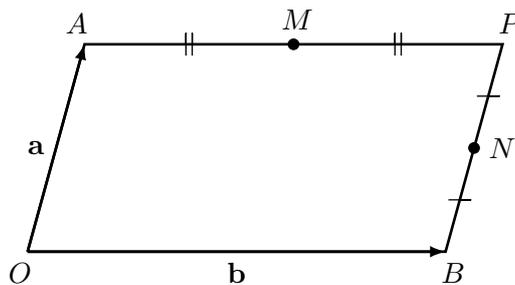
Question 2 Consider the diagram below.



Find the following in terms of \mathbf{a} and \mathbf{b} .

- (a) \overrightarrow{OM} (b) \overrightarrow{MB}

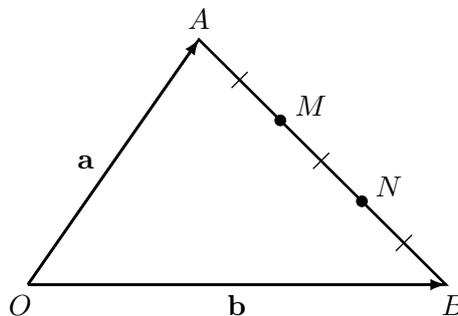
Question 3 Consider the diagram below.



Find the following in terms of \mathbf{a} and \mathbf{b} .

- (a) \overrightarrow{OM} (b) \overrightarrow{MN} (c) \overrightarrow{ON} (d) \overrightarrow{MB}

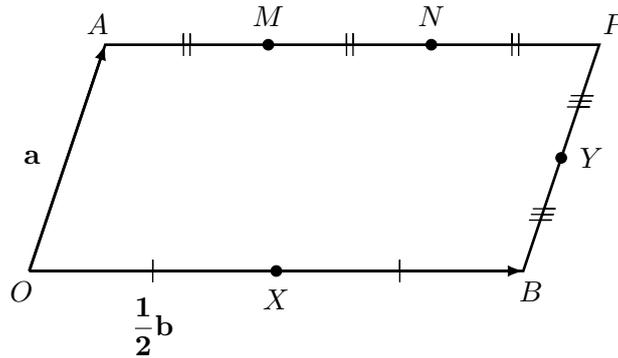
Question 4 Consider the diagram below.



Find the following in terms of \mathbf{a} and \mathbf{b} .

- (a) \overrightarrow{OM} (b) \overrightarrow{ON}

Question 5 Consider the diagram below.



Find the following in terms of **a** and **b**.

- (a) \overrightarrow{OM} (b) \overrightarrow{ON} (c) \overrightarrow{OY} (d) \overrightarrow{MY} (e) \overrightarrow{MB}
 (f) \overrightarrow{MX} (g) \overrightarrow{NY} (h) \overrightarrow{NB} (i) \overrightarrow{NX} (j) \overrightarrow{XY}

Question 6 [Weighted point]

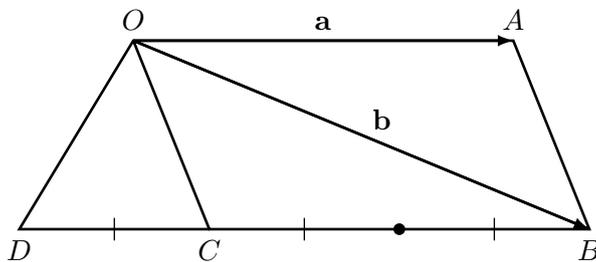
The diagram below shows a vector \overrightarrow{PQ} and a point X that divides PQ in the ratio $m : n$.



Let $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$. Prove that

$$\overrightarrow{OX} = \frac{n\mathbf{p} + m\mathbf{q}}{m + n}$$

Question 7 The diagram below shows a parallelogram $OACB$ and a point D such that $DC : DB = 1 : 3$.

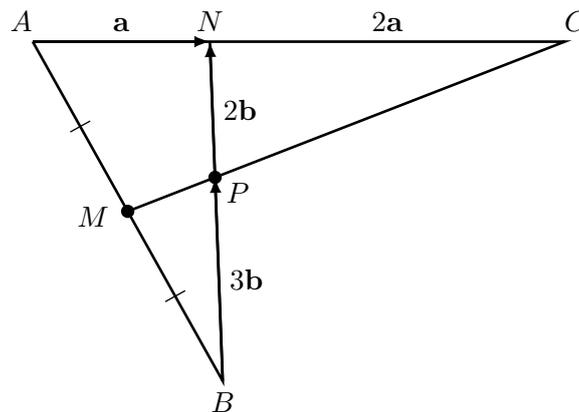


Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

- (a) Show that $\overrightarrow{DB} = \frac{3}{2}\mathbf{a}$. (b) Show that $\overrightarrow{DA} = \frac{5}{2}\mathbf{a} - \mathbf{b}$.



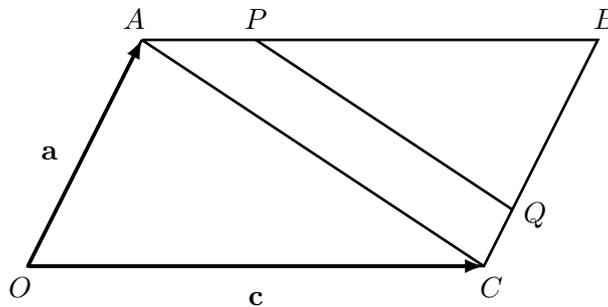
Question 8 Consider the diagram below.



Let $\overrightarrow{AN} = \mathbf{a}$, $BP : PN = 3 : 2$, and $AN : AC = 1 : 3$. Let $\overrightarrow{BP} = 3\mathbf{b}$ and M be the midpoint of AB .

- (a) Find \overrightarrow{AB} in terms of \mathbf{a} and \mathbf{b} . (b) Find \overrightarrow{MP} in terms of \mathbf{a} and \mathbf{b} .
 (c) Find \overrightarrow{PC} in terms of \mathbf{a} and \mathbf{b} . (d) Deduce that M , P and C are collinear.

Question 9 The diagram below shows a parallelogram $OABC$. The points P and Q are chosen so that $AP : PB = 1 : 3$ and similarly $CQ : QB = 1 : 3$.



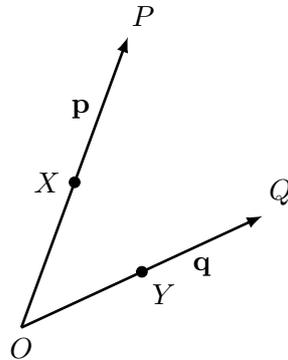
- (a) Show that $\overrightarrow{PQ} = \frac{3}{4}\mathbf{c} - \frac{3}{4}\mathbf{a}$.
 (b) Deduce that \overrightarrow{PQ} is parallel to \overrightarrow{AC} .

Question 10 Let \overrightarrow{AB} and \overrightarrow{CD} be parallel vectors that are equal in length.

- (a) Show that $\mathbf{b} - \mathbf{a} = \mathbf{d} - \mathbf{c}$.
 (b) Deduce that $\overrightarrow{AC} = \overrightarrow{BD}$.
 (c) Hence, what kind of quadrilateral is $ABCD$?

Question 11 [Classic proof in vector geometry]

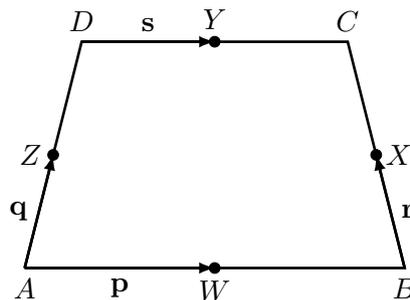
The diagram below shows vectors $\mathbf{p} = \overrightarrow{OP}$ and $\mathbf{q} = \overrightarrow{OQ}$ on the xy -plane. Let X and Y be the midpoints of the interval OP and OQ respectively.



- Write down an expression for \overrightarrow{PQ} in terms of \mathbf{p} and \mathbf{q} .
- Express \overrightarrow{OX} and \overrightarrow{OY} in terms of \mathbf{p} and \mathbf{q} respectively.
- Deduce that $\overrightarrow{XY} = \frac{1}{2}\overrightarrow{PQ}$.
- Hence, complete the following statement.

“The line joining the midpoints of two sides of a triangle is $\frac{1}{2}$ _____ to, and $\frac{1}{2}$ _____ the length of the third side”

Question 12 Let $ABCD$ be a quadrilateral, and let the midpoints of AB , BC , CD and AD be W , X , Y and Z respectively.

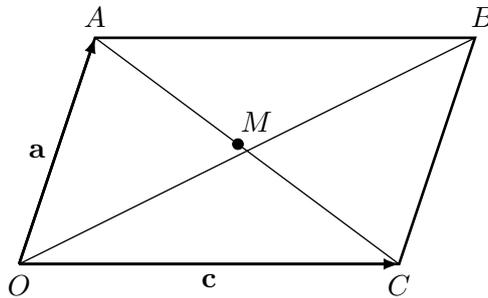


Let $\mathbf{p} = \overrightarrow{AW}$, $\mathbf{q} = \overrightarrow{AZ}$, $\mathbf{r} = \overrightarrow{BX}$, and $\mathbf{s} = \overrightarrow{DY}$

- Simplify $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$.
- Hence, show that $\mathbf{p} + \mathbf{r} = \mathbf{s} + \mathbf{q}$.
- Prove that $\overrightarrow{WX} = \overrightarrow{ZY}$.
- Deduce that $WXYZ$ forms a parallelogram.

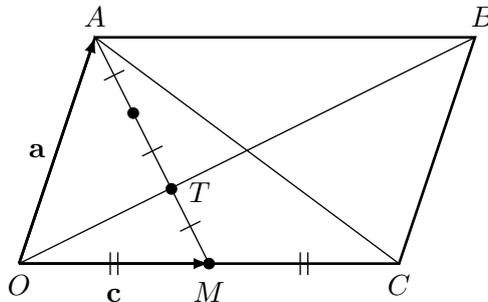
Question 13 [Diagonals of a parallelogram bisect each other]

The diagram below shows a parallelogram $OABC$.



Let M be the midpoint of AC . Since the goal is to prove that the diagonals bisect each other, we cannot *yet* say that M is the midpoint of BO too. Let $\vec{OA} = \mathbf{a}$ and $\vec{OC} = \mathbf{c}$.

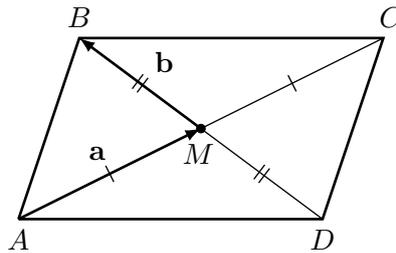
- Write down a vector expression for \vec{OM} in terms of \vec{OA} and \vec{AC} .
- Hence, show that $\vec{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$.
- Explain why this implies that the diagonals of a parallelogram bisect each other.

Question 14 The diagram below shows a parallelogram $OABC$. Let M be the midpoint of OC .

The point T lies on AM so that $AT : TM = 2 : 1$. Let $\vec{OA} = \mathbf{a}$ and $\vec{OM} = \mathbf{c}$.

- Construct a vector expression for \vec{OT} in terms of \vec{OA} and \vec{AM} .
- Hence, find \vec{OT} in terms of \mathbf{a} and \mathbf{c} .
- Explain why this implies that T lies on \vec{OB} .
Hint: Find \vec{OB} in terms of \mathbf{a} and \mathbf{c} .
- Write down the ratio $OT : TB$.

Question 15 The diagram below shows a quadrilateral $ABCD$ with diagonals bisecting each other.

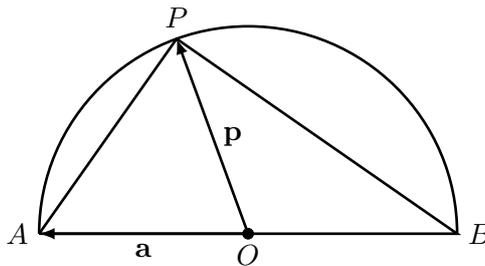


Let $\overrightarrow{AM} = \mathbf{a}$ and $\overrightarrow{MB} = \mathbf{b}$.

- Write down an expression for \overrightarrow{AB} in terms of \mathbf{a} and \mathbf{b} .
- Write down an expression for \overrightarrow{DC} in terms of \mathbf{a} and \mathbf{b} .
- Explain the significance of your result.

Question 16 [Thales' Theorem]

Let P be any point on the circumference of a semi-circle with centre O and diameter AB , as shown in the diagram below.



Let $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{p} = \overrightarrow{OP}$.

- Find $\overrightarrow{AP} \cdot \overrightarrow{BP}$.
- Deduce that $\angle APB = \frac{\pi}{2}$.

Question 17 [Area of a parallelogram]

Define position vectors $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$. Let θ be the angle between \mathbf{u} and \mathbf{v} .

- Show that the area of the parallelogram traced out by \mathbf{u} and \mathbf{v} is $|\mathbf{u}||\mathbf{v}|\sin\theta$.
- Show that

$$\cos\theta = \frac{ac + bd}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}}$$

- Show that

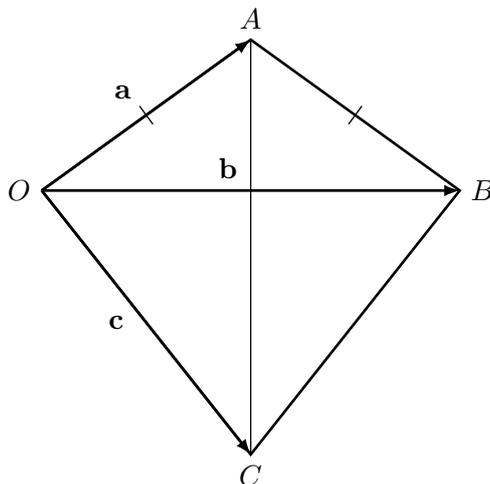
$$\sin\theta = \frac{|ad - bc|}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}}$$

- Deduce that the area of the parallelogram is $|ad - bc|$.

Question 18 Consider a rhombus $OABC$ with vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Prove that the diagonals are perpendicular.

Question 19 [Diagonals of a kite are perpendicular]

The diagram below shows a kite $OABC$.



(a) Explain why

$$\mathbf{a} \cdot \mathbf{a} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$$

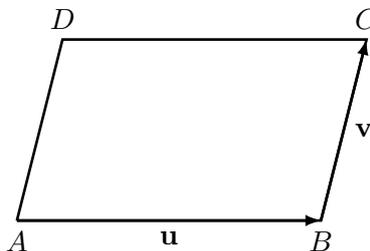
(b) Hence, show that $\mathbf{b} \cdot \mathbf{b} = 2(\mathbf{a} \cdot \mathbf{b})$.

(c) Similarly, show that $\mathbf{b} \cdot \mathbf{b} = 2(\mathbf{b} \cdot \mathbf{c})$.

(d) Deduce that \overrightarrow{OB} is perpendicular to \overrightarrow{AC} .

Question 20 [Parallelogram law]

Let $ABCD$ be a parallelogram.



Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$.

(a) Find vector expressions for \overrightarrow{AC} and \overrightarrow{BD} in terms of \mathbf{u} and \mathbf{v} .

(b) Hence, show that

$$|AC|^2 + |BD|^2 = 2|AB|^2 + 2|CD|^2$$

Question 21 ['If and only if' statements]

Let \mathbf{x} and \mathbf{y} be any two non-zero vectors in general.

(a) Prove that

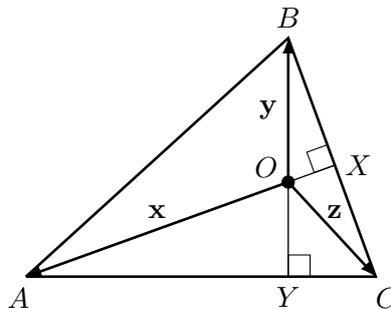
$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{4}(|\mathbf{x} + \mathbf{y}|^2 - |\mathbf{x} - \mathbf{y}|^2)$$

Hint: $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$

- (b) Consider the parallelogram traced out by \mathbf{x} and \mathbf{y} . Use the identity above to show that if the diagonals are of equal length, then the parallelogram is actually a rectangle.
- (c) Prove the converse of the statement above. That is, if the parallelogram is actually a rectangle, then the diagonals are of equal length.

Question 22 [Altitudes of a triangle are concurrent]

The diagram below shows a triangle ABC with altitudes AX and BY that intersect at O .



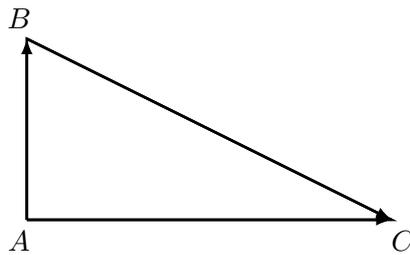
Let $\vec{OA} = \mathbf{x}$, $\vec{OB} = \mathbf{y}$, and $\vec{OC} = \mathbf{z}$

- (a) Show that $\vec{AC} = \mathbf{z} - \mathbf{x}$.
- (b) Write down a similar result for \vec{BC} .
- (c) By considering two dot products, show that

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z}$$

- (d) Deduce that \vec{OC} is perpendicular to \vec{AB} .
- (e) Explain why this implies that the altitudes are concurrent.

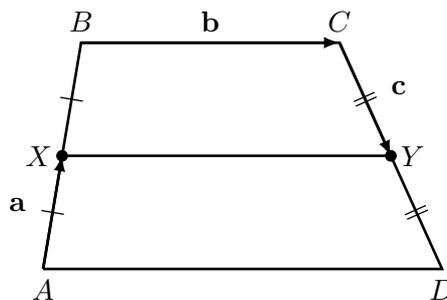
Question 23 The diagram below shows a triangle ABC .



- (a) Prove that if $|\vec{AB}|^2 + |\vec{AC}|^2 = |\vec{BC}|^2$, then the triangle is right-angled.
 (b) Prove that if the triangle is right-angled, then

$$|\vec{AB}|^2 + |\vec{AC}|^2 = |\vec{BC}|^2$$

Question 24 Let $ABCD$ be a trapezium, and let the midpoints of AB and DC be X and Y respectively.

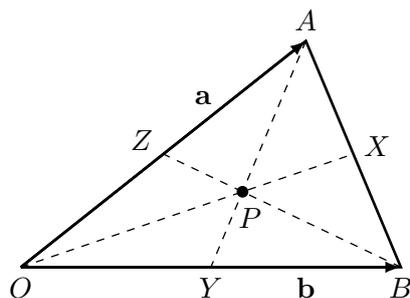


Suppose that $\vec{AD} = k\vec{BC}$. Let $\vec{AX} = \mathbf{a}$, $\vec{BC} = \mathbf{b}$, and $\vec{CY} = \mathbf{c}$.

- (a) Show that $\vec{XY} = \left(\frac{k+1}{2}\right)\vec{BC}$.
 (b) Hence, what can we say about \vec{XY} in relation to \vec{AD} and \vec{BC} .
 (c) If $\vec{AD} = 2\vec{BC}$, how much longer is XY compared to BC ?

Question 25 [Special case of the centroid]

The diagram below shows a triangle OAB where X , Y and Z are the midpoints of AB , OB and OA respectively.



Let P be the intersection of \overrightarrow{OX} and \overrightarrow{BZ} . Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Suppose that $\overrightarrow{OP} = k\overrightarrow{OX}$ for some constant k . By symmetry, we should also have $\overrightarrow{BP} = k\overrightarrow{BZ}$ for the same constant k .

- (a) Show that $\overrightarrow{OX} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.
- (b) Show that $\overrightarrow{BZ} = \frac{1}{2}\mathbf{a} - \mathbf{b}$.
- (c) Use the fact that $\overrightarrow{BZ} = \frac{1}{2}\mathbf{a} - \mathbf{b}$ to show that $\overrightarrow{OP} = \mathbf{b} + \frac{k}{2}(\mathbf{a} - 2\mathbf{b})$.
- (d) Write down another expression for \overrightarrow{OP} .
- (e) By equating the two expressions, show that $k = \frac{2}{3}$.
- (f) Hence, show that

$$\overrightarrow{OP} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$$

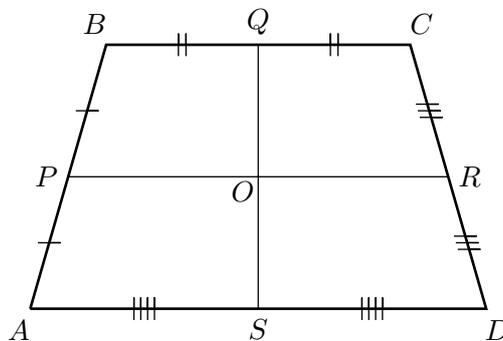
- (g) Express \overrightarrow{YP} and \overrightarrow{YA} in terms of \mathbf{a} and \mathbf{b} .
- (h) Deduce that the medians OX , BZ and AY are concurrent at P .

Challenge Problems

Problem 1 Show that the sum of the vectors from the midpoint of each side of a triangle to the opposite vertex is zero. In other words those intervals themselves form a connected triangle.

Problem 2 Let $ABCD$ be an arbitrary parallelogram. Show that the line joining A to the midpoint of CD intersects the other diagonal BD at one of the trisection points of the diagonal.

Problem 3 Prove that the lines joining midpoints of opposite sides of a quadrilateral bisect each other.



Exercise 2G

Displacement and velocity



Fundamentals

Fundamentals 1

- (a) The d _____ of a particle is the change in position, and can be expressed as a vector. The d _____ from A to B is expressed as ____.
- (b) The v _____ of a particle is the rate of change of position with respect to time.
- (c) Both d _____ and v _____ can be expressed as v _____ since they both have m _____ and d _____.
- (d) The m _____ of a v _____ v _____ is called s _____.

Fundamentals 2

- (a) When a particle is subject to two or more velocities, we can instead subject the particle to the s _____ of those velocities. This way, we need only to deal with one vector instead of several.
- (b) The s _____ of these velocity vectors is called the r _____ velocity.
- (c) The magnitude of the r _____ velocity is called the t _____ velocity of the particle, since it is where the particle will be going overall.
- (d) The d _____ of the r _____ velocity indicates the d _____ that the particle will go overall.

Question 1 Find the displacement vector in the form $a\mathbf{i} + b\mathbf{j}$ that will move a particle from point A to point B on the plane.

- (a) $A(1, 2), B(5, 7)$ (b) $A(-3, 4), B(2, -1)$ (c) $A(-2, -6), B(-4, -3)$

Question 2 Find the speed of the particle with the following velocity vectors.

- (a) $3\mathbf{i} + 4\mathbf{j} \text{ ms}^{-1}$ (b) $-5\mathbf{i} + 12\mathbf{j} \text{ ms}^{-1}$ (c) $2\mathbf{i} + 2\mathbf{j} \text{ ms}^{-1}$

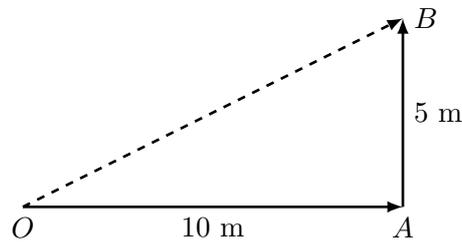
Question 3 A particle has position vector $\mathbf{i} + 2\mathbf{j}$ and constant velocity vector $3\mathbf{i} + 4\mathbf{j}$. Find the position vector after

- (a) one second. (b) two seconds. (c) t seconds.

Question 4 A particle has position vector $-2\mathbf{i} + 3\mathbf{j}$ and constant velocity vector $5\mathbf{i} + 12\mathbf{j} \text{ ms}^{-1}$. Find the particle's distance from the origin after

- (a) one second. (b) two seconds. (c) t seconds.

Question 5 Bob is standing at point O . He runs 10 metres east to A , then 5 metres north to B , and completes the journey in 5 seconds.



- Calculate the distance that Bob is from O .
- Find the bearing of Bob from O .
- Complete the following statement about Bob's displacement.

Bob's displacement is _____ metres from O , in the direction of _____ $^{\circ}$ T

- Write down the total distance travelled by Bob, and hence write down Bob's average *speed*.
- Calculate Bob's average *velocity*.
- Complete the following statement about Bob's speed and velocity.

Bob travelled a total of ___ metres in ___ seconds, so his average *speed* is ___ metres per second. However, overall his displacement was _____ in the span of ___ seconds, so his average *velocity* is ___ metres per second.

Question 6 Use a similar technique to the previous question to find the displacement, average *speed* and average *velocity* in each of the scenarios below. Bob travels

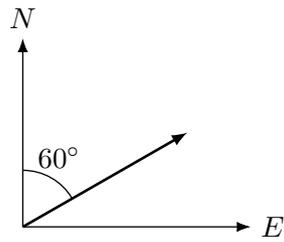
- 40 metres east, then 30 metres south in 10 seconds.
- 30 metres north, then 15 metres east in 5 seconds.
- 20 metres east, then 10 metres south, then 20 metres east in 20 seconds.
- 15 metres south, then 20 metres east, then 5 metres north in 5 seconds.

Question 7 [Drills necessary for this exercise]

Find the magnitude and direction (expressed as a true bearing) of the following vectors.

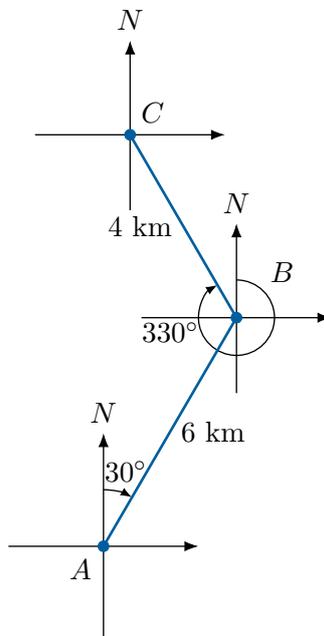
- | | | | |
|---------------------------------------|--|--|--|
| (a) \mathbf{i} | (b) $2\mathbf{j}$ | (c) $-4\mathbf{i}$ | (d) $-3\mathbf{j}$ |
| (e) $2\mathbf{i} + 2\mathbf{j}$ | (f) $4\mathbf{i} - 4\mathbf{j}$ | (g) $-5\mathbf{i} + 5\mathbf{j}$ | (h) $-3\mathbf{i} - 3\mathbf{j}$ |
| (i) $\mathbf{i} + \sqrt{3}\mathbf{j}$ | (j) $\mathbf{i} - \sqrt{3}\mathbf{j}$ | (k) $-\mathbf{i} + \sqrt{3}\mathbf{j}$ | (l) $-\mathbf{i} - \sqrt{3}\mathbf{j}$ |
| (m) $\sqrt{3}\mathbf{i} + \mathbf{j}$ | (n) $-\sqrt{3}\mathbf{i} + \mathbf{j}$ | (o) $\sqrt{3}\mathbf{i} - \mathbf{j}$ | (p) $-\sqrt{3}\mathbf{i} - \mathbf{j}$ |

Question 8 Bob walks 40 metres at a bearing of $060^\circ T$.



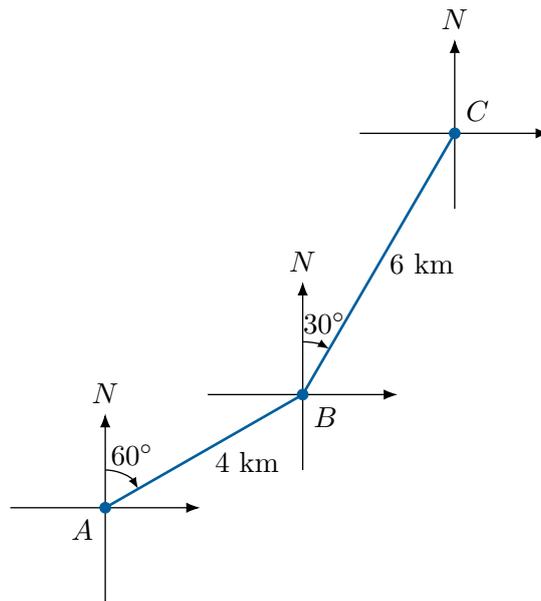
- Label the horizontal and vertical components of his motion.
- Hence, write down his displacement vector in the form $x\mathbf{i} + y\mathbf{j}$.

Question 9 Bob walks along the following route.



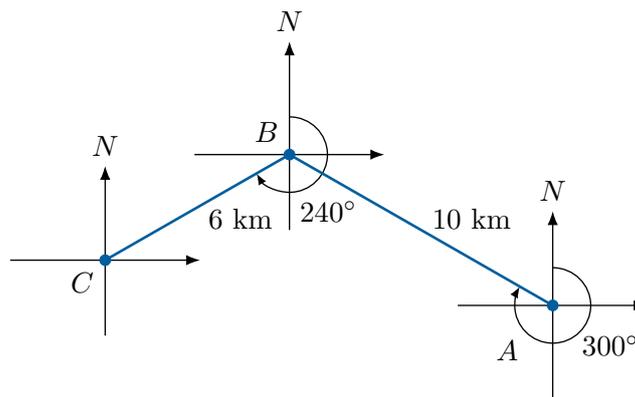
- Write down an expression for \overrightarrow{AB} in the form $x\mathbf{i} + y\mathbf{j}$.
- Write down an expression for \overrightarrow{BC} in the form $x\mathbf{i} + y\mathbf{j}$.
- Hence, find Bob's displacement vector \overrightarrow{AC} from A.

Question 10 Bob walks along the following route.



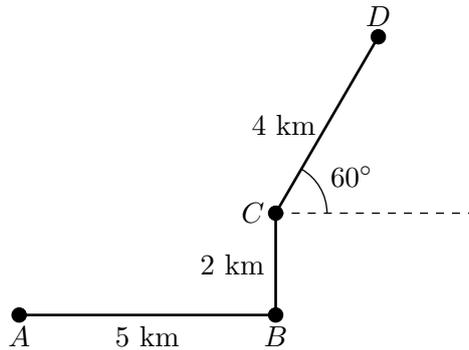
- Find the displacement vector from B to C .
- Find the displacement vector from A to C .
- Find the magnitude of this vector. What does this mean physically?
- Find the direction of this vector. What does this mean physically?
- If Bob had an average speed of 5 km h^{-1} , what is his average velocity from A to C ?

Question 11 Bob jogs along the following route.



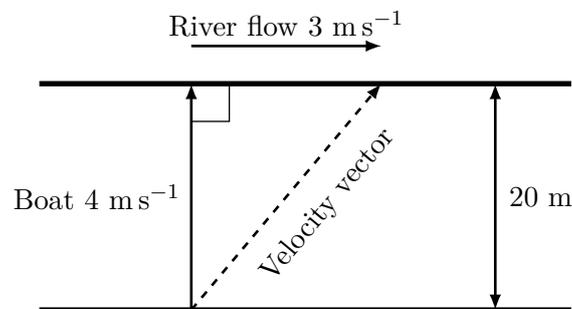
- Find Bob's displacement vector from A .
- If Bob had an average speed of 10 km h^{-1} , what is his average velocity?

Question 12 Bob runs a particular route as shown below, at a constant speed of 3 metres per second.



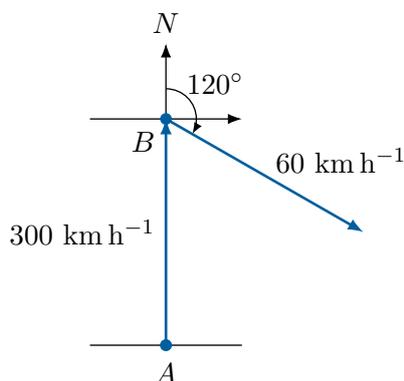
- Write down the vectors \vec{AB} , \vec{BC} and \vec{CD} in the form $x\mathbf{i} + y\mathbf{j}$.
- Hence, write down Bob's final displacement \vec{AD} in the form $x\mathbf{i} + y\mathbf{j}$.
- How long did it take for Bob to complete his journey?
- Hence, calculate Bob's average velocity.

Question 13 Matty rides a boat across a 20 metre-wide river that flows at a speed of 3 metres per second. The boat is able to travel at a constant speed of 4 metres per second in still water. He points the boat to go straight across to the other side, but drifts horizontally as a result of the flow of the river, as shown in the diagram below.



- Write down Matty's velocity vector in the form $x\mathbf{i} + y\mathbf{j}$.
- What is the magnitude of Matty's velocity vector?
- What is the direction of Matty's velocity vector, to the nearest degree?
- Hence, how far did Matty actually travel, correct to 1 decimal place?
- How long did the trip take?
- If Matty begins from $(0, 0)$, at what point will he meet the other end of the river?
- In what direction should he have pointed the boat, so that his overall trajectory was vertical?

Question 14 A plane is charted to fly north from Town A to Town B , which are 1000 kilometres apart. It normally has at a speed of 300 km h^{-1} when there is no wind, but the wind blows at a bearing of 120°T at a speed of 60 km h^{-1} .



- Draw the resultant vector.
- Use the cosine rule to find the magnitude of the resultant vector.
- Find the bearing of the resultant velocity vector.

Question 15 The same pilot from the previous question aims the plane in a north-westerly direction at some bearing $N\theta^\circ W$ so that his resultant vector is exactly north.

- Draw out the scenario.
- Find the value of θ .
- How long will the journey take?

⚙️ Challenge Problems

Problem 1 During a flood, Adam rides a boat across a 50 metre-wide river that flows east at a speed of 5 metres per second to the right. The boat is able to travel at a constant speed of 10 metres per second in still water. He points the boat to go north across to a dock to the other side, but the flow of the river will force him to drift horizontally and miss the dock.

- In what direction should he point the boat so that his overall motion is north?
- How much longer, expressed as a percentage, will it take for Adam to cross the river as a result of the flood compared to still water?

Problem 2 A plane is charted to fly at a bearing of 120° from Town A to Town B , which are 800 kilometres apart. The plane normally has at a speed of 400 km h^{-1} when there is no wind, but the wind blows at a bearing of 060°T at a speed of 40 km h^{-1} . In what direction should the pilot fly the plane to stay on track, and how long will the journey take?

Exercise 2H

Resolving forces and equilibrium



Fundamentals

Fundamentals 1

- When several forces are applied to an object, the n ___ result can be expressed as a single force.
- This single force is called the r _____ force, and it is the s ___ of all the other forces.
- The r _____ force is denoted by _____.
- If ___ = 0, then the object is said to be in e _____.

Fundamentals 2

A object in equilibrium has

- net horizontal force equal to ___.
- net vertical force equal to ___.

Fundamentals 3

- Any object with some mass m experiences g _____ acceleration denoted by g , which is approximately ___ m s^{-2} .
- Using Newton's _____ Law, which states that $\sum F = ma$, a object with mass m experiences a gravitational f _____ of ___, which is also sometimes called the w _____ force.

Fundamentals 4

- When an object is at rest on a surface, it exerts a force on the surface and due to Newton's _____ Law, the surface exerts a force back onto the object.
- This force is called the n _____ force, and it is named as such because the direction in which it acts is p _____ to the surface.

Fundamentals 5

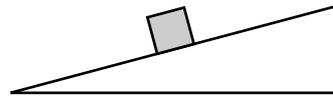
- An object of mass m hung vertically from a string has a w _____ force acting downwards.
- If the object is in equilibrium, then there must be another force acting u _____. This force is called the t _____ force, and it is exerted by the string.

Question 1 The scenarios below show an object at rest on a surface experiencing a gravitational acceleration of g . Label the weight and normal force on the diagram.

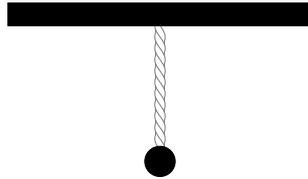
(a)



(b)

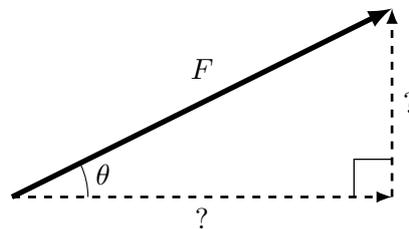


Question 2 The diagram below shows a particle of mass m suspended vertically from a ceiling by a string. The particle experiences a gravitational acceleration of g .

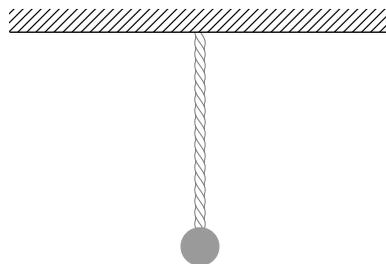


- Label all the forces in the diagram.
- What is the net vertical force?
- Hence $T = \underline{\hspace{2cm}}$.

Question 3 Label the following component vectors of F .



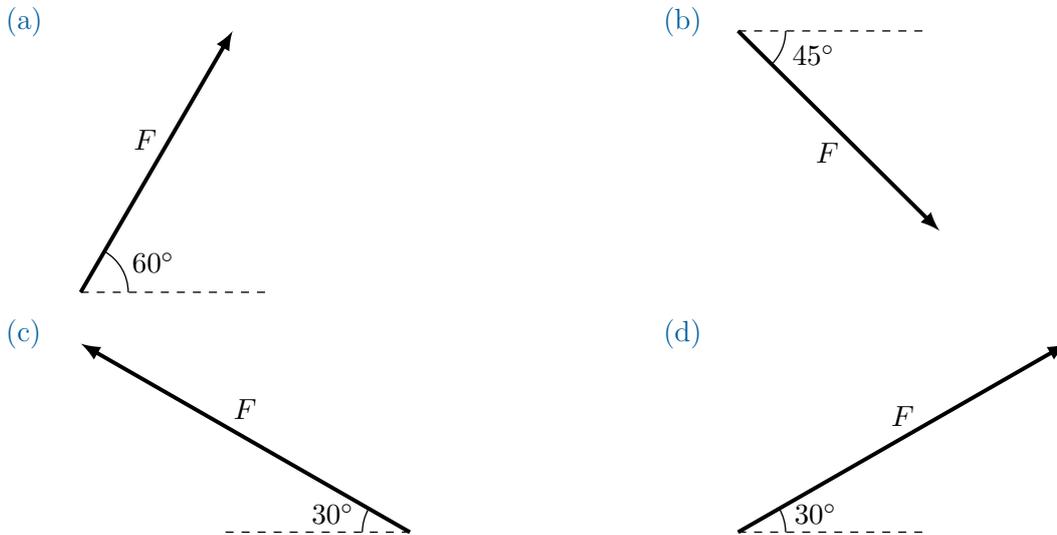
Question 4 The diagram below shows a weight suspended vertically from a ceiling by a rope with tension T . The weight experiences a downwards force of 40N due to gravity.



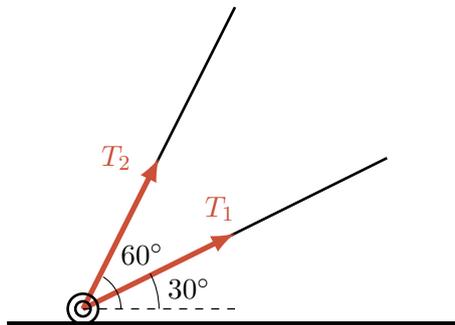
- Since the object is in e _____, the resultant force must be ____.
- In this case, the u _____ force must be balanced with d _____ force.
- Hence, write down how much tension is exerted by the rope.

Question 5 [Drills necessary for this exercise]

For each of the following, resolve the vector into horizontal and vertical components, and label them in the form $F \cos \theta$ or $F \sin \theta$.



Question 6 The diagram below shows two strings attached to an eye bracket on the ground.

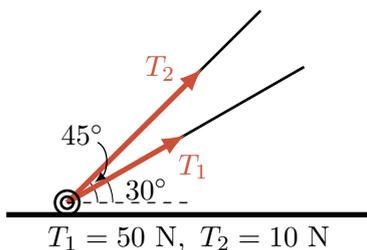


One string is inclined at an angle of 60° from the horizontal with tension force 20N, and the other string is inclined at an angle of 30° from the horizontal with tension force 40N.

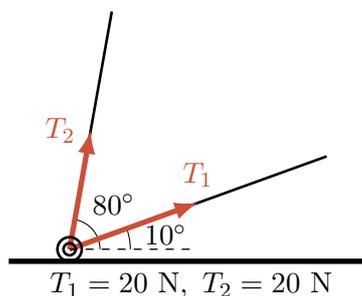
- Resolve T_1 and T_2 into horizontal and vertical component vectors.
- Find expressions for vectors T_1 and T_2 in the form $x \mathbf{i} + y \mathbf{j}$.
- Hence, find an expression for the resultant vector $\sum \mathbf{F} = T_1 + T_2$.
- Find the magnitude of $\sum \mathbf{F}$ and hence, how much force is exerted on the bracket overall.
- Find the direction of $\sum \mathbf{F}$ and hence, the direction in which the bracket is being pulled overall.

Question 7 Each diagram below shows two ropes attached to an eye bracket on the ground, and the angle of inclination of the ropes from the horizontal plane. For each scenario, calculate the resultant force $\Sigma \mathbf{F}$ and hence, the magnitude and direction in which the eye bracket is being pulled.

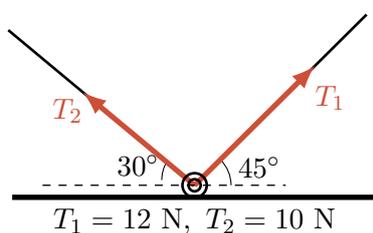
(a)



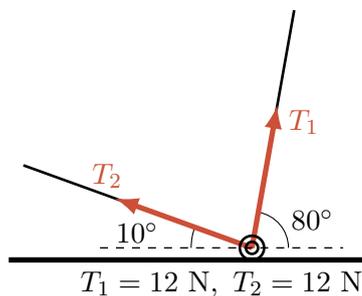
(b)



(c)



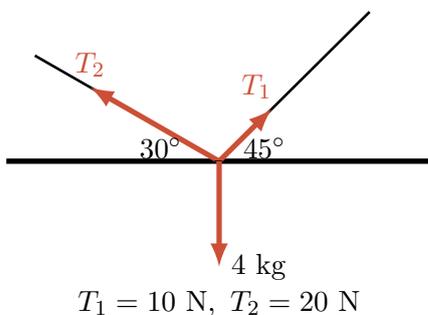
(d)



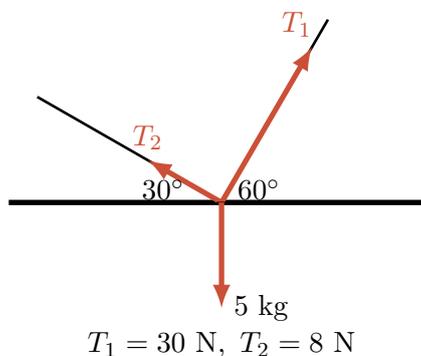
Question 8 [Resultant vector from three forces]

Find the magnitude and direction of the resultant vector $\Sigma \mathbf{F}$ in each of the following scenarios.

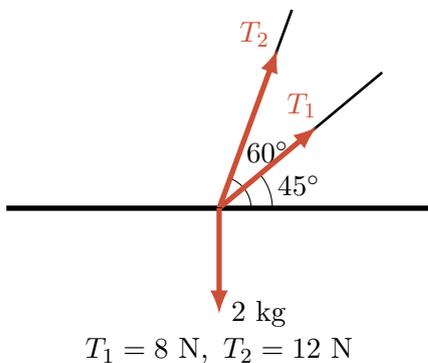
(a)



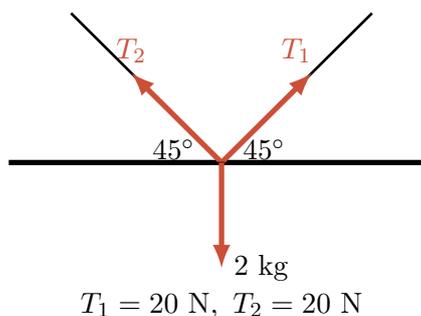
(b)



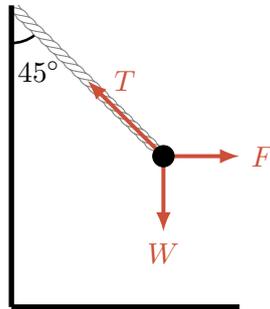
(c)



(d)

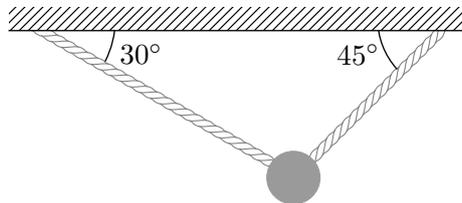


Question 9 The diagram below shows a battering ram of mass 200 kilograms suspended by a rope from a fixed point above the wall. The ram is pulled back via a horizontal force F so that the angle between the rope and the wall is 45° . Assume that acceleration due to gravity is 10 m s^{-2} .



- Find W .
- Resolve forces and hence find F and T .
- You found that $F = W$. Would this be true if the angle were something *other* than 45° ?

Question 10 The diagram below shows a 5 kilogram weight suspended from a ceiling by two ropes with tension vectors T_1 and T_2 . The ropes make angles of 30° and 45° from the horizontal respectively.



- State the magnitude of the net horizontal force $\sum \mathbf{F}_x$.
- State the magnitude of the net vertical force $\sum \mathbf{F}_y$.
- Resolve T_1 and T_2 into components.
- By considering the horizontal forces, show that

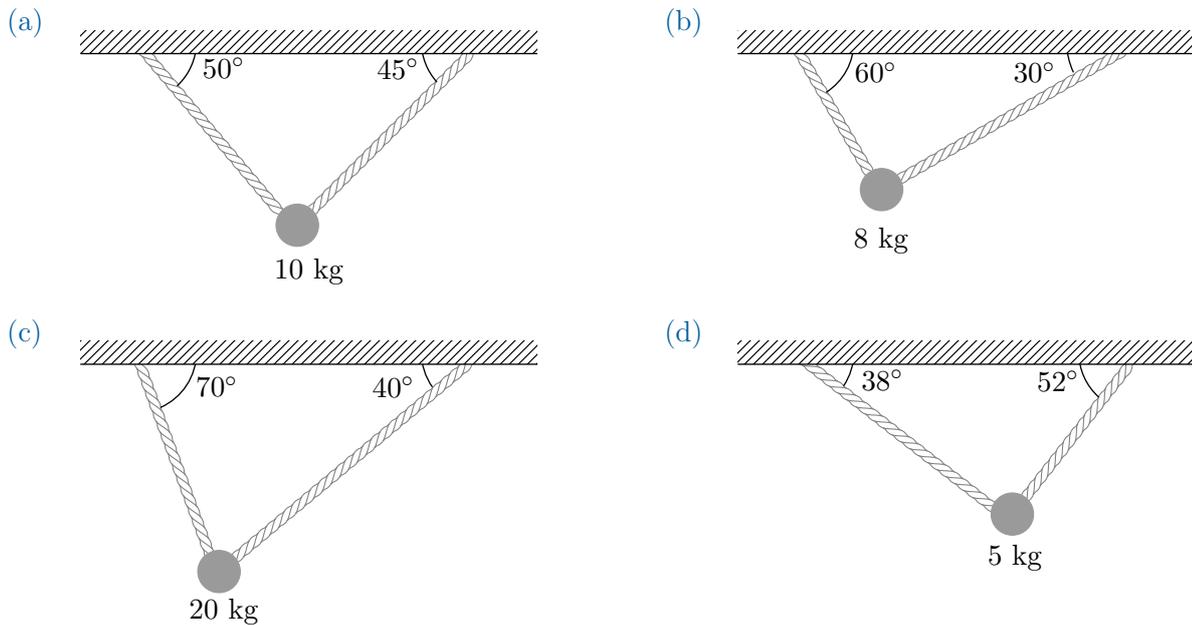
$$|T_1| \cos 30^\circ = |T_2| \cos 45^\circ$$

- By considering the vertical forces, show that

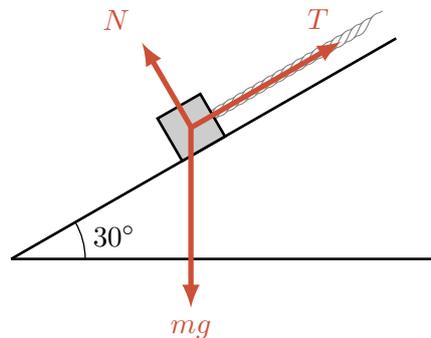
$$|T_1| \sin 30^\circ + |T_2| \sin 45^\circ = 50$$

- Hence, find $|T_1|$ and $|T_2|$.

Question 11 The diagrams below show a weight suspended from a ceiling by two ropes with tension vectors T_1 and T_2 . In each of the scenarios below, find $|T_1|$ and $|T_2|$.



Question 12 The diagram below shows an object of mass 5 kilograms on a smooth plane (no friction) inclined at an angle of 30° from the horizontal plane.



To prevent the object sliding down, a rope that experiences a tension force of T is attached to the object. The object also experiences a normal force of N .

- (a) Resolve the triangles containing T and N into horizontal and vertical components and label them.
- (b) Resolve the forces to show that

$$N \cos 30^\circ + T \sin 30^\circ = 50$$

$$N \sin 30^\circ - T \cos 30^\circ = 0$$

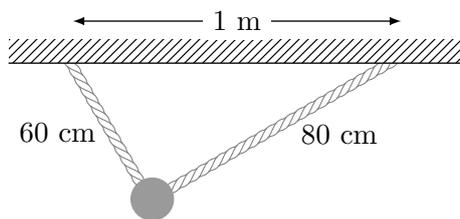
- (c) Hence, find the magnitudes of the normal and tension forces.

(a) Show that

$$\begin{aligned} 3|T_1| - 4|T_2| &= 0 \\ 4|T_1| + 3|T_2| &= 50 \end{aligned}$$

(b) Find $|T_1|$ and $|T_2|$

Question 16 The diagram below shows an object of mass 3 kilograms suspended from a ceiling by two ropes with tension vectors T_1 and T_2 with lengths 60 cm and 80 cm respectively. The ropes are attached to hooks on the ceiling that are 1 metre apart.



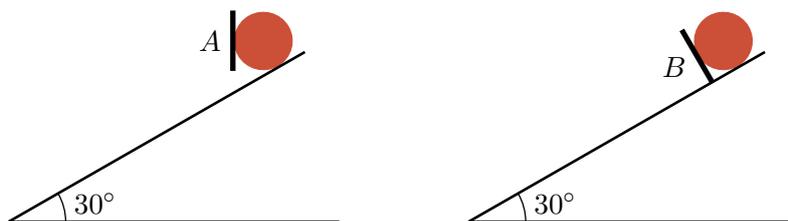
(a) Show that

$$\begin{aligned} 3|T_1| - 4|T_2| &= 0 \\ 4|T_1| + 3|T_2| &= 150 \end{aligned}$$

(b) Find $|T_1|$ and $|T_2|$

Question 17 [Another way of thinking about components]

A ball of mass 3 kilograms is placed on top of a plane inclined at an angle of 30° . A stopping pad is placed at either position A or position B , and holds it at rest.



Find the amount of force that the stopping pad experiences at

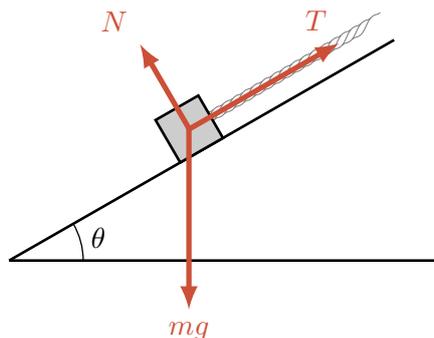
(a) position A .

(b) position B .

Challenge Problems

Note: Many of the results below seem surprisingly simple considering the techniques used to acquire them. Some readers may suspect that there may be easier methods. In fact, there are and they are explored in some of the investigation tasks at the end of this chapter.

Problem 1 The diagram below shows an object of mass m on a smooth plane inclined at some angle from the horizontal plane.

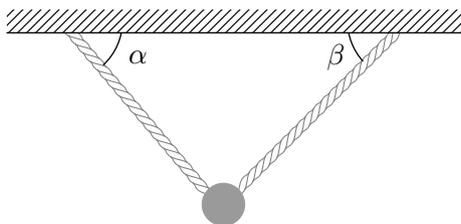


To prevent the object sliding down, a rope that experiences a tension force of T is attached to the object. The object also experiences a normal force of N . Show that

(a) $N = mg \cos \theta$

(b) $T = mg \sin \theta$

Problem 2 The diagram below shows an m kilogram object suspended from a ceiling by two ropes experiencing tensions of T_1 and T_2 . The ropes make angles of α and β from the horizontal respectively.



Show that

(a) $T_1 = \frac{mg \cos \beta}{\sin(\alpha + \beta)}$

(b) $T_2 = \frac{mg \cos \alpha}{\sin(\alpha + \beta)}$

Exercise 21

Projectile motion



Fundamentals

Fundamentals 1

The following are assumptions about projectile motion.

- (a) The particle is a single p_____.
- (b) Air resistance is n_____.
- (c) The only force acting on the particle is g_____.

Fundamentals 2

- (a) A p_____ vector is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.
- (b) A v_____ vector is given by $\mathbf{v}(t) = \text{_____}$.
- (c) An a_____ vector is given by $\mathbf{a}(t) = \text{_____}$.

Fundamentals 3

Suppose a particle is projected from the origin with some initial velocity V and initial angle θ . Describe the main steps involved in finding the

- (a) time of flight.
- (b) horizontal range.
- (c) time to reach maximum height.
- (d) maximum height.
- (e) speed at time $t = t_0$.
- (f) angle at time $t = t_0$.

Fundamentals 4

A particle is projected from the origin with initial angle θ and speed $V \text{ m s}^{-1}$. Complete the following equations of motion.

$$\mathbf{a}(t) = \text{_____}$$

$$\mathbf{v}(t) = \text{_____}$$

$$\mathbf{r}(t) = \text{_____}$$

Question 1 A particle is projected from the origin so that the position of the particle at time t is given by the position vector

$$\mathbf{r}(t) = 80t \mathbf{i} + (60t - 5t^2) \mathbf{j}$$

- Find the position vector of the particle after one second.
- Find the position vector of the particle after two seconds.
- When will the particle have a horizontal displacement of 240 metres?
- When will the particle have a vertical displacement of 160 metres?
- When will the particle land back on the ground?

Question 2 A particle is projected from the origin so that the position of the particle at time t is given by the position vector

$$\mathbf{r}(t) = 20\sqrt{3}t \mathbf{i} + (20t - 5t^2) \mathbf{j}$$

Find the following.

- Find an expression for the velocity vector $\mathbf{v}(t)$.
- Find the initial velocity vector $\mathbf{v}(0)$.
- Find $|\mathbf{v}(0)|$ and state the physical significance of this.
- Find the direction of $|\mathbf{v}(0)|$ and state the physical significance of this.

Question 3 A particle is projected from the origin so that the position of the particle at time t is given by the position vector

$$\mathbf{r}(t) = 10t \mathbf{i} + (10\sqrt{3}t - 5t^2) \mathbf{j}$$

Find the following.

- Show that it takes $t = 2\sqrt{3}$ seconds for the particle to land back on the ground.
- Find an expression for $\mathbf{v}(2\sqrt{3})$.
- Find the magnitude and direction of $\mathbf{v}(2\sqrt{3})$, and state the significance of your result. Measure your angle with respect to the horizontal axis.
- Find the initial velocity and angle, and compare it with your landing velocity and angle.

Question 4 A particle is projected from the origin so that the position of the particle at time t is given by the position vector

$$\mathbf{r}(t) = 10\sqrt{2}t \mathbf{i} + (10\sqrt{2}t - 5t^2) \mathbf{j}$$

- Find the time taken to reach the maximum height.
- Hence, find the maximum height of the particle.
- Find the time of flight.
- Compare your answer with your result in (a). What do you notice? Explain why this was to be expected.
- Find the horizontal range of the particle.

Question 5 A particle is projected from the top of a 20 metre high cliff so that the position of the particle at time t is given by the position vector

$$\mathbf{r}(t) = 10\sqrt{3}t\mathbf{i} + (10t - 5t^2 + h)\mathbf{j}$$

- Find the value of h .
- Find the time of flight, correct to 1 decimal place.
- Hence, find the speed and angle at which the particle hits the ground.

Question 6 A particle is projected from the origin with initial velocity 50 m s^{-1} and initial angle 60° from the horizontal. Assume that $g = 10 \text{ m s}^{-2}$.

- Write down the acceleration vector $\mathbf{a}(t)$.
- Derive the velocity vector $\mathbf{v}(t)$.
- Hence, derive the position vector $\mathbf{r}(t)$.

Question 7 Derive the position vectors for the particles with the following initial velocities and angles, assuming that the projection point is the origin. Assume that $g = 10 \text{ m s}^{-2}$.

- (a) $V = 30, \theta = 30^\circ$ (b) $V = 60, \theta = 45^\circ$ (c) $V = 40, \theta = 60^\circ$

Question 8 A particle is projected *horizontally* from the top of a 40 metre cliff with initial velocity 20 m s^{-1} . Assume that $g = 10 \text{ m s}^{-2}$.

- What is the initial angle of the particle?
- Hence, derive the equations of motion.
- Find the horizontal range of the particle.
- Find the speed and angle at which the particle hits the ground.

Question 9 A particle is projected from the top of a 5 metre cliff with initial velocity 40 m s^{-1} and initial angle θ such that $\tan \theta = \frac{3}{4}$. Assume that $g = 10 \text{ m s}^{-2}$.

- Find $\sin \theta$ and $\cos \theta$.
- Hence, derive the equations of motion.
- Find the horizontal range of the particle.
- Find the speed and angle at which the particle hits the ground.
- Find the maximum height of the particle above the ground.

Question 10 [Find the angle, given a velocity]

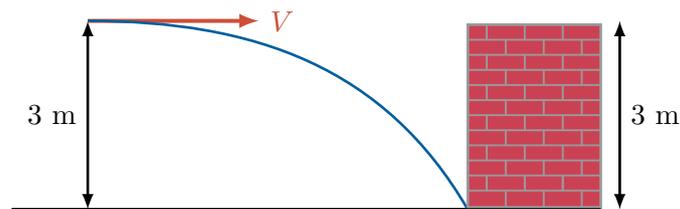
A cannon has fixed muzzle velocity of 120 m s^{-1} , and the target is 1200 metres away. Assume that acceleration due to gravity is 10 m s^{-2} .

- Derive the equations of motion.
- At what angles should the cannon be aimed from the horizontal, correct to the nearest minute, so that it hits the target?
- Can the cannon hit a target 1500 metres away?

Question 11 [Find the velocity, given an angle]

A hose is aimed at a fixed angle of 30° . What should the speed of the stream be so that the water hits a target 20 metres away? Assume that $g = 10 \text{ m s}^{-2}$.

Question 12 A water cannon is attached to a 3 metre high truck, and aimed horizontally. The target is a wall that is 10 metres away and 3 metres high. Assume that $g = 10 \text{ m s}^{-2}$.

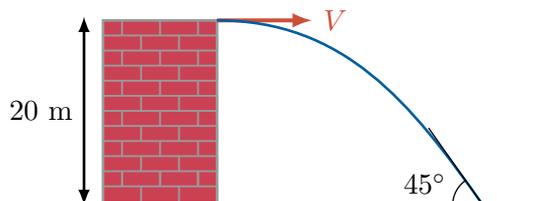


- What should the speed of the water stream be, so that the stream hits the base of the wall?
- If the speed is doubled, find how much further up the wall the stream will hit.
- How much stronger should the speed be from (a), so that the stream hits half-way up the wall?

Question 13 A plane flies horizontally at a speed of 360 km h^{-1} and elevation of 1 kilometre. Assume that $g = 10 \text{ m s}^{-2}$.

- Find the speed of the plane in metres per second.
- Derive the equations of motion.
- The plane drops a package to a stranded person so it drops exactly where they are standing. At what horizontal distance from the person should the plane drop the parcel for this to occur?
- The pilot has a monitor that measures exactly the angle of depression of the person from the position of the plane. What should the reading be on the monitor, to the nearest minute, so that the pilot knows exactly when to release the parcel?

Question 14 A particle is to be projected horizontally with initial speed $V \text{ m s}^{-1}$ from a 20 metre high building so that it lands at an angle of 45° from the horizontal. Assume that $g = 10 \text{ m s}^{-2}$.



- (a) Find the time of flight.
 (b) Show that $\mathbf{v}(2) = V \mathbf{i} - 20 \mathbf{j}$.
 (c) Hence, find the initial speed of the particle.

Question 15 A particle is projected from the origin with initial angle θ and speed $V \text{ m s}^{-1}$. The particle attains a maximal height of 20 metres and a horizontal range of $80\sqrt{3}$ metres. Find the value of V and θ . Assume that $g = 10 \text{ m s}^{-2}$.

Question 16 A particle is projected from the origin with initial angle θ and speed $V \text{ m s}^{-1}$. Two seconds after projection, the particle just clears a wall 4 metres high and 40 metres away. Find the

- (a) initial speed and angle of projection. (b) horizontal range.

Question 17 A particle is projected from the origin with initial angle 60° and speed 20 m s^{-1} . After T seconds, the particle is travelling in a direction with angle 30° . Find the value of T . Assume that $g = 10 \text{ m s}^{-2}$.

Question 18 A particle is projected from the origin with initial speed $V \text{ m s}^{-1}$ and initial angle θ . As a result, the particle attains a maximal range of R_V . Bob claims that if the initial speed is doubled, the maximal range also doubles. Let g be the acceleration due to gravity.

- (a) Find R_V in terms of V and θ .
 (b) Write down an expression for the horizontal range if the velocity were doubled instead, and deduce whether Bob's answer is correct or incorrect. If his statement is incorrect, then write down the correct statement instead.

Question 19 [General results for horizontal initial angle]

A particle is projected horizontally from a cliff of height h metres with initial speed $V \text{ m s}^{-1}$. Let g be the acceleration due to gravity. Show that

- (a) the time of flight is $T = \sqrt{\frac{2h}{g}}$.
 (b) the horizontal range is $R = V\sqrt{\frac{2h}{g}}$.
 (c) the angle from the horizontal at which it hits the ground is given by $\tan \alpha = -\frac{\sqrt{2gh}}{V}$.
 (d) the speed at which it hits the ground is $\sqrt{V^2 + 2gh}$.

Question 20 [Check your understanding]

Three particles are projected horizontally from vertical cliff of height h . Particle A is dropped, particle B is projected at 10 m s^{-1} and particle C is projected at 20 m s^{-1} . What can we say about the time of flight of each particle? Assume that $g = 10 \text{ m s}^{-2}$.

Question 21 [General results]

Prove each of the following results about a particle that is projected from the origin with initial speed $V \text{ m s}^{-1}$ and initial angle θ . Let g be the acceleration due to gravity.

- (a) The time of flight is $T = \frac{2V \sin \theta}{g}$.
- (b) The horizontal range is $R = \frac{V^2 \sin 2\theta}{g}$.
- (c) The time to reach maximum height is $T = \frac{V \sin \theta}{g}$.
- (d) The maximum height is $H = \frac{V^2 \sin^2 \theta}{2g}$.

Question 22 [More proofs]

Prove the following facts using any of the results from the previous question.

- (a) The maximum possible horizontal range is attained when $\theta = 45^\circ$.
- (b) The maximum possible height, if the angle can vary, is $\frac{V^2}{2g}$.
- (c) If a target on the ground can be hit with initial angle θ , then the same target can also be hit using angle $90^\circ - \theta$.

Question 23 A particle is projected from the origin with initial angle θ and speed V such that the horizontal range is equal to the maximum height. Let g be the acceleration due to gravity.

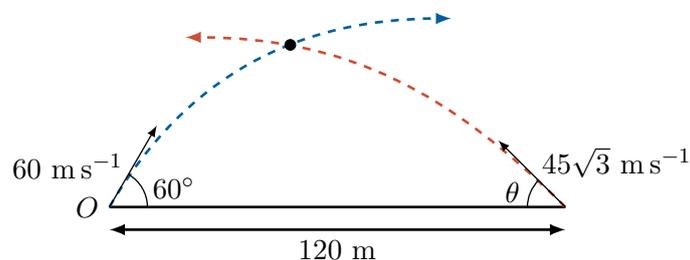
- (a) Find the maximum height.
- (b) Find the horizontal range.
- (c) Hence, show that $\theta = \tan^{-1}(4)$.
- (d) What can we say about initial speed for the condition to occur?

Question 24 A particle is projected from the origin with initial speed $V \text{ m s}^{-1}$ and initial angle θ . As a result, the particle attains a maximal range of R . Let the time of flight be T . Let g be the acceleration due to gravity. Show that

$$T^2 = \frac{2R}{g} \tan \theta$$

Question 25 [Collision problem]

A particle P is projected from a point O with an initial velocity of 60 m s^{-1} at an angle of 60° from the horizontal. At the same instant, a second particle Q is projected in the opposite direction with an initial speed of $45\sqrt{3} \text{ m s}^{-1}$ from the same horizontal plane 120 metres away from O .



After T seconds, the two particles collide. Assume that $g = 10 \text{ m s}^{-2}$.

- (a) Show that the equations of motion of each particle is

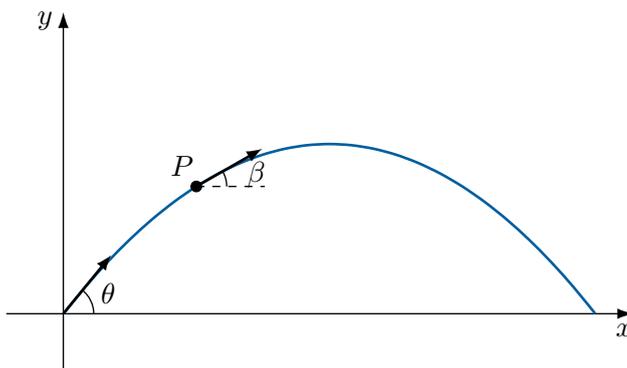
$$\text{Particle } P \quad \mathbf{r}(t) = 30t \mathbf{i} + (30t\sqrt{3} - 5t^2) \mathbf{j}$$

$$\text{Particle } Q \quad \mathbf{r}(t) = 45t\sqrt{3} \cos \theta \mathbf{i} + (45t\sqrt{3} \sin \theta - 5t^2) \mathbf{j}$$

- (b) Find the angle of projection of particle Q .
 (c) Find the time at which the particles collide.

Challenge Problems

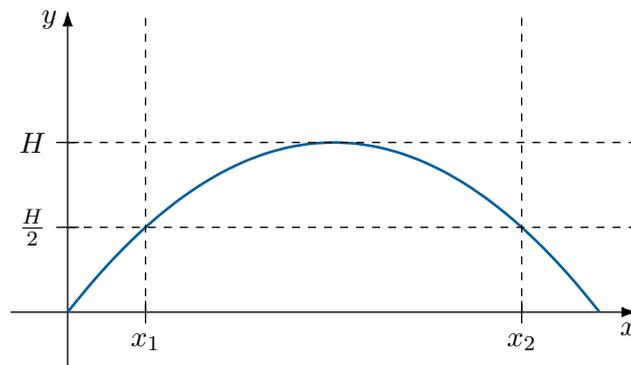
Problem 1 A particle is projected from the origin with initial speed $V \text{ m s}^{-1}$ and initial angle θ . When the particle reaches a point P on the upwards flight, the direction of flight makes an angle of β with the horizontal, as shown in the diagram below. Let g be the acceleration due to gravity.



Show that the time taken to travel to P is $T = \frac{V \sin(\theta - \beta)}{g \cos \beta}$.

Problem 2 [Application of sum and product of roots]

A particle is projected from the origin with initial speed $V \text{ m s}^{-1}$ and initial angle θ . Assume that acceleration due to gravity is $g \text{ m s}^{-2}$.



There are two points x_1 and x_2 where the particle has half of the maximum height. Let g be the acceleration due to gravity.

- (a) Show that the Cartesian equation of the trajectory is

$$y = x \tan \theta - \frac{gx^2}{2V^2} (1 + \tan^2 \theta)$$

- (b) Show that x_1 and x_2 are solutions of the quadratic

$$2g^2x^2(1 + \tan^2 \theta) - 4gxV^2 \tan \theta + V^4 \sin^2 \theta = 0$$

- (c) Show that the horizontal distance between x_1 and x_2 is $\frac{R}{\sqrt{2}}$, where R is the horizontal range of the particle.

Hint: Expand $(x_2 - x_1)^2$

Chapter 2 Review

Vectors

Review

Question 1 Let $A = (-2, 4)$, $B = (1, -3)$, $C = (6, -1)$, and let O be the origin. Write down the vector that corresponds to the following. Express your answer using $a\mathbf{i} + b\mathbf{j}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ notation.

- (a) \overrightarrow{OA} (b) \overrightarrow{OC} (c) \overrightarrow{AB}
 (d) \overrightarrow{BA} (e) \overrightarrow{AC} (f) \overrightarrow{BC}

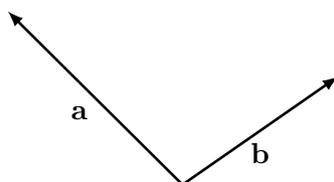
Question 2 Find the value of a and b so that $\begin{bmatrix} 2a - 3b \\ a + b \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$

Question 3 If \mathbf{u} is the position vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and \mathbf{v} is the position vector $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ determine

- (a) $\mathbf{u} - \mathbf{v}$ (b) $\mathbf{v} - 3\mathbf{u}$ (c) $2\mathbf{u} + 3\mathbf{v}$

Question 4 Find the value of k such that $\langle k, 6 \rangle = \langle 6, 18 \rangle$

Question 5 The diagram below shows two vectors \mathbf{a} and \mathbf{b} . Draw the following vectors

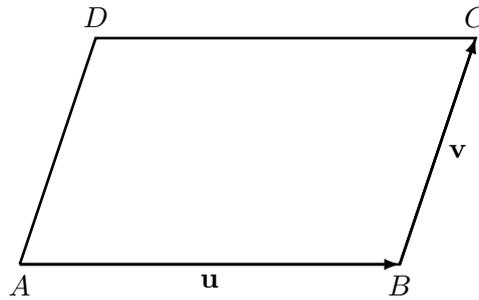


- (a) $2\mathbf{a}$ (b) $-\frac{1}{2}\mathbf{b}$ (c) $3\mathbf{b}$
 (d) $\mathbf{a} + \mathbf{b}$ (e) $\mathbf{a} - \mathbf{b}$ (f) $-\mathbf{a} - \mathbf{b}$

Question 6 Define points $A(-2, -1)$, $B(0, 3)$, $C(3, 2)$ and $D(1, -2)$

- (a) Calculate \overrightarrow{AB} and \overrightarrow{DC} .
 (b) Hence, explain why \overrightarrow{AB} and \overrightarrow{CD} are parallel.
 (c) Similarly, show that \overrightarrow{BC} and \overrightarrow{AD} are parallel.
 (d) State the type of quadrilateral that $ABCD$ forms.

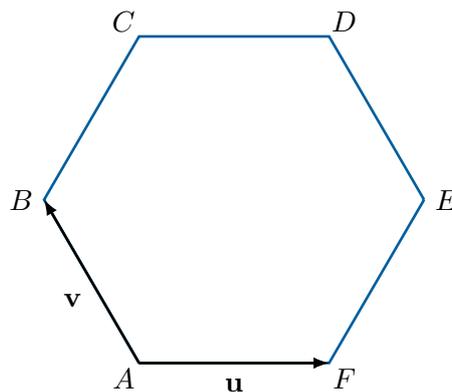
Question 7 The diagram below shows a parallelogram $ABCD$.



Let $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$. Write down the following in terms of \mathbf{u} and \mathbf{v} .

- | | | |
|---------------------------|---------------------------|---------------------------|
| (a) \overrightarrow{AD} | (b) \overrightarrow{DA} | (c) \overrightarrow{CD} |
| (d) \overrightarrow{AC} | (e) \overrightarrow{BD} | (f) \overrightarrow{CA} |

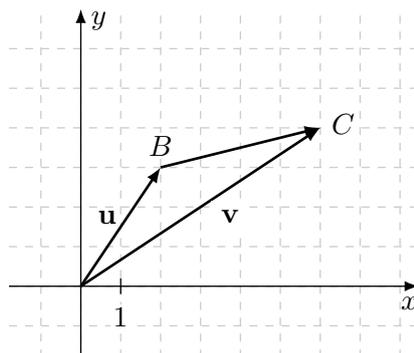
Question 8 The diagram below shows a regular hexagon $ABCDEF$. Let $\mathbf{u} = \overrightarrow{AF}$ and $\mathbf{v} = \overrightarrow{AB}$



Express the following vectors in terms of \mathbf{u} and \mathbf{v}

- | | | |
|---------------------------|---------------------------|---------------------------|
| (a) \overrightarrow{CD} | (b) \overrightarrow{ED} | (c) \overrightarrow{BE} |
| (d) \overrightarrow{FC} | (e) \overrightarrow{FA} | (f) \overrightarrow{FB} |
| (g) \overrightarrow{CE} | (h) \overrightarrow{FE} | (i) \overrightarrow{AE} |

Question 9 Let \vec{OB} and \vec{OC} be position vectors as shown.



- Write down \vec{BC} in terms of \mathbf{u} and \mathbf{v} , $x\hat{i} + y\hat{j}$, $\begin{bmatrix} \quad \\ \quad \end{bmatrix}$.
- Draw the following vector $-\frac{1}{2}\mathbf{v}$ and write it in form $x\hat{i} + y\hat{j}$.
- Draw the following vector $\mathbf{u} + \mathbf{v}$ and write it in form $x\hat{i} + y\hat{j}$.
- If points P and Q are midpoints of \vec{OB} and \vec{OC} , write down \vec{PQ} in terms of \mathbf{u} and \mathbf{v} and $x\hat{i} + y\hat{j}$.
- State the geometric relationship between \vec{PQ} and \vec{BC} .

Question 10 Consider two position vectors $A(3, 4)$ and $B(-4, 3)$. Let P be a point on the line passing through A and B such that $\vec{AP} = m\vec{AB}$. Find

- an expression for \vec{AB} .
- the distance between A and B .
- the position vector of P in terms of m .
- the value of m if P lies on the y axis.
- the value of m if P lies on the line $y = x$.

Question 11 Let $\mathbf{u} = \langle -3, 1 \rangle$ and $\mathbf{v} = \langle 4, 2 \rangle$. Find the value a in each of the following such that $\mathbf{u} + a\mathbf{v}$ is

- parallel to the x axis.
- parallel to the y axis.
- parallel to the vector $\langle 5, 5 \rangle$.

Question 12 P , Q and R are points with position vectors \mathbf{p} , \mathbf{q} and \mathbf{r} , where $p = 2\hat{i} + 6\hat{j}$, $q = 4\hat{i} + 2\hat{j}$ and $r = 2\hat{i} - 4\hat{j}$

- (a) Find \overrightarrow{QP} and \overrightarrow{QR}
- (b) Find $|\overrightarrow{QP}|$ and $|\overrightarrow{QR}|$
- (c) Find $\overrightarrow{QP} \cdot \overrightarrow{QR}$
- (d) Show that $\cos \angle PQR = -\frac{1}{\sqrt{2}}$
- (e) Hence find the magnitude of $\angle PQR$
- (f) Calculate $\mathbf{p} \cdot \mathbf{p}$ and $\mathbf{q} \cdot \mathbf{r}$

Question 13

- (a) Find the unit vector \mathbf{v} in the direction of the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
- (b) Express the vector $\begin{bmatrix} -6 \\ 8 \end{bmatrix}$ in terms of \mathbf{v} .

Question 14

- (a) If $|\mathbf{u}| = 5$, $|\mathbf{v}| = 2$ and the angle between \mathbf{u} and \mathbf{v} is 45° , what is $\mathbf{u} \cdot \mathbf{v}$?
- (b) If $|\mathbf{u}| = 5$, $|\mathbf{v}| = 2$ and the angle between \mathbf{u} and \mathbf{v} is 135° , what is $\mathbf{u} \cdot \mathbf{v}$?

Question 15 Find the vector \mathbf{u} which has a magnitude of 26 units and is in the same direction as $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$.

Question 16 Consider two vectors $u = 2\hat{i} - 3\hat{j}$ and $v = 5\hat{i} + \hat{j}$, find

- (a) the scalar projection of \mathbf{u} onto \mathbf{v} .
- (b) the scalar projection of \mathbf{v} onto \mathbf{u} .
- (c) the vector projection of \mathbf{u} onto \mathbf{v} .
- (d) the vector projection of \mathbf{v} onto \mathbf{u} .

Question 17 Let A and B be points with position vectors \mathbf{u} and \mathbf{v} such that $|\mathbf{u}| = 6$, $|\mathbf{v}| = 8$ and $\mathbf{u} \cdot \mathbf{v} = 24$.

- (a) Show that $\cos \theta = \frac{1}{2}$
- (b) Find $|\overrightarrow{AB}|$

Question 18 Let A and B be the points defined by the position vectors $\mathbf{a} = 3\hat{i} + 4\hat{j}$ and $\mathbf{b} = 12\hat{i} - 5\hat{j}$.

- Find \overrightarrow{BA} , \overrightarrow{AB} .
- Find $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$ and $|\overrightarrow{BA}|$.
- Find $\angle AOB$.
- Find $\angle OAB$.
- Find $\angle ABO$.
- Find the unit vector parallel to \overrightarrow{AB} .

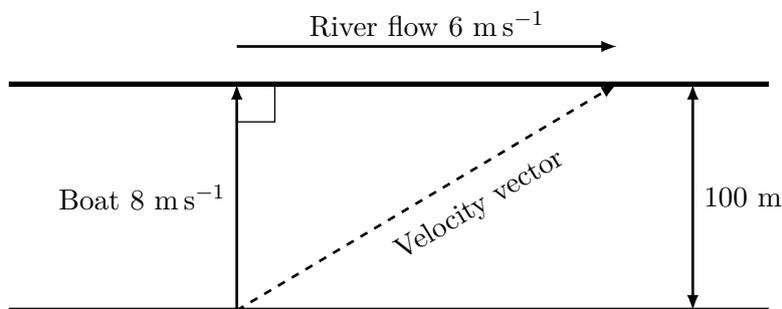
Question 19 Use vector methods to prove that

- in a rectangle, diagonals bisect each other and are equal in length.
- the diagonals of a parallelogram bisect each other.
- the midpoint of the hypotenuse of a right-angled triangle is equidistant from all three vertices of the triangle.
- the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length.
- the midpoints of the sides of any arbitrary quadrilateral form a parallelogram.

Question 20 Bob and Mary set out on two boats B and M with position vectors $3\hat{i} + 6\hat{j}$ and $3\hat{i} + \hat{j}$ respectively. The velocity vector of Bob's boat is $3\hat{i} + \hat{j}$ and the velocity vector of Mary's boat is $2\hat{i} + 4\hat{j}$.

- Draw up the position vectors of both boats B and M .
- Write down the vector \overrightarrow{BM} after t hours.
- Find the position of Bob and Mary's boat after 1 hour.
- How far apart will Bob and Mary's boat be after 1 hour?
- Find the distance between the boats after t hours.

Question 21 Bob rides his boat across a 100 metre-wide river that flows at 6 m s^{-1} . His boat is able to travel at a constant speed of 8 m s^{-1} in still water. He points the boat to go straight to the other side but his boat drifts as a result of the flow of the river.

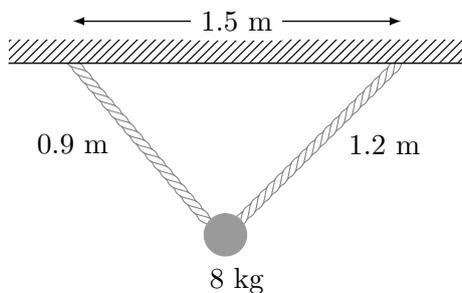


- Write down Bob's velocity vector in the form $x\mathbf{i} + y\mathbf{j}$.
- What is the magnitude of Bob's velocity vector?
- What is the direction of Bob's velocity vector, to the nearest degree?
- Hence, how far did Bob actually travel, correct to 1 d.p?
- How long did the trip take?
- If Bob begins from $(0, 0)$, at what point will he meet the other end of the river?
- In what direction should he have pointed the boat, so that his overall trajectory was vertical?

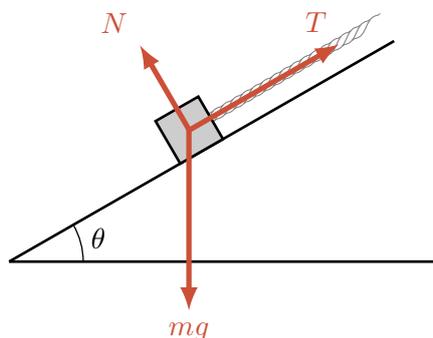
Question 22 A helicopter can fly at 160 km h^{-1} in still air. The wind is blowing at 20 km h^{-1} from the east.

- On what bearing should the helicopter head in order to fly directly to a point 100 km due north?
- On what bearing should the helicopter head in order to fly directly to a point 100 km due south?

Question 23 A mass of 8 kg is suspended from two strings of length 0.9 m and 1.2 m. The ends of the string on the horizontal plane are 1.5 m apart. Find the tension in the string.



Question 24 The diagram below shows an object of mass m on a smooth plane inclined at some angle from the horizontal plane.

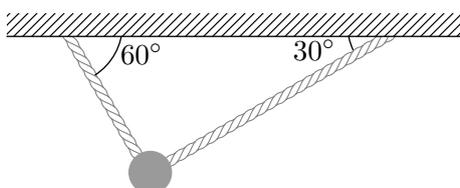


Find the tension in the string and the magnitude of the force exerted on the body by the plane if

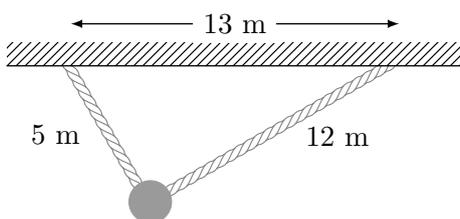
(a) $m = 20 \text{ kg}$ and $\theta = 30^\circ$

(b) $m = 10 \text{ kg}$ and $\theta = 45^\circ$

Question 25 A particle of mass 10 kg is suspended by two ropes with tension vectors T_1 and T_2 attached to two points in the same horizontal plane. Find $|T_1|$ and $|T_2|$



Question 26 The diagram below shows an object of unit mass suspended from a ceiling by two ropes with tension vectors T_1 and T_2 with lengths 5 m and 12 m respectively. The ropes are attached to hooks on the ceiling that are 13 metres apart.



(a) Show that

$$5|T_1| - 12|T_2| = 0$$

$$12|T_1| + 5|T_2| = 130$$

(b) Find $|T_1|$ and $|T_2|$

Question 27 T seconds after a particle is projected, it reaches a point P on its upward path, where the direction of the flight is 30° to the horizontal. If $V = 40 \text{ m s}^{-1}$, angle of projection is 60° and $g = 10 \text{ m s}^{-2}$, show that $T = \frac{4\sqrt{3}}{3}$

Question 28 An object is projected horizontally from the top of a building 125 m high at a speed of 20 m s^{-1} . Using $g = 10 \text{ m s}^{-2}$, find

- (a) the equations of motion.
- (b) the time when the object hits the ground and its distance from the base of the building.

Question 29 A cricket ball is hit with a velocity of 20 m s^{-1} at an angle of 30° with the horizontal.

- (a) Write down the equations of motion.
- (b) Find the greatest height attained.
- (c) Find the time of flight.
- (d) Find the horizontal distance travelled.

Question 30 A particle is projected and it reaches a maximum height of 100 m and its horizontal range is 1600 m. Find the angle of projection and the initial velocity of the particle. Use $g = 10 \text{ m s}^{-2}$

 Investigation Task

The monkey and the hunter

Any physics textbook discussing projectile motion will probably mention some form of the ‘Monkey and Hunter’ problem. This investigation task will allow the student to explore this interesting concept further.

Question 1 Write a short paragraph, with some diagrams, to explain what the scenario is describing, and what is the main result.

Question 2 Derive all relevant formulae and prove the main result using techniques of projectile motion learned in this chapter.

Question 3 There is a reason why almost all elementary physics textbooks contain some discussion of this problem. Why is it so interesting and important that it would be so commonly mentioned?

 Investigation Task

Work

In physics, one of the basic applications of the dot product is in calculating *work*.

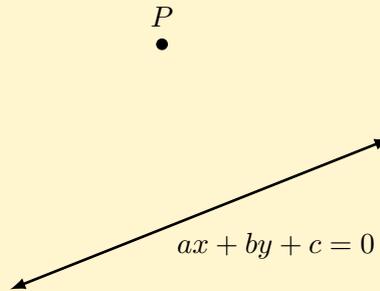
Write a two page document explaining everything there is to know about *work* and the dot product. Your answer should include:

- the definition of ‘work’.
- the basic formula for work that is taught at an early high school level.
- a discussion of the kinds of scenarios where the basic formula no longer works.
- a discussion and proof of the relationship between work and the dot product.
- some example calculations of work. One of the examples should cover the the amount of work done by a vector to move a particle from point A to point B on the xy -plane.
- how the physical concept of ‘work’ can be used to better understand the dot product between two vectors, and why we would expect the dot product of perpendicular vectors to be zero.

 Investigation Task

Perpendicular distance formula

We know how to find the distance from a point to another point using the distance formula. But how would you find the distance from a point to a *line*?



This is done using the perpendicular distance formula, which states that the distance from the point $P(x_0, y_0)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Your task is to write a one page document about this formula. Your document should include:

- the full derivation of the formula.
- worked examples.
- a discussion of what happens if the absolute value is not there.
- a discussion of how a student can determine whether the result will be positive or negative from the diagram, if the absolute value were not used in the formula.
- a discussion of how the formula can be used to determine whether a line is tangent to a circle.
- a discussion of how to find the distance between two parallel lines.

 Investigation Task

Static equilibrium

In Exercise 2H, you solved many equilibrium problems by resolving all forces into vertical and horizontal components. Then, you equated left-and-right vectors and up-and-down vectors using the fact that the particle is in equilibrium. Finally, you used simultaneous equations to find the magnitude of the normal force and the tension force.

If you look at the **Challenge** section of Exercise 2H, you may notice that a lot of the results are suspiciously simple. So simple in fact that it entices the reader into thinking that there may be an alternative method. This investigation task will allow the reader to explore an alternative method of doing these equilibrium problems.

Question 1 In many of the problems you have done involving tension and normal, the diagrams often have three vectors being tension, normal and gravitational force. These three vectors are drawn *from* the object, pointing *outwards*. Explain why if we re-arranged all of these vectors (without changing the length or angle), they will form a *closed* triangle. That is, they will perfectly connect tip-to-tail and enclose a triangle with no gaps or overlaps.

Question 2

- Pick a random standard equilibrium problem (you may use any from Exercise 2H) and draw the force diagram to scale. For example, a vector with magnitude 20N should be double the length of a vector with magnitude 10N.
- Draw a translated version of each vector to show that they indeed form a closed triangle.
- Find all the angles inside the triangle and hence find an alternative way of finding the lengths of the vectors.

Question 3 Repeat (or attempt for the first time!) the **Challenge** section of Exercise 2H using this new technique. Each one should come out in a few lines after drawing the new diagram and finding all the angles inside it.

3

TRIGONOMETRIC EQUATIONS

- Revision of solving trigonometric equations
- Solving equations using compound angle formulae
- Solving equations involving sums to products
- Auxiliary angle formulae
- Solving equations involving t-formulae

Question 4 Solve the following equations for the domain $0 \leq x \leq 2\pi$

(a) $\tan\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{3}}$ (b) $\cot\left(x + \frac{\pi}{3}\right) = -1$
 (c) $\sin\left(x - \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ (d) $\cos(x - \pi) = \frac{1}{\sqrt{2}}$

Question 5 Solve the following for $0 \leq x \leq 2\pi$

(a) $2 \sin\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$ (b) $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Question 6 Solve the following equations for the domain $0 \leq x \leq 2\pi$

(a) $4 \cos^2 x = 1$ (b) $2 \sin^2 x = 1$
 (c) $2 \sin^2 x = \sin x$ (d) $\cos^2 x - \cos x \sin x = 0$
 (e) $2 \cos^2 x - 1 = \cos x$ (f) $\sec x - 4 \cos x = 0$

Question 7 Solve the following equations for the domain $0 \leq x \leq 2\pi$

(a) $\cos^2 x = 5 \cos x$ (b) $\sin^2 x = 4$
 (c) $\cos^2 x - \cos x - 6 = 0$ (d) $\sin^2 x - 5 \sin x + 4 = 0$

Question 8 Solve the following equations for the domain $0 \leq x \leq \pi$

(a) $\sin 2x = \cos 2x$ (b) $\cos 2x = \sec 2x$ (c) $\sqrt{3} \sin 2x + \cos 2x = 0$

Question 9 Solve the following equations for the domain $0 \leq x \leq 2\pi$

(a) $3 \sin^2 x + 5 \sin x - 2 = 0$ (b) $\tan^2 x - 2 \tan x - 3 = 0$
 (c) $2 \cos^2 x + \sin x = 1$ (d) $3 \operatorname{cosec}^2 x = \cot x + 5$

Question 10 Solve the following equations for given the domain

(a) $\sin^2 x = \sin x$ for $-\pi \leq x \leq \pi$
 (b) $2 \cos^2 x - 3 \cos x = 2$ for $-\pi \leq x \leq \pi$
 (c) $\tan^2 x + 2 \tan x - 3 = 0$ for $0 \leq x \leq 2\pi$
 (d) $3 \tan^3 x + 3 \tan^2 x + \tan x + 1 = 0$ for $-\pi \leq x \leq \pi$

Question 11 Solve the following equations for the domain $0 \leq x \leq 2\pi$

(a) $\cos\left(x + \frac{\pi}{3}\right) = -1$ (b) $\sin\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
 (c) $\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$ (d) $\operatorname{cosec}\left(x + \frac{\pi}{3}\right) = \sqrt{2}$

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Question 12 Solve the following equations for the domain $0 \leq x \leq 2\pi$

- (a) $\sin 3x = -\frac{1}{2}$ (b) $\sin 2x = -\frac{1}{\sqrt{2}}$ (c) $2 \cos 3x = \sqrt{2}$
 (d) $\tan 3x = 1$ (e) $\sec^2 x = \sec x$ (f) $\sin x = \cos x$

Question 13 Solve the following equations for the domain for $-2\pi \leq x \leq 2\pi$

- (a) $\sin\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$ (b) $\sin^2\left(\frac{x}{2}\right) = 1$

Question 14 Solve the following equations for the domain $0 \leq x \leq 2\pi$

- (a) $\sin\left(2x + \frac{\pi}{2}\right) = -\frac{1}{\sqrt{2}}$ (b) $\operatorname{cosec}\left(2x + \frac{\pi}{2}\right) = -\sqrt{2}$
 (c) $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ (d) $\tan\left(2x + \frac{\pi}{4}\right) = 1$

Question 15 Solve the following equations for the domain $-\pi \leq x \leq \pi$

- (a) $\tan 2x = \sin 2x$ (b) $\cot x = 3 \tan x$ (c) $2 \sin^2 2x = \sin 2x$

Question 16 A simplified model for the depth of water in metres at the entrance to a harbour is given by

$$y = 12 - 6 \cos\left(\frac{\pi}{4}t\right)$$

where t is measured in hours from 10 am.

- (a) Sketch the graph of $y = 12 - 6 \cos\left(\frac{\pi}{4}t\right)$.
 (b) What are the depths of water at low tide and high tide?
 (c) At what time will the next two *low* tides occur?
 (d) At what time will the next two *high* tides occur?
 (e) A ship needs a depth of 12 metres to enter the harbour. If low tide occurred at 10 am when can the ship enter the harbour?

Challenge Problems

Problem 1 Solve the following for $0 \leq x \leq 2\pi$.

- (a) $\tan^2 x + \sec x = 1$ (b) $\sqrt{2} \sin x \cos x - \sqrt{2} \cos x + \sin x - 1 = 0$
 (c) $\sec^2 x = 3 \tan x + 5$ (d) $\cos^2 x - 2 \sin x \cos x - 3 \sin^2 x = 0$
 (e) $\sec^2 x + \tan x - 3 = 0$ (f) $2 \cos^2 x - \sin x - 1 = 0$

Exercise 3B

Solving equations using compound angle formulae

Fundamentals

Fundamentals 1

Write down the expansion of the following.

- (a) $\sin(A + B)$ (b) $\sin(A - B)$ (c) $\cos(A + B)$
 (d) $\cos(A - B)$ (e) $\tan(A + B)$ (f) $\tan(A - B)$

Fundamentals 2

Write down the expansion of the following.

- (a) $\sin(2A)$ (b) $\cos(2A)$ (c) $\tan(2A)$

Fundamentals 3

- (a) If more than two trigonometric ratios are present, convert to one ratio.
 (b) If equations involve angles of different magnitude like $\sin(2x)$ and $\sin(x)$, make sure that the arguments (inside of the bracket) are all the same. The compound or double angle formulae will be useful here.

Question 1 Using identities for double angles solve for $0 \leq x \leq 2\pi$

- (a) $\sin 2x = 2 \cos x$ (b) $\sin 2x = 2 \cos^2 x$ (c) $\cos 2x = 2 \sin^2 x$
 (d) $\sin x \cos x = 1$ (e) $\sin 2x = \tan x$ (f) $\tan 2x = 2 \tan x$
 (g) $\cos 2x = \sin x$ (h) $\cos 2x - \cos x = 0$ (i) $\tan 2x + 2 \tan x = 0$
 (j) $\sin x + \cos 2x - 1 = 0$ (k) $\cos 2x + \cos x = 0$ (l) $8 \sin^2 x \cos^2 x - 1 = 0$

Question 2 Prove the following identities.

- (a) $\cos 3x = 4 \cos^3 x - 3 \cos x$ (b) $\sin 3x = 3 \sin x - 4 \sin^3 x$ (c) $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

Question 3 Use the identities from the previous question to solve the following for $0 \leq x \leq 2\pi$

- (a) $\cos 3x = \cos x$ (b) $\sin 3x = \sin x$ (c) $\tan 3x = \tan x$
 (d) $\cos 3x + 2 \cos x = 0$ (e) $\sin 2x - \sin 3x = 0$ (f) $\tan 3x - \tan 2x = 0$

Question 4 Solve $\cos 4x - 8 \sin^2 x \cos^2 x = 0$ for $0 \leq x \leq 2\pi$.

Exercise 3C

Solving equations involving sums to products



Fundamentals

Fundamentals 1

Write down the expansions of the following.

- | | |
|-------------------|-------------------|
| (a) $\sin(A + B)$ | (b) $\sin(A - B)$ |
| (c) $\cos(A + B)$ | (d) $\cos(A - B)$ |

Fundamentals 2

Write down the corresponding products-to-sums formulae.

- | | |
|---------------------|---------------------|
| (a) $\sin A \cos B$ | (b) $\cos A \sin B$ |
| (c) $\cos A \cos B$ | (d) $\sin A \sin B$ |

Question 1 Let $0 \leq x \leq 2\pi$.

- | | |
|--|--|
| (a) Expand $\cos(3x - x) + \cos(3x + x)$ | (b) Hence, solve $\cos 2x + \cos 4x = 0$ |
|--|--|

Question 2 Let $0 \leq x \leq 2\pi$.

- | | |
|--|---|
| (a) Expand $\sin(3x + 2x) - \sin(3x - 2x)$ | (b) Hence, solve $\sin 5x - \sin x = 0$ |
|--|---|

Question 3 Let $0 \leq x \leq 2\pi$.

- | |
|---|
| (a) Express $\sin 3x \cos x$ as the sum or difference of two trigonometric expressions. |
| (b) Hence, solve $\sin 4x + \sin 2x = 0$ |

Question 4 [Sum-to-product formulae]

Use your product to sum formulae and the pair of substitutions $P = A + B$ and $Q = A - B$ to express the following as products of trigonometric functions.

- | | | | |
|-----------------------|-----------------------|-----------------------|-----------------------|
| (a) $\sin P + \sin Q$ | (b) $\sin P - \sin Q$ | (c) $\cos P + \cos Q$ | (d) $\cos P - \cos Q$ |
|-----------------------|-----------------------|-----------------------|-----------------------|

Question 5 Express the following as products.

- | | |
|-------------------------|-------------------------|
| (a) $\sin 4x - \sin 2x$ | (b) $\cos 4x + \cos 2x$ |
| (c) $\sin 6x + \sin 4x$ | (d) $\cos 3x + \cos 7x$ |

Question 6 Convert sums and differences to products and then solve the equations for $0 \leq x \leq \pi$

- (a) $\sin 4x - \sin 2x = 0$ (b) $\sin 3x - \sin 5x = 0$ (c) $\sin 7x + \sin 3x = 0$
 (d) $\cos 5x - \cos 3x = 0$ (e) $\cos 3x + \cos 7x = 0$ (f) $\cos 6x - \cos 4x = 0$

Question 7 Convert sums and differences to products and then solve the equations for $0 \leq x \leq \pi$

- (a) $\sin 5x + \sin x = \sin 3x$ (b) $\cos 5x + \cos 3x = \cos 4x$ (c) $\cos 7x + \cos x = \cos 3x$

Question 8 Express the following sums as products of two trigonometric functions.

- (a) Express $\cos 5x + \cos 3x$ and $\cos 7x + \cos x$ as the product of two trigonometric functions.
 (b) Hence, express

$$\cos 7x + \cos 5x + \cos 3x + \cos x$$

as the product of three trigonometric functions.

- (c) Hence, solve $\cos 7x + \cos 5x + \cos 3x + \cos x = 0$ for $0 \leq x \leq 2\pi$

Question 9 Express the following sums as products of two trigonometric functions.

- (a) Express $\sin 4x + \sin x$ and $\sin 3x + \sin 2x$ as the product of two trigonometric functions.
 (b) Hence, express

$$\sin 4x + \sin 3x + \sin 2x + \sin x$$

as the product of three trigonometric functions.

- (c) Hence, solve $\sin 4x + \sin 3x + \sin 2x + \sin x = 0$ for $0 \leq x \leq 2\pi$

Question 10

- (a) Show that

$$\sin 2x + \sin 4x = \sin 6x$$

can be expressed as $\sin 3x \sin 2x \sin x = 0$

- (b) Hence, solve $\sin 2x + \sin 4x = \sin 6x$ for x in the domain $0 \leq x \leq \pi$.

Question 11

- (a) Show that

$$\sin 5x + \sin 3x = \sin 4x + \sin 2x$$

can be expressed as $\cos\left(\frac{7x}{2}\right) \cos x \sin\left(\frac{x}{2}\right) = 0$

- (b) Hence, solve $\sin 5x + \sin 3x = \sin 4x + \sin 2x$ for x in the domain $0 \leq x \leq \pi$.

⚙️ Challenge Problems

Problem 1 [Telescoping sum]

Consider the series

$$S_n = \sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x$$

- (a) Express $\sin x \sin x$ as the sum of two cosine functions.
- (b) Express $\sin x \sin 3x$ as the sum of two cosine functions.
- (c) Express $\sin x \sin 5x$ as the sum of two cosine functions.
- (d) Express $\sin x \sin(2n-1)x$ as the sum of two cosine functions.
- (e) By considering $\sin x S_n$, show that

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$$

- (f) Solve

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n-1)x = 0$$

in the domain $0 \leq x \leq 2\pi$.

Hint: It looks like $\sin x \neq 0$ i.e. $x \neq 0, \pi, 2\pi, \dots$ but think about that more carefully.

Exercise 3D

Auxiliary angle formulae



Fundamentals

Fundamentals 1

Expressions in the form

$$a \sin x + b \cos x$$

are generally not helpful because it is hard to see features of the curve and also there are two trigonometric functions to work with. Instead, we can use the method of a _____ angles to convert it into one of the forms

$$\begin{aligned} R \sin(x \pm \alpha) \\ R \cos(x \pm \alpha) \end{aligned}$$

Fundamentals 2

Complete the following set of steps to convert $a \sin x + b \cos x$ into the form $R \sin(x + \alpha)$.

$$a \sin x + b \cos x \equiv R \sin(x + \alpha)$$

- Expand $R \sin(x + \alpha)$.
- Equate the coefficients of $\sin x$ and $\cos x$. Call these results (1) and (2) respectively.
- Calculate $(1)^2 + (2)^2$ to find R in terms of a and b .
- Calculate $\frac{(2)}{(1)}$ to find α in terms of a and b .

Question 1 [Drills for finding R]

Find R in the following questions using $R = \sqrt{a^2 + b^2}$.

- | | | |
|--------------------------------|--------------------------------|----------------------------|
| (a) $\sin x + \cos x$ | (b) $\sqrt{3} \sin x + \cos x$ | (c) $\cos x - \sin x$ |
| (d) $\sin x - \sqrt{3} \cos x$ | (e) $3 \sin x - 4 \cos x$ | (f) $5 \sin x + 12 \cos x$ |

Question 2 State the expansion of

- | | | | |
|--------------------------|--------------------------|--------------------------|--------------------------|
| (a) $R \sin(x + \alpha)$ | (b) $R \sin(x - \alpha)$ | (c) $R \cos(x + \alpha)$ | (d) $R \cos(x - \alpha)$ |
|--------------------------|--------------------------|--------------------------|--------------------------|

Question 3 In each of the following, determine the most appropriate form from either $R \cos(x \pm \alpha)$ or $R \sin(x \pm \alpha)$, state the value of R and α , and hence express it in auxiliary angle form.

- (a) $\sqrt{3} \sin x + \cos x$ (b) $\cos x - \sqrt{3} \sin x$ (c) $\sin x - \sqrt{3} \cos x$
 (d) $\sqrt{3} \cos x + \sin x$ (e) $\sin x - \cos x$ (f) $\sin x + \cos x$
 (g) $\cos x - \sin x$ (h) $\cos x + \sin x$ (i) $3 \sin x + \sqrt{3} \cos x$
 (j) $\sqrt{3} \cos x + 3 \sin x$ (k) $\sqrt{3} \sin x - 3 \cos x$ (l) $3 \cos x - \sqrt{3} \sin x$

Question 4 Let $0 \leq x \leq 360^\circ$

- (a) Express $3 \sin x + 4 \cos x$ in the form $R \sin(x + \alpha)$.
 (b) Hence, solve $3 \sin x + 4 \cos x = 5$.

Question 5 Let $0 \leq x \leq 360^\circ$.

- (a) Express $2 \sin x - 3 \cos x$ in the form $R \sin(x - \alpha)$.
 (b) Hence, solve $2 \sin x - 3 \cos x = 1$.

Question 6 By first expressing $\sqrt{3} \sin x + \cos x$ in an appropriate auxiliary angle form, solve the following in the domain $0 \leq x \leq 2\pi$.

- (a) $\sqrt{3} \sin x + \cos x = 2$ (b) $\sqrt{3} \sin x + \cos x = 0$ (c) $\sqrt{3} \sin x + \cos x = -1$

Question 7 Solve the following equations in the domain $0 \leq x \leq 2\pi$ by first converting to auxiliary angle form.

- (a) $\sqrt{3} \sin x + \cos x = 1$ (b) $\cos x - \sqrt{3} \sin x = -1$ (c) $\sin x - \sqrt{3} \cos x = 0$
 (d) $\sqrt{3} \cos x + \sin x = \sqrt{3}$ (e) $\sin x - \cos x = 1$ (f) $\sin x + \cos x = \frac{1}{\sqrt{2}}$
 (g) $\cos x - \sin x = \sqrt{\frac{3}{2}}$ (h) $\cos x + \sin x = -1$ (i) $3 \sin x + \sqrt{3} \cos x = 3$
 (j) $\sqrt{3} \cos x + 3 \sin x = -\sqrt{3}$ (k) $\sqrt{3} \sin x - 3 \cos x = 2\sqrt{3}$ (l) $3 \cos x - \sqrt{3} \sin x = 2\sqrt{3}$

Question 8 [Maximum and minimum values]

Consider the function

$$f(x) = \sqrt{3} \cos x - \sin x$$

over the domain $0 \leq x \leq 2\pi$.

- (a) Convert $f(x)$ into the form $R \cos(x + \alpha)$.
 (b) Hence, state the maximum and minimum values.
 (c) Verify your answer by differentiating the original form of $f(x)$.
 (d) Find when the maximum and minimum value occur by solving $f(x) = \pm 2$.

Question 9 Find the value of R and hence, state the maximum and minimum values of the following.

- (a) $\sin x + \cos x$ (b) $4 \cos x - 3 \sin x$ (c) $6 \sin x - 8 \cos x$
 (d) $2 + 3 \sin x + 4 \cos x$ (e) $1 + \sin x - \cos x$ (f) $5 \cos x + 12 \sin x - 2$

Question 10 [Auxiliary angle method can be used to help produce sketches]

Consider the equation

$$f(x) = \sqrt{3} \sin x + \cos x,$$

in the domain $0 \leq x \leq 2\pi$

- (a) Express $f(x)$ in the form $R \sin(x + \alpha)$ for $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
 (b) Hence, describe how this curve has been transformed from the standard sine curve $y = \sin x$.
 (c) Sketch the curve.

Question 11 By first converting to an appropriate auxiliary angle form, sketch the graph of the following.

- (a) $y = \cos x - \sqrt{3} \sin x$ (b) $y = \sin x + \cos x$ (c) $y = \sqrt{3} \sin x - 3 \cos x$

Question 12 Consider the function $f(x) = \cos x + \sqrt{3} \sin x$ in the domain $0 \leq x \leq 2\pi$.

- (a) Sketch $y = f(x)$.
 (b) Solve $f(x) = 1$.
 (c) Hence, use your sketch to solve the inequality $\cos x + \sqrt{3} \sin x > 1$.

Question 13 Solve $2 \cos 3x - \sin 3x = \sqrt{5}$ for $0 \leq x \leq 2\pi$

Question 14 Without using calculus, find the range of $f(x) = \frac{1}{1 + \sin x + \cos x}$.

⚙️ Challenge Problems

Problem 1 Without using calculus, find the range of

$$f(\theta) = \cos \theta + \cos \left(\theta + \frac{\pi}{3} \right)$$

Problem 2 Without using calculus, find the range of

$$f(\theta) = \frac{1}{\sin \theta \cos \theta + \cos^2 \theta}$$

Question 1 Consider the trigonometric equation

$$2 \sin x + \cos x = 1$$

for $0^\circ \leq x \leq 360^\circ$. Let $t = \tan\left(\frac{x}{2}\right)$.

(a) Use t -formula substitutions to show that this equation reduces to

$$t^2 - 2t = 0$$

(b) Hence, show that x satisfies $\tan\left(\frac{x}{2}\right) = 0$ or 2 , where $0^\circ \leq x \leq 360^\circ$

(c) Solve for x .

(d) Test $x = 180^\circ$ and determine if this is a solution.

(e) Write down your solution set.

Question 2 Consider the trigonometric equation

$$\sin x + \cos x = -1$$

for $0 \leq x \leq 2\pi$. Let $t = \tan\left(\frac{x}{2}\right)$.

(a) Show that $\tan\left(\frac{x}{2}\right) = -1$, where $0 \leq x \leq \pi$, and hence solve for x .

(b) Test $x = \pi$ and determine if this is a solution.

(c) Write down your solution set.

Question 3 Solve each of the following equations using the substitution $t = \tan\left(\frac{x}{2}\right)$, in the domain $0 \leq x \leq 2\pi$.

(a) $2 \sin x + \cos x = -1$

(b) $2 \cos x - 3 \sin x = -2$

(c) $3 \sin x + \cos x = 1$

(d) $3 \sin x - 4 \cos x = 4$

(e) $4 \cos x - 3 \sin x = 3$

(f) $3 \cos x + 4 \sin x = -3$

Challenge Problems

Problem 1 [Connection to the auxiliary-angle formula technique]

Consider the equation $a \sin x + b \cos x = c$. Let $t = \tan\left(\frac{x}{2}\right)$.

(a) Show that $(b+c)t^2 - 2at + (c-b) = 0$.

(b) Hence, show that there will be a double root if $c^2 = a^2 + b^2$.

(c) Use the auxiliary-angle technique to give a geometric interpretation of this result.

Chapter 3 Review

Trigonometric Equations

Review

Question 1 Using identities for double angles solve for the domain $-\pi \leq x \leq \pi$

(a) $\cos^2 x + \frac{\sqrt{3}}{2} \sin 2x = 0$

(b) $1 + \cos x = \cos \frac{x}{2}$

(c) $\sin 2x = -\sin x$

(d) $\sin 4x = \cos 2x$

(e) $\cos 4x = \sin 2x$

Question 2 Using identities for double angles solve for $0 \leq x \leq 2\pi$

(a) $\cos x = \cos 2x$

(b) $\sin x = \cos 2x$

(c) $\cos^2 x = \cos^2 \frac{x}{2}$

(d) $\sin 4x = \sin 2x$

(e) $\tan 2x + \tan x = 0$

Question 3 Solve for $0 \leq x \leq \pi$

(a) $\sin \left(x - \frac{\pi}{3} \right) = \cos x$

(b) $\sin \left(x - \frac{\pi}{4} \right) = \cos \left(x + \frac{\pi}{3} \right)$

Question 4

(a) Show that $\cos 3x = 4 \cos^3 x - 3 \cos x$.

(b) Hence, solve $\cos 3x = 2 \cos x$ in the domain $0 \leq x \leq 2\pi$.

Question 5 Find the maximum and minimum values of the following

(a) $\sin x - \sqrt{3} \cos x$

(b) $3 \cos x + \sqrt{3} \sin x$

Question 6

(a) Express $\sqrt{3} \sin 2x - \cos 2x$ in the form $R \sin(2x - \alpha)$ where α is in radians

(b) Express $3 \cos x - 4 \sin x$ in the form $A \cos(x + \alpha)$ where α is in degrees

Question 7 Solve for $0^\circ \leq x \leq 360^\circ$

(a) $\cos x + \sqrt{3} \sin x = \sqrt{2}$

(b) $2 \cos x - \sqrt{5} \sin x = 1$

(c) $\sqrt{3} \sin x - \cos x = -\sqrt{3}$

Question 8 Solve each of the following equations in the domain $0^\circ \leq x \leq 360^\circ$ using a t -formula substitution.

(a) $5 \sin x - 5 \cos x = 1$

(b) $3 \sin x + 4 \cos x = 3$

(c) $2 \sin x - 5 \cos x + 2 = 0$

(d) $\tan 2x = \cot x$

Question 9 Convert sums and differences to products and then solve each of the following equations for $0 \leq x \leq \pi$

- (a) $\sin 3x + \sin x = 0$ (b) $\cos 3x + \cos x = 0$
 (c) $\cos 3x + \cos x = \cos 2x$ (d) $\sin 2x + \sin 3x + \sin 4x = 0$

Question 10

- (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
 (b) Hence, solve $8x^3 - 6x - \sqrt{3} = 0$.
 (c) Deduce that $\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{11\pi}{18}\right)\cos\left(\frac{13\pi}{18}\right) = \frac{\sqrt{3}}{8}$.
 (d) Find the exact value of $\tan^2\left(\frac{\pi}{18}\right) + \tan^2\left(\frac{11\pi}{18}\right) + \tan^2\left(\frac{13\pi}{18}\right)$.

Question 11

- (a) Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.
 (b) Hence, find the roots of the polynomial equation $t^3 + 3t^2 - 3t - 1 = 0$.
 (c) Express $t^3 + 3t^2 - 3t - 1$ as the product of a linear and quadratic factor.
 (d) Hence, find the exact values of $\tan\left(\frac{7\pi}{12}\right)$ and $\tan\left(\frac{11\pi}{12}\right)$.

 Investigation Task

General Solutions

When solving trigonometric equations, a domain has always been given so that the student has a finite number of solutions. However, how would you present the solution set in general i.e. for all real x ? In those cases, there are *infinitely* many solutions, and it is of course impractical to list them out. In these cases, a *general solution* can be used to express the solution set.

This investigation task will allow students to explore the concept of a general solution, and apply it to solve some equations.

Question 1 Find the general solution of the following.

(a) $\sin x = \frac{1}{2}$

(b) $\cos x = -\frac{1}{2}$

(c) $\tan x = \sqrt{3}$

Question 2 Find the general solution of the following.

(a) $\sin 2x = \frac{1}{2}$

(b) $\cos 3x = -\frac{1}{2}$

(c) $\tan 4x = \sqrt{3}$

Question 3 Write a three page document that explains everything there is to know about general solutions. Your answer should include derivations of the formulae, any necessary diagrams and some worked examples. Your document should be sufficiently detailed so that a student who studies it thoroughly should be able to solve any standard general solution textbook problems.

4

FURTHER CALCULUS

- Differentiation of inverse trigonometric functions
- Integration involving inverse trigonometric functions
- Review of integration techniques
- Integration of squares of sine and cos
- Integration by substitution

Exercise 4A

Differentiation of inverse trigonometric functions



Fundamentals

Fundamentals 1

The relationship

$$\frac{dy}{dx} = \frac{1}{\dots}$$

is very useful when finding derivatives of inverse functions.

Fundamentals 2

Write down the derivative of each of the following

- | | | |
|-----------------------|-----------------------|-----------------------|
| (a) $\sin^{-1} x$ | (b) $\cos^{-1} x$ | (c) $\tan^{-1} x$ |
| (d) $\sin^{-1}(f(x))$ | (e) $\cos^{-1}(f(x))$ | (f) $\tan^{-1}(f(x))$ |

Fundamentals 3

Differentiate the following

- | | | |
|--|--|--|
| (a) $\sin^{-1} ax$ | (b) $\cos^{-1} ax$ | (c) $\tan^{-1} ax$ |
| (d) $\sin^{-1}(ax + b)$ | (e) $\cos^{-1}(ax + b)$ | (f) $\tan^{-1}(ax + b)$ |
| (g) $\sin^{-1}\left(\frac{ax}{b}\right)$ | (h) $\cos^{-1}\left(\frac{ax}{b}\right)$ | (i) $\tan^{-1}\left(\frac{ax}{b}\right)$ |

Fundamentals 4

Write down the domain and range of each of the following functions.

- | | | |
|---|---|---|
| (a) $\sin^{-1}(x)$ | (b) $\cos^{-1}(x)$ | (c) $\tan^{-1}(x)$ |
| (d) $\sin^{-1}\left(\frac{x}{a}\right)$ | (e) $\cos^{-1}\left(\frac{x}{a}\right)$ | (f) $\tan^{-1}\left(\frac{x}{a}\right)$ |

Question 1 Differentiate the following with respect to x .

- | | | |
|---|---|---|
| (a) $\sin^{-1}\left(\frac{x}{2}\right)$ | (b) $\sin^{-1}\left(\frac{x}{6}\right)$ | (c) $\sin^{-1}\left(\frac{x}{5}\right)$ |
| (d) $\sin^{-1}(2x)$ | (e) $\sin^{-1}(4x)$ | (f) $\sin^{-1}(6x)$ |
| (g) $\cos^{-1}\left(\frac{x}{3}\right)$ | (h) $\cos^{-1}\left(\frac{x}{8}\right)$ | (i) $\cos^{-1}\left(\frac{x}{4}\right)$ |
| (j) $\cos^{-1}(3x)$ | (k) $\cos^{-1}(5x)$ | (l) $\cos^{-1}(7x)$ |

Question 2 State the values of x for which the derivatives in Question 1 are defined.

Question 3 Differentiate the following with respect to x .

- (a) $\tan^{-1}(2x)$ (b) $\tan^{-1}(3x)$ (c) $\tan^{-1}(4x)$
 (d) $\tan^{-1}\left(\frac{x}{2}\right)$ (e) $\tan^{-1}\left(\frac{x}{5}\right)$ (f) $\tan^{-1}\left(\frac{x}{10}\right)$

Question 4 Differentiate the following with respect to x .

- (a) $\sin^{-1}(3x - 1)$ (b) $\sin^{-1}(6 + x)$ (c) $\sin^{-1}(2x - 5)$
 (d) $\sin^{-1}(1 - 6x)$ (e) $\sin^{-1}\frac{3x}{2}$ (f) $\sin^{-1}\frac{2x}{5}$

Question 5 Differentiate the following with respect to x .

- (a) $\cos^{-1}(x - 1)$ (b) $\cos^{-1}(3 + x)$ (c) $\cos^{-1}(3x - 1)$
 (d) $\cos^{-1}(1 - 2x)$ (e) $\cos^{-1}\left(\frac{2x}{3}\right)$ (f) $\cos^{-1}\left(\frac{5x}{2}\right)$

Question 6 Differentiate the following with respect to x .

- (a) $\tan^{-1}(x + 1)$ (b) $\tan^{-1}(6 - x)$ (c) $\tan^{-1}(2x - 4)$
 (d) $\tan^{-1}(2 - 3x)$ (e) $\tan^{-1}\left(\frac{2x}{3}\right)$ (f) $\tan^{-1}\left(\frac{4x}{5}\right)$

Question 7 Find the equation of the tangent to each curve at the point given.

- (a) $y = \sin^{-1}(x)$ at $x = 0$ (b) $y = \cos^{-1}(x)$ at $x = 0$
 (c) $y = \tan^{-1}(x)$ at $x = 1$ (d) $y = \sin^{-1}\left(\frac{x}{2}\right)$ at $x = -\sqrt{2}$
 (e) $y = 2\cos^{-1}\left(\frac{x}{3}\right)$ at $x = 0$ (f) $y = 3\tan^{-1}(x) + \frac{3\pi}{4}$ at $x = -1$
 (g) $y = 2\sin^{-1}(3x) + \frac{\pi}{2}$ at $x = 0$ (h) $y = 3\cos^{-1}\left(\frac{2x}{3}\right) - 2\pi$ at $x = -\frac{3}{4}$

Question 8 Differentiate

- (a) $\sin^{-1}(x^2)$ (b) $\cos^{-1}(x^3)$ (c) $\tan^{-1}(3x^2)$
 (d) $\sin^{-1}(1 - x^2)$ (e) $\cos^{-1}(1 - 2x^2)$ (f) $\tan^{-1}(1 - x^2)$

Question 9 Differentiate with respect to x

- (a) $\cos^{-1}\left(\frac{1}{x}\right)$ (b) $\sin^{-1}\left(\frac{2}{x}\right)$ (c) $\tan^{-1}\left(\frac{6}{x}\right)$

Question 10 Differentiate the following with respect to x .

- (a) $\sin^{-1}\sqrt{1 - 2x}$ (b) $\cos^{-1}\sqrt{1 - x}$ (c) $\tan^{-1}\sqrt{1 - x}$

Question 11 Find the derivative of each function, and express in simplest form.

- | | | |
|------------------------|----------------------------|----------------------------|
| (a) $\cos^{-1}(e^x)$ | (b) $\tan^{-1} \sin(2x)$ | (c) $\sin^{-1}(e^{2x})$ |
| (d) $\tan^{-1}(e^x)$ | (e) $\tan^{-1} \sqrt{e^x}$ | (f) $\sin^{-1} \sqrt{e^x}$ |
| (g) $\sin^{-1}(\ln x)$ | (h) $\ln(\sin^{-1} x)$ | (i) $\ln(\tan^{-1} x)$ |
| (j) $\tan^{-1}(\ln x)$ | (k) $\tan^{-1}(\cos x)$ | (l) $e^{\cos^{-1} x}$ |

Question 12 Differentiate the following with respect to x .

- | | | |
|-------------------------|------------------------|----------------------------|
| (a) $x \sin^{-1}(x)$ | (b) $x \tan^{-1}(x)$ | (c) $x^2 \cos^{-1}(x)$ |
| (d) $x^2 \tan^{-1}(3x)$ | (e) $e^x \sin^{-1}(x)$ | (f) $e^{2x} \tan^{-1}(3x)$ |

Question 13 Differentiate the following with respect to x .

- | | | |
|------------------------------|--------------------------------|------------------------------|
| (a) $\frac{\sin^{-1}(x)}{x}$ | (b) $\frac{\cos^{-1}(x)}{x^2}$ | (c) $\frac{\tan^{-1}(x)}{x}$ |
|------------------------------|--------------------------------|------------------------------|

Question 14 Differentiate the following with respect to x .

- | | |
|-----------------------------------|-----------------------------|
| (a) $(\sin^{-1} x)^2$ | (b) $(\tan^{-1} x)^3$ |
| (c) $(\sin^{-1} x)(\cos^{-1} x)$ | (d) $(1 + x^2) \tan^{-1} x$ |
| (e) $\tan^{-1} x + x \tan^{-1} x$ | |

Question 15 [Be careful when simplifying!]

Show that

- | | |
|---|---|
| (a) $\frac{d}{dx}(\sin(\sin^{-1} x)) = 1$ | (b) $\frac{d}{dx}(\sin^{-1}(\sin x)) = \frac{\cos x}{ \cos x }$ |
| (c) $\frac{d}{dx}(\cos(\cos^{-1} x)) = 1$ | (d) $\frac{d}{dx}(\cos^{-1}(\cos x)) = \frac{\sin x}{ \sin x }$ |
| (e) $\frac{d}{dx}(\tan(\tan^{-1} x)) = 1$ | (f) $\frac{d}{dx}(\tan^{-1}(\tan x)) = 1$ |

Question 16 Consider the curve

$$y = \sin(\sin^{-1} x)$$

- | | |
|--|----------------------------|
| (a) Write down the domain and range. | (b) Find $\frac{dy}{dx}$. |
| (c) Simplify $y = \sin(\sin^{-1} x)$. | (d) Sketch the function. |

Question 17 Consider the curve

$$y = \sin^{-1}(\sin x)$$

- Write down the domain and range.
- Show that $y' = \frac{\cos x}{|\cos x|}$.
- State the values of x where $y' > 0$ and where $y' < 0$.
- Simplify $y = \sin(\sin^{-1} x)$.
- Sketch the function.

Question 18 Consider the curve

$$y = \sin(\cos^{-1} x)$$

- Write down the domain.
- Find $\frac{dy}{dx}$ and state the values of x for which it is defined.
- Show that $y = \sqrt{1 - x^2}$. Given this, was your domain in (b) to be expected?

Question 19

- In the domain $0 < x < \frac{\pi}{2}$, find the value of $\frac{d}{dx}(\sin^{-1}(\cos x))$ and $\frac{d}{dx}(\cos^{-1}(\sin x))$.

- Hence show that

$$\sin^{-1}(\cos x) = \cos^{-1}(\sin x),$$

in the domain $0 < x < \frac{\pi}{2}$.

Question 20 Define the function

$$f(x) = \sin^{-1} x + \cos^{-1} x$$

- Find $f'(x)$ and explain the geometric significance of your finding.
- Find $f(0)$. Explain why we can then assume that the rest of $f(x)$ has the same value.
- Write down the domain of x .
- Sketch the function

Question 21 Define the function $f(x) = \tan^{-1} x + \tan^{-1} \left(\frac{1}{x} \right)$.

- Find $f'(x)$.
- Write down the domain of x .
- Show that the function is odd.
- By calculating $f\left(\frac{\pi}{4}\right)$, sketch the function

Question 22 Consider the curve $y = x \sin^{-1} x$.

- Find the domain and range.
- Show that the function is even.
- Find y' .
- Find any turning points and determine their nature.
- Discuss the behaviour of the gradient function as $x \rightarrow -1^+$ and as $x \rightarrow 1^-$
- Sketch the curve.

Question 23 Consider the curve $y = \sin^{-1} \sqrt{x}$

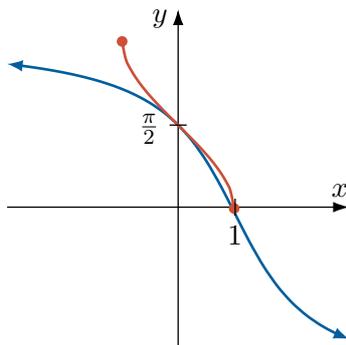
- Find y'
- Hence show that $y' = \frac{1}{\sin 2y}$

Question 24 Let

$$f(x) = 2 \cos^{-1} \left(\frac{x}{\sqrt{2}} \right) + \sin^{-1} (x^2 - 1).$$

Show that $f'(x) = 0$ for $x > 0$.

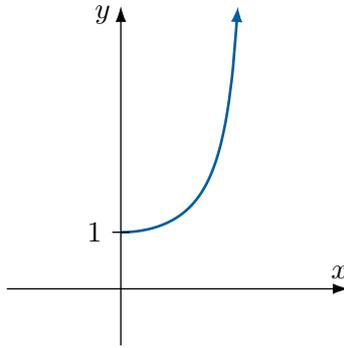
Question 25 The diagram below shows the graphs of $y = \cos^{-1} x$ and $y = 2 \tan^{-1}(1 - x)$.



- Show that the point $\left(0, \frac{\pi}{2}\right)$ is a point of intersection of the curves.
- Show that they have a common tangent at this point.
- Find the equation of the common tangent.

Challenge Problems

Problem 1 Consider the curve $y = \sec x$ in the interval $0 \leq x < \frac{\pi}{2}$ shown below.



- Find $\frac{dy}{dx}$ and show that in this interval it is an increasing function, hence it has an inverse.
- Write down the range of the function
- Show that $\frac{dx}{dy} = \frac{1}{y\sqrt{y^2-1}}$
- Hence, show that $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

Problem 2 Let $y = \tan^{-1} \sqrt{3-x^2}$

- State the domain
- Find $\frac{dy}{dx}$
- Describe the gradient of the tangent as $x \rightarrow \pm\sqrt{3}$
- Hence, sketch the curve

Exercise 4B

Integration involving inverse trigonometric functions

Fundamentals

Fundamentals 1

Write down the formula for the following.

$$(a) \int \frac{1}{\sqrt{1-x^2}} dx \quad (b) \int \frac{1}{\sqrt{a^2-x^2}} dx \quad (c) \int \frac{-1}{\sqrt{a^2-x^2}} dx$$

$$(d) \int \frac{1}{1+x^2} dx \quad (e) \int \frac{a}{a^2+x^2} dx \quad (f) \int \frac{1}{a^2+x^2} dx$$

Fundamentals 2

Find the formula for the following.

$$(a) \int \frac{1}{\sqrt{a^2-b^2x^2}} dx \quad (b) \int \frac{-1}{\sqrt{a^2-b^2x^2}} dx$$

$$(c) \int \frac{1}{a^2+b^2x^2} dx \quad (d) \int \frac{f'(x)}{\sqrt{1-(f(x))^2}} dx$$

$$(e) \int \frac{-f'(x)}{\sqrt{1-(f(x))^2}} dx \quad (f) \int \frac{f'(x)}{1+(f(x))^2} dx$$

Question 1 Find the following indefinite integrals:

$$(a) \int \frac{1}{\sqrt{64-x^2}} dx \quad (b) \int \frac{1}{\sqrt{25-x^2}} dx \quad (c) \int \frac{-1}{\sqrt{9-x^2}} dx$$

$$(d) \int \frac{1}{\sqrt{2-x^2}} dx \quad (e) \int \frac{-1}{\sqrt{8-x^2}} dx \quad (f) \int \frac{1}{\sqrt{\frac{1}{9}-x^2}} dx$$

Question 2 Find the following indefinite integrals:

$$(a) \int \frac{1}{9+x^2} dx \quad (b) \int \frac{5}{25+x^2} dx \quad (c) \int \frac{4}{16+x^2} dx$$

$$(d) \int \frac{3}{3+x^2} dx \quad (e) \int \frac{4}{2+x^2} dx \quad (f) \int \frac{1}{\frac{1}{16}+x^2} dx$$

Question 3 Find the following.

$$(a) \int \frac{1}{\sqrt{1-16x^2}} dx \quad (b) \int \frac{1}{\sqrt{1-6x^2}} dx \quad (c) \int \frac{1}{\sqrt{25-4x^2}} dx$$

$$(d) \int \frac{1}{\sqrt{9-25x^2}} dx \quad (e) \int \frac{1}{\sqrt{1-(x-2)^2}} dx \quad (f) \int \frac{1}{\sqrt{9-(2x+1)^2}} dx$$

Question 4 Find the following.

(a) $\int \frac{1}{1+9x^2} dx$

(b) $\int \frac{6}{9+4x^2} dx$

(c) $\int \frac{1}{1+(x-2)^2} dx$

(d) $\int \frac{2}{1+(2x-1)^2} dx$

Question 5 Complete the square of the denominator and hence find

(a) $\int \frac{1}{x^2+4x+5} dx$

(b) $\int \frac{1}{x^2+4x+8} dx$

(c) $\int \frac{1}{x^2+6x+10} dx$

(d) $\int \frac{2}{x^2-8x+20} dx$

Question 6 Complete the square of the denominator and hence find

(a) $\int \frac{1}{\sqrt{6x-x^2-8}} dx$

(b) $\int \frac{1}{\sqrt{4x-x^2}} dx$

(c) $\int \frac{1}{\sqrt{x-x^2}} dx$

(d) $\int \frac{1}{\sqrt{-24+30x-9x^2}} dx$

Question 7 Evaluate

(a) $\int_0^1 \frac{1}{1+x^2} dx$

(b) $\int_0^2 \frac{1}{4+x^2} dx$

(c) $\int_0^{\frac{1}{3}} \frac{1}{1+9x^2} dx$

(d) $\int_0^1 \frac{6}{9+3x^2} dx$

(e) $\int_0^2 \frac{1}{1+(x-1)^2} dx$

(f) $\int_1^2 \frac{1}{3+(x-2)^2} dx$

Question 8 Evaluate

(a) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx$

(b) $\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} dx$

(c) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}} dx$

(d) $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}} dx$

(e) $\int_0^4 \frac{1}{\sqrt{4x-x^2}} dx$

(f) $\int_0^{\frac{1}{6}} \frac{1}{\sqrt{6x-9x^2}} dx$

Question 9

(a) Simplify $\frac{(1+x^2)-1}{1+x^2}$ and hence show that $\int_0^1 \frac{x^2}{1+x^2} dx = 1 - \frac{\pi}{4}$

(b) Simplify $\frac{(x^4-1)-1}{1+x^2}$ and hence show that $\int_0^1 \frac{x^4}{1+x^2} dx = \frac{\pi}{4} - \frac{2}{3}$

(c) Differentiate $y = (\sin^{-1} x)^2$ hence find $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x dx}{\sqrt{1-x^2}}$

(d) Differentiate $y = \cos^{-1}(4-x)$ and hence find $\int_3^5 \frac{1}{\sqrt{-x^2+8x-15}} dx$

(e) Differentiate $y = \tan^{-1} \frac{x^3}{2}$ and hence find $\int_0^1 \frac{6x^2}{4+x^6} dx$

Question 10 Use the formula for $\int \frac{f'(x)}{1 + (f(x))^2} dx$ to find the following.

- (a) $\int \frac{2x}{1 + x^4} dx$ (b) $\int \frac{x^2}{1 + x^6} dx$ (c) $\int \frac{x}{4 + 9x^4} dx$
 (d) $\int \frac{e^x}{1 + e^{2x}} dx$ (e) $\int \frac{e^{2x}}{1 + e^{4x}} dx$ (f) $\int \frac{e^x}{25 + 16e^{2x}} dx$
 (g) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx$ (h) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x} dx$ (i) $\int_0^{\frac{\pi}{6}} \frac{4 \cos x}{1 + 16 \sin^2 x} dx$

Question 11 Use the formula for $\int \frac{f'(x)}{\sqrt{1 - (f(x))^2}} dx$ to find the following.

- (a) $\int \frac{x}{\sqrt{1 - x^4}} dx$ (b) $\int \frac{x^2}{\sqrt{1 - x^6}} dx$ (c) $\int \frac{x}{\sqrt{16 - 9x^4}} dx$
 (d) $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$ (e) $\int \frac{e^{2x}}{\sqrt{9 - e^{4x}}} dx$ (f) $\int \frac{e^x}{\sqrt{16 - 25e^{2x}}} dx$

Question 12 Without evaluating directly, write down and explain your result.

- (a) $\int_{-1}^1 \sin^{-1} x dx$ (b) $\int_{-1}^1 \tan^{-1} x dx$

Question 13 Find the equation of the curve $y = f(x)$ if

- (a) it passes through $\left(0, \frac{\pi}{4}\right)$ and $\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$
 (b) it passes through $\left(2, \frac{\pi}{8}\right)$ and $\frac{dy}{dx} = \frac{1}{4 + x^2}$
 (c) it passes through $\left(0, -\frac{\pi}{2}\right)$ and $\frac{dy}{dx} = \frac{1}{\sqrt{4 - x^2}}$
 (d) it passes through $(0, 0)$ and $\frac{dy}{dx} = \frac{-2}{\sqrt{25 - 4x^2}}$

Question 14

- (a) Differentiate $y = x \sin^{-1} x + \sqrt{1 - x^2}$ hence evaluate $\int_0^1 \sin^{-1} x dx$
 (b) Differentiate $y = x \cos^{-1} x$ hence evaluate $\int_0^{\frac{\sqrt{3}}{2}} \cos^{-1} x dx$
 (c) Differentiate $y = x \tan^{-1} x$ and hence evaluate $\int_0^1 \tan^{-1} x dx$
 (d) Differentiate $x^2 \tan^{-1} x$ and hence evaluate $\int_0^1 x \tan^{-1} x dx$

Question 15 [Inverse sine and inverse cosine]

A teacher gives Bob and Mary the integral

$$\int \frac{-1}{\sqrt{25-x^2}} dx$$

Bob uses the formula for inverse cosine whereas Mary first takes out the minus sign, then uses the formula for inverse sine.

- Find Bob's answer.
- Find Mary's answer.
- Explain why their answers are different, despite being valid methods.

Challenge Problems

Problem 1 Find

$$\int \sqrt{\frac{1-x}{1+x}} dx$$

by first rationalising the numerator.

Problem 2 Find $\int_{-1}^1 \cos^{-1} x dx$.

Problem 3 Find the following.

(a) $\int \frac{\tan^{-1} x}{1+x^2} dx$

(b) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

(c) $\int \frac{1}{(1+x^2)\tan^{-1} x} dx$

(d) $\int \frac{1}{\sqrt{1-x^2}\sin^{-1} x} dx$

Problem 4 Find the following.

(a) $\int \frac{1}{x\sqrt{x} + \sqrt{x}} dx$

(b) $\int \frac{1}{e^x + e^{-x}} dx$

Problem 5

(a) Show that $\frac{1}{\sqrt{e^{2x}-1}} = \frac{e^{-x}}{\sqrt{1-e^{-2x}}}$.

(b) Hence, find $\int \frac{1}{\sqrt{e^{2x}-1}} dx$.

Exercise 4C

Review of integration techniques

Fundamentals

Fundamentals 1

Write down the formula for the following.

$$(a) \int x^n dx \qquad (b) \int (ax + b)^n dx \qquad (c) \int f'(x)(f(x))^n dx$$

Fundamentals 2

Write down the formula for the following.

$$(a) \int e^{ax+b} dx \qquad (b) \int f'(x)e^{f(x)} dx \qquad (c) \int a^x dx$$

Fundamentals 3

Write down the formula for the following.

$$(a) \int \sin(ax + b) dx \qquad (b) \int \cos(ax + b) dx \qquad (c) \int \tan(ax + b) dx$$

$$(d) \int f'(x) \sin(f(x)) dx \qquad (e) \int f'(x) \cos(f(x)) dx \qquad (f) \int f'(x) \tan(f(x)) dx$$

Fundamentals 4

Write down the formula for the following.

$$(a) \int \frac{1}{x} dx \qquad (b) \int \frac{1}{ax + b} dx \qquad (c) \int \frac{f'(x)}{f(x)} dx$$

Fundamentals 5

Write down the formula for the following.

$$(a) \int \frac{1}{\sqrt{a^2 - x^2}} dx \qquad (b) \int \frac{-1}{\sqrt{a^2 - x^2}} dx \qquad (c) \int \frac{1}{a^2 + x^2} dx$$

$$(d) \int \frac{f'(x)}{\sqrt{1 - (f(x))^2}} dx \qquad (e) \int \frac{-f'(x)}{\sqrt{1 - (f(x))^2}} dx \qquad (f) \int \frac{f'(x)}{1 + (f(x))^2} dx$$

Question 1

(a) $\int x\sqrt{x} dx$

(b) $\int \frac{1}{x^3} dx$

(c) $\int \frac{1}{x\sqrt{x}} dx$

(d) $\int (3x+1)^4 dx$

(e) $\int \sqrt{3x+1} dx$

(f) $\int \frac{1}{3x+1} dx$

(g) $\int \frac{1}{1-3x} dx$

(h) $\int \frac{1}{\sqrt{3x+1}} dx$

(i) $\int \frac{x}{\sqrt{3x^2+1}} dx$

(j) $\int \frac{1}{\sqrt{1-3x}} dx$

(k) $\int \frac{3x+1}{x} dx$

(l) $\int \frac{3x+1}{3} dx$

(m) $\int \frac{3x^2+1}{x} dx$

(n) $\int \frac{3x^2+1}{x^2} dx$

(o) $\int x^2(3x+1) dx$

(p) $\int \frac{1}{(3x+1)^2} dx$

(q) $\int \frac{1}{3x^2+1} dx$

(r) $\int \frac{x}{3x^2+1} dx$

(s) $\int \frac{x+1}{x^2+2x+5} dx$

(t) $\int \frac{5x+15}{x^2+6x-5} dx$

(u) $\int \frac{x+2}{3x^2+7x+2} dx$

Question 2

(a) $\int \frac{1}{4+x^2} dx$

(b) $\int \frac{x}{4+x^2} dx$

(c) $\int \frac{1}{1+4x^2} dx$

(d) $\int \frac{4x}{4-x^2} dx$

(e) $\int \frac{1}{\sqrt{4-x^2}} dx$

(f) $\int \frac{x}{\sqrt{4-x^2}} dx$

(g) $\int x\sqrt{4-x^2} dx$

(h) $\int (1-2x)\sqrt{x^2-x+5} dx$

(i) $\int x^2\sqrt{4x^3-1} dx$

(j) $\int \frac{1}{4+9x^2} dx$

(k) $\int \frac{3x}{4+9x^2} dx$

(l) $\int \frac{12-4x}{x^2-6x+5} dx$

(m) $\int \frac{12-4x}{(x^2-6x+5)^4} dx$

(n) $\int \frac{1}{4+(3x+1)^2} dx$

(o) $\int \frac{1}{\sqrt{4-(3x+1)^2}} dx$

Question 3

(a) $\int e^{4x} dx$

(b) $\int e^{4x}(e^{4x}+1) dx$

(c) $\int e^{1-4x} dx$

(d) $\int xe^{4x^2} dx$

(e) $\int 4xe^{1-4x^2} dx$

(f) $\int \frac{e^{4x}+1}{e^{4x}} dx$

(g) $\int \frac{e^{4x}}{e^{4x}+1} dx$

(h) $\int \frac{e^{2x}}{e^{4x}+1} dx$

(i) $\int e^{4x}(e^{4x}+1)^5 dx$

(j) $\int \frac{e^{4x}}{\sqrt{e^{4x}+1}} dx$

(k) $\int 2^x dx$

(l) $\int 3^{2x} dx$

(m) $\int 3^{2-4x} dx$

(n) $\int x2^{x^2} dx$

(o) $\int (e^{2x}-e^{-2x})^2 dx$

Question 4

- | | | |
|---|---|---|
| (a) $\int \sin \frac{x}{2} dx$ | (b) $\int \sin^2 \frac{x}{2} dx$ | (c) $\int \cos \frac{x}{2} dx$ |
| (d) $\int \sec^2 \frac{x}{2} dx$ | (e) $\int \tan \frac{x}{2} dx$ | (f) $\int \tan^2 \frac{x}{2} dx$ |
| (g) $\int \tan^2 \frac{x}{2} \sec^2 \frac{x}{2} dx$ | (h) $\int \sin^2 \frac{x}{2} \cos \frac{x}{2} dx$ | (i) $\int \cos^2 \frac{x}{2} \sin \frac{x}{2} dx$ |

Question 5

- | | | |
|--|---|--|
| (a) $\int \frac{\sin 2x}{\cos 2x} dx$ | (b) $\int \frac{\sin x}{\cos^3 x} dx$ | (c) $\int \frac{\sin x}{\cos^2 x} dx$ |
| (d) $\int \frac{\cos x}{4 + \sin^2 x} dx$ | (e) $\int \frac{\cos x}{4 + \sin x} dx$ | (f) $\int \frac{\sin x}{1 - \cos x} dx$ |
| (g) $\int \frac{\sin x}{\sqrt{\cos x}} dx$ | (h) $\int \sin x \sqrt{\cos x} dx$ | (i) $\int \frac{\sin x}{\sqrt{3 - \cos x}} dx$ |
| (j) $\int \sin x \cos^3 x dx$ | (k) $\int \cos x \sin^3 x dx$ | (l) $\int \sec^2 x \tan^3 x dx$ |
| (m) $\int x \sin(x^2) dx$ | (n) $\int \frac{1}{\sqrt{x}} \cos(\sqrt{x}) dx$ | (o) $\int \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right) dx$ |

Question 6 Express as a single trigonometric function then find the anti-derivative of the function.

- | | | |
|---|---------------------------------------|---|
| (a) $2 \sin 3x \cos 3x$ | (b) $\sin 2x \cos 2x$ | (c) $\sin \frac{x}{2} \cos \frac{x}{2}$ |
| (d) $\cos^2 4x - \sin^2 4x$ | (e) $2 \cos^2 5x - 1$ | (f) $2 \cos^2 \frac{x}{2} - 1$ |
| (g) $\cos^2(3x)$ | (h) $\sin^2(2x)$ | (i) $\cos^2\left(\frac{3x}{2}\right)$ |
| (j) $\frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$ | (k) $\frac{2 \tan 3x}{1 - \tan^2 3x}$ | (l) $\sin 5x \cos 3x - \cos 5x \sin 3x$ |

Question 7 Use the formulae for trigonometric products to express the following as sum or difference of trigonometric functions and then find the primitive

- | | | |
|------------------------|---|---|
| (a) $2 \sin 4x \cos x$ | (b) $2 \cos 5x \sin 3x$ | (c) $\sin 4x \sin 2x$ |
| (d) $\cos 5x \cos 2x$ | (e) $\sin \frac{5x}{2} \cos \frac{7x}{2}$ | (f) $\cos \frac{9x}{2} \cos \frac{3x}{2}$ |

Question 8 Evaluate the following.

(a) $\int_0^{\frac{\pi}{6}} \sin 3x \sin 2x \, dx$

(b) $\int_0^{\frac{\pi}{3}} \sin x \cos 2x \, dx$

(c) $\int_0^{\frac{\pi}{3}} \sin 3x \cos 2x \, dx$

(d) $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} 2 \cos 3x \cos x \, dx$

(e) $\int_0^{\frac{\pi}{6}} \cos 3x \cos x \, dx$

(f) $\int_0^{\frac{\pi}{2}} \sin\left(x + \frac{\pi}{4}\right) \cos\left(x - \frac{\pi}{4}\right) \, dx$

(g) $\int_0^{\frac{\pi}{2}} \cos\left(x + \frac{\pi}{4}\right) \cos\left(x - \frac{\pi}{4}\right) \, dx$

(h) $\int_0^{\frac{\pi}{2}} \sin\left(x + \frac{\pi}{4}\right) \sin\left(x - \frac{\pi}{4}\right) \, dx$

Question 9

(a) Differentiate $\ln\left(\frac{x-1}{x+1}\right)$ and hence find $\int_2^4 \frac{dx}{(x-1)(x+1)}$.

(b) Differentiate $x^3 - \ln(x^3 + 1)$ and hence find $\int_0^1 \frac{3x^5}{x^3 + 1} \, dx$.

(c) Differentiate $\ln(x + \sqrt{x^2 - 9})$ and hence find $\int_5^9 \frac{1}{\sqrt{x^2 - 9}} \, dx$.

Question 10 Show that $\int_0^1 \frac{1+x}{1+x^2} \, dx = \frac{\pi}{4} + \frac{1}{2} \ln 2$

Question 11

(a) Find $\int_0^{\frac{\pi}{4}} (\tan^4 x + \tan^2 x) \, dx$

(b) Find $\int_0^{\frac{\pi}{4}} (\tan^6 x + \tan^4 x) \, dx$

(c) Hence, find $\int_0^{\frac{\pi}{4}} \tan^6 x \, dx$

Challenge Problems

Problem 1

(a) $\int \frac{1 + \tan x}{1 - \tan x} \, dx$

(b) $\int \frac{\cos x}{\sqrt{4 - \sin^2 x}} \, dx$

(c) $\int \frac{1}{1 + \cos 2x} \, dx$

(d) $\int \frac{1}{e^x + 1} \, dx$

(e) $\int \frac{e^x - 1}{e^x + 1} \, dx$

(f) $\int \frac{e^{2x}}{e^x + 1} \, dx$

Problem 2

(a) $\int \frac{2x+1}{x+1} \, dx$

(b) $\int \frac{2x+1}{4x+1} \, dx$

(c) $\int \frac{x^2 + x}{x^2 + 1} \, dx$

Exercise 4D

Integration of squares of sine and cos

Fundamentals

Fundamentals 1

Complete the following identities.

(a) $\cos 2x = \underline{\hspace{2cm}}$ (b) $\cos^2 x = \underline{\hspace{2cm}}$ (c) $\sin^2 x = \underline{\hspace{2cm}}$

Fundamentals 2

Complete the following.

(a) $\int \sin^2 x \, dx = \int \underline{\hspace{2cm}} \, dx = \underline{\hspace{2cm}} + C$

(b) $\int \cos^2 x \, dx = \int \underline{\hspace{2cm}} \, dx = \underline{\hspace{2cm}} + C$

(c) $\int \sin^2(nx) \, dx = \int \underline{\hspace{2cm}} \, dx = \underline{\hspace{2cm}} + C$

(d) $\int \cos^2(nx) \, dx = \int \underline{\hspace{2cm}} \, dx = \underline{\hspace{2cm}} + C$

Question 1 Find the primitive function of

(a) $\sin^2 2x$ (b) $\sin^2 6x$ (c) $\sin^2 \pi x$
 (d) $\sin^2 \frac{x}{3}$ (e) $8 \sin^2 \frac{2x}{3}$ (f) $\sin^2(2x - 3)$

Question 2 Find the primitive function of

(a) $\cos^2 2x$ (b) $\cos^2 8x$ (c) $\cos^2 \pi x$
 (d) $10 \cos^2 10x$ (e) $4 \cos^2 \frac{x}{2}$ (f) $\cos^2 \left(\frac{\pi x}{2} \right)$

Question 3 Find the exact value of the following definite integrals.

(a) $\int_{\frac{\pi}{2}}^{\pi} \sin^2 x \, dx$ (b) $\int_0^{2\pi} \frac{1}{2\pi} \sin^2 x \, dx$ (c) $\int_0^{\pi} \sin^2 2x \, dx$
 (d) $\int_0^{\pi} \sin^2 \frac{x}{2} \, dx$ (e) $\int_0^{\frac{\pi}{12}} 8 \sin^2 x \, dx$ (f) $\int_0^{\frac{\pi}{8}} \cos^2 x \, dx$
 (g) $\int_0^{\frac{\pi}{2}} \cos^2 3x \, dx$ (h) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \left(x - \frac{\pi}{4} \right) \, dx$ (i) $2 \int_0^{\pi} \cos^2 x + 1 \, dx$

Question 4 Find the exact value of the definite integrals

(a) $\int_0^{\frac{\pi}{3}} (1 + \sin 3x)^2 dx$ (b) $\int_0^{\frac{\pi}{4}} (1 - \cos 4x)^2 dx$ (c) $\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)^2 dx$

Question 5 Find $\int_0^{\frac{\pi}{2}} \cos^4 x - \sin^4 x dx$.

Question 6 Show that $\int_{-\frac{2\pi}{3}}^{\frac{4\pi}{3}} \cos^2 \left(x - \frac{\pi}{3}\right) dx = \pi$

Challenge Problems

Problem 1

- (a) Show that $\sin^2 x \cos^2 x = \frac{1}{8}(1 - \cos 4x)$
 (b) Hence, find $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x dx$.

Problem 2

 Find the following.

- (a) $\int \sin^2 x \cos^2 x dx$ (b) $\int_0^{\frac{\pi}{6}} \sin^2 x \cos^2 x dx$
 (c) $\int 8 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} dx$ (d) $\int_0^{\frac{\pi}{4}} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} dx$

Problem 3 Show that $\int \cos^4 x dx = \frac{1}{32}(12x + 8 \sin 2x + \sin 4x) + C$

Hint: $\cos^4 x = (\cos^2 x)^2$

Exercise 4E

Integration by substitution

Fundamentals

Fundamentals 1

Many integrals are algebraically identical, despite looking entirely different. For example the integral $\int \frac{\ln x}{x} dx$ can be transformed into $\int u du$ despite looking nothing like it at all. This can be explained using a process called integration by s _____.

Fundamentals 2

Complete the steps below for substitution problems involving indefinite integrals.

- We can use either $\text{---} = h(x)$ or $\text{---} = h(u)$ to reduce the integral to a standard form.
- The integral $\int f(x) dx$ can then be expressed as $\int g(\text{---}) d\text{---}$ and integrated as usual.
- Remember to c _____ back to the original variable once you integrate.

Fundamentals 3

Complete the steps below for substitution problems involving definite integrals.

- When a substitution is used to evaluate a d _____ integral, the l _____ of integration must be converted to refer to the new variable.
- Your working out should begin with

$$\begin{aligned} x = x_2 &\longrightarrow u = u_2 \\ x = x_1 &\longrightarrow u = \text{---} \end{aligned}$$

- Hence, you should have

$$\int_{x_1}^{x_2} f(x) dx = \int_{\square}^{\square} \text{---} du$$

where $u = h(x)$ so $u_1 = h(\text{---})$ and $u_2 = h(\text{---})$.

Fundamentals 4

Remember that if you are given the option to evaluate the integral *without* a substitution, you can always use the r _____ c _____ rule.

Question 1 Complete the following steps to calculate $\int x(x^2 + 1)^4 dx$ using $u = x^2 + 1$.

(a) Find the differential term.

$$\begin{aligned}\frac{du}{dx} &= \text{---} \\ du &= \text{---} dx \\ \therefore x dx &= \text{---} du\end{aligned}$$

(b) Express the integral in terms of u , and hence find the integral.

$$\begin{aligned}\int x(x^2 + 1)^4 dx &= \int x(x^2 + 1)^4 dx \\ &= \int u^4 \frac{1}{2} du \\ &= \text{---} \int u^4 du \\ &= \text{---} \times \text{---} u^5 + C \\ &= \text{---} (\text{---})^5 + C\end{aligned}$$

Question 2 Complete the following steps to calculate $\int \frac{x}{\sqrt{3x^2 - 5}} dx$ using $u = 3x^2 - 5$.

(a) Calculate the differential term.

$$\begin{aligned}\frac{du}{dx} &= \text{---} \\ du &= \text{---} dx \\ \therefore x dx &= \text{---} du\end{aligned}$$

(b) Express the integral in terms of u , and hence find the integral.

$$\begin{aligned}\int \frac{x}{\sqrt{3x^2 - 5}} dx &= \int \frac{x}{\sqrt{3x^2 - 5}} dx \\ &= \int \frac{1}{\sqrt{u}} \frac{1}{6} du \\ &= \text{---} \int \frac{1}{\sqrt{u}} du \\ &= \text{---} \times \text{---} \sqrt{u} + C \\ &= \text{---} \sqrt{\text{---}} + C\end{aligned}$$

Question 3 Find the following integrals using the given substitution.

- (a) $\int 2x(x^2 + 4)^3 dx$ using $u = x^2 + 4$. (b) $\int x^2(x^3 + 1)^3 dx$ using $u = x^3 + 1$.
- (c) $\int x(4 - 3x^2)^5 dx$ using $u = 4 - 3x^2$. (d) $\int \frac{x}{(x^2 + 1)^2} dx$ using $u = x^2 + 1$.
- (e) $\int \frac{x^2}{(1 - 2x^3)^3} dx$ using $u = 1 - 2x^3$. (f) $\int \frac{2x - 1}{(x^2 - x)^4} dx$ using $u = x^2 - x$.

Question 4 Find the following integrals using the given substitution.

- (a) $\int (x - 1)(x^2 - 2x + 2)^5 dx$ using $u = x^2 - 2x + 2$.
- (b) $\int (1 - x)(x^2 - 2x + 2)^4 dx$ using $u = x^2 - 2x + 2$.
- (c) $\int \frac{x - 2}{(x^2 - 4x + 2)^4} dx$ using $u = x^2 - 4x + 2$.
- (d) $\int \frac{3 - 2x}{(x^2 - 3x + 5)^4} dx$ using $u = x^2 - 3x + 5$.

Question 5 Find the following integrals using the given substitution.

- (a) $\int x(x + 4)^5 dx$ using $u = x + 4$ (b) $\int x^2(x - 1)^8 dx$ using $u = x - 1$
- (c) $\int x(4 - x)^3 dx$ using $u = 4 - x$ (d) $\int \frac{x}{x + 6} dx$ using $u = x + 6$
- (e) $\int \frac{x}{(2x - 1)^2} dx$ using $u = 2x - 1$ (f) $\int \frac{x}{(x + 2)^3} dx$ using $u = x + 2$

Question 6 Find the following integrals using the given substitution.

- (a) $\int x\sqrt{x^2 - 4} dx$ using $u = x^2 - 4$ (b) $\int x^2\sqrt{1 + 4x^3} dx$ using $u = 1 + 4x^3$
- (c) $\int \frac{x^3}{\sqrt{x^4 + 2}} dx$ using $u = x^4 + 2$ (d) $\int \frac{x}{\sqrt{1 + 3x^2}} dx$ using $u = 1 + 3x^2$
- (e) $\int \frac{x^2}{\sqrt{1 - x^3}} dx$ using $u = 1 - x^3$ (f) $\int \frac{x + 1}{\sqrt{x^2 + 2x - 6}} dx$ using $u = x^2 + 2x - 6$
- (g) $\int x\sqrt{x - 4} dx$ using $u = x - 4$ (h) $\int \frac{x}{\sqrt{4x + 1}} dx$ using $u = \sqrt{4x + 1}$
- (i) $\int \frac{x}{\sqrt{1 + x}} dx$ using $u = 1 + x$ (j) $\int \frac{x}{(1 + x)\sqrt{1 + x}} dx$ using $u = 1 + x$

Question 7 Find the following integrals using the given substitution.

(a) $\int x e^{x^2} dx$ using $u = x^2$

(b) $\int x^2 e^{x^3} dx$ using $u = x^3$

(c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ using $u = \sqrt{x}$

(d) $\int \frac{e^{\frac{1}{x}}}{x^2} dx$ using $u = \frac{1}{x}$

(e) $\int e^x \sqrt{e^x - 1} dx$ using $u = e^x - 1$

(f) $\int \frac{e^x}{\sqrt{e^x + 1}} dx$ using $u = e^x + 1$

(g) $\int \frac{e^x}{1 + e^x} dx$ using $u = 1 + e^x$

(h) $\int \frac{e^{2x}}{1 + e^{2x}} dx$ using $u = e^{2x}$

Question 8 Find

(a) $\int \frac{\ln x}{x} dx$ using $u = \ln x$

(b) $\int \frac{(\ln x)^3}{x} dx$ using $u = \ln x$

(c) $\int \frac{\sqrt{\ln x}}{x} dx$ using $u = \ln x$

(d) $\int \frac{1}{x \ln x} dx$ using $u = \ln x$

Question 9 Find

(a) $\int \sin x \cos x dx$ using $u = \sin x$

(b) $\int \sin x \cos x dx$ using $u = \cos x$

(c) $\int \sin x \cos^5 x dx$ using $u = \cos x$

(d) $\int \sec^2 x \tan x dx$ using $u = \tan x$

(e) $\int \sec^2 x \tan^6 x dx$ using $u = \tan x$

(f) $\int \tan x dx$ using $u = \cos x$

(g) $\int \frac{\sin x}{5 + \cos x} dx$ using $u = 5 + \cos x$

(h) $\int \cos x e^{\sin x} dx$ using $u = \sin x$

(i) $\int \frac{\cos x}{\sin^3 x} dx$ using $u = \sin x$

(j) $\int \frac{\sin x}{\sqrt{\cos x}} dx$ using $u = \cos x$

Question 10 [Involving inverse trigonometric functions]

Find

(a) $\int \frac{1}{\sqrt{9 - (2x - 1)^2}} dx$ using $u = 2x - 1$

(b) $\int \frac{1}{4 + (3x + 2)^2} dx$ using $u = 3x + 2$

(c) $\int \frac{x}{1 + x^4} dx$ using $u = x^2$

(d) $\int \frac{x}{\sqrt{1 - x^4}} dx$ using $u = x^2$

(e) $\int \frac{1}{x \sqrt{1 - (\ln x)^2}} dx$ using $u = \ln x$

(f) $\int \frac{e^x}{\sqrt{4 - e^{2x}}} dx$ using $u = e^x$

(g) $\int \frac{e^x}{1 + e^{2x}} dx$ using $u = e^x$

(h) $\int \frac{dx}{\sqrt{x}(1+x)}$ using $u = \sqrt{x}$

Question 11 Complete the following steps to calculate $\int_0^3 \frac{x}{\sqrt{x^2+16}} dx$ using $u = x^2 + 16$.

(a) Find the differential term.

$$\begin{aligned}\frac{du}{dx} &= \text{---} \\ du &= \text{---} dx \\ \therefore x dx &= \text{---} du\end{aligned}$$

(b) Calculate the new limits.

(c) Express the integral in terms of u , and hence find the integral.

$$\begin{aligned}\int_0^3 \frac{x}{\sqrt{x^2+16}} dx &= \int_{\square}^{\square} \text{---} du \\ &= \frac{1}{2} \int_{\square}^{\square} \text{---} du \\ &= [\text{---}]_{\square}^{\square} \\ &= \end{aligned}$$

Question 12 Evaluate

(a) $\int_1^2 2x(2x-1)^3 dx$ using $u = 2x-1$

(b) $\int_0^2 \frac{3x}{(3x+1)^2} dx$ using $u = 3x+1$

(c) $\int_0^1 x\sqrt{3x^2+1} dx$ using $u = 3x^2+1$

(d) $\int_0^4 x\sqrt{4-x} dx$ using $u = 4-x$

(e) $\int_3^{11} \frac{x}{\sqrt{x-2}} dx$ using $u = \sqrt{x-2}$

(f) $\int_{-4}^0 \frac{x}{\sqrt{1-2x}} dx$ using $u = \sqrt{1-2x}$

Question 13 Use $u = 1-x$ to show that

$$\int_0^1 x(1-x)^n dx = \frac{1}{(n+1)(n+2)}$$

Question 14 Find the following.

(a) $\int_0^{\ln 8} \frac{e^x}{\sqrt{1+e^x}} dx$ using $u = e^x$

(b) $\int_0^2 xe^{2x^2-4} dx$ using $u = 2x^2-4$

Question 15 Evaluate the following

(a) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}} dx$ using $u = \cos x$

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x} dx$ using $u = \tan x$

(c) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin x} dx$ using $u = \sin x$

(d) $\int_0^{\frac{\pi}{2}} \sin x e^{\cos x} dx$ using $u = \cos x$

(e) $\int_0^{\frac{\pi}{3}} \sec^2 x \tan^4 x dx$ using $u = \tan x$

(f) $\int_0^{\frac{\pi}{8}} \sin^2 2x \cos 2x dx$ using $u = \sin 2x$

Question 16

(a) $\int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$ using $u = \tan^{-1} x$

(b) $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1 - x^2}} dx$ using $u = \sin^{-1} x$

Question 17 [Different techniques can yield ‘different’ answers]

A teacher gives Bob and Mary and Jane the integral

$$\int \sin x \cos x dx$$

Bob uses the substitution $u = \cos x$. Mary uses the substitution $u = \sin x$. Jane decides a substitution is not necessary, and uses double-angle formulae instead.

- Find Bob’s answer.
- Find Mary’s answer.
- Find Jane’s answer.
- Explain why they all are different, despite being valid methods.

Question 18 [Finding the difference between two valid answers]

Consider the integral

$$I = \int \frac{1}{\sqrt{2x - x^2}} dx$$

- Show that $I = \sin^{-1}(x - 1) + C_1$ by first completing the square in the denominator.
- Show that $I = 2 \sin^{-1}\left(\sqrt{\frac{x}{2}}\right) + C_2$ by using $u = \sqrt{x}$.
- Explain why

$$f(x) = 2 \sin^{-1}\left(\sqrt{\frac{x}{2}}\right) - \sin^{-1}(x - 1)$$

is a constant function, and state the domain of x .

- Hence, find the value of the constant by which the two expressions for I differ.

Question 19 [Trigonometric substitution]

Evaluate the following

(a) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$ using $x = \sin \theta$

(b) $\int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{x^2\sqrt{1+x^2}} dx$ using $x = \tan \theta$

(c) $\int_1^2 \frac{\sqrt{4-x^2}}{x^2} dx$ using $x = 2 \sin \theta$

(d) $\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$ using $x = \tan \theta$

(e) $\int_0^{\frac{1}{4}} \sqrt{\frac{x}{1-x}} dx$ using $x = \sin^2 \theta$

(f) $\int_{\frac{1}{2}}^1 \sqrt{\frac{1-x}{1+x}} dx$ using $x = \cos 2\theta$

Question 20 [Verifying a well-known formula]

(a) Use $x = r \sin \theta$ to find $\int_0^r \sqrt{r^2 - x^2} dx$

(b) Explain how this was to be expected.

Question 21 [Trigonometric substitution problems are much harder without limits!]

Consider the integral

$$\int \frac{x^2}{\sqrt{1-x^2}} dx,$$

using $x = \sin \theta$.

(a) Show that

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \frac{\theta}{2} - \frac{1}{2} \sin \theta \cos \theta + C$$

when left in terms of θ .(b) Use the fact that $\sin \theta = \frac{x}{1}$ to construct a right-angled triangle, and use Pythagoras' theorem to find the missing side.

(c) Hence, show that

$$\int \frac{x^2}{\sqrt{1-x^2}} dx = \frac{1}{2} \sin^{-1} x - \frac{x}{2} \sqrt{1-x^2} + C$$

Question 22 Use a similar technique to the previous question to find the following indefinite integrals.

(a) $\int \frac{x}{\sqrt{1-x^2}} dx$ using $x = \sin \theta$

(b) $\int \frac{1}{x^2\sqrt{1-x^2}} dx$ using $x = \cos \theta$

(c) $\int \frac{x}{\sqrt{1+x^2}} dx$ using $x = \tan \theta$

(d) $\int \frac{1}{(1+x^2)\sqrt{1+x^2}} dx$ using $x = \tan \theta$

Challenge Problems

Problem 1 Find

(a) $\int \frac{e^{2x}}{1+e^x} dx$ using $u = e^x$

(b) $\int \frac{e^{-2x}}{e^{-x}+1} dx$ using $u = e^{-x} + 1$

Problem 2 Find $\int \frac{1}{x+\sqrt{x}} dx$ using $u = \sqrt{x}$.

Problem 3 Find $\int \frac{1}{1+\sqrt{x}} dx$ using $u = \sqrt{x}$.

Problem 4 Find $\int \frac{1}{x\sqrt{x^2-1}} dx$ using $u = \frac{1}{x}$.

Problem 5 Find $\int \frac{1}{x\sqrt{x+x^2}} dx$ using $u = \sqrt{x}$.

Problem 6

(a) Use the substitution $x = \sec \theta$ to show that

$$\int \frac{1}{\sqrt{x^2-1}} dx = \ln(x + \sqrt{x^2-1}) + C$$

(b) Use the substitution $x = \tan \theta$ to show that

$$\int \frac{1}{\sqrt{x^2+1}} dx = \ln(x + \sqrt{x^2+1}) + C$$

Chapter 4 Review

Further Calculus

Review

Question 1 Differentiate

- (a) $\sin^{-1} \frac{x}{4}$ (b) $\cos^{-1} \frac{x}{2}$ (c) $\tan^{-1} \frac{x}{6}$
 (d) $\tan^{-1} \frac{3x}{2}$ (e) $\cos^{-1} 2x$ (f) $\sin^{-1} \frac{3x}{4}$

Question 2 Differentiate

- (a) $\sin^{-1} \sqrt{x}$ (b) $\tan^{-1} \sqrt{x}$ (c) $\tan^{-1} x^2$
 (d) $\sin^{-1} x^2$ (e) $\tan^{-1} \frac{2}{x}$

Question 3 Differentiate

- (a) $(\cos^{-1} 2x)^3$ (b) $(\tan^{-1} x)^2$ (c) $\sin^{-1} (e^x)$
 (d) $e^{\sin^{-1} x}$ (e) $\ln(\cos^{-1} x)$ (f) $\cos^{-1}(\ln x)$

Question 4 Differentiate and simplify

- (a) $\cos(\tan^{-1} x)$ (b) $\tan(\sin^{-1} x)$

Question 5 Write down the equation of the tangent to each curve at the point indicated

- (a) $y = \sin^{-1} \frac{x}{2}$ at $x = -\sqrt{3}$
 (b) $y = 3 \cos^{-1} \frac{x}{4}$ at $x = 2$

Question 6 Sketch the graph of $y = \sin^{-1}(x) + \sin^{-1}(-x)$.

Question 7 Consider $y = \cos(\sin^{-1} x)$

- (a) Write down the domain. (b) Find $\frac{dy}{dx}$. (c) Show that $y = \sqrt{1-x^2}$.

Question 8

- (a) If $y = \tan^{-1} x$ show that $(1+x^2)y'' + 2xy' = 0$
 (b) If $y = \sin(2 \sin^{-1} x)$ show that $(1-x^2)y'' - xy' + 4y = 0$
 (c) If $y = \tan(2 \tan^{-1} x)$ show that $(1+x^2)y'' + 2xy' - 4yy' = 0$

Question 9 Find

(a) $\int \frac{1}{16+x^2} dx$

(b) $\int \frac{1}{8+x^2} dx$

(c) $\int \frac{3}{49+x^2} dx$

(d) $\int \frac{1}{1+4x^2} dx$

(e) $\int \frac{2}{49+4x^2} dx$

(f) $\int \frac{1}{7+3x^2} dx$

Question 10 Find

(a) $\int \frac{1}{\sqrt{100-x^2}} dx$

(b) $\int \frac{1}{\sqrt{64-x^2}} dx$

(c) $\int \frac{-1}{\sqrt{10-x^2}} dx$

(d) $\int \frac{1}{\sqrt{1-10x^2}} dx$

(e) $\int \frac{-1}{\sqrt{4-25x^2}} dx$

(f) $\int \frac{1}{\sqrt{9-2x^2}} dx$

Question 11 Without finding any primitives evaluate

(a) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^{-1} x dx$

(b) $\int_{-2}^2 \tan^{-1} x dx$

(c) $\int_{-1}^1 \cos^{-1} x dx$

(d) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{-1} x dx$

Question 12 Find the exact value of the following.

(a) $\int_0^3 \frac{1}{9+x^2} dx$

(b) $\int_0^1 \frac{1}{\sqrt{4-x^2}} dx$

(c) $\int_{-2}^2 \frac{1}{\sqrt{16-x^2}} dx$

(d) $\int_0^{\frac{2}{3}} \frac{3}{\sqrt{4-9x^2}} dx$

(e) $\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2}$

(f) $\int_{\frac{2}{\sqrt{3}}}^{2\sqrt{3}} \frac{dx}{x^2+4}$

Question 13 Find the equation of the curve if

(a) It passes through $(-2, 0)$ and its derivative $\frac{dy}{dx} = \frac{2}{4+x^2}$

(b) It passes through $(0, -\frac{\pi}{2})$ and its derivative $\frac{dy}{dx} = -\frac{3}{\sqrt{16-9x^2}}$

Question 14

(a) Differentiate $\sin^{-1} \frac{x^3}{2}$ and hence find $\int \frac{x^2}{\sqrt{4-x^6}} dx$

(b) Differentiate $\tan^{-1} \frac{x^3}{2}$ and hence find $\int \frac{x^2}{4+x^6} dx$

Question 15

- (a) Differentiate $\sqrt{1-x^2} \sin^{-1} x$ and hence evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$
- (b) Differentiate $\cos^{-1}(e^x)$ and hence evaluate $\int_{-\frac{1}{2} \ln 2}^{-\ln 2} \frac{e^x}{\sqrt{1-e^{2x}}} dx$
- (c) Differentiate $\left(\tan^{-1} \frac{x}{2}\right)^2$ and hence evaluate $\int_0^2 \frac{\tan^{-1} \frac{x}{2}}{4+x^2} dx$

Question 16 Find

- (a) $\int \frac{1}{(2x-1)^2} dx$ (b) $\int \frac{1}{1-2x} dx$ (c) $\int \frac{1}{\sqrt{2-x}} dx$
- (d) $\int \frac{4x}{x^2+4} dx$ (e) $\int \frac{4}{x^2+4} dx$ (f) $\int \frac{9x}{\sqrt{8+9x^2}} dx$

Question 17 Recalling primitives of squares of the six trigonometric functions

- (a) $\int \sin^2 nx dx$ (b) $\int \cos^2 nx dx$ (c) $\int \sec^2 nx dx$
- (d) $\int \tan^2 nx dx$ (e) $\int \operatorname{cosec}^2 nx dx$ (f) $\int \cot^2 nx dx$

Question 18 Find

- (a) $\int (1 - \sin 3x)^2 dx$ (b) $\int \sin^2 6x dx$ (c) $\int \cos^2 \frac{x}{4} dx$

Question 19 Find the following using either the suggested substitution, or the reverse chain rule.

- (a) $\int (3x+1)^5 dx$ using $u = 3x+1$ (b) $\int x(x^2+5)^3 dx$ using $u = x^2+5$
- (c) $\int \frac{x^2}{(x^3-5)^3} dx$ using $u = x^3-5$ (d) $\int \frac{x}{\sqrt{x^2-4}} dx$ using $u = x^2-4$
- (e) $\int \frac{x^3}{\sqrt{x^4-4}} dx$ using $u = x^4-4$ (f) $\int \frac{x}{\sqrt{1-x^2}} dx$ using $u = 1-x^2$
- (g) $\int \frac{x^2}{x+2} dx$ using $u = x+2$ (h) $\int x\sqrt{3-x} dx$ using $u = 3-x$

Question 20 Use the formulae for trigonometric products and express the following as sum or difference of trigonometric functions and then find the primitive

- (a) $2 \cos 3x \sin x$ (b) $2 \sin 5x \cos x$ (c) $2 \sin 2x \cos x$

Question 21 Use the above technique to show that

$$(a) \int_0^{\frac{\pi}{6}} \sin 4x \cos 2x \, dx = \frac{7}{24}$$

$$(b) \int_0^{\frac{\pi}{4}} \sin 3x \cos 5x \, dx = -\frac{1}{4}$$

Question 22

$$(a) \int \cos x \sin^3 x \, dx \text{ using } u = \sin x$$

$$(b) \int \cos 2x \sin^2 2x \, dx \text{ using } u = \sin 2x$$

$$(c) \int \frac{\sec^2 x}{\tan x} \, dx \text{ using } u = \tan x$$

$$(d) \int \frac{\sin 3x}{1 + \cos 3x} \, dx \text{ using } u = \cos 3x$$

$$(e) \int \frac{\sin 3x}{1 + \cos^2 3x} \, dx \text{ using } u = \cos 3x$$

$$(f) \int \sec^2 x \tan x \, dx \text{ using } u = \tan x$$

Question 23

$$(a) \int_0^1 x\sqrt{1-x^2} \, dx \text{ using } u = 1-x^2$$

$$(b) \int_0^4 3x\sqrt{9+x^2} \, dx \text{ using } u = 9+x^2$$

$$(c) \int_0^1 \frac{e^x}{1+e^{2x}} \, dx \text{ using } u = e^x$$

$$(d) \int_0^1 \frac{e^{3x}}{e^{3x}+1} \, dx \text{ using } u = e^{3x}$$

$$(e) \int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \text{ using } u = \sqrt{x}$$

$$(f) \int_0^1 \frac{x}{\sqrt{3x^2+1}} \, dx \text{ using } u = 3x^2+1$$

$$(g) \int_0^{\frac{\pi}{4}} \frac{2 \cos x}{1+4 \sin^2 x} \, dx \text{ using } u = 2 \sin x$$

$$(h) \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos x} \, dx \text{ using } u = \cos x$$

Question 24 Use reverse chain rule to show

$$(a) \int_0^{\frac{\pi}{3}} \sin x \cos x \, dx = \frac{3}{8}$$

$$(b) \int_0^{\frac{\pi}{4}} \sin 2x \cos 2x \, dx = \frac{1}{4}$$

$$(c) \int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \, dx = \frac{1}{3}$$

$$(d) \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan^3 x} \, dx = \frac{1}{3}$$

Question 25 Show that

$$(a) \int_0^{\frac{\pi}{2}} 2 \sin^2 x \, dx = \frac{\pi}{2}$$

$$(b) \int_0^{\frac{\pi}{2}} \sin^2 2x \, dx = \frac{\pi}{4}$$

$$(c) \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2 \cos^2 \frac{x}{2} \, dx = 1 + \frac{\pi}{3}$$

$$(d) \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 16 \sin^2 x \cos^2 x \, dx = \frac{\pi+2}{4}$$

$$(e) \int_0^{\pi} \sec^2 \frac{x}{4} \, dx = 4$$

$$(f) \int_0^{\frac{2\pi}{3}} (1 - \sin 3x)^2 \, dx = \pi$$

$$(g) \int_0^{\frac{\pi}{4}} (1 + \sin 4x)^2 \, dx = 1 + \frac{3\pi}{8}$$

$$(h) \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1 + \tan x} \, dx = \ln 2$$

Question 26 Evaluate

(a) $\int_0^1 \sqrt{1-x^2} dx$ using $x = \sin \theta$

(b) $\int_0^2 \sqrt{4-x^2} dx$ using $x = 2 \sin \theta$

Question 27 A curve has a gradient function $\frac{4e^{2x}}{1+e^{4x}}$ and passes through the point $\left(0, \frac{\pi}{2}\right)$. Use the substitution $u = e^{2x}$ to find its equation

 Investigation Task

Further study into integration by substitution

A standard integration by substitution problem is something like evaluating

$$\int_0^1 x\sqrt{3x^2 + 1} dx$$

using $u = 3x^2 + 1$. As you know, we change the limits using so when $x = 0$, $u = 1$ and when $x = 1$, $u = 4$. No worries at all here.

Now, consider a slightly harder problem involving a trigonometric substitution like

$$\int_1^2 \frac{\sqrt{4-x^2}}{x^2} dx$$

using $x = 2 \sin \theta$. When we find the new limits, we loosely say things like when $x = 1$, $\sin \theta = \frac{1}{2}$ so $\theta = \frac{\pi}{6}$. But when you think about it closely, how do we *know* that θ had to become $\frac{\pi}{6}$ specifically? Why not $\theta = \frac{5\pi}{6}$ or any other equivalent value of θ for that matter? Isn't $\theta = \frac{5\pi}{6}$ just as valid a solution of $\sin \theta = \frac{1}{2}$?

Your task is to write a two to three page document that will de-mystify this whole scenario. Your answer should include:

- What happens if we pick these 'alternative' values of θ instead.
- Why do we pick the first-quadrant values?
- What is truly happening in the background when we do integration by substitution?
- Why was this not a problem for the first example provided, but suddenly a problem for the second example?

 Investigation Task**Derivative of other inverses.**

In this course, we learn how to differentiate the inverse sine, cosine and tangent functions. But what about the inverses of the other trigonometric functions? This investigation task will give students the opportunity to study the mathematics behind obtaining the derivative of inverse trigonometric functions a bit more deeply.

Question 1 Suppose you are given a function $f(x)$ and asked to produce the derivative of the inverse function $f^{-1}(x)$. There is one formula that you learn in this course that will be *crucial* for your task. State what this formula is.

Question 2 Derive, giving full explanation and justifications, the formulas for the derivative of

(a) $y = \sin^{-1} x$

(b) $y = \cos^{-1} x$

(c) $y = \tan^{-1} x$

Question 3 Draw sketches and state the domain and range of

(a) $y = \sec^{-1} x$

(b) $y = \operatorname{cosec}^{-1} x$

(c) $y = \cot^{-1} x$

Question 4 Derive, giving full explanation and justifications, the formulas for the derivative of

(a) $y = \sec^{-1} x$

(b) $y = \operatorname{cosec}^{-1} x$

(c) $y = \cot^{-1} x$

 Investigation Task

Integration by parts

Integration by parts is a technique used to integrate expressions that often mix two different ‘types’ of functions. For example, it can be used to find $\int x e^x dx$, $\int e^x \sin x dx$ and $\int \ln x dx$. Students may recognise some of these integrals and may even remember that they have done them in some exercises earlier in this chapter.

In this book, there were a number of ‘differentiate and hence integrate’-type problems used to find such integrals. Students doing them may feel somewhat robbed of the answer since by being given the expression to differentiate, the student was virtually already given the answer. This investigation task aims to hopefully appease this and to allow the student to solve such integrals themselves without being given a term to differentiate.

Question 1 Write down the formula for integration by parts, and prove it.

Question 2 Use integration by parts to find the following integrals.

(a) $\int x \ln x dx$ (b) $\int x e^x dx$ (c) $\int x \sin x dx$

Question 3 Use integration by parts to find $\int \ln x dx$ by using a clever choice of u and dv .

Question 4 Use integration by parts twice to find the following.

(a) $\int e^x \sin x dx$ (b) $\int \sec^3 x dx$

 Investigation Task

t-formula substitutions

In Year 11 you first learned the t -formula substitution, which is sometimes called the *Weierstrass substitution*. You used it to prove trigonometric identities and solve trigonometric equations. Given that in this chapter you have learned integration by substitution, it is only natural to expect that we can also use it to find certain integrals. In fact, this very much is the case!

Question 1 The standard substitution is $t = \tan\left(\frac{x}{2}\right)$. Show that

$$dx = \frac{2}{1+t^2} dt$$

using *two* different methods.

Question 2 Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find

$$(a) \int \frac{1}{1+\cos x} dx \qquad (b) \int \frac{1}{1+\sin x} dx \qquad (c) \int \frac{1}{3+5\cos x} dx$$

Question 3 Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to find

$$(a) \int_0^{\frac{\pi}{2}} \frac{1}{5+4\cos x} dx \qquad (b) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x} dx \qquad (c) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x + \tan x} dx$$

5

VOLUMES OF REVOLUTION

- Areas involving x and y -axes
- Volumes of revolution (x -axis)
- Volumes of revolution
- Volumes involving two curves

Exercise 5A

Areas involving x and y-axes

Fundamentals

Fundamentals 1

In most area questions it is helpful to visualise the r _____ required to find the area. This is so you know whether the question involves c _____ regions, area between two curves, or so you know which parts will yield n _____ area.

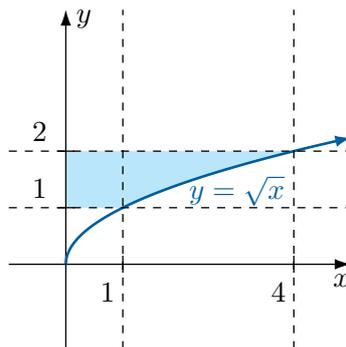
Fundamentals 2

- (a) The net signed area bound by a curve $x = g(y)$ and the y -axis from $y = c$ to $y = d$ is

$$\text{Area} = \int_{\square}^{\square} \text{---} dy = \int_{\square}^{\square} \text{---} dy$$

- (b) Regions to the right of the y -axis have p _____ net signed area, whereas to the left of y -axis counted as n _____ net signed area.
- (c) When working out the area of the region between a curve $y = f(x)$ and the y -axis it is necessary to make _____ the subject of the equation first.

Question 1 Complete the steps given to find the area of the shaded region below.



- (a) Write down the formula for the area bound by the curve and the y -axis.
- (b) If $y = \sqrt{x}$, then $x = \text{---}$.
- (c) Find the limits in terms of y .

$$\begin{aligned} x = 4 &\rightarrow y = \text{---} \\ x = 1 &\rightarrow y = \text{---} \end{aligned}$$

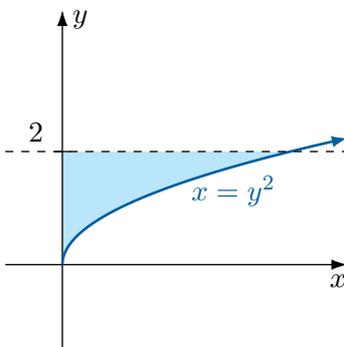
(d) Complete the following.

$$\begin{aligned} \text{Area} &= \int_{\square}^{\square} \text{---} dy \\ &= [\text{---}]_{\square}^{\square} \\ &= \end{aligned}$$

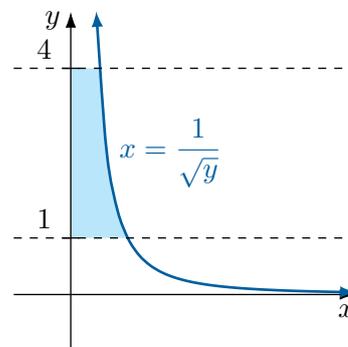
Question 2 [Drills for areas involving the y -axis]

Find the area of the shaded regions below.

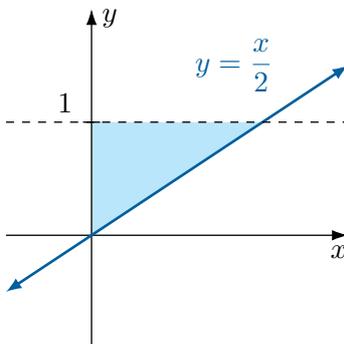
(a) $x = y^2$, $y = 2$



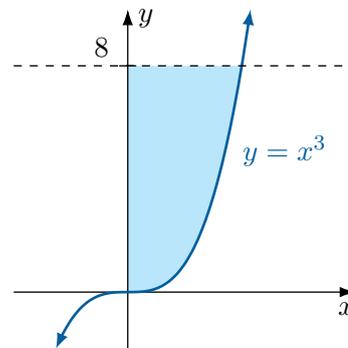
(b) $x = \frac{1}{\sqrt{y}}$, $y \in [1, 4]$



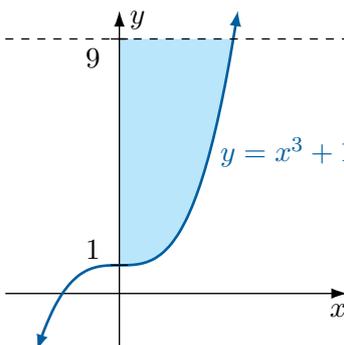
(c) $y = \frac{x}{2}$, $y \in [0, 1]$



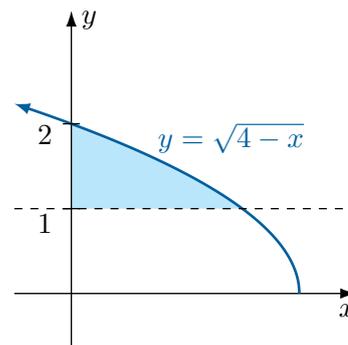
(d) $y = x^3$, $y \in [0, 8]$



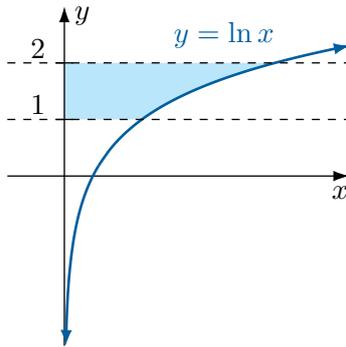
(e) $y = x^3 + 1$, $y \in [1, 9]$



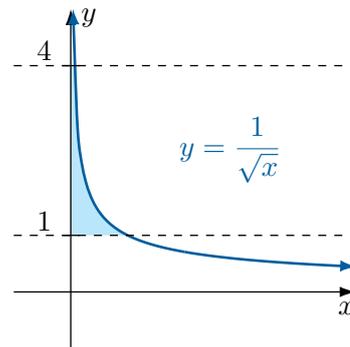
(f) $y = \sqrt{4-x}$, $y \in [1, 2]$



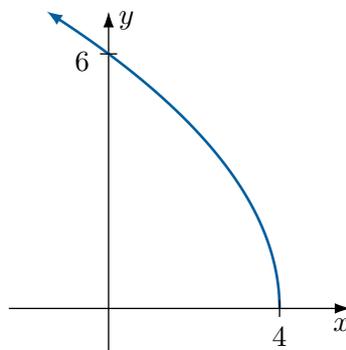
(g) $y = \ln x$, $y \in [1, 2]$



(h) $y = \frac{1}{\sqrt{x}}$, $y \in [1, 4]$



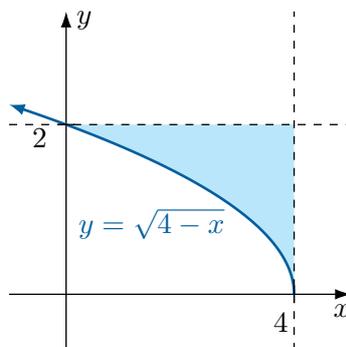
Question 3 A section of the curve $y = 3\sqrt{4-x}$ is drawn below. Find the area of the region bounded by the curve and the coordinate axes by considering the region between the curve and the

(a) x -axis(b) y -axis

Question 4

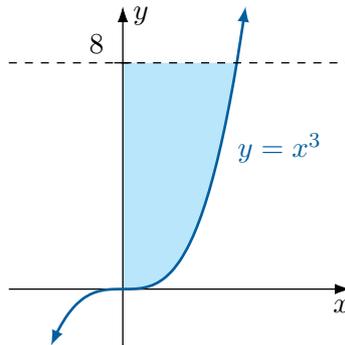
- Find the area of the region enclosed by $y = \sqrt{x}$, the x -axis and the line $x = 16$.
- Find the area of the region enclosed by $y = \sqrt{x}$, the y -axis and the line $y = 4$.
- Find the sum of the results in parts (a) and (b), and explain what this area represents

Question 5 The diagram below shows $y = \sqrt{4-x}$ and a region bounded by the lines $x = 4$ and $y = 2$.



Find the area of the shaded region.

Question 6 Consider the shaded region below.

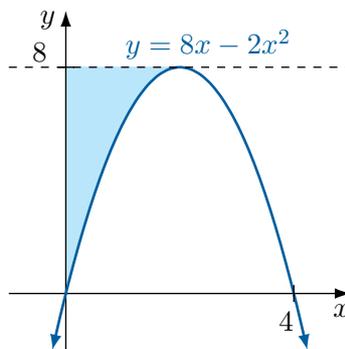


Write down and evaluate the integral that represents the shaded region by considering the region bounded by the curve and the

(a) y -axis.

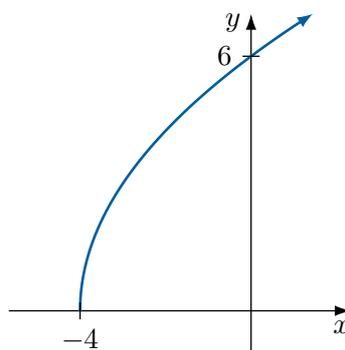
(b) x -axis.

Question 7 The diagram below shows a sketch of $y = 8x - 2x^2$.



Find the area of the shaded region.

Question 8 Bob is asked to find the area of the region bound by the curve $y = 3\sqrt{x+4}$ and the coordinate axes.



He produces the diagram above, follows the usual steps and obtains the integral

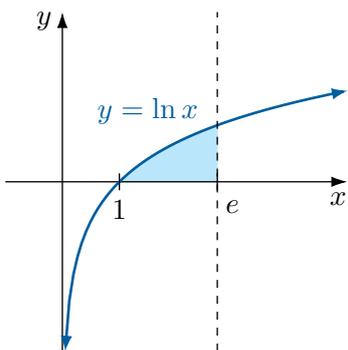
$$A = \int_0^6 \left(\frac{y^2}{9} - 4 \right) dy$$

Explain why this integral does *not* give the required area.

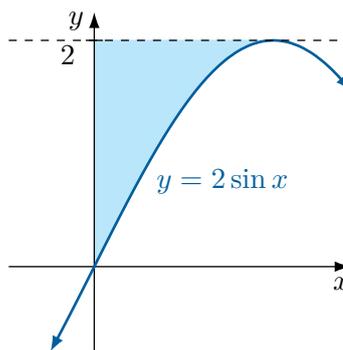
- (a) Write down an integral representing the area of the shaded region.
- (b) Write down an integral representing the area of \mathcal{B} and hence find the area.
- (c) Find the area of the rectangle shown in dotted lines.
- (d) Hence, find the area of $|\mathcal{A}|$.

Question 13 Use a similar technique to the above question to find the area of the following regions.

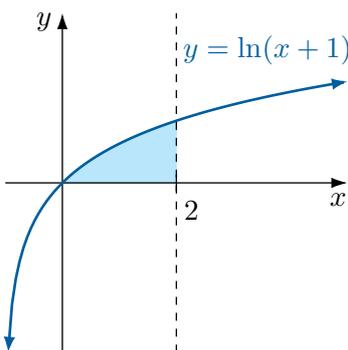
(a) $y = \ln x$ for $x \in [1, e]$



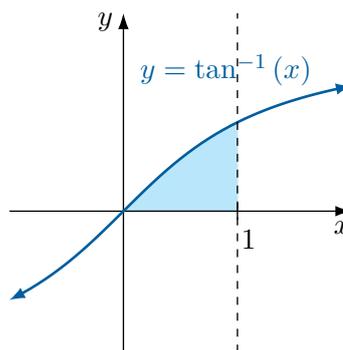
(b) $y = 2 \sin x$ and y -axis for $y \in [0, 2]$



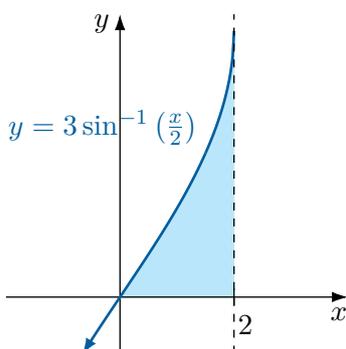
(c) $y = \ln(x + 1)$ for $x \in [0, 2]$



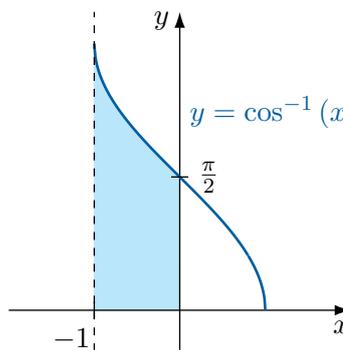
(d) $y = \tan^{-1}(x)$ for $x \in [0, 1]$



(e) $y = 3 \sin^{-1}\left(\frac{x}{2}\right)$ for $x \in [0, 2]$



(f) $y = \cos^{-1}(x)$ for $x \in [-1, 0]$.



Question 14 By drawing a sketch where necessary and shading the appropriate region, evaluate the following integrals.

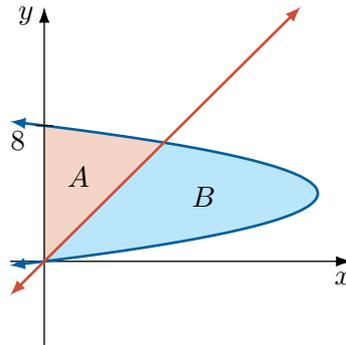
(a) $\int_0^1 \sin^{-1} x \, dx$

(b) $\int_0^1 \cos^{-1} x \, dx$

(c) $\int_0^1 \tan^{-1} x \, dx$

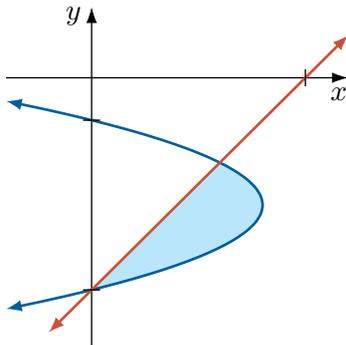
(d) $\int_1^2 \ln x \, dx$

Question 15 The curves $y = x$ and $x = 8y - y^2$ are drawn below.



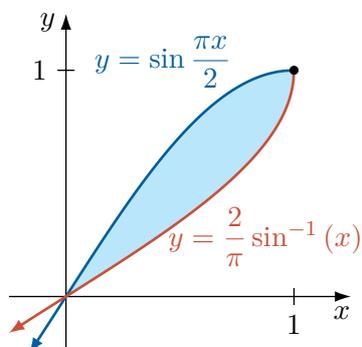
- Find where the curves intersect.
- Find the area of region A .
- Find the area of region B .

Question 16 The curves $x = y + 5$ and $x = -y^2 - 6y - 5$ are drawn. Find the area of the shaded region.



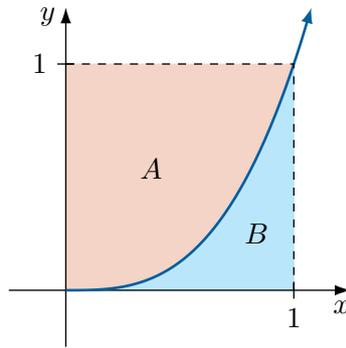
Challenge Problems

Problem 1 The diagram below shows a sketch of $y = \sin \frac{\pi x}{2}$ and $y = \frac{2}{\pi} \sin^{-1}(x)$ in the domain $0 \leq x \leq 1$.



Find the area of the region enclosed by the curves.

Problem 2 The diagram below shows a sketch of $y = x^n$ that splits the unit square into two regions A and B .



Find the value of n if Area A : Area B is in the ratio

(a) $2 : 1$

(b) $3 : 1$

(c) $p : q$

Problem 3 [Special case of Young's Inequality]

Let $y = f(x)$ be a one-to-one function over all real x that passes through the origin such that $f(a) = b$. Show that

$$\int_0^a f(x) dx + \int_0^b f^{-1}(x) dx = ab$$

Problem 4 Let $y = f(x)$ be a one-to-one function over all real x such that $f(a) = c$ and $f(b) = d$. Show that

$$\int_a^b f(x) dx + \int_c^d f^{-1}(x) dx = bd - ac$$

Exercise 5B

Volumes of revolution (x-axis)

Fundamentals

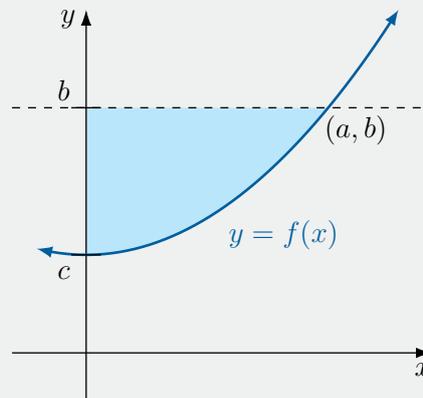
Fundamentals 1

The region between $y = f(x)$ and the x -axis from $x = a$ to $x = b$ is rotated about the x -axis to form a solid. The volume of the solid is

$$\text{Volume} = \int_a^b \text{---} dx = \int_a^b \text{---} dx$$

Fundamentals 2

The shaded region bound by $y = f(x)$, the y -axis and the line $y = b$ is rotated about the x -axis to form a solid.



Write down an expression for the volume of this solid.

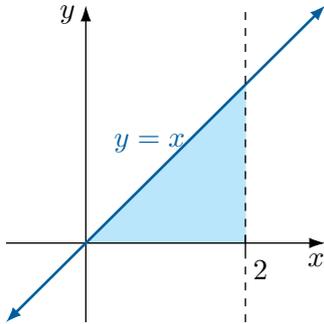
Question 1 [Introductory volumes questions]

Describe the shape of the solid formed from the following descriptions, and find the volume using any previously learned formulae.

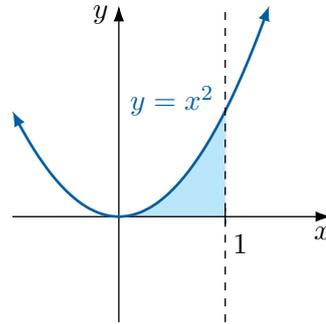
- The region bounded by $y = 3$ and the x -axis for $x \in [0, 4]$ is rotated about the x -axis.
- The region bounded by $y = 3$ and the x -axis for $x \in [0, 4]$ is rotated about the y -axis.
- The region bounded by $y = 3x$ and the x -axis for $x \in [0, 3]$ is rotated about the x -axis.
- The region bounded by $y = 3x$ and the x -axis for $x \in [0, 3]$ is rotated about the y -axis.
- The region bound by $y = 2$ and the x -axis for $x \in [1, 4]$ is rotated about the y -axis.

Question 2 Find the volume of the solid formed when the following regions are rotated about the x -axis.

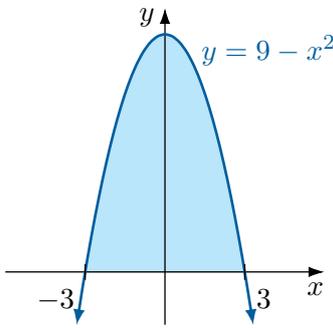
(a) $y = x$, $x \in [0, 2]$



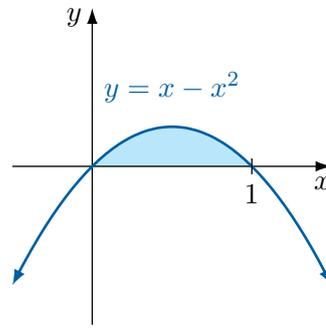
(b) $y = x^2$, $x \in [0, 1]$



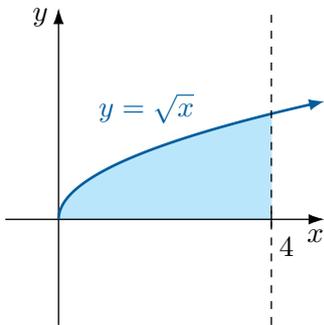
(c) $y = 9 - x^2$, $x \in [-3, 3]$



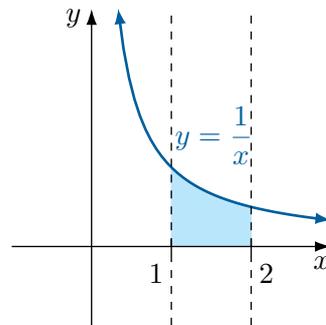
(d) $y = x - x^2$, $x \in [0, 1]$



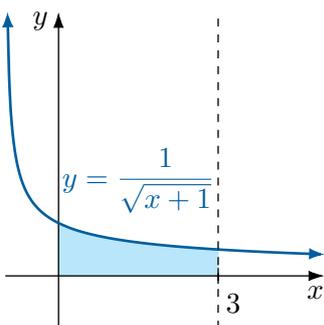
(e) $y = \sqrt{x}$, $x \in [0, 4]$



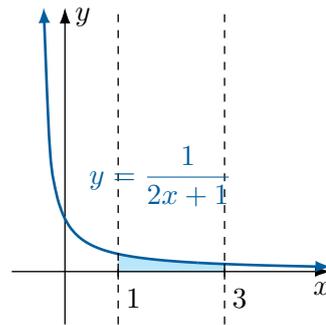
(f) $y = \frac{1}{x}$, $x \in [1, 2]$



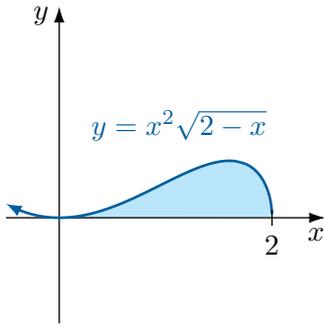
(g) $y = \frac{1}{\sqrt{x+1}}$, $x \in [0, 3]$



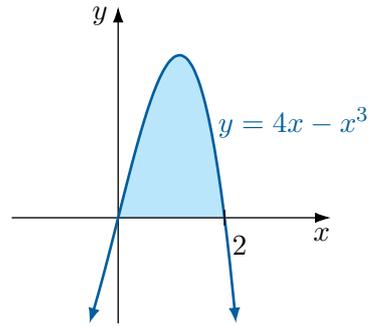
(h) $y = \frac{1}{2x+1}$, $x \in [1, 3]$



(i) $y = x^2\sqrt{2-x}$, $x \in [0, 2]$

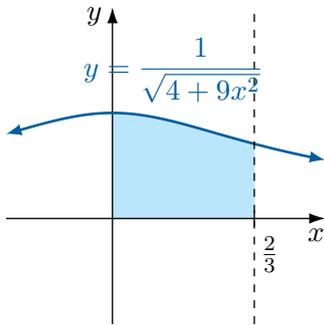


(j) $y = 4x - x^3$, $x \in [0, 2]$

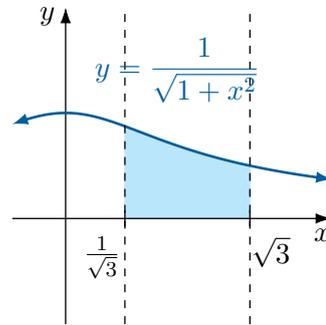

Question 3 [Involving inverse trigonometric functions]

 Find the volume of the solid formed when the following regions are rotated about the x -axis.

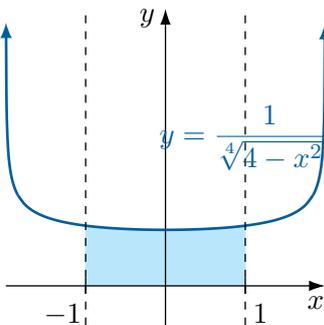
(a) $y = \frac{1}{\sqrt{4+9x^2}}$, $0 \leq x \leq \frac{2}{3}$



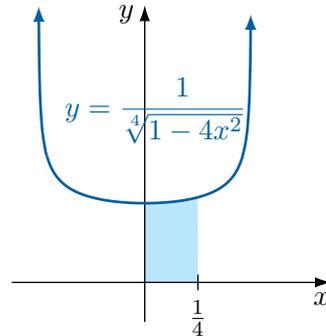
(b) $y = \frac{1}{\sqrt{1+x^2}}$, $\frac{1}{\sqrt{3}} \leq x \leq \sqrt{3}$



(c) $y = \frac{1}{\sqrt[4]{4-x^2}}$, $-1 \leq x \leq 1$



(d) $y = \frac{1}{\sqrt[4]{1-4x^2}}$, $0 \leq x \leq \frac{1}{2}$


Question 4 [Involving trigonometric functions]

 Find the volume of solid generated when the region enclosed by the curve and the x -axis in the given domain, is rotated about the x -axis.

(a) $y = \sin x$, $x \in [0, \pi]$

(b) $y = \cos x$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

(c) $y = 3 \sin(2x)$, $x \in \left[0, \frac{\pi}{2}\right]$

(d) $y = 4 \sin\left(\frac{x}{2}\right)$, $x \in [0, 2\pi]$

(e) $y = \tan x$, $x \in \left[0, \frac{\pi}{4}\right]$

(f) $y = 2 \tan(3x)$, $x \in \left[0, \frac{\pi}{9}\right]$

Question 5 [Involving exponential functions]

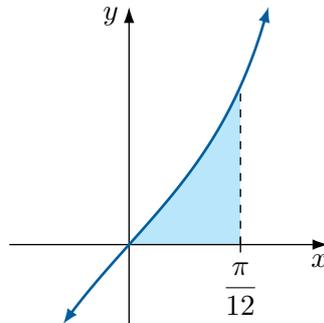
Find the volume of the solid generated when the curve $y = e^{-\frac{x}{2}}$ is rotated about the x -axis-between

(a) $x = 0$ and $x = 2$

(b) $x = -\ln 4$ and $x = \ln 4$

Question 6 [Area versus volume]

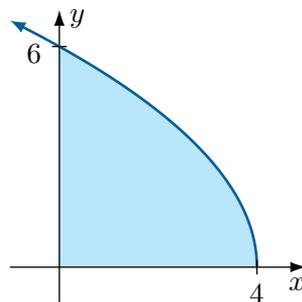
A small section of $y = \tan 3x$ is drawn below.



(a) Find $\int_0^{\frac{\pi}{12}} \tan 3x \, dx$ and describe what this represents on the diagram.

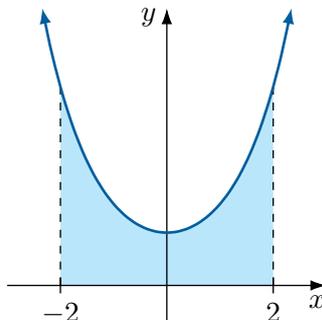
(b) Find $\pi \int_0^{\frac{\pi}{12}} \tan^2 3x \, dx$ and describe what this represents on the diagram.

Question 7 A section of the curve $y = 3\sqrt{4-x}$ is drawn below.



Find the volume of the solid generated when the region between the curve and the coordinate axes is rotated about the x -axis.

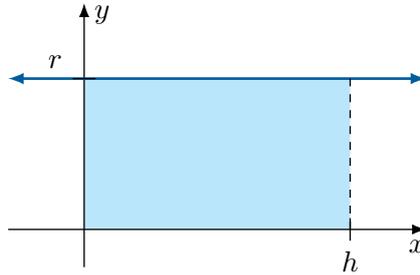
Question 8 The curve $y = \frac{e^x + e^{-x}}{2}$ is drawn below.



Find the volume of the solid generated when the region bound by the curve and the x -axis in the domain $x \in [-2, 2]$ is rotated about the x -axis.

Question 9 [Volume of a cylinder]

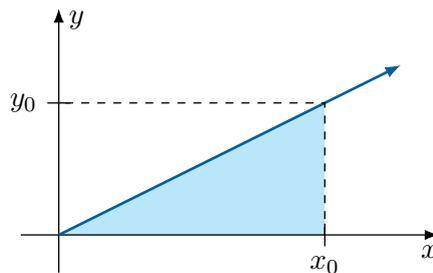
The diagram below shows a graph of $y = r$ for some $r > 0$ in the domain $x \in [0, h]$.



The region bound by the curve and the x -axis in the given domain is rotated about the x -axis to form a solid. Find the volume of the solid and hence show that the volume of a cylinder with radius r and height h is $V = \pi r^2 h$.

Question 10 [Volume of a cone]

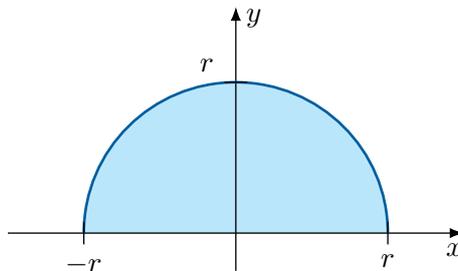
The diagram below shows a graph of the linear function $y = kx$ for some constant k , from the origin up to a point (x_0, y_0) .



- The shaded region is rotated about the x -axis to form a cone of radius r and height h . What should (x_0, y_0) be to represent this scenario?
- Hence, find k in terms of h and r .
- Find the volume of the solid formed, and verify that the volume of the cone is $V = \frac{\pi}{3} r^2 h$.

Question 11 [Volume of a sphere]

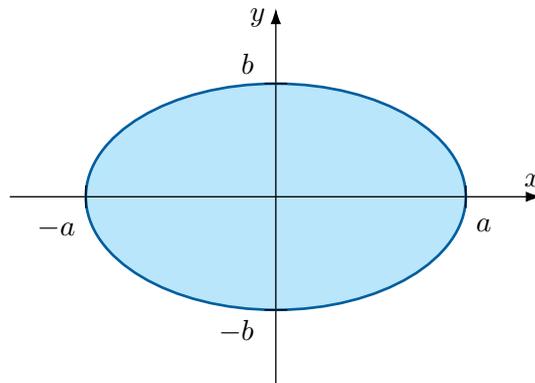
The diagram below shows the semi-circle $y = \sqrt{r^2 - x^2}$.



The region bounded by the curve and the x -axis is rotated about the x -axis to form a solid. Show that the volume of a sphere of radius r is $V = \frac{4}{3} \pi r^3$.

Question 12 [Volume of an ellipsoid]

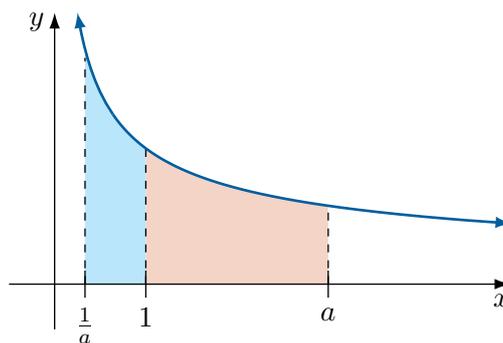
The diagram below shows an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



The region bounded by the ellipse is rotated about the x -axis to form an ellipsoid.

- Show that $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.
- Hence, show that the volume of the ellipsoid formed is $V = \frac{4}{3}\pi ab^2$.
- Deduce the formula for the volume of a sphere from the above result.

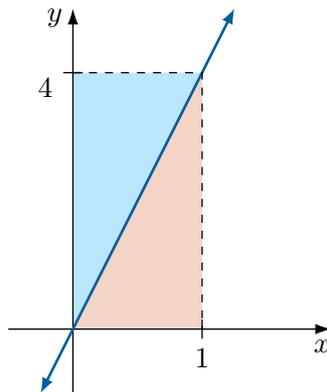
Question 13 The diagram below shows a sketch of $y = \frac{2}{\sqrt{x}}$. The region bound by the curve and the x -axis in the domains $x \in \left[\frac{1}{a}, 1\right]$ and $x \in [1, a]$ is rotated around the x -axis to form two solids.



Show that the two solids will always have equal volume for any $a > 0$.

Question 14 Bob is given the task of finding the volume of the solid formed by rotating the curve $y = \sin^{-1} x$ for $y \in \left[0, \frac{\pi}{2}\right]$ about the y -axis. Mary claims that he may as well rotate the curve $y = \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$ about the x -axis. Explain Mary's claim, and find the volume of the solid formed.

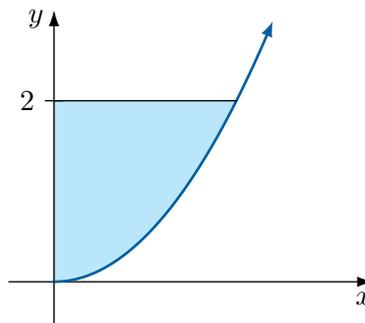
Question 15 The diagram below shows the line $y = 4x$ in the domain $0 \leq x \leq 1$.



Find the volume of the solid formed when the region bound by the

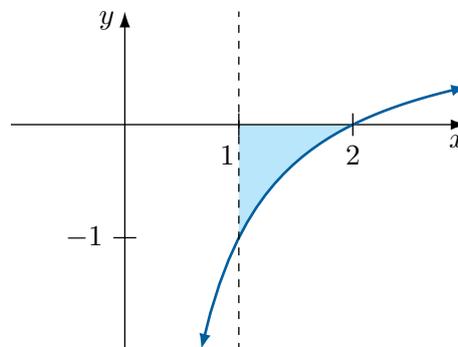
- (a) lines $y = 4x$, $y = 4$ and y -axis is rotated about the x -axis.
 (b) lines $y = 4x$, $x = 1$ and x -axis is rotated about the y -axis.

Question 16 The diagram below shows a sketch of $y = 2x^2$ and the region bound by the curve, the y -axis and $y = 2$.



The shaded region is rotated about the x -axis to form a solid. Find the volume of the solid formed.

Question 17 The diagram below shows a sketch of the curve $y = 1 - \frac{2}{x}$ and the region bound by the curve and the x -axis in the domain $x \in [1, 2]$.

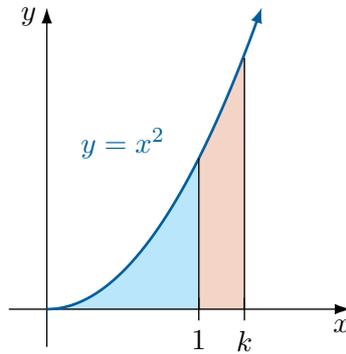


Find the volume of the solid formed when the shaded region is rotated about the

- (a) x -axis. (b) y -axis.

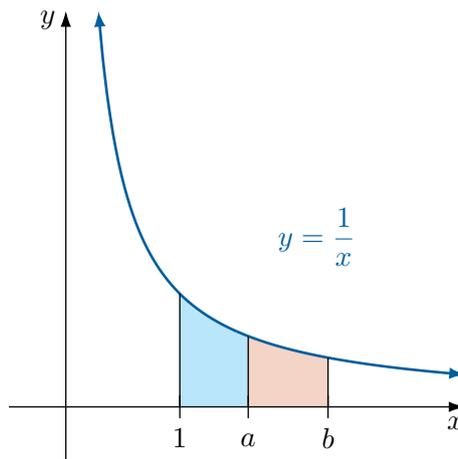
Challenge Problems

Problem 1 The diagram below shows a graph of $y = x^2$ for $x > 0$.



The regions in the domains $x \in [0, 1]$ and $x \in [1, k]$ are rotated about the x -axis to form two equal volumes. Find the value of k .

Problem 2 The diagram below shows a graph of $y = \frac{1}{x}$ for $x > 0$ and $a > 1$.



The regions in the domains $x \in [1, a]$ and $x \in [a, b]$ are rotated about the x -axis to form two volumes.

- Show that if the two volumes are equal, then $b = \frac{a}{2-a}$.
- Deduce that $1 < a < 2$.

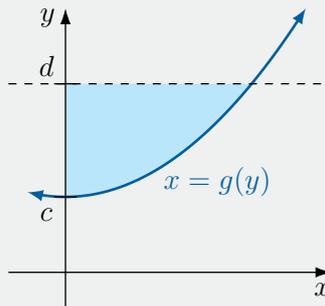
Exercise 5C

Volumes of revolution

Fundamentals

Fundamentals 1

The shaded region is bound by the curve $x = g(y)$ and the y -axis from $y = c$ to $y = d$.

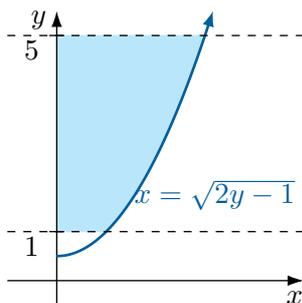


The shaded region is rotated about the y -axis to form a solid. The volume of the solid is

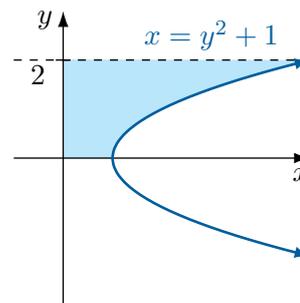
$$\text{Volume} = \int_{\square}^{\square} \text{---} dx = \int_{\square}^{\square} \text{---} dx$$

Question 1 Find the volume of the solid formed when the following regions are rotated about the y -axis.

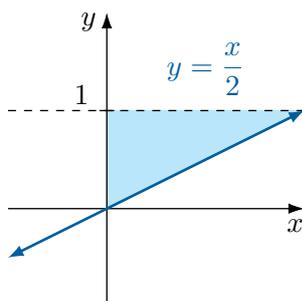
(a) $x = \sqrt{2y-1}$, $y \in [1, 5]$



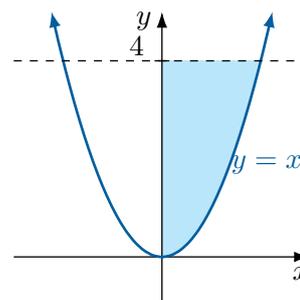
(b) $x = y^2 + 1$, $y \in [0, 2]$



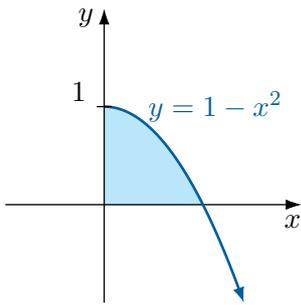
(c) $y = \frac{x}{2}$, $y \in [0, 1]$



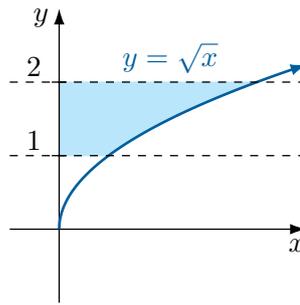
(d) $y = x^2$, $y \in [0, 4]$



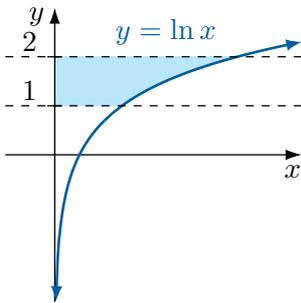
(e) $y = 1 - x^2$, $y \in [0, 1]$



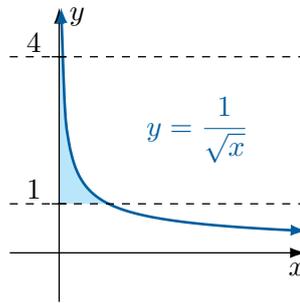
(f) $y = \sqrt{x}$, $y \in [1, 2]$



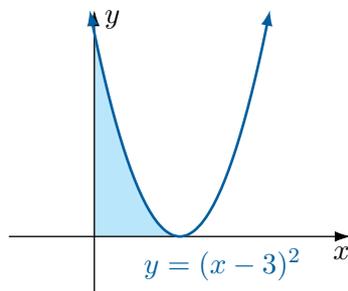
(g) $y = \ln x$, $y \in [1, 2]$



(h) $y = \frac{1}{\sqrt{x}}$, $y \in [1, 4]$

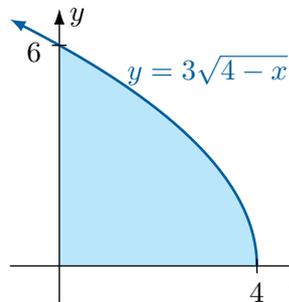


Question 2 The diagram below shows the region bound by $y = (x - 3)^2$ and the coordinate axes.



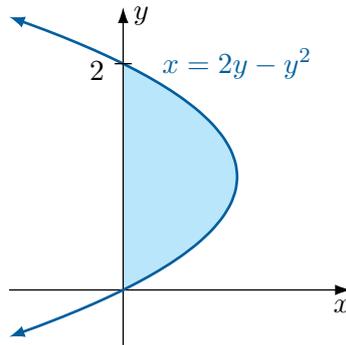
The region is rotated about the y -axis to form a solid. Find the volume of the solid.

Question 3 A section of the curve $y = 3\sqrt{4 - x}$ is drawn below.



Find the volume of the solid generated when the region between the curve and the coordinate axes is rotated about the y -axis.

Question 4 The diagram below shows the region bound by $x = 2y - y^2$ and the y -axis.



The region is rotated about the y -axis to form a solid. Find the volume of the solid.

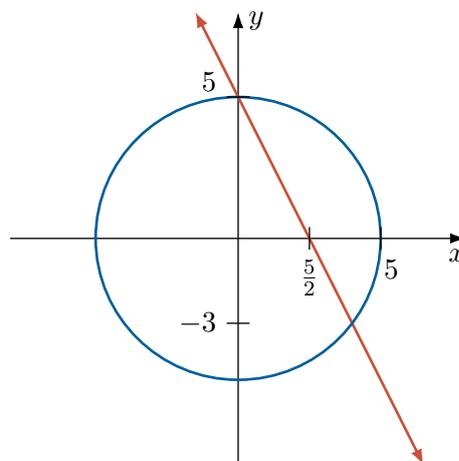
Question 5 [Volume of a cone]

Consider the line $y = \frac{x}{2}$.

- Find the volume of the cone formed when the region defined by $y = \frac{x}{2}$ for $y \in [0, 2]$ is rotated about the y -axis.
- Use the formula for the volume of a cone to verify your answer.

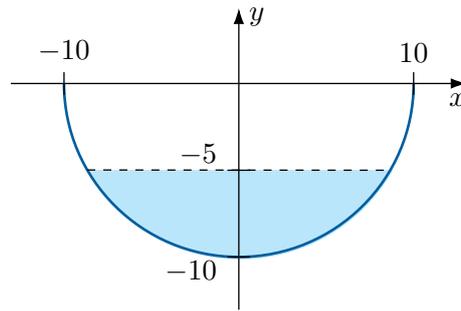
Question 6

- Show that the circle and line intersect at $(4, -3)$.
- Find the volume of solid generated when the region enclosed by $x = \frac{1}{2}(5 - y)$ and the y -axis in the domain $-3 \leq y \leq 5$ is rotated about the y -axis.



- What percentage of the sphere does the volume of the cone cover?

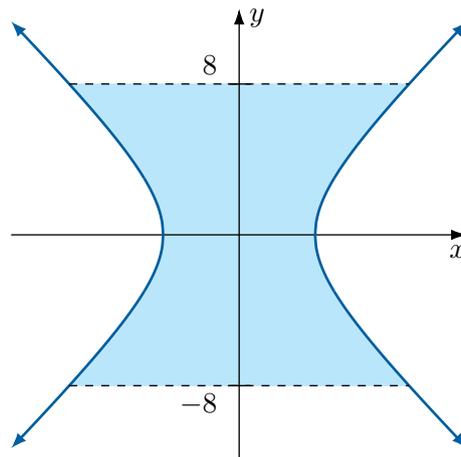
Question 7 A bowl of radius 10 is formed by rotating $y = -\sqrt{100 - x^2}$ about the y -axis.



The bowl is filled with water to a depth of 5 units.

- (a) Find the capacity of the bowl.
- (b) To what percentage of the total capacity is the bowl filled?

Question 8 The diagram below shows the curve $x^2 - y^2 = 16$. The region bounded by the curve and $y = \pm 8$ is rotated about the y -axis to form a drum.



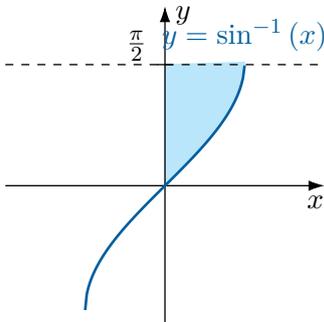
Find the volume of the drum.

Question 9 Find the volume of the solid generated when the ellipse $x^2 + 4y^2 = 16$ is rotated about the y -axis between $y = -2$ and $y = 2$

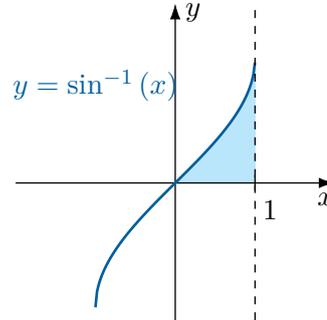
Question 10 [Involving inverse trigonometric functions]

The shaded regions below are rotated about the y -axis to form a solid. Find the volume of the solid formed.

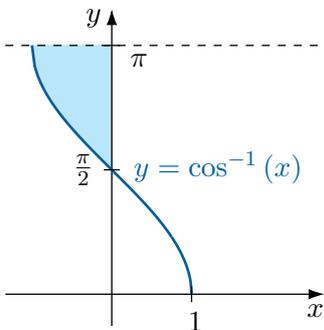
(a) $y = \sin^{-1} x$ and y -axis for $0 \leq y \leq \frac{\pi}{2}$



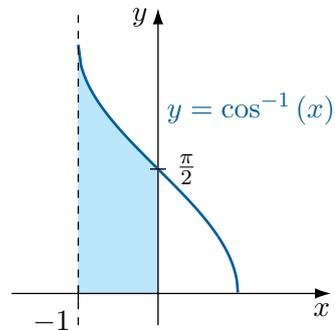
(b) $y = \sin^{-1} x$ and x -axis for $0 \leq x \leq 1$



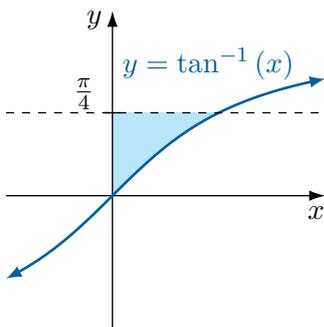
(c) $y = \cos^{-1} x$ and y -axis for $\frac{\pi}{2} \leq y \leq \pi$



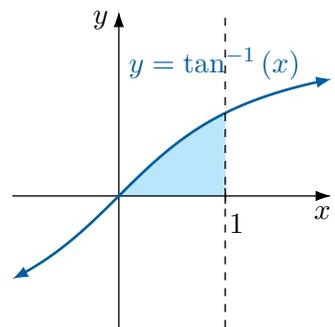
(d) $y = \cos^{-1} x$ and x -axis for $-1 \leq x \leq 0$



(e) $y = \tan^{-1} x$ and y -axis for $0 \leq y \leq \frac{\pi}{4}$



(f) $y = \tan^{-1} x$ and x -axis for $0 \leq x \leq 1$

**Question 11** [Subtraction of volumes necessary]

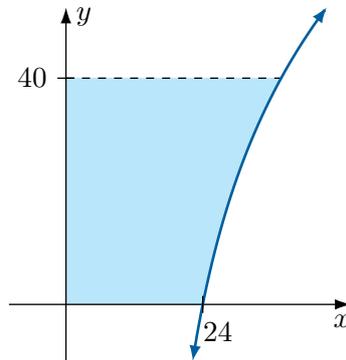
Find the volume of the solid generated when the region enclosed by the curve

(a) $y = 2 \sin^{-1}(x)$ and the x -axis for $0 \leq x \leq 1$ is rotated about the y -axis.

(b) $y = 4 \cos^{-1}(3x)$ and the x -axis for $-\frac{1}{3} \leq x \leq 0$ is rotated about the y -axis.

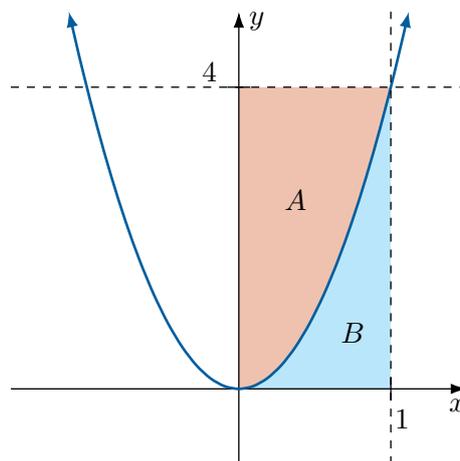
(c) $y = 2 \tan^{-1}(2x)$, the x -axis for $0 \leq x \leq \frac{1}{2}$ is rotated around the y -axis.

Question 12 A bucket formed by rotating the curve $y = 40 \ln\left(\frac{x-16}{8}\right)$ for $y \in [0, 40]$ about the y -axis.



Find the capacity of the bucket.

Question 13 Region A is bounded by the curve $y = 4x^2$, the y -axis and the line $y = 4$. Region B is bounded by the curve $y = 4x^2$, the x -axis and the line $x = 1$.



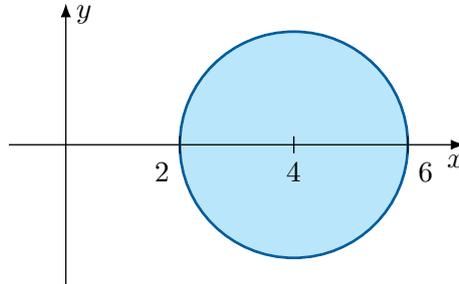
Find the volume of the solid generated by rotating the region

- | | |
|-----------------------------|-----------------------------|
| (a) B about the x -axis | (b) A about the y -axis |
| (c) B about the y -axis | (d) A about the x -axis |

Challenge Problems

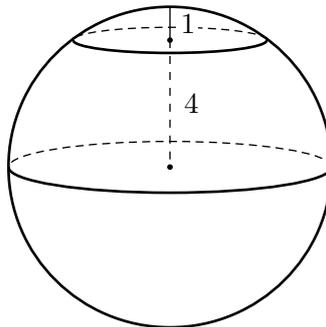
Problem 1 [Volume of a torus]

The region inside the circle $(x - 4)^2 + y^2 = 4$ is rotated about the y -axis to form a *torus*.



- Write down the two sections of the circle that you will need to consider to calculate the required volume.
- Show that the volume of the solid is given by $V = 32\pi \int_0^2 \sqrt{4 - y^2} dy$
- Evaluate this integral.

Problem 2 The diagram below shows a sphere of radius 5 and a hemispherical cap of depth 1 unit.



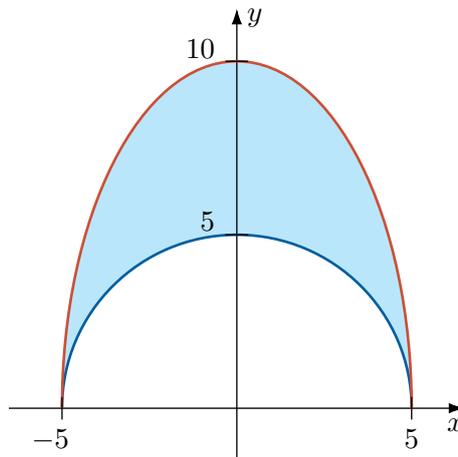
- Show that the volume of the cap is $\frac{14\pi}{3}$.
- Write down the integral that represents the volume of the rest of the sphere, and find it.
- Verify that the sum of your answers is the volume of a sphere of radius 5 units

Problem 3 [Volume of a hemispherical cap]

Consider now the general case involving a circle of radius r , where the cap is sliced to a depth of h units from the top of the sphere.

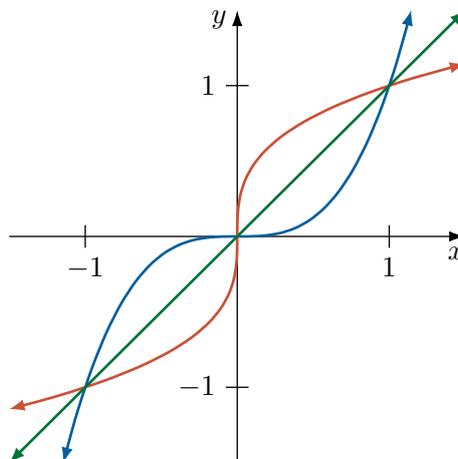
- Show that the volume of the cap is given by $V_{\text{cap}} = \frac{\pi}{3}(3rh^2 - h^3)$.
- State the value of h so that the volume of the cap is exactly half of the sphere.

Question 4 The curves $y = \sqrt{25 - x^2}$ and $y = 2\sqrt{25 - x^2}$ are drawn below.



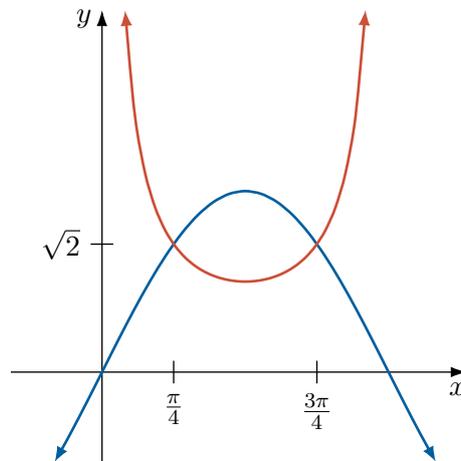
- (a) Write down the expression you will use to determine the volume of the solid formed when the region enclosed by the curves is rotated about the x -axis
- (b) Evaluate the volume of this region formed

Question 5 A section of the graphs of $y = x^3$, $y = x$ and $x = y^3$ are drawn below



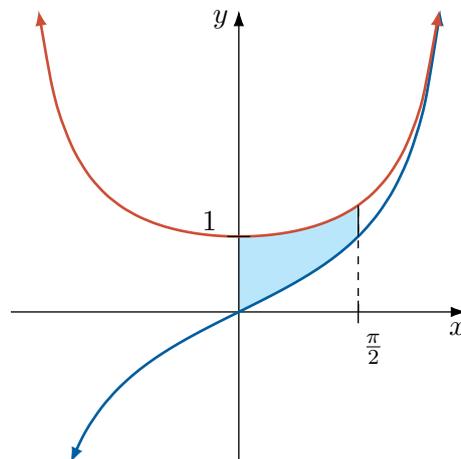
- (a) Find the area of the region enclosed by $y = x^3$ and $y = x$
- (b) Find the volume of the solid formed by rotating the region enclosed in (a) about the
 - (i) x -axis
 - (ii) y -axis
- (c) Find the area of the region enclosed by $y = x^3$ and $x = y^3$.
- (d) Find the volume of the solid formed by rotating the region enclosed in (a) about the
 - (i) x -axis
 - (ii) y -axis

Question 6 The diagram below shows the graphs of $y = 2 \sin x$ and $y = \operatorname{cosec} x$



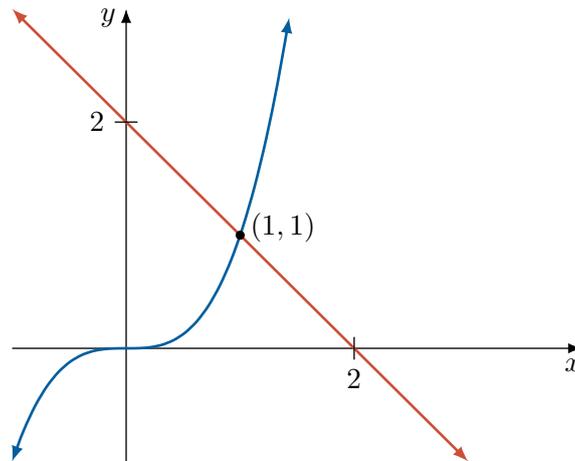
- (a) Show that these curves intersect at $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$
- (b) The region enclosed by the curves is rotated about the x -axis. Show that the volume formed is π^2

Question 7 The diagram below shows the graph of $y = \tan\left(\frac{x}{2}\right)$ and $y = \sec\left(\frac{x}{2}\right)$. The region bounded by the curves in the domain $0 \leq x \leq \frac{\pi}{2}$ is rotated about the x -axis.



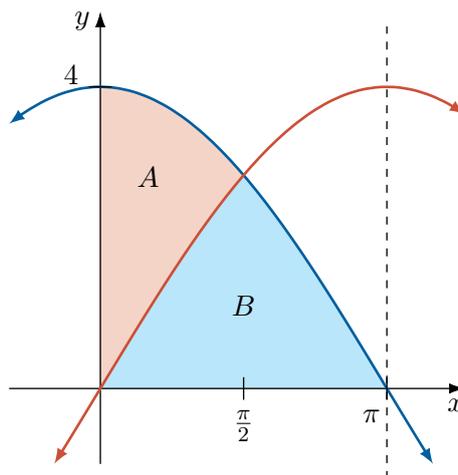
Find the volume of the solid formed.

Question 8 The diagram below shows the curves $y = x^3$ and $y = 2 - x$.



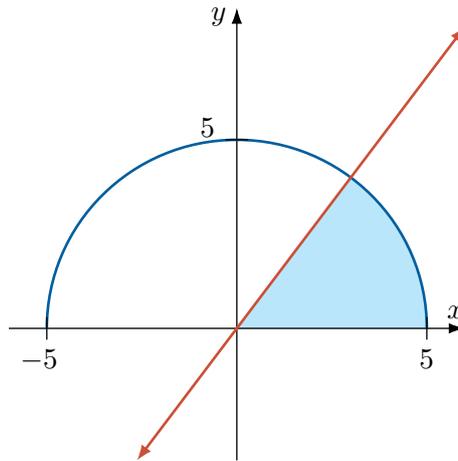
- Show that the two curves intersect at $(1, 1)$.
- Find the area of the region enclosed by the two curves and x -axis
- Find the area of the region enclosed by the two curves and y -axis
- The region in (b) is rotated about the x -axis. Find the volume of the solid generated.
- The region in (c) is rotated about the y -axis. Find the volume of the solid generated.

Question 9 A section of the graphs $y = 4 \sin\left(\frac{x}{2}\right)$ and $y = 4 \cos\left(\frac{x}{2}\right)$ between $x = 0$ and $x = \pi$ is drawn. Section A is the region bounded by the curves and the y -axis and section B is the region bounded by the curves and the x -axis in this domain.



- Find the volume of the solid generated when region A is rotated about the x -axis.
- Find the volume of the solid generated when region B is rotated about the x -axis.

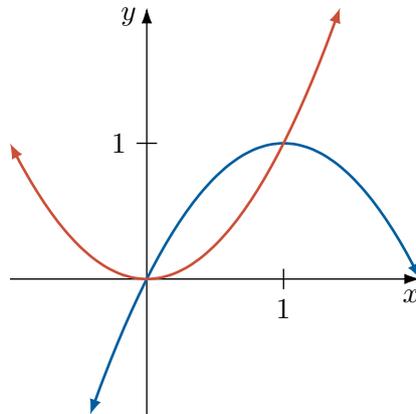
Question 10 The diagram below shows the curve $y = \sqrt{25 - x^2}$ and the line $y = \frac{4x}{3}$.



- (a) The region bounded by $y = \sqrt{25 - x^2}$, $y = \frac{4x}{3}$ and the x -axis is rotated about the x -axis.
- Write down the expression you need to integrate in order that you calculate the volume of the solid formed.
 - Find the volume of the solid.
- (b) The region bounded by $y = \sqrt{25 - x^2}$, $y = \frac{4x}{3}$ and the x -axis is rotated about the y -axis.
- Write down the expression you need to integrate in order that you calculate the volume of the solid formed.
 - Find the volume of the solid.
- (c) The region bounded by $y = \sqrt{25 - x^2}$, $y = \frac{4x}{3}$ and the y -axis is rotated about the x -axis.
- Write down the expression you need to integrate in order that you calculate the volume of the solid formed.
 - Find the volume of the solid.
- (d) The region bounded by $y = \sqrt{25 - x^2}$, $y = \frac{4x}{3}$ and the y -axis rotated about the y -axis.
- Write down the expression you need to integrate in order that you calculate the volume of the solid formed.
 - Find the volume of the solid.

Challenge Problems

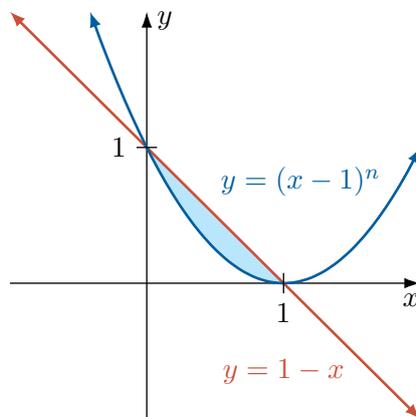
Problem 1 Given the curves $y = x^2$ and $y = 1 - (x - 1)^2$ drawn below



Find the volume of the solid formed when the region bound by the two curves is rotated about the

- (a) x -axis. (b) y -axis.

Problem 2 The diagram below shows the region bound by $y = 1 - x$ and $y = (x - 1)^n$ for even $n \geq 2$.



The region is rotated about the x -axis to form a solid.

- (a) Show that the volume of the solid formed is

$$V_n = \pi \left(\frac{1}{3} - \frac{1}{2n+1} \right)$$

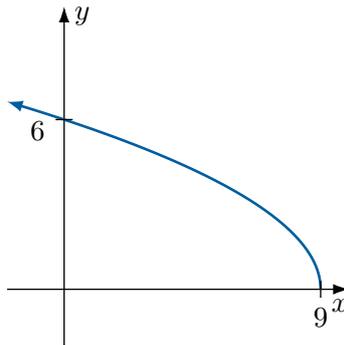
- (b) Find the limit of V_n as $n \rightarrow \infty$.
 (c) Describe the significance of your result.

Chapter 5 Review

Volumes of Revolution

Review

Question 1 A section of the curve $y = 2\sqrt{9-x}$ is drawn below



Find the area of the region bounded by the curve and the x and y -axis by considering

- (a) the region bounded by the curve and the x -axis.
- (b) the region bounded by the curve and the y -axis.

Question 2 Find the area bounded by the curves $y = 2x - x^2$ and $y = -x$

Question 3 Find the area bounded by $y = \frac{4}{4+x^2}$, the x -axis and the lines $x = 2$ and $x = -2$

Question 4 Find the volume of the solid generated when the line $y = 2x$ between $x = 0$ and $x = 3$ is rotated about the x -axis

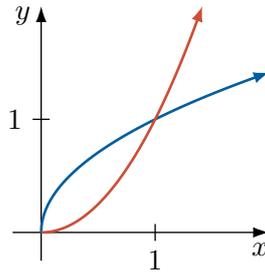
Question 5 Find the volume of the solid generated when the area bounded by $x = y^2$ and the line $x = 9$ is rotated about

- (a) the x -axis
- (b) the y -axis

Question 6 Find the volume generated when the region between the line $x + 2y = 4$, x and y -axis is rotated about the

- (a) x -axis.
- (b) y -axis.

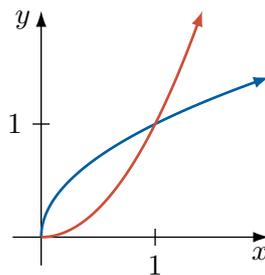
Question 7 The graphs of $y = \sqrt{x}$ and $y = x^2$ are drawn below.



Find the volume of the solid generated when the area of the region enclosed by

- (a) the curve $y = \sqrt{x}$, the y -axis and the line $y = 1$ is rotated about the x -axis
- (b) the curve $y = x^2$, the y -axis and the line $y = 1$ is rotated about the x -axis
- (c) the curve $y = \sqrt{x}$, the x -axis and the line $x = 1$ is rotated about the y -axis
- (d) the curve $y = x^2$, the x -axis and the line $x = 1$ is rotated about the y -axis

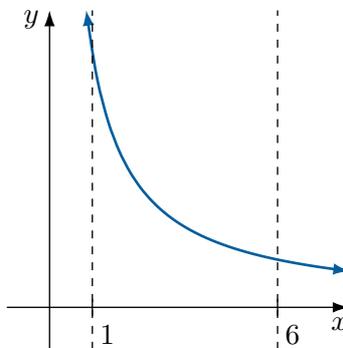
Question 8 The graphs of $y = \sqrt{x}$ and $y = x^2$ are drawn below.



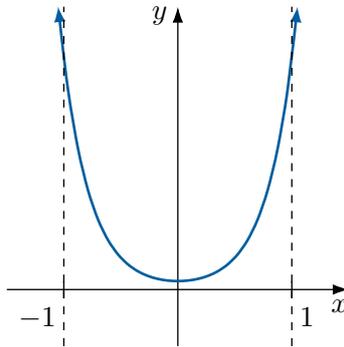
Find the volume of the solid formed by rotating the region enclosed by the two curves about the

- (a) x -axis
- (b) y -axis

Question 9 A section of the curve $y = \frac{6}{x}$ is drawn below



Question 16 Find the volume generated when the area bounded by $y = \frac{e^{4x} + e^{-4x}}{10}$, x -axis, $x = -1$, $x = 1$ is rotated about x -axis



Question 17

- (a) The curve $y = \sqrt{\cos 3x}$ is rotated about the x -axis between $-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$. Find the volume of the solid formed.
- (b) The curve $y = \sqrt{\sin \frac{x}{4}}$ is rotated about the x -axis between $0 \leq x \leq 4\pi$. Find the volume of the solid formed.

Question 18 Find the volume of the solid formed by rotating the region enclosed by $y = \ln x$ between $x = 1$ and $x = 2$ about the y -axis

Question 19 Sketch $y = 1 - \tan x$ for $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

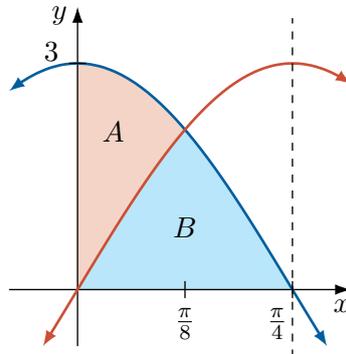
- (a) Find the area bounded by the curve, the x -axis and $x = -\frac{\pi}{4}$
- (b) Find the volume when the area above is rotated about the x -axis

Question 20 The portion of the curve $y = \frac{1}{\sqrt{25 + x^2}}$ from 0 to 5 is rotated about the x -axis through a complete revolution. Find the volume generated in exact form

Question 21

- (a) The portion of the curve $y = \sin^{-1} 2x$ from $0 \leq y \leq \frac{\pi}{2}$ is rotated about the y -axis through a complete revolution. Find the volume generated in exact form
- (b) Find the volume of the solid generated when the region enclosed by the curve $y = 2 \cos^{-1}(x)$, the y -axis and $y = \pi$ is rotated around the y -axis

Question 22 A section of the graphs $y = 3 \sin 2x$ and $y = 3 \cos 2x$ between $x = 0$ and $x = \frac{\pi}{4}$ is drawn. Section A is the region bounded by the curves and the y -axis and section B is the region bounded by the curves and the x -axis in this domain.



- (a) Find the volume generated when region A is rotated about the x -axis
 (b) Find the volume generated when region B is rotated about the x -axis

Question 23 The curve $y = 2 \cos\left(\frac{\pi x}{3}\right)$ and the line $y = x$ intersect at the point $(1, 1)$. The region bound by the curve, the line $y = x$ and the y -axis is rotated about the x -axis to form a solid. Find the volume of the solid.

Question 24 A bucket is formed by rotating the curve with equation

$$y = 30 \ln\left(\frac{x - 15}{10}\right)$$

for $0 \leq y \leq 30$ about the y -axis. Find the capacity of the bucket.

 Investigation Task

Centre of mass

An important application of integration is in engineering. One of the first things you will learn in any standard engineering course is the *centre of mass*. In order to keep the mathematics within the scope of the syllabus, let's assume that we are finding the centre of mass of a plate that has *uniform density*.

Question 1 Explain what a *centre of mass* is, and state the formula for the centre of mass of a uniform plate defined by the region between two curves $f(x)$ and $g(x)$.

Question 2 Use graphing software to plot each of the regions below. Take a screenshot of the region and guess intuitively where you would expect the centre of mass to be.

- The region bounded by $y = 1 - x^2$ and the x -axis.
- The region bounded by $y = \sqrt{x}$ in the domain $x \in [0, 4]$.
- The region bounded by $y = x^2$ and $y = \sqrt{x}$.

Question 3 Calculate the actual centre of mass of all the regions in the above question and compare it with your original guess.

Question 4 Write a four page document aimed at Year 12 Mathematics Extension 1 students who have just completed the *Integration* topic. The document should

- contain the derivation and the intuitions behind the derivation of the formulae.
- contain two worked examples.
- be sufficiently detailed so that a student reading it can do their own calculations.
- explain mathematically why the formula for the centre of mass is independent of the density, if the plate is uniformly dense.
- list some practical applications of the centre of mass.

6

THE BINOMIAL DISTRIBUTION

- Bernoulli random variables
- Bernoulli trials
- Binomial distributions
- Sample proportions
- Approximately normal distributions

Question 1 People in a city were randomly surveyed to see if they are in favour of the current Prime Minister. The results show that 60% are in favour and 40% are not in favour. Let X be a random variable that indicates whether a person is in favour or not.

- (a) Express your findings in the form

$$P(X = x) = \begin{cases} \text{---}, & \text{if } x = \text{---} \\ \text{---}, & \text{if } x = \text{---} \end{cases}$$

- (b) Calculate μ .
- (c) Hence, if multiple surveys of 200 people are conducted, on average how many people will be in favour?

Question 2 Bob plays a game where he flips a coin. If it lands on heads he wins \$10 but if it lands on tails, he loses \$6. Let X represent the amount of money that Bob wins in a single flip of the coin.

- (a) Complete the formula

$$P(X = x) = \begin{cases} \text{---}, & \text{for } x = \text{---} \\ \text{---}, & \text{for } x = \text{---} \end{cases}$$

- (b) Find Bob's expected winnings.
- (c) Find the variance and hence standard deviation of his winnings.

Question 3 Mary plays a game where she rolls two dice. If it is a double, she wins \$15 but otherwise she loses \$2. Let X represent the amount of money that Mary wins in a single roll.

- (a) Complete the formula

$$P(X = x) = \begin{cases} \text{---}, & \text{for } x = \text{---} \\ \text{---}, & \text{for } x = \text{---} \end{cases}$$

- (b) Find Mary's expected winnings.
- (c) Find the variance and hence standard deviation of her winnings.

Question 4 Bob plays a game where he attempts to score a goal. If he scores, he wins \$50. It costs \$5 to attempt the game. The probability of Bob scoring is 0.05. Let X represent the amount of money that Bob wins in a single attempt.

- (a) Find μ and hence explain why the game is unfavourable for Bob.
- (b) Suppose that Bob's probability of scoring is fixed at 0.05. What should the entry fee for the game be so that the game is fair for Bob?
- (c) Suppose that the cost to play is fixed at \$5. What should the probability of Bob scoring be, so that the game is fair for Bob?

Question 5 [Different methods to calculate expectation]

A teacher gives Bob and Mary an expectation question where a person rolls a die. If a 'six' is rolled, the person wins \$5, otherwise they win nothing. The game costs \$2 to enter.

Bob calculates the expectation directly from below.

$$P(X = x) = \begin{cases} 3, & \text{if } x = \frac{1}{6} \\ -2, & \text{if } x = \frac{5}{6} \end{cases}$$

Mary instead calculates the expected value from

$$P(X = x) = \begin{cases} 5, & \text{if } x = \frac{1}{6} \\ 0, & \text{if } x = \frac{5}{6} \end{cases}$$

and then subtracts the fee of \$2 after calculating the expectation. Who is correct?

Question 6 [Proof for $E(X)$ of a Bernoulli random variable]

Consider the Bernoulli random variable X defined so that

$$P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}$$

- Write down the formula for the expected value of a discrete random variable.
- Hence, calculate $E(X)$ and show that $E(X) = p$

Question 7 [Proof for $\text{Var}(X)$ of a Bernoulli random variable]

Consider the Bernoulli random variable X defined so that

$$P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}$$

- Write down the formula for the variance of a discrete random variable.
- Hence, calculate $\text{Var}(X)$ and show that $\text{Var}(X) = p - p^2$.

⚙️ Challenge Problems

Problem 1 The probability function of a Bernoulli random variable is sometimes expressed as

$$f(x) = p^x(1 - p)^{1-x}$$

Explain how this works.

Question 1 Consider the random variable $X \sim \mathcal{B}(3, 0.2)$. Find the value of the following.

- (a) $P(X = 0)$ (b) $P(X = 1)$ (c) $P(X = 2)$ (d) $P(X = 3)$

Question 2 Consider the random variable $X \sim \mathcal{B}(8, 0.3)$. Find the value of the following.

- (a) $P(X \geq 7)$ (b) $P(X \leq 2)$ (c) $P(X \geq 2)$ (d) $P(X \leq 7)$

Question 3 An archer aims at a target. The probability of the archer hitting the target is $\frac{2}{3}$, and every shot is an independent event. He takes six shots at the target.

- (a) If the random variable is $X \sim \mathcal{B}(n, p)$, write down n and p .
 (b) Hence, find the probability that the archer hits the target exactly four times.

Question 4 A factory produces phones and the probability of a phone being faulty is 10%. A sample of twenty phones is inspected.

- (a) If the random variable is $X \sim \mathcal{B}(n, p)$, write down n and p .
 (b) Hence, find the probability that exactly five of the phones were faulty.

Question 5 A coin is biased so that $P(H) = 0.2$. The coin is flipped ten times.

- (a) If the random variable is $X \sim \mathcal{B}(n, p)$, write down n and p .
 (b) Hence, find the probability that there are exactly six tails.

Question 6 A fair coin is flipped five times. Find the probability that there are

- (a) exactly 3 heads. (b) at least one head. (c) at least four heads.

Question 7 A manufacturing company produces toys. Typically, about 2% of all toys produced are faulty. In a batch of 100, find the probability that

- (a) no toys are faulty. (b) at least two toys are faulty.

Question 8 A bag contains 4 white balls and 6 black balls. Five balls are drawn, and the ball is replaced after each draw.

- (a) Find the probability that three of the five drawn balls are black.
 (b) Find the probability that the only black balls are the first three?

Question 9 A multiple choice exam has 10 questions, each of which having four possible options. Cody did not study at all and guesses all questions in the test. What is the probability that Cody

- (a) scored zero? (b) scored full marks?
 (c) scored 4 out of 10. (d) scored more than 8 correctly?
 (e) scored more than 2 correctly? (f) scored at most 2 correctly?

Question 10 A lottery scratchie ticket has a probability 0.1 of being a winner. Hugh buys four lottery tickets. Find the probability that

- (a) exactly two of them were winners. (b) all of them were winners.
 (c) none of them were winners. (d) at least one of them was a winner.

Question 11 A survey of a town found that 40% of people drink coffee every day. A sample of six people were selected from the town. What is the probability that more than half of them drank coffee every day?

Question 12 [Checking your understanding]

A teacher gives Bob the standard scenario of rolling a fair die five times. Bob has to find the probability of rolling ‘six’ exactly three times. Explain what is wrong with the following responses from Bob.

- (a) $\left(\frac{1}{6}\right)^3$ (b) $\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$

Question 13 [Symmetry]

A teacher gives Bob and Mary the following question.

“A coin is flipped ten times. Find the probability that there were at least five heads”

Bob takes a brute-force approach

$$\begin{aligned} P(X \geq 5) &= P(X = 5) + P(X = 6) + \cdots + P(X = 10) \\ &= \binom{10}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 + \binom{10}{6} \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 + \cdots + \binom{10}{10} \left(\frac{1}{2}\right)^{10} \end{aligned}$$

Mary instead takes a more elegant approach and simply writes down

$$P(X \geq 5) = \frac{1}{2} \left(1 - \binom{10}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 \right) + \binom{10}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5$$

- (a) Are both methods correct?
 (b) Explain Mary’s approach.
 (c) Would Mary’s approach still work if it was guessing answers in a multiple choice exam with four options instead? Explain your answer.

Question 14 [Don’t blindly use the formula!]

A fair die is rolled ten times. Find the probability that the number ‘six’ is rolled exactly

- (a) four times.
 (b) four times, but all in the first four rolls.
 (c) four times, but with the first roll being a ‘six’.
 (d) four times, but with the first and last roll being ‘six’.

Question 15 In a particular city, the probability of a day being cloudy is $\frac{1}{5}$ and the cloudiness of days are independent events. Find the probability that in a week

- (a) exactly three days are cloudy.
- (b) exactly five days are clear.
- (c) exactly five days are cloudy, with Monday being one of them.
- (d) exactly four days are cloudy, with Monday and Tuesday being two of them.

Question 16 A fair die is rolled n times.

- (a) What is the probability of rolling at least one 'six'?
- (b) Find the value of n so that the probability of rolling at least one 'six' is greater than 90%.

Question 17 [Technology]

Bob spins a counter n times, where each spin has a 0.2 chance of landing on the prize.

- (a) Show that the probability of Bob winning at most once

$$0.8^n + 0.2n(0.8)^{n-1}$$

- (b) Bob wishes to find the number of times he should spin the counter so that the probability of him winning at least twice exceeds 90%. Show that

$$0.8^n + 0.2n(0.8)^{n-1} < 0.1$$

- (c) Use graphing software to sketch $y = 0.8^x + 0.2x(0.8)^{x-1}$ and hence find the least number of spins that Bob should take.



Challenge Problems

Problem 1 Bill spins a counter that has a 0.1 chance of landing on a prize. If he makes n attempts, how many times should Bill spin the counter so that the probability of winning at least one prize is at least 95%?

Problem 2 A biased coin is flipped 6 times. It is known that the coin is nine times more likely to flip H four times, than scoring H two times. Find the probability for each face of the coin.

Problem 3 [Compound probabilities]

In any week of the year, the probability that a day is wet is $\frac{1}{3}$. Assume that each day is an independent event.

- (a) Find the probability that most of the week was rainy.
- (b) Find the probability that during a ten-week period, exactly three of them had mostly wet days.

Problem 4 A competition consists of a week of daily rounds. Every day, archers must take ten shots and hit the target at least nine out of the ten shots, otherwise they are eliminated from the entire competition. Leon is consistent and hits the target 80% of the time, and every shot is an independent event. What is the probability that Leon gets eliminated in the last day of the competition?

Problem 5 A biased coin has probability $p \neq \frac{1}{2}$ of landing on heads. Bob is just as likely to get more than half of his flips being 'heads' regardless of whether he flips the coin two or four times. What is the value of p ?

Exercise 6C

Binomial distributions

Fundamentals

Fundamentals 1

Consider the binomial random variable $X \sim \mathcal{B}(n, p)$.

- Write down the formula for the mean.
- Write down the formula for the variance.
- Write down the formula for the standard deviation.

Question 1 For each of the following binomial distributions below, calculate μ and $\text{Var}(X)$.

- $X \sim \mathcal{B}(10, 0.2)$
- $X \sim \mathcal{B}(20, 0.6)$
- $X \sim \mathcal{B}\left(120, \frac{1}{3}\right)$
- $X \sim \mathcal{B}\left(50, \frac{1}{4}\right)$

Question 2 A binomial distribution $X \sim \mathcal{B}(n, p)$ has $\mu = 6$ and $\text{Var}(X) = 4$.

- Find the parameters n and p .
- Hence, find $P(X = 3)$.

Question 3 [Technology]

A fair die is rolled 12 times. Let X be the number of sixes that get rolled.

- Find the mean and standard deviation of X .
- State the values of X such that $\mu - \sigma \leq X \leq \mu + \sigma$.
- Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$.
- Repeat the above but if the die were rolled 120 times instead.
- What familiar looking result do you end up with? What is the possible reason for this?

Question 4 [Technology]

A fair coin is flipped 30 times. Let X be the number of heads flipped.

- Find μ and σ .
- Hence, state the values of X that lie in the interval $[\mu - 2\sigma, \mu + 2\sigma]$.
- Use computer assisted software to calculate $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$ and show that it is approximately equal to 0.95.

Question 5 A fair coin is flipped 10 times. Let X be the number of heads flipped.

- (a) State the value of μ and hence the most likely number of heads.
- (b) Find $\text{Var}(X)$ and hence σ .

Question 6 Bob flips three fair coins and wins 2^X dollars, where X is the number of heads flipped. However, the game costs \$4 to play.

- (a) Find the expected gross winnings per game, excluding the cost.
- (b) How much should Bob expect to win or lose overall, after 200 games?

Question 7 Mary has five attempts at hitting a target when she throws a ball. Each throw is an independent event with probability 0.6 of hitting the target. For every time she hits the target, she wins \$5. The game costs \$20 to have the five attempts.

- (a) Show that the game is *not* in Mary's favour.
- (b) What should the probability of Mary hitting the target be, so that the game is at least fair for Mary?

Challenge Problems

Problem 1 Consider $X \sim \mathcal{B}(5, p)$.

- (a) Show that if $P(X = 1) = P(X = 4)$, then $p = \frac{1}{2}$.
- (b) Show that if $P(X = 2) = P(X = 3)$, then $p = \frac{1}{2}$.
- (c) How could the above results have been found intuitively?

Problem 2 Mary plays a game with a machine that randomly generates 5-digit numbers from 1, 2, 3, ..., 9. If the number has more odd than even digits, then she wins \$5. The game costs \$1 to play. Is the game favourable for Mary? In other words, after playing the game repeatedly, will she expect to walk away with a profit or loss?

Problem 3 Let $X \sim \mathcal{B}(n, p)$ have mean μ and standard deviation σ .

- (a) Show that $p = 1 - \frac{\text{Var}(X)}{\mu}$.
- (b) Deduce that $\mu > \text{Var}(X)$.
- (c) As $p \rightarrow 1$, do the values of μ and $\text{Var}(X)$ get closer or further apart?
- (d) As $p \rightarrow 0$, do the values of μ and $\text{Var}(X)$ get closer or further apart?

Exercise 6D

Sample proportions

Fundamentals

Fundamentals 1

In the previous exercise, the focus was on the exact number of outcomes satisfying a particular condition. Sometimes, we prefer the focus to be on the p _____ of outcomes satisfying a particular condition, like in a survey. In these cases, we use another random variable called the s _____ proportion.

Fundamentals 2

- (a) A p _____ proportion is the proportion of people with a particular attribute with respect to the entire p _____. The value of p is c _____ for a given population.
- (b) A s _____ proportion is the proportion of people with a particular attribute with respect to a sample of people.
- (c) The formula for the s _____ proportion is

$$\hat{p} = \text{_____}$$

- (d) The value of \hat{p} can v _____, and hence may have differing values across many samples.
- (e) For this reason, \hat{p} is a r _____ v _____ and the distribution is also a b _____ distribution.

Fundamentals 3

- (a) The mean of the distribution of the sample proportion is given by $\bar{x}_{\hat{p}} = \text{_____}$.
- (b) The standard deviation is given by $\sigma_{\hat{p}} = \text{_____}$.

Question 4 In a school, 70% of students study some form of mathematics in Year 12. A sample of five students is taken. Let p be the population proportion and let \hat{p} be the sample proportion of students who study mathematics in Year 12. Find the probability that from the sample

- (a) exactly two students study mathematics.
- (b) more than half of the students study mathematics.
- (c) less than half of the students study mathematics.

Question 5 In a population, the population proportion of a particular trait is p . Random samples of size n are taken from the population, and the proportion of the sample who have that trait are recorded and follow a binomial distribution.

- (a) Find the mean and standard deviation of the distribution of the sample proportion if $p = 0.4$ and $n = 20$.
- (b) Find the size of the samples taken if the standard deviation and mean of the distribution of the sample proportion are both 0.2.
- (c) Find the population proportion if the standard deviation is 0.04 and the sample size is 100.

Question 6 A census showed that 30% of students at a school play a musical instrument. A random sample of 60 students is taken. Let \hat{p} represent the binomially distributed sample proportion. Calculate the

- (a) mean for the distribution of the sample proportion.
- (b) variance for the distribution of the sample proportion.
- (c) standard deviation for the distribution of the sample proportion.

Question 7 A biased coin has the probability of flipping heads to be 0.6. A 'round' of flips consists of n flips. Let \hat{p} be the random variable of the proportion of flips in a round that result in heads. How many flips need to be in round, so that the standard deviation of \hat{p} is 0.1?



Challenge Problems

Problem 1 [\hat{P} has maximal probability when $\hat{p} = p$]

Let $X \sim \mathcal{B}(n, p)$ so that $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

- (a) Write down a similar expression for $P(X = k - 1)$.
 (b) Show that

$$\frac{P(X = k)}{P(X = k - 1)} = \frac{n - k + 1}{k} \times \frac{p}{1 - p}$$

- (c) Show that if $P(X = k) \geq P(X = k - 1)$, then $k \leq np + p$.
 (d) Explain why the value of x that maximises $P(X = x)$ is $x = np$.
 (e) Deduce that the value of \hat{P} is maximised when $\hat{p} = p$.

Problem 2 [Mean and standard deviation of the sample proportion]

Let $X \sim \mathcal{B}(n, p)$ and let the sample proportion random variable be \hat{P} . Assume that the population is large.

- (a) Explain briefly why we can assume that $\hat{P} = \frac{X}{n}$.
 (b) You may have noticed from previous questions that the mean μ of \hat{P} was always the population proportion p . Use the fact that $E(aX + b) = aE(X) + b$ to prove that

$$E(\hat{P}) = p$$

- (c) Use the fact that $\text{Var}(aX + b) = a^2 \text{Var}(X)$ to prove that

$$\text{Var}(\hat{P}) = \frac{p(1-p)}{n}$$

Exercise 6E

Approximately normal distributions

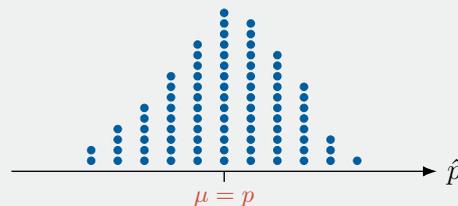
Fundamentals

Fundamentals 1

In earlier binomial probability and sample proportion problems, the sample sizes were sufficiently small so that the probabilities can be calculated manually on a case-by-case basis. Realistically, sample sizes are often large enough so that it becomes impractical to manually calculate probabilities by summation. However, we can use the fact that if n is sufficiently large, the binomial distribution and the distribution of the sample proportion both are sufficiently approximated by a normal distribution. With this comes the benefit of being able to use z -score tables.

Fundamentals 2

Several samples are taken from a large population to survey for a particular trait, which is known to have population proportion p . The sample proportion \hat{p} having that trait is recorded on a frequency dot plot.



What kind of distribution do the values of \hat{p} appear to be forming?

Fundamentals 3

- Suppose a large population has population proportion p of some attribute. When several samples are collected and their sample proportions \hat{p} calculated, the value of \hat{p} should cluster around ____.
- The values of \hat{p} will form approximately a normal distribution centred at ____.
- The values of \hat{p} will be approximately normal if $np \geq$ ____ and $n(1 - p) \geq$ ____.

Fundamentals 4

Follow the steps below to calculate probabilities using the normal distribution.

- Verify that n is sufficiently large and that the distribution can be approximated by a normal distribution.
- Calculate the mean and standard deviation of the distribution.
- Calculate the z -score of the boundaries required, and use the appropriate table to calculate the required probability, as done in Mathematics Advanced.

Question 1 [Useful skill for this exercise]

A census showed that 80% of households cook on the weekends. A random sample of 40 families is taken. Find the z -score (correct to 2 decimal places) of the sample if in the sample

- (a) 34 families cooked on the weekends. (b) 29 families cooked on the weekends.

Question 2 [Guided question]

A biased coin has probability 0.4 of flipping heads. The coin is tossed 160 times. Let X be the binomially distributed random variable representing the number of heads flipped.

- (a) Find $E(X)$ and $\text{Var}(X)$.
 (b) Briefly explain why we can use a normal distribution.
 (c) Find the z -score corresponding to 70 heads, correct to two decimal places.
 (d) Complete the following and use the z -score table to find the probability that at most 70 heads were flipped.

$$P(X \leq 70) = P(z \leq \underline{\quad})$$

$$= \underline{\hspace{2cm}}$$

- (e) Find the probability that at least 72 heads were flipped.
 (f) Find the probability that at most 62 heads were flipped.

Question 3 A recent census showed that 15% of adults worked from home. A survey of 120 adults is conducted. Let \hat{p} represent the binomially distributed sample proportion of people who work from home.

- (a) Find the mean and standard deviation of the sample proportion.
 (b) Find the probability that in the survey, at most 24 people work from home.
 (c) Find the probability that in the survey, at most 15 people work from home.

Question 4 When a biased coin is flipped, the probability of flipping heads is 0.6. The coin is flipped 100 times. Let X be the binomially distributed random variable representing the number of heads flipped.

- (a) Find $E(X)$ and $\text{Var}(X)$. (b) Find $P(62 < X < 66)$.
 (c) Find $P(55 < X < 62)$. (d) Find $P(50 < X < 55)$.

Question 5 [Guided question for sample proportions]

In a university, 10% of all students live on campus. A sample of 90 students is taken. Let \hat{p} be the sample proportion of students who live on campus.

- Find the mean and standard deviation for the distribution of the sample proportion.
- Find the value of \hat{p} that corresponds to 12 students living on campus.
- Hence, find the z -score that corresponds to that value of \hat{p} .
- Hence, find the probability that from the sample, at least 12 students live on campus.

Question 6 In a city, 80% of people have a driver's licence. A sample of 50 people is taken. Let p be the population proportion and let \hat{p} be the sample proportion of people who have a licence. Find the probability that from the sample

- more than 85% of people own a licence.
- at most 36 people own a licence.

Question 7 A random sample of 160 is taken from a population in which the probability that a person owns a vehicle is $p = 0.56$. Find

- $P(\hat{p} > 0.62)$
- $P(0.52 \leq \hat{p} \leq 0.62)$

Question 8 A census showed that 80% of households had a working smoke alarm. The sample proportion of households having a working smoke alarm is binomially distributed. In a sample of 300 households, find the probability that at least 75% of households had a working smoke alarm.

Question 9 A biased coin is tossed 100 times. The probability of a head coming up in any one throw is 0.4. What is the probability of getting between 35 to 45 heads inclusive?

Challenge Problems

Problem 1 A global company survey is conducted and 24% of employees go to the gym regularly. A random sample of 800 employees is taken, and the sample proportion of employees who go to the gym regularly is binomially distributed. What is the probability that anywhere from 21% to 23% of the sample go to the gym regularly?

Problem 2 In a company 55% of employees are male. A sample of 80 employees is taken. Let p be the population proportion and let \hat{p} be the sample proportion of male employees. Find the probability that anywhere from 36 to 46 people inclusive from the sample are male.

Chapter 6 Review

The Binomial Distribution

Review

Question 1 Bob plays a game where he rolls two die. If he gets a double, he wins \$4. The game costs \$1 to enter.

- (a) Complete the formula

$$P(X = x) = \begin{cases} \text{---}, & \text{if } x = \text{---} \\ \text{---}, & \text{if } x = \text{---} \end{cases}$$

- (b) Find Bob's expectation.
 (c) After 30 games, how many dollars should Bob expect to have won or lost?

Question 2 Bob plays a game where he attempts to score a goal. If he scores, he wins \$20. It costs \$3 to attempt the game. The probability of Bob scoring is 0.1. Let X represent the amount of money that Bob wins in a single attempt.

- (a) Find μ and hence explain why the game is unfavourable for Bob.
 (b) Suppose that Bob's probability of scoring is fixed at 0.1. What should the entry fee for the game be so that the game is fair for Bob?
 (c) Suppose that the cost to play is fixed at \$3. What should the probability of Bob scoring be, so that the game is fair for Bob?

Question 3 Find the mean and variance of a Bernoulli random variable X with parameter

- (a) $p = \frac{1}{2}$ (b) $p = \frac{2}{3}$ (c) $p = \frac{1}{4}$

Question 4 Consider the random variable $X \sim \mathcal{B}(6, 0.2)$. Find the value of the following.

- (a) $P(X \geq 5)$ (b) $P(X \leq 1)$ (c) $P(X \geq 1)$ (d) $P(X \leq 5)$

Question 5 A fair coin is flipped four times. Find the probability that there are

- (a) exactly 3 heads. (b) at least one head. (c) at least three heads.

Question 6 A multiple choice exam has 5 questions, each of which having four possible options. Jono did not study at all and guesses all questions in the test. What is the probability that Jono

- (a) scored zero? (b) scored full marks?
 (c) scored 4 out of 5. (d) scored 4 or more correctly?
 (e) scored more than 1 correctly? (f) scored at most 1 correctly?

Question 7 A biased coin such that $P(H) = \frac{2}{3}$ is flipped six times. Find the probability that H is flipped exactly

- (a) four times.
- (b) four times, but all in the first four flips.
- (c) four times, but with the first flip being H .
- (d) four times, but with the first three flips being H .

Question 8 A fair die is rolled n times.

- (a) What is the probability of rolling at least one 'six'.
- (b) Find the value of n so that the probability of rolling at least one 'six' is greater than 99%.

Question 9 How many times should a pair of dice be rolled so that the probability of scoring at least one 'double six' exceeds 50%?

Question 10 There are five families each with four children. Find the probability that

- (a) a particular family has two boys and two girls.
- (b) exactly two families have two boys and two girls.
- (c) at most one family has two boys and two girls.

Question 11 A box contains 8 identical balls except for their colour, where four are blue, three are white and one is red.

- (a) If two balls are selected at random without replacement, find the probability of getting two blue balls.
- (b) Two balls are now selected at random on five occasions. After each selection of two balls, they are replaced and the process is repeated. What is the probability of getting two blue balls on exactly three occasions?

Question 12 Bob is given a biased coin where $P(H) = \frac{1}{3}$. He flips it four times and wins X^2 dollars, where X is the total number of 'heads' scored. However, the game costs \$2 to play.

- (a) Find the expected gross winnings per game.
- (b) How much should Bob expect to win or lose overall, after 30 games?

Question 13 In a school, 40% of students take the bus to school. A sample of 5 students is taken. Let p be the population proportion and let \hat{p} be the sample proportion of students who take the bus to school. Find

- (a) $P(\hat{P} = 0.2)$
- (b) $P(\hat{P} = 0.4)$
- (c) $P(\hat{P} = 0.8)$

Question 14 For each of the following binomial distributions below, calculate the μ and $\text{Var}(X)$.

(a) $X \sim \mathcal{B}\left(20, \frac{2}{5}\right)$ (b) $X \sim \mathcal{B}\left(100, \frac{1}{10}\right)$ (c) $X \sim \mathcal{B}\left(90, \frac{2}{3}\right)$

Question 15 Suppose $X \sim \mathcal{B}(n, p)$ has a mean of 12 and variance of 9. Find n and p .

Question 16 In a law firm, there are 10 lawyers and 6 consultants. On any given day, the probability that a lawyer will have a 7AM meeting is 12% and the probability that a consultant will have a 7AM meeting is 4.5%. On any random day, find the probability that exactly 3 lawyers and 2 consultants will have 7AM meetings.

Question 17 Elle makes 10 attempts to score a goal. She has probability 0.8 of scoring, and each attempt is an independent event. For every goal she scores, she wins \$10. It costs \$60 to have the ten attempts.

- (a) Show that the game is in Elle's favour.
 (b) Suppose Elle misses the first shot. Is the game still favourable for her at this point?

Question 18 How many times does an unbiased coin need to be tossed before the standard deviation is less than 5% of the number of trials?

Question 19 A recent census shows that 50% of athletes stretch before training. Samples of 20 athletes is taken. Let \hat{p} be the sample proportion of athletes who stretch before training. Find the probability that in the sample

- (a) more than 50% stretch before training. (b) at most 80% stretch before training.
 (c) at least 40% stretch before training. (d) no more than 30% stretch before training.

Question 20 At a sports school, the proportion of students who swim competitively is 16%. A random sample of 200 students is taken. Let \hat{p} represent the sample proportion of those who swim competitively.

- (a) Find $E(\hat{p})$ and $\text{Var}(\hat{p})$.
 (b) What is the probability that fewer than 14% of the sample swim competitively?
 (c) What is the probability that anywhere from 12% to 18% of the sample swim competitively?

Question 21 A survey of families showed that 65% of households had both parents being employed. In a random sample of 40 families, find the probability that between 22 to 24 families had both parents being employed.

 Investigation Task

Central Limit Theorem

In the Extension 1 course, we make a claim that for large n , the binomial distribution is approximately normal. But upon deeper thought, this is not at all obvious and several questions are raised.

- What does ‘approximately’ mean in this case?
- Will *any* binomial distribution be ‘approximately normal’?
- Are there certain circumstances for which it is approximately normal?
- Is there some measure of how ‘approximate’ the normal distribution is to the binomial distribution?

Write a short informative article explaining in detail the connection between the binomial distribution and the normal distribution. Your answer should address the questions raised above, as well as address the role that the *Central Limit Theorem* plays in the connection between the binomial and normal distribution.

 Investigation Task

Continuity Correction

When reading any text that explains the connection between the normal and binomial distributions, you will almost certainly see the term *continuity correction* being used multiple times. Research what this term means, and the significance of it when going from a binomial distribution to a normal distribution. Write a short paper that is comprehensive enough so that if a Year 12 Extension 1 student reads it, they will be able to fully understand what continuity correction is, what it is used for and how it works.

Your response should address/include the following:

- What are the formulas involved in continuity correction?
- An example of a probability calculated with and without continuity correction.
- What kinds of scenarios rely more on continuity correction to obtain the correct probability?

 Investigation Task

Mean and variance of $\mathcal{B}(n, p)$

In the course we use the facts that if $X \sim \mathcal{B}(n, p)$, then $E(X) = np$ and $\text{Var}(X) = np(1 - p)$. However, the proof of these results are not required by the student. In actual fact they are quite accessible by the typical Extension 1 student with the assistance of sigma notation.

Sigma notation offers us an easier way of managing sums without having to explicitly write down many terms. One of the simplest examples is shown below.

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (n-1)^2 + n^2$$

You will need this notation to assist you in the proofs below. For each proof:

- show all algebraic steps.
- state any formulas being used.
- justify any non-obvious steps.

Question 1 Derive in full detail the formula for expectation $E(X) = np$ for the binomial distribution $X \sim \mathcal{B}(n, p)$.

Question 2 Derive in full detail the formula for variance $\text{Var}(X) = np(1 - p)$ for the binomial distribution $X \sim \mathcal{B}(n, p)$.

7

DIFFERENTIAL EQUATIONS

- Introduction to differential equations
- Direction fields
- First order differential equations
- Applications of differential equations

Exercise 7A

Introduction to differential equations

Fundamentals

Fundamentals 1

- (a) A differential equation is an equation that involves a d _____.
- (b) A s _____ of a differential equation is a function $f(x)$ that satisfies the differential equation.
- (c) The s _____ of a differential equation are not unique because they all differ by a c _____. In other words they form a f _____ of curves $f(x) + _$.

Fundamentals 2

- (a) When a differential equation is solved, there are i _____-many solutions that differ by a constant.
- (b) Sometimes, the constant can be found if the question provides an i _____ value (x_0, y_0) . The resulting solution that satisfies the differential equation and the i _____ value is called the p _____ solution.
- (c) When no i _____ condition is provided i.e. the constant is left as a constant, the solution obtained is instead called a g _____ solution.

Fundamentals 3

- (a) A f _____-order differential equation is a differential equation involving up to and including the f _____ derivative.
- (b) A s _____-order differential equation is a differential equation involving up to and including the s _____ derivative.

Fundamentals 4

To show that a function satisfies a differential equation, calculate the necessary d _____ and then substitute them into the differential equation and show that the l _____-hand side is equal to the r _____-hand side.

Question 1 Show that the following functions satisfy the given differential equation and initial conditions.

(a) $y = e^{2x}$
 $y' = 2y$
 $y(0) = 1$

(b) $y = e^{-3x}$
 $y' = -3y$
 $y(0) = 1$

(c) $y = \sqrt{2x+1}$
 $y' = \frac{1}{y}$
 $y(0) = 1$

(d) $y = \sqrt{x^2+1}$
 $y' = \frac{x}{y}$
 $y(\sqrt{3}) = 2$

(e) $y = 5 + 2e^{-2x}$
 $y' = 10 - 2y$
 $y(-\ln 2) = 13$

(f) $y = 1 + 3e^x$
 $y' = y''$
 $y(\ln 2) = 7$

(g) $y = \sin(2x)$
 $y'' = -4y$
 $y\left(\frac{\pi}{12}\right) = \frac{1}{2}$

(h) $y = 1 + 2\cos(3x)$
 $y'' = 9 - 9y$
 $y\left(\frac{\pi}{3}\right) = -1$

(i) $y = 2\sin(3x) - 4\cos(3x)$
 $y'' = -9y$
 $y\left(\frac{\pi}{2}\right) = -2$

Question 2

(a) Show that $y = \frac{1}{x}$ satisfies the differential equation $xy' + y = 0$

(b) Show that $y = \frac{1}{x^2}$ satisfies the differential equation $xy' + 2y = 0$

(c) Show that $y = \frac{1}{x^3}$ satisfies the differential equation $xy' + 3y = 0$

(d) Conjecture a differential equation for $y = \frac{1}{x^n}$ and show that it works.

Question 3 Show that $y = \tan x$ satisfies the differential equation

$$y' = 1 + y^2$$

Question 4 Show that $y = xe^{-x}$ satisfies the differential equation

$$xy' + (x-1)y = 0$$

Question 5 Show that $y = x \sin x$ satisfies the differential equation

$$y'' + y = 2 \cos x$$

Question 6 Show that $y = \frac{\sin x}{x}$ satisfies the differential equation

$$xy'' + 2y' + xy = 0$$

Question 7 Show that $y = x \ln x$ satisfies the differential equation

$$x^2 y'' - xy' + y = 0$$

Question 8 Show that $y = \frac{x}{x+1}$ satisfies the differential equation

$$(x + x^2)y' - y = 0$$

Question 9 Show that $y = \frac{x}{\ln x}$ satisfies the differential equation

$$y' = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

Question 10 [Second order linear ordinary differential equation]

Show that $y = Ae^{-x} + Be^{2x}$ satisfies the differential equation

$$y'' - y' - 2y = 0,$$

for any real values of A and B .

Question 11 [Non-uniqueness of solutions]

Show that the equations below satisfy the differential equation $y'' - 2y' + y = 0$.

(a) $y = e^x$

(b) $y = xe^x$

Question 12 [Newton's Law of Cooling]

Show that $T = B + Ae^{-kt}$ satisfies the differential equation

$$\frac{dT}{dt} = -k(T - B)$$

for any real constants A , B and k .

Question 13 [Non-uniqueness of the differential equation]

Show that $y = Ae^{kx}$ satisfies the differential equations below.

(a) $y' - ky = 0$

(b) $y'' - k^2y = 0$

Question 14 [Hyperbolic sine and cosine]

Show that the equations below satisfy the differential equation $y'' - y = 0$.

(a) $y = \frac{e^x + e^{-x}}{2}$

(b) $y = \frac{e^x - e^{-x}}{2}$



Question 15 Show that

$$y = A \cos(nx) + B \sin(nx)$$

is a solution of $y'' + n^2y = 0$ for any real values of A and B .

⚙ Challenge Problems

Problem 1 Show that if $b^2 - 4ac < 0$, then $y = e^{kx}$ will never be a solution of

$$ay'' + by' + cy = 0$$

for any real values of k .

Problem 2 Show that $y = \ln(\ln x)$ satisfies the differential equation

$$x \left(\frac{d^2y}{dx^2} \right) + x \left(\frac{dy}{dx} \right)^2 + \frac{dy}{dx} = 0$$

Problem 3 Show that $y = \sqrt{x}(A + B \ln x)$ satisfies the differential equation

$$4x^2y'' + y = 0$$

for any real values of A and B .

Exercise 7B

Direction fields

Fundamentals

Fundamentals 1

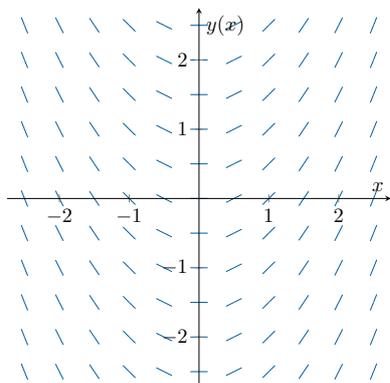
- (a) A differential equation $\frac{dy}{dx} = f(x)g(y)$ can have i_____ many solutions.
- (b) When a point (x_0, y_0) is substituted into the differential equation, a g_____ is obtained and a short line segment can be drawn at that point showing that g_____.
- (c) A d_____ field is the diagram obtained when a number of these line segments are constructed on the same set of axes.
- (d) These small line segments trace out trajectories that correspond to s_____ of the differential equation for various i_____ values.

Fundamentals 2

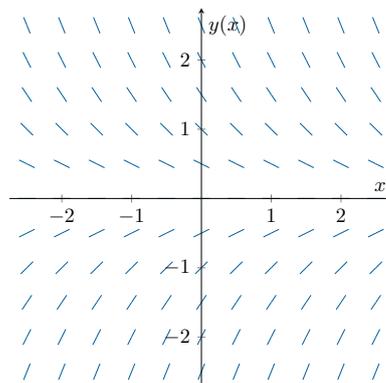
- (a) A direction field contains all the solutions of a differential equation on the one diagram. The particular solution from initial values (x_0, y_0) may be found by tracing the trajectory of the direction field from a given i_____ value (x_0, y_0) . The resultant curve is called the s_____ curve.
- (b) Depending on the location of the initial value and the differential equation, the solution curves may be similar-looking and p_____ to each other, or they may be different curves entirely.

Question 1 For each of the following direction fields, sketch a possible non-trivial solution curve satisfying the given initial condition.

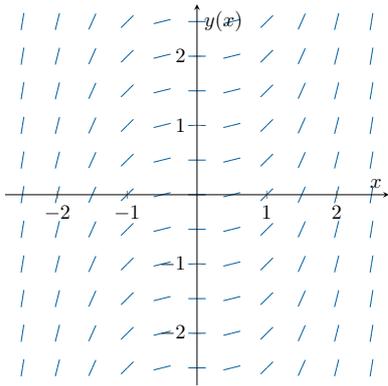
(a) $y(1) = 1$



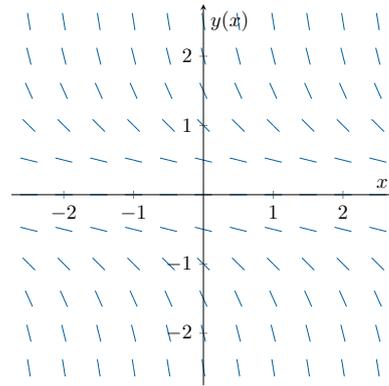
(b) $y(1) = -1$



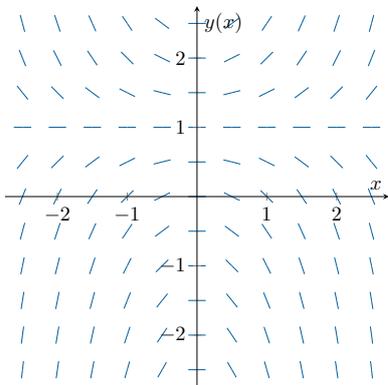
(c) $y(0) = 0$



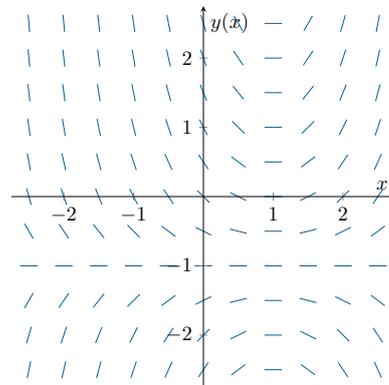
(d) $y(0) = 1$



(e) $y(-1) = 0$



(f) $y(0) = 0$



Question 2 Sketch the direction field for the following differential equations.

(a) $y' = 1$ (b) $y' = 2$ (c) $y' = -\frac{1}{2}$ (d) $y' = 0$

Question 3 Consider the differential equation $y' = y$.

- (a) Find the gradient of the solution curve at $(1, 1)$.
- (b) Plot line segments showing the gradients for when $y = 1$, $y = 2$ and $y = -1$ on the same set of axes.
- (c) Hence, complete the direction field of the differential equation $y' = y$.
- (d) Sketch the solution curve for the initial value $(1, 1)$
- (e) Sketch the solution curve for the initial value $(-1, -1)$

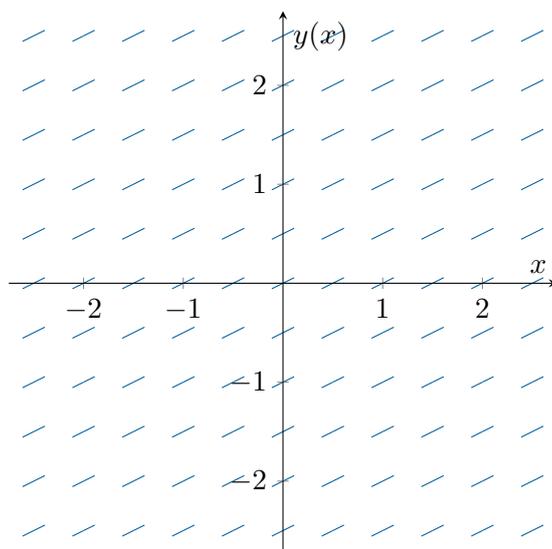
Question 4 Sketch the direction field for the following differential equations.

(a) $y' = x$ (b) $y' = \frac{1}{x}$ (c) $y' = e^x$
 (d) $y' = -x^2$ (e) $y' = \frac{1}{x^2}$ (f) $y' = x^2 - 1$

Question 5 Sketch the direction field for the following differential equations.

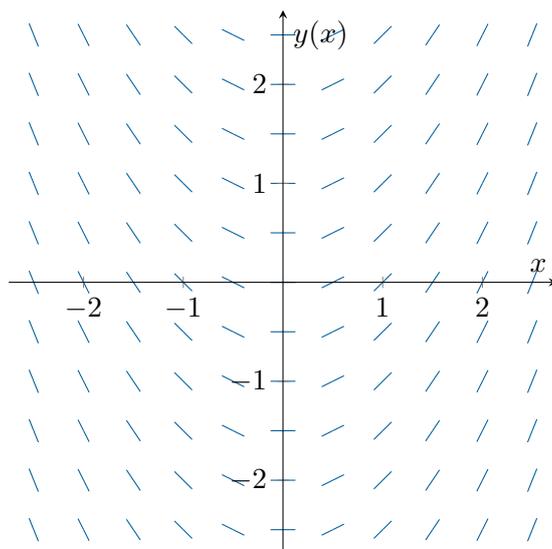
- (a) $y' = -y$ (b) $y' = y^2$ (c) $y' = \frac{1}{y}$
 (d) $y' = \frac{1}{y^2}$ (e) $y' = -e^y$ (f) $y' = \cos\left(\frac{\pi}{3}y\right)$

Question 6 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



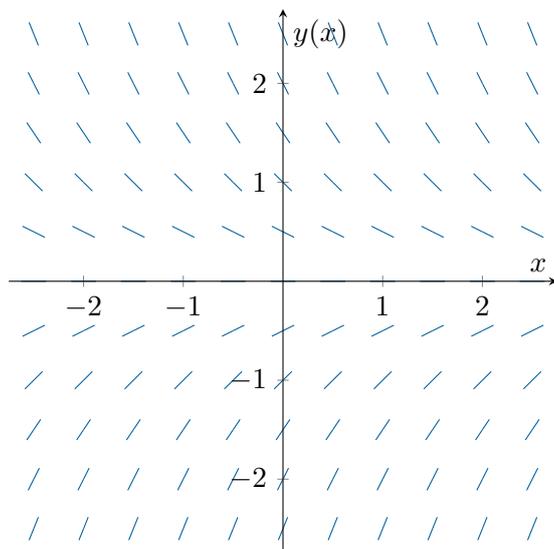
- (a) $y' = 2$ (b) $y' = \frac{1}{2}$ (c) $y' = -2$ (d) $y' = -\frac{1}{2}$

Question 7 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



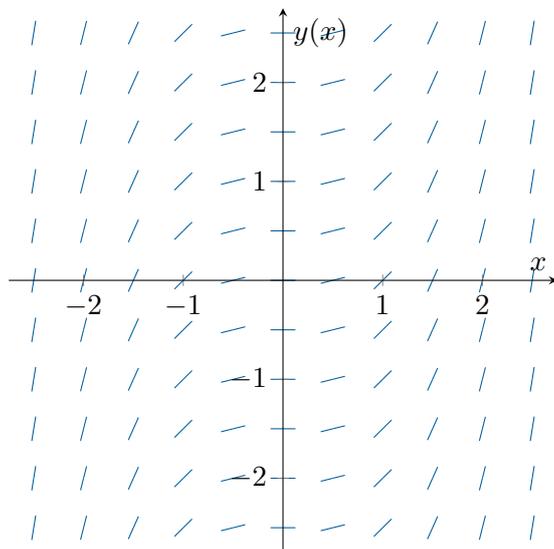
- (a) $y' = x$ (b) $y' = -x$ (c) $y' = y$ (d) $y' = -y$

Question 8 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



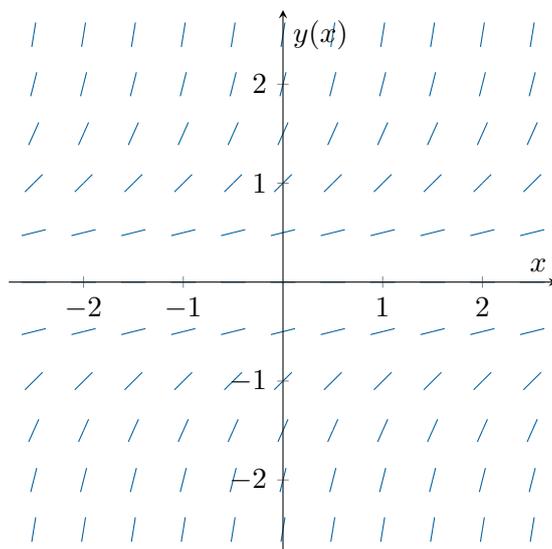
- (a) $y' = x$ (b) $y' = -x$ (c) $y' = y$ (d) $y' = -y$

Question 9 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



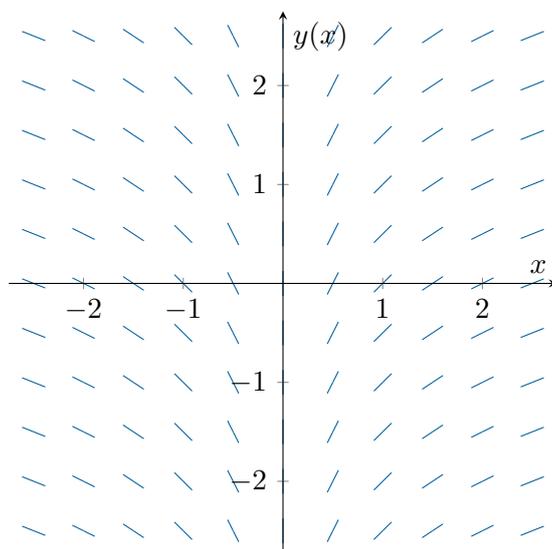
- (a) $y' = x^2$ (b) $y' = -x^2$ (c) $y' = y^2$ (d) $y' = -y^2$

Question 10 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



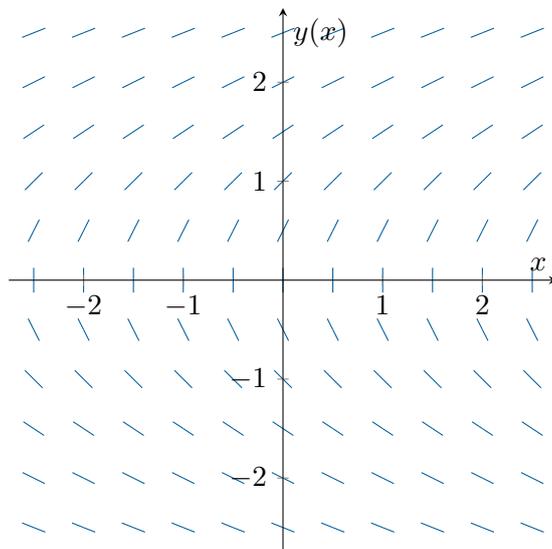
- (a) $y' = x^2$ (b) $y' = -x^2$ (c) $y' = y^2$ (d) $y' = -y^2$

Question 11 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



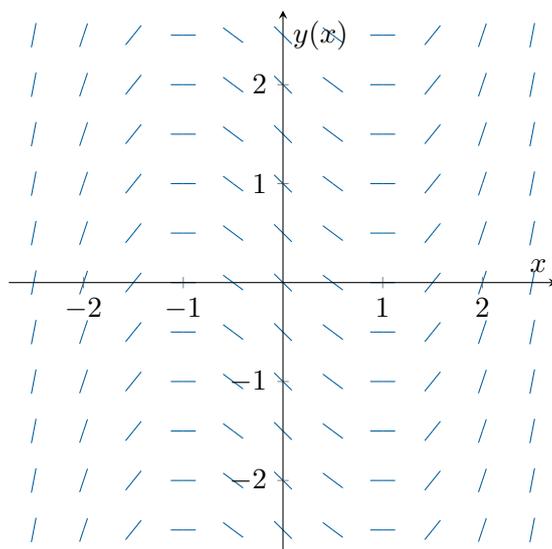
- (a) $y' = \frac{1}{x}$ (b) $y' = -\frac{1}{x}$ (c) $y' = \frac{1}{y}$ (d) $y' = -\frac{1}{y}$

Question 12 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



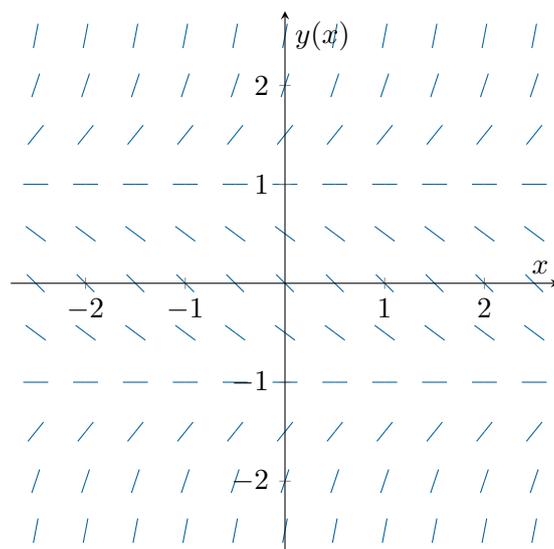
- (a) $y' = \frac{1}{x}$ (b) $y' = -\frac{1}{x}$ (c) $y' = \frac{1}{y}$ (d) $y' = -\frac{1}{y}$

Question 13 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



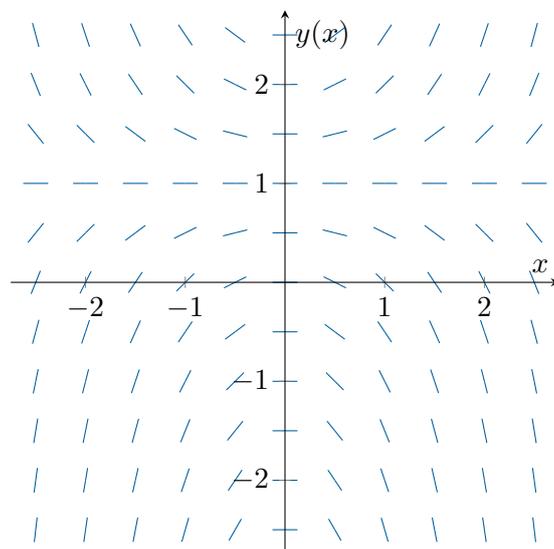
- (a) $y' = x^2 + 1$ (b) $y' = -(x^2 + 1)$ (c) $y' = x^2 - 1$ (d) $y' = 1 - x^2$

Question 14 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



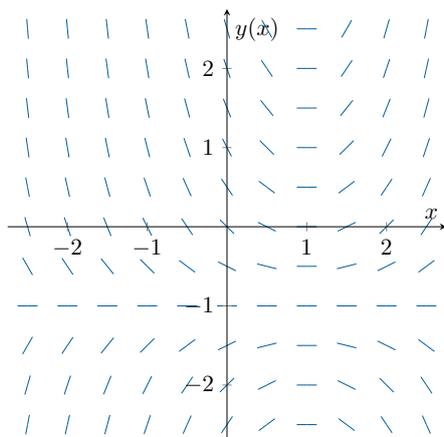
- (a) $y' = y^2 + 1$ (b) $y' = 1 - y^2$ (c) $y' = y^2 - 1$ (d) $y' = -(y^2 + 1)$

Question 15 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



- (a) $y' = x(y - 1)$ (b) $y' = y(x - 1)$ (c) $y' = x(x - 1)$ (d) $y' = y(y - 1)$

Question 16 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



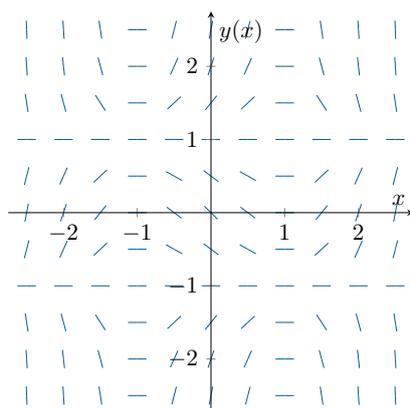
- (a) $y' = (x+1)(y-1)$ (b) $y' = (y+1)(x-1)$ (c) $y' = (x+1)(x-1)$ (d) $y' = (y+1)(y-1)$

Challenge Problems

Problem 1 Sketch the direction field for the following differential equations.

- (a) $y' = 1 + x$ (b) $y' = x + y$ (c) $y' = x - y$
 (d) $y' = xy$ (e) $y' = \frac{x}{y}$ (f) $y' = \frac{y}{x}$

Problem 2 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



- (a) $y' = \frac{x^2 - 1}{y^2 - 1}$ (b) $y' = \frac{y^2 - 1}{x^2 - 1}$
 (c) $y' = (x^2 - 1)(y^2 - 1)$ (d) $y' = -(x^2 - 1)(y^2 - 1)$

Exercise 7C

First order differential equations

Fundamentals

Fundamentals 1

Below are the steps to solve differential equations in the form $\frac{dy}{dx} = f(x)$.

- I_____ to get $y = \int f(x) dx = F(x) + C$.
- S_____ the initial condition to calculate the value of C .
- Write down your final answer $y = \text{_____}$ for the value of C that you found.

Fundamentals 2

Below are the steps to solve differential equations in the form $\frac{dy}{dx} = g(y)$.

- F_____ both sides to get $\frac{dx}{dy} = \frac{1}{g(y)}$
- I_____ to get $x = \int \frac{1}{g(y)} dy = H(y) + C$.
- S_____ the initial condition to calculate the value of C .
- R_____ to make y the subject.

Fundamentals 3

Below are the steps to solve differential equations in the form $\frac{dy}{dx} = f(x)g(y)$.

- Use s_____ of variables to re-arrange it to the form $\frac{dy}{g(y)} = f(x) dx$
- Integrate both sides with respect to their variables to get $G(y) = \text{_____}$. We only need to add the $+C$ to one side.
- S_____ the initial condition to calculate the value of C .
- Re-arrange to make _____ the subject.

Fundamentals 4

When solving differential equations, a g_____ solution is the one obtained when integrating without finding the value of $+C$. A p_____ solution is the one found when i_____ values are substituted to determine a single value of $+C$.

Question 1 Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x+1},$$

with the initial condition $y(0) = 1$.

(a) Integrate both sides and express your answer in the form

$$y = f(x) + C,$$

for some function $f(x)$ and constant C .

(b) Find the value of C .

(c) Hence, write down the equation of the solution.

(d) Find the solution if the initial condition instead was $y(e-1) = 1$.

Question 2 Use a similar technique to **Question 1** to find the particular solutions of the following differential equations with the given initial values.

(a) $\frac{dy}{dx} = 4x + 1$, $y(0) = 3$

(b) $\frac{dy}{dx} = \frac{1}{x^2}$, $y(1) = 1$

(c) $\frac{dy}{dx} = \frac{1}{(2x+1)^2}$, $y(-1) = \frac{7}{2}$

(d) $\frac{dy}{dx} = \frac{1}{\sqrt{1-2x}}$, $y(-4) = 0$

(e) $\frac{dy}{dx} = e^{-2x}$, $y(0) = 1$

(f) $\frac{dy}{dx} = 2xe^{x^2}$, $y(-1) = e$

(g) $\frac{dy}{dx} = \frac{1}{2x+1}$, $y\left(\frac{1}{2}\right) = 0$

(h) $\frac{dy}{dx} = \frac{x}{x^2+1}$, $y(0) = 0$

(i) $\frac{dy}{dx} = \cos(2x)$, $y\left(\frac{3\pi}{4}\right) = -1$

(j) $\frac{dy}{dx} = 2 \tan(2x)$, $y\left(\frac{\pi}{12}\right) = \log 2$

(k) $\frac{dy}{dx} = \frac{1}{4-x^2}$, $y(1) = \frac{\log 3}{2}$

(l) $\frac{dy}{dx} = \frac{1}{9+x^2}$, $y(\sqrt{3}) = \frac{\pi}{6}$

Question 3 Consider the differential equation

$$\frac{dy}{dx} = 1 + y,$$

with the initial condition $y(0) = 1$.

(a) By first flipping both sides and then integrating, show that

$$x = f(y) + C,$$

for some function $f(y)$ and constant C .

(b) Find the value of C .

(c) Show that $x = \ln\left(\frac{1+y}{2}\right)$

(d) Hence, show that $y = 2e^x - 1$.

Question 4 Use a similar technique to **Question 3** to find the particular solutions of the following differential equations with the given initial values.

- | | |
|---|---|
| (a) $\frac{dy}{dx} = 2y$, $y(0) = 3$ | (b) $\frac{dy}{dx} = 2y + 1$, $y(0) = 0$ |
| (c) $\frac{dy}{dx} = \frac{1}{2y}$, $y(0) = 2$ | (d) $\frac{dy}{dx} = \sqrt{y}$, $y(0) = 4$ |
| (e) $\frac{dy}{dx} = e^{2y}$, $y(0) = 0$ | (f) $\frac{dy}{dx} = \frac{2}{e^{3y}}$, $y(0) = 0$ |
| (g) $\frac{dy}{dx} = \sqrt{1 - y^2}$, $y(0) = \frac{1}{2}$ | (h) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 1$ |
| (i) $\frac{dy}{dx} = \tan y$, $y(0) = \frac{\pi}{2}$ | (j) $\frac{dy}{dx} = \sqrt{y + 1}$, $y(0) = 0$ |

Question 5 Consider the differential equation

$$\frac{dy}{dx} = e^x \cos^2 y$$

with the initial condition $y(0) = 0$.

- Show that $\sec^2 y \, dy = e^x \, dx$.
- Integrate both sides and substitute the initial condition to show that $\tan(y) = e^x - 1$
- Hence, show that $y = \tan^{-1}(e^x - 1)$
- State the limiting value of y as $x \rightarrow \infty$.

Question 6 Use a similar technique to **Question 5** to find the particular solutions of the following differential equations with the given initial values.

- | | |
|---|--|
| (a) $\frac{dy}{dx} = xy$, $y(1) = \frac{1}{\sqrt{e}}$ | (b) $\frac{dy}{dx} = x^2y$, $y(0) = 3$ |
| (c) $\frac{dy}{dx} = xy^2$, $y(2) = -\frac{2}{5}$ | (d) $\frac{dy}{dx} = \frac{y}{x}$, $y(1) = \pi$ |
| (e) $\frac{dy}{dx} = \frac{2x + 1}{y^2}$, $y(2) = 3$ | (f) $\frac{dy}{dx} = \frac{y^2 + 1}{x}$, $y\left(e^{\frac{\pi}{3}}\right) = \sqrt{3}$ |
| (g) $\frac{dy}{dx} = 2e^{x-y}$, $y(1) = \ln(1 + 2e)$ | (h) $\frac{dy}{dx} = \frac{e^x}{2y}$, $y(2) = e$ |
| (i) $\frac{dy}{dx} = 2xe^y$, $y\left(\frac{3}{5}\right) = 2 \ln\left(\frac{5}{4}\right)$ | (j) $\frac{dy}{dx} = (2x + 3x^2)y$, $y(-1) = e$ |
| (k) $\frac{dy}{dx} = \frac{2x}{\cos y}$, $y\left(\frac{1}{2}\right) = -\frac{\pi}{6}$ | (l) $\frac{dy}{dx} = 2x \cot y$, $y(0) = \frac{\pi}{3}$ |

Question 7 Solve $y'' = x^2$, where $y(0) = 2$ and $y'(0) = 4$.

Question 8 Solve $y' = 3y$ and show that the solution is in the form

$$y = Ae^{3x},$$

for some real constant A .

Question 9 Solve $y' = \frac{x}{y}$ and show that the solution is in the form

$$y = \pm\sqrt{x^2 + A},$$

for some real constant A .

Question 10 Solve $y' = \frac{x^2}{y^2}$ and show that the solution is in the form

$$y = \sqrt[3]{x^3 + A},$$

for some real constant A .

Question 11 Solve $y' = 2x\sqrt{1 - y^2}$ and show that the solution is in the form

$$y = \sin(x^2 + C),$$

for some real constant C .

Question 12 Solve

$$2xy + 6x + (x^2 - 4)y' = 0$$

and show that the solution is in the form

$$y = \frac{A}{x^2 - 4} - 3$$

for some real constant A .

Question 13 Consider the differential equation $y'' + 2y' = 0$. Use the substitution $u = y'$ and solve two differential equations to obtain $y = Ae^{-2x} + B$ for some constants A and B .

Challenge Problems

Problem 1 [Solving the logistic equation via a substitution]

Consider the differential equation

$$\frac{dP}{dt} = kP(1 - P)$$

(a) Let $u = \frac{1}{P}$. Show that $\frac{du}{dt} = -\frac{1}{P^2} \frac{dP}{dt}$.

(b) Show that the differential equation is equivalent to $u' + ku = k$.

(c) Hence, show that the solution is

$$P = \frac{1}{1 - Ae^{-kt}}$$

Problem 2

- (a) Show that

$$\frac{1}{2} \left(\frac{1}{1+y} + \frac{1}{1-y} \right) = \frac{1}{1-y^2}$$

- (b) Hence, show that the general solution of
- $\frac{dy}{dx} = 1 - y^2$
- is

$$y = \frac{Ae^{2x} - 1}{Ae^{2x} + 1}$$

Problem 3 [Orthogonal Trajectories]

Consider the family of parabolas $y = kx^2$.

- (a) Show that $\frac{dy}{dx} = \frac{2y}{x}$.
- (b) Explain why the solution curves of $\frac{dy}{dx} = -\frac{x}{2y}$ will always be perpendicular to the family of curves $y = kx^2$ for any real value of k .
- (c) Hence, find the family of curves that will always be perpendicular to $y = kx^2$ at any point of intersection, and state what kind of curve it is.

- (g) The rate of acceleration of a particle falling towards Earth is proportional to the reciprocal of the square of the distance x of the particle from the Earth's centre.
- (h) The acceleration of a particle is proportional to the distance of the particle from a fixed point x_0 .

Question 2 [Exponential growth and decay]

The differential equation for basic growth and decay is

$$\frac{dP}{dt} = kP,$$

where k is a constant. Solve this differential equation and show that the general solution is

$$P = Ae^{kt},$$

where A is a constant.

Question 3 [Modified growth and decay]

The differential equation for basic growth and decay is

$$\frac{dP}{dt} = -k(P - B),$$

where k and B are constants. Solve this differential equation and show that the general solution is

$$P = B + Ae^{-kt},$$

where A is a constant.

Question 4 A species with population P after t years from a certain date increases proportionally to the size of the population at that time. The species has initial population 2000.

- (a) Write down a differential equation modelling the given scenario.
- (b) Solve the differential equation to show that the population is in the form $P = 2000e^{kt}$, where k is a constant.
- (c) After five years, the population doubles. Show that $k = \frac{1}{5} \ln 2$.
- (d) Find the population after one year.
- (e) Find how long it takes for the population to reach 8000.

Question 5 [Half-life of a radioisotope]

The rate of decay of the mass M of a radioisotope is proportional to the mass at that time t . Let the initial mass of the radioisotope be M_0 .

- (a) By constructing and solving an appropriate differential equation, show that the particular solution is $M = M_0e^{-kt}$.
- (b) Show that it takes $T = \frac{1}{k} \ln(2)$ years for the mass to be exactly half of what it was originally.
- (c) A mass that is originally 1000 kilograms takes 200 years to decay to 500 kilograms. From this point in time, how long will this same mass take to decay to 250 kilograms?

Question 6 A cup full of coffee initially has temperature 70°C , and it sits in a room of temperature 20°C . The temperature of the coffee cup can be modelled by the differential equation

$$\frac{dT}{dt} = -k(T - B),$$

where B is the temperature of the room and T is the temperature of the coffee cup after t minutes.

(a) Solve the differential equation and show that the solution is

$$T = 20 + 50e^{-kt},$$

where k is a constant.

(b) What is the limiting temperature of the coffee?

Question 7 A frozen pie with initial temperature $T = 0^\circ\text{C}$ is left in an oven set to 180°C . After ten minutes, the pie has temperature $T = 30^\circ\text{C}$. The rate that the temperature of the pie increases is proportional to the difference in temperature between the oven and the pie.

(a) Construct a differential equation modelling the scenario, and solve it to show that the particular solution is

$$T = 180 - 180e^{-kt},$$

where $k = \frac{1}{10} \ln\left(\frac{6}{5}\right)$.

(b) The pie is ready to serve when the interior temperature is 60°C . After how many minutes, to the nearest minute, will the pie be ready?

Question 8 An invasive rabbit population of 2000 is introduced into a region. The rate of the rabbit population is given by

$$\frac{dP}{dt} = \frac{P}{20} \left(1 - \frac{P}{50000}\right),$$

where t is time in months after the introduction of the rabbits.

(a) Show that $\frac{50000}{P(50000 - P)} = \frac{1}{P} + \frac{1}{50000 - P}$.

(b) Hence, solve the differential equation to show that

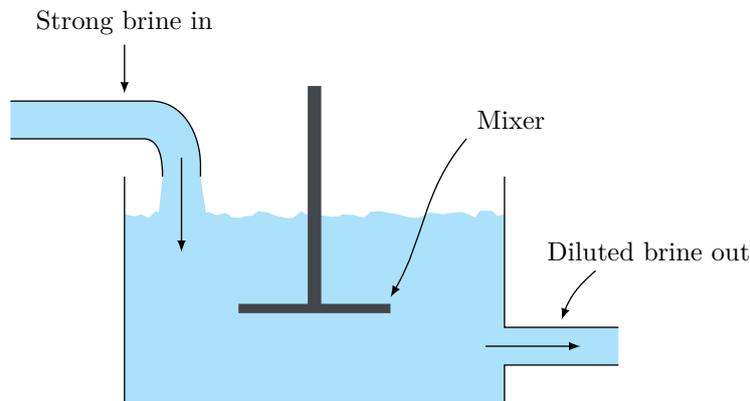
$$P = \frac{50000}{1 + 24e^{-\frac{t}{20}}}$$

(c) Hence, write down the limiting rabbit population.

(d) How many months did it take to exceed quadruple its original population?

Question 9 [Brine problem]

The diagram below shows a simple model of a brine mixer.



Initially, the tank contains 500 litres of pure water. A strong brine with concentration 5 grams per litre is pumped into the tank continuously at a rate of R litres per minute, and then mixed instantaneously. At the same time, the mixture is being pumped out at a rate of R litres per minute so that the tank always has 500 litres of liquid at any given time. Let the amount of salt in the tank after t minutes be S .

- (a) How many grams of salt are entering the system per minute?
 (b) How many grams of salt are leaving the system after t minutes, in terms of S and R ?
 (c) Show that

$$\frac{dS}{dt} = -\frac{R}{500}(S - 2500)$$

- (d) Solve the differential equation and hence show that

$$S = 2500 + Ae^{-\frac{Rt}{500}}$$

- (e) How much salt will be in the tank, if the system is left to run for an extended period of time?
 (f) Find the value of A .
 (g) After 1 hour, the salt concentration in the tank is 4 grams per litre. What was the value of R ?

Question 10 [Pharmaceutical drugs]

A drug is administered into the bloodstream at a constant rate r via an intravenous (IV) bag. As the concentration of the drug in the bloodstream increases, the body metabolises the drug at a rate proportional to the concentration $C(t)$ in the blood stream, where t is time in hours. A patient is administered the drug after having some initial concentration C_0 in their bloodstream.

- (a) Explain why

$$\frac{dC}{dt} = r - kC,$$

where k is a positive constant.

- (b) Solve the differential equation and show that

$$C = \frac{r}{k} - \left(\frac{r - kC_0}{k} \right) e^{-kt}$$

- (c) State the limiting concentration in the patient's bloodstream as t gets large.
 (d) Explain what happens to the concentration in Jacks' blood when $r < kC_0$ and when $r > kC_0$.

Question 11 [Current in an electric circuit]

The current I in an electric circuit is modelled by the differential equation

$$RI + L \frac{dI}{dt} = V,$$

where L , R and V are constants.

- (a) Show that the general solution is given by

$$I = \frac{V}{R} \left(1 - Be^{-\frac{R}{L}t} \right),$$

where B is some constant.

- (b) State the limiting value of the current as t becomes large.

Question 12 [Logistic equation]

The differential equation for a population with finite resources is

$$\frac{dP}{dt} = kP(1 - P),$$

where k is a constant.

- (a) Simplify $\frac{1}{P} + \frac{1}{1 - P}$.

- (b) Hence, solve the differential equation and show that the general solution is

$$P = \frac{A}{A + e^{-kt}},$$

where A is a constant.

- (c) State the limiting value of P as $t \rightarrow \infty$.

Challenge Problems

Problem 1 [Resisted motion]

When a particle is released from rest, the acceleration is given by

$$a = g - kv$$

where g is acceleration due to gravity, v is the velocity in metres per second, and k is a positive constant.

- (a) Solve the differential equation to show that

$$v = \frac{g}{k}(1 - e^{-kt})$$

- (b) State the limiting value of v as $t \rightarrow \infty$.
 (c) Describe the physical significance of this.
 (d) Bob observes that this limiting velocity could also have been found by letting $a = 0$ and solving for v . Explain how this is also a valid method of finding the limiting velocity.

Problem 2 [Gompertz equation]

The *Gompertz equation* can be used to model tumour growth. It is described by the differential equation

$$\frac{dV}{dt} = a(\ln b - \ln V)V$$

where a and b are constants, and V is the volume of the tumour in mm^3 .

- (a) Show that the general solution is in the form

$$V = Ae^{Be^{-at}}$$

for some constants A and B .

Hint: Use a formula for the form $\int \frac{f'(V)}{f(V)} dV$

- (b) Find the particular solution in terms of V_0 , a and B if $V(0) = V_0$.
 (c) State the limiting volume of the tumour, in terms of V_0 and B .

Problem 3 [Escape velocity]

A particle is fired from the Earth's surface with initial velocity u , and the acceleration of the particle is given by

$$a = -\frac{k}{x^2},$$

where x is the distance of the particle from the centre of the Earth, and k is a constant. Let R be the radius of the Earth.

- (a) Prove the general result that $v \frac{dv}{dx} = \frac{dv}{dt}$.

Hint: Turn v from the left hand side into something that involves t .

- (b) If acceleration due to gravity has a magnitude of g on the Earth's surface, show that

$$a = -\frac{gR^2}{x^2}$$

- (c) Use the result in part (a) to show that

$$v^2 = u^2 - 2gR^2 \left(\frac{1}{R} - \frac{1}{x} \right)$$

- (d) Show that if $u^2 = 2gR$, then $v \neq 0$.
 (e) Explain the significance of this result.

Chapter 7 Review

Differential Equations

Review

Question 1 Show that the following functions satisfy the given differential equations.

- (a) $y = \sin(2x)$, $y'' = -4y$ (b) $y = x^2e^x$, $xy' = (x+2)y$
 (c) $y = -\frac{3}{x}$, $x^2y'' + 3xy' + y = 0$ (d) $y = x - \frac{1}{x}$, $xy' + y - 2x = 0$

Question 2 Show that $y = \frac{A + \ln x}{x}$ satisfies the differential equation

$$x^2y' + xy = 1,$$

for any real values of A .

Question 3 Show that $y = Ae^{2x} + Be^{-3x}$ satisfies the differential equation

$$y'' + y' - 6y = 0,$$

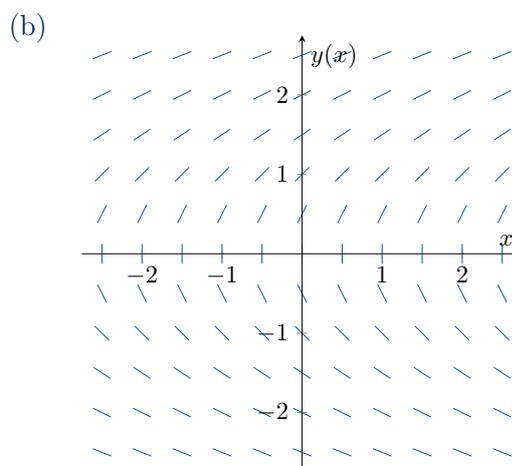
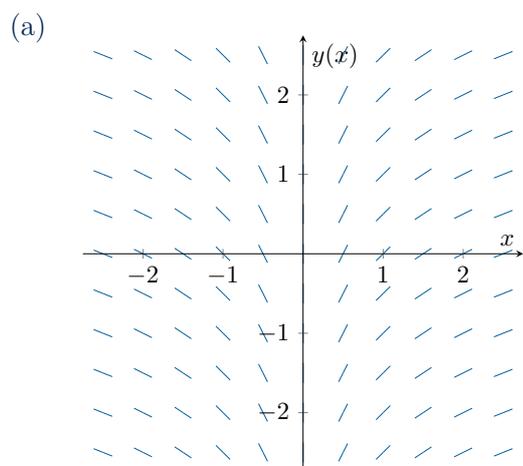
for any real values of A and B .

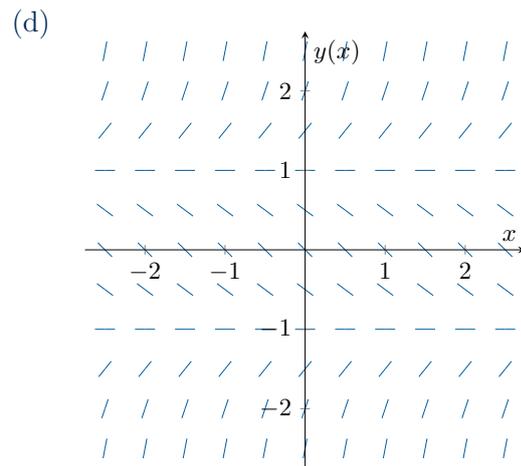
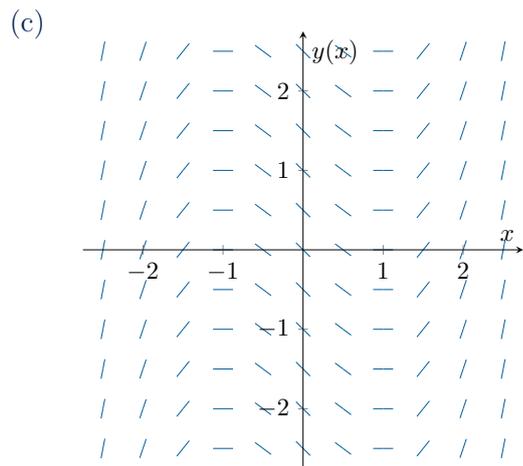
Question 4 Show that $y = (A + Bx)e^{-x}$ satisfies the differential equation

$$y'' + 2y' + y = 0,$$

for any real values of A and B .

Question 5 For each of the following direction fields, sketch a possible solution curve.





Question 6 Sketch the direction field for the following differential equations.

(a) $y' = -x$

(b) $y' = -\frac{1}{x}$

(c) $y' = x^2$

(d) $y' = -y$

(e) $y' = -\frac{1}{y}$

(f) $y' = e^y$

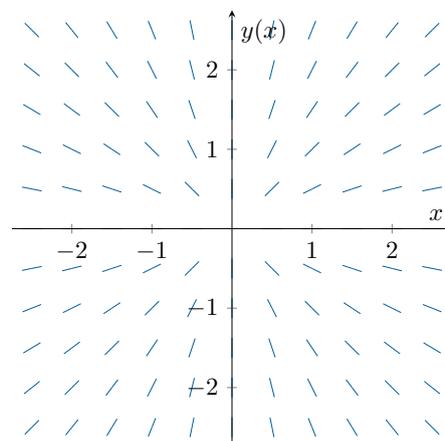
Question 7 Sketch the direction field for the following differential equations.

(a) $y' = y - x$

(b) $y' = -\frac{y}{x}$

(c) $y' = xy$

Question 8 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



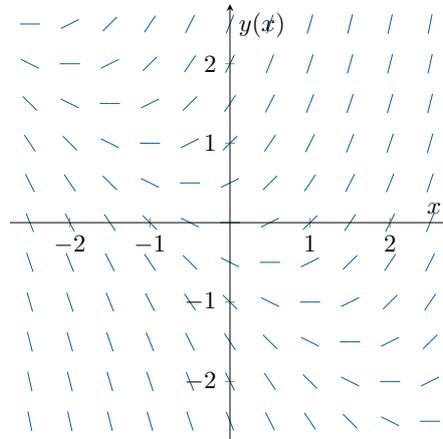
(a) $y' = \frac{x}{y}$

(b) $y' = -\frac{x}{y}$

(c) $y' = \frac{y}{x}$

(d) $y' = -\frac{y}{x}$

Question 9 The diagram below shows a direction field of a differential equation. Which of the following is a possible differential equation?



- (a) $y' = x + y$ (b) $y' = x - y$ (c) $y' = y - x$ (d) $y' = -x - y$

Question 10 Find the particular solutions of the following differential equations with the given initial values.

- (a) $\frac{dy}{dx} = 1 - 2x$, $y(1) = 3$ (b) $\frac{dy}{dx} = -\frac{2}{(x+3)^2}$, $y(3) = \frac{7}{3}$
 (c) $\frac{dy}{dx} = -\frac{3}{x}$, $y(-2) = -3 \ln 2$ (d) $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$, $y(1) = 0$

Question 11 Find the particular solutions of the following differential equations with the given initial values.

- (a) $\frac{dy}{dx} = e^{-y}$, $y(0) = 0$ (b) $\frac{dy}{dx} = \cot y$, $y(0) = 0$
 (c) $\frac{dy}{dx} = 3 - 2y$, $y(0) = 0$ (d) $\frac{dy}{dx} = \frac{1}{\sqrt{y}}$, $y(0) = 1$

Question 12 Find the particular solutions of the following differential equations with the given initial values.

- (a) $\frac{dy}{dx} = 4xy$, $y(0) = 1$ (b) $\frac{dy}{dx} = \frac{2x}{y}$, $y(1) = \sqrt{3}$
 (c) $\frac{dy}{dx} = \frac{2x}{e^y}$, $y(-2) = \ln 3$ (d) $\frac{dy}{dx} = \frac{x}{ye^{x^2}}$, $y(0) = 2$
 (e) $\frac{dy}{dx} = \cos^2(y)$, $y(-1) = 0$ (f) $\frac{dy}{dx} = 4x^3y$, $y(1) = e$
 (g) $\frac{dy}{dx} = 12x^3y$, $y(0) = 2$ (h) $\frac{dy}{dx} = -\frac{y}{x}$, $y(1) = \sqrt{2}$
 (i) $\frac{dy}{dx} = -\frac{\sin x}{y}$, $y(0) = 1$ (j) $\frac{dy}{dx} = xy + y$, $y(0) = 1$

Question 13 Solve $y' = 1 - 2y$ and show that the solution is in the form

$$y = A + Be^{-2x},$$

for some real constants A and B .

Question 14 Solve $y' = \frac{x^2}{y}$ and show that the solution is in the form

$$y = A\sqrt{x^3 + B},$$

for some real constants A and B .

Question 15 Solve $y' = -2x(1 + y)$ and show that the solution is in the form

$$y = Ae^{-x^2} - 1,$$

for some real constant A .

Question 16 Solve $y'' = -18 \cos(3x)$, where $y(0) = 2$ and $y'(0) = 0$.

Question 17 The mass of a kitten when it is born is 0.2 kilograms. After that, the rate at which it grows is given by the equation

$$\frac{dM}{dt} = \frac{20}{\pi(1 + 4t^2)},$$

where t is time in years.

(a) Show that the mass of the kitten is modelled by the equation

$$M = 0.2 + \frac{10}{\pi} \tan^{-1}(2t)$$

(b) Hence, find how much the kitten will weight eventually as it grows.

(c) What percentage of the limiting weight does the kitten grow within it's first year, correct to the nearest percent?

Question 18 A large tank has a leak and the rate of change of the volume in the tank is given by

$$\frac{dV}{dt} = -kV^2$$

where t is time in minutes. Initially, the tank has 500 litres of water in it.

- Solve the differential equation and show that $V = \frac{500}{1 + 500kt}$
- One minute later, a tenth of the water had leaked out. Show that $k = \frac{1}{4500}$.
- Hence, find how long it takes for three quarters of the total water to leak out?

Question 19 A population of bacteria grows at a rate proportional to the population at that time. Initially, the population was 100 and it doubles every 4 hours.

- Write down a differential equation that models the scenario above.
- Solve the differential equation to show that $P = 100e^{kt}$, where $k = \frac{1}{4} \ln 2$.
- Hence, show that it takes exactly $t = 16 \log_2(10)$ hours to reach a population of a million.

Question 20 At a small fishing farm, the population of fish grows at a rate that is proportional to the population at that time, where the constant of proportionality is $\frac{1}{2}$. Also, fish are farmed continuously throughout the year at a rate of F fish per year on average. Initially, the farm has 5000 fish.

- Write down the differential equation modelling the above scenario.
- Solve the differential equation to show that

$$P = 2F + (5000 - 2F)e^{\frac{t}{2}}$$

- Hence, find the maximal number of fish that can be farmed per year without causing the population to go to zero eventually.

Question 21 The rate of decay of the mass M of a radioisotope is proportional to the mass at that time t . Initially, there are 1000 kilograms of the radioisotope.

- By constructing and solving an appropriate differential equation, show that the particular solution is $M = 1000e^{-kt}$.
- It takes 500 years for a quarter of the mass to decay. Approximately how long will it take for half of the mass to decay?

Question 22 A cup of boiling water has initial temperature 100°C , and it sits in a room of temperature 30°C . The temperature of the coffee cup can be modelled by the differential equation

$$\frac{dT}{dt} = k(B - T),$$

where B is the temperature of the room and T is the temperature of the water after t minutes.

- (a) What is the limiting temperature of the coffee?
 (b) Solve the differential equation and show that the solution is

$$T = 30 + 70e^{-kt},$$

where k is a constant.

- (c) It took 20 minutes for the water to cool by 40° . How long approximately will it take for the water to cool down to half of its original temperature?

Question 23 In a particular region, there are initially 1000 of an endangered bird species, and a population recovery project is implemented to raise the population of the birds. The rate of increase of the bird population is given by

$$\frac{dP}{dt} = \frac{P}{5} \left(1 - \frac{P}{10000} \right),$$

where t is time in months after the implementation of the program.

- (a) Show that $\frac{10000}{P(10000 - P)} = \frac{1}{P} + \frac{1}{10000 - P}$.
 (b) Hence, solve the differential equation to show that

$$P = \frac{10000}{1 + 9e^{-\frac{t}{5}}}$$

- (c) Hence, write down the limiting bird population.
 (d) How many months did it take for the bird population to reach the half-way point between the initial population and the maximum population?

 Investigation Task

Integrating Factor

The techniques of solving differential equations in this course are limited to nice separable differential equations. However, the vast majority of differential equations out there are not nice and separable. In any study of differential equations beyond high school, a student will encounter the idea of an *integrating factor*, which is a more advanced technique of solving differential equations. This investigation task aims to develop this theory and to show the student a new technique of solving differential equations.

Question 1

- (a) Write down the form of the first-order differential equation where an integrating factor may be used.
- (b) Write down the formula for the integrating factor.
- (c) Apply the integrating factor to the general form in (a) and use it to obtain a general solution.
- (d) Explain the motivation behind the idea of the integrating factor, and how someone ‘discovering’ it for the first time may have had the idea to do this in the first place.

Question 2 Produce four first-order differential equations suitable for solving via the integrating factor, and obtain their general solutions.

Question 3 Write a one page summary that is sufficiently comprehensive such that a Year 12 Extension 1 student who has completed the Differential Equations topic can identify and solve their own problems requiring the use of the integrating factor. Include all necessary formulae, derivations and worked examples.

 Investigation Task

SIR Model

In the study of epidemiology, one of the basic models for the spread of disease is the SIR model. It is a fascinating application of first-order differential equations that forms the groundwork of mathematical modelling in epidemiology.

Question 1 Research what the SIR model is and write a three-page article that answers the following questions.

- (a) What does SIR stand for?
- (b) What are the relevant differential equations?
- (c) What are the assumptions in-built into the model?
- (d) What are the important graphs relevant to the model, and how do they behave depending on each other?

Question 2 This section is about *herd immunity* and the mathematics involved behind it.

- (a) Explain in a short paragraph what herd immunity is, and why it is so important for society. Your answer should mention specifically what groups of people rely on herd immunity the most.
- (b) Explain what a *contact number* is and how it is relevant to the idea of herd immunity.
- (c) Suppose the contact number for a disease is 15 and that vaccination has a 100% success rate. What proportion of the population must be vaccinated to prevent an epidemic forming?
- (d) Answer the above question, but if the vaccine is only 90% effective instead.

Question 3 Construct a presentation explaining to an audience of Year 12 Mathematics Extension 1 students everything there is to know about the SIR model that is accessible using tools within the course, starting from the base definitions all the way up to and including contact numbers, herd immunity and minimal vaccination rates to prevent epidemics.

 Investigation Task

Euler's Method

In this course, we have two ways of constructing trajectories of differential equations, given an initial value. The first is to explicitly solve the differential equation with the initial value, and to then directly obtain the equation of the trajectory. The second is to obtain (somehow) a direction field, plot the initial value, and follow the direction field to form a trajectory. The issue with the first method is that we may not necessarily know how to solve the differential equation in the first place. The issue with the second method is that you may not have any computer assisted software to produce a direction field for you.

This investigation task is to show the student a way to approximate trajectories using a technique called *Euler's Method*, which is the 'trapezoidal rule' of differential equations, so-to-speak.

The task is to write a document that explains Euler's Method with sufficient detail so that a Year 12 Mathematics Extension 1 student reading it will be able to create trajectories of their own. Your answer should include

- All necessary formulae.
- The derivation and intuition behind the derivations.
- At least three worked examples.
- Plenty of graphs demonstrating the diagrams with each iteration.

 Investigation Task

Second order differential equations

In this chapter, the student studies a fairly limited set of differential equations. The study of differential equations is highly extensive, and there are many different types out there beyond those within the scope of the syllabus only, yet still are accessible to a Year 12 Extension 1 student.

Question 1 Give a detailed explanation of what a differential equation is, and give various examples of them and the different types. Your answer should mention:

- order
- linearity
- homogeneity
- particular solutions
- general solutions

Question 2 Give three examples of second-order differential equations, and verify by differentiation that the solution indeed satisfies the differential equations. Aim to demonstrate solution equations involving different types of functions e.g. trigonometric functions, logarithmic functions.

Question 3 Consider the differential equation

$$ay'' + by' + cy = 0$$

- Research what a *characteristic polynomial* and how it is relevant to the above differential equation.
- Explain what the solutions become if the solutions of the characteristic polynomial are real distinct numbers α and β .
- Repeat the above question, but for the case where the solution is a double root $x = \alpha$.

1. Mathematical Induction

Exercise 1A

Induction involving sums

F1

first, 1

assume, $k + 1$

Prove, assumption

$P(k + 1)$, $P(2)$, $P(3)$

Q1

(a) 1, 5 (b) 1, -3 (c) 1, 17

(d) 1, 5 (e) $\frac{1}{4}$, $\frac{2}{7}$

Q2

(a) $2k + 2$ (b) $2k + 1$

(c) $2k + 3$ (d) $(k + 1)2^k$

(e) $\frac{1}{(3k + 1)(3k + 4)}$ (f) $\frac{1}{(2k + 1)(2k + 3)}$

Q3

See full worked solutions.

Q4

(a) LHS = $1^2 = 1$

$$\text{RHS} = \frac{1}{6}(1)(2)(3) = 1$$

(b) $(k + 1)^2$, $\frac{1}{6}(k + 1)(k + 2)(2k + 3)$

(c) $1^2 + 2^2 + 3^2 + \dots + k^2 + (k + 1)^2$,
 $\frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2$

(d) $n = 60$, 73810

(e) $S_{30} - S_9 = 9170$

Q5

See full worked solutions.

Q6

See full worked solutions.

Q7

See full worked solutions.

Q8

(a)

(i) $\frac{2}{3}$ (ii) $\frac{3}{4}$ (iii) $\frac{4}{5}$

(b) $\frac{n}{n + 1}$

(c) See full worked solutions.

(d)

(i) $\frac{4}{50}$ (ii) $\frac{1}{25}$

Q9

(a) $(k + 1)!(1 + k + 1) = (k + 1)!(k + 2) = (k + 2)!$

(b) See full worked solutions.

Q10

See full worked solutions.

Q11

See full worked solutions.

Q12

See full worked solutions.

Q13

(a) See full worked solutions.

(b) $\frac{3}{4}$

Q14

(a) $\sin(A + B) - \sin(A - B)$

(b) LHS = $\cos x$

$$\text{RHS} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x$$

(c) $\cos(2k + 1)x$, $\frac{\sin(2k + 2)x}{2 \sin x}$

(d) $(2k + 1 + 1)$, $(2k + 1 - 1)$

Q15

See full worked solutions.

P1

See full worked solutions.

P2

See full worked solutions.

P3

- (a) See full worked solutions.
 (b) See full worked solutions.
 (c) Equate the previous two parts.
 (d) Use the sum of AP
 $1 + 2 + 3 + \dots + n = \frac{n}{2}(n + 1)$ and square both sides.

P4

- (a) $(n + 1)(4n^2 + 14n + 9)$
 (b) See full worked solutions.

P5

See full worked solutions.

Exercise 1B**Induction involving divisibility****F1**

$P(1), P(k), P(k + 1)$

F2

pM

F3

- (a) $1, k + 2$ (b) $2, k + 2$

Q1

- (a) $k^3 + 2k = 3N$ (b) $4^k + 2 = 6N$
 (c) $k^2 + k = 2N$ (d) $9^k + 7 = 8N$
 (e) $5^k + 3 = 4N$ (f) $3^{2n} - 1 = 8N$

Q2

- (a) 4
 (b) $4N$, where N is an integer
 (c) $4M + 1, 4M + 1, 20M + 4, (5M + 1)$

Q3

See full worked solutions.

Q4

- (a) 5
 (b) $5M, 5N$ where N is a positive integer
 (c) $8, 3, 8, 3, 5M + 3^k, 40M + 5(3^k), 8M + 3^k$
 (d) See full worked solutions.

Q5

See full worked solutions.

Q6

- (a) 8
 (b) $8M, 8N$. We use $n = k + 2$ because the statement is only true for odd integers i.e. $n = 1, 3, 5, \dots$. Since our base case is $n = 1$, we need $n = k + 2$ to reach all the odd integers.
 (c) $9, 8M - 5^k, 25, 72M + 16(5^k), 9M + 2(5^k),$

Q7

See full worked solutions.

Q8

See full worked solutions.

Q9

(a)	n	1	2	3	4	5	6	7
	$2^n - (-1)^n$	3	3	9	15	33	63	129

- (b) $2^n - (-1)^n$ is always a multiple of 3
 (c) See full worked solutions.

P1

See full worked solutions.

P2

See full worked solutions.

P3

See full worked solutions.

P4

See full worked solutions.

Chapter Review**R1**

See full worked solutions.

R2

- (a) See full worked solutions.
 (b) $n = 505, 510555$
 (c) $S_{125} - S_{24} = 30199$

R3

See full worked solutions.

R4

See full worked solutions.

R5

- (a) See full worked solutions.
 (b) $\frac{1}{3}$

R6

- (a) See full worked solutions.
 (b) $\frac{1}{4}$

R7

- (a) See full worked solutions.
 (b) $\frac{1}{2}$

R8

See full worked solutions.

R9

See full worked solutions.

R10

See full worked solutions.

2. Vectors

Exercise 2A

Introduction to vectors

F1

- (a) magnitude, direction (b) equal
 (c) position (d) direction
 (e) direction, parallel

F2

- (a) horizontal, vertical (b) horizontal, vertical
 (c) $\begin{bmatrix} x \\ y \end{bmatrix}$ (d) $\langle x, y \rangle$
 (e) a

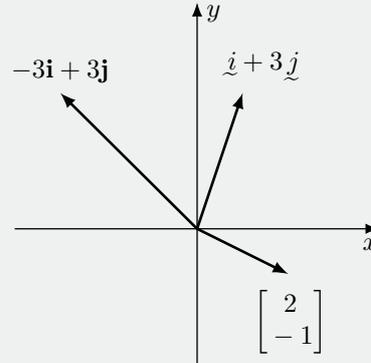
F3

$$\begin{aligned} \vec{AB} &= \langle x_2 - x_1, y_2 - y_1 \rangle \\ &= (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} \\ &= \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \end{aligned}$$

Q1

- (a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$
 (c) $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$

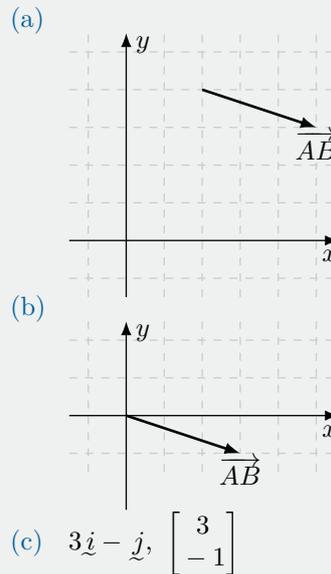
Q2



Q3

- (a) $2\underline{i} + 3\underline{j}$, $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (b) $4\underline{i} - 3\underline{j}$, $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$
 (c) $-2\underline{i} - 3\underline{j}$, $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$ (d) $-4\underline{i} + 3\underline{j}$, $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Q4



Q5

- (a) \vec{IJ} (b) \vec{KL} (c) \vec{GH}

Q6

- (a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

Q7

- (a) $2\hat{i} + 3\hat{j}$ (b) $4\hat{i} - 7\hat{j}$
 (c) $-4\hat{i} + 7\hat{j}$ (d) $2\hat{i} - 3\hat{j}$

Q8

- (a) $\begin{bmatrix} 8 \\ -6 \end{bmatrix}$ (b) $\begin{bmatrix} -5 \\ -2 \end{bmatrix}$
 (c) $\begin{bmatrix} 0 \\ 13 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Q9

- (a) $a = 4, b = -3$
 (b) $a = 5, b = -2$

P1

When we multiply both components by the *same* constant k , the components remain in the same proportion. This is equivalent to just changing the length of the vector, but the ratio of the components i.e. the angle is still the same.

P2

- (a) The magnitude of the vector is just the distance of the point (x, y) from the origin. Using distance formula, this is $\sqrt{x^2 + y^2}$.
 (b) From **Problem 1**, the vector parallel to \mathbf{a} is just $k\mathbf{a}$ for some constant k . The only difference is that it may be longer or shorter now. To obtain a vector of unit length, we need to 'de-scale' \mathbf{a} by a factor of $\frac{1}{\sqrt{x^2 + y^2}}$, since it currently has a length of $\sqrt{x^2 + y^2}$. Hence, $\frac{1}{\sqrt{x^2 + y^2}}\mathbf{a}$ is the same as \mathbf{a} , but just de-scaled to the point where it has length exactly one.

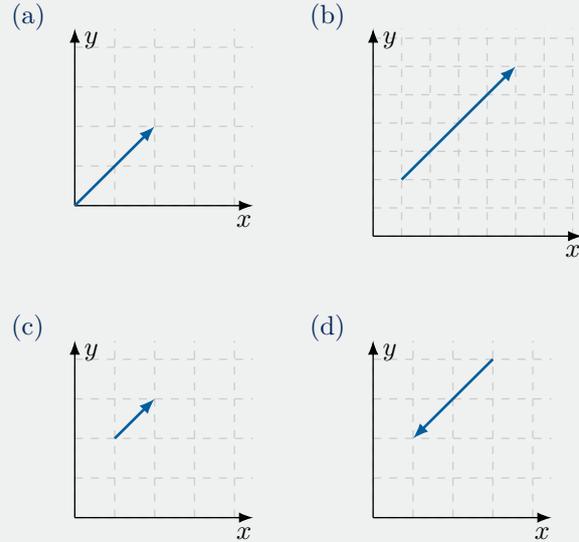
Exercise 2B

Addition, subtraction and scalar multiples of vectors

F1

- (a) scalar (b) longer
 (c) shorter (d) flipped
 (e) parallel (f) v , parallel
 (g) parallel, $\mathbf{u} = k\mathbf{v}$

F2



F3

- (a) \vec{AC} (b) $\vec{A_1A_n}$ (c) polygon

Q1

- (a) $-\hat{i} - 2\hat{j}$ (b) $-4\hat{i} + 6\hat{j}$
 (c) $-\hat{i} + 5\hat{j}$ (d) $3\hat{i} - \hat{j}$
 (e) $-4\hat{i} + 13\hat{j}$ (f) $7\hat{i}$

Q2

- (a) $\begin{bmatrix} -4 \\ -4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$
 (d) $\begin{bmatrix} -6 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} -10 \\ -5 \end{bmatrix}$ (f) $\begin{bmatrix} -16 \\ -6 \end{bmatrix}$

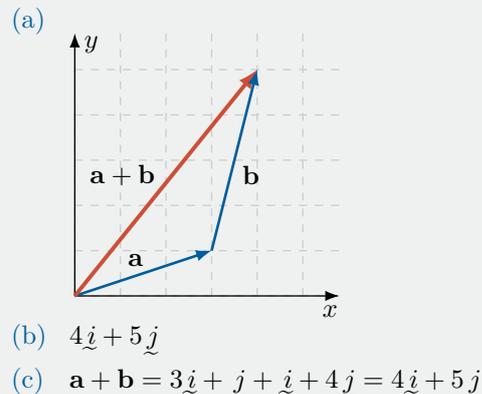
Q3

- (a) $\langle 19, 42 \rangle$ (b) $\langle 24, 16 \rangle$ (c) $\langle -4, -8 \rangle$

Q4

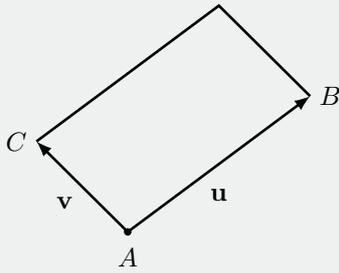
- (a) $k = 1$ (b) $k = -1$ (c) $k = -\frac{3}{2}$

Q5

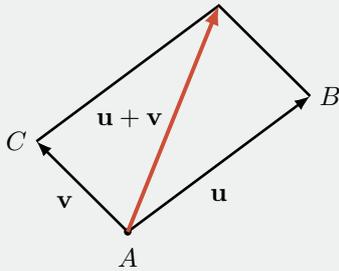


Q6

(a)



(b)



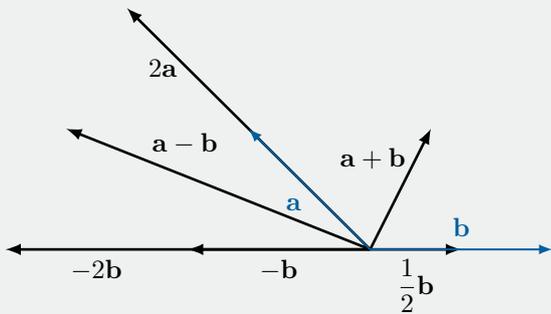
(c) parallelogram, opposite

Q7

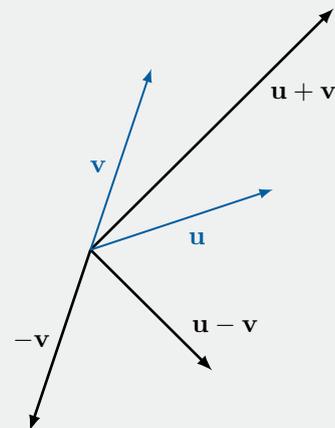
(a) $\vec{BC} = \mathbf{v} - \mathbf{u}$ (b) $\vec{CB} = \mathbf{u} - \mathbf{v}$

(c) \mathbf{u}, \mathbf{v}

Q8



Q9



Q10

0

Q11

(a) $k = 4$ (b) $k = -3$ (c) $k = \frac{1}{3}$

Q12

(a) $\frac{1}{2}\mathbf{u}$ (b) $-\frac{1}{2}\mathbf{u}$ (c) \mathbf{u}
 (d) $-\mathbf{u}$ (e) $\frac{3}{2}\mathbf{u}$ (f) $-\frac{3}{2}\mathbf{u}$

Q13

(a) $\vec{AB} = 2\hat{i} + 4\hat{j}$, $\vec{DC} = 2\hat{i} + 4\hat{j}$
 (b) They are equal vectors and hence parallel.
 (c) $\vec{BC} = 3\hat{i} - \hat{j} = \vec{AD}$
 (d) Parallelogram

Q14

(a) \mathbf{v} (b) $-\mathbf{v}$ (c) $-\mathbf{u}$
 (d) $\mathbf{u} + \mathbf{v}$ (e) $\mathbf{v} - \mathbf{u}$ (f) $-\mathbf{u} - \mathbf{v}$

Q15

(a) $\mathbf{u} + \mathbf{v}$ (b) $\mathbf{v} - \mathbf{u}$

Q16

(a) $a = -\frac{1}{3}$ (b) $a = \frac{2}{5}$
 (c) $a = \frac{5}{7}$ (d) $a = \frac{4}{21}$

Q17

See full worked solutions.

Q18

$a = 3, b = -2$

Q19

(a) $\frac{1}{2}(\mathbf{u} - \mathbf{v})$ (b) $\frac{1}{2}(\mathbf{u} + \mathbf{v})$
 (c) $\frac{1}{2}(\mathbf{v} - \mathbf{u})$ (d) $-\frac{1}{2}(\mathbf{u} + \mathbf{v})$

P1

(a) \mathbf{v}
 (b) \mathbf{p}
 (c) Observe that $\vec{UV} = 3\vec{UP}$. Then, eliminate \vec{UP} from the two expressions you obtained in previous parts.

P2

- (a) See full worked solutions.
- (b) $\frac{1}{2}\vec{AB} + \vec{BC} = -\mathbf{v}$
- (c) From the previous two parts we have a pair of simultaneous equations. Solve to make \vec{AB} the subject.
- (d) $\vec{BC} = -\frac{2}{3}(\mathbf{u} + 2\mathbf{v})$

Exercise 2C**Magnitude of a vector****F1**

- (a) $|\mathbf{u}|$ (b) distance (c) $\sqrt{x^2 + y^2}$

F2

- (a) unit (b) unit, $\hat{\mathbf{u}}$ (c) $\hat{\mathbf{u}}, \hat{\mathbf{u}}, \frac{1}{|\mathbf{u}|}$

Q1

- (a) $\sqrt{26}$ (b) $\sqrt{26}$ (c) $\sqrt{13}$
 (d) $\sqrt{20}$ (e) $\sqrt{53}$ (f) $\sqrt{53}$
 (g) 1 (h) 1

Q2

- (a) $\hat{a} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$ (b) $\hat{b} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$
 (c) $\hat{c} = -\frac{5}{13}\hat{i} + \frac{12}{13}\hat{j}$ (d) $\hat{d} = \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j}$
 (e) $\hat{e} = \hat{i}$ (f) $\hat{f} = -\hat{j}$

Q3

See full worked solutions.

Q4

- (a) $\vec{AB} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \vec{DC}$
 (b) $\vec{BC} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \vec{AD}$
 (c) Parallelogram
 (d) $|\vec{AB}| = |\vec{AD}| = \sqrt{26}$. A parallelogram with adjacent sides equal is a rhombus.
 (e) $|\vec{AC}| \neq |\vec{BD}|$

Q5

See full worked solutions.

Q6

- (a) $kx\hat{i} + ky\hat{j}$
 (b) $\sqrt{k^2x^2 + k^2y^2}$
 (c) $|k\mathbf{v}| = \sqrt{k^2x^2 + k^2y^2} = k\sqrt{x^2 + y^2} = k|\mathbf{v}|$
 (d) $|6\hat{i} + 8\hat{j}| = 10$, and $2|3\hat{i} + 4\hat{j}| = 2 \times 5 = 10$

P1

- (a) 1
 (b) \mathbf{u} was already in unit vector form, so of course the magnitude is just one.

P2

- (a) From the diagram, clearly the third side cannot be longer than the sum of the other two. The inequality in the question just characterises this description but algebraically.
- (b) Express the previous result in terms of u_1, u_2, v_1, v_2 . Calculate the magnitudes of the vectors and express it in terms of square roots. Square both sides, simplify and isolate the lone square root term, then square both sides again.

Exercise 2D**The dot product and applications****F1**

- (a) scalar, dot (b) $a_1b_1 + a_2b_2$
 (c) 0 (d) $|a|$

F2

- (a) $|\mathbf{a}||\mathbf{b}|$ (b) $|\mathbf{a}||\mathbf{b}|, |\mathbf{a}||\mathbf{b}|$
 (c) $|\mathbf{a}||\mathbf{b}|$ (d) $-|\mathbf{a}||\mathbf{b}|$
 (e) 0 (f) perpendicular

Q1

- (a) -13 (b) -2 (c) 12
 (d) 13 (e) 26 (f) 8

Q2

- (a) 6 (b) -6 (c) 0

Q3

- (a) $\cos \theta = -\frac{1}{\sqrt{5}\sqrt{13}}, \theta \approx 97^\circ 08'$
 (b) $\cos \theta = \frac{9}{\sqrt{5}\sqrt{17}}, \theta \approx 12^\circ 32'$
 (c) $\cos \theta = -\frac{5}{\sqrt{17}\sqrt{13}}, \theta \approx 109^\circ 39'$

Q4

- (a) $\mathbf{a} \cdot \mathbf{b} = 0$ (b) $k = 2$

Q5

- (a) $k = 2$ (b) $k = 0$ (c) $k = -7$

Q6

Using the dot product we have $\cos \theta = \frac{2ab}{a^2 + b^2}$. Flip both sides to obtain the result.

Q7

$\mathbf{u} \cdot \mathbf{v} = |1||\sqrt{2}| \cos 45^\circ = 1$

Q8

- (a) $\overrightarrow{AP} = \begin{bmatrix} x-r \\ y \end{bmatrix}, \overrightarrow{BP} = \begin{bmatrix} x+r \\ y \end{bmatrix}$
 (b) 0
 (c) Any angles subtended by the diameter in a semi-circle must be 90°

Q9

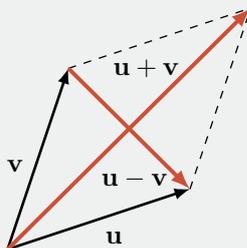
See full worked solutions.

Q10

See full worked solutions.

Q11

- (a) They are the diagonals of the parallelogram



- (b) 0
 (c) See full worked solutions.
 (d) See full worked solutions.
 (e) The diagonals of a parallelogram are perpendicular. This means that the parallelogram is actually a rhombus and hence the sides are equal.

Q12

- (a) $\mathbf{u} \cdot \mathbf{v} = 1 \times 1 \times \cos(\alpha - \beta) = \cos(\alpha - \beta)$
 (b) $\mathbf{v} = \langle \cos \beta, \sin \beta \rangle$
 (c) See full worked solutions.

P1

- (c) $|\mathbf{w}| \cos \beta = \mathbf{u} \cdot \mathbf{v} + |\mathbf{u}||\mathbf{v}|$
 (d) Equate the previous two parts.
 (e) The vector \mathbf{w} is equally inclined from either vector i.e. bisects the angle between \mathbf{u} and \mathbf{v} .

P2

- (a) $\mathbf{a} - \mathbf{b}$
 (b) These are two different expressions for the same length $|AB|$. The left hand side is using the cosine rule, and the right hand side is just the square of the length of the vector \overrightarrow{BA} .
 (c) $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2$
 (d) Combine the results from the previous two parts and the required result follows.

P3

- (a) $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$
 (b) $(\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2 |\mathbf{v}|^2 \cos^2 \theta \leq |\mathbf{u}|^2 |\mathbf{v}|^2$ since $\cos \theta \leq 1$. Equality occurs when \mathbf{u} and \mathbf{v} are parallel.
 (c) Equality occurs when $\frac{a}{c} = \frac{b}{d}$

P4

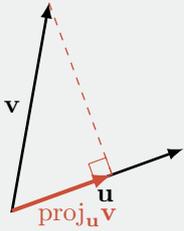
- (a) The sum of the squares of the diagonals is double the sum of the squares of the adjacent sides.
 (b) Use the fact that $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ repeatedly.

Exercise 2E

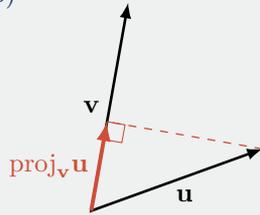
Vector projections

F1

(a)



(b)



F2

(a) $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$

(b) $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = (\mathbf{u} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}}$

Q1

(a) $\frac{3}{\sqrt{10}}$

(b) $\frac{3}{10}(3\hat{i} + \hat{j})$

(c) $\frac{3}{\sqrt{13}}$

(d) $\frac{3}{13}(2\hat{i} - 3\hat{j})$

Q2

(a) $\frac{4}{\sqrt{5}}$

(b) $\frac{4}{5}(2\hat{i} + \hat{j})$

(c) $\frac{4}{\sqrt{13}}$

(d) $\frac{4}{13}(3\hat{i} - 2\hat{j})$

Q3

(a) $\frac{9}{5}(\hat{i} + 2\hat{j})$

(b) $-\frac{1}{5}(\hat{i} + 2\hat{j})$

(c) $-2\hat{i} - \hat{j}$

(d) $\frac{23}{5}(4\hat{i} - 3\hat{j})$

Q4

(a) $\frac{27}{17}(\hat{i} + 4\hat{j})$

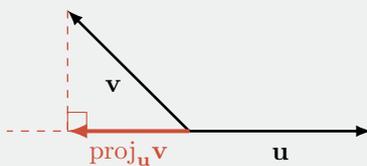
(b) $-\frac{1}{5}(-3\hat{i} + \hat{j})$

(c) $\frac{1}{5}(-4\hat{i} + 3\hat{j})$

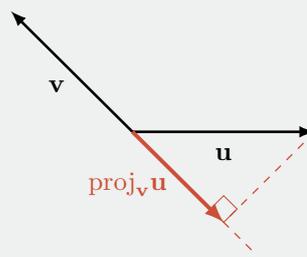
(d) $\frac{23}{29}(2\hat{i} - 5\hat{j})$

Q5

(a)



(b)



Q6

(a) $|\mathbf{u}||\mathbf{v}|$

(b) It is the scalar projection of \mathbf{v} onto \mathbf{u} .(c) Make $|\mathbf{v}| \cos \theta$ the subject of (a).

(d) $\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$

(e) The vector projection is the length of the vector i.e. the scalar projection, multiplied by the unit vector. Hence

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \times \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$$

Q7

(a) It is the shortest vector from any point on \mathbf{u} to the tip of the vector \mathbf{v} when they are connected tail-to-tail.(b) $\mathbf{u} \cdot \mathbf{w} = 0$. This should have been expected. We calculated the dot product of a vector \mathbf{u} with a perpendicular vector \mathbf{w} , so we should expect the result to be zero since the dot product of perpendicular vectors is zero.(c) It means perpendicular, so this is the perpendicular component of the projection of \mathbf{v} onto \mathbf{u} .

Q8

(a) See full worked solutions.

(b) 1

(c) The area this time is 5 units squared. The results are different and we should not expect them to be the same anyway since the vectors do not have the same projections onto each other. Their projections are different because their lengths are different.

Q9

(a) See full worked solutions.

(b) See full worked solutions.

(c) The result in (a) is symmetric in terms of a and b . In other words, if we swap a and b , we get exactly the same result. So, if we calculate it all over again but swapping the vectors, we can expect to get the same result.

- (d) The two vectors have the same length, so we can expect that they form the same projections onto each other. Consequently we can then expect the areas to be the same.

P1

- (a) $|\mathbf{u}| = |\mathbf{v}|$ OR $\mathbf{u} \cdot \mathbf{v} = 0$. To prove this, let $\text{scal}_{\mathbf{u}} \mathbf{v} = \text{scal}_{\mathbf{v}} \mathbf{u}$.
- (b) $\mathbf{u} = \mathbf{v}$ OR $\mathbf{u} \cdot \mathbf{v} = 0$. Similarly, to prove this, let $\text{proj}_{\mathbf{u}} \mathbf{v} = \text{proj}_{\mathbf{v}} \mathbf{u}$. The only way the results can be equal is if their dot product is zero, or if they were actually the same vector in the first place. A quick diagram shows this to be reasonably intuitive as well.

P2

Let $\mathbf{v} = \langle x, y \rangle$. Letting $\text{scal}_{\mathbf{u}} \mathbf{v} = 2$ and simplifying this, it eventually reduces to $3x + 4y = 10$. Hence, any vector $\langle x, y \rangle$ where $3x + 4y = 10$ will work. An example is $\langle 2, 1 \rangle$ and a quick computation confirms that the scalar projection is 2.

P3

$$\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|} \right) \left(\frac{\mathbf{u}}{|\mathbf{u}|} \right) = \left(\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \mathbf{v} \right) \left(\frac{\mathbf{u}}{|\mathbf{u}|} \right) = (\hat{\mathbf{u}} \cdot \mathbf{v}) \hat{\mathbf{u}}$$

Exercise 2F

Proofs using vectors

F1

- (a) parallel (b) $\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = 0$ (c) parallel

F2

- (a) Show that $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$
- (b) Show that $\overrightarrow{AB} = k\overrightarrow{CD}$ for some constant k .
- (c) Show that $\overrightarrow{AB} = k\overrightarrow{BC}$ for some constant k .
- (d) Show that $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$ or that $\overrightarrow{AM} = \overrightarrow{MB}$
- (e) Show that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$, usually by showing that $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v}$ where $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{CD} = \mathbf{v}$
- (f) Show that they share the same midpoint. This can be done by finding the midpoint M of one interval \overrightarrow{AB} , then showing that $\overrightarrow{CM} = \overrightarrow{MD}$ or a similar result to the same effect.

F3

- (a) $|\mathbf{a}|^2$ (b) $\mathbf{b} \cdot \mathbf{a}$ (c) $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

F4

- (a) collinear (b) position

Q1

- (a) $\overrightarrow{PM}, \overrightarrow{PQ}, \mathbf{q} - \mathbf{p}$ (b) $\overrightarrow{OM} = \mathbf{p} + \frac{1}{3}(\mathbf{q} - \mathbf{p})$

Q2

- (a) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$ (b) $\frac{1}{2}(\mathbf{b} - \mathbf{a})$

Q3

- (a) $\mathbf{a} + \frac{1}{2}\mathbf{b}$ (b) $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$
- (c) $\mathbf{b} + \frac{1}{2}\mathbf{a}$ (d) $\frac{1}{2}\mathbf{b} - \mathbf{a}$

Q4

- (a) $\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ (b) $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$

Q5

- (a) $\mathbf{a} + \frac{1}{3}\mathbf{b}$ (b) $\mathbf{a} + \frac{2}{3}\mathbf{b}$
- (c) $\mathbf{b} + \frac{1}{2}\mathbf{a}$ (d) $\frac{2}{3}\mathbf{b} - \frac{1}{2}\mathbf{a}$
- (e) $\frac{2}{3}\mathbf{b} - \mathbf{a}$ (f) $\frac{1}{6}\mathbf{b} - \mathbf{a}$
- (g) $\frac{1}{3}\mathbf{b} - \frac{1}{2}\mathbf{a}$ (h) $\frac{1}{3}\mathbf{b} - \mathbf{a}$
- (i) $-\mathbf{a} - \frac{1}{6}\mathbf{b}$ (j) $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

Q6

$$\overrightarrow{OX} = \mathbf{p} + \frac{m}{m+n}(\mathbf{q} - \mathbf{p})$$

Q7

- (a) $\overrightarrow{DB} = \frac{1}{2}\mathbf{a} + \mathbf{a} = \frac{3}{2}\mathbf{a}$
- (b) $\overrightarrow{DA} = \overrightarrow{DB} + \overrightarrow{BO} + \overrightarrow{OA}$

Q8

- (a) $\overrightarrow{AB} = \mathbf{a} - 5\mathbf{b}$
- (b) $\overrightarrow{MP} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
- (c) $\overrightarrow{PC} = 2\mathbf{a} + 2\mathbf{b}$
- (d) $\overrightarrow{MP} = \frac{1}{4}\overrightarrow{PC}$

Q9

- (a) See full worked solutions.
 (b) $\overrightarrow{PQ} = \frac{3}{4}(\mathbf{c} - \mathbf{a}) = \frac{3}{4}\overrightarrow{AC}$

Q10

- (a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{CD} = \mathbf{d} - \mathbf{c}$
 (b) From the previous part we can re-arrange to get $\mathbf{c} - \mathbf{a} = \mathbf{d} - \mathbf{b}$, which immediately gives the result.
 (c) Parallelogram

Q11

- (a) $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$
 (b) $\overrightarrow{OX} = \frac{1}{2}\mathbf{p}$, $\overrightarrow{OY} = \frac{1}{2}\mathbf{q}$
 (c) $\overrightarrow{XY} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{q} - \mathbf{p}) = \frac{1}{2}\overrightarrow{PQ}$
 (d) parallel, half

Q12

- (a) $\mathbf{0}$
 (b) From (a), $2\mathbf{p} + 2\mathbf{r} - 2\mathbf{s} - 2\mathbf{q} = \mathbf{0}$ and the result follows.
 (c) $\overrightarrow{WX} = \overrightarrow{WB} + \overrightarrow{BX} = \mathbf{p} + \mathbf{r}$.
 $\overrightarrow{ZY} = \overrightarrow{ZD} + \overrightarrow{DY} = \mathbf{q} + \mathbf{s} = \mathbf{p} + \mathbf{r} = \overrightarrow{WX}$.
 (d) By symmetry, we can also expect $\overrightarrow{YX} = \overrightarrow{ZW}$, and hence $WXYZ$ forms a parallelogram.

Q13

- (a) $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$
 (b) Use the fact that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$.
 (c) The midpoint of \overrightarrow{OB} is $\frac{1}{2}(\mathbf{a} + \mathbf{c})$. But this is precisely the midpoint M of \overrightarrow{AC} . Hence both diagonals share the same midpoint i.e. the diagonals bisect each other.

Q14

- (a) $\overrightarrow{OA} + \frac{2}{3}\overrightarrow{AM} = \overrightarrow{OT}$
 (b) $\overrightarrow{OT} = \frac{1}{3}(\mathbf{a} + 2\mathbf{c})$
 (d) 1 : 2

Q15

- (a) $\overrightarrow{AB} = \mathbf{a} + \mathbf{b}$
 (b) $\overrightarrow{DC} = \mathbf{a} + \mathbf{b}$
 (c) If the diagonals of a quadrilateral bisect each other, the quadrilateral must be a parallelogram

Q16

- (a) 0. Note that $|\mathbf{a}| = |\mathbf{p}|$
 (b) Since the dot product is zero, AP is perpendicular to BP .

Q17

- (a) The area of the parallelogram is double the area of the triangle spanned by the vectors. Use the familiar formula $A = \frac{1}{2}ab \sin C$ for the area of the triangle.
 (b) Use $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$
 (c) Use $\sin^2 \theta = 1 - \cos^2 \theta$ or draw the corresponding right-angled triangle for (b) and find the sine ratio using Pythagoras' theorem.
 (d) Combine the results from (a) and (c).

Q18

Show that $\overrightarrow{AC} \cdot \overrightarrow{OB} = 0$

Q19

- (a) Recall that $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$. We are given that \overrightarrow{OA} and \overrightarrow{AB} have the same length.
 (b) Expand and simplify the previous part.
 (c) Repeat both previous parts, but with the other pair of equal sides.
 (d) Equating the above results, we get $\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, so $\mathbf{b}(\mathbf{c} - \mathbf{a}) = \mathbf{0}$, so $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

Q20

- (a) $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$, $\overrightarrow{BD} = \mathbf{v} - \mathbf{u}$
 (b) See full worked solutions.

Q21

- (a) See full worked solutions.
 (b) If the diagonals are equal in length, then $|\mathbf{x} + \mathbf{y}| = |\mathbf{x} - \mathbf{y}|$ and hence the dot product is zero, so the vectors are perpendicular i.e. form a rectangle.

- (c) If the parallelogram is actually a rectangle, then the dot product is zero since \mathbf{x} and \mathbf{y} are perpendicular. From the identity we deduce $|\mathbf{x} + \mathbf{y}| = |\mathbf{x} - \mathbf{y}|$ and hence, the diagonals are equal.

Q22

- (a) $\vec{OA} + \vec{AC} = \vec{OC}$
 (b) $\vec{BC} = \mathbf{z} - \mathbf{y}$
 (c) Use the fact that \vec{AX} is perpendicular to \vec{BC} and similarly \vec{BY} is perpendicular to \vec{AC} .
 (d) Combine the dot product results from above to show that $\vec{OC} \cdot \vec{AB} = 0$
 (e) Since \vec{OC} is perpendicular to \vec{AB} , we can conclude that \vec{OC} produced to AB is the third altitude. But since it shares the point O with the other two altitudes, then the altitudes must be concurrent.

Q23

- (a) Let $\vec{AB} = \mathbf{u}$ and $\vec{AC} = \mathbf{v}$. Express the given result as dot products, expand them and simplify to eventually show that $\mathbf{u} \cdot \mathbf{v} = 0$
 (b) Express the left-hand side as dot products, expand and use the now given fact that $\mathbf{u} \cdot \mathbf{v} = 0$ to show that it becomes the right-hand side.

Q24

- (a) Show that $\vec{AD} = 2\mathbf{a} + \mathbf{b} + 2\mathbf{c}$ and use the fact that $\vec{AD} = k\mathbf{b}$ and $\vec{XY} = \mathbf{a} + \mathbf{b} + \mathbf{c}$.
 (b) They are all parallel lines.
 (c) 1.5 times longer

Q25

- (a) $\vec{OX} = \vec{OA} + \vec{AX}$
 (b) $\vec{BZ} = \vec{BO} + \vec{OZ}$
 (c) $\vec{OP} = \vec{OB} + \vec{BP}$, where $\vec{BP} = k\vec{BZ}$.
 (d) $\vec{OP} = \frac{k}{2}(\mathbf{a} + \mathbf{b})$
 (e) See full worked solutions.
 (f) Substitute $k = \frac{2}{3}$ into \vec{OP} .
 (g) $\vec{YP} = \frac{1}{3}\mathbf{a} - \frac{1}{6}\mathbf{b}$, $\vec{YA} = \mathbf{a} - \frac{1}{2}\mathbf{b}$
 (h) See full worked solutions.

P1

See full worked solutions.

P2

See full worked solutions.

P3

See full worked solutions.

Exercise 2G

Displacement and velocity

F1

- (a) displacement, displacement, \vec{AB}
 (b) velocity
 (c) displacement, velocity, vectors, magnitude, direction
 (d) magnitude, velocity, vector, speed

F2

- (a) sum
 (b) sum, resultant
 (c) resultant, true
 (d) direction, resultant, direction

Q1

- (a) $\vec{AB} = 4\mathbf{i} + 5\mathbf{j}$ (b) $\vec{AB} = 5\mathbf{i} - 5\mathbf{j}$
 (c) $\vec{AB} = -2\mathbf{i} + 3\mathbf{j}$

Q2

- (a) 5 m s^{-1} (b) 13 m s^{-1} (c) $2\sqrt{2} \text{ m s}^{-1}$

Q3

- (a) $4\mathbf{i} + 6\mathbf{j}$
 (b) $7\mathbf{i} + 10\mathbf{j}$
 (c) $\mathbf{i} + 2\mathbf{j} + t(3\mathbf{i} + 4\mathbf{j}) = (1 + 3t)\mathbf{i} + (2 + 4t)\mathbf{j}$

Q4

- (a) $|3\mathbf{i} + 15\mathbf{j}| = \sqrt{234}$ metres
 (b) $|8\mathbf{i} + 27\mathbf{j}| = \sqrt{793}$ metres
 (c) $|(-2 + 5t)\mathbf{i} + (3 + 12t)\mathbf{j}| = \sqrt{169t^2 + 52t + 13}$ metres

Q5

- (a) $5\sqrt{5}$ metres
 (b) $063^\circ 26'T$
 (c) Bob's displacement is $5\sqrt{5}$ metres from O , in the direction of $063^\circ 26'T$
 (d) 15 metres, 3 metres per second
 (e) $\sqrt{5}$ metres per second
 (f) Bob travelled a total of 15 metres in 5 seconds, so his average *speed* is 3 metres per second. However, overall his displacement was $5\sqrt{5}$ metres in the span of 5 seconds, so his average *velocity* is $\sqrt{5}$ metres per second.

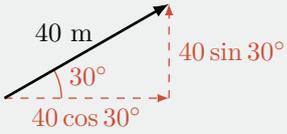
Q6

- (a) Displacement 50 m, average speed 7 m s^{-1} , average velocity 5 m s^{-1}
 (b) Displacement $15\sqrt{5}$ m, average speed 9 m s^{-1} , average velocity $3\sqrt{5} \text{ m s}^{-1}$
 (c) Displacement $10\sqrt{17}$ m, average speed 2.5 m s^{-1} , average velocity $\frac{\sqrt{17}}{2} \text{ m s}^{-1}$
 (d) Displacement $10\sqrt{5}$ m, average speed 8 m s^{-1} , average velocity $2\sqrt{5} \text{ m s}^{-1}$

Q7

- (a) 1, $090^\circ T$ (b) 2, $000^\circ T$
 (c) 4, $270^\circ T$ (d) 3, $180^\circ T$
 (e) $2\sqrt{2}$, $045^\circ T$ (f) $4\sqrt{2}$, $135^\circ T$
 (g) $5\sqrt{2}$, $315^\circ T$ (h) $3\sqrt{2}$, $225^\circ T$
 (i) 2, $030^\circ T$ (j) 2, $150^\circ T$
 (k) 2, $330^\circ T$ (l) 2, $210^\circ T$
 (m) 2, $060^\circ T$ (n) 2, $300^\circ T$
 (o) 2, $120^\circ T$ (p) 2, $240^\circ T$

Q8

- (a) 
 (b) $40 \cos 30^\circ \mathbf{i} + 40 \sin 30^\circ \mathbf{j} = 20\sqrt{3} \mathbf{i} + 20 \mathbf{j}$

Q9

- (a) $\overrightarrow{AB} = 6 \cos 60^\circ \mathbf{i} + 6 \sin 60^\circ \mathbf{j} = 3 \mathbf{i} + 3\sqrt{3} \mathbf{j}$
 (b) $\overrightarrow{BC} = -4 \cos 60^\circ \mathbf{i} + 4 \sin 60^\circ \mathbf{j} = -2 \mathbf{i} + 2\sqrt{3} \mathbf{j}$
 (c) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \mathbf{i} + 5\sqrt{3} \mathbf{j}$

Q10

- (a) $6 \cos 60^\circ \mathbf{i} + 6 \sin 60^\circ \mathbf{j} = 3 \mathbf{i} + 3\sqrt{3} \mathbf{j}$
 (b) $(2\sqrt{3} + 3) \mathbf{i} + (2 + 3\sqrt{3}) \mathbf{j}$
 (c) 9.67 km. It is the distance of Bob's displacement from A .
 (d) $48^\circ 4'$. It is the direction of Bob's displacement from the horizontal from A . Hence, his bearing is $41^\circ 56'T$.
 (e) 4.84 km h^{-1}

Q11

- (a) $\overrightarrow{AB} = -5\sqrt{3} \mathbf{i} + 5 \mathbf{j}$, $\overrightarrow{BC} = -3\sqrt{3} \mathbf{i} - 3 \mathbf{j}$,
 $\overrightarrow{AC} = -8\sqrt{3} \mathbf{i} + 2 \mathbf{j}$
 (b) $|\overrightarrow{AC}| = 14$. Overall time taken is 1.6 hours so average velocity is 8.75 km h^{-1}

Q12

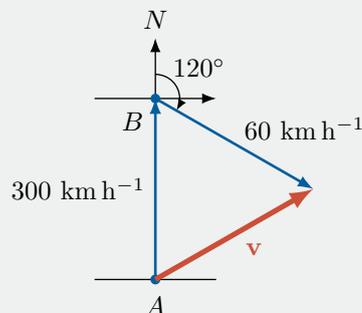
- (a) $\overrightarrow{AB} = 5 \mathbf{i}$, $\overrightarrow{BC} = 2 \mathbf{j}$, $\overrightarrow{CD} = 2 \mathbf{i} + 2\sqrt{3} \mathbf{j}$
 (b) $7 \mathbf{i} + (2 + 2\sqrt{3}) \mathbf{j}$
 (c) 1.02 hours
 (d) 8.71 km h^{-1}

Q13

- (a) $3 \mathbf{i} + 4 \mathbf{j}$ (b) 5 m s^{-1} (c) $037^\circ T$
 (d) 25 m (e) 5 seconds (f) (15, 20)
 (g) $311^\circ 25'T$

Q14

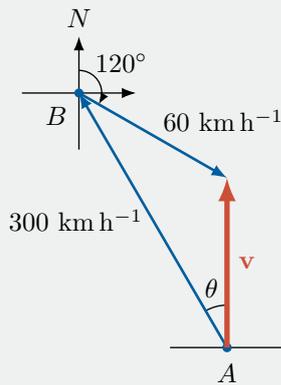
- (a)



- (b) $10\sqrt{756} \approx 275 \text{ km h}^{-1}$
 (c) $010^\circ 54'T$

Q15

(a)



(b) $9^{\circ}58'$

(c) 3 hours 46 minutes

P1

(a) 330°T

(b) 15.47% longer than still water

P2

The pilot should fly at a bearing of $124^{\circ}58'\text{T}$, and the flight will take 1 hour 55 minutes.

Exercise 2H

Resolving forces and equilibrium

F1

(a) net

(b) resultant, sum

(c) resultant, $\sum F$

(d) $\sum F$, equilibrium

F2

(a) 0

(b) 0

F3

(a) gravitational, 9.8

(b) Second, force, mg , weight

F4

(a) Third

(b) normal, perpendicular

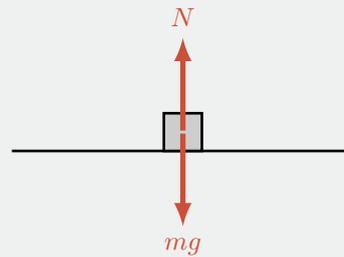
F5

(a) weight

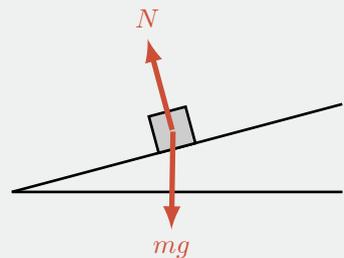
(b) upwards, tension

Q1

(a)

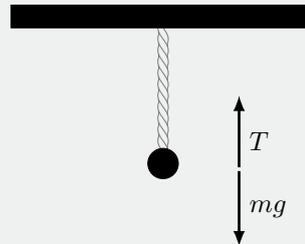


(b)



Q2

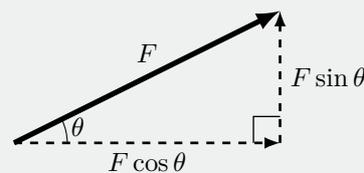
(a)



(b) 0

(c) mg

Q3



Q4

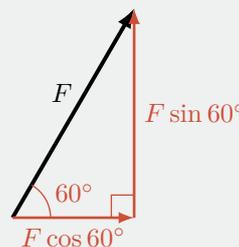
(a) equilibrium, zero

(b) upwards, downwards

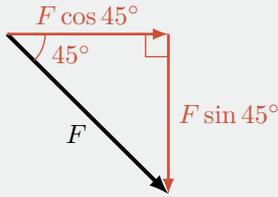
(c) 40N

Q5

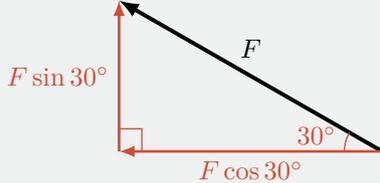
(a)



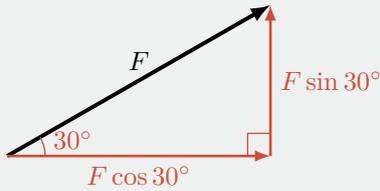
(b)



(c)

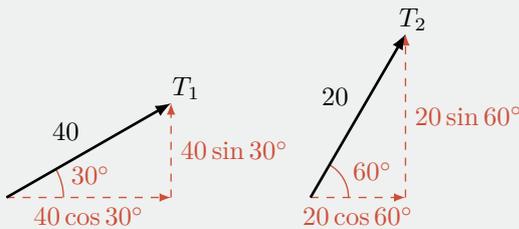


(d)



Q6

(a)



(b) $T_1 = 20\sqrt{3}\mathbf{i} + 20\mathbf{j}$, $T_2 = 10\mathbf{i} + 10\sqrt{3}\mathbf{j}$

(c) $\Sigma \mathbf{F} = (10 + 20\sqrt{3})\mathbf{i} + (20 + 10\sqrt{3})\mathbf{j}$

(d) 58.19 Newtons

(e) $39^\circ 54'$ from the horizontal, which is a bearing of $50^\circ 6'$ T

Q7

(a) $\Sigma F = (25\sqrt{3} + 5\sqrt{2})\mathbf{i} + (25 + 5\sqrt{2})\mathbf{j}$,
59.72 Newtons, $32^\circ 29'$ from horizontal

(b) $\Sigma F = (20 \cos 10^\circ + 20 \cos 80^\circ)\mathbf{i}$
 $+ (20 \sin 10^\circ + 20 \sin 80^\circ)\mathbf{j}$
32.77 Newtons, 45° from horizontal

(c) $\Sigma F = (6\sqrt{2} - 5\sqrt{3})\mathbf{i} + (6\sqrt{2} + 5)\mathbf{j}$,
13.49 Newtons, $90^\circ 45'$ from horizontal

(d) $\Sigma F = (12 \cos 80 - 12 \cos 10)\mathbf{i}$,
 $+ (12 \sin 80 + 12 \sin 10)\mathbf{j}$
 $12\sqrt{2}$ Newtons, 125° from horizontal

Q8

(a) $\Sigma F = (5\sqrt{2} - 10\sqrt{3})\mathbf{i} + (5\sqrt{2} - 30)\mathbf{j}$,
25.12 N, $204^\circ 55'$ T

(b) $\Sigma F = (15 - 4\sqrt{3})\mathbf{i} + (15\sqrt{3} - 46)\mathbf{j}$, 21.59 N,
 $154^\circ 2'$ T

(c) $\Sigma F = (4\sqrt{2} + 6)\mathbf{i} + (4\sqrt{2} + 6\sqrt{3} - 20)\mathbf{j}$,
12.31 N, $108^\circ 43'$ T

(d) $\Sigma F = (20\sqrt{2} - 20)\mathbf{j}$, 8.28 N, 0° T

Q9

(a) $W = mg = 2000$ Newtons

(b) $T = 2000\sqrt{2}$, $F = 2000$

(c) No. In this diagram, the angle is 45° , so the horizontal and vertical components of T are equal i.e. F and W are equal. But if the angle were something else, this would no longer hold.

Q10

(a) $\Sigma F_x = 0$ N

(b) $\Sigma F_y = 50$ N

(c) $T_1 = |T_1| \cos 30^\circ \mathbf{i} + |T_1| \sin 30^\circ \mathbf{j}$
 $T_2 = |T_2| \cos 45^\circ \mathbf{i} + |T_2| \sin 45^\circ \mathbf{j}$

(d) The horizontal forces are equal but in opposite directions.

(e) The vertical forces are equal but in opposite directions.

(f) $|T_1| = 36.6$ Newtons, $|T_2| = 44.8$ Newtons

Q11

(a) $|T_1| = 70.98$, $|T_2| = 64.52$

(b) $|T_1| = 40\sqrt{3}$, $|T_2| = 40$

(c) $|T_1| = 163.04$, $|T_2| = 72.79$

(d) $|T_1| = 30.78$, $|T_2| = 39.40$

Q12

(a) $\mathbf{T} = T \cos 30^\circ \mathbf{i} + T \sin 30^\circ \mathbf{j}$,
 $\mathbf{N} = N \sin 30^\circ \mathbf{i} + N \cos 30^\circ \mathbf{j}$

(b) $\Sigma F_x = \Sigma F_y = 0$

(c) $N = 25\sqrt{3}$, $T = 25$

Q13

(a) $N = 25\sqrt{3}$

(b) $T = 25$

Q14

- (a) $N + T = 10\sqrt{2}$
 $N - T = 0$
 $N = 5\sqrt{2}, T = 5\sqrt{2}$
- (b) $N \cos 20^\circ + T \sin 20^\circ = 10$
 $N \sin 20^\circ - T \cos 20^\circ = 0$
 $N = 10 \cos 20^\circ, T = 10 \sin 20^\circ$
- (c) $N \cos 35^\circ + T \sin 35^\circ = 10$
 $N \sin 35^\circ - T \cos 35^\circ = 0$
 $N = 10 \cos 35^\circ, T = 10 \sin 35^\circ$
- (d) $N \cos 40^\circ + T \sin 40^\circ = 10$
 $N \sin 40^\circ - T \cos 40^\circ = 0$
 $N = 10 \cos 40^\circ, T = 10 \sin 40^\circ$

Q15

- (a) See full worked solutions.
- (b) $|T_1| = 8, |T_2| = 6$

Q16

- (a) See full worked solutions.
- (b) $|T_1| = 120, |T_2| = 90$

Q17

- (a) $10\sqrt{3}$ Newtons (b) 15 Newtons

P1

See full worked solutions.

P2

See full worked solutions.

Exercise 21**Projectile motion****F1**

- (a) particle
 (b) negligible
 (c) gravity

F2

- (a) position
 (b) velocity, $x'(t)\mathbf{i} + y'(t)\mathbf{j}$
 (c) acceleration, $x''(t)\mathbf{i} + y''(t)\mathbf{j}$

F3

- (a) Let $y(t) = 0$ and solve for t .
 (b) Substitute the time value from (a) into $x(t)$.
 (c) Let $y'(t) = 0$ and solve for t .
 (d) Substitute the time value from (c) into $y(t)$.
 (e) Calculate $|\mathbf{v}(t_0)| = \sqrt{(x'(t_0))^2 + (y'(t_0))^2}$
 (f) Calculate the direction of $\mathbf{v}(t_0)$ using
 $\tan \alpha = \frac{y'(t_0)}{x'(t_0)}$

F4

$$\mathbf{a}(t) = -g\mathbf{j}$$

$$\mathbf{v}(t) = V \cos \theta \mathbf{i} + (-gt + V \sin \theta)\mathbf{j}$$

$$\mathbf{r}(t) = Vt \cos \theta \mathbf{i} + \left(-\frac{1}{2}gt^2 + Vt \sin \theta\right)\mathbf{j}$$

Q1

- (a) $80\mathbf{i} + 55\mathbf{j}$ (b) $160\mathbf{i} + 100\mathbf{j}$
 (c) $t = 3$ (d) $t = 4, 8$
 (e) $t = 12$

Q2

- (a) $\mathbf{v}(t) = 20\sqrt{3}\mathbf{i} + (20 - 10t)\mathbf{j}$
 (b) $\mathbf{v}(0) = 20\sqrt{3}\mathbf{i} + 20\mathbf{j}$
 (c) 40 m s^{-1} , initial speed
 (d) 30° , initial projection angle from the horizontal

Q3

- (a) Let the \mathbf{j} -component of $\mathbf{r}(t)$ be zero.
 (b) $10\mathbf{i} - 10\sqrt{3}\mathbf{j}$
 (c) The magnitude is 20 m s^{-1} and this is the speed at which the particle lands. The angle is 120° from the horizontal, and this is the angle at which the particle lands.
 (d) The initial speed is 20 m s^{-1} , and the initial angle is $\frac{\pi}{3}$. The landing velocity is the same as the initial speed, and the landing angle is just the negative of the initial angle. This is not a surprise since the trajectory is perfectly parabolic and symmetric, so this was to be expected.

Q4

- (a) $t = \sqrt{2}$
 (b) 10
 (c) $t = 2\sqrt{2}$
 (d) It is exactly double of the previous result, which is to be expected since the trajectory is symmetric.

(e) 40

Q5

- (a) $h = 20$ m
 (b) $t = 1 + \sqrt{5} \approx 3.2$ sec
 (c) $20\sqrt{2}$ m s⁻¹, 127°46'

Q6

- (a) $\mathbf{a}(t) = -10\mathbf{j}$
 (b) $\mathbf{v}(t) = 25\mathbf{i} + (25\sqrt{3} - 10t)\mathbf{j}$
 (c) $\mathbf{r}(t) = 25t\mathbf{i} + (25\sqrt{3}t - 5t^2)\mathbf{j}$

Q7

- (a) $\mathbf{r}(t) = 15\sqrt{3}t\mathbf{i} + (15t - 5t^2)\mathbf{j}$
 (b) $\mathbf{r}(t) = 30\sqrt{2}t\mathbf{i} + (30\sqrt{2}t - 5t^2)\mathbf{j}$
 (c) $\mathbf{r}(t) = 20t\mathbf{i} + (20\sqrt{3}t - 5t^2)\mathbf{j}$

Q8

- (a) 0°
 (b) $\mathbf{a}(t) = -10\mathbf{j}$, $\mathbf{v}(t) = 20\mathbf{i} - 10t\mathbf{j}$,
 $\mathbf{r}(t) = 20t\mathbf{i} + (40 - 5t^2)\mathbf{j}$
 (c) $40\sqrt{2}$ m
 (d) $20\sqrt{3}$ m s⁻¹, $\theta = -54^\circ 44'$

Q9

- (a) $\sin \theta = \frac{3}{5}$, $\cos \theta = \frac{4}{5}$
 (b) $\mathbf{a}(t) = -10\mathbf{j}$, $\mathbf{v}(t) = 32\mathbf{i} + (24 - 10t)\mathbf{j}$,
 $\mathbf{r}(t) = 32t\mathbf{i} + (-5t^2 + 24t + 5)\mathbf{j}$
 (c) 160 m
 (d) $10\sqrt{17}$ m s⁻¹, $-39^\circ 6'$
 (e) 33.8 m

Q10

- (a) $\mathbf{a}(t) = -10\mathbf{j}$,
 $\mathbf{v}(t) = 120 \cos \theta \mathbf{i} + (120 \sin \theta - 10t)\mathbf{j}$,
 $\mathbf{r}(t) = 120t \cos \theta \mathbf{i} + (120t \sin \theta - 5t^2)\mathbf{j}$
 (b) 28°13' or 61°47'
 (c) No. The maximal range is 1440 metres

Q11

$$\mathbf{a}(t) = -10\mathbf{j}, \mathbf{v}(t) = \frac{\sqrt{3}}{2}V\mathbf{i} + \left(\frac{V}{2} - 10t\right)\mathbf{j},$$

$$\mathbf{r}(t) = \frac{\sqrt{3}Vt}{2}\mathbf{i} + \left(\frac{Vt}{2} - 5t^2\right)\mathbf{j}$$

$$V \approx 15.2 \text{ m s}^{-1}$$

Q12

- (a) 12.91 m s⁻¹
 (b) 2.25 m
 (c) 18.26 m s⁻¹

Q13

- (a) 100 metres per second
 (b) $\mathbf{a}(t) = -10\mathbf{j}$, $\mathbf{v}(t) = 100\mathbf{i} - 10t\mathbf{j}$,
 $\mathbf{r}(t) = 100t\mathbf{i} + (1000 - 5t^2)\mathbf{j}$
 (c) $1000\sqrt{2}$ metres
 (d) 53°44'

Q14

- (a) $t = 2$
 (b) $\mathbf{v}(t) = V\mathbf{i} - 10t\mathbf{j}$
 (c) $V = 20$ m s⁻¹

Q15

$$V = 40 \text{ m s}^{-1}, \theta = 30^\circ$$

Q16

- (a) $V = 4\sqrt{34}$, $\tan \theta = \frac{3}{5}$, so $\theta \approx 30^\circ 58'$
 (b) 48 metres

Q17

$$T = \frac{2\sqrt{3}}{3}$$

Q18

- (a) $R_V = \frac{V^2 \sin 2\theta}{g}$
 (b) $R_{2V} = \frac{4V^2 \sin 2\theta}{g}$, which is actually four times the original horizontal range, not two times as Bob claims.

Q19

See full worked solutions.

Q20

They are all the same because the general time of flight is $T = \sqrt{\frac{2h}{g}}$, which relies only on h i.e. the initial velocity does not change anything.

Q21

See full worked solutions.

Q22

- (a) Recall the horizontal range formula
 $R = \frac{V^2 \sin 2\theta}{g}$. The maximum range occurs when $\sin 2\theta = 1$ i.e. $2\theta = 90^\circ$
- (b) Recall the maximum height formula
 $H = \frac{V^2 \sin^2 \theta}{2g}$. The maximum possible height occurs when $\sin^2 \theta = 1$.
- (c) Let $R(\theta) = \frac{V^2}{g} \sin 2\theta$ be the horizontal range for any given θ . Find $R(90^\circ - \theta)$ and simplify, and it actually just becomes $R(\theta)$, meaning the horizontal ranges are equal.

Q23

- (a) $H = \frac{V^2 \sin^2 \theta}{2g}$
- (b) $R = \frac{V^2 \sin 2\theta}{g}$
- (c) Let $H = R$ and use the fact that $\sin 2\theta = 2 \sin \theta \cos \theta$
- (d) As long as the angle satisfies $\theta = \tan^{-1}(4)$, the condition will still hold and the initial speed is irrelevant.

Q24

See full worked solutions.

Q25

- (a) See full worked solutions.
- (b) $\theta = 41^\circ 49'$
- (c) $t \approx 1.36$ seconds. Use the fact that $x_P + x_Q = 120$ and substitute the now known value of θ .

P1

See full worked solutions.

P2

See full worked solutions.

Chapter Review

R1

- (a) $-2\hat{i} + 4\hat{j}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}$
- (b) $6\hat{i} - \hat{j}, \begin{bmatrix} 6 \\ -1 \end{bmatrix}$
- (c) $3\hat{i} - 7\hat{j}, \begin{bmatrix} 3 \\ -7 \end{bmatrix}$
- (d) $-3\hat{i} + 7\hat{j}, \begin{bmatrix} -3 \\ 7 \end{bmatrix}$

(e) $8\hat{i} - 5\hat{j}, \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

(f) $5\hat{i} + 2\hat{j}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

R2

$a = 3, b = -1$

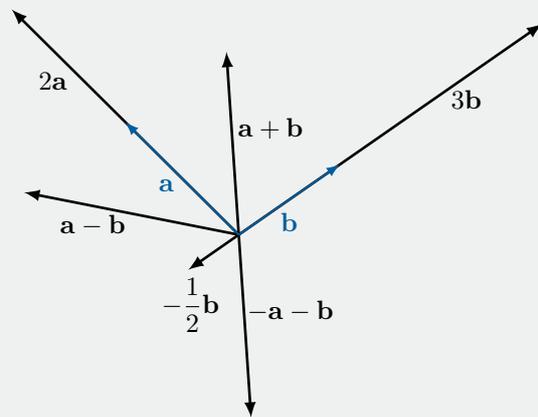
R3

- (a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 \\ -7 \end{bmatrix}$ (c) $\begin{bmatrix} -4 \\ 12 \end{bmatrix}$

R4

$k = 2$

R5



R6

- (a) $2\hat{i} + 4\hat{j}, 2\hat{i} + 4\hat{j}$
- (b) They are equal vectors
- (c) $3\hat{i} - \hat{j}$ both
- (d) parallelogram

R7

- (a) \mathbf{v} (b) $-\mathbf{v}$ (c) $-\mathbf{u}$
- (d) $\mathbf{u} + \mathbf{v}$ (e) $\mathbf{v} - \mathbf{u}$ (f) $-\mathbf{u} - \mathbf{v}$

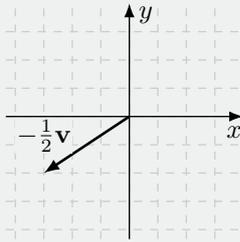
R8

- (a) \mathbf{u} (b) \mathbf{v} (c) $2\mathbf{u}$
- (d) $2\mathbf{v}$ (e) $-\mathbf{u}$ (f) $\mathbf{v} - \mathbf{u}$
- (g) $\mathbf{u} - \mathbf{v}$ (h) $\mathbf{v} + \mathbf{u}$ (i) $\mathbf{v} + 2\mathbf{u}$

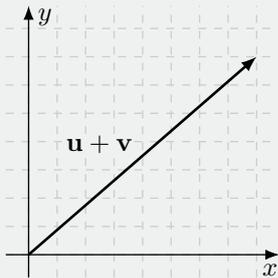
R9

(a) $\mathbf{v} - \mathbf{u}, 4\hat{i} + \hat{j}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) $-3\hat{i} - 2\hat{j}$



(c) $8\hat{i} + 7\hat{j}$



(d) $P(1, 1.5), Q(3, 2), \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}, 2\hat{i} + \frac{1}{2}\hat{j}$

(e) \overrightarrow{PQ} and \overrightarrow{BC} are parallel and \overrightarrow{PQ} is half of \overrightarrow{BC}

R10

(a) $-7\hat{i} - \hat{j}$

(b) $\sqrt{50}$

(c) $(3 - 7m)\hat{i} + (4 - m)\hat{j}$

(d) $\frac{3}{7}$

(e) $-\frac{1}{6}$

R11

(a) $-\frac{1}{2}$ (b) $\frac{3}{4}$ (c) 2

R12

(a) $\overrightarrow{QP} = -2\hat{i} + 4\hat{j}, \overrightarrow{QR} = -2\hat{i} - 6\hat{j}$

(b) $2\sqrt{5}, 2\sqrt{10}$

(c) -20

(d) $\cos \angle PQR = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{|\overrightarrow{QP}| |\overrightarrow{QR}|} = \frac{-20}{20\sqrt{2}}$

(e) $\frac{3\pi}{4}$

(f) 40, 0

R13

(a) $\frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ (b) $10\mathbf{v}$

R14

(a) $5\sqrt{2}$ (b) $-5\sqrt{2}$

R15

$\begin{bmatrix} 24 \\ 10 \end{bmatrix}$

R16

(a) $\frac{7}{\sqrt{26}}$ (b) $\frac{7}{\sqrt{13}}$

(c) $\frac{5}{26}(5\hat{i} + \hat{j})$ (d) $\frac{7}{13}(2\hat{i} - 3\hat{j})$

R17

(a) $48 \cos \theta = 24$

(b) $\sqrt{52}$

R18

(a) $\mathbf{a} - \mathbf{b} = -9\hat{i} + 9\hat{j}, \mathbf{b} - \mathbf{a} = 9\hat{i} - 9\hat{j}$

(b) 5, 13, $9\sqrt{2}$

(c) $75^\circ 45'$

(d) $81^\circ 52'$

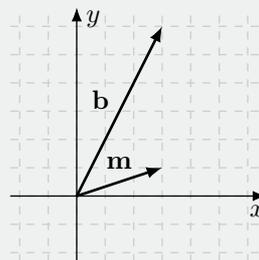
(e) $22^\circ 23'$

(f) $\frac{1}{9\sqrt{2}}(9\hat{i} - 9\hat{j})$

R19

See full worked solutions.

R20



(a)

(b) $\mathbf{m} - \mathbf{b} = -t\hat{i} + (-5 + 3t)\hat{j}$

(c) Bob's boat $6\hat{i} + 7\hat{j}$, Mary's boat $5\hat{i} + 5\hat{j}$

(d) $\sqrt{5}$ units

(e) $\sqrt{10t^2 - 30t + 25}$

R21

- (a) $6\hat{i} + 8\hat{j}$
 (b) 10 m s^{-1}
 (c) $36^\circ 52' \text{ T}$
 (d) 125 m
 (e) 12.5 s
 (f) $(75, 100)$
 (g) $306^\circ 52' \text{ T}$

R22

- (a) $07^\circ 11' \text{ T}$
 (b) $172^\circ 49' \text{ T}$

R23

$$|T_1| = 64, |T_2| = 48$$

R24

- (a) $T = 100, N = 100\sqrt{3}$
 (b) $T = N = 50\sqrt{2}$

R25

$$|T_1| = 50\sqrt{3}, |T_2| = 50$$

R26

- (a) See full worked solutions.
 (b) $|T_1| = \frac{120}{13}, |T_2| = \frac{50}{13}$

R27

See full worked solutions.

R28

- (a) $a(t) = -10\hat{j}, v(t) = 20\hat{i} - 10t\hat{j},$
 $r(t) = 20t\hat{i} + (-5t^2 + 125)\hat{j}$
 (b) $t = 5 \text{ s}, 100 \text{ m}$

R29

- (a) $\mathbf{a}(t) = -10\mathbf{j}, \mathbf{v}(t) = 10\sqrt{3}\mathbf{i} + (10 - 10t)\mathbf{j},$
 $\mathbf{r}(t) = 10t\sqrt{3}\mathbf{i} + (10t - 5t^2)\mathbf{j}$
 (b) 5 m
 (c) 2 seconds
 (d) $20\sqrt{3} \text{ m}$

R30

$$\alpha = 14^\circ 2', V = 20\sqrt{85} \text{ m s}^{-1}$$

3. Trigonometric Equations

Exercise 3A

Revision of solving trigonometric equations

F1

- (a) domain
 (b) radians, degrees

F2

one

F3

- (a) factorised
 (b) side, factorise, cancel, solutions

Q1

- (a) $x = 0, 2\pi$ (b) $x = \frac{\pi}{2}, \frac{3\pi}{2}$
 (c) $x = \frac{3\pi}{2}$ (d) $x = \frac{\pi}{2}$
 (e) $x = 0, \pi, 2\pi$ (f) $x = 0, \pi, 2\pi$

Q2

- (a) $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
 (b) $-\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}$
 (c) $-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$
 (d) $-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$
 (e) $-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$
 (f) $-\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

Q3

- (a) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
 (b) $\frac{\pi}{3}, \frac{5\pi}{3}$
 (c) No solution
 (d) $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$
 (e) $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

(f) $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

Q4

(a) $\frac{5\pi}{12}, \frac{17\pi}{12}$

(b) $\frac{5\pi}{12}, \frac{17\pi}{12}$

(c) $\frac{3\pi}{2}, \frac{11\pi}{6}$

(d) $\frac{3\pi}{4}, \frac{5\pi}{4}$

Q5

(a) $\frac{\pi}{3}, \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}$

(b) $0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

Q6

(a) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(b) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(c) $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$

(d) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}$

(e) $\frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi$

(f) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Q7

(a) $\frac{\pi}{2}, \frac{3\pi}{2}$

(b) No solution

(c) No solution

(d) $\frac{\pi}{2}$

Q8

(a) $\frac{\pi}{8}, \frac{5\pi}{8}$

(b) $0, \frac{\pi}{2}, \pi$

(c) $\frac{5\pi}{12}, \frac{11\pi}{12}$

Q9

(a) 2.802, 0.340 correct 3 d. p.

(b) $\frac{3\pi}{4}, \frac{7\pi}{4}, 1.249, 4.391$

(c) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(d) $\frac{\pi}{4}, \frac{5\pi}{4}, 2.159, 5.300$

Q10

(a) $0, \pi, -\pi, \frac{\pi}{2}$

(b) $-\frac{2\pi}{3}, \frac{2\pi}{3}$

(c) $\frac{\pi}{4}, \frac{5\pi}{4}, 1.89255,$

(d) $-\frac{\pi}{4}, \frac{3\pi}{4}$

5.03414

Q11

(a) $\frac{2\pi}{3}$

(b) $\frac{\pi}{2}, \frac{5\pi}{6}$

(c) $\frac{\pi}{6}, \frac{3\pi}{2}$

(d) $\frac{5\pi}{12}, \frac{23\pi}{12}$

Q12

(a) $\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$

(b) $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

(c) $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$

(d) $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

(e) $0, 2\pi$

(f) $\frac{\pi}{4}, \frac{5\pi}{4}$

Q13

(a) $\frac{2\pi}{3}, \frac{4\pi}{3}$

(b) $\pi, -\pi$

Q14

(a) $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

(b) $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

(c) $0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

(d) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

Q15

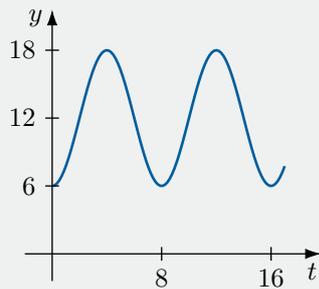
(a) $0, \pm\frac{\pi}{2}, \pm\pi$

(b) $\pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$

(c) $0, \pm\frac{\pi}{2}, \pm\pi, -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$

Q16

(a)



(b) Low tide is 6 m and high tide is 18 m.

(c) 6pm and 2am

(d) 2pm and 10pm

(e) At noon

P1

(a) $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

(b) $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}$

(c) $\frac{3\pi}{4}, \frac{7\pi}{4}, 4.46741, 1.32582$

(d) $\frac{3\pi}{4}, \frac{7\pi}{4}, 0.321751, 3.46334$

(e) $\frac{\pi}{4}, \frac{5\pi}{4}, 2.03, 5.18$

(f) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

Exercise 3B

Solving equations using compound angle formulae

F1

(a) $\sin A \cos B + \cos A \sin B$

(b) $\sin A \cos B - \cos A \sin B$

(c) $\cos A \cos B - \sin A \sin B$

(d) $\cos A \cos B + \sin A \sin B$

(e) $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

(f) $\frac{\tan A - \tan B}{1 + \tan A \tan B}$

F2

(a) $2 \sin A \cos A$

(b) $\cos^2 A - \sin^2 A, 2 \cos^2 A - 1, 1 - 2 \sin^2 A$

(c) $\frac{2 \tan A}{1 - \tan^2 A}$

F3

(a) one

(b) same, compound, double

Q1

(a) $\frac{\pi}{2}, \frac{3\pi}{2}$

(b) $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}$

(c) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(d) No solution

(e) $0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

(f) $0, \pi, 2\pi$

(g) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

(h) $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

(i) $0, \pi, 2\pi, \pm \tan^{-1} \sqrt{2}, \pi \pm \tan^{-1} \sqrt{2}$

(j) $0, \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}$

(k) $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

(l) $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

Q2

See full worked solutions.

Q3

(a) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(b) $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

(c) $0, \pi, 2\pi$

(d) $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

(e) $0, \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}, 2\pi$

(f) $0, \pi, 2\pi$

Q4

$$\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

Q5

$$\frac{-7\pi}{8}, \frac{-3\pi}{8}, \frac{-3\pi}{4}, \frac{\pi}{8}, \frac{\pi}{4}, \frac{5\pi}{8}$$

Q6

(a) $8x^3 - 6x - 1 = 2 \cos 3x - 1 = 0$

(b) Solve $\cos 3\theta = \frac{1}{2}$ for the first three values of θ , and the roots of the polynomial are $x = \cos \theta$.

(c) Use the sum of roots.

(d) Use the product of roots.

(e) Use the sum of the reciprocals of the roots.

Q7

(a) $x = \cos\left(\frac{2\pi}{9}\right), \cos\left(\frac{4\pi}{9}\right), \cos\left(\frac{8\pi}{9}\right)$.

(b) 8

P1

$$\frac{\pi}{24}, \frac{9\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}$$

P2

(a) $t = -1, \tan\left(\frac{\pi}{12}\right), \tan\left(\frac{5\pi}{12}\right)$

(b) Use the sum of roots.

(c) Use the product of roots.

(d) They satisfy the sum and product of roots of the quadratic.

(e) $\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$ and $\tan\left(\frac{5\pi}{12}\right) = 2 + \sqrt{3}$

Exercise 3C**Solving equations involving sums to products****F1**

(a) $\sin A \cos B + \cos A \sin B$

(b) $\sin A \cos B - \cos A \sin B$

(c) $\cos A \cos B - \sin A \sin B$

(d) $\cos A \cos B + \sin A \sin B$

F2

(a) $\frac{1}{2}(\sin(A+B) + \sin(A-B))$

(b) $\frac{1}{2}(\sin(A+B) - \sin(A-B))$

(c) $\frac{1}{2}(\cos(A+B) + \cos(A-B))$

(d) $\frac{1}{2}(\cos(A-B) - \cos(A+B))$

Q1

(a) $2 \cos x \cos 3x$

(b) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Q2

(a) $2 \sin 2x \cos 3x$

(b) $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$

Q3

(a) $\frac{1}{2}(\sin 4x + \sin 2x)$

(b) $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, 2\pi$

Q4

(a) $2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$

(b) $2 \cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$

(c) $2 \cos \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$

(d) $-2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$

Q5

(a) $2 \cos 3x \sin x$ (b) $2 \cos 3x \cos x$

(c) $2 \sin 5x \cos x$ (d) $2 \cos 5x \cos 2x$

Q6

(a) $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

(b) $0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi$

(c) $0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{4}, \frac{3\pi}{4}, \pi$

(d) $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

(e) $\frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

(f) $0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi$

Q7

(a) $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

(b) $\frac{\pi}{3}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

(c) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

Q8

(a) $\cos 5x + \cos 3x = 2 \cos 4x \cos x,$
 $\cos 7x + \cos x = 2 \cos 4x \cos 3x$

(b) $4 \cos x \cos 2x \cos 4x$

(c) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{2}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8},$
 $\frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

Q9

(a) $2 \sin \frac{5x}{2} \cos \frac{3x}{2}, 2 \sin \frac{5x}{2} \cos \frac{x}{2}$

(b) $4 \sin \frac{5x}{2} \cos x \cos \frac{x}{2}$

(c) The equation simplifies to
 $2 \sin 3x \cos x = 2 \sin 3x \cos 3x$
 $0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{3\pi}{2}, \frac{8\pi}{5}, 2\pi$

Q10

(a) See full worked solutions.

(b) $0, \frac{\pi}{2}, \pi, \frac{\pi}{3}, \frac{2\pi}{3}$

Q11

(a) See full worked solutions.

(b) $0, \frac{\pi}{2}, \pi, \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}$

P1

(a) $\frac{1}{2}(\cos 0 - \cos 2x)$

(b) $\frac{1}{2}(\cos 2x - \cos 4x)$

(c) $\frac{1}{2}(\cos 4x - \cos 6x)$

(d) $\frac{1}{2}(\cos(2n-2)x - \cos 2nx)$

(e) Sum the results from the previous parts and observe that the sum is self-cancelling, leaving only the first and last term.

$\sin x S_n = \frac{1}{2}(\cos 0 - \cos(2nx))$ and then make S_n the subject.

(f) $x = \frac{k\pi}{n}$, where $k = 0, 1, 2, 3, \dots, 2n$.

Exercise 3D

Auxiliary angle formulae

F1

auxiliary

F2

(a) $R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$

(b) Equating $\sin x$ gives $R \cos \alpha = a$. Equating $\cos x$ gives $R \sin \alpha = b$.

(c) $R = \sqrt{a^2 + b^2}$

(d) $\alpha = \tan^{-1} \left(\frac{b}{a} \right)$

Q1

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{2}$
 (d) 2 (e) 5 (f) 13

Q2

(a) $R \sin x \cos x + R \cos x \sin x$
 (b) $R \sin x \cos x - R \cos x \sin x$
 (c) $R \cos x \cos x - R \sin x \sin x$
 (d) $R \cos x \cos x + R \sin x \sin x$

Q3

(a) $R \sin(x + \alpha), R = 2, \alpha = \frac{\pi}{6}, 2 \sin \left(x + \frac{\pi}{6} \right)$
 (b) $R \cos(x + \alpha), R = 2, \alpha = \frac{\pi}{3}, 2 \cos \left(x + \frac{\pi}{3} \right)$
 (c) $R \sin(x - \alpha), R = 2, \alpha = \frac{\pi}{3}, 2 \sin \left(x - \frac{\pi}{3} \right)$
 (d) $R \cos(x - \alpha), R = 2, \alpha = \frac{\pi}{6}, 2 \cos \left(x - \frac{\pi}{6} \right)$
 (e) $R \sin(x - \alpha), R = \sqrt{2}, \alpha = \frac{\pi}{4},$
 $\sqrt{2} \sin \left(x - \frac{\pi}{4} \right)$
 (f) $R \sin(x + \alpha), R = \sqrt{2}, \alpha = \frac{\pi}{4},$
 $\sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$
 (g) $R \cos(x + \alpha), R = \sqrt{2}, \alpha = \frac{\pi}{4},$
 $\sqrt{2} \cos \left(x + \frac{\pi}{4} \right)$

- (h) $R \cos(x - \alpha)$, $R = \sqrt{2}$, $\alpha = \frac{\pi}{4}$,
 $\sqrt{2} \cos\left(x - \frac{\pi}{4}\right)$
- (i) $R \sin(x + \alpha)$, $R = 2\sqrt{3}$, $\alpha = \frac{\pi}{6}$,
 $2\sqrt{3} \sin\left(x + \frac{\pi}{6}\right)$
- (j) $R \cos(x - \alpha)$, $R = 2\sqrt{3}$, $\alpha = \frac{\pi}{3}$,
 $2\sqrt{3} \cos\left(x - \frac{\pi}{3}\right)$
- (k) $R \sin(x - \alpha)$, $R = 2\sqrt{3}$, $\alpha = \frac{\pi}{3}$,
 $2\sqrt{3} \sin\left(x - \frac{\pi}{3}\right)$
- (l) $R \cos(x + \alpha)$, $R = 2\sqrt{3}$, $\alpha = \frac{\pi}{6}$,
 $2\sqrt{3} \cos\left(x + \frac{\pi}{6}\right)$

Q4

- (a) $5 \sin(x + 53^\circ 8')$ (b) $36^\circ 52'$

Q5

- (a) $\sqrt{13} \sin(x - 56^\circ 19')$ (b) $72^\circ 25'$, $220^\circ 13'$

Q6

$$\sqrt{3} \sin x + \cos x = 2 \sin\left(x + \frac{\pi}{6}\right)$$

- (a) $\frac{\pi}{3}$
- (b) $\frac{5\pi}{6}, \frac{11\pi}{6}$
- (c) $\pi, \frac{5\pi}{3}$
- Q7**
- (a) $0, \frac{2\pi}{3}, 2\pi$ (b) $\frac{\pi}{3}, \pi$ (c) $\frac{\pi}{3}, \frac{4\pi}{3}$
- (d) $0, \frac{\pi}{3}, 2\pi$ (e) $\frac{\pi}{2}, \pi$ (f) $\frac{7\pi}{12}, \frac{23\pi}{12}$
- (g) $\frac{19\pi}{12}, \frac{23\pi}{12}$ (h) $\pi, \frac{3\pi}{2}$ (i) $\frac{\pi}{2}, \frac{\pi}{6}$
- (j) $\pi, \frac{5\pi}{3}$ (k) $\frac{5\pi}{6}$ (l) $\frac{11\pi}{6}$

Q8

- (a) $f(x) = 2 \cos\left(x + \frac{\pi}{6}\right)$
- (b) Maximum is 2, minimum is -2.
- (c) See full worked solutions.

- (d) Maximum is 2 when $x = \frac{11\pi}{6}$.
 Minimum is -2 when $x = \frac{5\pi}{6}$.

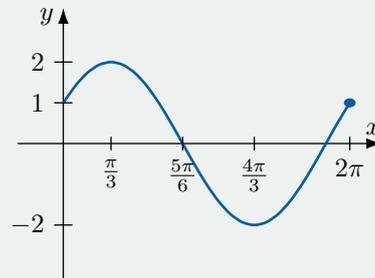
Q9

- (a) $R = \sqrt{2}, \sqrt{2}, -\sqrt{2}$ (b) $R = 5, 5, -5$
- (c) $R = 10, 10, -10$ (d) $R = 5, 7, -3$
- (e) $R = \sqrt{2}, 1 + \sqrt{2}$, (f) $R = 13, 11, -15$
 $1 - \sqrt{2}$

Q10

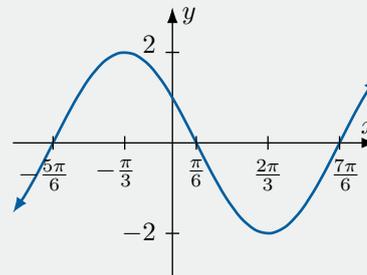
- (a) $f(x) = 2 \sin\left(x + \frac{\pi}{6}\right)$
- (b) Stretched vertically by a factor of 2,
 translated left by $\frac{\pi}{6}$

(c)

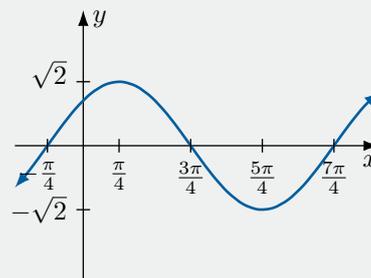


Q11

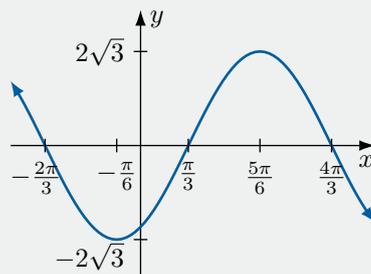
(a)



(b)

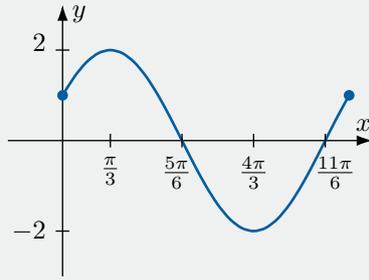


(c)



Q12

(a)



(b) $0, \frac{2\pi}{3}, 2\pi$

(c) $0 < x < \frac{2\pi}{3}$

Q13

1.93985, 4.03424, 6.12864

Q14

$f(x) \geq -1 + \sqrt{2}$ or $f(x) \leq -1 - \sqrt{2}$

P1

$-\sqrt{3} \leq f(\theta) \leq \sqrt{3}$. Expand and express as $\frac{3}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta$, then find the maximum and minimum as usual.

P2

$f(\theta) \geq 2(-1 + \sqrt{2})$ or $f(\theta) \leq -2(1 + \sqrt{2})$

Exercise 3E

Solving equations involving t-formulae

F1

(a) $\frac{2t}{1+t^2}$ (b) $\frac{1-t^2}{1+t^2}$ (c) $\frac{2t}{1-t^2}$

F2

polynomial

F3

(a) $t, 0$ (b) 0
 (c) $-\pi, \pi$ (d) π, π

Q1

- (a) See full worked solutions.
- (b) Solve the quadratic.
- (c) $x = 0, 126^\circ 52', 360^\circ$
- (d) Not a solution.
- (e) $x = 0, 126^\circ 52'$

Q2

- (a) $x = \frac{3\pi}{2}$
- (b) $x = \pi$ is a solution.
- (c) $x = \pi, \frac{3\pi}{2}$

Q3

- (a) $\pi, 5.36$ (b) $1.18, \pi$
- (c) $0, 2.4981, \pi$ (d) $1.85, \pi$
- (e) $0.28, \frac{3\pi}{2}$ (f) $\pi, 4.996$

P1

- (a) See full worked solutions.
- (b) Let $\Delta = 0$.
- (c) If $c = \sqrt{a^2 + b^2}$, then we are really just solving $R \sin(x + \alpha) = R$. This of course has double roots because geometrically, it corresponds to seeing where $y = R \sin(x + \alpha)$ intersects the line $y = R$. But since R is the amplitude of the sine curve, they intersect at the peaks only hence the double root.

Chapter Review

R1

- (a) $-\frac{\pi}{6} \pm \frac{\pi}{2}, \frac{5\pi}{6}$
- (b) $\pm\pi, \pm\frac{2\pi}{3}$
- (c) $0, \pm\pi, \pm\frac{2\pi}{3}$
- (d) $-\frac{11\pi}{12}, -\frac{7\pi}{12}, \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$
- (e) $\frac{\pi}{12}, \frac{5\pi}{12}, -\frac{7\pi}{12}, -\frac{11\pi}{12}, -\frac{\pi}{4}, \frac{3\pi}{4}$

R2

- (a) $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$
- (b) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
- (c) $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$
- (d) $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$
- (e) $0, \pi, 2\pi, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

R3

(a) $\frac{5\pi}{12}$ (b) $\frac{5\pi}{24}$

R4

(a) See full worked solutions.

(b) $\frac{\pi}{2}, \frac{3\pi}{2}$

R5

(a) 2, -2 (b) $2\sqrt{3}, -2\sqrt{3}$

R6

(a) $2\sin\left(2x - \frac{\pi}{6}\right)$ (b) $5\cos(x + 53^\circ 8')$

R7

(a) $15^\circ, 105^\circ$ (b) $22^\circ 21', 241^\circ 17'$

(c) $270^\circ, 330^\circ$

R8

(a) $53^\circ 8', 216^\circ 52'$

(b) $90^\circ, 343^\circ 44'$

(c) $46^\circ 24', 270^\circ$

(d) $30^\circ, 150^\circ, 90^\circ, 210^\circ, 330^\circ, 270^\circ$

R9

(a) $0, \frac{\pi}{2}, \pi$ (b) $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

(c) $\frac{\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}$ (d) $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

R10

(a) See full worked solutions.

(b) $x = \cos\left(\frac{\pi}{18}\right), \cos\left(\frac{11\pi}{18}\right), \cos\left(\frac{13\pi}{18}\right)$

(c) Use the product of roots.

(d) 9

R11

(a) See full worked solutions.

(b) $t = 1, \tan\left(\frac{7\pi}{12}\right), \tan\left(\frac{11\pi}{12}\right)$

(c) $t^3 + 3t^2 - 3t - 1 = (t - 1)(t^2 + 4t + 1)$

(d) $\tan\left(\frac{7\pi}{12}\right) = -2 - \sqrt{3},$

$\tan\left(\frac{11\pi}{12}\right) = -2 + \sqrt{3}$

4. Further Calculus

Exercise 4A

Differentiation of inverse trigonometric functions

F1

$\frac{dx}{dy}$

F2

(a) $\frac{1}{\sqrt{1-x^2}}$	(b) $-\frac{1}{\sqrt{1-x^2}}$
(c) $\frac{1}{1+x^2}$	(d) $\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
(e) $-\frac{f'(x)}{\sqrt{1-(f(x))^2}}$	(f) $\frac{f'(x)}{1+(f(x))^2}$

F3

(a) $\frac{a}{\sqrt{1-a^2x^2}}$	(b) $-\frac{a}{\sqrt{1-a^2x^2}}$
(c) $\frac{a}{1+(ax)^2}$	(d) $\frac{a}{\sqrt{1-(ax+b)^2}}$
(e) $-\frac{a}{\sqrt{1-(ax+b)^2}}$	(f) $\frac{a}{1+(ax+b)^2}$
(g) $\frac{a}{\sqrt{b^2-(ax)^2}}$	(h) $-\frac{a}{\sqrt{b^2-(ax)^2}}$
(i) $\frac{ab}{b^2+(ax)^2}$	

F4

(a) $-1 \leq x \leq 1, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(b) $-1 \leq x \leq 1, 0 \leq y \leq \pi$

(c) All real $x, -\frac{\pi}{2} < y < \frac{\pi}{2}$

(d) $-a \leq x \leq a, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(e) $-a \leq x \leq a, 0 \leq y \leq \pi$

(f) All real $x, -\frac{\pi}{2} < y < \frac{\pi}{2}$

Q1

- (a) $\frac{1}{\sqrt{4-x^2}}$ (b) $\frac{1}{\sqrt{36-x^2}}$
 (c) $\frac{1}{\sqrt{25-x^2}}$ (d) $\frac{2}{\sqrt{1-4x^2}}$
 (e) $\frac{4}{\sqrt{1-16x^2}}$ (f) $\frac{6}{\sqrt{1-36x^2}}$
 (g) $-\frac{1}{\sqrt{9-x^2}}$ (h) $-\frac{1}{\sqrt{64-x^2}}$
 (i) $-\frac{1}{\sqrt{16-x^2}}$ (j) $\frac{-3}{\sqrt{1-9x^2}}$
 (k) $\frac{-5}{\sqrt{1-25x^2}}$ (l) $\frac{-7}{\sqrt{1-49x^2}}$

Q2

- (a) $-2 < x < 2$ (b) $-6 < x < 6$
 (c) $-5 < x < 5$ (d) $-\frac{1}{2} < x < \frac{1}{2}$
 (e) $-\frac{1}{4} < x < \frac{1}{4}$ (f) $-\frac{1}{6} < x < \frac{1}{6}$
 (g) $-3 < x < 3$ (h) $-8 < x < 8$
 (i) $-4 < x < 4$ (j) $-\frac{1}{3} < x < \frac{1}{3}$
 (k) $-\frac{1}{5} < x < \frac{1}{5}$ (l) $-\frac{1}{7} < x < \frac{1}{7}$

Q3

- (a) $\frac{2}{1+4x^2}$ (b) $\frac{3}{1+9x^2}$ (c) $\frac{4}{1+16x^2}$
 (d) $\frac{2}{4+x^2}$ (e) $\frac{5}{25+x^2}$ (f) $\frac{10}{100+x^2}$

Q4

- (a) $\frac{3}{\sqrt{6x-9x^2}}$ (b) $\frac{1}{\sqrt{1-(6+x)^2}}$
 (c) $\frac{2}{\sqrt{1-(2x-5)^2}}$ (d) $-\frac{6}{\sqrt{1-(1-6x)^2}}$
 (e) $\frac{3}{\sqrt{4-9x^2}}$ (f) $\frac{2}{\sqrt{25-4x^2}}$

Q5

- (a) $-\frac{1}{\sqrt{2x-x^2}}$ (b) $-\frac{1}{\sqrt{1-(3+x)^2}}$
 (c) $-\frac{3}{\sqrt{1-(3x-1)^2}}$ (d) $\frac{1}{\sqrt{x-x^2}}$
 (e) $-\frac{2}{\sqrt{9-4x^2}}$ (f) $-\frac{5}{\sqrt{4-25x^2}}$

Q6

- (a) $\frac{1}{1+(x+1)^2}$ (b) $\frac{-1}{1+(6-x)^2}$
 (c) $\frac{2}{1+(2x-4)^2}$ (d) $\frac{-3}{1+(2-3x)^2}$
 (e) $\frac{6}{9+4x^2}$ (f) $\frac{20}{25+16x^2}$

Q7

- (a) $y = x$ (b) $y = \frac{\pi}{2} - x$
 (c) $y = \frac{x}{2} + \frac{\pi}{4} - \frac{1}{2}$ (d) $y = \frac{x}{\sqrt{2}} - \frac{\pi}{4} + 1$
 (e) $y = \pi - \frac{2x}{3}$ (f) $y = \frac{3x}{2} + \frac{3}{2}$
 (g) $y = 6x + \frac{\pi}{2}$ (h) $y = -\frac{4}{\sqrt{3}}x - \sqrt{3}$

Q8

- (a) $\frac{2x}{\sqrt{1-x^4}}$ (b) $-\frac{3x^2}{\sqrt{1-x^6}}$
 (c) $\frac{6x}{1+9x^4}$ (d) $-\frac{2}{\sqrt{2-x^2}}$
 (e) $\frac{2x}{\sqrt{x^2-x^4}}$ (f) $\frac{-2x}{x^4-2x^2+2}$

Q9

- (a) $\frac{1}{x\sqrt{x^2-1}}$ (b) $-\frac{2}{x\sqrt{x^2-4}}$
 (c) $-\frac{6}{x^2+36}$

Q10

- (a) $-\frac{1}{\sqrt{x}\sqrt{2-4x}}$ (b) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$
 (c) $\frac{1}{2(x-2)\sqrt{1-x}}$

Q11

- (a) $-\frac{e^x}{\sqrt{1-e^{2x}}}$ (b) $\frac{2\cos 2x}{1+\sin^2 2x}$
 (c) $\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ (d) $\frac{e^x}{1+e^{2x}}$
 (e) $\frac{\sqrt{e^x}}{2e^x+2}$ (f) $\frac{\sqrt{e^x}}{2\sqrt{1-e^x}}$
 (g) $\frac{1}{x\sqrt{1-\ln^2(x)}}$ (h) $\frac{1}{(\sin^{-1}x)\sqrt{1-x^2}}$
 (i) $\frac{1}{(\tan^{-1}x)(1+x^2)}$ (j) $\frac{1}{x(1+\ln^2(x))}$
 (k) $-\frac{\sin x}{(1+\cos^2 x)}$ (l) $-\frac{1}{\sqrt{1-x^2}}$

Q12

- (a) $\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$
 (b) $\tan^{-1} x + \frac{x}{1+x^2}$
 (c) $2x \cos^{-1} x - \frac{x^2}{\sqrt{1-x^2}}$
 (d) $2x \tan^{-1} 3x + \frac{3x^2}{1+9x^2}$
 (e) $e^x \sin^{-1} x + \frac{e^x}{\sqrt{1-x^2}}$
 (f) $2e^{2x} \tan^{-1} 3x + \frac{3e^{2x}}{1+9x^2}$

Q13

- (a) $\frac{1}{x\sqrt{1-x^2}} - \frac{\sin^{-1} x}{x^2}$
 (b) $\frac{-1}{x^2\sqrt{1-x^2}} - \frac{2\cos^{-1} x}{x^3}$
 (c) $\frac{1}{x(1+x^2)} - \frac{\tan^{-1} x}{x^2}$

Q14

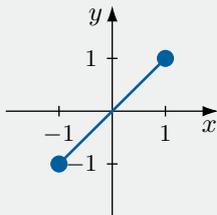
- (a) $\frac{2\sin^{-1} x}{\sqrt{1-x^2}}$ (b) $\frac{3(\tan^{-1} x)^2}{1+x^2}$
 (c) $\frac{\cos^{-1} x - \sin^{-1} x}{\sqrt{1-x^2}}$ (d) $2x \tan^{-1} x + 1$
 (e) $\tan^{-1} x + \frac{x+1}{1+x^2}$

Q15

- (a) Note that $\sin(\sin^{-1} x) = x$
 (b) Note that $\sqrt{\cos^2 x} = |\cos x|$ not just $\cos x$
 (c) Note that $\cos(\cos^{-1} x) = x$
 (d) Note that $\sqrt{\sin^2 x} = |\sin x|$ not just $\sin x$
 (e) Note that $\tan(\tan^{-1} x) = x$
 (f) Note that $\tan^{-1}(\tan x) = x$

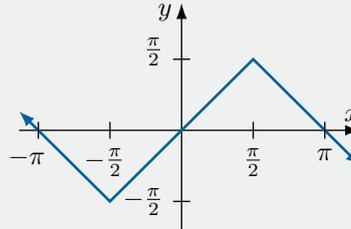
Q16

- (a) $-1 \leq x \leq 1, -1 \leq y \leq 1$
 (b) 1
 (c) $y = x$, but in the domain found
 (d)



Q17

- (a) All real $x, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
 (b) See full worked solutions.
 (c) $y' > 0$ when $\cos x > 0$
 $y' < 0$ when $\cos x < 0$
 (d) $y = \pm x$, depending on the domain.
 (e)



Q18

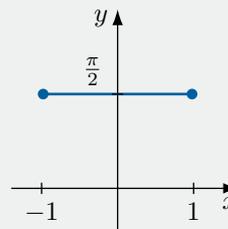
- (a) $-1 \leq x \leq 1$
 (b) $-\frac{x}{\sqrt{1-x^2}}, -1 < x < 1$
 (c) Yes, it was to be expected. This is because the curve really is just the semi-circle $y = \sqrt{1-x^2}$ expressed in a different way, and we know from the graph of the semi-circle that the derivative is only defined in the interval $-1 < x < 1$.

Q19

- (a) Both are equal to -1 .
 (b) See full worked solutions.

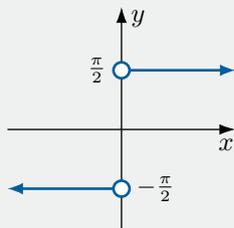
Q20

- (a) $f'(x) = 0$, so the graph is always horizontal.
 (b) $\frac{\pi}{2}$. Since the graph is horizontal and continuous, the graph has the same function value for the entirety of the graph.
 (c) $-1 \leq x \leq 1$
 (d)



Q21

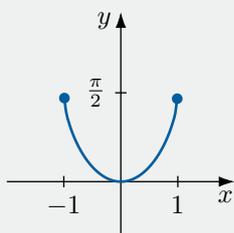
- (a) 0
- (b) All real x , but $x \neq 0$
- (c) $f(-x) = \tan^{-1}(-x) + \tan^{-1}\left(-\frac{1}{x}\right) = -\tan^{-1}(x) - \tan^{-1}\left(\frac{1}{x}\right) = -f(x)$
- (d)



Q22

- (a) $-1 \leq x \leq 1, 0 \leq y \leq \frac{\pi}{2}$
- (b) $f(-x) = (-x) \sin^{-1}(-x) = f(x)$
- (c) $y' = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$
- (d) $(0, 0)$, minimum
- (e) As $x \rightarrow -1^+$, $y' \rightarrow -\infty$ and as $x \rightarrow 1^-$, $y' \rightarrow \infty$

(f)



Q23

- (a) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$
- (b) See full worked solutions.

Q24

See full worked solutions.

Q25

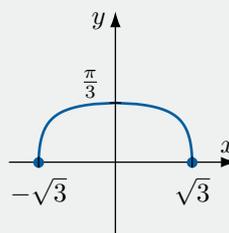
- (a) Substitute the point into both curves and show that it works for both.
- (b) Both curves have $\frac{dy}{dx} = -1$ at $x = 0$.
- (c) $y + x = \frac{\pi}{2}$

P1

- (a) $\frac{dy}{dx} = \sec x \tan x > 0$ for $0 \leq x < \frac{\pi}{2}$
- (b) $1 \leq y < \infty$ (in this domain)
- (c) See full worked solutions.
- (d) See full worked solutions.

P2

- (a) $-\sqrt{3} \leq x \leq \sqrt{3}$
- (b) $\frac{dy}{dx} = \frac{x}{(x^2 - 4)\sqrt{3 - x^2}}$
- (c) As $x \rightarrow \sqrt{3}$, $\frac{dy}{dx} \rightarrow -\infty$ and similarly as $x \rightarrow -\sqrt{3}$, $\frac{dy}{dx} \rightarrow \infty$
- (d)



Exercise 4B

Integration involving inverse trigonometric functions

F1

- (a) $\sin^{-1} x + C$
- (b) $\sin^{-1} \frac{x}{a} + C$
- (c) $\cos^{-1} \frac{x}{a} + C$
- (d) $\tan^{-1} x + C$
- (e) $\tan^{-1} \frac{x}{a} + C$
- (f) $\frac{1}{a} \tan^{-1} \frac{x}{a} + C$

F2

- (a) $\frac{1}{b} \sin^{-1} \frac{bx}{a} + C$
- (b) $\frac{1}{b} \cos^{-1} \frac{bx}{a} + C$
- (c) $\frac{1}{ab} \tan^{-1} \frac{bx}{a} + C$
- (d) $\sin^{-1}(f(x))$
- (e) $\cos^{-1}(f(x))$
- (f) $\tan^{-1}(f(x))$

Q1

- (a) $\sin^{-1}\left(\frac{x}{8}\right) + C$
- (b) $\sin^{-1}\left(\frac{x}{5}\right) + C$
- (c) $\cos^{-1}\left(\frac{x}{3}\right) + C$
- (d) $\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$
- (e) $\cos^{-1}\left(\frac{x}{2\sqrt{2}}\right) + C$
- (f) $\sin^{-1}(3x) + C$

Q2

- (a) $\frac{1}{3} \tan^{-1} \frac{x}{3} + C$ (b) $\tan^{-1} \frac{x}{5} + C$
 (c) $\tan^{-1} \frac{x}{4} + C$ (d) $\sqrt{3} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + C$
 (e) $2\sqrt{2} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right)$ (f) $4 \tan^{-1} (4x) + C$

Q3

- (a) $\frac{1}{4} \sin^{-1} 4x + C$ (b) $\frac{1}{\sqrt{6}} \sin^{-1} \sqrt{6}x + C$
 (c) $\frac{1}{2} \sin^{-1} \frac{2x}{5} + C$ (d) $\frac{1}{5} \sin^{-1} \frac{5x}{3} + C$
 (e) $\sin^{-1} (x-2) + C$ (f) $\frac{1}{2} \sin^{-1} \left(\frac{2x+1}{3} \right) + C$

Q4

- (a) $\frac{1}{3} \tan^{-1} 3x + C$ (b) $\tan^{-1} \frac{2x}{3} + C$
 (c) $\tan^{-1} (x-2) + C$ (d) $\tan^{-1} (2x-1) + C$

Q5

- (a) $\tan^{-1} (x+2) + C$ (b) $\frac{1}{2} \tan^{-1} \frac{x+2}{2} + C$
 (c) $\tan^{-1} (x+3) + C$ (d) $\tan^{-1} \frac{x-4}{2} + C$

Q6

- (a) $\sin^{-1} (x-3) + C$ (b) $\sin^{-1} \left(\frac{x-2}{2} \right) + C$
 (c) $\sin^{-1} (2x-1) + C$ (d) $\frac{1}{3} \sin^{-1} (3x-5) + C$

Q7

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi}{12}$
 (d) $\frac{\pi}{3\sqrt{3}}$ (e) $\frac{\pi}{2}$ (f) $\frac{\pi}{6\sqrt{3}}$

Q8

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$
 (d) $\frac{\pi}{18}$ (e) π (f) $\frac{\pi}{9}$

Q9

- (a) See full worked solutions.
 (b) See full worked solutions.
 (c) $\frac{\pi^2}{32}$
 (d) π
 (e) $\tan^{-1} \frac{1}{2}$

Q10

- (a) $\tan^{-1} (x^2) + C$
 (b) $\frac{1}{3} \tan^{-1} (x^3) + C$
 (c) $\frac{1}{12} \tan^{-1} \left(\frac{3x^2}{2} \right) + C$
 (d) $\tan^{-1} (e^x) + C$
 (e) $\frac{1}{2} \tan^{-1} (e^{2x}) + C$
 (f) $\frac{1}{20} \tan^{-1} \left(\frac{4e^x}{5} \right) + C$
 (g) $\frac{\pi}{4}$
 (h) $\frac{\pi}{4}$
 (i) $\tan^{-1} 2$

Q11

- (a) $\frac{1}{2} \sin^{-1} x^2 + C$ (b) $\frac{1}{3} \sin^{-1} x^3 + C$
 (c) $\frac{1}{6} \sin^{-1} \frac{3x^2}{4} + C$ (d) $\sin^{-1} e^x + C$
 (e) $\frac{1}{2} \sin^{-1} \frac{e^{2x}}{3} + C$ (f) $\frac{1}{5} \sin^{-1} \frac{5e^x}{4} + C$

Q12

Both integrals evaluate to zero because the integrand is an odd function.

Q13

- (a) $y = \sin^{-1} \left(\frac{x}{3} \right) + \frac{\pi}{4}$
 (b) $y = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right)$
 (c) $y = \sin^{-1} \left(\frac{x}{2} \right) - \frac{\pi}{2}$
 (d) $y = \cos^{-1} \left(\frac{2x}{5} \right) - \frac{\pi}{2}$

Q14

- (a) $\sin^{-1} x, \frac{\pi}{2} - 1$
- (b) $\cos^{-1} x - \frac{x}{\sqrt{1-x^2}}, \frac{\sqrt{3}\pi}{12} + \frac{1}{2}$
- (c) $\tan^{-1} x + \frac{x}{x^2+1}, \frac{\pi}{4} - \frac{1}{2} \ln 2$
- (d) $2x \tan^{-1} x + 1 - \frac{1}{1+x^2}, \frac{\pi}{4} - \frac{1}{2}$

Q15

- (a) $\cos^{-1} \left(\frac{x}{5} \right) + C$
- (b) $-\sin^{-1} \left(\frac{x}{5} \right) + C$
- (c) Their constants of integration are different. Using the identity $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$, either answer can be obtained from the other.

P1

$\sin^{-1} x + \sqrt{1-x^2} + C$

P2

One method is to draw $y = \cos^{-1} x$, shade the corresponding region and recognise that really it's just a rectangle when you re-arrange certain parts. Another would be to use the identity $\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$, to turn the integral into

$\int_{-1}^1 \frac{\pi}{2} - \sin^{-1}(x) dx$, giving a result of π .

P3

- (a) $\frac{1}{2} \left(\tan^{-1} x \right)^2 + C$
- (b) $\frac{1}{2} \left(\sin^{-1} x \right)^2 + C$
- (c) $\ln \left(\tan^{-1} x \right) + C$
- (d) $\ln \left(\sin^{-1} x \right) + C$

P4

- (a) $2 \tan^{-1}(\sqrt{x}) + C$
- (b) $\tan^{-1}(e^x) + C$

P5

- (a) See full worked solutions.
- (b) $-\sin^{-1}(e^{-x}) + C$

Exercise 4C

Review of integration techniques

F1

- (a) $\frac{1}{n+1} x^{n+1} + C$
- (b) $\frac{1}{a(n+1)} (ax+b)^{n+1} + C$
- (c) $\frac{1}{n+1} (f(x))^{n+1}$

F2

- (a) $\frac{1}{a} e^{ax+b} + C$
- (b) $e^{f(x)} + C$
- (c) $\frac{1}{\ln a} a^x + C$

F3

- (a) $-\frac{1}{a} \cos(ax+b) + C$
- (b) $\frac{1}{a} \sin(ax+b) + C$
- (c) $-\frac{1}{a} \ln |\cos(ax+b)| + C$
- (d) $-\cos(f(x)) + C$
- (e) $\sin(f(x)) + C$
- (f) $-\ln |\cos(f(x))| + C$

F4

- (a) $\ln|x| + C$
- (b) $\frac{1}{a} \ln|ax+b| + C$
- (c) $\ln|f(x)| + C$

F5

- (a) $\sin^{-1} \left(\frac{x}{a} \right)$
- (b) $\cos^{-1} \left(\frac{x}{a} \right)$
- (c) $\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$
- (d) $\sin^{-1}(f(x))$
- (e) $-\cos^{-1}(f(x))$
- (f) $\tan^{-1}(f(x))$

Q1

- (a) $\frac{2}{5}x^{\frac{5}{2}} + C$ (b) $-\frac{1}{2x^2} + C$
 (c) $-\frac{2}{\sqrt{x}} + C$ (d) $\frac{1}{15}(3x+1)^5 + C$
 (e) $\frac{2}{9}(3x+1)^{\frac{3}{2}} + C$ (f) $\frac{1}{3}\ln|3x+1| + C$
 (g) $-\frac{1}{3}\ln|1-3x| + C$ (h) $\frac{2}{3}\sqrt{3x+1} + C$
 (i) $\frac{1}{3}\sqrt{3x^2+1} + C$ (j) $-\frac{2}{3}\sqrt{1-3x} + C$
 (k) $3x + \ln|x| + C$ (l) $\frac{x^2}{2} + \frac{x}{3} + C$
 (m) $\frac{3x^2}{2} + \ln|x| + C$ (n) $3x - \frac{1}{x} + C$
 (o) $\frac{3x^4}{4} + \frac{x^3}{3} + C$ (p) $-\frac{1}{3(3x+1)} + C$
 (q) $\frac{1}{\sqrt{3}}\tan^{-1}(x\sqrt{3}) + C$ (r) $\frac{1}{6}\ln|3x^2+1| + C$
 (s) $\frac{1}{2}\ln|x^2+2x+5| + C$ (t) $\frac{5}{2}\ln|x^2+6x-5| + C$
 (u) $\frac{1}{3}\ln|3x+1| + C$

Q2

- (a) $\frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (b) $\frac{1}{2}\ln(x^2+4) + C$
 (c) $\frac{1}{2}\tan^{-1}2x + C$ (d) $-2\ln|x^2-4| + C$
 (e) $\sin^{-1}\frac{x}{2} + C$ (f) $-\sqrt{4-x^2} + C$
 (g) $-\frac{1}{3}(4-x^2)^{\frac{3}{2}} + C$ (h) $-\frac{2}{3}(x^2-x+5)^{\frac{3}{2}} + C$
 (i) $\frac{1}{18}(4x^3-1)^{\frac{3}{2}} + C$ (j) $\frac{1}{6}\tan^{-1}\frac{3x}{2} + C$
 (k) $\frac{1}{6}\ln(9x^2+4) + C$ (l) $-2\ln|x^2-6x+5| + C$
 (m) $\frac{\frac{2}{3}}{(x^2-6x+5)^3} + C$ (n) $\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C$
 (o) $\frac{1}{3}\sin^{-1}\left(\frac{3x+1}{2}\right) + C$

Q3

- (a) $\frac{e^{4x}}{4} + C$ (b) $\frac{1}{8}e^{4x}(e^{4x}+2) + C$
 (c) $-\frac{1}{4}e^{1-4x} + C$ (d) $\frac{1}{8}e^{4x^2} + C$
 (e) $-\frac{1}{2}e^{1-4x^2} + C$ (f) $x - \frac{e^{-4x}}{4} + C$

- (g) $\frac{1}{4}\ln|e^{4x}+1| + C$ (h) $\frac{1}{2}\tan^{-1}e^{2x} + C$
 (i) $\frac{1}{24}(e^{4x}+1)^6 + C$ (j) $\frac{1}{2}\sqrt{e^{4x}+1} + C$
 (k) $\frac{1}{\ln 2}2^x + C$ (l) $\frac{1}{2\ln 3}3^{2x} + C$
 (m) $-\frac{1}{4\ln 3}3^{2-4x} + C$ (n) $\frac{1}{2\ln 2}2^{x^2} + C$
 (o) $\frac{1}{4}(e^{4x}-8x-e^{-4x}) + C$

Q4

- (a) $-2\cos\frac{x}{2} + C$ (b) $\frac{1}{2}(x-\sin x) + C$
 (c) $2\sin\frac{x}{2} + C$ (d) $2\tan\frac{x}{2} + C$
 (e) $-2\ln\left|\cos\frac{x}{2}\right| + C$ (f) $2\tan\frac{x}{2} - x + C$
 (g) $\frac{2}{3}\tan^3\frac{x}{2} + C$ (h) $\frac{2}{3}\sin^3\frac{x}{2} + C$
 (i) $-\frac{2}{3}\cos^3\frac{x}{2} + C$

Q5

- (a) $-\frac{1}{2}\ln|\cos 2x| + C$ (b) $\frac{1}{2}\sec^2 x + C$
 (c) $\sec x + C$ (d) $\frac{1}{2}\tan^{-1}\frac{\sin x}{2} + C$
 (e) $\ln|4+\sin x| + C$ (f) $\ln|1-\cos x| + C$
 (g) $-2\sqrt{\cos x} + C$ (h) $-\frac{2}{3}\cos^{\frac{3}{2}} x + C$
 (i) $2\sqrt{3-\cos x} + C$ (j) $-\frac{1}{4}\cos^4 x + C$
 (k) $\frac{1}{4}\sin^4 x + C$ (l) $\frac{1}{4}\tan^4 x + C$
 (m) $-\frac{1}{2}\cos(x^2) + C$ (n) $2\sin(\sqrt{x}) + C$
 (o) $-\tan\left(\frac{1}{x}\right) + C$

Q6

- (a) $\sin 6x, -\frac{1}{6}\cos 6x + C$
 (b) $\frac{1}{2}\sin 4x, -\frac{1}{8}\cos 4x + C$
 (c) $\frac{1}{2}\sin x, -\frac{1}{2}\cos x + C$
 (d) $\cos 8x, \frac{1}{8}\sin 8x + C$
 (e) $\cos 10x, \frac{1}{10}\sin 10x + C$

- (f) $\cos x, \sin x + C$
- (g) $\frac{1}{2}(\cos 6x + 1), \frac{1}{12}(6x + \sin 6x) + C$
- (h) $\frac{1}{2}(1 - \cos 4x), \frac{1}{8}(4x - \sin 4x) + C$
- (i) $\frac{1}{2}(\cos 3x + 1), \frac{1}{6}(3x + \sin 3x) + C$
- (j) $\tan x, -\ln|\cos x| + C$
- (k) $\tan 6x, -\frac{1}{6}\ln|\cos 6x| + C$
- (l) $\sin 2x, -\frac{1}{2}\cos 2x + C$

Q7

- (a) $\sin 5x + \sin 3x, -\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x + C$
- (b) $\sin 8x - \sin 2x, \frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x + C$
- (c) $\frac{1}{2}(\cos 2x - \cos 6x), \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + C$
- (d) $\frac{1}{2}(\cos 3x + \cos 7x), \frac{1}{6}\sin 3x + \frac{1}{14}\sin 7x + C$
- (e) $\frac{1}{2}(\sin 6x - \sin x), -\frac{1}{12}\cos 6x + \frac{1}{2}\cos x + C$
- (f) $\frac{1}{2}(\cos 6x + \cos 3x), \frac{1}{6}\sin 3x + \frac{1}{12}\sin 6x + C$

Q8

- (a) $\frac{1}{5}$ (b) $\frac{1}{12}$ (c) $\frac{3}{10}$
- (d) $\frac{1}{2} + \frac{1}{\sqrt{2}}$ (e) $\frac{3\sqrt{3}}{16}$ (f) $\frac{1}{4}(2 + \pi)$
- (g) 0 (h) 0

Q9

- (a) $\frac{1}{2}\ln\frac{9}{5}$ (b) $1 - \ln 2$
- (c) $\ln\left|1 + \frac{2\sqrt{2}}{3}\right|$

Q10

See full worked solutions.

Q11

- (a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) $\frac{13}{15} - \frac{\pi}{4}$

P1

- (a) $-\ln|\cos x - \sin x| + C$
- (b) $\sin^{-1}\left(\frac{\sin x}{2}\right) + C$
- (c) $\frac{1}{2}\tan x + C$
- (d) $-\ln(1 + e^{-x}) + C$
- (e) $x + 2\ln(1 + e^{-x}) + C$
- (f) $e^x - x - \ln(1 + e^{-x}) + C$

P2

- (a) $2x - \ln|x + 1| + C$
- (b) $\frac{x}{2} + \frac{1}{8}\ln|4x + 1| + C$
- (c) $x + \frac{1}{2}\ln(x^2 + 1) - \tan^{-1}x + C$

Exercise 4D

Integration of squares of sine and cos

F1

- (a) $\cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- (b) $\frac{1}{2}(1 + \cos 2x)$
- (c) $\frac{1}{2}(1 - \cos 2x)$

F2

- (a) $\frac{1}{2}(1 - \cos 2x), \frac{1}{2}x - \frac{1}{4}\sin 2x$
- (b) $\frac{1}{2}(1 + \cos 2x), \frac{1}{2}x + \frac{1}{4}\sin 2x$
- (c) $\frac{1}{2}(1 - \cos 2nx), \frac{1}{2}x - \frac{1}{4n}\sin 2nx$
- (d) $\frac{1}{2}(1 + \cos 2nx), \frac{1}{2}x + \frac{1}{4n}\sin 2nx$

Q1

- (a) $\frac{x}{2} - \frac{1}{8}\sin 4x + C$ (b) $\frac{x}{2} - \frac{1}{24}\sin 12x + C$
- (c) $\frac{x}{2} - \frac{1}{4\pi}\sin 2\pi x + C$ (d) $\frac{x}{2} - \frac{3}{4}\sin\frac{2x}{3} + C$
- (e) $4x - 3\sin\frac{4x}{3} + C$ (f) $\frac{x}{2} - \frac{1}{8}\sin(4x - 6) + C$

274 Answers

Q2

$$(a) \frac{x}{2} + \frac{1}{8} \sin 4x + C \quad (b) \frac{x}{2} + \frac{1}{32} \sin 16x + C$$

$$(c) \frac{x}{2} + \frac{1}{4\pi} \sin 2\pi x + C \quad (d) 5x + \frac{1}{4} \sin 20x + C$$

$$(e) 2x + 2 \sin x + C \quad (f) \frac{x}{2} + \frac{1}{2\pi} \sin(\pi x) + C$$

Q3

$$(a) \frac{\pi}{4} \quad (b) \frac{1}{2} \quad (c) \frac{\pi}{2}$$

$$(d) \frac{\pi}{2} \quad (e) \frac{\pi}{3} - 1 \quad (f) \frac{\pi + 2\sqrt{2}}{16}$$

$$(g) \frac{\pi}{4} \quad (h) \frac{\pi}{2} \quad (i) 3\pi$$

Q4

$$(a) \frac{4}{3} + \frac{\pi}{2} \quad (b) \frac{3\pi}{8} \quad (c) \frac{\pi}{2}$$

Q5

0

Q6

See full worked solutions.

P1

(a) See full worked solutions.

$$(b) \frac{\pi}{16}$$

P2

$$(a) \frac{1}{32}(4x - \sin 4x) + C \quad (b) \frac{1}{16} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right)$$

$$(c) x - \frac{1}{2} \sin 2x + C \quad (d) \frac{1}{32}(\pi - 2)$$

P3

See full worked solutions.

Exercise 4E

Integration by substitution

F1

substitution

F2

$$(a) u, x \quad (b) \int g(u) du \quad (c) \text{convert}$$

F3

(a) definite, limits

(b) u_1

$$(c) \int_{x_1}^{x_2} f(x) dx = \int_{u_1}^{u_2} g(u) du, x_1, x_2$$

F4

reverse, chain

Q1

$$(a) \quad \frac{du}{dx} = 2x \\ du = 2x dx \\ \therefore x dx = \frac{1}{2} du$$

$$(b) \quad \int x(x^2 + 1)^4 dx = \frac{1}{2} \int u^4 du \\ = \frac{1}{2} \times \frac{1}{5} u^5 + C \\ = \frac{1}{10} (x^2 + 1)^5 + C$$

Q2

(a)

$$\frac{du}{dx} = 6x \\ du = 6x dx \\ \therefore x dx = \frac{1}{6} du$$

(b)

$$\int \frac{x}{\sqrt{3x^2 - 5}} dx = \frac{1}{6} \int \frac{du}{\sqrt{u}} \\ = \frac{1}{6} \times 2\sqrt{u} + C \\ = \frac{1}{3} \sqrt{3x^2 - 5} + C$$

Q3

$$(a) \frac{(x^2 + 4)^4}{4} + C \quad (b) \frac{(x^3 + 1)^4}{12} + C$$

$$(c) -\frac{(4 - 3x^2)^6}{36} + C \quad (d) \frac{-1}{2(x^2 + 1)} + C$$

$$(e) \frac{1}{12(1 - 2x^3)^2} + C \quad (f) \frac{-1}{3(x^2 - x)^3} + C$$

Q4

- (a) $\frac{1}{12}(x^2 - 2x + 2)^6 + C$
 (b) $-\frac{1}{10}(x^2 - 2x + 2)^5 + C$
 (c) $\frac{-1}{6(x^2 - 4x + 2)^3} + C$
 (d) $\frac{1}{3(x^2 - 3x + 5)^3} + C$

Q5

- (a) $\frac{1}{7}(x+4)^7 - \frac{2}{3}(x+4)^6 + C$
 (b) $\frac{1}{11}(x-1)^{11} + \frac{1}{5}(x-1)^{10} + \frac{1}{9}(x-1)^9 + C$
 (c) $\frac{1}{5}(4-x)^5 - (4-x)^4 + C$
 (d) $x - 6 \ln|x+6| + C$
 (e) $\frac{1}{4} \left(\ln|2x-1| - \frac{1}{2x-1} \right) + C$
 (f) $\frac{1}{(x+2)^2} - \frac{1}{x+2} + C$

Q6

- (a) $\frac{1}{3}(x^2 - 4)^{\frac{3}{2}} + C$
 (b) $\frac{1}{18}(1 + 4x^3)^{\frac{3}{2}} + C$
 (c) $\frac{1}{2}\sqrt{x^4 + 2} + C$
 (d) $\frac{1}{3}\sqrt{1 + 3x^2} + C$
 (e) $-\frac{2}{3}\sqrt{1 - x^3} + C$
 (f) $(x^2 + 2x - 6)^{\frac{1}{2}} + C$
 (g) $\frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + C$
 (h) $\frac{1}{12}(2x-1)\sqrt{4x+1} + C$
 (i) $\frac{2}{3}(x-2)\sqrt{x+1} + C$
 (j) $\frac{2}{\sqrt{x+1}} + 2\sqrt{x+1} + C$

Q7

- (a) $\frac{1}{2}e^{x^2} + C$ (b) $\frac{1}{3}e^{x^3} + C$
 (c) $2e^{\sqrt{x}} + C$ (d) $-e^{\frac{1}{x}} + C$
 (e) $\frac{2}{3}(e^x - 1)^{\frac{3}{2}} + C$ (f) $2\sqrt{e^x + 1} + C$
 (g) $\ln|1 + e^x| + C$ (h) $\frac{1}{2}\ln|e^{2x} + 1| + C$

Q8

- (a) $\frac{1}{2}(\ln|x|)^2 + C$ (b) $\frac{1}{4}(\ln|x|)^4 + C$
 (c) $\frac{2}{3}(\ln|x|)^{\frac{3}{2}} + C$ (d) $\ln|\ln|x|| + C$

Q9

- (a) $\frac{\sin^2 x}{2} + C$ (b) $\frac{-\cos^2 x}{2} + C$
 (c) $-\frac{1}{6}\cos^6 x + C$ (d) $\frac{1}{2}\tan^2 x + C$
 (e) $\frac{1}{7}\tan^7 x + C$ (f) $\ln|\sec x| + C$
 (g) $-\ln|5 + \cos x| + C$ (h) $e^{\sin x} + C$
 (i) $-\frac{1}{2\sin^2 x} + C$ (j) $-2\sqrt{\cos x} + C$

Q10

- (a) $\frac{1}{2}\sin^{-1}\left(\frac{2x-1}{3}\right) + C$ (b) $\frac{1}{6}\tan^{-1}\left(\frac{3x+2}{2}\right) + C$
 (c) $\frac{1}{2}\tan^{-1}(x^2) + C$ (d) $\frac{1}{2}\sin^{-1}(x^2) + C$
 (e) $\sin^{-1}(\ln x) + C$ (f) $\sin^{-1}\left(\frac{e^x}{2}\right) + C$
 (g) $\tan^{-1}(e^x) + C$ (h) $2\tan^{-1}\sqrt{x} + C$

Q11

- (a) $\frac{du}{dx} = 2x$
 $du = 2x dx$
 $\therefore x dx = \frac{1}{2}du$

- (b) $x = 3 \rightarrow u = 25$
 $x = 0 \rightarrow u = 16$

(c)

$$\begin{aligned} \int_0^3 \frac{x}{\sqrt{x^2+16}} dx &= \int_{16}^{25} \frac{\frac{1}{2}}{\sqrt{u}} du \\ &= \frac{1}{2} \int_{16}^{25} \frac{1}{\sqrt{u}} du \\ &= [\sqrt{u}]_{16}^{25} \\ &= 1 \end{aligned}$$

Q12

- (a) $\frac{171}{5}$ (b) $\frac{1}{3} \ln 7 - \frac{2}{7}$ (c) $\frac{7}{9}$
 (d) $\frac{128}{15}$ (e) $\frac{76}{3}$ (f) $-\frac{10}{3}$

Q13

See full worked solutions.

Q14

- (a) $6 - 2\sqrt{2}$ (b) $\frac{e^4 - e^{-4}}{4}$

Q15

- (a) 0 (b) $\frac{\ln 3}{2}$ (c) $\ln 2$
 (d) $e - 1$ (e) $\frac{9\sqrt{3}}{5}$ (f) $\frac{1}{12\sqrt{2}}$

Q16

- (a) $\frac{\pi^2}{32}$ (b) $\frac{\pi^2}{8}$

Q17

- (a) $-\frac{1}{2} \cos^2 x + C_1$
 (b) $\frac{1}{2} \sin^2 x + C_2$
 (c) $-\frac{1}{4} \cos(2x) + C_3$
 (d) They all differ by a constant. Using trigonometric identities, you can obtain any form from the other.

Q18

- (a) See full worked solutions.
 (b) See full worked solutions.
 (c) Equating the previous two results, we have
 $2 \sin^{-1} \left(\sqrt{\frac{x}{2}} \right) - \sin^{-1}(x-1) = C_1 - C_2 = k$
 for some constant k . The domain is
 $0 < x < 2$.
 (d) Substitute $x = 1$ to get $f(1) = \frac{\pi}{2}$

Q19

- (a) $1 - \frac{\sqrt{3}}{2}$ (b) $2 - \sqrt{2}$ (c) $\sqrt{3} - \frac{\pi}{3}$
 (d) $\frac{\sqrt{3}}{2}$ (e) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (f) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

Q20

- (a) $\frac{\pi r^2}{4}$
 (b) The integral represents the region in the first quadrant bounded by a semi-circle of radius r centred at the origin, and the x -axis. So the area can be found using a quarter circle.

Q21

- (a) See full worked solutions.
 (b) $\sqrt{1-x^2}$
 (c) See full worked solutions.

Q22

- (a) $-\sqrt{1-x^2} + C$
 (b) $-\frac{\sqrt{1-x^2}}{x} + C$
 (c) $\sqrt{1+x^2} + C$
 (d) $\frac{x}{\sqrt{1+x^2}} + C$

P1

- (a) $e^x - \ln|1+e^x| + C$
 (b) $\ln(e^{-x}+1) - e^{-x} + C$

P2

$$2 \ln(1 + \sqrt{x}) + C$$

P3

$$2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C$$

P4

$$-\sin^{-1}\left(\frac{1}{x}\right) + C$$

P5

$$-2\sqrt{\frac{1+x}{x}} + C$$

P6

See full worked solutions.

Chapter Review

R1

(a) $\frac{1}{\sqrt{16-x^2}}$ (b) $-\frac{1}{\sqrt{4-x^2}}$

(c) $\frac{6}{36+x^2}$ (d) $\frac{6}{4+9x^2}$

(e) $-\frac{2}{\sqrt{1-4x^2}}$ (f) $\frac{3}{\sqrt{16-9x^2}}$

R2

(a) $\frac{1}{2\sqrt{x(1-x)}}$ (b) $\frac{1}{2\sqrt{x(x+1)}}$

(c) $\frac{2x}{1+x^4}$ (d) $\frac{2x}{\sqrt{1-x^4}}$

(e) $-\frac{2}{4+x^2}$

R3

(a) $-\frac{6(\cos^{-1} 2x)^2}{\sqrt{1-4x^2}}$ (b) $\frac{2 \tan^{-1} x}{1+x^2}$

(c) $\frac{e^x}{\sqrt{1-e^{2x}}}$ (d) $\frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$

(e) $-\frac{1}{(\cos^{-1} x)\sqrt{1-x^2}}$ (f) $-\frac{1}{x\sqrt{1-\ln^2|x|}}$

R4

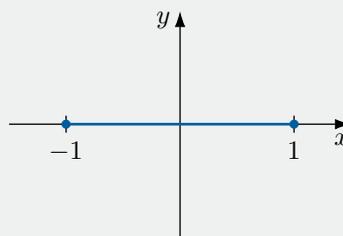
(a) $-\frac{x}{(1+x^2)^{\frac{3}{2}}}$ (b) $\frac{1}{(1-x^2)^{\frac{3}{2}}}$

R5

(a) $y = x - \frac{\pi}{3} + \sqrt{3}$

(b) $y = -\frac{\sqrt{3}x}{2} + \pi + \sqrt{3}$

R6



R7

(a) $-1 \leq x \leq 1$

(b) $-\frac{x}{\sqrt{1-x^2}}$

(c) Let $\alpha = \sin^{-1} x$ so $x = \sin \alpha$. The required curve is just $y = \cos \alpha$.

R8

See full worked solutions.

R9

(a) $\frac{1}{4} \tan^{-1} \frac{x}{4} + C$

(b) $\frac{1}{2\sqrt{2}} \tan^{-1} \frac{x}{2\sqrt{2}} + C$

(c) $\frac{3}{7} \tan^{-1} \frac{x}{7} + C$

(d) $\frac{1}{2} \tan^{-1} 2x + C$

(e) $\frac{1}{7} \tan^{-1} \frac{2x}{7} + C$

(f) $\frac{1}{\sqrt{21}} \tan^{-1} \frac{\sqrt{3}x}{\sqrt{7}} + C$

R10

(a) $\sin^{-1} \frac{x}{10} + C$

(b) $\sin^{-1} \frac{x}{8} + C$

(c) $\cos^{-1} \frac{x}{\sqrt{10}} + C$

(d) $\frac{1}{\sqrt{10}} \sin^{-1}(x\sqrt{10}) + C$

(e) $\frac{1}{5} \cos^{-1} \frac{5x}{2} + C$

(f) $\frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x\sqrt{2}}{3} \right) + C$

R11

(a) 0 (b) 0 (c) π (d) $\frac{\pi}{2}$

R12

- (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$
 (d) $\frac{\pi}{2}$ (e) $\frac{\pi}{8}$ (f) $\frac{\pi}{12}$

R13

- (a) $y = \tan^{-1} \frac{x}{2} + \frac{\pi}{4}$
 (b) $y = \cos^{-1} \frac{3x}{4} - \pi$

R14

- (a) $\frac{3x^2}{\sqrt{4-x^6}}, \frac{1}{3} \sin^{-1} \frac{x^3}{2} + C$
 (b) $\frac{6x^2}{4+x^6}, \frac{1}{6} \tan^{-1} \frac{x^3}{2} + C$

R15

- (a) $1 - \frac{2x \sin^{-1} x}{2\sqrt{1-x^2}}, \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$
 (b) $-\frac{e^x}{\sqrt{1-e^{2x}}}, -\frac{\pi}{12}$
 (c) $4 \tan^{-1} \frac{x}{2} \times \frac{1}{4+x^2}, \frac{\pi^2}{64}$

R16

- (a) $\frac{1}{2-4x} + C$ (b) $-\frac{1}{2} \ln |1-2x| + C$
 (c) $-2\sqrt{2-x} + C$ (d) $2 \ln(x^2+4) + C$
 (e) $2 \tan^{-1} \frac{x}{2} + C$ (f) $\sqrt{8+9x^2} + C$

R17

- (a) $\frac{x}{2} - \frac{1}{4n} \sin 2nx + C$ (b) $\frac{x}{2} + \frac{1}{4n} \sin 2nx + C$
 (c) $\frac{1}{n} \tan nx + C$ (d) $\frac{1}{n} \tan nx - x + C$
 (e) $-\frac{1}{n} \cot nx + C$ (f) $-\frac{1}{n} \cot nx - x + C$

R18

- (a) $\frac{1}{12}(8 \cos 3x - \sin 6x + 18x) + C$
 (b) $\frac{x}{2} - \frac{1}{24} \sin 12x + C$
 (c) $\frac{x}{2} + \sin \frac{x}{2} + C$

R19

- (a) $\frac{1}{18}(3x+1)^6 + C$
 (b) $\frac{1}{8}(x^2+5)^4 + C$
 (c) $-\frac{1}{6(x^3-5)^2} + C$
 (d) $\sqrt{x^2-4} + C$
 (e) $\frac{1}{2}\sqrt{x^4-4} + C$
 (f) $-\sqrt{1-x^2} + C$
 (g) $\frac{1}{2}x^2 - 2x + 4 \ln|x+2| + C$
 (h) $\frac{2}{5}(3-x)^{\frac{5}{2}} - 2(3-x)^{\frac{3}{2}} + C$

R20

- (a) $\sin 4x - \sin 2x, \frac{1}{2} \cos 2x - \frac{1}{4} \cos 4x + C$
 (b) $\sin 6x + \sin 4x, -\frac{1}{6} \cos 6x - \frac{1}{4} \cos 4x + C$
 (c) $\sin 3x + \sin x, -\cos x - \frac{1}{3} \cos 3x + C$

R21

See full worked solutions.

R22

- (a) $\frac{1}{4} \sin^4 x + C$
 (b) $\frac{1}{6} \sin^3 2x + C$
 (c) $\ln |\tan x| + C$
 (d) $-\frac{1}{3} \ln |1 + \cos 3x| + C$
 (e) $-\frac{1}{3} \tan^{-1}(\cos 3x) + C$
 (f) $\frac{1}{2} \tan^2 x + C$ or $\frac{1}{2} \sec^2 x + C_1$

R23

- (a) $\frac{1}{3}$
 (b) 98
 (c) $\tan^{-1} e - \frac{\pi}{4}$
 (d) $\frac{1}{3} \ln \frac{e^3+1}{2}$

(e) $2(e^2 - 1)$

(f) $\frac{1}{3}$

(g) $\tan^{-1} \sqrt{2}$

(h) $\ln 2$

R24

See full worked solutions.

R25

See full worked solutions.

R26

(a) $\frac{\pi}{4}$ (b) π

R27

$y = 2 \tan^{-1} e^{2x}$

5. Volumes of Revolution

Exercise 5A

Areas involving x and y-axes

F1

region, compound, negative

F2

(a) $\int_c^d x \, dy = \int_c^d g(y) \, dy$

(b) positive, negative

(c) x

Q1

(a) $A = \int_{y_1}^{y_2} f(y) \, dy$

(b) y^2

(c) When $x = 4$, $y = 2$ and when $x = 1$, $y = 1$.

(d)

$$\begin{aligned} \text{Area} &= \int_1^2 y^2 \, dy \\ &= \left[\frac{1}{3} y^3 \right]_1^2 \\ &= \frac{7}{3} \end{aligned}$$

Q2

(a) $\frac{8}{3}$ (b) 2 (c) 1

(d) 12 (e) 12 (f) $\frac{5}{3}$

(g) $e^2 - e$ (h) $\frac{3}{4}$

Q3

(a) 16 (b) 16

Q4

(a) $\frac{128}{3}$

(b) $\frac{64}{3}$

(c) Total area is 64. It is the area of the rectangle enclosed by $0 \leq x \leq 16$ and $0 \leq y \leq 4$.

Q5

$\frac{8}{3}$

Q6

(a) $A = \int_0^8 y^{\frac{1}{3}} \, dy = 12$

(b) $A = 16 - \int_0^2 x^3 \, dx = 12$

Q7

$16 - \int_0^2 (8x - 2x^2) \, dx = \frac{16}{3}$

Q8

Answer to the above is -16 because it is the signed area on the left of the y -axis, so it is expected to be negative. The actual area is just the positive version of this, so 16.

Q9

- (a) $\frac{45}{4}$ (b) $\frac{45}{4}$ (c) 24

Q10

- (a) $\frac{32}{3}$ (b) $\frac{1}{6}$ (c) $\frac{32}{3}$ (d) $\frac{9}{2}$

Q11

- (a) $3 \ln 3 - 2$
 (b) $4 \ln 2 - 2$
 (c) 1
 (d) $\frac{\pi}{4} - \frac{1}{2} \ln 2$

Q12

- (a) $|\mathcal{A}| = \int_1^4 \ln x \, dx$
 (b) $|\mathcal{B}| = \int_0^{1 \ln 4} e^y \, dy = 3$
 (c) $4 \ln 4 = 8 \ln 2$
 (d) $|\mathcal{A}| = A_{\text{rectangle}} - |\mathcal{B}| = 8 \ln 2 - 3$

Q13

- (a) 1 (b) $\pi - 2$
 (c) $3 \ln 3 - 2$ (d) $\frac{\pi}{4} - \frac{1}{2} \ln 2$
 (e) $3\pi - 6$ (f) $\pi - 1$

Q14

- (a) $\frac{\pi}{2} - 1$ (b) 1
 (c) $\frac{\pi}{4} - \frac{1}{2} \ln 2$ (d) $2 \ln 2 - 1$

Q15

- (a) (0, 0), (7, 7) (b) $\frac{169}{6} = 28\frac{1}{6}$
 (c) $\frac{343}{6} = 57\frac{1}{6}$

Q16

$$A = \int_{-5}^{-2} (-y^2 - 7y - 10) \, dy = \frac{9}{2}$$

P1

$$\frac{4}{\pi} - 1$$

P2

- (a) $n = 2$ (b) $n = 3$ (c) $n = \frac{p}{q}$

P3

See full worked solutions.

P4

See full worked solutions.

Exercise 5B

Volumes of revolution (x-axis)

F1

$$\pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$$

F2

$$V = \pi ab^2 - \pi \int_0^a [f(x)]^2 \, dx$$

Q1

- (a) Cylinder, 36π
 (b) Cylinder, 48π
 (c) Cone, 81π
 (d) Cylinder with a cone removed from inside it, 54π
 (e) Cylinder with a cylinder removed from inside it, 30π

Q2

- (a) $\frac{8\pi}{3}$ (b) $\frac{\pi}{5}$ (c) $\frac{1296\pi}{5}$
 (d) $\frac{\pi}{30}$ (e) 8π (f) $\frac{\pi}{2}$
 (g) $2\pi \ln 2$ (h) $\frac{2\pi}{21}$ (i) $\frac{32\pi}{15}$
 (j) $\frac{1024\pi}{105}$

Q3

- (a) $\frac{\pi^2}{24}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi^2}{3}$ (d) $\frac{\pi^2}{12}$

Q4

- (a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^2}{12}$
 (c) $\frac{9\pi^2}{4}$ (d) $16\pi^2$
 (e) $\pi - \frac{\pi^2}{4}$ (f) $4\pi \left(\frac{1}{\sqrt{3}} - \frac{\pi}{9} \right)$

Q5

- (a) $\pi - \frac{\pi}{e^2}$ (b) $\frac{15\pi}{4}$

Q6

- (a) $A = \frac{1}{6} \ln 2$ represents the area of the shaded region.
 (b) $V = \frac{\pi}{12}(4 - \pi)$ represents the volume if the shaded region were rotated about the x -axis.

Q7

72π

Q8

$$\frac{\pi}{2} \left(\frac{1}{2}e^4 + 4 - \frac{1}{2}e^{-4} \right)$$

Q9

See full worked solutions.

Q10

- (a) x_0 is the height of the cone i.e. h .
 y_0 is the radius of the cone i.e. r . So
 $(x_0, y_0) = (h, r)$.
 (b) $k = \frac{r}{h}$
 (c) See full worked solutions.

Q11

See full worked solutions.

Q12

- (a) See full worked solutions.
 (b) $V = \pi \int_{-a}^a \frac{b^2}{a^2}(a^2 - x^2) dx$
 (c) The sphere is a special case of the ellipsoid when $a = b = r$, so $V = \frac{4}{3}\pi(r)(r)^2 = \frac{4}{3}\pi r^3$

Q13

Show that $\pi \int_{\frac{1}{a}}^1 f^2(x) dx = \pi \int_1^a f^2(x) dx = 4\pi \ln a$

Q14

Both solids have volume $\frac{\pi^2}{4}$. Both methods actually produce the exact same solid except they are oriented differently.

Q15

- (a) $V = \pi \int_0^1 16 - 16x^2 dx = \frac{32\pi}{3}$
 (b) $V = \pi \int_0^4 \left(1 - \frac{y^2}{16}\right) dy = \frac{8\pi}{3}$

Q16

$$\frac{16\pi}{5}$$

Q17

- (a) $\pi(3 - 4 \ln 2)$ (b) π

P1

$$k = \sqrt[5]{2}$$

P2

- (a) See full worked solutions.
 (b) We are given $a > 1$. From the previous part, $2 - a > 0$ since b is positive, hence $a < 2$.
 Combining these inequalities yields
 $1 < a < 2$.

Exercise 5C

Volumes of revolution

F1

$$\pi \int_c^d x^2 dy = \pi \int_c^d [g(y)]^2 dy$$

Q1

- (a) 20π (b) $\frac{206\pi}{15}$
 (c) $\frac{4\pi}{3}$ (d) 8π
 (e) $\frac{\pi}{2}$ (f) $\frac{31\pi}{5}$
 (g) $\frac{\pi}{2}e^2(e^2 - 1)$ (h) $\frac{21\pi}{64}$

Q2

$$V = \pi \int_0^9 (\sqrt{y} + 3)^2 dy = \frac{459\pi}{2}$$

Q3

$$V = \pi \int_0^6 \left(4 - \frac{y^2}{9}\right)^2 dy = \frac{256\pi}{5}$$

Q4

$$V = \pi \int_0^2 (2y - y^2)^2 dy = \frac{16\pi}{15}$$

Q5

- (a) $V = \frac{32\pi}{3}$
 (b) $V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(4)^2(2) = \frac{32\pi}{3}$

Q6

- (a) Substitute $x = 4$ into both curves and both result in $y = -3$. Do not attempt to solve simultaneously here.
- (b) $\frac{128\pi}{3}$
- (c) 25.6%

Q7

- (a) $\frac{2000\pi}{3}$
- (b) $V_{\text{water}} = \frac{625\pi}{3}$, so 31.25%

Q8

$$\frac{1792\pi}{3}$$

Q9

$$\frac{128}{3}\pi$$

Q10

- (a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi^2}{4}$
- (d) $\frac{3\pi^2}{4}$ (e) $\pi - \frac{\pi^2}{4}$ (f) $\frac{\pi^2}{2} - \pi$

Q11

- (a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^2}{3}$ (c) $\frac{\pi^2}{4} - \frac{\pi}{2}$

Q12

$$1280(8e + e^2 - 1)\pi \text{ cm}^3$$

Q13

- (a) $\frac{16\pi}{5}$ (b) 2π (c) 2π (d) $\frac{64\pi}{5}$

P1

- (a) $x = 4 + \sqrt{4 - y^2}$ and $x = 4 - \sqrt{4 - y^2}$
- (b) See full worked solutions.
- (c) $32\pi^2$

P2

- (a) $V = \pi \int_4^5 25 - y^2 dy = \frac{14\pi}{3}$
- (b) $V = \pi \int_{-5}^4 25 - y^2 dy = 162\pi$
- (c) $V = \frac{500\pi}{3}$

P3

- (a) $V_{\text{cap}} = \int_{r-h}^r r^2 - y^2 dy$
- (b) $h = r$

Exercise 5D

Volumes involving two curves

F1

- (a) $V = \pi \int_a^b [f(x)^2 - g(x)^2] dx$
- (b) $V = \pi \int_a^b [p(y)^2 - q(y)^2] dy$

F2

$$V = \int_0^a [f(x)]^2 dx + \int_a^b [g(x)]^2 dx$$

Q1

- (a) (0, 0), (4, 2)
- (b) $\frac{4}{3}$
- (c) (i) $\frac{8\pi}{3}$ (ii) $\frac{64\pi}{15}$

Q2

- (a) (0, 0), (4, 16)
- (b) $\frac{64}{3}$
- (c) (i) $\frac{1536\pi}{5}$ (ii) $\frac{384\pi}{5}$

Q3

- (a) (-3, 9), (3, 9)
- (b) 108
- (c) (i) 2721.6π (ii) 121.5π

Q4

- (a) $V = 2\pi \int_0^5 [4(25 - x^2) - (25 - x^2)] dx$
- (b) $V = 500\pi$

Q5

- (a) $\frac{1}{2}$
- (b)

(i) $V = 2\pi \int_0^1 (x^2 - x^6) dx = \frac{8\pi}{21}$

(ii) $V = 2\pi \int_0^1 (y^{\frac{2}{3}} - y^2) dy = \frac{8\pi}{15}$

(c) 1

(d)

(i) $\frac{32\pi}{35}$ (ii) $\frac{32\pi}{35}$

Q6

See full worked solutions.

Q7

$\frac{\pi^2}{2}$

Q8

(a) Substitute $x = 1$ into both curves.

(b) $\frac{3}{4}$

(c) $\frac{5}{4}$

(d) $\frac{10\pi}{21}$

(e) $\frac{14\pi}{15}$

Q9

(a) 16π (b) $8\pi(\pi - 2)$

Q10

(a)

(i) $V = \pi \int_0^3 \frac{16x^2}{9} dx + \pi \int_3^5 (25 - x^2) dx$

(ii) $\frac{100\pi}{3}$

(b)

(i) $\pi \int_0^4 \left(25 - \frac{25y^2}{16}\right) dy$

(ii) $\frac{200\pi}{3}$

(c)

(i) $\pi \int_0^3 \left(25 - \frac{25x^2}{9}\right) dx$

(ii) 50π

(d)

(i) $V = \pi \int_0^4 \frac{9y^2}{16} dy + \pi \int_4^5 (25 - y^2) dy$

(ii) $\frac{50\pi}{3}$

P1

(a) $V = \pi \int_0^1 (2x - x^2)^2 - x^4 dx = \frac{\pi}{3}$

(b) $\pi \int_0^1 y - (1 - \sqrt{1-y})^2 dy = \frac{\pi}{3}$

P2

(a) See full worked solutions.

(b) $\frac{\pi}{3}$

(c) As n gets large, the curve gets closer to the x -axis so the region becomes more like the triangle already enclosed by $y = 1 - x$ and the coordinate axes. When this triangular region is rotated, it forms a cone of radius 1 and height 1, so it is no surprise that the $V_\infty = V_{\text{cone}} = \frac{\pi}{3}$

Chapter Review

R1

(a) 36 (b) 36

R2

4.5

R3

π

R4

36π

R5

(a) $\frac{81\pi}{2}$ (b) $\frac{1944\pi}{5}$

R6

(a) $\frac{16\pi}{3}$ (b) $\frac{32\pi}{3}$

R7

(a) $\frac{\pi}{2}$ (b) $\frac{4\pi}{5}$ (c) $\frac{4\pi}{5}$ (d) $\frac{\pi}{2}$

R8

(a) $\frac{3\pi}{10}$ (b) $\frac{3\pi}{10}$

R9

(a) $6 \ln 6$ (b) 30π

R10

(a) 2π (b) $\frac{256\sqrt{2}\pi}{15}$

R11

- (a) $10 \ln 10 - 9$
 (b) $(100 \ln 10 - 130.5)\pi$

R12

- (a) $\frac{4}{3}$ (b) $\frac{2\pi^2}{3}$

R13

- (a) 16
 (b) $32\pi \ln 2$

R14

$$\frac{\pi^2}{2}$$

R15

$$\frac{392\pi}{25}$$

R16

$$\frac{\pi}{50} \left(2 - \frac{e^{-8}}{8} + \frac{e^8}{8} \right)$$

R17

- (a) $\frac{2\pi}{3}$ (b) 8π

R18

$$\pi(4 \ln 2 - 1.5)$$

R19

- (a) $\frac{\pi}{2}$ (b) 2π

R20

$$\frac{\pi^2}{20}$$

R21

- (a) $\frac{\pi^2}{16} u^3$ (b) $\frac{\pi^2}{2}$

R22

- (a) $\frac{9\pi}{4}$ (b) $\frac{9\pi}{8}(\pi - 2)$

R23

$$\frac{5\pi}{3} + \frac{3\sqrt{3}}{2}$$

R24

$$750(12e + 2e^2 - 5)\pi \text{ cm}^3$$

6. The Binomial Distribution

Exercise 6A

Bernoulli random variables

F1

random

F2

- (a) success, fail (b) 1, 0, fail

F3

- (a) 1
 (b) $1 - p$
 (c) $P(X = x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}$

F4

- (a) $E(X) = \mu = x_1 p_1 + x_2 p_2 + x_3 p_3 + \cdots + x_n p_n$
 (b) $\text{Var}(X) = x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \cdots + x_n^2 p_n - \mu^2$

F5

- (a) $\mu = p$
 (b) $\text{Var}(X) = p(1 - p)$
 (c) $\sigma = \sqrt{p(1 - p)}$

Q1

- (a) $P(X = x) = \begin{cases} 0.6, & \text{if } x = 1 \\ 0.4, & \text{if } x = 0 \end{cases}$
 (b) 0.6
 (c) 120

Q2

- (a) $P(X = x) = \begin{cases} \frac{1}{2}, & \text{for } x = 10 \\ \frac{1}{2}, & \text{for } x = -6 \end{cases}$
 (b) \$2
 (c) $\text{Var}(X) = 64, \sigma = 8$

Q3

- (a) $P(X = x) = \begin{cases} \frac{1}{6}, & \text{for } x = 15 \\ \frac{5}{6}, & \text{for } x = -2 \end{cases}$
 (b) $\$ \frac{5}{6}$
 (c) $\text{Var}(X) = \frac{1445}{36}, \sigma = \frac{\sqrt{1445}}{6}$

Q4

- (a) $\mu = -2.5$, which is negative and hence the game is unfavourable.
- (b) \$2.50
- (c) 0.1

Q5

Both Bob and Mary are correct.

Q6

- (a) $\mu = x_1p_1 + x_2p_2 + x_3p_3 + \dots + x_np_n$
- (b) $E(x) = 1(p) + 0(1-p) = p$

Q7

- (a) $\text{Var}(X) = x_1^2p_1 + x_2^2p_2 + \dots + x_n^2p_n - \mu^2$
- (b) $\text{Var}(X) = 1^2(p) + 0^2(1-p) - p^2 = p - p^2$

P1

$f(1) = p$ and $f(0) = 1 - p$, which are exactly the probabilities in a Bernoulli distribution.

Exercise 6B

Bernoulli trials

F1

- (a) binomial
- (b) $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, 0, n$

F2

- (a) $\binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$
- (b) $\binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k}$

F3

parameters, n, p

F4

- (a) $X = 1, 2, 3, \dots, 10$
- (b) $X = 0, 1, 2$
- (c) $X = 8, 9, 10$
- (d) $X = 0, 1, 2, 3, \dots, 9$

F5

- (a) $1 - P(X = 0)$
- (b) $P(X = 0) + P(X = 1) + P(X = 2)$
- (c) $1 - P(X = 9) - P(X = 10)$
- (d) $P(X = 9) + P(X = 10)$

Q1

- (a) $\binom{3}{0} (0.2)^3 (0.8)^0$
- (b) $\binom{3}{1} (0.2)^2 (0.8)^1$
- (c) $\binom{3}{2} (0.2)^1 (0.8)^2$
- (d) $\binom{3}{3} (0.2)^0 (0.8)^3$

Q2

- (a) $\binom{8}{7} (0.3)^7 (0.7)^1 + \binom{8}{8} (0.3)^8 (0.7)^0$
- (b) $\binom{8}{0} (0.3)^0 (0.7)^8 + \binom{8}{1} (0.3)^1 (0.7)^7 + \binom{8}{2} (0.3)^2 (0.7)^6$
- (c) $1 - \binom{8}{0} (0.3)^0 (0.7)^8 - \binom{8}{1} (0.3)^1 (0.7)^7$
- (d) $1 - \binom{8}{8} (0.3)^8 (0.7)^0$

Q3

- (a) $n = 6, p = \frac{2}{3}$
- (b) $\binom{6}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$

Q4

- (a) $n = 20, p = 0.1$
- (b) $\binom{20}{5} (0.1)^5 (0.9)^{15}$

Q5

- (a) $n = 10, p = 0.2$
- (b) $\binom{10}{4} (0.2)^4 (0.8)^6$

Q6

- (a) $\binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$
- (b) $1 - \binom{5}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5$
- (c) $\binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0$

Q7

- (a) $\binom{100}{0} (0.02)^0 (0.98)^{100}$
- (b) $1 - \binom{100}{0} (0.02)^0 (0.98)^{100} - \binom{100}{1} (0.02)^1 (0.98)^{99}$

Q8

$$(a) \binom{5}{3} \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 \quad (b) \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2$$

Q9

$$(a) \binom{10}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10}$$

$$(b) \binom{10}{10} \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^0$$

$$(c) \binom{10}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^6$$

$$(d) \binom{10}{9} \left(\frac{1}{4}\right)^9 \left(\frac{3}{4}\right)^1 + \binom{10}{10} \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^0$$

$$(e) 1 - \binom{10}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10} - \binom{10}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^9$$

$$(f) \binom{10}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10} + \binom{10}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^9 + \binom{10}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^8$$

Q10

$$(a) \binom{4}{2} (0.1)^2 (0.9)^2$$

$$(b) \binom{4}{4} (0.1)^4 (0.9)^0$$

$$(c) \binom{4}{0} (0.1)^0 (0.9)^4$$

$$(d) 1 - \binom{4}{0} (0.1)^0 (0.9)^4$$

Q11

$$\binom{6}{4} (0.4)^4 (0.6)^2 + \binom{6}{5} (0.4)^5 (0.6)^1 +$$

$$\binom{6}{6} (0.4)^6 (0.6)^0$$

Q12

(a) Bob only considered the probabilities of getting three sixes. But there is still probability associated with the other two *not* being sixes.

(b) Bob did not account for the many ways that three sixes and two not-sixes can be acquired.

Q13

(a) Yes.

(b) She observed that $P(X \geq 6) = P(X \leq 4)$ by symmetry of the binomial distribution, since $p = q = \frac{1}{2}$. So by removing $P(X = 5)$ from

the total probability she is left with $2 \times P(\geq 6)$. Then, she adds back in $P(X = 5)$.

(c) No. In the new scenario $p \neq q$ so the distribution is no longer symmetric.

Q14

$$(a) \binom{10}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 \quad (b) \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6$$

$$(c) \binom{9}{3} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 \quad (d) \binom{8}{2} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6$$

Q15

$$(a) \binom{7}{3} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^4 \quad (b) \binom{7}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^5$$

$$(c) \binom{6}{4} \left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^2 \quad (d) \binom{5}{2} \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^3$$

Q16

$$(a) 1 - \left(\frac{5}{6}\right)^n \quad (b) n = 13$$

Q17

(a) Let $P(X = x)$ be the probability of winning x times.

$$P(\text{at most once}) = P(X = 0) + P(X = 1)$$

(b) $P(\text{win at least twice})$
 $= 1 - P(X = 0) - P(X = 1)$
 > 0.9

(c) 18 spins

P1

29

P2

$$P(H) = \frac{3}{4} \text{ and } P(T) = \frac{1}{4}$$

P3

$$(a) \frac{379}{2187}$$

$$(b) \binom{10}{3} p^3 (1-p)^7 \approx 0.165, \text{ where } p \text{ is the}$$

answer from the previous part.

P4

$p^6(1-p)$, where

$$p = \binom{10}{9} (0.8)^9 (0.2)^1 + \binom{10}{10} (0.8)^{10}$$

P5

$\frac{1}{3}$

Exercise 6C

Binomial distributions

F1

- (a) $E(X) = np$
- (b) $\text{Var}(X) = np(1-p)$
- (c) $\sigma = \sqrt{np(1-p)}$

Q1

- (a) $\mu = 2, \text{Var}(X) = 1.6$
- (b) $\mu = 12, \text{Var}(X) = 4.8$
- (c) $\mu = 40, \text{Var}(X) = \frac{80}{3}$
- (d) $\mu = 12.5, \text{Var}(X) = 9.375$

Q2

(a) $n = 18, p = \frac{1}{3}$ (b) $\binom{18}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{15}$

Q3

- (a) $\mu = 2, \sigma = 1.29$
- (b) $X = 1, 2, 3$
- (c) 0.763
- (d) $\mu = 20, \sigma = 4.08, P(16 \leq X \leq 24) \approx 0.673$
- (e) We get roughly 68%, which reminds us of the empirical rules of the normal distribution. We get this because the binomial distribution is approximately normal for large n .

Q4

- (a) $\mu = 15, \sigma = 2.74$
- (b) $10 \leq X \leq 20$
- (c) 0.957

Q5

- (a) $\mu = 5$. Most likely number is 5
- (b) $\text{Var}(X) = 2.5, \sigma = \sqrt{2.5}$

Q6

- (a) \$3.375
- (b) Bob should expect to lose \$125 after 200 games.

Q7

- (a) $E(x) = -\$5$, which is negative.
- (b) $\frac{4}{5}$

P1

- (a) See full worked solutions.
- (b) See full worked solutions.
- (c) The probabilities are symmetric since they are equidistant from the extremities of X , so the base probability p must be symmetric i.e. $p = \frac{1}{2}$.

P2

The expectation is \$2.02, which is positive. Hence, she should walk away with a profit over a large number of games.

P3

- (a) See full worked solutions.
- (b) Use the fact that $p > 0$.
- (c) Further
- (d) Closer

Exercise 6D

Sample proportions

F1

proportion, sample

F2

- (a) population, population, constant
- (b) sample
- (c) sample,

$$\hat{p} = \frac{\# \text{ in sample with a particular attribute}}{\text{Sample size}}$$
- (d) vary
- (e) random, variable, binomial

F3

(a) p (b) $\sqrt{\frac{pq}{n}}$

Q1

(a) $\hat{p} = \frac{5}{10} = \frac{1}{2}$ (b) $\hat{p} = \frac{8}{10} = \frac{4}{5}$

(c) $\hat{p} = \frac{6}{10} = \frac{3}{5}$ (d) $\hat{p} = \frac{7}{10}$

Q2

(a) $p = 0.6$

(b) $\hat{p} = 0, 0.25, 0.5, 0.75, 1$

(c) 0.75

(d)

$$P(X = 3) = P(\hat{P} = 0.75)$$

$$= \binom{4}{3} \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^1$$

$$\approx 0.3456$$

\hat{p}	0	0.25	0.5	0.75	1
$P(\hat{P} = \hat{p})$	0.0256	0.1536	0.3456	0.3456	0.1296

(e)

(f)

(i) 0.3456 (ii) 0.8208 (iii) 0.8704

Q3

(a) 0.4

(b) $\hat{p} = 0, 0.2, 0.4, 0.6, 0.8, 1$

\hat{p}	0	0.2	0.4	0.6	0.8	1
$P(\hat{P} = \hat{p})$	0.0778	0.2592	0.3456	0.2304	0.0768	0.0102

(d) 0.4

(e)

(i) 0.3456 (ii) 0.337 (iii) 0.3174

Q4

(a) $\binom{5}{2} (0.7)^2 (0.3)^3 = 0.1323$

(b) $\binom{5}{3} (0.7)^3 (0.3)^2 + \binom{5}{4} (0.7)^4 (0.3) +$

$\binom{5}{5} (0.7)^5 = 0.8369$

(c) $\binom{5}{0} (0.3)^5 + \binom{5}{1} (0.7)(0.3)^4 +$

$\binom{5}{2} (0.7)^2 (0.3)^3 = 0.1631$

Q5

(a) $\mu = 0.4, \sigma = \sqrt{\frac{3}{250}}$

(b) $n = 4$

(c) $p = 0.2$ or $p = 0.8$

Q6

(a) $\mu = 0.3$ (b) $\frac{7}{2000}$ (c) $\sqrt{\frac{7}{2000}}$

Q7

24 flips

P1

(a) $P(X = k - 1) = \binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}$

(b) See full worked solutions.

(c) See full worked solutions.

(d) See full worked solutions.

(e) See full worked solutions.

P2

(a) \hat{P} represents a sample proportion whereas X represents the actual numbers in the sample with a desired attribute. To turn the actual number into a proportion, we divide by the total n .

(b) See full worked solutions.

(c) See full worked solutions.

Exercise 6E

Approximately normal distributions

F1

sample, normal, z

F2

Normal distribution.

F3

(a) p (b) normal, p (c) 10, 10

F4

(a) large

(b) mean, standard deviation

(c) z

Q1

- (a) 0.79 (b) -1.19

Q2

- (a) $E(X) = 64$, $\text{Var}(X) = 38.4$
- (b) There are a sufficiently large number of tosses.
- (c) 0.97
- (d) $P(\leq 70) = P(z \leq 0.97) = 0.8340$
- (e) 0.0985
- (f) 0.3745

Q3

- (a) $\bar{x}_{\hat{p}} = 0.15$, $\sigma_{\hat{p}} = 0.0326$
- (b) 0.937
- (c) 0.22

Q4

- (a) $E(X) = 60$, $\text{Var}(X) = 24$
- (b) $P(0.41 < z < 1.22) = 0.2297$
- (c) $P(-1.02 < z < 0.41) = 0.5052$
- (d) $P(-2.04 < z < -1.02) = 0.1332$

Q5

- (a) $\mu = 0.1$, $\sigma = 0.0316$
- (b) $\hat{p} = \frac{2}{15}$
- (c) 1.055
- (d) 0.1469

Q6

- (a) 0.1894 (b) 0.0793

Q7

- (a) 0.063 (b) 0.7831

Q8

$$P(z \geq -2.16) = 0.9846$$

Q9

$$P(-1.02 \leq z \leq 1.02) = 0.6922$$

P1

$$P(-1.99 \leq z \leq -0.66) = 0.2313$$

P2

$$P(-1.80 \leq z \leq 0.45) = 0.6377$$

Chapter Review**R1**

- (a) $P(X = x) = \begin{cases} 3, & \text{if } x = \frac{1}{6} \\ -1, & \text{if } x = \frac{5}{6} \end{cases}$
- (b) $-\$ \frac{1}{3}$
- (c) Bob should expect to have lost around \$10.

R2

- (a) $\mu = -\$1$, which is negative and hence the game is unfavourable.
- (b) \$2
- (c) 0.15

R3

- (a) $\mu = \frac{1}{2}$, $\text{Var}(X) = \frac{1}{4}$
- (b) $\mu = \frac{2}{3}$, $\text{Var}(X) = \frac{2}{9}$
- (c) $\mu = \frac{1}{4}$, $\text{Var}(X) = \frac{3}{16}$

R4

- (a) $\binom{6}{5} (0.2)^5 (0.8)^1 + \binom{6}{6} (0.2)^6 (0.8)^0$
- (b) $\binom{6}{0} (0.2)^0 (0.8)^6 + \binom{6}{1} (0.2)^1 (0.8)^5$
- (c) $1 - \binom{6}{0} (0.2)^0 (0.8)^6$
- (d) $1 - \binom{6}{6} (0.2)^6 (0.8)^0$

R5

- (a) $\binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$
- (b) $1 - \binom{4}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$
- (c) $\binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 + \binom{4}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0$

R6

- (a) $\binom{5}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5$
 (b) $\binom{5}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0$
 (c) $\binom{5}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^1$
 (d) $\binom{5}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^1 + \binom{5}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0$
 (e) $1 - \binom{5}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5$
 (f) $\binom{5}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^5 + \binom{5}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^4$

R7

- (a) $\binom{6}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$
 (b) $\left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$
 (c) $\binom{5}{3} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$
 (d) $\binom{3}{1} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$

R8

- (a) $1 - \left(\frac{5}{6}\right)^n$ (b) 26

R9

25 times

R10

- (a) $\binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$
 (b) $\binom{5}{2} \left(\frac{3}{8}\right)^2 \left(\frac{5}{8}\right)^3$
 (c) $\binom{5}{0} \left(\frac{3}{8}\right)^0 \left(\frac{5}{8}\right)^5 + \binom{5}{1} \left(\frac{3}{8}\right)^1 \left(\frac{5}{8}\right)^4$

R11

- (a) $\frac{3}{14}$ (b) $\binom{5}{3} \left(\frac{3}{14}\right)^3 \left(\frac{11}{14}\right)^2$

R12

- (a) $-\$ \frac{10}{9}$
 (b) Bob should expect to lose $\$ \frac{100}{3}$ after 30 games.

R13

- (a) 0.2592 (b) 0.3456 (c) 0.0768

R14

- (a) $\mu = 8, \text{Var}(X) = 4.8$
 (b) $\mu = 10, \text{Var}(X) = 9$
 (c) $\mu = 60, \text{Var}(X) = 20$

R15

$$n = 48, p = \frac{1}{4}$$

R16

$P(L = 3) = 0.08474, P(C = 2) = 0.0253$, so
 $P(L = 3) \times P(C = 2) \approx 0.00214$

R17

- (a) $E(x) = \$20$, which is positive.
 (b) If she misses the first shot, it is still favourable since the expectation is \$72, which still exceeds \$60.

R18

$$n > 100$$

R19

- (a) 0.4119 (b) 0.9963
 (c) 0.8133 (d) 0.0367

R20

- (a) $E(\hat{p}) = 0.16, \text{Var}(\hat{p}) = \frac{21}{31250}$
 (b) $P(z \leq -0.77) = 0.2206$
 (c) $P(-1.54 \leq z \leq 0.77), 0.7176$

R21

$$P(-1.33 \leq z \leq -0.66) = 0.1628$$

7. Differential Equations

Exercise 7A

Introduction to differential equations

F1

- (a) derivative
 (b) solution
 (c) solution, constant, family, C

F2

- (a) infinitely
- (b) initial, initial, particular
- (c) initial, general

F3

- (a) first, first
- (b) second, second

F4

derivatives, left, right

Q1

See full worked solutions.

Q2

- (a) $y' = -\frac{1}{x^2}$
- (b) $y' = -\frac{2}{x^3}$
- (c) $y' = -\frac{3}{x^4}$
- (d) $y' = -\frac{n}{x^{n+1}}, xy' + ny = 0$

Q3

$y' = \sec^2 x$

Q4

$y' = e^{-x} - xe^{-x}$

Q5

$y' = \sin x + x \cos x, y'' = 2 \cos x - x \sin x$

Q6

$y' = \frac{x \cos x - \sin x}{x^2}, y'' = -\frac{(x^2 - 2) \sin x + 2x \cos x}{x^3}$

Q7

$y' = 1 + \ln x, y'' = \frac{1}{x}$

Q8

$y' = \frac{1}{(x+1)^2}$

Q9

$y' = \frac{\ln(x) - 1}{(\ln x)^2}$

Q10

$y' = -Ae^{-x} + 2Be^{2x}, y'' = Ae^{-x} + 4Be^{2x}$

Q11

- (a) $y' = e^x, y'' = e^x$
- (b) $y' = e^x + xe^x, y'' = 2e^x + xe^x$

Q12

$\frac{dT}{dt} = -Ake^{-kt}$

Q13

- (a) $y' = Ake^{kx}$
- (b) $y'' = Ak^2e^{kx}$

Q14

(a) $y' = \frac{e^x - e^{-x}}{2}, y'' = \frac{e^x + e^{-x}}{2}$

(b) $y' = \frac{e^x + e^{-x}}{2}, y'' = \frac{e^x - e^{-x}}{2}$

Q15

$y' = -An \sin(nx) + Bn \cos(nx),$
 $y'' = -An^2 \cos(nx) - Bn^2 \sin(nx)$

P1

$y' = ke^{kx}, y'' = k^2e^{kx}$

P2

$\frac{dy}{dx} = \frac{1}{x \ln x}$

P3

$y' = \frac{2B + A + B \ln x}{2\sqrt{x}}, y'' = -\frac{A + B \ln x}{4x\sqrt{x}}$

Exercise 7B

Direction fields

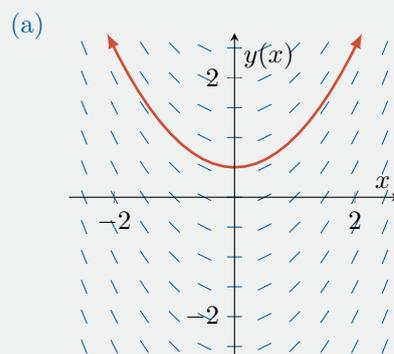
F1

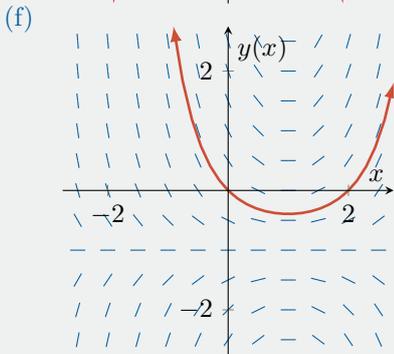
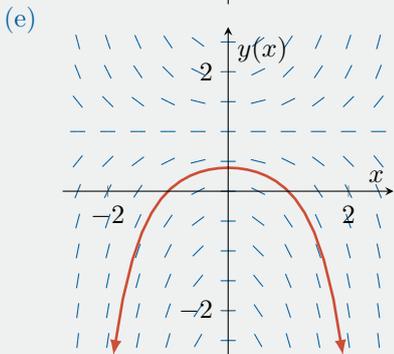
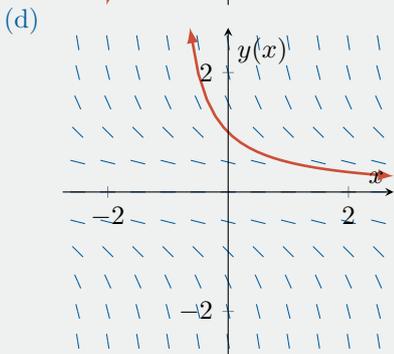
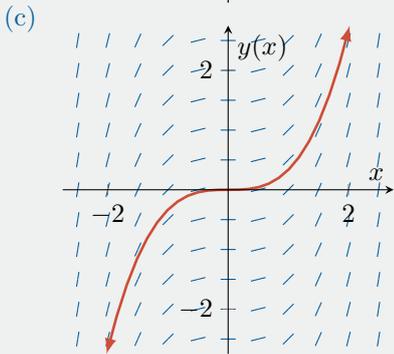
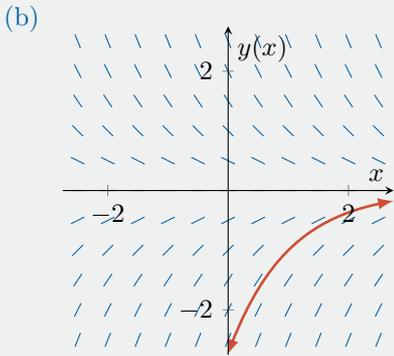
- (a) infinitely
- (b) gradient, gradient
- (c) direction
- (d) solutions, initial

F2

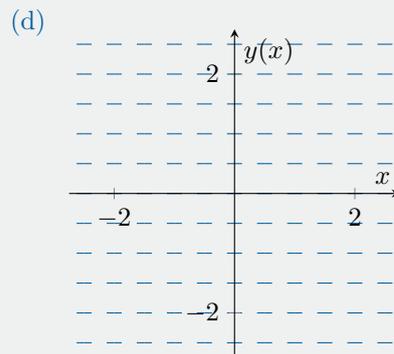
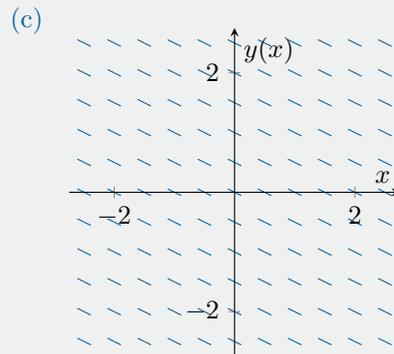
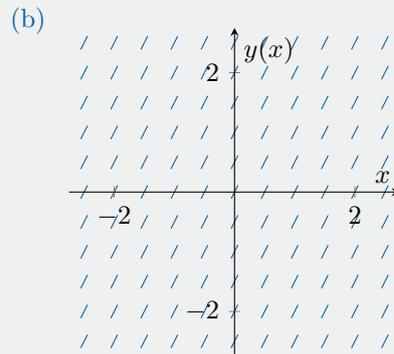
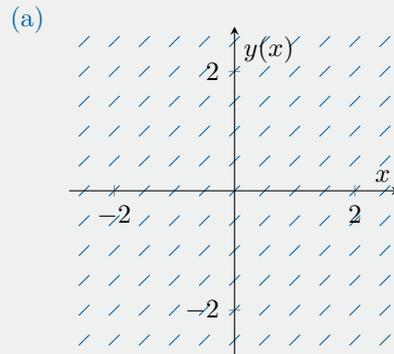
- (a) initial, solution
- (b) parallel

Q1





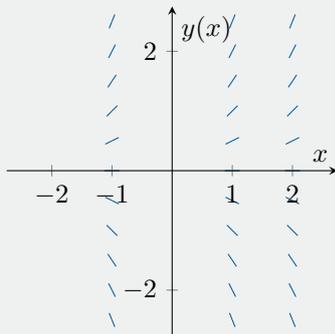
Q2



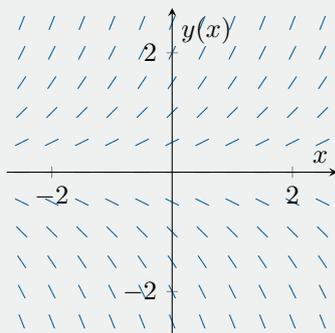
Q3

(a) $m = 1$

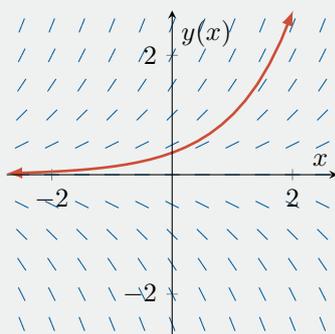
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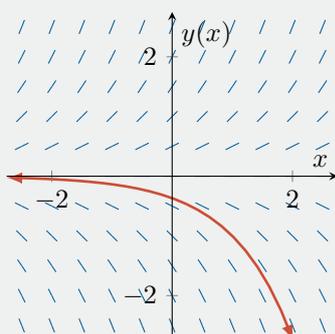
(c)



(d)

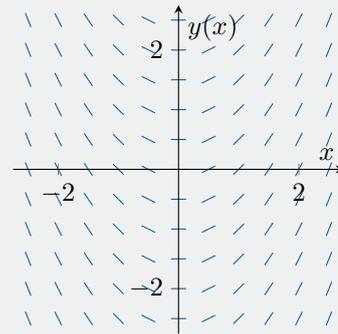


(e)

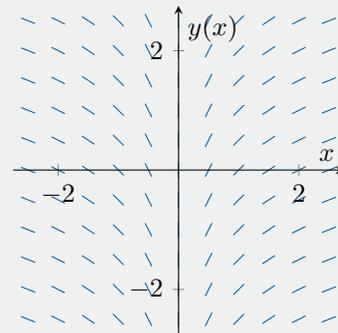


Q4

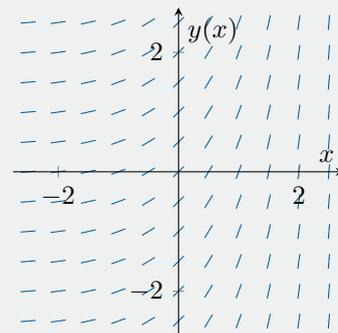
(a)



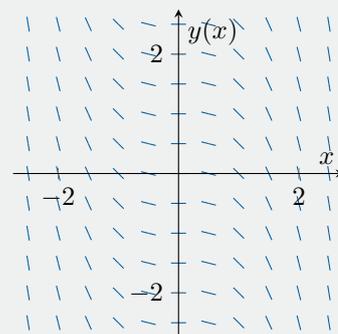
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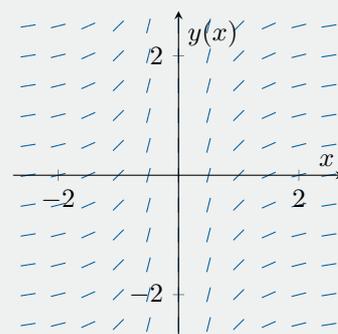
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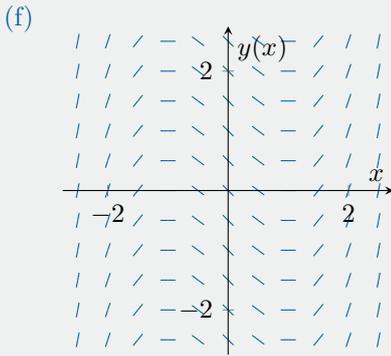


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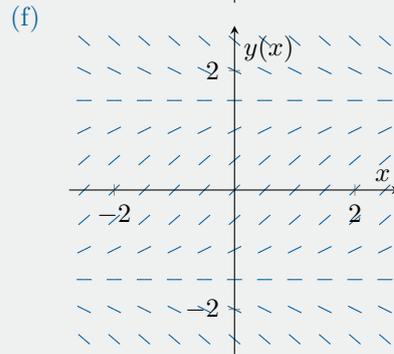
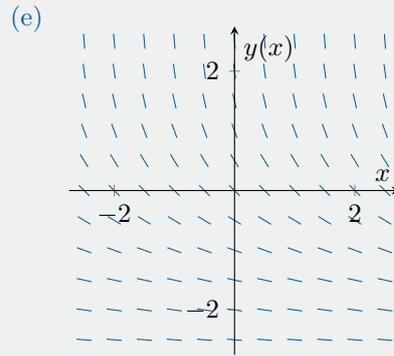
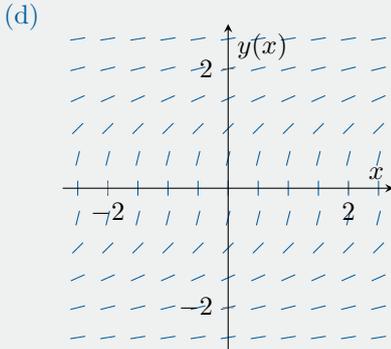
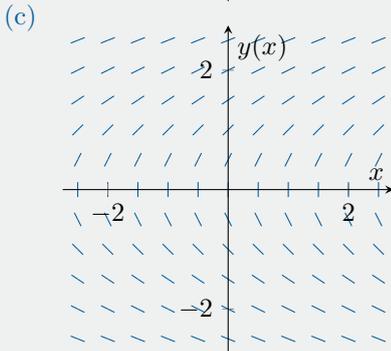
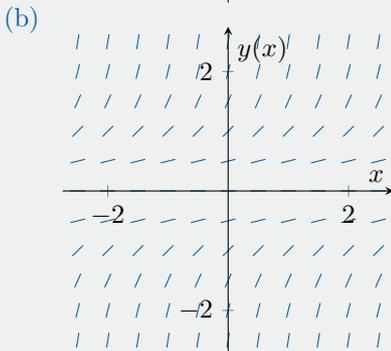
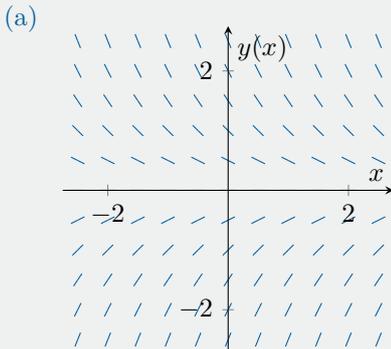


(e)





Q5



Q6

(B)

Q7

(A)

Q8

(D)

Q9

(A)

Q10

(C)

Q11

(A)

Q12

(C)

Q13

(C)

Q14

(C)

Q15

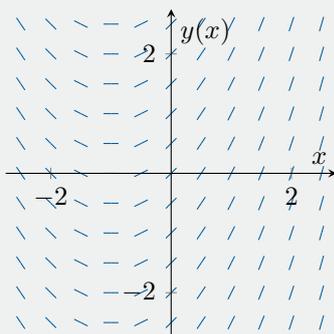
(A)

Q16

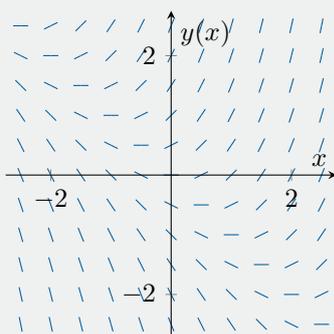
(B)

P1

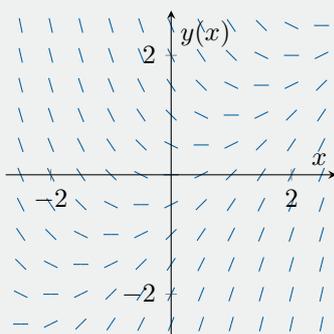
(a)



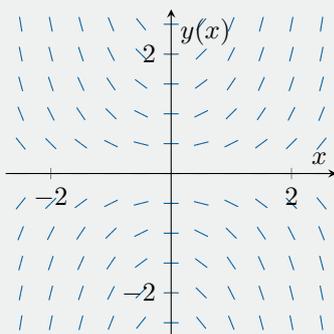
(b)



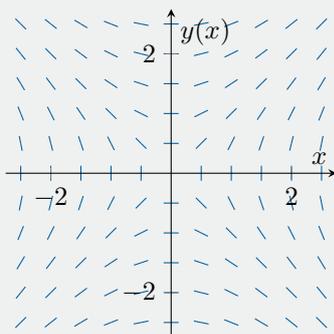
(c)



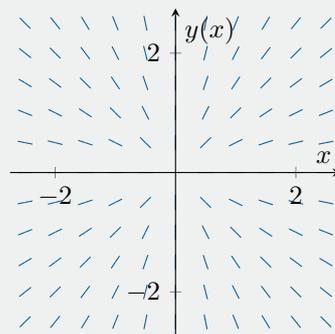
(d)



(e)



(f)



P2

(D)

Exercise 7C

First order differential equations

F1

(a) Integrate (b) Substitute (c) $F(x) + C$

F2

(a) Flip (b) Integrate
(c) Substitute (d) Re-arrange

F3

(a) separation (b) $F(x) + C$
(c) Substitute (d) y

F4

general, particular, initial

Q1

(a) $y = \ln(x + 1) + C$ (b) $C = 1$
(c) $y = \ln(x + 1) + 1$ (d) $y = \ln(x + 1)$

Q2

(a) $y = 2x^2 + x + 3$
(b) $y = 2 - \frac{1}{x}$
(c) $y = 3 - \frac{1}{2(2x + 1)}$
(d) $y = 3 - \sqrt{1 - 2x}$
(e) $y = \frac{3 - e^{-2x}}{2}$
(f) $y = e^{x^2}$

(g) $y = \frac{1}{2} \ln \left(x + \frac{1}{2} \right)$

(h) $\frac{1}{2} \ln(x^2 + 1)$

(i) $\frac{1}{2} (\sin(2x) - 1)$

(j) $y = \frac{1}{2} \ln 3 - \log(\cos 2x)$

(k) $y = \frac{1}{4} \left(\ln \left(\frac{2+x}{2-x} \right) + \log 3 \right)$

(l) $y = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + \frac{\pi}{9}$

Q3

(a) $x = \ln(1 + y) + C$

(b) $C = -\ln 2$

(c) See full worked solutions.

(d) See full worked solutions.

Q4

(a) $y = 3e^{2x}$ (b) $y = \frac{1}{2}e^{2x} - \frac{1}{2}$

(c) $y = 2e^{\frac{x}{2}}$ (d) $y = \frac{1}{4}(x+4)^2$

(e) $y = -\frac{1}{2} \ln(1-2x)$ (f) $y = \frac{1}{3} \ln(6x+1)$

(g) $y = \sin \left(x + \frac{\pi}{6} \right)$ (h) $y = \tan \left(x + \frac{\pi}{4} \right)$

(i) $y = \sin^{-1}(e^x)$ (j) $y = \frac{1}{4}x^2 + x$

Q5

(a) $\frac{dy}{\cos^2 y} = e^x dx$ but $\frac{1}{\cos^2 y} = \sec^2 y$

(b) See full worked solutions.

(c) Inverse tan both sides of the previous result

(d) $y \rightarrow \frac{\pi}{2}$

Q6

(a) $y = e^{\frac{x^2}{2}-1}$

(b) $y = 3e^{\frac{x^3}{3}}$

(c) $y = -\frac{2}{x^2+1}$

(d) $y = \pi x$

(e) $y = \sqrt[3]{3(x^2+x+3)}$

(f) $y = \tan(\ln x)$

(g) $y = \ln(2e^x + 1)$

(h) $y = e^{\frac{x}{2}}$

(i) $y = -\ln(1-x^2)$

(j) $y = e^{1+x^2+x^3}$

(k) $y = \sin^{-1} \left(x^2 - \frac{3}{4} \right)$

(l) $y = \cos^{-1} \left(\frac{1}{2}e^{-x^2} \right)$

Q7

$y = \frac{1}{12}x^4 + 4x + 2$

Q8

See full worked solutions.

Q9

See full worked solutions.

Q10

See full worked solutions.

Q11

See full worked solutions.

Q12

See full worked solutions.

Q13

See full worked solutions.

P1

See full worked solutions.

P2

(a) $\text{LHS} = \frac{1}{2} \times \frac{2}{1-y^2} = \frac{1}{1-y^2}$

(b) See full worked solutions.

P3

(a) See full worked solutions.

(b) The gradients are negative reciprocals of each other.

(c) $\frac{x^2}{2} + y^2 = A$ for some constant A . It represents a family of ellipses.**Exercise 7D****Applications of differential equations****F1**

linear, 1

F2

(a) Linear

(b) Linear

(c) Not linear

(d) Not linear

F3

$-kP$, first, linear

F4

- (a) Exponential growth equation
- (b) Exponential decay equation
- (c) Newton's law of cooling
- (d) Logistic equation

Q1

- (a) $\frac{dP}{dt} = kP$
- (b) $\frac{dP}{dt} = -k\sqrt{P}$
- (c) $\frac{dT}{dt} = k(T - E)$
- (d) $\frac{dT}{dt} = k(E - T)$
- (e) $\frac{dM}{dt} = -kM$
- (f) $\frac{dP}{dt} = k(\alpha - \beta)$
- (g) $\frac{d^2x}{dt^2} = -\frac{k}{x^2}$
- (h) $\frac{d^2x}{dt^2} = k(x - x_0)$

Q2

See full worked solutions.

Q3

See full worked solutions.

Q4

- (a) $\frac{dP}{dt} = kP$
- (b) See full worked solutions.
- (c) See full worked solutions.
- (d) $P = 2297$
- (e) 10 years

Q5

- (a) See full worked solutions.
- (b) See full worked solutions.
- (c) 200 years

Q6

- (a) See full worked solutions.
- (b) 20°C

Q7

- (a) $\frac{dT}{dt} = k(B - T)$
- (b) 23 minutes

Q8

- (a) See full worked solutions.
- (b) See full worked solutions.
- (c) 50000
- (d) 31 months

Q9

- (a) $5R$ grams per minute
- (b) $\frac{RS}{500}$ grams per minute
- (c) See full worked solutions.
- (d) See full worked solutions.
- (e) 2500 grams
- (f) $A = -2500$
- (g) 13.4 litres per minute

Q10

- (a) The concentration increases at a fixed rate of r , but the drug is removed at a rate proportional to the concentration, hence $-kC$. Combined, this gives $r - kC$.
- (b) See full worked solutions.
- (c) $C \rightarrow \frac{r}{k}$
- (d) If $r < kC_0$, the concentration decreases. If $r > kC_0$, the concentration increases.

Q11

- (a) $\int \frac{1}{V - RI} dI = \int \frac{1}{L} dt$
- (b) $\frac{1}{R}$

Q12

- (a) $\frac{1}{P(1 - P)}$
- (b) See full worked solutions.
- (c) 1

Q13

- (a) See full worked solutions.
- (b) $P = \frac{1}{k} ((kP_0 - m)e^{kt} + m)$
- (c)
 - (i) $m < kP_0$
 - (ii) $m = kP_0$
 - (iii) $m > kP_0$

Q14

- (a) See full worked solutions.
 (b) $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$
 (c) Initial height is 4.
 (d) $k = 8 - \sqrt{2}$
 (e) 4 minutes and 35 seconds

P1

- (a) See full worked solutions.
 (b) $v \rightarrow \frac{g}{k}$
 (c) This is the limiting velocity of the particle, otherwise known as the terminal velocity.
 (d) As the velocity approaches the limit, it approaches a constant value. This means that the derivative of the velocity i.e. acceleration approaches zero.

P2

- (a) See full worked solutions.
 (b) $V = V_0 e^{B(e^{-at} - 1)}$
 (c) $V \rightarrow V_0 e^{-B}$

P3

- (a) See full worked solutions.
 (b) When $x = R$, $a = -g$.
 (c) See full worked solutions.
 (d) See full worked solutions.
 (e) It means that if $u^2 = 2gR$, then the particle will never return to the Earth's surface after projection i.e. it will escape the Earth's gravitational pull.

Chapter Review

R1

- (a) $y' = 2 \cos(2x)$, $y'' = -4 \sin(2x)$
 (b) $y' = 2xe^x + x^2e^x$
 (c) $y' = \frac{1 - \ln x}{x^2}$, $y'' = \frac{2 \ln x - 3}{x^3}$
 (d) $y' = 1 + \frac{1}{x^2}$

R2

$$y' = \frac{1 - A - \ln x}{x}$$

R3

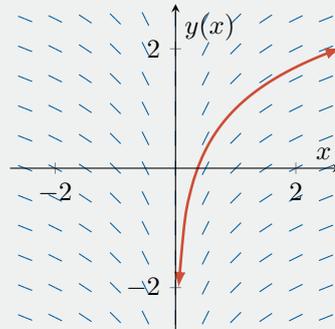
$$y' = 2Ae^{2x} - 3Be^{-3x}, \quad y'' = 4Ae^{2x} + 9Be^{-3x}$$

R4

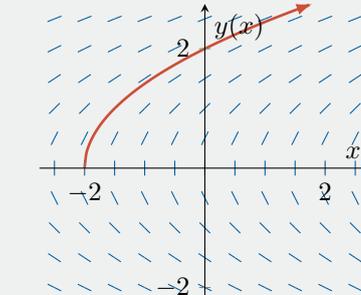
$$y' = -e^{-x}(a - b + bx), \quad y'' = e^{-x}(a - 2b + bx)$$

R5

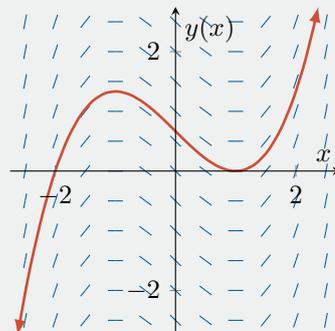
(a)



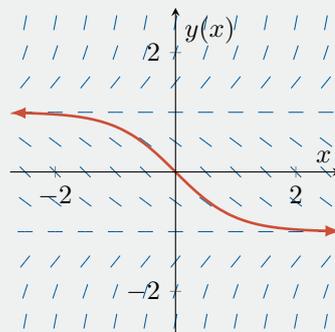
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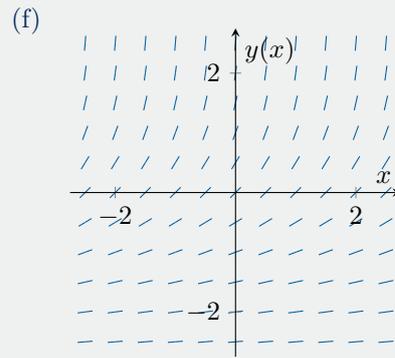
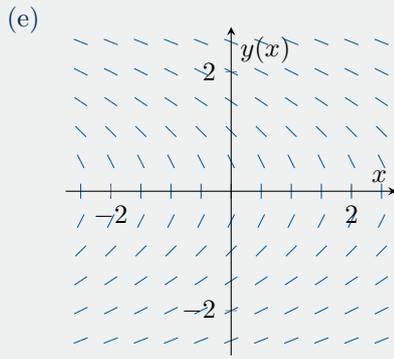
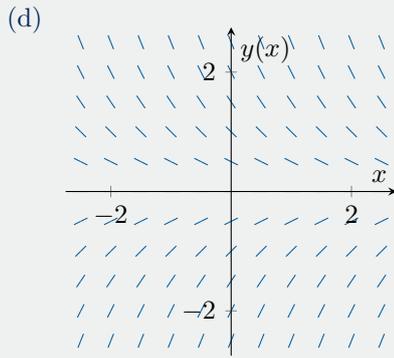
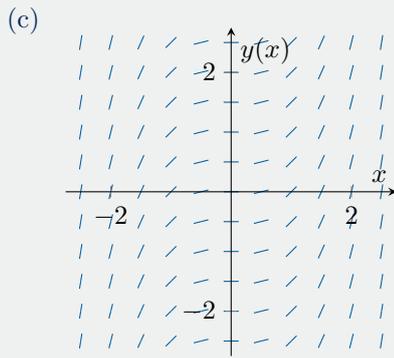
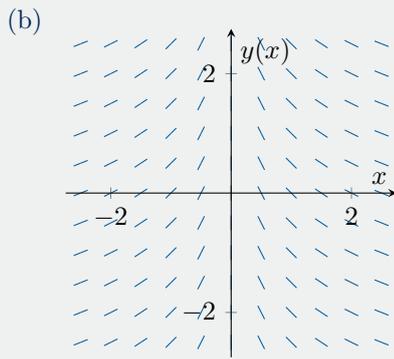
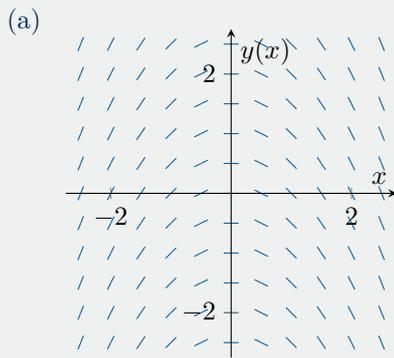
(c)



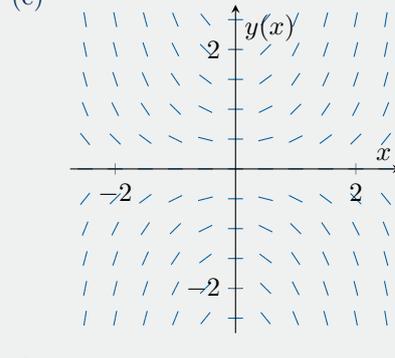
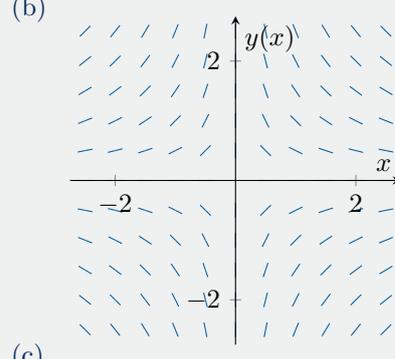
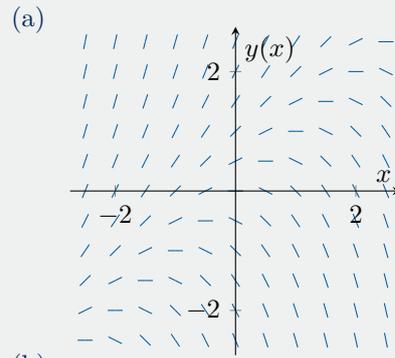
(d)



R6



R7



R8

(C)

R9

(A)

R10

(a) $y = 3 + x - x^2$

(b) $y = \frac{2}{x+3} + 2$

(c) $y = -3 \ln(-x)$

(d) $y = \sin^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{6}$

R11

(a) $y = \ln(x+1)$ (b) $y = \cos^{-1}(e^{-x})$

(c) $y = \frac{3}{2}(1 - e^{-2x})$ (d) $y = \left(\frac{3x}{2} + 1\right)^{\frac{2}{3}}$

R12

(a) $y = e^{2x^2}$

(b) $y = \sqrt{2x^2 + 1}$

(c) $y = \ln(x^2 - 1)$

(d) $y = \sqrt{5 - e^{-x^2}}$

(e) $y = \tan^{-1}(x+1)$

(f) $y = e^{x^4}$

(g) $y = 2e^{3x^4}$

(h) $y = \frac{\sqrt{2}}{x}$

(i) $y = \sqrt{2 \cos(x) - 1}$

(j) $y = e^{\frac{x^2+2x}{2}}$

R13

See full worked solutions.

R14

See full worked solutions.

R15

See full worked solutions.

R16

$y = 2 \cos(3x)$

R17

(a) See full worked solutions.

(b) 5.2 kilograms

(c) 71.6%

R18

(a) See full worked solutions.

(b) See full worked solutions.

(c) 27 minutes

R19

(a) $\frac{dP}{dt} = kP$

(b) See full worked solutions.

(c) See full worked solutions.

R20

(a) $\frac{dP}{dt} = \frac{1}{2}P - F$

(b) See full worked solutions.

(c) $F = 2500$ **R21**

(a) $\frac{dM}{dt} = -kM$

(b) 250 years

R22

(a) 30°C

(b) See full worked solutions.

(c) 29.57 minutes

R23

(a) See full worked solutions.

(b) See full worked solutions.

(c) 10000

(d) 12 months