MATHS Quest

YEAR 12

MATHS C

FOR QUEENSLAND

SECOND EDITION

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Nick Simpson ▸ Catherine Smith

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jacaranda Maths Quest

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jacaranda MQ Maths Quest

Nick **Simpson**
Catherine **Smith**

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Introduction

Maths Quest Maths C Year 12 for Queensland 2nd edition is one of the exciting Maths Quest resources specifically designed for the Queensland senior mathematics syllabuses beginning in 2009. It has been written and compiled by practising Queensland Maths C teachers. It breaks new ground in Mathematics textbook publishing.

This resource contains:
- a student textbook with accompanying student website (eBookPLUS)
- a teacher edition with accompanying teacher website (eGuidePLUS)
- a solutions manual containing fully worked solutions to all questions contained in the student textbook.

Student textbook

Full colour is used throughout to produce clearer graphs and headings, to provide bright, stimulating photos and to make navigation through the text easier.

Clear, concise *theory sections* contain *worked examples, graphics calculator tips* and *remember boxes*.

Worked examples in a Think/Write format provide clear explanation of key steps and suggest how solutions can be presented.

Exercises contain many carefully graded skills and application problems, including multiple-choice questions. Cross-references to relevant worked examples appear beside the first ‘matching’ question throughout the exercises.

Investigations, often suggesting the use of technology, provide further discovery learning opportunities.

Each chapter concludes with a *summary* and *chapter review* exercise containing questions that help consolidate students’ learning of new concepts.

As part of the chapter review, there is also a Modelling and problem solving section. This provides students with further opportunities to practice their skills.

Technology is fully integrated within the resource. To support the use of graphics calculators, instructions for two models of calculator are presented in worked examples and graphics calculator tips throughout the text. The two models of graphics calculator featured are the Casio *fx*-9860G AU and the TI-Nspire CAS. (Note that the screen shots shown in this text for the TI-Nspire CAS calculator were produced using OS 1.7. Screen displays may vary depending on the operating system in use.)

For those students using the TI-89 model of graphics calculator, an appendix containing matching instructions has been included at the back of the book.

The *Maths Quest for Queensland* series also features the use of spreadsheets with supporting Excel files supplied on the student website.

Student website — eBookPLUS

The accompanying student website contains an electronic version of the textbook as well as the resources listed below.

WorkSHEETs — editable Word 97 documents that may be completed on screen, or printed and completed later.

SkillSHEETs — printable pages that contain additional examples and problems designed to help students revise required concepts.

Test yourself activities — multiple-choice quizzes for students to test their skills after completing each chapter.

Casio Classpad 330 calculator instructions and screen shots for all matching examples and Graphic calculator tips that appear in the text book.

Teacher edition

The teacher edition textbook contains everything in the student textbook and more. To support teachers assisting students in the class, answers appear in red next to most questions in the exercises and investigations. Each chapter is annotated with relevant syllabus information.

Teacher website — eGuidePLUS

The accompanying teacher website contains everything in the student website plus the following resources:

- two tests per chapter (with fully worked solutions)
- fully worked solutions to WorkSHEETs
- a syllabus planning document
- assessment tasks (and answers)
- fully worked solutions to all questions in the student textbook.

Solutions manual

Maths Quest Maths C Year 12 for Queensland Solutions Manual contains the fully worked solutions to every question and investigation in the *Maths Quest Maths C Year 12 for Queensland 2nd edition* student textbook.

Fully worked solutions are available for all titles in the *Maths Quest for Queensland* senior series.

Maths Quest is a rich collection of teaching and learning resources within one package.

About eBookPLUS

jacaranda *plus*

Next generation teaching and learning

This book features eBookPLUS: an electronic version of the entire textbook and supporting multimedia resources. It is available for you online at the JacarandaPLUS website (**www.jacplus.com.au**).

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Minimum requirements

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Troubleshooting

- Go to the JacarandaPLUS help page at **www.jacplus.com.au/jsp/help.jsp**.
- Contact John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

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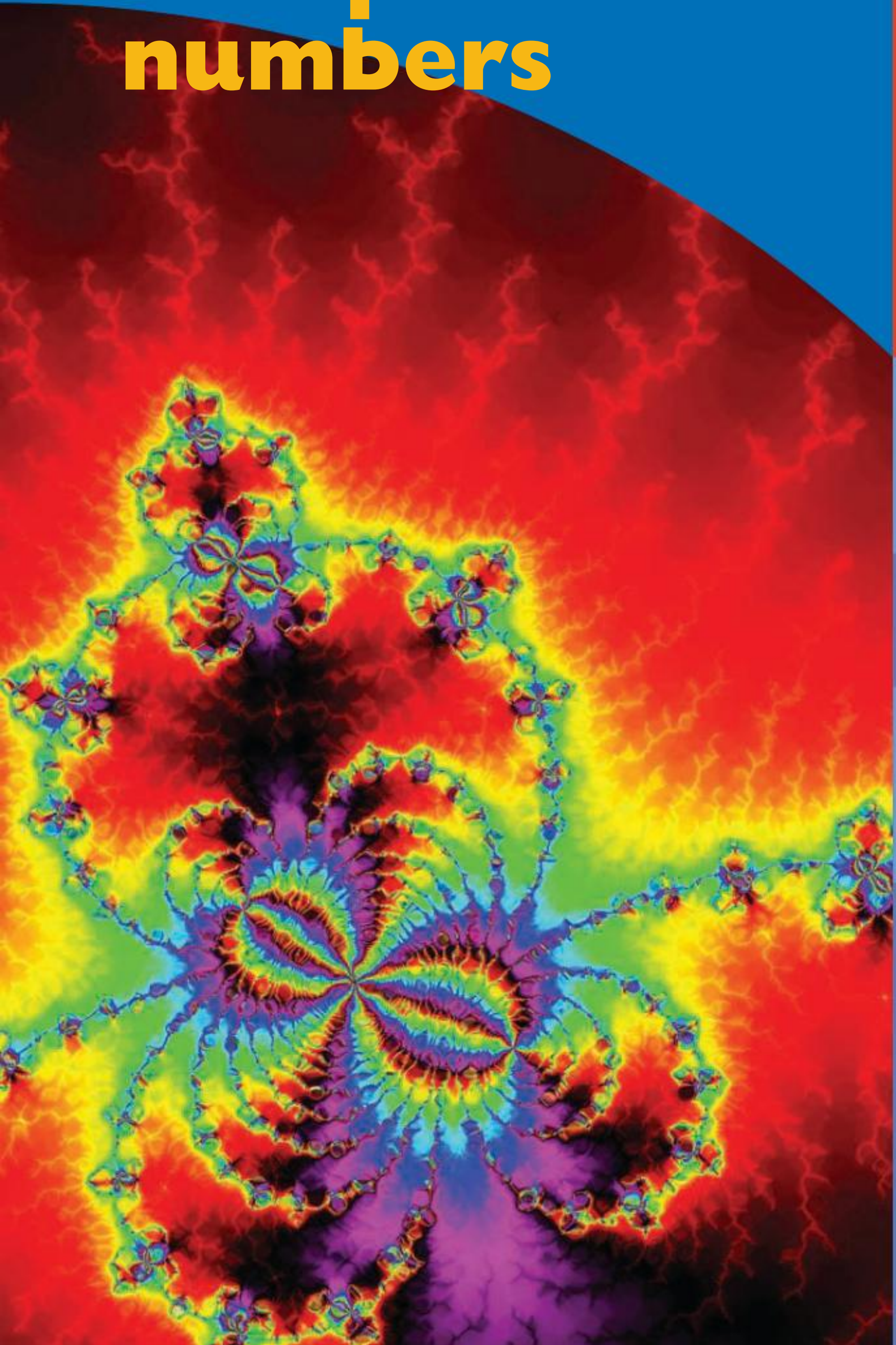
1

Complex numbers



syllabus reference

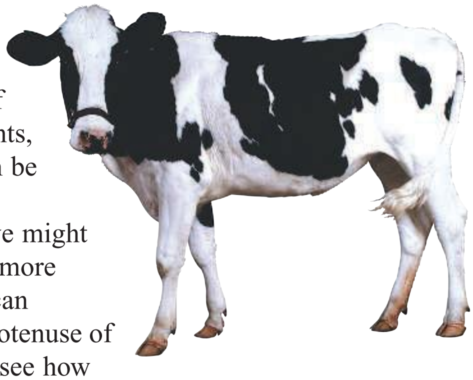
Core topic:

Real and complex number systems

In this chapter

1A Operations on and representations of complex numbers
1B Factorisation of polynomials in *C*
1C Solving equations in *C*
1D Graphs of complex relations: rays, lines and circles

Operations on and representations of complex numbers



What meaning do we give to numbers? The number '5' can be applied to 5 cows or 5 metres. The number $\frac{3}{4}$ is a little more difficult to give a meaning to. For, while $\frac{3}{4}$ of \$12 and $\frac{3}{4}$ metres are reasonable measurements, what about $\frac{3}{4}$ of a cow? The number '–5' can be given some meaning. On a cold morning, the temperature might be 5 degrees below 0 or we might owe someone \$5. The concept of –5 cows is more problematic. What about $\sqrt{5}$? Although we can construct this number as the length of the hypotenuse of a triangle with sides 2 and 1, it is difficult to see how $\sqrt{5}$ cows or \$ $\sqrt{5}$ makes sense.

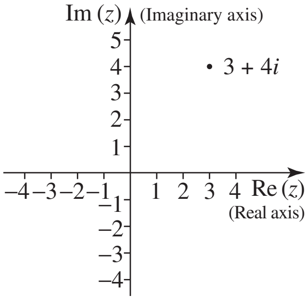
Although some numbers are abstract and it is difficult to form a representation for them, this does not mean that these numbers are not useful. In Year 11, the concept of a complex number was introduced.

A complex number is of the form $a + bi$ where a and b are real numbers and $i^2 = -1$. In this number, a is the real part and bi is the imaginary part.

While it is difficult to conceive of $3 + 4i$ cows or \$$(3 + 4i)$, a number such as $3 + 4i$ has uses in a variety of contexts in physics and mathematics. In physics, complex numbers are used to study variations in voltage and current in an AC circuit. In mathematics, complex numbers are used to solve equations, represent transformations and to create objects such as the Mandelbrot set.

The Year 11 text introduced the concept of a complex number and operations involving complex numbers. In Year 12 we begin by revising these topics.

Review of operations on complex numbers



Real numbers are represented on a number line. This representation helps us to understand certain properties of these numbers. We see that 4 is bigger than 3 and that $\sqrt{2}$ lies between 1 and 2. Complex numbers are represented in two dimensions using the Argand diagram, and this image helps us to visualise the complex number. For example, the complex number $3 + 4i$ can be represented on the Argand diagram as shown at right.

You will recall that complex numbers can be added, subtracted, multiplied and divided. However, subtraction and division are not treated as distinct operations. Subtraction of a number is achieved by adding the negative of that number and division is achieved by multiplying by the reciprocal of the number.

If $z = a + bi$ and $w = c + di$ then:

Addition: $z + w = a + bi + c + di = (a + c) + (b + d)i$

Scalar multiplication: $k \cdot z = ka + kbi$

Multiplication: $z \cdot w = (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$

Inverse of z:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{a+bi}\\ &= \frac{1}{a+bi}\cdot\frac{a-bi}{a-bi}\\ &= \frac{a-bi}{a^2+b^2}\\ &= \frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\end{aligned}$$

Conjugate: If $z = a + bi$, the conjugate of z, $\overline{z} = a - bi$

WORKED Example 1

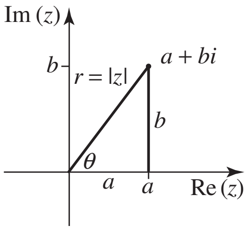
If $p = 2 + 3i$ and $q = 4 - 2i$, evaluate each of the following without using a calculator.

a $3p$ **b** $2p - 3q$ **c** $p \cdot q$ **d** $\frac{1}{p}$

THINK	WRITE
a ① Substitute for p.	**a** $3p = 3(2 + 3i)$
② Multiply each part of p by 3.	$= 6 + 9i$
b ① Substitute for p and q.	**b** $2p - 3q = 2(2 + 3i) - 3(4 - 2i)$
② Multiply by the scalars to expand the brackets.	$= 4 + 6i - 12 + 6i$
③ Simplify the real and imaginary parts.	$= -8 + 12i$
c ① Substitute for p and q.	**c** $p \cdot q = (2 + 3i)(4 - 2i)$
② Use the Distributive Law to expand the brackets.	$= 2 \cdot 4 + 2 \cdot -2i + 3i \cdot 4 + 3i \cdot -2i$ $= 8 - 4i + 12i - 6i^2$
③ Simplify the real and imaginary parts. Remember that $i^2 = -1$.	$= 8 + 6 - 4i + 12i$ $= 14 + 8i$
d ① Substitute for p.	**d** $\frac{1}{p} = \frac{1}{2+3i}$
② Multiply the top and bottom by the conjugate of $2 + 3i$; that is, $2 - 3i$.	$= \frac{1}{2+3i}\cdot\frac{2-3i}{2-3i}$
③ Evaluate the products.	$= \frac{2-3i}{13}$
④ Write as separate real and complex components.	$= \frac{2}{13} - \frac{3}{13}i$

Polar form

We have seen that a complex number can be located on the Argand diagram by giving its real (Re) and imaginary (Im) components. This form of representation is called *standard form*, *Cartesian form* or *rectangular form*. Another useful way of representing a location on the Argand diagram is in *polar form*.



Consider the complex number $z = a + bi$ as pictured in the Argand diagram at right.

The magnitude of the complex number z is called mod (z) and written as r or $|z|$.

$$r = \sqrt{a^2 + b^2}$$

In other words,

$$|a + bi| = \sqrt{a^2 + b^2}$$

The angle θ is called the *argument* of z, arg (z), and is given by $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, where θ is measured in an anticlockwise direction from the positive x-axis.

It is usual though to refer to the principal argument of z, Arg (z), which is the angle made with the positive x-axis and lying between $-\pi$ and π. (Note the use of the capital letter A.)

Noting that $a = |z| \cos \theta$ and $b = |z| \sin \theta$ it is clear that:

$$z = r \cos \theta + r \sin \theta i.$$

Thus in polar form, sometimes called mod-Arg form, z can be written as:

$$z = r \text{ cis } \theta \text{ where } -\pi < \theta \leq \pi.$$

Note: cis θ is short for cos $\theta + i \cdot$ sin θ.

WORKED Example 2

If $v = 2 - 3i$ and $w = 1 + 2i$, calculate each of the following.

a $|v|$ **b** $|\bar{v}|$ **c** $|v + 2w|$

THINK	WRITE
a Calculate $\lvert v\rvert$ (or the magnitude of v) by using the rule: $\lvert v\rvert = \lvert a + bi\rvert = \sqrt{a^2 + b^2}$ where $a = 2$ and $b = -3$.	**a** $\lvert v\rvert = \lvert 2 - 3i\rvert$ $= \sqrt{2^2 + (-3)^2}$ $= \sqrt{13}$
b 1 Find the conjugate of v.	**b** $\bar{v} = 2 + 3i$
2 Calculate $\lvert\bar{v}\rvert$ (or the magnitude of $\bar{v}$) by using the rule $\lvert a + bi\rvert = \sqrt{a^2 + b^2}$.	$\lvert\bar{v}\rvert = \lvert 2 + 3i\rvert$ $= \sqrt{2^2 + 3^2}$ $= \sqrt{13}$
c 1 First calculate $v + 2w$.	**c** $v + 2w = 2 - 3i + 2(1 + 2i)$ $= 2 - 3i + 2 + 4i$ $= 4 + i$
2 Calculate $\lvert v + 2w\rvert$ by using the rule $\lvert a + bi\rvert = \sqrt{a^2 + b^2}$.	$\lvert v + 2w\rvert = \lvert 4 + i\rvert$ $= \sqrt{4^2 + 1^2}$ $= \sqrt{17}$

WORKED Example 3

a **Express $2 - 3i$ in polar form.**

b **Express $5 \text{ cis } \frac{2\pi}{3}$ in Cartesian form.**

THINK | **WRITE**

a ① To write a complex number in polar form, r and θ are needed for r cis θ. Find r by calculating mod $(2 - 3i)$.

$$|2 - 3i| = \sqrt{2^2 + (-3)^2}$$
$$= \sqrt{13}$$

② Find θ in radians. Remember $\theta = \tan^{-1} \frac{b}{a}$ for $a + bi$.

$$\theta = \tan^{-1}\left(\frac{-3}{2}\right) \quad \text{(4th quadrant)}$$
$$= -0.98$$

③ Write in polar form; that is, in the form r cis θ.

$$2 - 3i = \sqrt{13} \text{ cis } (-0.98)$$

b Use the relationship $r \text{ cis } \theta = r \cos \theta + i \cdot r \sin \theta$.

$$5 \text{ cis } \frac{2\pi}{3} = 5 \cos \frac{2\pi}{3} + i \cdot 5 \sin \frac{2\pi}{3}$$
$$= 5 \cdot -\tfrac{1}{2} + i \cdot 5 \cdot \tfrac{\sqrt{3}}{2}$$
$$= -\tfrac{5}{2} + \tfrac{5\sqrt{3}}{2} i \quad \text{(or } -2.5 + 4.3i\text{)}$$

Graphics Calculator tip! Complex numbers

The graphics calculator can be used to good effect in performing calculations on complex numbers.

For the CASIO *fx*-9860G AU

To enter the complex number $3 + 4i$, press:

- MENU
- 1 (RUN)
- OPTN
- F3 (CPLX)

Enter $3 + 4i$, using F1 for i.

3+4i

3+4i

For the TI-Nspire CAS

To enter the complex number $3 + 4i$, on a Calculator page, complete the entry line as:

$3 + 4i$

Then press ENTER enter.

Note that the complex number 'i' is found on a grey button on the far left column of the keypad.

3+4·i

3+4·i

Worked example 4 shows how to evaluate complex numbers using a graphics calculator.

WORKED Example 4

Use your graphics calculator to evaluate each of the following.

a $(3 + 4i) \cdot (2 - i)$ **b** $(3 + 4i)^{-1}$ **c** $|3 + 4i|$ **d** Arg $(3 + 4i)$

THINK

WRITE/DISPLAY

For the CASIO *fx*-9860G AU

① As the complex number $3 + 4i$ is used frequently, it is worthwhile storing it as a variable instead of frequently re-entering it. To do this, complete the entry line as:
$3 + 4i \rightarrow A$
then press EXE.

```
3+4i→A
                3+4i
```

② To answer parts **a**, **b**, **c** and **d**, using F1, F2 and F3 complete the entry lines as:
$A \cdot (2 - i)$
A^{-1}
Abs(A)
Arg(A)
Press EXE after each line.

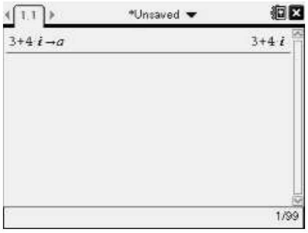
```
               10+5i
A^-1
          0.12-0.16i
Abs (A)
                   5
Arg (A)
         0.927295218
```

③ Write the answers.

a $(3 + 4i) \cdot (2 - i) = 10 + 5i$
b $(3 + 4i)^{-1} = 0.12 - 0.16i$
c $|3 + 4i| = 5$
d Arg $(3 + 4i) = 0.9273$

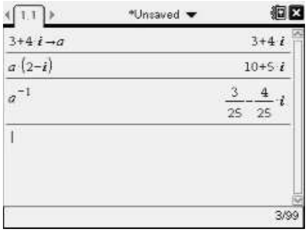
For the TI-Nspire CAS

① As the complex number $3 + 4i$ is used frequently, it is worthwhile storing it as a variable instead of frequently re-entering it. To do this, complete the entry line as:
$3 + 4i \rightarrow a$
then press ENTER enter.
Note: To find →, press:
- Ctrl ctrl
- STO sto▸ var



```
3+4·i→a                        3+4·i
                                1/99
```

② To answer parts **a** and **b**, complete the entry lines as:
$a \cdot (2 - i)$
a ^−1
Press ENTER enter after each line.



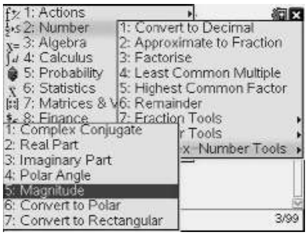
```
3+4·i→a                        3+4·i
a·(2−i)                       10+5·i
```

a^{-1} $\quad \frac{3}{25}-\frac{4}{25}\cdot i$

```
                                3/99
```

③ To calculate $|3 + 4i|$ press:
- MENU menu
- 2: Number 2
- 9: Complex Number Tools 9
- 5: Magnitude 5

Complete the entry line as:
$|a|$
then press ENTER enter.



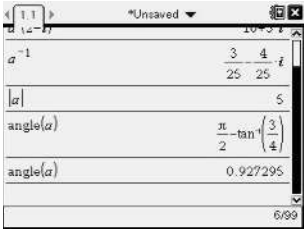
THINK

④ To calculate Arg $(3 + 4i)$, press:
- MENU (menu)
- 2: Number ⟨2⟩
- 9: Complex Number Tools ⟨9⟩
- 4: Polar Angle ⟨4⟩

Complete the entry line as:
angle(a)
then press ENTER ⟨enter⟩.

⑤ Write the answers.

WRITE/DISPLAY



a^{-1} $\frac{3}{25}-\frac{4}{25}\cdot i$

$|a|$ 5

angle(a) $\frac{\pi}{2}-\tan^{-1}\left(\frac{3}{4}\right)$

angle(a) 0.927295

6/99

a $(3 + 4i) \cdot \; (2 - i) = 10 + 5i$

b $(3 + 4i)^{-1} = \dfrac{3}{25} - \dfrac{4}{25}i$

c $|3 + 4i| = 5$

d Arg $(3 + 4i) = 0.9273$

De Moivre's theorem

De Moivre's theorem is very useful for multiplying and dividing complex numbers. One version of de Moivre's theorem is:

If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ then

$$z_1 \cdot \; z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2).$$

A logical consequence of this involves the calculation of a power of z.

$$z^n = z \cdot \; z \cdot \; \ldots \cdot \; z \text{ (}n\text{ times)}$$

If $z = r \text{ cis } \theta$ then $z^n = r^n \text{ cis } n\theta$.

In other words, $(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta$.

WORKED Example 5

If $v = 3 \text{ cis } \dfrac{3\pi}{4}$ and $w = 4 \text{ cis } \dfrac{2\pi}{3}$, calculate each of the following.

a $v \cdot \; w$

b $\dfrac{v}{w}$

c v^4

THINK

a ① Use de Moivre's theorem for multiplying complex numbers.

WRITE

a $v \cdot \; w = 3 \text{ cis } \dfrac{3\pi}{4} \cdot \; 4 \text{ cis } \dfrac{2\pi}{3}$

$= 3 \cdot \; 4 \text{ cis } \left(\dfrac{3\pi}{4} + \dfrac{2\pi}{3}\right)$

$= 12 \text{ cis } \dfrac{17\pi}{12}$

② Subtract 2π from $\dfrac{17\pi}{12}$ so that the angle lies between $-\pi$ and π.

$= 12 \text{ cis } \left(-\dfrac{7\pi}{12}\right)$

Continued over page

THINK	**WRITE**
b Use de Moivre's theorem for dividing complex numbers.	**b** $\frac{v}{w} = \frac{3}{4} \text{ cis } \left(\frac{3\pi}{4} - \frac{2\pi}{3}\right)$ $= 0.75 \text{ cis } \frac{\pi}{12}$
c ① Use de Moivre's theorem for calculating the result of raising a complex number to a power.	**c** $v^4 = 3^4 \text{ cis } \left(4 \cdot \frac{3\pi}{4}\right)$ $= 81 \text{ cis } 3\pi$
② Subtract 2π from 3π so that the angle lies between $-\pi$ and π.	$= 81 \text{ cis } \pi$
③ Write cis π in expanded form to simplify further.	$= 81(\cos \pi + i \cdot \sin \pi)$ $= 81(-1 + 0)$ $= -81$

WORKED Example 6

a **Express $\sqrt{3} + i$ in polar form.**

b **Find one value of $\sqrt{\sqrt{3} + i}$ and express it in polar form.**

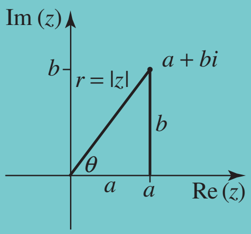
c **Express $\sqrt{\sqrt{3} + i}$ in standard form.**

THINK	**WRITE**
a ① To write a complex number in polar form, r and θ are needed for r cis θ. Find r by calculating $\sqrt{a^2 + b^2}$ where $a = \sqrt{3}$ and $b = 1$.	**a** $r = \sqrt{(\sqrt{3})^2 + 1^2}$ $= \sqrt{4}$ $= 2$
② Find θ in radians. Remember $\theta = \tan^{-1} \frac{b}{a}$ for $a + bi$.	$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $= \frac{\pi}{6}$
③ Write the answer in the form r cis θ.	$\sqrt{3} + i = 2 \text{ cis } \frac{\pi}{6}$
b ① Substitute the result found in part **a**.	**b** $\sqrt{\sqrt{3} + i} = \sqrt{2 \text{ cis } \frac{\pi}{6}}$
② Change the square root to a power of $\frac{1}{2}$.	$= \left(2 \text{ cis } \frac{\pi}{6}\right)^{\frac{1}{2}}$

THINK	WRITE
③ Use de Moivre's theorem for raising a complex number in polar form to a power. That is, $(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta$.	$= 2^{\frac{1}{2}} \text{ cis } \left(\frac{1}{2} \cdot \frac{\pi}{6}\right)$ $= \sqrt{2} \text{ cis } \frac{\pi}{12}$
c ① Use your answer from part **b**. This is in polar form.	**c** $\sqrt{\sqrt{3}+i} = \sqrt{2} \text{ cis } \frac{\pi}{12}$
② Write $\text{cis } \frac{\pi}{12}$ in expanded form.	$= \sqrt{2}\left(\cos \frac{\pi}{12} + i \cdot \sin \frac{\pi}{12}\right)$ $= \sqrt{2} \cos \frac{\pi}{12} + i \cdot \sqrt{2} \sin \frac{\pi}{12}$
③ Evaluate and simplify to obtain an expression in standard form.	$= 1.37 + 0.37i$

remember

1. A complex number has the form $a + bi$ where $a, b \in R$ and $i^2 = -1$.
2. If $z = a + bi$ and $w = c + di$ then operations are as follows.
   - (a) Addition: $z + w = a + bi + c + di = (a + c) + (b + d)i$
   - (b) Scalar multiplication: $k \cdot z = ka + kbi$
   - (c) Multiplication: $z \cdot w = (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$
   - (d) Inverse of z: $\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$
   - (e) Conjugate: If $z = a + bi$, the conjugate of z, $\bar{z} = a - bi$
3. Polar form



4. The magnitude of the complex number z is called mod (z) and is written as r or $|z|$.

$$r = \sqrt{a^2 + b^2}$$

5. (a) The angle θ is called the argument of z, arg (z), and is given by $\theta = \tan^{-1}\left(\frac{b}{a}\right)$, where θ is measured in an anticlockwise direction from the positive x-axis.

(b) It is usual though, to refer to the principal argument of z, Arg (z), which is the angle made with the positive x-axis and lying between $-\pi$ and π.

(c) Thus in polar form, sometimes called mod-Arg form, z can be written as:

$$z = r \text{ cis } \theta \text{ where } -\pi < \theta \leq \pi.$$

(cis θ is short for $\cos\theta + \boldsymbol{i} \cdot \sin\theta$.)

6. De Moivre's theorem

If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ then:

$$z_1 \cdot z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2)$$

$$z^n = r^n \text{ cis } n\theta$$

eBook plus

Interactivity:
Complex numbers: product and quotient
int-0344

EXERCISE 1A Operations on and representations of complex numbers

WORKED Example 1

1 If $u = 3 - 2i$ and $v = 2 + 4i$, evaluate each of the following without using a calculator.

a $4u$ **b** $u + v$ **c** $2u + 3v$
d $u - v$ **e** $u \cdot v$ **f** $u \mid v$
g u^2

WORKED Example 2

2 If $v = -2 + i$ and $w = 1 + 4i$, evaluate each of the following.

a $|v|$ **b** $|v + 2w|$ **c** $\overline{v}$
d $\overline{w + 2v}$ **e** $|\overline{w}|$ **f** $v\overline{v}$
g $2w \cdot \overline{2w}$ **h** v^{-1} **i** w^{-1}

3 If $z = a + bi$, show that:

a $z^{-1} = \dfrac{\overline{z}}{|z|^2}$ **b** $z\overline{z} = |z|^2$

WORKED Example 3a

4 Express each of the following complex numbers in polar form.

a $3 + 3i$ **b** $3 - 3i$ **c** $-3 + 3i$
d $2\sqrt{3} + 4i$ **e** $2\sqrt{3} - 4i$ **f** $4 + 3i$
g $-4 - 3i$ **h** $-4 + 3i$ **i** $3 + 4i$

WORKED Example 3b

5 Express each of the following complex numbers in Cartesian form.

a $4 \text{ cis } \dfrac{\pi}{3}$ **b** $3 \text{ cis } \dfrac{\pi}{6}$ **c** $20 \text{ cis } \dfrac{2\pi}{3}$
d $8 \text{ cis } \left(-\dfrac{\pi}{4}\right)$ **e** $i \text{ cis } \pi$ **f** $4 \text{ cis } 1$
g $3 \text{ cis } 2$ **h** $2 \text{ cis } (-1)$

WORKED Example 4

6 Repeat question **1** using a graphics calculator.

7 Repeat question **2** using a graphics calculator.

WORKED Example 5

8 If $u = 4 \text{ cis } \dfrac{\pi}{4}$ and $v = 8 \text{ cis } \dfrac{2\pi}{3}$, calculate each of the following leaving your answer in polar form.

a $u \cdot v$ **b** $u \mid v$ **c** $v \mid u$

d u^2 **e** v^3

WORKED Example 6

9 Express each of the following in the form indicated:

a $2 + i$ in polar form **b** $\sqrt{2 + i}$ in polar form

c $\sqrt{2 + i}$ in standard form.

10 Express each of the following in standard form.

a $\dfrac{1}{\sqrt{1 + 2i}}$ **b** $\sqrt{i}$ **c** $\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)^{32}$

11 **a** Draw an Argand diagram and mark on it the numbers: $u = 2 + i$, $v = 1 + 2i$ and $w = -2i$.

b On the same diagram mark the points: iu, iv and iw.

c Describe, geometrically, the relationship between a complex number z and iz on the Argand diagram.

12 **a** Draw an Argand diagram and mark on it the numbers: $u = 2 + i$, $v = 1 + 2i$ and $w = -2i$.

b On the same diagram mark the points $-u$, $-v$ and $-w$.

c Describe, geometrically, the relationship between a complex number z and $-z$ on the Argand diagram.

13 **a** As in question **11**, draw an Argand diagram and mark on it the numbers: $u = 2 + i$, $v = 1 + 2i$ and $w = -2i$.

b On the same diagram mark the points $-iu$, $-iv$ and $-iw$.

c Describe, geometrically, the relationship between a complex number z and $-iz$ on the Argand diagram.

14 Copy and complete the following table.

Operation on z	**Geometric relationship to z**
a $\overline{z}$	
b iz	
c $-z$	
d $-iz$	

Complex numbers and matrices

Let $H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. If we take $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to be the identity, verify that $H^2 = -I$.

From this you can see that H acts like the imaginary number i.

Take a complex number such as $3 + 4i$ and identify this number with the matrix sum.

$$\begin{aligned} 3I + 4H &= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \end{aligned}$$

In this way it is possible to develop all the properties of complex numbers using matrices.

For example:

$$(3 + 4i) \cdot (2 - i) = 6 + 4 + 8i - 3i = 10 + 5i$$

or

$$\begin{bmatrix} 10 & -5 \\ 5 & 10 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$$

1 Consider the complex number $z = 3 + 4i$.

- **a** To what matrix A does this number correspond?
- **b** Calculate A^2. To what complex number does this correspond?
- **c** Calculate A^{-1}. To what complex number does this relate? How does it relate to z?

2 Some of the properties of a matrix are:

Transpose
Inverse
Determinant

For matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ show how each of these properties connect with operations on complex numbers.

a Transpose **b** Inverse **c** Determinant

Investigating $e^{i\theta}$

Functions such as e^x, cos x and sin x have polynomial expansions.

1. The expansion for e^x is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad \frac{x^n}{n!} +$$

By substituting $x = i\theta$ in this equation, show that:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \quad + (-1)^n \frac{\theta^{2n}}{(2n)!} + \quad \right) +$$

$$i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \quad + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \quad \right)$$

2. The expansions for cos x and sin x are:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \quad (-1)^n \frac{x^{2n}}{(2n)!} +$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \quad (-1)^n \frac{x^{2n+1}}{(2n+1)!} +$$

1 Use these expansions and the result obtained above to show that:

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad [1]$$

2 Using equation [1] show that:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

3 Find a similar expression for sin θ.

4 Show that $\cos i\theta = \frac{e^{\theta} + e^{-\theta}}{2}$. This function is called hyperbolic cosine or cosh θ.

5 For further research: some of the problems presented here are involved in the algorithm that calculators use to compute 2^x. Can you discover how this algorithm works?

$e^{i\theta}$ and de Moivre's theorem

The result $e^{i\theta} = \cos\theta + i\sin\theta$ is a form of de Moivre's theorem. There are many interesting connections in mathematics, but perhaps none is as mysterious as one obtained using this form of de Moivre's theorem.

$$e^{i\pi} = -1$$

This equation connects four fundamental entities in arithmetic: 1, π, e and i.

1 Use the result $e^{i\theta} = \cos\theta + i\sin\theta$ to prove that if $v = r_1 \text{ cis } \theta_1$ and $w = r_2 \text{ cis } \theta_2$ then:

a $v \cdot w = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$

b $\frac{v}{w} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2)$

c $v^n = r_1{}^n \text{ cis } n\theta_1$.

2 Noting that $e^{i2\theta} = (e^{i\theta})^2$, show that:

a $\cos 2\theta = \cos^2\theta - \sin^2\theta$

b $\sin 2\theta = 2\sin\theta\cos\theta$.

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3 Express each of the following in standard form.

a i^i

b $e^{1 - \pi i}$

c $\log_e (2\sqrt{3} - 2i)$

Factorisation of polynomials in *C*

A polynomial in z is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z + a_0,$$

where $n \in N$ is the degree (highest power) of $P(z)$ and a_n (with $a_n \neq 0$) are the coefficients.

If $a_n \in R$ — that is, all the coefficients are real — then $P(z)$ is said to be a *polynomial over R*. Similarly, if at least one of the a_n is complex, $P(z)$ is said to be a *polynomial over C*.

For example, $P(z) = 3z^4 - 5z^2 + 6$ is a polynomial of degree 4 over R and $P(z) = 2iz^3 + 3z^2 - 8i$ is a polynomial of degree 3 over C.

The fundamental theorem of algebra

If a polynomial $P(x)$ has $(x - a)$ as a factor, then $P(x)$ can be written $P(x) = (x - a)\,Q(x)$. From this it is clear that $P(a) = (a - a)\,Q(a) = 0$. The converse can also be shown to be true; that is, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$. This result is known as the *factor theorem*.

Factor theorem: If $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$ and conversely.

In 1799 the German mathematician Carl Friedrich Gauss proved that every polynomial over C has a solution that is a complex number.

That is, if $P_n(z)$ is a polynomial of degree n over C, then there exists a $z_0 \in C$ such that $P_n(z_0) = 0$. This important result can be used to show that a polynomial of degree n, with $n \in N$, has n solutions.

The proof relies on a repeated application of the fundamental theorem of algebra and the factor theorem.

Firstly, the fundamental theorem of algebra guarantees that there is a $z_0 \in C$ such that $P_n(z_0) = 0$. The factor theorem states that if $P_n(z_0) = 0$ for some z_0 then $(z - z_0)$ is a factor of $P_n(z)$ so that $P_n(z) = (z - z_0)P_{n-1}(z)$, where $P_{n-1}(z)$ is a polynomial of degree $n - 1$.

Now by applying the fundamental theorem of algebra to $P_{n-1}(z)$ there is a $z_1 \in C$ such that $P_{n-1}(z_1) = 0$ and the factor theorem ensures that $P_{n-1}(z) = (z - z_1)P_{n-2}(z)$.

Hence $P_n(z) = (z - z_1)(z - z_0)P_{n-2}(z)$. By applying this method to each successive polynomial we can state the following:

There are n linear factors for all $P_n(z)$ over C; that is

$$P_n(z) = (z - z_n)(z - z_{n-1})(z - z_{n-2}) \ldots (z - z_2)(z - z_1)P_0(z),$$

where $P_0(z)$ is a constant.

Note: Although n solutions are obtained, the fundamental theorem of algebra does not prescribe that they are necessarily distinct.

WORKED Example 7

Factorise $z^2 + 4z + 13$ over C.

THINK	WRITE
① Use the 'complete the square' method.	$z^2 + 4z + 13 = (z^2 + 4z + 4) - 4 + 13$ $= (z + 2)^2 + 9$ $= (z + 2)^2 - 9i^2$ $= (z + 2)^2 - (3i)^2$
② Express as the 'difference of squares'.	$= (z + 2 - 3i)(z + 2 + 3i)$

WORKED Example 8

Factorise $6z^4 + z^2 - 1$ over C.

THINK	WRITE
① Express the polynomial as a quadratic.	Let $w = z^2$ so $6z^4 + z^2 - 1$ becomes $6w^2 + w - 1$.
② Factorise the quadratic.	$6w^2 + w - 1$ $= (3w - 1)(2w + 1)$
③ Write the expression in brackets as a quadratic.	$= (3z^2 - 1)(2z^2 + 1)$ $= (3z^2 - 1)(2z^2 - i^2)$
④ Factorise each quadratic by writing it as the 'difference of squares'.	$= (\sqrt{3}z - 1)(\sqrt{3}z + 1)(\sqrt{2}z - i)(\sqrt{2}z + i)$

WORKED Example 9

If $f(z) = z^3 + 7z^2 + 16z + 10$, find all factors of $f(z)$ over C
a by hand b using the TI-Nspire CAS calculator.

THINK / **WRITE/DISPLAY**

a By hand

1. Find $z = z_0$ so that $f(z_0) = 0$.

   $f(-1) = (-1)^3 + 7(-1)^2 + 16(-1) + 10 = 0$,
   so $(z + 1)$ is a factor of $f(z)$.

2. Use the factor theorem. $f(z) = (z + 1)Q(z)$, where $Q(z)$ is a polynomial of degree 2.

   Let $f(z) = (z + 1)(z^2 + pz + q)$.
   So $z^3 + 7z^2 + 16z + 10 = (z + 1)(z^2 + pz + q)$
   $z^3 + 7z^2 + 16z + 10 = z^3 + pz^2 + qz + z^2 + pz + q$

3. Compare the two expressions for $f(z)$.

   Comparing the left-hand side and right-hand side we see that:
   $q = 10$
   and $pz^2 + z^2 = 7z^2$
   so $p = 6$.

4. Factorise the quadratic equation by completing the square.

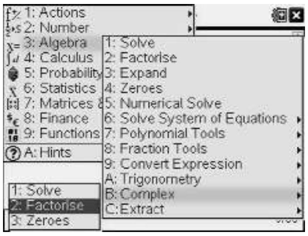
   $$\begin{aligned}\text{So } f(z) &= (z + 1)(z^2 + 6z + 10)\\ &= (z + 1)[(z + 3)^2 + 1]\\ &= (z + 1)(z + 3 - i)(z + 3 + i)\end{aligned}$$

5. State all factors of $f(z)$.

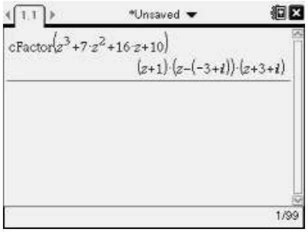
   The three factors of $f(z)$ are $z + 1$, $z + 3 - i$ and $z + 3 + i$.

b Using the TI-Nspire CAS

1. To find factors of $f(z)$ over C, on a Calculator page press:
   - MENU menu
   - 3: Algebra 3
   - B: Complex B
   - 2: Factor 2



2. Complete the entry line as: cFactor($z^3 + 7z^2 + 16z + 10$) then press ENTER enter.



$\text{cFactor}(z^3+7\cdot z^2+16\cdot z+10)$
$(z+1)\cdot(z-(-3+i))\cdot(z+3+i)$

3. Write the answer.

   $f(z) = z^3 + 7z^2 + 16z + 10$
   $f(z) = (z + 1)(z + 3 - i)(z + 3 + i)$

Conjugate root theorem

In the previous three examples the polynomials contained only real coefficients and the complex factors occurred in conjugate pairs.

The conjugate root theorem states: ‘If a polynomial has real coefficients, its roots are either real numbers or occur as pairs of conjugate complex numbers.’

To prove the theorem, we aim to show that for some z, if $P(z) = 0$, then $P(\bar{z}) = 0$. First, by the fundamental theorem of algebra, $P(z) = 0$ for some z so $\overline{P(z)} = 0$. Now, if $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z + a_0$, then:

$$\overline{P(z)} = \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \overline{a_{n-2} z^{n-2}} + \ldots + \overline{a_1 z} + \overline{a_0}$$

(using the property that $\overline{z + w} = \bar{z} + \bar{w}$)

$$= \bar{a}_n \bar{z}^n + \bar{a}_{n-1} \bar{z}^{n-1} + \bar{a}_{n-2} \bar{z}^{n-2} + \ldots + a_1 \bar{z} + \bar{a}_0$$

(using the property that $\overline{zw} = \bar{z} \cdot \bar{w}$)

$$= a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + a_{n-2} \overline{z^{n-2}} + \ldots + a_1 \bar{z} + a_0$$

(since a_n, $n \in N$ are real coefficients)

$$= a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + a_{n-2} \bar{z}^{n-2} + \ldots + a_1 \bar{z} + a_0 \quad (\text{since } \bar{z}^n = \bar{z}^n)$$

$$= P(\bar{z}).$$

Since we assumed $P(z) = 0$ then $P(\bar{z}) = 0$ so $\bar{z}$ is also a solution.

The conjugate root theorem provides an immediate way of determining zeros (roots) of the polynomial if at least one zero (root) is known.

WORKED Example 10

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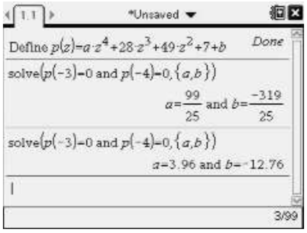
Tutorial:
Worked example 10
int-0372

If $5 + 5\sqrt{2}i$ is a zero of $P(z) = z^3 - 8z^2 + 55z + 150$, find the other two zeros over C.

THINK	WRITE
① Since all the coefficients of the polynomial are real, use the conjugate root theorem to find one of the other roots.	Let $z_1 = 5 + 5\sqrt{2}i$. Then $z_2 = 5 - 5\sqrt{2}i$ is another zero.
② Define the third zero.	Let z_3 be the third zero. Then $P(z) = (z - z_1)(z - z_2)(z - z_3)$ $= (z - 5 - 5\sqrt{2}i)(z - 5 + 5\sqrt{2}i)(z - z_3)$
③ Expand the two brackets containing the two zeros.	$= [(z - 5)^2 + 50](z - z_3)$ $= (z^2 - 10z + 75)(z - z_3)$
④ Compare the two expressions for $P(z)$.	So $z^3 - 8z^2 + 55z + 150 = (z^2 - 10z + 75)(z - z_3)$. Comparing the left- and right-hand sides: $150 = 75 \cdot -z_3$ so $z_3 = -2$.
⑤ State the other two zeros of $P(z)$ over C.	Therefore the two zeros required are -2 and $5 - 5\sqrt{2}i$.

WORKED Example 11

Find the values of a and b ($a, b \in R$) if –3 and –4 are zeros of $P(z) = az^4 + 28z^3 + 49z^2 + 7 + b$. Find the values a by hand b using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a By hand	**a**
1 Obtain expressions for $P(-3)$ and $P(-4)$.	$P(-3) = a(-3)^4 + 28(-3)^3 + 49(-3)^2 + 7 + b$ $= 81a + b - 308$ $P(-4) = a(-4)^4 + 28(-4)^3 + 49(-4)^2 + 7 + b$ $= 256a + b - 1001$
2 Write down the pair of simultaneous equations in a and b.	Since $P(-3) = 0$ and $P(-4) = 0$ we have the two equations $81a + b - 308 = 0$ [1] $256a + b - 1001 = 0$ [2]
3 Solve the simultaneous equations for a by subtracting equation [1] from equation [2].	$175a - 693 = 0$ $a = 3.96$
4 Substitute $a = 3.96$ into equation [1] to find b.	Substituting $a = 3.96$ into equation [1]: $320.76 + b - 308 = 0$ $b = -12.76$
5 State the values of a and b.	So $a = 3.96$, $b = -12.76$.
b Using the TI-Nspire CAS	**b**
1 On a Calculator page, complete the entry line as: Define $p(z) = a \cdot z^4 + 28z^3 + 49z^2 + 7 + b$ Press ENTER enter. To solve for a and b, complete the entry line as: solve($p(-3) = 0$ and $p(-4) = 0$, $\{a, b\}$) Press ENTER enter.	 Define $p(z)=a\cdot z^4+28\cdot z^3+49\cdot z^2+7+b$ Done solve($p(-3)=0$ and $p(-4)=0,\{a,b\}$) $a=\frac{99}{25}$ and $b=\frac{-319}{25}$ solve($p(-3)=0$ and $p(-4)=0,\{a,b\}$) $a=3.96$ and $b=-12.76$ 3/99
2 Write the answer.	$P(z) = az^4 + 28z^3 + 49z^2 + 7 + b$ If $P(-3) = 0$ and $P(-4) = 0$, solving for a and b gives $a = 3.96$ and $b = -12.76$.

WORKED Example 12

Find the values of a and b ($a, b \in R$) if $z - 2$ is a factor of $P(z) = iz^3 + a(1 - i)z^2 - (4 + i)z + bi$.

THINK	WRITE
1 Obtain an expression for $P(2)$ in the form $x + yi$.	$P(2) = i(2)^3 + a(1 - i)(2)^2 - (4 + i)(2) + bi$ $= 8i + 4a - 4ai - 8 - 2i + bi$ $= (4a - 8) + (6 - 4a + b)i$

THINK	**WRITE**
② Since $P(2) = 0$, equate $P(2)$ to $0 + 0i$.	$P(2) = (4a - 8) + (6 - 4a + b)i$ $= 0 + 0i$
③ Equate the real parts of $P(2)$ and solve for a.	$4a - 8 = 0$ $a = 2$
④ Equate the imaginary parts of $P(2)$ and solve for b.	$6 - 4a + b = 0$ $6 - 8 + b = 0$ $b = 2$
⑤ State the solution for a and b.	Solving for a and b gives $a = 2$ and $b = 2$.

remember

1. Every polynomial over C has a solution that is a complex number (fundamental theorem of algebra).
2. A polynomial of degree n has n solutions.
3. A polynomial with real coefficients has solutions that are real numbers or that occur as pairs of conjugate complex numbers (conjugate root theorem).
4. Factor theorem states: if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$ and conversely.

EXERCISE 1B

Factorisation of polynomials in C

WORKED Example 7

1 Factorise each of the following over C.

a $z^2 + 4$ **b** $z^2 + 7$ **c** $z^2 + 8z + 25$

d $z^2 - 3z + 4$ **e** $4z^2 - 4z + 17$ **f** $-9z^2 + 24z - 32$

WORKED Example 8

2 Factorise each of the following over C.

a $z^4 - 81$ **b** $z^4 - 2z^2 - 3$ **c** $z^4 + 20z^2 + 64$ **d** $z^4 + 3z^2 - 10$

WORKED Example 9

3 Find all of the factors $f(z)$ over C for each of the following.

a $f(z) = z^3 - 4z^2 + 2z + 28$ **b** $f(z) = z^3 + z^2 + 2z - 4$

c $f(z) = z^3 - z^2 - z + 10$ **d** $f(z) = 2z^3 + 3z^2 - 14z - 15$

e $f(z) = z^4 - 2z^2 - 16z - 15$ **f** $f(z) = z^6 - 1$

WORKED Example 10

4 For each of the following polynomials, find the other two zeros of $P(z)$ if:

a $1 + i$ is a zero of $P(z) = z^3 + 4z^2 - 10z + 12$

b $-2 + i$ is a zero of $P(z) = 2z^3 + 9z^2 + 14z + 5$

c $4 - i$ is a root of $P(z) = z^3 - 10z^2 + 33z - 34$.

5 If $z - 3i$ is a factor of $2z^4 - 4z^3 + 21z^2 - 36z + 27$, find the remaining factors.

6 **multiple choice**

The zeros of $P(z) = z^2 + (3 + 2i)z + 6i$ are:

A $2i, -3$ **B** $3i, 2$ **C** $3i, -2$ **D** $-3, -2$ **E** $-2i, -3$

7 **multiple choice**

If $P(z) = z^3 - (1 + 2i)z^2 + 2(1 + i)z - 2$ and $P(z) = (z - 1)Q(z)$, where $Q(z)$ is a polynomial, then $Q(i)$ is:

A 4 **B** 2 **C** 0 **D** 3 **E** 1

8 **multiple choice**

If $P(z)$ is a polynomial of degree 4 with all of its coefficients real with ai and bi $(a, b \in R)$ as two of its zeros, then the term that does not contain z is:

A ab **B** $a - b$ **C** $a + b$ **D** a^3b^3 **E** a^2b^2

9 **multiple choice**

If $P(z) = z^3 + 2z^2 - 6z + a$ and $P(1 - i) = 0$, then a is equal to:

A 4 **B** −6 **C** 8 **D** 6 **E** 5

10 Find the value of a in each of the following given that:

a $(z + 2)$ is a factor of $z^3 + 3z^2 + az + 8$

b $(z + i)$ is a factor of $z^3 + az^2 + z - 4$

c $(z + 1 - 2i)$ is a factor of $2z^3 + 3z^2 + 8z + a$

d $2i$ is a root of $z^3 - 2z^2i + az - 32i$.

WORKED Example 11

11 Find the values of a and b $(a, b \in R)$ given that:

a -3 and 2 are zeros of $z^3 + az^2 + 8z + b$

b 4 and 1 are zeros of $z^4 + az^3 + bz^2 - 7z + 12$

c $2i$ and $3i$ are roots of $P(z) = z^3 + aiz^2 + bz - 12i$.

WORKED Example 12

12 Find the values of a and b $(a, b \in R)$ if:

a $(z + 1)$ is a factor of $z^3 - 2iz^2 + aiz + b$

b $(z - i)$ is a factor of $az^3 - 3z^2 + biz + 12i$

c $(z + 2i)$ is a factor of $z^3 + aiz^2 + 2iz + (1 + i)b$.

13 Explain why at least one of the zeros of a polynomial of degree n (where n is an odd natural number) is a real number.

14 Write down a polynomial of degree 3, whose coefficients are all real, that has $4i$ and 2 as two of its zeros.

15 Find the values of a, $(a \in R)$ for which ai is a solution to:

a $P(z) = z^3 + 3z^2 + 36z + 108$ **b** $P(z) = z^3 + 6iz^2 - 11z - 6i$.

16 Factorise $z^3 + i$ over C.

17 **a** Show that $P(1) = 0$ for $P(z) = z^4 - (1 + 3i)z^3 + 3(i - 1)z^2 + (7 + i)z - 4 - i$.

b Find the polynomial $Q(z)$ if $P(z) = (z - 1)Q(z)$.

c Determine the values of $a \in C$, $b \in R$ if $Q(z)$ is of the form $Q(z) = (z - a)^3 + b$.

18 Factorise $z^4 + 2z^3 + 8z^2 + 10z + 15$ over C given that $z + \sqrt{5}\,i$ is a factor.

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Digital doc: **WorkSHEET 1.1**

19 Factorise $P(z) = 9z^3 + (9i - 12)z^2 + (5 - 12i)z + 5i$ over C if $P(-i) = 0$.

20 Determine the value of $a \in R$ if $-\sqrt{3}i$ is to be a zero of $a + z^2 = \dfrac{a - 11}{z^2}$.

Solving equations in *C*

Solving polynomial equations using factorisation

A polynomial equation $P(z) = 0$ can be solved by first factorising $P(z)$ so that:

$$\text{if } P(z) = (z - z_1)(z - z_2)\qquad(z - z_n) = 0$$

$$\text{then } z = z_1, z_2, \quad , z_n.$$

Solution to quadratic equations

A quadratic equation of the form $az^2 + bz + c = 0$, where $a, b, c \in R$, can be solved using the quadratic formula: $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

WORKED Example 13

Solve the quadratic equation $x^2 - 2x + 2 = 0$ over C.

THINK

① Use either the 'completing the square' method or the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

② Substitute i^2 for -1.

③ Simplify the expression.

④ State the solutions for x.

WRITE

$$x^2 - 2x + 2 = 0$$
$$(x - 1)^2 - 1^2 + 2 = 0$$
$$(x - 1)^2 + 1 = 0$$
$$(x - 1)^2 = -1$$
$$x - 1 = \pm\sqrt{-1}$$
$$x = 1 \pm \sqrt{i^2}$$
$$= 1 \pm i$$

or

$a = 1$, $b = -2$, $c = 2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 \pm \sqrt{4i^2}}{2}$$

$$x = \frac{2 \pm 2i}{2}$$

$$x = 1 \pm i$$

The two solutions for x are $x = 1 + i$, $x = 1 - i$.

Solving cubic equations

WORKED Example 14

Solve the equation $z^3 + 3z^2 + 8z - 12 = 0$ over C a by hand b using the TI-Nspire CAS calculator.

THINK

a By hand

① Use the factor theorem to find a solution to the cubic equation.

WRITE/DISPLAY

a

$$P(1) = (1)^3 + 3(1)^2 + 8(1) - 12 = 0$$

$z = 1$ is a solution.

Continued over page →

THINK | **WRITE/DISPLAY**

② Express the first solution as a factor of the cubic.

$z - 1$ is a factor.

③ Determine the quadratic factor.

$z^3 + 3z^2 + 8z - 12 = (z - 1)(z^2 + 4z + 12)$

④ Solve the quadratic expression using the quadratic formula.

If $z^2 + 4z + 12 = 0$

$$\begin{aligned} z &= \frac{-4 \pm \sqrt{4^2 - 4(1)(12)}}{2 \cdot 1} \\ &= \frac{-4 \pm \sqrt{-32}}{2} \\ &= \frac{-4 \pm 4\sqrt{-2}}{2} \\ &= -2 \pm 2\sqrt{-2} \\ &= -2 \pm 2\sqrt{2i^2} \\ &= -2 \pm 2\sqrt{2}i \end{aligned}$$

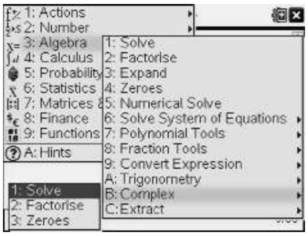
⑤ State the three solutions to the cubic equation.

The three solutions are 1, $-2 + 2\sqrt{2}i$ and $-2 - 2\sqrt{2}i$.

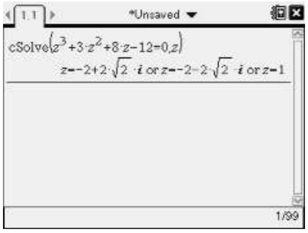
b **Using the TI-Nspire CAS**

① On a Calculator page, press:
- MENU (menu)
- 3: Algebra (3)
- B: Complex (B)
- 1: Solve (1)

b



② Complete the entry line as:
cSolve($z^3 + 3z^2 + 8z - 12 = 0$, z)
then press ENTER (enter).



```
cSolve(z^3+3·z^2+8·z-12=0,z)
        z=-2+2·√2·i or z=-2-2·√2·i or z=1
```

③ Write the answer.

$z^3 + 3z^2 + 8z - 12 = 0$
Solving for z over C gives
$z = 1$ or $z = -2 + 2\sqrt{2}i$ or $z = -2 - 2\sqrt{2}i$.

WORKED Example 15

Solve the equation $x^4 + 13x^2 + 36 = 0$ over C.

THINK | **WRITE**

① Obtain a quadratic equation by putting $y = x^2$.

$x^4 + 13x^2 + 36 = 0$
$y^2 + 13y + 36 = 0$, with $y = x^2$

② Factorise the quadratic equation.

$(y + 4)(y + 9) = 0$

③ Substitute for y.

$(x^2 + 4)(x^2 + 9) = 0$

THINK

4. Solve for x using each expression contained in brackets.

WRITE

$x^2 + 4 = 0$, $x^2 + 9 = 0$

$x^2 = -4$, $x^2 = -9$

$x = \pm\sqrt{-4}$, $x = \pm\sqrt{-9}$

$x = \pm\sqrt{4i^2}$, $x = \pm\sqrt{9i^2}$

$x = \pm 2i$, $x = \pm 3i$

The four solutions for x are $x = 2i, -2i, 3i, -3i$.

Square root of z

Given $z^2 = a + bi$ with $z = x + yi$ and $a, b, x, y \in R$, we seek values for x and y so that:

$$x + yi = \sqrt{a + bi}.$$

Then $$(x + yi)^2 = a + bi.$$

By expanding $(x + yi)^2$ we obtain $(x^2 - y^2) + 2xyi = a + bi$.

Equating real and imaginary parts yields two equations:

$$x^2 - y^2 = a \text{ and } 2xy = b.$$

Solving this pair of simultaneous equations gives $x^2 = \dfrac{a + \sqrt{a^2 + b^2}}{2}$. (Verify this.)

(Notice the requirement to take the positive square root, since x is real.)

The value of y can now be determined by substituting for x^2 in $y^2 = x^2 - a$.

WORKED Example 16

Find $\sqrt{3 + 4i}$ in Cartesian form.

THINK

1. Determine the values of a and b in the expression $z^2 = a + bi$.
2. Substitute a and b in $x^2 = \dfrac{a + \sqrt{a^2 + b^2}}{2}$ to find x.
3. Use $y^2 = x^2 - a$ to find y.

WRITE

$z = \sqrt{3 + 4i}$ so $z^2 = 3 + 4i$. Take $a = 3$ and $b = 4$.

$$x^2 = \frac{3 + \sqrt{3^2 + 4^2}}{2}$$

$x^2 = 4$

$x = \pm 2$

$y^2 = 4 - 3$

$y^2 = 1$

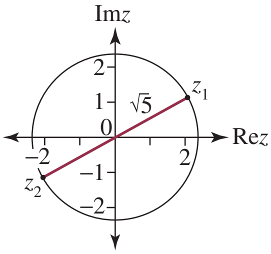
$y = \pm 1$

Therefore the two roots z_1, z_2 are $z_1 = 2 + i$ and $z_2 = -2 - i$.

So $\sqrt{3 + 4i}$ is $2 + i$ or $-2 - i$.

Notes

1. The line segment connecting z_1 to z_2 'halves' the complex number plane.
2. The root z_2 can be obtained by rotating z_1 through an angle of 180° corresponding to multiplication by i^2. That is, $z_2 = i^2 z_1$.
3. The modulus of z_1 and z_2 is $\sqrt{5}$ and z_1 and z_2 lie on the circle of radius $\sqrt{5}$ units.



An alternative method to finding the square root and the cube root of z involves expressing z in polar form. This is outlined below.

Solving equations of the form $z^n = w$, where $w \in C$

Values of z that satisfy $z^n = w$, where $z, w \in C$ and $n \in N$, can be found using de Moivre's theorem. We seek solutions to $z^n = w$ for $n = 2$ and $n = 3$ and for the particular case $w = a$ where $a \in R$. The steps involved are outlined as follows.

If $z^n = w$,

1. Express w in polar form $r \text{ cis } \theta$.
2. Express $r \text{ cis } \theta = r \text{ cis}(\theta + 2k\pi)$, $k \in z$ (since the solution repeats every 2π).
3. Therefore, $z = [r \text{ cis}(\theta + 2k\pi)]^{\frac{1}{n}}$, $k \in z$, (by de Moivre's theorem).
4. Hence $z = r^{\frac{1}{n}} \text{ cis}\left(\dfrac{\theta + 2k\pi}{n}\right)$, $k \in z$, using k until n solutions are found (that is, until the solutions start repeating).

Cube root of z

WORKED Example 17

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Tutorial:
Worked example 17
int-0373

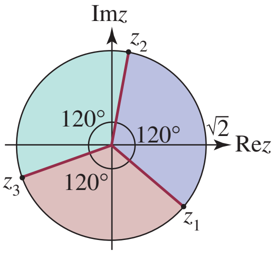
Use de Moivre's theorem to solve $z^3 = -2 - 2i$, in polar form.

THINK	WRITE
① Express $w = -2 - 2i$ in polar form.	If $w = -2 - 2i = r \text{ cis } \theta$: $r = \sqrt{(-2)^2 + (-2)^2}$ $= \sqrt{8}$ $\tan \theta = \dfrac{-2}{-2} = 1$ $\theta = \pi + \tan^{-1} 1$ (3rd quadrant) $= \pi + \dfrac{\pi}{4}$ $= \dfrac{5\pi}{4}$ So $w = \sqrt{8} \text{ cis}\left(\dfrac{5\pi}{4} + 2k\pi\right)$
② Let $z^3 = w$.	Let $z^3 = \sqrt{8} \text{ cis}\left(\dfrac{5\pi}{4} + 2k\pi\right)$
③ Determine z using de Moivre's theorem.	$z = \left[\sqrt{8} \text{ cis}\left(\dfrac{5\pi}{4} + 2k\pi\right)\right]^{\frac{1}{3}}$ $= \sqrt{2} \text{ cis}\left(\dfrac{5\pi}{12} + \dfrac{2k\pi}{3}\right)$

THINK	**WRITE**
④ Evaluate the solutions by rule.	$k = 0, \quad z = \sqrt{2}\ \text{cis}\dfrac{5\pi}{12}$ $k = 1, \quad z = \sqrt{2}\ \text{cis}\left(\dfrac{5\pi}{12} + \dfrac{2\pi}{3}\right)$ $= \sqrt{2}\ \text{cis}\dfrac{13\pi}{12}$ $= \sqrt{2}\ \text{cis}\left(-\dfrac{11\pi}{12}\right)$ $k = 2, \quad z = \sqrt{2}\ \text{cis}\left(\dfrac{5\pi}{12} + \dfrac{4\pi}{3}\right)$ $= \sqrt{2}\ \text{cis}\dfrac{21\pi}{12}$ $= \sqrt{2}\ \text{cis}\left(-\dfrac{3\pi}{12}\right)$ $= \sqrt{2}\ \text{cis}\left(-\dfrac{\pi}{4}\right)$
⑤ State the solutions.	The three cube roots of $-2 - 2i$ in polar form are $\sqrt{2}\ \text{cis}\left(\dfrac{5\pi}{12}\right)$, $\sqrt{2}\ \text{cis}\left(-\dfrac{11\pi}{12}\right)$ and $\sqrt{2}\ \text{cis}\left(-\dfrac{\pi}{4}\right)$.

Notes

1. On the complex number plane, z_1, z_2 and z_3 are equidistant from the origin and are equally spaced $\left(\dfrac{2\pi}{3} \text{ or } 120°\right)$ on the circle of radius $\sqrt{2}$ units.
2. The three solutions should be converted to Cartesian form if required.



Solving $z^n = a$, where $a \in R$

Since $1 = 1 + 0i = \text{cis } 0$, it follows that:

$$\begin{aligned} a &= a \cdot 1 = a \text{ cis } 0 \\ &= a \text{ cis}(0 + 2k\pi),\ k \in J \end{aligned}$$

Therefore solving $z^n = a$ becomes $z^n = a \text{ cis } 2k\pi$, $k \in J$, and using de Moivre's theorem as outlined previously it follows that:

If $z^n = a$, $a \in R$,

then $$z = a^{\frac{1}{n}} \text{ cis } \frac{2k\pi}{n}$$

where $k \in J$ until n solutions are found.

WORKED Example 18

Find the solutions to $z^4 = 9$ in Cartesian form.

THINK	WRITE
① Express the equation in polar form.	$z^4 = 9$ $= 9 \text{ cis } 2k\pi$
② Use de Moivre's theorem to find z in polar form.	$\Rightarrow z = (9 \text{ cis } 2k\pi)^{\frac{1}{4}}$ $= 9^{\frac{1}{4}} \text{ cis } \frac{2k\pi}{4}$ $= \sqrt{3} \text{ cis } \frac{k\pi}{2}$
③ Substitute $k = 0$, 1, 2 and 3 to find the four solutions.	$k = 0, z = \sqrt{3} \text{ cis } 0$ $k = 1, z = \sqrt{3} \text{ cis } \frac{\pi}{2}$ $k = 2, z = \sqrt{3} \text{ cis } \pi$ $k = 3, z = \sqrt{3} \text{ cis } \frac{3\pi}{2}$
④ Express the solutions in Cartesian form.	The four solutions in Cartesian form are $\sqrt{3}$, $\sqrt{3}\, i$, $-\sqrt{3}$, and $-\sqrt{3}\, i$.

The exact values of cos $\frac{2\pi}{5}$ and sin $\frac{2\pi}{5}$

Consider the values of z for which $z^5 = 1$.

1. Substitute $z = x + yi$ in $z^5 = 1$ and expand the left-hand side.
2. Equate the real parts to obtain an equation in x and y.
3. By expressing $|z|$ in terms of x and y obtain an equation for y^2 in terms of x^2.
4. Combine the equations from parts **2** and **3** to obtain an equation involving powers of x only.
5. Using a computer package, or otherwise, solve the equation to get two exact positive solutions for x.
6. We know that the five solutions for $z^5 = 1$ are given by cis $\frac{2k\pi}{5}$, $k \in N$. Use this information with the appropriate answer from **5** to obtain $\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$.
7. Use Pythagoras' theorem together with your answer for $\cos \frac{2\pi}{5}$ to show that $\sin \frac{2\pi}{5} = \frac{\sqrt{2}\sqrt{5 + \sqrt{5}}}{4}$.

remember

1. A polynomial $P(z) = (z - z_1)(z - z_2) \dots (z - z_n)$ has solutions $z = z_1, z_2, \dots z_n$.
2. Use the 'complete the square' method or $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve equations in z that can be expressed in quadratic form.
3. Substitute $-i^2$ for 1 to change expressions of the form $a^2 + b^2$ to the 'difference of two squares': $a^2 + b^2 = (a + bi)(a - bi)$.
4. If $z^n = r \text{ cis } \theta$, then: $z = r^{\frac{1}{n}} \text{ cis} \left(\dfrac{\theta + 2k\pi}{n}\right)$, $k \in J$, using k until n solutions are found (that is, until the solutions start repeating).
5. If $z^n = a$, $a \in R$, then: $z = a^{\frac{1}{n}} \text{ cis } \dfrac{2k\pi}{n}$, where $k \in J$ until n solutions are found.

EXERCISE 1C Solving equations in C

WORKED Example 13

1 Solve the following quadratic equations over C.

a $x^2 + 2x + 5 = 0$ b $x^2 - 8x + 25 = 0$ c $x^2 - 14x + 149 = 0$

d $4x^2 - 12x + 13 = 0$ e $4x^2 - \sqrt{32}\,x + 4 = 0$

WORKED Example 14

2 Solve the following equations over C.

a $z^3 - z^2 - z + 10 = 0$ b $z^3 - 2z^2 + 3z - 2 = 0$ c $2z^3 - 7z^2 + 10z - 8 = 0$

d $3z^3 - 13z^2 + 5z - 4 = 0$ e $4z^3 - 20z^2 + 34z - 20 = 0$

3 For $f(z) = z - 4$, $g(z) = z^2 - z + 1$ and $h(z) = z^3 - 5z^2 + 5z - 4$ show that $f(z) \cdot g(z) = h(z)$ and hence determine the values of z such that $h(z) = 0$.

WORKED Example 15

4 Solve these equations over C.

a $x^4 + 25x^2 + 144 = 0$ b $z^4 - 3z^2 - 4 = 0$

c $9z^4 + 35z^2 - 4 = 0$ d $4x^4 + 12x^2 + 9 = 0$

5 **multiple choice**

The solutions to the equation $(z - 3)^2 + 4 = 0$ are:

A $z = 2 + 3i, z = 2 - 3i$ B $z = 3 - 2i, z = 3 + 2i$ C $z = 3 + 4i, z = 3 - 4i$

D $z = 4 - 3i, z = 4 + 3i$ E $z = 9 + 16i, z = 9 - 16i$

WORKED Example 16

6 Find the square roots of each of the following in Cartesian form.

a $1 + \sqrt{3}i$ b $11 + 60i$ c $16 + 63i$

7 Find $\sqrt{i}$ in Cartesian form.

8 **multiple choice**

If one of the roots of $a(1 + i)$ is $\sqrt{a} \text{ cis } \dfrac{\pi}{8}$, the other root is:

A $\sqrt{a} \text{ cis } \dfrac{7\pi}{8}$ B $-\sqrt{a} \text{ cis } \dfrac{7\pi}{8}$ C $\sqrt{a} \text{ cis } -\dfrac{7\pi}{8}$

D $\sqrt{a} \text{ cis } \dfrac{9\pi}{8}$ E $-\sqrt{a} \text{ cis } \dfrac{9\pi}{8}$

WORKED Example 17

9 Use de Moivre's theorem to solve the following equations, in polar form.

a $z^2 = \sqrt{3} - i$ **b** $z^2 = 4 + 4i$ **c** $z^3 = -4 + 4\sqrt{3}\,i$

d $z^3 = i$ **e** $z^3 = -1 - i$

10 Find $(-125i)^{\frac{1}{3}}$ and determine the value of the sum of the roots.

11 a Find the cube root of 64.

b Show the results on an Argand diagram.

WORKED Example 18

12 Solve the following equations in Cartesian form.

a $z^4 = 16$ **b** $z^4 = 25$ **c** $z^6 = 64$ **d** $z^6 = 27$

13 Find all z satisfying $z^5 = 1$. Express your answers in polar form.

Current and voltage

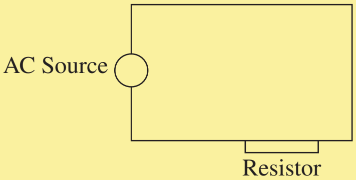
In Junior Science we learn that when a voltage, V, is applied across a resistor of resistance R, then the current flowing through the resistor, I, can be found using Ohm's Law:

$$V = IR.$$

In electrical circuits, the resistor may be in the form of a light or a thin wire.

Let us consider what happens in a circuit when an alternating current (AC) is flowing.

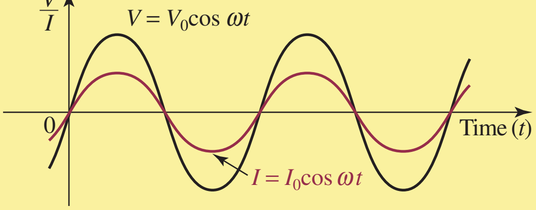
An AC voltage source produces the electric current, I, which varies according to the rule $I = I_0 \cos \omega t$. I_0 is the maximum current, t is the time in seconds and ω is the angular frequency.



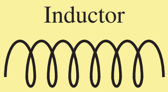
The voltage across the resistor is given by $V = R \cdot\ I_0 \cos \omega t$.

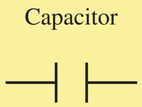
This is written $V = V_0 \cos \omega t$.

Notice that the voltage and current are in phase; that is, they achieve maximum and minimum values at the same time.



Two other types of element used in circuits are inductors and capacitors. The measure of the effect of an inductor is called inductance (L) and the measure of the effect of a capacitor is called capacitance, C. An inductor is essentially a coil of wire and a capacitor can be thought of as a set of parallel plates.





The relationship between voltage and current for flow through these components is not as straightforward as that for the resistor.

Complex numbers are used by engineers to analyse the relationship between current and voltage in a circuit such as this.

If the current is $I_0\cos \omega t$, then it can be shown that the voltage needed to sustain this current is:

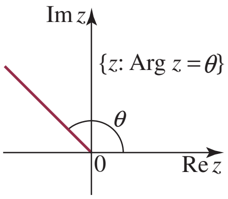
$$\text{Re}\left[\left(i\omega L + R + \frac{1}{i\omega C}\right)I_0 e^{i\omega t}\right]$$

1 Derive an expression for the voltage needed to sustain a current of $I = 5$ cos (130t)A in a circuit with a resistor, inductor and capacitor as shown above, if $R = 80\ \Omega$, $L = 0.15$ H and $C = 3 \cdot\ 10^{-6}$ F, where A is the unit for current (amps).

Graphs of complex relations: rays, lines and circles

Rays

Consider $\{z: \text{Arg } z = \theta\}$. This expression defines the set of points which make an angle of θ with the positive real axis.
This is represented on the complex plane by a *ray* which starts at the origin and makes an angle of θ with the positive real axis.



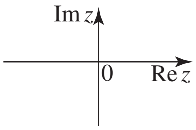
WORKED Example 19

Sketch $\left\{z: \text{Arg } z = -\frac{5\pi}{6}\right\}$ on the complex plane.

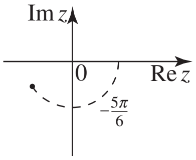
THINK

① Draw a complex plane.

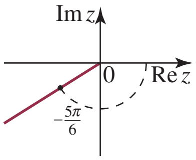
WRITE



② Mark a point which is approximately $-\frac{5\pi}{6}$ radians from the positive real axis (3rd quadrant).



③ Draw a ray from the origin through this point.



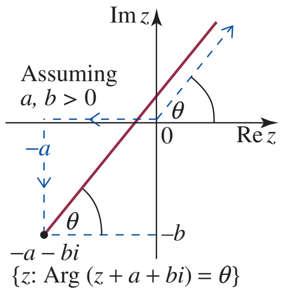
Translation of rays

The translation of a complex relation is similar to translations on the Cartesian plane. For example, the relation $y + 1 = (x + 2)^2$ is the same as $y = x^2$ translated -2 units in the x-direction and -1 unit in the y-direction.

Consider the ray defined by $\{z: \text{Arg } (z + a + bi) = \theta, a, b \in R\}$.

This is the same set as $\{z: \text{Arg } z = \theta\}$ translated $-a$ units in the 'real' direction and $-b$ units in the 'imaginary' direction.

Note: The ray starts at the point $(-a - bi)$, making an angle of θ with the positive side of the horizontal line passing through $-b$ on the imaginary axis.



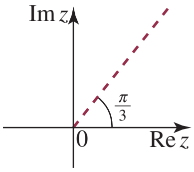
WORKED Example 20

Show $\left\{z\text{: Arg }(z + 4 - 2i) = \dfrac{\pi}{3}\right\}$ on an Argand diagram.

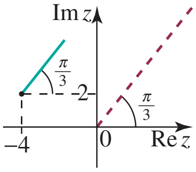
THINK

① Sketch $\left\{z\text{: Arg } z = \dfrac{\pi}{3}\right\}$ on a complex plane.

WRITE

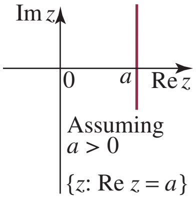


② Translate the ray –4 units in the real direction and +2 units in the imaginary direction.

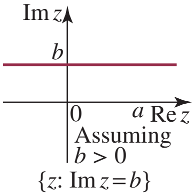


Lines

Consider $\{z\text{: Re } z = a, a \in R\}$. This defines the set of points whose real part is a (that is, $x = a$). It is represented on the complex plane by a vertical line passing through the point $a + 0i$.



Now consider $\{z\text{: Im } z = b, b \in R\}$. This defines the set of points whose imaginary part is b (that is, $y = b$). It is represented on the complex plane by a horizontal line passing through the point $0 + bi$.



WORKED Example 21

Sketch the graphs of the following on the complex plane.

a $\{z\text{: Re } z = \frac{7}{2}\}$ **b** $\{z\text{: Im } z = -2\}$

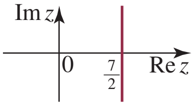
THINK

a Sketch a vertical line through the point $\frac{7}{2} + 0i$ on the complex plane.

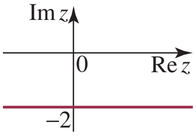
b Sketch a horizontal line through the point $0 - 2i$ on the complex plane.

WRITE

a

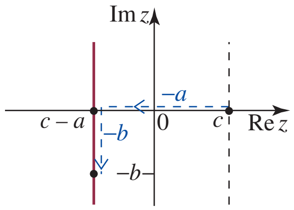


b



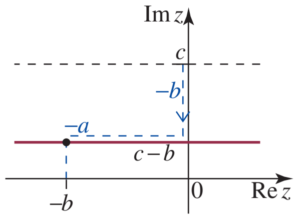
Translation of lines

The graph of $\{z\text{: Re }(z + a + bi) = c$, where $a, b, c \in$ R$\}$ is the same as the graph of $\{z\text{: Re } z = c\}$ translated $-a$ units in the real direction and $-b$ units in the imaginary direction.
Note: Since the graph is a vertical line the translation in the imaginary direction is meaningless.



$\{z\text{: Re }(z + a + bi) = c\}$
Assuming $a, b, c > 0$ and $a > c$

Likewise the graph of $\{z\text{: Im }(z + a + bi) = c$, where $a, b, c \in$ R$\}$ is the same as the graph of $\{z\text{: Im } z = c\}$ translated $-a$ units in the real direction and $-b$ units in the imaginary direction.
Note: Since the graph is a horizontal line the translation in the real direction is meaningless.



$\{z\text{: Im }(z + a + bi) = c\}$
Assuming $a, b, c > 0$ and $b < c$

WORKED Example 22

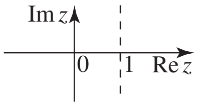
Show on an Argand diagram:
a $\{z\text{: Re }(z - 2 + 3i) = 1\}$ **b** $\{z\text{: Im }(z - 5i) = -3\}$.

THINK

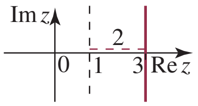
WRITE

a ① Sketch the graph of $\{z\text{: Re } z = 1\}$ on a complex plane.

a

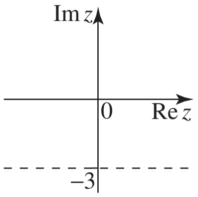


② Translate the line +2 units in the real direction (and −3 units in the imaginary direction).

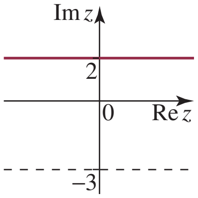


b ① Sketch the graph of $\{z\text{: Im } z = -3\}$ on a complex plane.

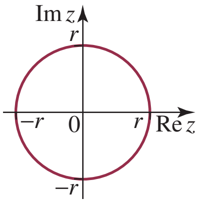
b



② Translate the line +5 units in the imaginary direction.



Circles



Since $|z|$ is defined as the distance of a point from the origin on the complex plane, it follows that $\{z: |z| = r\}$ is the set of all the points on the complex plane that are r units from the origin.

$\{z: |z| = r\}$ defines a circle of radius r and centre $0 + 0i$.

WORKED Example 23

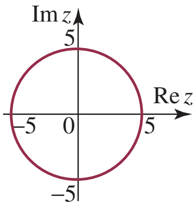
Illustrate $\{z: |z| = 5\}$ on an Argand diagram.

THINK

① The equation defines a circle of radius 5 and centre $0 + 0i$.

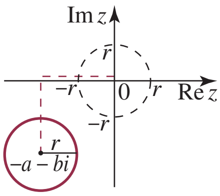
② Sketch this circle on the complex plane.

WRITE



Translation of circles

Consider $\{z: |z + a + bi| = r, a, b \in R\}$. This defines the same set of points as $\{z: |z| = r\}$ but translated $-a$ units in the real direction and $-b$ units in the imaginary direction.



Assuming $a, b > 0$

$\{z: |z + a + bi| = r\}$ defines a circle of radius r and centre $(-a - bi)$.

WORKED Example 24

Sketch each of the following on a separate complex plane.

a $\{z: |z + 1| = 2\}$ **b** $\{z: |z - 2 + 3i| = 3\}$

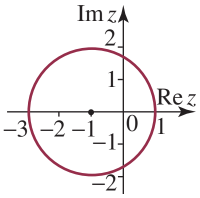
THINK

a This represents a circle of radius 2 and centre $-1 + 0i$.

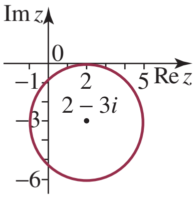
b This represents a circle of radius 3 and centre $2 - 3i$.

WRITE

a



b



remember

1. The graph of $\{z\text{: Arg } z = \theta\}$ is a ray starting at the origin and making an angle of θ with the positive real axis.
2. The graph of $\{z\text{: Arg } (z + a + bi) = \theta\}$ is the same as $\{z\text{: Arg } z = \theta\}$ translated $-a$ units in the real direction and $-b$ units in the imaginary direction.
3. The graph of $\{z\text{: Re } (z + a + bi) = c\}$ is a vertical line passing through the point $(c - a) + 0i$.
4. The graph of $\{z\text{: Im } (z + a + bi) = c\}$ is a horizontal line passing through the point $0 + (c - b)i$.
5. The relation $\{z\text{: } |z + a + bi| = r\}$ defines a circle of radius r and centre $-a -bi$.

EXERCISE 1D Graphs of complex relations: rays, lines and circles

WORKED Example 19

1 Sketch each of the following on separate complex planes.

a $\left\{z\text{: Arg } z = \frac{\pi}{6}\right\}$ **b** $\left\{z\text{: Arg } z = \frac{3\pi}{4}\right\}$ **c** $\left\{z\text{: Arg } z = -\frac{\pi}{4}\right\}$

d $\left\{z\text{: Arg } z = -\frac{2\pi}{3}\right\}$ **e** $\{z\text{: Arg } z = \pi\}$ **f** $\left\{z\text{: Arg } z = -\frac{\pi}{6}\right\}$

WORKED Example 20

2 Show each of the following on separate Argand diagrams.

a $\left\{z\text{: Arg } (z - 2) = \frac{\pi}{2}\right\}$ **b** $\left\{z\text{: Arg } (z + 3) = \frac{2\pi}{3}\right\}$

c $\left\{z\text{: Arg } (z - 1) = -\frac{3\pi}{4}\right\}$ **d** $\left\{z\text{: Arg } (z + 2i) = \pi\right\}$

e $\left\{z\text{: Arg } (z - 3i) = -\frac{5\pi}{6}\right\}$ **f** $\left\{z\text{: Arg } (z + i) = -\frac{\pi}{4}\right\}$

g $\left\{z\text{: Arg } (z + 2 - i) = \frac{\pi}{3}\right\}$ **h** $\left\{z\text{: Arg } (z - 3 + 2i) = -\frac{\pi}{8}\right\}$

i $\left\{z\text{: Arg } (z - 4 + 3i) = \frac{\pi}{2}\right\}$ **j** $\left\{z\text{: Arg } (z + 2) = -\frac{\pi}{5}\right\}$

3 **multiple choice**

Consider the ray represented by $\left\{z\text{: Arg } z = \frac{3\pi}{4}\right\}$.

a If the ray is translated –2 units in the imaginary direction it could be described by:

A $\left\{z\text{: Arg } (z - 2i) = \frac{3\pi}{4}\right\}$ **B** $\left\{z\text{: Arg } (z + 2) = -\frac{3\pi}{4}\right\}$

C $\left\{z\text{: Arg } (z + 2i) = \frac{3\pi}{4}\right\}$ **D** $\left\{z\text{: Arg } (z + 2i) = -\frac{3\pi}{4}\right\}$

E $\left\{z\text{: Arg } (z - 2) = \frac{3\pi}{4}\right\}$

b If the ray is translated –3 units in the real direction and +2 units in the imaginary direction then it can be described by:

A $\left\{z\text{: Arg }(z - 3 + 2i) = \frac{3\pi}{4}\right\}$

B $\left\{z\text{: Arg }(z + 3 + 2i) = \frac{3\pi}{4}\right\}$

C $\left\{z\text{: Arg }(z - 3 - 2i) = \frac{3\pi}{4}\right\}$

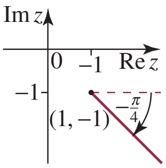
D $\left\{z\text{: Arg }(z + 3 - 2i) = -\frac{3\pi}{4}\right\}$

E $\left\{z\text{: Arg }(z + 3 - 2i) = \frac{3\pi}{4}\right\}$

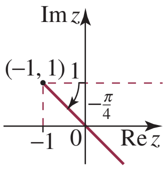
4 **multiple choice**

Which one of the following graphs represents $\left\{z\text{: Arg }(z + 1 - i) = -\frac{\pi}{4}\right\}$?

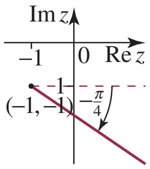
A



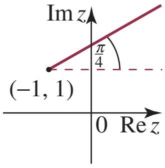
B



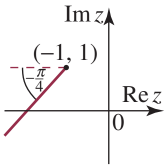
C



D



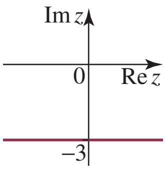
E



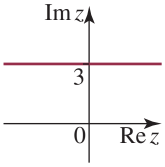
5 **multiple choice**

The graph which represents $\{z\text{: Re } z = -3\}$ is:

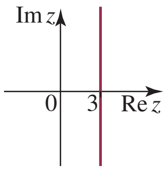
A



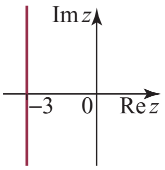
B



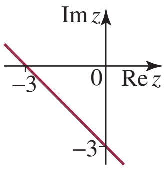
C



D



E



WORKED Example 21a

WORKED Example 21b

6 Sketch the graph of each of the following on a separate complex plane.

a $\{z\text{: Re } z = 5\}$

b $\{z\text{: Re } z = -2\}$

c $\{z\text{: Re } z = 4\}$

d $\left\{z\text{: Re } z = \frac{5}{2}\right\}$

e $\{z\text{: Im } z = -4\}$

f $\left\{z\text{: Im } z = \frac{3}{2}\right\}$

g $\{z\text{: Im } z = 1\}$

h $\{z\text{: Im } z = -3\}$

WORKED Example 22a

WORKED Example 22b

7 Show each of the following on a separate Argand diagram.

a $\{z: \text{Re}\,(z - 2) = 3\}$ **b** $\{z: \text{Re}\,(z - i) = 2\}$

c $\{z: \text{Re}\,(z + 3) = -2\}$ **d** $\{z: \text{Re}\,(z - 1 + 2i) = 1\}$

e $\{z: \text{Im}\,(z + 3i) = 2\}$ **f** $\{z: \text{Im}\,(z - 2i) = -1\}$

g $\{z: \text{Im}\,(z + 2) = 3\}$ **h** $\{z: \text{Im}\,(z - 2 - 4i) = -2\}$

8 Sketch each of the following on a separate complex plane.

a $\left\{z: \text{Arg}\, z = \frac{2\pi}{3}\right\}$ **b** $\{z: \text{Re}\,(z - 5) = -3\}$

c $\{z: \text{Im}\,(z - 2 + i) = 2\}$ **d** $\left\{z: \text{Arg}\,(z - 1 + 3i) = \frac{\pi}{2}\right\}$

e $\{z: \text{Re}\,(z + 3 + 2i) = 1\}$ **f** $\{z: \text{Im}\,(z + 5i) = 3\}$

g $\left\{z: \text{Arg}\,(z + i) = \frac{3\pi}{8}\right\}$ **h** $\{z: \text{Im}\,(z - 3i) = 0\}$

WORKED Example 23

9 Illustrate each of the following on a separate Argand diagram.

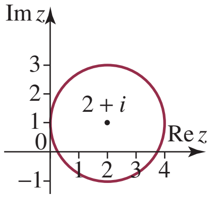
a $\{z: |z| = 1\}$ **b** $\{z: |z| = 4\}$

c $\{z: |z| = 3\}$ **d** $\{z: |z| = 10\}$

e $\{z: |z| = 9\}$ **f** $\{z: |z| = 2\}$

10 **multiple choice**

The equation of the circle at right is:



A $|z + 2 + i| = 2$

B $|z - 2 - i| = 2$

C $|z - 2 + i| = 2$

D $|z - 2 - i| = 4$

E $|z + i + 2| = 4$

11 **multiple choice**

The equation of a circle with centre $-3 + 2i$ and radius 4 is:

A $|z - 3 + 2i| = 16$ **B** $|z - 3 - 2i| = 4$ **C** $|z + 3 + 2i| = 4$

D $|z + 3 - 2i| = 4$ **E** $|z + 3 - 2i| = 16$

WORKED Example 24

12 Sketch the graph of each of the following on separate complex planes.

a $\{z: |z - 3| = 1\}$ **b** $\{z: |z + 2| = 2\}$

c $\{z: |z - 4i| = 3\}$ **d** $\{z: |z + i| = 4\}$

e $\{z: |z - 2 - i| = 3\}$ **f** $\{z: |z + 3 - 2i| = 1\}$

g $\{z: |z + 4 + 3i| = 5\}$ **h** $\{z: |z - 5 + 5i| = 2\}$

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Digital doc:
WorkSHEET 1.2

summary

Introduction to complex numbers

- We define the 'imaginary number' i as having the property that $i^2 = -1$.
- A complex number $z = a + bi$ with $a, b \in R$ and $C = \{z: z = a + bi, a, b \in R\}$ defines the set of complex numbers.
- The real part of z is a and is written as Re z.
- The imaginary part of z is b and is written as Im z.

Basic operations using complex numbers

- If z and w are two complex numbers such that $z = a + bi$ and $w = c + di$ for $a, b, c, d \in R$ then:
  1. $z = w$ iff $a = c$ and $b = d$
  2. $z + w = (a + c) + (b + d)i$
  3. $z - w = (a - c) + (b - d)i$
  4. $kz = ka + kbi$, for $k \in R$
  5. $z \cdot w = (ac - bd) + (ad + bc)i$.
  6. $\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$.
  7. The conjugate $\bar{z}$ of z is $\bar{z} = a - bi$.

The polar form of complex numbers

- The magnitude (modulus or absolute value) of $z = a + bi$ is the length of the line segment from (0, 0) to z. It is denoted by $|z|$, $|x + yi|$ or mod z.
- $|z| = \sqrt{a^2 + b^2}$.
- The argument of z, arg z, is the angle measurement anticlockwise of the positive Real axis and arg $z = \theta$ where $\tan \theta = \frac{b}{a}$.
- $z = a + bi$ can be expressed in polar form as $z = r \cos \theta + r \sin \theta i = r \text{ cis } \theta$.
- Arg z is the angle θ in the range $-\pi < \theta \leq \pi$.

De Moivre's theorem for complex numbers in polar form

- If $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$, then:
  1. $z_1 \cdot z_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$
  2. $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2)$.
- For $z = r \text{ cis } \theta$, $z^n = r^n \text{ cis } n\theta$, $n \in J$.

Factorisation of polynomials in *C*

- The fundamental theorem of algebra states that each polynomial over C has a solution that is a complex number.
- A polynomial of degree n has n solutions.
- The conjugate root theorem states that polynomials with real coefficients have solutions that are real numbers or that occur as pairs of conjugate complex numbers.

Solving equations in C

- A polynomial $P(z) = (z - z_1)(z - z_2) \ldots (z - z_n)$ has solutions $z = z_1, z_2, \ldots, z_n$.
- Use the 'complete the square' method or $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve equations in z that can be expressed in quadratic form.
- Substitute $-i^2$ for 1 to change expressions of the form $a^2 + b^2$ to the 'difference of two squares': $a^2 + b^2 = (a + bi)(a - bi)$.
- If $z^n = r \operatorname{cis} \theta$, then:

  $z = r^{\frac{1}{n}} \operatorname{cis}\left(\dfrac{\theta + 2k\pi}{n}\right)$, $k \in J$, using k until n solutions are found (that is, until the solutions start repeating).
- If $z^n = a$, $a \in R$, then:

  $z = a^{\frac{1}{n}} \operatorname{cis} \dfrac{2k\pi}{n}$, where $k \in J$ until n solutions are found.

Graphs of complex relations: rays, lines and circles

- The graph of $\{z: \operatorname{Arg} z = \theta\}$ is a ray starting at the origin and making an angle of θ with the positive real axis.
- The graph of $\{z: \operatorname{Arg}(z + a + bi) = \theta\}$ is the same as $\{z: \operatorname{Arg} z = \theta\}$ translated $-a$ units in the real direction and $-b$ units in the imaginary direction.
- The graph of $\{z: \operatorname{Re}(z + a + bi) = c\}$ is a vertical line passing through the point $(c - a) + 0i$.
- The graph of $\{z: \operatorname{Im}(z + a + bi) = c\}$ is a horizontal line passing through the point $0 + (c - b)i$.
- The relation $\{z: |z + a + bi| = r\}$ defines a circle of radius r and centre $-a - bi$.

CHAPTER review

1A **1** If $u = 4 - 3i$ and $v = 2 + 5i$, evaluate each of the following.

a $2u$ **b** $u + v$ **c** $u \cdot v$

d $\frac{u}{v}$ **e** $3u + 2v$ **f** $|3u + 2v|$

1A **2** If $z = 3 + 4i$ and $w = -1 + 2i$, evaluate each of the following.

a $|z|$ **b** $|w|$ **c** $\overline{z}$

d $|z + w|$ **e** $z\overline{w}$ **f** w^{-1}

1A **3** Express $2 + 2i$ in polar form.

1A **4** Express $3 \text{ cis } \frac{\pi}{6}$ in Cartesian form.

1A **5** If $u = 3 \text{ cis } \frac{3\pi}{4}$ and $v = 2 \text{ cis } \frac{\pi}{3}$, calculate each of the following, leaving your answer in polar form.

a $u \cdot v$ **b** $\frac{u}{v}$ **c** v^3 **d** $\sqrt{u}$

1A **6** Express each of the following in standard form.

a $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^6$ **b** $\frac{1}{\sqrt{3} - 3i}$

1B **7** **multiple choice**

When factorised over C, $x^2 + 6x + 1$ is equal to:

A $(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$ **B** $(x + 6 + \sqrt{2})(x + 6 + \sqrt{2})$

C $(x - 3 - 2\sqrt{2}\,i)(x - 3 + 2\sqrt{2}\,i)$ **D** $(x + 3 + 2\sqrt{2}\,i)(x - 3 - 2\sqrt{2}\,i)$

E $(x + 3 + 2\sqrt{2}\,i)(x + 3 - 2\sqrt{2}\,i)$

1B **8** **multiple choice**

If $2i$ is a zero of $z^3 - 3z^2 + 4z - 12$, then the other two zeros are:

A $-2i$ and 3 **B** $-2i$ and -3 **C** $-i$ and 3 **D** $-i$ and -3 **E** $-3i$ and -3

1B **9** Factorise each of the following polynomials over C.

a $P(z) = z^2 - 16z + 89$ **b** $P(z) = 2z^3 + 6z^2 - 7z + 4$

1B **10** Find the value $a \in R$ if $P(3i) = 0$ for $P(z) = z^4 + z^3 + az^2 + 9z - 9$.

11 **multiple choice** 1C

What is $\sqrt{5-5\sqrt{3}i}$ in Cartesian form?

A $\pm(5-5\sqrt{3}i\ )$ **B** $5\sqrt{3}-5i$ **C** $\pm(5+5\sqrt{3}\ i)$

D $\pm\left(\frac{\sqrt{30}}{2}-\frac{\sqrt{10}}{2}\ i\right)$ **E** $\sqrt{15}-\sqrt{5}i$

12 **multiple choice** 1C

The solutions to $z^3=\sqrt{3}+i$ in polar form are:

A $2\text{ cis }\frac{\pi}{6},\ 2\text{ cis }\frac{5\pi}{6},\ 2\text{ cis }\frac{3\pi}{2}$ **B** $\sqrt{2}\text{ cis}\frac{\pi}{6},\ \sqrt{2}\text{ cis}\frac{5\pi}{6},\ \sqrt{2}\text{ cis}\left(-\frac{\pi}{2}\right)$

C $\sqrt[3]{2}\text{ cis}\frac{\pi}{18},\ \sqrt[3]{2}\text{ cis}\frac{13\pi}{18},\ \sqrt[3]{2}\text{ cis}\left(-\frac{11\pi}{18}\right)$ **D** $4\text{ cis }\frac{\pi}{6}$

E $\sqrt{2}\text{ cis}\frac{\pi}{3},\ \sqrt{2}\text{ cis}\left(-\frac{2\pi}{3}\right)$

13 **multiple choice** 1C

How many degrees apart are two consecutive roots of $z^8=1$ on the unit circle?

A 180 **B** 90 **C** 135 **D** 225 **E** 45

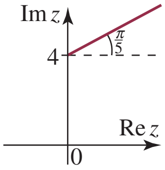
14 Solve each of these equations over C. 1C

a $x^2-2\sqrt{5}x+13 = 0$ **b** $z^2=8+15i$ **c** $z^3+8=0$

15 **multiple choice** 1D

The graph that correctly represents $\left\{z\text{: Arg }(z-4i)=-\frac{\pi}{5}\right\}$ is:

A



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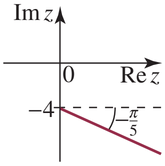

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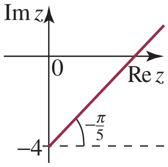

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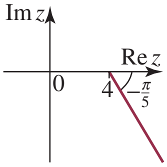
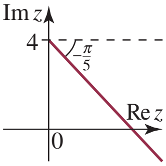


E



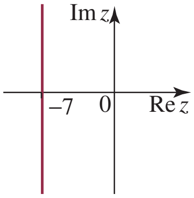
16 **multiple choice** 1D

The straight line shown at right can be described by:

A $\{z\text{: Re }z=7\}$ **B** $\{z\text{: Re }z=-7\}$

C $\{z\text{: Im }z=-7\}$ **D** $\{z\text{: Im }z=7\}$

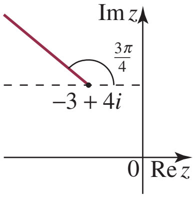
E $\{z\text{: Re }(z-7)=0\}$



1D

17 **multiple choice**

The ray shown below can be described by:



A $\left\{z\text{: Arg }(z-3+4i)=\frac{3\pi}{4}\right\}$

B $\left\{z\text{: Arg }(z-3)=\frac{3\pi}{4}\right\}$

C $\left\{z\text{: Arg }(z+3-4i)=\frac{3\pi}{4}\right\}$

D $\left\{z\text{: Re }(z+3-4i)=\frac{3\pi}{4}\right\}$

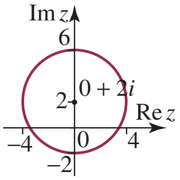
E $\left\{z\text{: Arg }(z+3-4i)=-\frac{3\pi}{4}\right\}$

1D

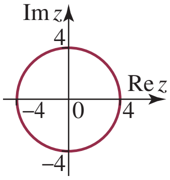
18 **multiple choice**

The Argand diagram below that correctly represents $\{z\text{: } |z-2i| = 4\}$ is:

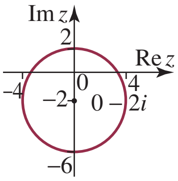
A



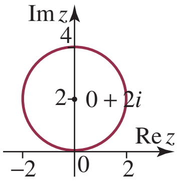
B



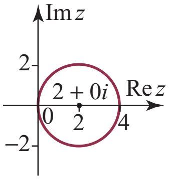
C



D



E



1D

19 **multiple choice**

The circle at right is described by the relation:

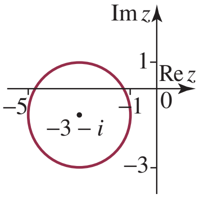
A $\{z\text{: } |z-3-i| = 2\}$

B $\{z\text{: } |z-3-i| = 4\}$

C $\{z\text{: } |z+3-i| = 4\}$

D $\{z\text{: } |z+3+i| = 2\}$

E $\{z\text{: } |z+3+i| = 4\}$



1D

20 Sketch the graph of $\left\{z\text{: Arg }(z-5-3i)=-\frac{2\pi}{5}\right\}$ on an Argand diagram.

1D

21 Sketch the graph of $\{z\text{: Im }(z+20+3i)=-1\}$ on the complex plane.

1D

22 Illustrate $\{z\text{: } |z-3+3i| = 1\}$ on the complex plane.

Modelling and problem solving

1 Let $z = 2 \text{ cis } \frac{\pi}{3}$ and $w = 2 \text{ cis } \frac{\pi}{4}$.

a Express $\frac{z}{w}$ in the form $r \cos \theta + r \sin \theta i$.

b Express z and w in Cartesian form.

c Express $\frac{z}{w}$ in Cartesian form.

d Using the results of parts **a** and **c**, find the exact values for:

i $\cos \frac{\pi}{12}$ **ii** $\sin \frac{\pi}{12}$ **iii** $\tan \frac{\pi}{12}$.

e By letting $z = 2 \text{ cis } \frac{\pi}{4}$ and $w = 2 \text{ cis } \frac{\pi}{6}$ and following parts **a** to **c** for zw instead of $\frac{z}{w}$, deduce that $\tan \frac{5\pi}{12} = 2 + \sqrt{3}$.

2 a Factorise $z^2 + 64$ over C.

b Express $z^4 + 64$ as a pair of quadratic factors in C.

c Show that:

i $(2 + 2i)^2 = 8i$

ii $(2 - 2i)^2 = -8i$.

d Hence factorise $z^4 + 64$ into linear factors over C.

e Factorise $z^4 + 64$ over R.

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Chapter 1

eBookplus ACTIVITIES

1A Operations on and representations of complex numbers

Interactivity

- Int-0344: Use the interactivity to see how complex numbers are represented on an Argand diagram. (*page 10*)

Digital doc

- SkillSHEET 1.1: Practise using index laws. (*page 14*)

1B Factorisation of polynomials in *C*

Tutorial

- Int-0372: Watch a tutorial on finding the zeros of a cubic polynomial over the complex number field. (*page 17*)

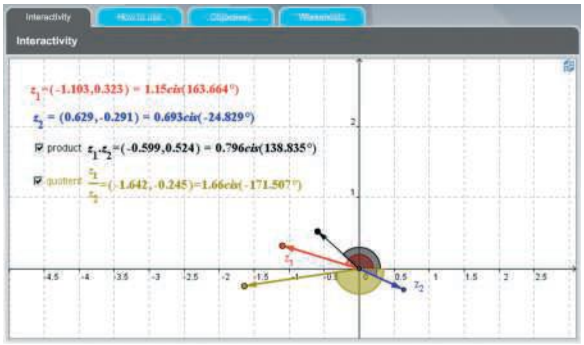
Digital doc

- WorkSHEET 1.1: Add, subtract, multiply and divide complex numbers, writing the expression with a real denominator and determine real and imaginary components of complex numbers. (*page 20*)

1C Solving equations in *C*

Tutorial

- Int-0373: Watch a tutorial on how to use de Moivre's theorem to solve a cubic equation. (*page 24*)



z_1=(-1.103,0.323) = 1.15*cis*(163.664°)

z_2 = (0.629,-0.291) = 0.693*cis*(-24.829°)

☑ product $z_1.z_2$=(-0.599,0.524) = 0.796*cis*(138.835°)

☑ quotient $\frac{z_1}{z_2}$=(-1.642,-0.245)=1.66*cis*(-171.507°)

1D Graphs of complex relations: rays, lines and circles

Digital doc

- WorkSHEET 1.2: Factorise polynomials over *C* and sketch complex regions on Argand diagrams. (*page 35*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 41*)

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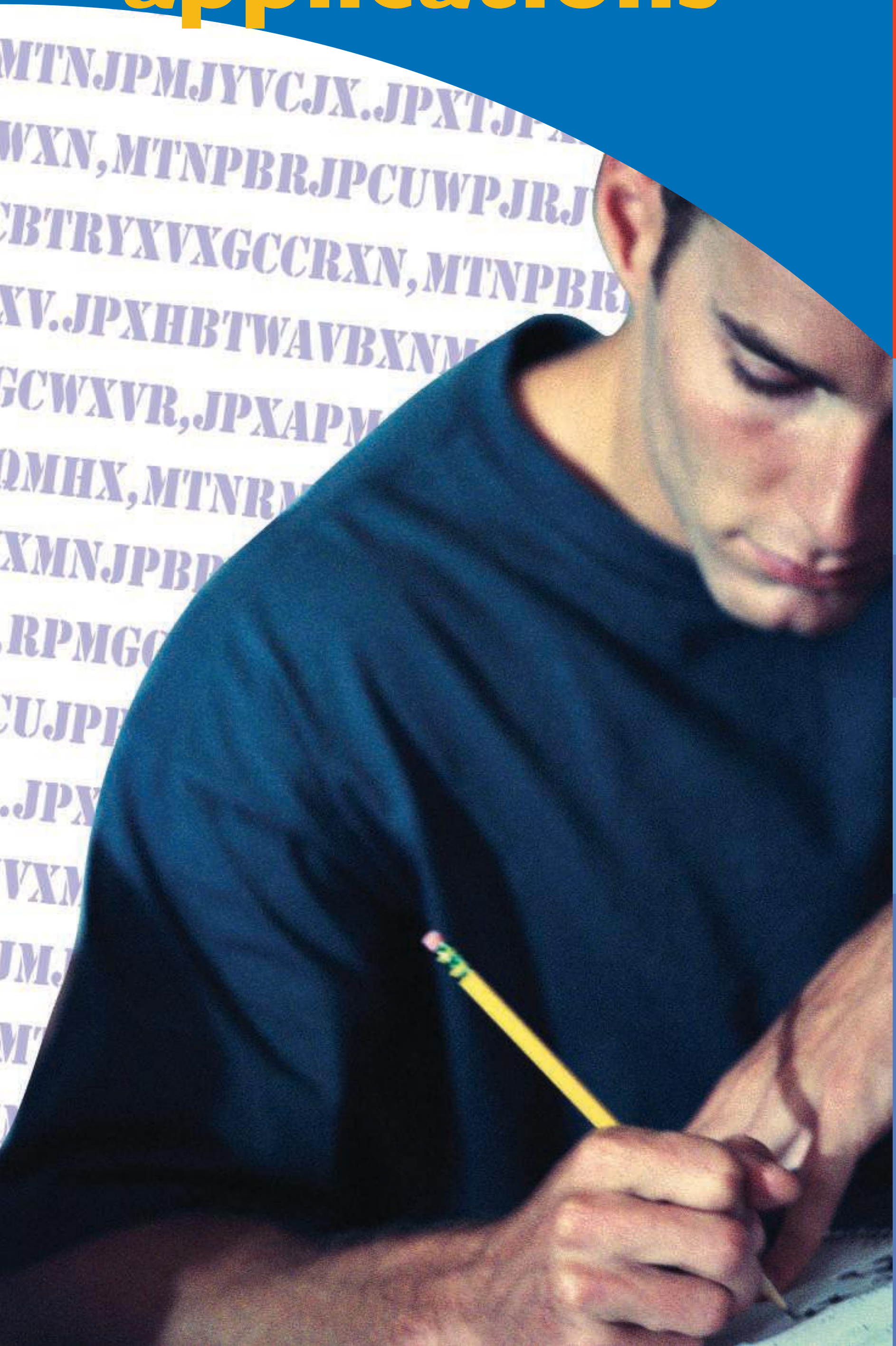
Matrices and applications

2



syllabus reference

Core topic:
Matrices and applications

In this chapter

2A Review of matrix operations
2B Markov chains and eigenvalues
2C Leontief matrices and Leslie matrices
2D Cryptology

Review of matrix operations

There are many contenders for the title of the greatest invention — from the wheel to sliced bread. Some have suggested the development of the numeral '0' as an important milestone in the growth of civilisation.

In ancient cultures, icons were used to stand for certain numbers. If ♣ represented 10 then ♣♣♣ would represent 30.

Roman numeration displayed an advance on such a system in one respect — the position of the numerals altered their role. Recall that X stands for 10 and C stands for 100. The number XC represents 90 whereas CX is 110. Clearly, the position of the numerals is vitally important.

Consider the advancement that the use of '0' represents. Although 0, in itself, is worth nothing, it allows us to assign a different meaning to each of the following numbers: 21, 201, 2010, 2001; and so on. Zero is a placeholder, and is the key to the sophisticated system of numeration that we now use.

Extending this idea, that information can be recorded using numbers in specific places, leads to the concept of the *array* or *matrix* of numbers.

In the local football competition, the Avengers have played 6 games — winning 3, drawing 2 and losing 1. We record this information as follows:

$$6 \quad 3 \quad 2 \quad 1$$

We write it in such a way so as not to confuse it with 6321.

Suppose a second team in the competition, the Barracudas, have played 5 games — winning 2, drawing 3 and losing 0.

We could write this information as:

$$5 \quad 2 \quad 3 \quad 0.$$

Naturally, there appears to be a connection between these results; so we write them as:

$$\begin{matrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{matrix} \quad \text{or} \quad \begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix}.$$

This is an example of a matrix. This matrix has 2 rows and 4 columns. We say that it has dimension 2 · 4 or that it is a 2 · 4 matrix.

Within a matrix we can identify elements within rows (i) and columns (j).

Consider $A = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix}$. Within matrix A, the symbol A_{ij} represents the element in the ith row and the jth column. For example, the element $A_{2\,4} = 0$.

Operations with matrices can be performed as shown in the following worked example.

WORKED Example 1

If $A = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & -2 & 4 \\ 4 & 2 & -1 & 1 \end{bmatrix}$, compute each of the following.

a $2A$ **b** $A + B$ **c** $A - 3B$

THINK	**WRITE**
a To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.	**a** $2A = \begin{bmatrix} 12 & 6 & 4 & 2 \\ 10 & 4 & 6 & 0 \end{bmatrix}$
b ① Check that the two matrices are of the same size. They are. ② Add the corresponding elements.	**b** $A + B = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -2 & 4 \\ 4 & 2 & -1 & 1 \end{bmatrix}$ $= \begin{bmatrix} 9 & 3 & 0 & 5 \\ 9 & 4 & 2 & 1 \end{bmatrix}$
c To subtract one matrix from another, subtract the corresponding elements.	**c** $A - 3B = \begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 9 & 0 & -6 & 12 \\ 12 & 6 & -3 & 3 \end{bmatrix}$ $= \begin{bmatrix} -3 & 3 & 8 & -11 \\ -7 & -4 & 6 & -3 \end{bmatrix}$

Recall that the matrix $\begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix}$ came originally from recording two teams' performance during a season:

$$\begin{array}{l|cccc|} & \text{Games} & \text{Wins} & \text{Draws} & \text{Losses} \\ \text{Avengers} & 6 & 3 & 2 & 1 \\ \text{Barracudas} & 5 & 2 & 3 & 0 \end{array}$$

Karyn and Peter are members of the committee of the football association and are discussing possible scoring systems for the seasons.

Karyn suggests 1 point for playing the game, and then 3 points for a win, 1 for a draw, and 0 for a loss.

Peter suggests 0 points for playing the game, and 2 points for a win, 1 for a draw, and –1 for a loss.

These values can be written in a matrix.

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$$

The points of each team can be calculated as follows.

Avengers on Karyn's system:	Avengers on Peter's system:
$6 \cdot 1 + 3 \cdot 3 + 2 \cdot 1 + 1 \cdot 0$	$6 \cdot 0 + 3 \cdot 2 + 2 \cdot 1 + 1 \cdot -1$
(17)	(7)
Barracudas on Karyn's system:	Barracudas on Peter's system:
$5 \cdot 1 + 2 \cdot 3 + 3 \cdot 1 + 0 \cdot 0$	$5 \cdot 0 + 2 \cdot 2 + 3 \cdot 1 + 0 \cdot -1$
(14)	(7)

This calculation suggests a method for matrix multiplication.

$$\begin{bmatrix} 6 & 3 & 2 & 1 \\ 5 & 2 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 \cdot 1 + 3 \cdot 3 + 2 \cdot 1 + 1 \cdot 0 & 6 \cdot 0 + 3 \cdot 2 + 2 \cdot 1 + 1 \cdot -1 \\ 5 \cdot 1 + 2 \cdot 3 + 3 \cdot 1 + 0 \cdot 0 & 5 \cdot 0 + 2 \cdot 2 + 3 \cdot 1 + 0 \cdot -1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 7 \\ 14 & 7 \end{bmatrix}$$

In general, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then the matrix multiplication of $A \cdot B$ becomes:

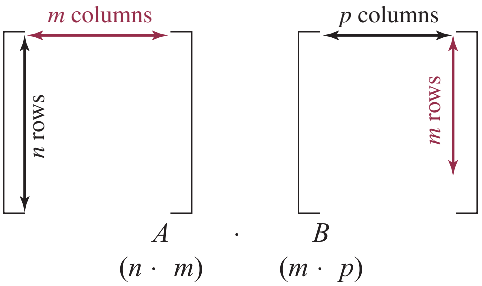
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Each row of matrix A is multiplied by each column of matrix B.

Properties of matrix multiplication

Note the following properties of matrix multiplication.

1. The product AB is possible only if matrix A has the same number of columns as B has rows.



2. In general, if A is an $n \cdot m$ matrix and B is a $m \cdot p$ matrix, the element in the ith row and the jth column of $A \cdot B$ is:

$$(AB)_{ij} = A_{i1} \cdot B_{1j} + A_{i2} \cdot B_{2j} + \quad A_{im} \cdot B_{mj}.$$

3. In general, if A is an $n \cdot\ m$ matrix, and B is an $m \cdot\ p$ matrix, the product AB is an $n \cdot\ p$ matrix.
4. In general, even if AB can be found, BA may not be able to be calculated. Further, if both AB and BA exist, AB need not equal BA; that is, matrix multiplication is not commutative.
5. The $2 \cdot\ 2$ identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is written as I and has the property that $IA = AI = A$.

   In higher orders, the identity matrix is a square matrix with 1s along the leading diagonal and 0s elsewhere.

WORKED Example 2

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Tutorial:
Worked example 2
int-0512

If $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$, calculate each of the following products.

a AB **b** BA

THINK	**WRITE**
a ① Check whether AB exists. Does the number of columns in A equal the number of rows in B?	**a** The product AB exists as there are 3 columns in A and 3 rows in B.
② Write the product.	$AB = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 3 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$
③ Multiply the rows of matrix A by the columns of matrix B.	$= \begin{bmatrix} 8+0-2 & 10+0+1 \\ 16-2-6 & 20+0+3 \\ 0-3+2 & 0+0-1 \end{bmatrix}$ $= \begin{bmatrix} 6 & 11 \\ 8 & 23 \\ -1 & -1 \end{bmatrix}$
b Check whether BA exists. Does the number of columns in B equal the number of rows in A?	**b** The product BA does not exist as there are 2 columns in B and 3 rows in A.

Determinant and inverse

A matrix which has the same number of rows and columns is called a *square matrix*. If A is a $2 \cdot\ 2$ matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant, written $|A|$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ or det A, is given by

$$|A| = ad - bc.$$

If $|A| = 0$, then A is said to be a *singular matrix*.

WORKED Example 3

a **Calculate the determinant of $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$.**

b **If $A = \begin{bmatrix} 1 - x & 2 \\ -4 & x + 1 \end{bmatrix}$, calculate the value of x for which A is a singular matrix, that is, $|A| = 0$.**

THINK	WRITE
a Use the formula for the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; that is, $ad - bc$.	**a** $\det = 3 \cdot 5 - (-2) \cdot 1$ $= 15 + 2$ $= 17$
b ① Use the formula for the determinant; that is, $ad - bc$.	**b** $|A| = (1 - x)(x + 1) + 8$ $= -x^2 + 9$
② Solve the equation, $|A| = 0$.	If $|A| = 0$, $-x^2 + 9 = 0$ $x = \pm 3$
③ Write the answer.	A is a singular matrix when $x = -3$ or 3.

It becomes quite complex to calculate the determinant of a square matrix that is of higher order than $2 \cdot 2$. We consider the mechanics of such a calculation in a later investigation. Your graphics calculator will calculate the determinant for higher order matrices.

WORKED Example 4

Use your graphics calculator to calculate the determinant of $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$.

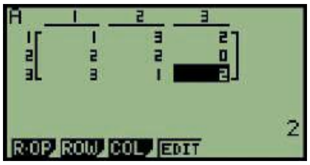
THINK

For the CASIO *fx*-9860G AU

① To define Matrix A, press:
- MENU
- 1 (RUN)
- F1 (MAT)

Enter Mat A as a $3 \cdot 3$ matrix and enter the values into the matrix as shown, pressing EXE after each entry.

WRITE/DISPLAY



```
A     1    2    3
1[    1    3    2]
2[    2    2    0]
3[    3    1    2]
                   2
```

THINK | **WRITE/DISPLAY**

2 To find the determinant of Matrix A, press:
- EXIT
- EXIT
- OPTN
- F2 (MAT)
- F3 (Det)

Complete the entry line as:
Det Mat A
Then press EXE.


```
Det Mat A
                -16
```

3 Write the answer.

The determinant of Matrix A is –16.

For the TI-Nspire CAS

1 To define Matrix A, on a Calculator page, press:
- MENU menu
- 1: Actions 1
- 1: Define 1

Complete the entry line as:

Define $a = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

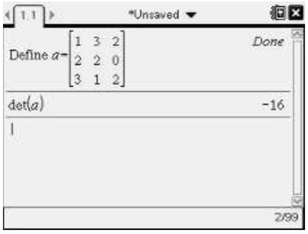
Then press ENTER enter.
Note: The Matrix template can be found by pressing
- Ctrl ctrl
- ·


Define $a = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ *Done*

1/99

2 To find the determinant of Matrix A, press:
- MENU menu
- 7: Matrix & Vector 7
- 3: Determinant 3

Complete the entry line as:
det(a)
then press ENTER enter.


Define $a = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ *Done*

det(a) -16

2/99

3 Write the answer.

The determinant of Matrix A is –16.

If A^{-1} is the inverse of A then $A \cdot A^{-1} = I$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse, A^{-1} can be found as follows.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Another way of writing this is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Note: If det $A = 0$ then the inverse does not exist.

WORKED Example 5

Calculate the inverse of matrix P:

a where $P = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ **b** where $P = \begin{bmatrix} a & a+1 \\ 2 & 2 \end{bmatrix}$.

THINK

a ① Calculate the determinant ($ad - bc$).

② Recall the formula for the inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det P}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

b ① Calculate the determinant.

② Recall the formula for the inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det P}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

WRITE

a $\det P = 2 \cdot 5 - 4 \cdot 3$

$= -2$

$$P^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2.5 & 2 \\ 1.5 & -1 \end{bmatrix}$$

b $\det P = a \cdot 2 - (a+1) \cdot 2$

$= 2a - 2a - 2$

$= -2$

$$P^{-1} = \frac{1}{-2}\begin{bmatrix} 2 & -(a+1) \\ -2 & a \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 2 & -a-1 \\ -2 & a \end{bmatrix}$$

The calculation of inverses of higher order can be done on your graphics calculator as shown in the next worked example.

WORKED Example 6

Use your graphics calculator to find the inverse of $Q = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$.

THINK

For the CASIO *fx*-9860G AU

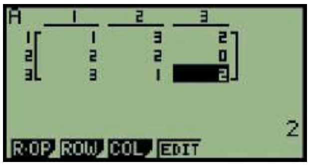
① To define Matrix Q, press:
- MENU
- 1 (RUN)
- F1 (MAT)

Enter Mat A as a 3 · 3 matrix and enter the values into the matrix as shown, pressing EXE after each entry.

② To determine the inverse of Matrix A, press:
- EXIT
- EXIT
- OPTN
- F2 (Mat)
- F1 (Mat)

Complete the entry line as:
Mat A^{-1}

WRITE/DISPLAY



$$\begin{array}{c|ccc} A & 1 & 2 & 3 \\ \hline 1 & 1 & 3 & 2 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 1 & 2 \end{array}$$

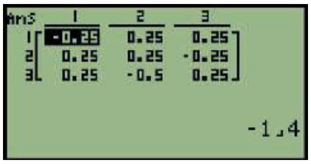
2

Mat A⁻¹

THINK

③ Press EXE to display the inverse of Matrix A.

WRITE/DISPLAY


```
Ans    1      2      3
 1 [ -0.25   0.25   0.25 ]
 2 [  0.25   0.25  -0.25 ]
 3 [  0.25  -0.5    0.25 ]
                      -1⌟4
```

④ Write the answer.

$$Q^{-1} = \begin{bmatrix} -0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 \\ 0.25 & -0.50 & 0.25 \end{bmatrix}$$

For the TI-Nspire CAS

① To define Matrix Q, on a Calculator page, press:

- MENU menu
- 1: Actions 1
- 1: Define 1

Complete the entry line as:

$$\text{Define } q = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

then press ENTER enter.

Note: The Matrix template can be found by pressing

- Ctrl ctrl
- ·

and using the NavPad to select the matrix template.

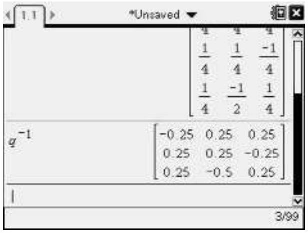
Define $q=\begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ Done

1/99

② To determine the inverse of Matrix Q, complete the entry line as:

q^{-1}

then press ENTER enter.



$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{-1}{2} & \frac{1}{4} \end{bmatrix}$

q^{-1} $\begin{bmatrix} -0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 \\ 0.25 & -0.5 & 0.25 \end{bmatrix}$

3/99

③ Write the answer.

$$Q^{-1} = \begin{bmatrix} -0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 \\ 0.25 & -0.50 & 0.25 \end{bmatrix}$$

WORKED Example 7

Solve the following matrix equation: $2X + \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

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Tutorial:
Worked example 7
int-0305

THINK

① Use the same strategies you would use to solve an equation involving real numbers. Express 2 as a matrix.

Remember that $2 = 2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

WRITE

$$2X + \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Continued over page

THINK	**WRITE**
② Add the ‘X’ terms on the left-hand side of the equation.	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
③ Multiply both sides by the inverse of $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.	$\frac{1}{1} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} X = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
④ Divide both sides by −1.	$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
⑤ Multiply the matrices.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$
⑥ Solve the equation. Remember that $IX = X$.	Therefore $X = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$.

remember

Review of matrix operations

1. Addition: To add two matrices of the same size, add the corresponding elements.
2. Scalar multiplication: To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.
3. Matrix multiplication: To multiply $A \cdot B$, the number of rows in B must be the same as the number of columns in A.

   $$A_{ij} = A_{i1} \cdot B_{1j} + A_{i2} \cdot B_{2j} +$$

   For example, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$.
4. Inverse matrix: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the inverse, $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where det A is the determinant of matrix A.

   $$\det = |A| = ad - bc.$$
5. Identity matrix: An identity matrix, I, is a square matrix where elements in the leading diagonal are 1 and all other elements are 0. For example, the $2 \cdot 2$ identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the properties:
   (a) $IA = AI = A$
   (b) $A^{-1}A = AA^{-1} = I$.

EXERCISE 2A Review of matrix operations

WORKED Example 1

1 If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, calculate, if possible, each of the following.

WORKED Example 2

a $A - 2B$
b $A + C$
c AC
d BA
e CA
f $A(A + 2I)$

2 If $P = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$, $Q = [2 \quad -2 \quad 1]$ and $R = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, calculate, if possible, each of the following.

a $3P$
b P^2
c PQ
d QP
e PR
f RP
g QR

WORKED Example 3a

3 Without the use of a graphics calculator, calculate the determinant of each of the following.

a $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$
b $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
c $\begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$
d $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

e $\begin{bmatrix} 12 & -8 \\ -9 & 6 \end{bmatrix}$
f $\begin{bmatrix} -3 & -4 \\ 2 & 2 \end{bmatrix}$
g $\begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}$
h $\begin{bmatrix} a & 3 \\ 1 & 2 \end{bmatrix}$

i $\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}$
j $\begin{bmatrix} a & -a \\ a & 1-a \end{bmatrix}$

WORKED Example 3b

4 If $A = \begin{bmatrix} x+2 & 1 \\ 4 & x-1 \end{bmatrix}$, calculate the value of x for which A is a singular matrix; that is, $|A| = 0$.

WORKED Example 4

5 Check your results to question **3** using a graphics calculator.

WORKED Example 5

6 For each of the matrices in question **3**, calculate, if possible, the inverse matrix.

7 Use your graphics calculator to find the determinant of each of the following.

a $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 6 \end{bmatrix}$
b $\begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

WORKED Example 6

8 Use your graphics calculator to find the inverse of each matrix in question **7**.

WORKED Example 7

9 Solve for X in each of the following.

a $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b $3X + \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 5 \end{bmatrix}$

c $3X + \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

10 Consider the following set of 2 · 2 matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad D = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix} \quad E = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

a Complete this multiplication table showing all possible products.

·	*I*	*A*	*B*	*C*	*D*	*E*
I						
A						
B						
C						
D						
E						

b Give each of the following in terms of I, A, B, C, D or E.

i A^2 **ii** A^3 **iii** A^4 **iv** A^5 **v** A^6 **vi** A^{400}

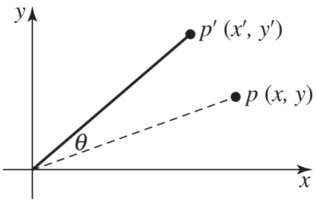
11 Recall that a group consists of a set, S, and an operation, $*$, with the following properties:

- S is closed under $*$. That is, for every $a, b \in S$, $a * b \in S$.
- An identity exists. There is an $e \in S$ such that for every $a \in S$, $a * e = e * a = a$.
- Each element has an inverse. For every $a \in S$, there is $b \in S$ such that $a * b = b * a = e$.
- The operation $*$ is associative. For every a, b and $c \in S$, $a * (b * c) = (a * b) * c$.

a Show that the set of 2 · 2 matrices forms a group under addition. Assume the operation is associative.

b Show that the set of 2 · 2 matrices with non-zero determinants forms a group under multiplication. Assume the operation is associative.

12 If a point, *P*, is rotated anticlockwise about the origin through an angle of $\theta°$, the coordinates of P' can be found using the matrix:



$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

a Recall that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. Write the matrix for a *clockwise* rotation of $\theta°$.

b Verify by matrix multiplication that $R_\theta \cdot R_{-\theta} = I$.

c Verify, by finding R_θ^{-1}, that $R_\theta^{-1} = R_{-\theta}$.

d Calculate $R_\theta \cdot R_\theta$.

e Use the result from **d** to show that:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \text{ and } \sin 2\theta = 2\sin\theta\cos\theta.$$

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Digital doc:
WorkSHEET 2.1

Determinants

As stated earlier, calculating the determinant of a square matrix that is of higher order than 2 · 2 is quite complex; so, for this investigation, all calculations should be made on your graphics calculator.

Consider the matrix $P = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$.

1 Find the determinant of *P*.

2 The matrix *Q* is obtained from *P* by multiplying the first row by 3. That is,

$$Q = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}.$$

a Calculate the determinant of *Q*.

b What do you notice?

3 Can you predict the determinant of the matrix R which is also obtained from P?

$$R = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$

4 Consider the matrix S which is obtained from P by interchanging rows 1 and 2.

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

- **a** Calculate the determinant of S.
- **b** What do you notice?

5 Consider the matrix T which is obtained from S by adding twice row 2 to row 3. That is, $T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 3 & 7 & 11 \end{bmatrix}$.

- **a** Calculate the determinant of T.
- **b** What do you notice?

6 Summarise your observations of the effects of multiplying and adding rows and columns of a matrix on the determinant.

For interest

The following is an extension of information on the determinant of a 3 · 3 matrix.

The determinant of a 3 · 3 matrix can be calculated as follows:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = a\begin{vmatrix} e & f \\ i & j \end{vmatrix} - b\begin{vmatrix} d & f \\ h & j \end{vmatrix} + c\begin{vmatrix} d & e \\ h & i \end{vmatrix}$$

Notice that the determinant of a 3 · 3 matrix is expressed in terms of the determinant of 2 · 2 matrices. In general the determinant of an n · n matrix is expressed in terms of determinants of $(n - 1)$ · $(n - 1)$ matrices:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \det A_{ij}$$

where A_{ij} is the matrix A with row i and column j removed.

The inverse of a 3 · 3 matrix is found as follows.

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T$$

where $|A_{ij}|$ is the determinant of the matrix A with row i and column j removed.

Markov chains and eigenvalues

Interactivity: Transition matrices int-0270

Markov chains

Andrei Markov was a Russian mathematician whose name is given to a technique that calculates the probability associated with the state of a system as it makes various transitions. It answers questions such as, 'What is the probability that James will be late to work today given that he was late yesterday?' or 'What is the likelihood that the next note in a piece of music will be an A, given that the previous note was a G'? or, 'What can be said about the long-term behaviour of the piece of music?'.

In Palmdale only two brands of soft drink are sold — *Blue* or *Red*. Suppose, at the start of the first month, 50% of the population drank *Blue* and 50% of the population drank *Red*. The people of Palmdale do not always stick to the same drink. At the end of the month some of them change to the other drink. Suppose at the end of each month 40% of people who drink *Blue* change to *Red* and 70% of those who drink *Red* change to *Blue*.

As time goes by the proportions are as follows.

	Start	**End of month 1**	**End of month 2**
Blue	50%	$50\% \cdot 60\% + 70\% \cdot 50\% = 65\%$	$65\% \cdot 60\% + 70\% \cdot 35\% = 63.5\%$
Red	50%	$50\% \cdot 30\% + 40\% \cdot 50\% = 35\%$	$35\% \cdot 30\% + 40\% \cdot 65\% = 36.5\%$

What is the long-term behaviour of the way in which the market share is distributed? Even the calculation of the behaviour for three months is complicated. The proportion of market share each month is called the *state* of the system. The sequence of states from month to month forms a *Markov chain*.

A *transition matrix* can be used to simplify the calculations. As its name suggests, the matrix assists in the calculation of the transition in drinking habits from one month to the next.

$$\begin{array}{ccc} & & \text{From} \\ & & \begin{array}{cc} \textit{Blue} & \textit{Red} \end{array} \\ \text{To} & \begin{array}{c} \textit{Blue} \\ \\ \textit{Red} \end{array} & \begin{bmatrix} 0.6 & 0.7 \\ \\ 0.4 & 0.3 \end{bmatrix} \end{array}$$

From *Red* to *Blue*

$$T = \begin{bmatrix} 9.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$$

Note that the columns of a transition matrix always add up to 1. At any time, the state is represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where x is the proportion of *Blue* and y is the proportion of *Red*. Thus, if the initial state, S_0, is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, then the state after:

$$1 \text{ month, } S_1\text{, is: } \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$2 \text{ months, } S_2\text{, is: } \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.635 \\ 0.365 \end{bmatrix}.$$

Often we are interested in the long-term behaviour of the market or the *steady state*, as it is often called. The steady state will be a value of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}.$$

This means the market share remains the same from one month to the next.

In this case $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives $x = 0.6x + 0.7y$.

Recalling that x and y are proportions that add to 1.

$$x + y = 1$$

Solving the simultaneous equations gives:

$$x = 0.6364$$
$$y = 0.3636.$$

That is, the steady state or long-term behaviour of the market is that *Blue* has 63.64% of the market and *Red* has 36.36%.

Steady state

A Markov chain has the transition matrix, $M = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.6 & 0.3 \\ 0.4 & 0.2 & 0.2 \end{bmatrix}$.

1 Use your graphics calculator to calculate the state after 4 stages, S_4, given an initial state $p = \begin{bmatrix} 0.4 \\ 0.3 \\ 0.3 \end{bmatrix}$. You should have obtained $\begin{bmatrix} 0.3101 \\ 0.4275 \\ 0.2623 \end{bmatrix}$.

2 Calculate the long-term behaviour by calculating $M^{50}p$.
Note: There is no significance in raising M to the 50th power. Any large number would have served the purpose.

3 Check that this is indeed the steady-state solution by using other initial states.

WORKED Example 8

The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & x \end{bmatrix}$.

a Manually calculate the value of x.

b If the state of the system is $\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$**, calculate the state of the system 2 stages later, using a graphics calculator.**

THINK	WRITE/DISPLAY
a Form an equation by adding the elements of the third column. The sum of each column must be 1.	**a** $0.5 + 0.4 + x = 1$ $x = 0.1$
b ① The state of the system will be $p = M^2p_0$ where p_0 is the initial state of the system. (Now use a graphics calculator to make the calculations.)	**b** $p = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2 \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$

Continued over page

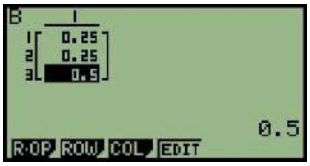
THINK

WRITE/DISPLAY

For the CASIO *fx*-9860G AU

② To enter the values for the matrices, press:
- MENU
- F1 (Mat)

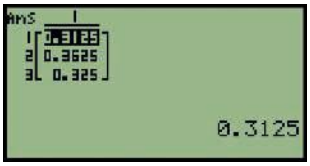
For Mat A: enter 3 · 3 and enter the values.
For Mat B: enter 3 · 1 and enter the values.



B
1 0.25
2 0.25
3 0.5
0.5

③ To multiply the matrices, press:
- EXIT
- EXIT
- OPTN
- F2 (MAT)
- F1 (Mat)

Complete the entry line as:
Mat A^2 · Mat B
then press EXE.



Ans
1 0.3125
2 0.3625
3 0.325
0.3125

④ Write the answer.

The steady state will be $p = \begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.3250 \end{bmatrix}$.

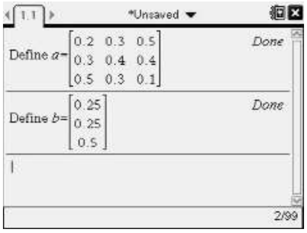
For the TI-Nspire CAS

② On a Calculator page, complete the entry lines as:

$$\text{Define } a = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}$$

$$\text{Define } b = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$

Press ENTER after each line.

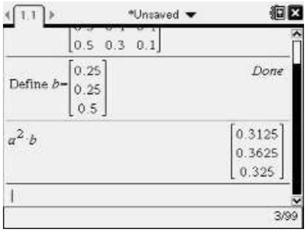


Define $a = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}$ *Done*

Define $b = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$ *Done*

2/99

③ To multiply the matrices, complete the entry line as:
$a^2 \cdot b$
then press ENTER.



$\begin{bmatrix} 0.5 & 0.3 & 0.1 \end{bmatrix}$

Define $b = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$ *Done*

$a^2 \cdot b$ $\begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.325 \end{bmatrix}$

3/99

④ Write the answer.

The steady state will be $p = \begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.3250 \end{bmatrix}$.

WORKED Example 9

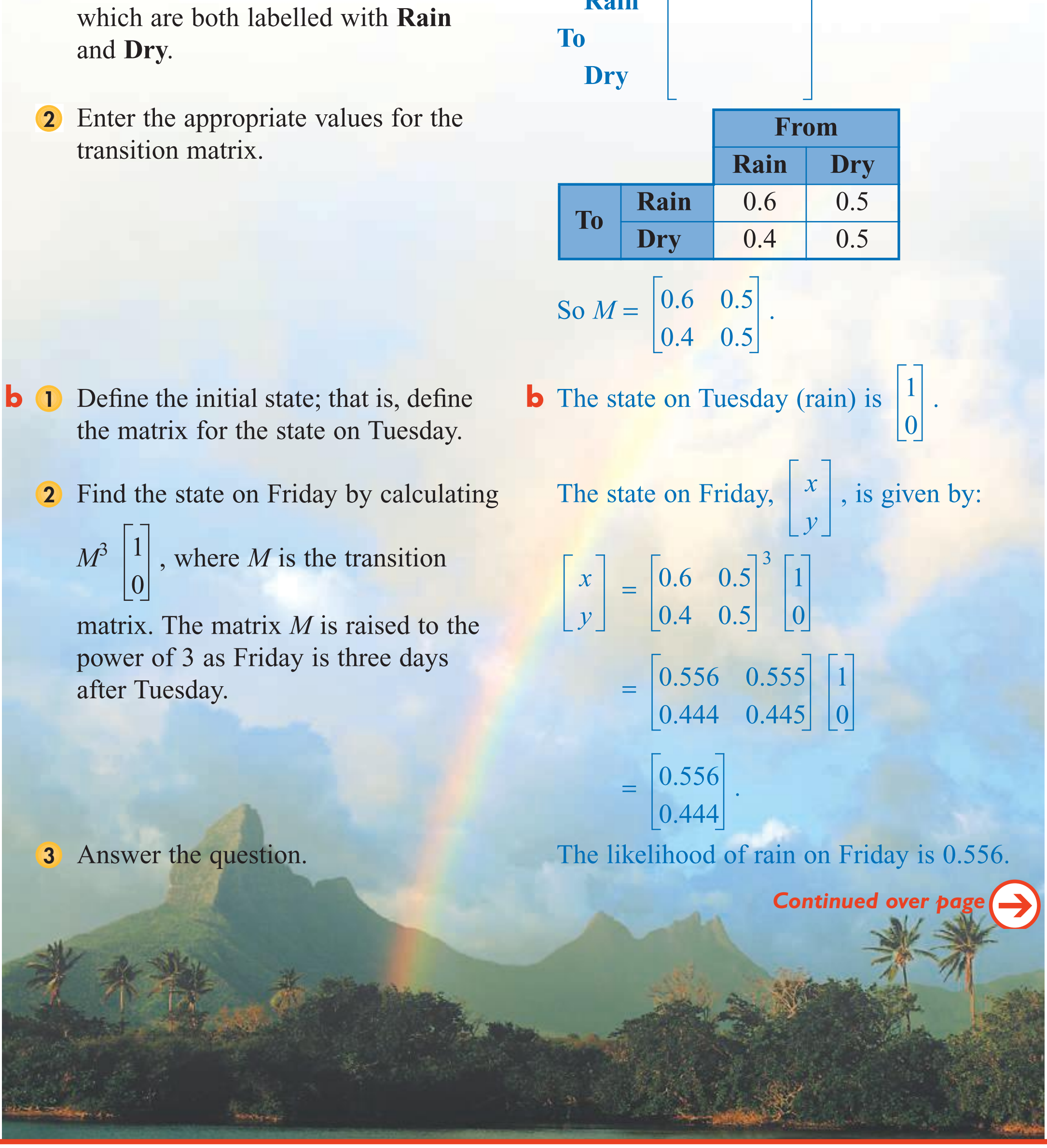
eBook plus
Tutorial:
Worked example 9
int-0515

The Nee Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 60%. If it is not raining on a particular day, the chance that it will rain on the following day is 50%.

a **Construct a transition matrix to represent this situation.**

b **Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.**

c **In the long term, on what proportion of days does it rain in the Nee Islands?**



THINK	WRITE
a 1 Label the columns and rows with the possible states. Write **From** above the columns and **To** beside the rows which are both labelled with **Rain** and **Dry**.	**a** $\begin{array}{cc} & \textbf{From} \\ & \begin{array}{cc}\textbf{Rain} & \textbf{Dry}\end{array} \\ \textbf{To}\ \begin{array}{c}\textbf{Rain}\\ \textbf{Dry}\end{array} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\end{array}$
2 Enter the appropriate values for the transition matrix.	$\begin{array}{\|c\|c\|c\|c\|}\hline & & \multicolumn{2}{c\|}{\textbf{From}} \\ \hline & & \textbf{Rain} & \textbf{Dry} \\ \hline \textbf{To} & \textbf{Rain} & 0.6 & 0.5 \\ \hline & \textbf{Dry} & 0.4 & 0.5 \\ \hline\end{array}$ So $M = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$.
b 1 Define the initial state; that is, define the matrix for the state on Tuesday.	**b** The state on Tuesday (rain) is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
2 Find the state on Friday by calculating $M^3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where M is the transition matrix. The matrix M is raised to the power of 3 as Friday is three days after Tuesday.	The state on Friday, $\begin{bmatrix} x \\ y \end{bmatrix}$, is given by: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.556 & 0.555 \\ 0.444 & 0.445 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.556 \\ 0.444 \end{bmatrix}$.
3 Answer the question.	The likelihood of rain on Friday is 0.556.

Continued over page →

THINK	**WRITE**
c ① Find the long-term term behaviour by considering a large (say $n = 50$) power of the transition matrix, M.	**c** Consider $M^{50} = \begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}^{50}$ $= \begin{bmatrix}0.5556 & 0.5556\\0.4444 & 0.4444\end{bmatrix}$
② Check that M^{51} is the same. They are the same (to 4 decimal places), which indicates that a steady-state solution is arrived at.	and $M^{51} = \begin{bmatrix}0.6 & 0.5\\0.4 & 0.5\end{bmatrix}^{51}$ $= \begin{bmatrix}0.5556 & 0.5556\\0.4444 & 0.4444\end{bmatrix}$
③ Choose a starting value, say $\begin{bmatrix}1\\0\end{bmatrix}$, and calculate the long-term proportions. *Note:* Any starting value could have been used.	If the starting value is $\begin{bmatrix}1\\0\end{bmatrix}$, the long-term behaviour is given by $M^{50}\begin{bmatrix}1\\0\end{bmatrix}$ $= \begin{bmatrix}0.5556 & 0.5556\\0.4444 & 0.4444\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$ $= \begin{bmatrix}0.5556\\0.4444\end{bmatrix}$.
④ Answer the question.	So, the long-term probability of rain is 0.5556 and dry is 0.4444.

Eigenvalues

For a matrix, A, if there is a column matrix (vector), X, such that:

$$AX = \lambda X \qquad [1]$$

then X is called an *eigenvector* of A and λ is called an *eigenvalue* of the matrix, A.

Eigenvectors of a matrix, A, are important because they represent objects that are in a sense unchanged by a transformation represented by the matrix, A. In physics, economics and biology, eigenvectors are important features in mathematical modelling.

In the previous section on Markov chains, the values for steady state were to be found by solving the equation $\begin{bmatrix}x\\y\end{bmatrix} = T\begin{bmatrix}x\\y\end{bmatrix}$ or using the notation we have adopted here, letting $X = \begin{bmatrix}x\\y\end{bmatrix}$, we obtain

$$X = TX.$$

Thus, the steady state, X, is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

Equation [1] is equivalent to:

$$AX - \lambda X = 0$$

$$AX - \lambda IX = 0 \text{ where } I \text{ is the identity matrix.}$$

$$(A - \lambda I)X = 0 \qquad [2]$$

It follows from equation [2] that:

$$\det(A - \lambda I) = 0. \qquad [3]$$

If A is a 2 · 2 matrix, equation [3] is a quadratic equation in λ, if A is a 3 · 3 matrix then equation [3] is an equation in λ of degree 3, and so on.

WORKED Example 10

If $A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$, find:

a **eigenvalues associated with the matrix**

b **eigenvectors associated with the matrix.**

THINK	WRITE
a ① To find the eigenvalues, λ, use the determinant equation, det $(A - \lambda I) = 0$.	**a** Consider the equation $\det(A - \lambda I) = 0$.
② Find the matrix for $A - \lambda I$.	$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $= \begin{bmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{bmatrix}$
③ Substitute for $A - \lambda I$ in det $(A - \lambda I) = 0$.	So $\begin{vmatrix} 1-\lambda & 6 \\ 2 & 2-\lambda \end{vmatrix} = 0$
④ Find the determinant and solve for λ.	$(1 - \lambda)(2 - \lambda) - 12 = 0$ $\lambda^2 - 3\lambda - 10 = 0$ $(\lambda - 5)(\lambda + 2) = 0$
⑤ Write the eigenvalues.	Thus, the eigenvalues are 5 and –2.
b ① State the relationship linking the eigenvectors with the eigenvalues.	**b** $AX = \lambda X$
② Substitute $\lambda = 5$ and $\lambda = -2$.	$AX = 5X$ or $AX = -2X$
③ Use $\lambda = 5$ first. Substitute values for matrix A and assign unknowns to matrix X.	Let $X = \begin{bmatrix} a \\ b \end{bmatrix}$ then with $\lambda = 5$, $AX = 5X$.
④ Substitute for A and X.	$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 5\begin{bmatrix} a \\ b \end{bmatrix}$
⑤ Consider the first row of the product only. (You will also obtain the same result with the second row.)	$a + 6b = 5a$ $6b = 4a$ $b = \frac{4a}{6}$ $= \frac{2a}{3}$

Continued over page

THINK	**WRITE**
⑥ As you cannot find *a* and *b* explicitly, state the relationship between them.	*X* is any vector of the form $\begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$.
⑦ Repeat this procedure for $\lambda = -2$. Again, substitute for *A* and *X*.	$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$
⑧ Consider the first row of the product only.	$a + 6b = -2a$ $6b = -3a$ $b = \frac{-3a}{6}$ $= -\frac{a}{2}$
⑨ State the eigenvectors for $\lambda = -2$.	*X* is any vector of the form $\begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}$.
⑩ Answer the question.	The eigenvectors for the given matrix are any vectors of the form $\begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}$ or $\begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}$.

remember

1. A transition matrix, *M*, is used to calculate the likelihood of certain states given the states at a previous stage.
2. The sequence of states, S_i, is called a Markov chain.
3. The values of *M* are arranged as follows:

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{\textbf{From}} \\ & & A & B \\ \textbf{To} & A & & \\ & B & & \end{array}$$

4. If a system can be in one of *r* states, the $r \cdot\ 1$ column matrix (or vector) *p* gives the likelihood of each of the states.
5. If a system is initially in a state *p* after *n* stages the system is in a state given by $M^n p$.
6. The sum of columns in a transition matrix is 1.
7. The sum of probabilities in the state matrix is 1.
8. The long-term behaviour of the system can be found by considering $M^{50}p$, first checking that M^{51} is the same as M^{50}.
9. For a matrix, *A*, if there is a column matrix (vector), *X*, such that:

   $$AX = \lambda X$$

   then *X* is called an eigenvector of *A* and λ is called an eigenvalue of the matrix, *A*.

EXERCISE 2B Markov chains and eigenvalues

WORKED Example 8

1 The transition matrix for a system is $\begin{bmatrix} 0.3 & 0.2 & 0.50 \\ 0.3 & 0.2 & 0.25 \\ 0.4 & x & 0.25 \end{bmatrix}$.

a Calculate the value of x.

b If the current state of the system is $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$, calculate the state of the system at each of the following stages:

i 1 stage later **ii** 2 stages later **iii** 3 stages later **iv** 4 stages later.

c If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:

i 1 stage later **ii** 2 stages later **iii** 3 stages later **iv** 4 stages later.

d If the current state of the system is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, calculate the state of the system at each of the following stages:

i 1 stage later **ii** 2 stages later **iii** 3 stages later **iv** 4 stages later.

2 The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.2 & c \\ 0.3 & b & 0.25 \\ a & 0.1 & 0.20 \end{bmatrix}$.

a Calculate the values of a, b and c.

b If the current state of the system is $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$, calculate the state of the system after each of the following stages:

i 1 stage later **ii** 2 stages later **iii** 3 stages later **iv** 4 stages later.

c If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:

i 1 stage later **ii** 2 stages later **iii** 3 stages later **iv** 4 stages later.

WORKED Example 10

3 If $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, find the:

a eigenvalues associated with the matrix

b eigenvectors associated with the matrix.

4 If $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$, find the:

a eigenvalues associated with the matrix

b eigenvectors associated with the matrix.

5 If matrix M is a 2 · 2 transition matrix, then the columns of M will each add to give 1. Show that $\lambda = 1$ is always an eigenvalue for such a transition matrix.

6 a Show that the eigenvalues of a matrix, $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ are found by solving the following equation for λ:

$$\lambda^2 - 5\lambda - 2 = 0.$$

b Show that the matrix A is a solution to the following equation.

$$A^2 - 5A - 2I = 0$$

Making music

Suppose we make some rules about how a melody works.

- All notes are of equal time.
- If the current note is C then the next note could be a D or E with a probability of $\frac{2}{3}$ and $\frac{1}{3}$ respectively.
- Let us continue to assign rules in this way and summarise the results in a matrix, M.

Current event

Next event

$$\begin{array}{c c} & \begin{array}{ccccc} C & D & E & F & G \end{array} \\ \begin{array}{c} C \\ D \\ E \\ F \\ G \end{array} & \begin{bmatrix} 0 & \frac{3}{10} & 0 & \frac{1}{4} & 0 \\ \frac{2}{3} & \frac{3}{10} & \frac{5}{11} & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{3}{10} & \frac{5}{11} & 0 & \frac{1}{2} \\ 0 & \frac{1}{10} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{11} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

Create some tunes!

1 Start with a note, say C. Calculate the probabilities of notes that follow C from the matrix. In this case P(D) = $\frac{2}{3}$ and P(E) = $\frac{1}{3}$.

2 Generate a random integer between 1 and 3. If it is a 1 or a 2 select D as the next note, otherwise select E as the next note.

3 Continue the melody by choosing the next note based on the note you obtained in the previous stage.

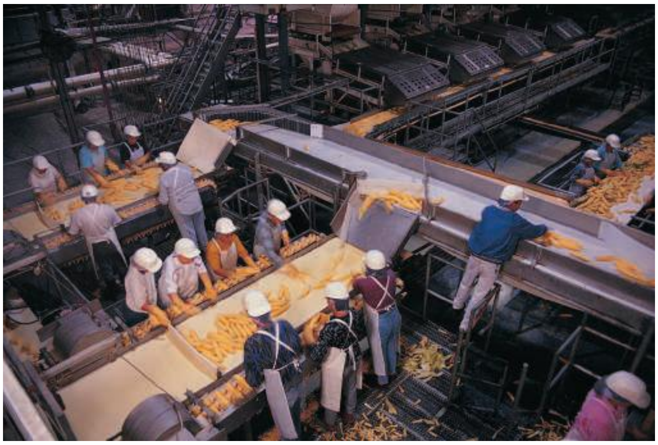
Leontief matrices and Leslie matrices

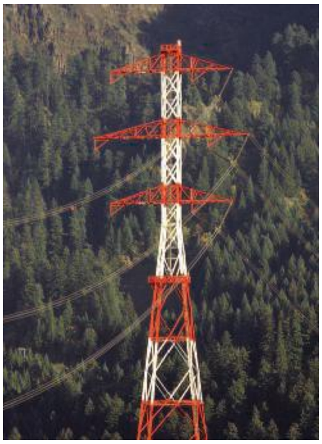
Leontief matrices

Wassily Leontief was born in St Petersburg in 1906. Although he trained in Russia and Germany his main contribution to economic theory took place while he was a professor of economics at Harvard University in the United States. He received the Nobel Prize for economics in 1973 and died in 1999. Leontief developed input–output analysis, a technique greatly valued by economists involved in planning and development.

Consider an industry or economy that produces energy (E), raw materials (R) and manufactured goods (M). There is an interrelationship between these three activities. To produce energy you require raw materials and manufactured goods. Likewise, to produce raw materials requires energy and manufactured goods.

The Leontief model looks at an economy such as this and relates input and output. It answers questions such as: 'what inputs (how much of each of E, R and M) are needed to produce a certain output?'.





Matrices are used to record the relationship between the industries.

Suppose the relationship can be quantified and represented by a consumption matrix, C.

$$\begin{array}{cc} & \begin{array}{ccc} E & R & M \end{array} \\ C = & \left[\begin{array}{ccc} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{array}\right] \begin{array}{c} E \\ R \\ M \end{array} \end{array}$$

This matrix should be interpreted as follows. The figure of 0.4 in the second row, first column means that every unit of E produced requires 0.4 units of R.

Output and input can be represented using 3 · 1 matrices, p and q. For example,

$p = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ represents an output of 3 units of E, 4 units of R and 7 units of M.

The input, q, required to give an output, p, can be obtained by using the equation:

$$q = Cp.$$

WORKED Example 11

Consider the consumption matrix $C = \begin{array}{c c} \begin{array}{ccc} E & R & M \end{array} & \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} & \begin{array}{c} E \\ R \\ M \end{array} \end{array}$

a What is the meaning of the value 0.4 in the third row of the matrix?

b Calculate the inputs needed to produce 1 unit of *E*, 0 units of *M* and 0 units of *R*.

c Calculate the inputs needed to produce 2 units of *E*, 2 units of *R* and 3 units of *M*.

THINK	WRITE
a Consider the row and column for this element. The row is linked to *M* and the column is linked to *R*.	**a** This entry means it takes 0.4 units of *M* to make 1 unit of *R*.
b 1 Write the output as a 3 · 1 matrix.	**b** Let $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the amount of $E = 1$ with the quantities of *R* and *M* being 0.
2 Use the equation relating input to output, that is, $q = Cp$.	$q = Cp$
3 Multiply the consumption matrix, *C*, by matrix *p*.	$= \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}$
4 Write the answer in a sentence.	The input q is 0.2 units of *E*, 0.4 units of *R* and 0.3 units of *M*.
c 1 Write the output as a 3 · 1 matrix.	**c** $q = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$
2 Use the equation $q = Cp$.	$q = Cp$
3 Multiply the consumption matrix, *C*, by matrix *p*.	$p = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ $= \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$
4 Write the answer in a sentence.	The input needed is 1.9 units of *E*, 1.9 units of *R* and 1.7 units of *M*.

In the previous worked example, you will notice that the products of the process $p = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ exceeded the inputs $q = \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$. This surplus, $p - q = \begin{bmatrix} 0.1 \\ 0.1 \\ 1.3 \end{bmatrix}$, can be used to meet external demand, d.

In a well-balanced economy, it is possible to choose a production, p, such that it matches a given demand, d, and

$$p - q = d.$$

Note that, as $q = Cp$, this equation is equivalent to:

$$p - Cp = d.$$

This gives $\qquad (I - C)p = d$

and $\qquad p = (I - C)^{-1}d. \qquad [1]$

You may think that, provided $(I - C)^{-1}$ exists, equation [1] ensures that we can always find production levels, p, such that the demand will always be met. This need not be so, for although the values can be computed in theory, in practice some of the elements of p may turn out to be negative. This would not make sense and thus the process would fail for that value of d.

WORKED Example 12

Given the consumption matrix $C = \begin{array}{c} \begin{array}{ccc} E & R & M \end{array} \\ \left[\begin{array}{ccc} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{array}\right] \end{array} \begin{array}{c} \\ E \\ R \\ M \end{array}$**:**

a find the production, p, that will match the demand $d = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$

b show that it is always possible to find values for production that will exactly match a given demand, d.

THINK

a ① Use equation [1] to find production, given demand.

② Substitute values for C and d. (Now use a graphics calculator to do the calculations.)

For the CASIO *fx*-9860G AU

③ To enter the values for the matrices, press:

- MENU
- F1 (Mat)

For Mat C: enter 3 · 3 and enter the values.
For Mat D: enter 3 · 1 and enter the values.
For Mat I: enter 3 · 3 and enter the values.
Complete the entry line as:
(Mat I – Mat C)$^{-1}$ · Mat D

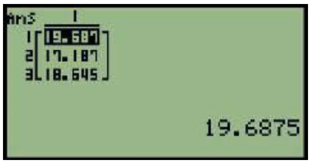
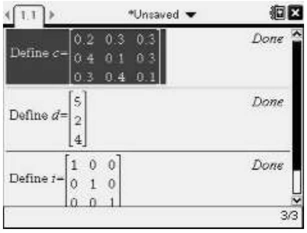
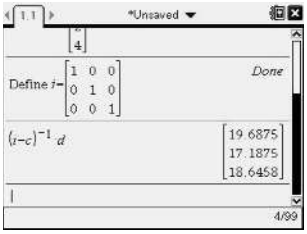
WRITE/DISPLAY

a $p = (I - C)^{-1}d.$

$$p = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}\right]^{-1} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

(Mat I−Mat C)⁻¹×Mat D

Continued over page

THINK	**WRITE/DISPLAY**
④ To display the matrix, press EXE.	 Ans $\begin{matrix}1\\2\\3\end{matrix}\begin{bmatrix}19.687\\17.187\\18.645\end{bmatrix}$ 19.6875
⑤ Write the answer.	When the demand is: $E = 5$, $R = 2$ and $M = 4$ the production that exactly matches it is: $E = 19.69$, $R = 17.19$ and $M = 18.65$.
For the TI-Nspire CAS ③ On a Calculator page, complete the entry line as: Define $c = \begin{bmatrix}0.2 & 0.3 & 0.3\\0.4 & 0.1 & 0.3\\0.3 & 0.4 & 0.1\end{bmatrix}$ Define $d = \begin{bmatrix}5\\2\\4\end{bmatrix}$ Define $i = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ Press ENTER after each line.	 Define $c=\begin{bmatrix}0.2 & 0.3 & 0.3\\0.4 & 0.1 & 0.3\\0.3 & 0.4 & 0.1\end{bmatrix}$ Done Define $d=\begin{bmatrix}5\\2\\4\end{bmatrix}$ Done Define $i=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ Done 3/3
④ To perform the required calculation, complete the entry line as: $(i - c)^{-1} \cdot d$ then press ENTER.	 Define $i=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ Done $(i-c)^{-1} \cdot d$ $\begin{bmatrix}19.6875\\17.1875\\18.6458\end{bmatrix}$ 4/99
⑤ Write the answer.	When the demand is: $E = 5$, $R = 2$ and $M = 4$ the production that exactly matches it is: $E = 19.69$, $R = 17.19$ and $M = 18.65$.
b ① Use the equation, $p = (I - C)^{-1} d$ to find p.	**b** $(I - C)^{-1} = \begin{bmatrix}2.40 & 1.35 & 1.25\\1.56 & 2.19 & 1.25\\1.49 & 1.42 & 2.08\end{bmatrix}$
② Recognise that the values of production need to be positive.	As long as $(I - C)^{-1}$ exists and has positive entries, it will be always possible to find values of p to match the demand, d.

The next worked example uses a two-industry economy and illustrates how we can form a consumption matrix from information about the economy.

WORKED Example 13

Consider the following two industry input–output models. To produce one tonne of coal requires 0.2 tonnes of coal and 0.3 tonnes of steel. To produce one tonne of steel requires 0.4 tonnes of coal and 0.2 tonnes of steel.

a Construct the consumption matrix for this economy.

b What production is required to meet the external demand of 80 tonnes of coal and 20 tonnes of steel?

THINK	WRITE
a Construct the consumption matrix using *C* and *S* across the top and down the side.	**a** $C = \begin{array}{cc} \begin{matrix} C & \quad S \end{matrix} & \\ \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} & \begin{matrix} C \\ S \end{matrix} \end{array}$
b ① Recall the formula relating production, *p*, to demand, *d*.	**b** $p = (I - C)^{-1}d$
② Substitute values for *C* and *d*.	$= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}\right]^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$ $= \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix}$ $= \begin{bmatrix} 138.5 \\ 76.9 \end{bmatrix}$
③ Write the answer in words.	The production that meets the demand is 138.5 tonnes of coal and 76.9 tonnes of steel.

Leslie matrices

There are many factors that affect the size of populations of animal species. The first of these is the survival rate; that is, the proportion of animals that survive until the next year (or whatever time period is being used). Clearly, a more accurate picture of the survival rate can be obtained if we use survival rates specific to the different age groups. For example, a saltwater crocodile may produce many offspring in a year but not many of the babies survive their first year. However, once they reach 1 year of age it is much more likely that they will get to 2 years of age.

A second factor influencing the population of a species is its fecundity or how many babies are produced per year. Again, fecundity rates that are specific to particular age groups will yield a more accurate analysis.

To see how matrices can be used to model animal populations, consider the following example.

A wildlife officer is interested in the number of a certain species of rat in a habitat and has compiled the following information on this species.

We have a population breakdown by age:

Age (x) (at the end of the year)	Number (N) (of females)
1	55
2	32
3	25

It is assumed that the rats do not live longer than three years and that there are the same number of males and females.

Fecundity rates, F_x, (the number of female offspring per female per year) have been calculated:

Age (x)	Fecundity (F_x)
1	0.6
2	2.4
3	0.5

The survival rates, S_x, of each age group have also been estimated:

Age (x)	Survival rate (S_x)
1	0.5
2	0.6

Given this information and an initial starting population, it is possible to predict the future population.

Note that the 1-year-old rats that survive the year become 2-year-old rats in the next year.

As each age group of females produce babies, the number of 1-year-old rats in the next year can be calculated as:

$$\begin{aligned}\text{Number of 1-year-old rats} &= \text{number of 1-year-olds} \cdot F_1\\ &+ \text{number of 2-year-olds} \cdot F_2\\ &+ \text{number of 3-year-olds} \cdot F_3\end{aligned}$$

The number of 2-year-old rats in the next year = 0.5 · number of 1-year-olds.

The number of 3-year-old rats in the next year = 0.6 · number of 2-year-olds.

It is useful to formulate the relationships within a population of rats in any year, and between the populations of rats over several years, using matrices. The population of rats at the end of year 1 can be written:

$$N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$

A matrix called the *Leslie matrix* can be used to estimate the next year's population given the current population. The Leslie matrix is given by:

$$L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$$

The estimated rat population at the end of the second year is N_2, where $N_2 = L \cdot N_1$.

A Leslie matrix has fecundity rates across the top row, survival rates along the sub-diagonal and 0s elsewhere.

$L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$ where F_i is the fecundity rate of the ith age group and S_i is the survival rate of the ith age group.

We can also calculate the number of females after any year using the Leslie matrix and the number of females after year 1.

Consider the number of females after years 2, 3, 4, , r.

After year 2: $N_2 = L \cdot N_1$
After year 3: $N_3 = L \cdot N_2 = L \cdot L \cdot N_1 = L^2 \cdot N_1$
After year 4: $N_4 = L \cdot N_3 = L \cdot L^2 \cdot N_1 = L^3 \cdot N_1$
After year r: $N_r = L^{r-1} \cdot \mathrm{N}_1$

So $N_r = L \cdot N_{r-1}$ and $N_r = L^{r-1} \cdot N_1$.

WORKED Example 14

Using the Leslie matrix, $L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$, and the female rat population in year 1,

$N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$:

a give the meaning of the number 2.4 in the matrix
b calculate the female rat population in year 2
c calculate the total rat population after 6 years, assuming equal numbers of male and female rats.

THINK	WRITE
a The first row of the Leslie matrix gives fecundity rates. The second column refers to 2-year-olds.	**a** The number 2.4 refers to the number of female rats born on average to a 2-year-old rat.
b ① The population in the next year is found by multiplying the Leslie matrix by the population this year.	**b** $N_2 = L \cdot N_1$

Continued over page →

THINK	WRITE
② Perform the matrix multiplications.	$N_2 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$ $= \begin{bmatrix} 122.3 \\ 27.5 \\ 19.2 \end{bmatrix}$
③ Write the answer.	The total female population is $122.3 + 27.5 + 19.2 = 169$.
c ① Calculate the population after 6 years using the formula $N_r = L^{r-1} \cdot N_1$.	**c** $N_6 = L^5 \cdot N_1$ $N_6 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}^5 \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$
② Perform the matrix multiplications.	$= \begin{bmatrix} 521.9 \\ 171.9 \\ 73.3 \end{bmatrix}$
③ Add the female populations.	The total female population is $521.9 + 171.9 + 73.3 = 767.1$.
④ Calculate the total rat population assuming that there are equal numbers of males and females.	Number of males and females $= 2 \cdot 767.1$ $= 1534.2$
⑤ Answer the question.	The total rat population is 1534 rats.

remember

1. A Leontief matrix relates input and output in an economy.
   (a) p is an $n \cdot 1$ matrix whose elements are the products of a process.
   (b) q is an $n \cdot 1$ matrix whose elements are the inputs needed to produce p.
   (c) d is an $n \cdot 1$ matrix whose elements are the external demands in the economy.
   (d) C is an $n \cdot n$ matrix with C_{ij} giving the amount of product i needed to make one unit of product j.
   $q = Cp$ where q is the input needed to give a production, p.
   (e) A production, p, to match a given demand, d, may be found using the equation:
   $p = (I - C)^{-1}d$.
2. A Leslie matrix has fecundity rates across the top row, survival rates along the sub-diagonal and 0s elsewhere. Given a Leslie matrix L, and an initial female population N_1, then the female population after r years is: $N_r = L^{r-1} \cdot N_1$.

EXERCISE 2C Leontief matrices and Leslie matrices

WORKED Example 11

1 Given the consumption matrix $C = \begin{array}{cc} \begin{array}{ccc} G & H & T \end{array} & \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} & \begin{array}{c} G \\ H \\ T \end{array} \end{array}$, calculate the inputs needed to produce:

a 1 unit of *H*

b 12 units of *G*, 15 units of *H* and 30 units of *T*.

2 Given the consumption matrix $C = \begin{array}{cc} \begin{array}{ccc} G & H & T \end{array} & \\ \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} & \begin{array}{c} G \\ H \\ T \end{array} \end{array}$, calculate the inputs needed to produce:

a 3 units of *G*

b 8 units of *G*, 25 units of *H* and 10 units of *T*.

WORKED Example 12

3 Given the consumption matrix at right: $C = \begin{array}{cc} \begin{array}{ccc} G & H & T \end{array} & \\ \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} & \begin{array}{c} G \\ H \\ T \end{array} \end{array}$

a calculate the production needed to match the demand, $d = \begin{bmatrix} 30 \\ 45 \\ 25 \end{bmatrix}$

b show that a suitable production can be found to match any demand.

4 Given the consumption matrix $C = \begin{array}{cc} \begin{array}{ccc} G & H & T \end{array} & \\ \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} & \begin{array}{c} G \\ H \\ T \end{array} \end{array}$:

a calculate the production needed to match the demand $d = \begin{bmatrix} 30 \\ 45 \\ 25 \end{bmatrix}$

b show that a suitable production can be found to match any demand.

5 Consider the following three-industry economy:
The external demands for A, B and C are 20, 30 and 10 respectively. The exchange matrix is: $$M = \begin{array}{cc} \begin{array}{ccc} A & B & C \end{array} & \\ \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.5 & 0.6 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \end{array}.$$

a What is the meaning of the entry 0.5 in the matrix?
b Calculate the required output for each of the three industries.

WORKED Example 13

6 A two-industry economy consists of coal and steel. To produce one tonne of coal requires 0.3 tonnes of coal and 0.4 tonnes of steel. To produce one tonne of steel requires 0.45 tonnes of coal and 0.2 tonnes of steel.
a Construct the consumption matrix.
b What production is required to meet the demand of 45 tonnes of coal and 25 tonnes of steel?

7 Consider the following hypothetical three-industry economy.

To manufacture 1 unit of *A* requires:	0.2 units of A	0.3 units of B	0.1 units of C
To manufacture 1 unit of *B* requires:	0.1 units of A	0.2 units of B	0.2 units of C
To manufacture 1 unit of *C* requires:	0.2 units of A	0.1 units of B	0.3 units of C

a Construct the consumption matrix for this economy.

b What inputs are required to produce 10 units of A, 20 units of B and 60 units of C?

c How much of each should be produced to meet the external demand of 45 units of A, 30 units of B and 40 units of C?

8 Suppose that only 15% of baby saltwater crocodiles live to 1 year of age and that 60% of 1-year-olds survive to 2 years of age. In one season, 40 crocodile eggs hatch. How many of these survive to:
a 1 year of age?
b 2 years of age?

WORKED Example 14

9 The change in the rat population in a certain habitat is described by the following Leslie matrix:

$$L = \begin{bmatrix} 0.7 & 1.4 & 0.7 \\ 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

The initial female rat population is $N_1 = \begin{bmatrix} 33 \\ 18 \\ 8 \end{bmatrix}$.

a Give the meaning of the number 1.4 in the matrix.
b Give the meaning of the number 0.5 in the matrix.
c Calculate the female rat population for year 3.
d Calculate the total rat population after 9 years, assuming equal numbers of male and female rats.

10 The numbers of a particular species of bird have been monitored over three years and the following Leslie matrix has been developed to model the population. The birds are aged 1, 2 or 3 years.

$$L = \begin{bmatrix} 0 & 3.1 & 2.2 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

a How many new birds do 1-year-old birds produce on average?
b How many new birds do 2-year-old birds produce on average?
c What proportion of 1-year-old birds survive the first year?
d If the female population in year 1 is $N_1 = \begin{bmatrix} 245 \\ 101 \\ 32 \end{bmatrix}$, calculate the population of female birds in year 3.

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Cryptology

Cryptology is the science of rendering a communication between two people, A and B, meaningless to anyone intercepting the message. It is an area of mathematics that has grown rapidly, particularly with the rise of the internet and the need for secure transactions and communications on the web. Mathematicians have accomplished all recent developments in cryptography.

Enciphering is the coding of a message (plain text) into a form that is meaningless to a third party. *Deciphering* is the reverse process; taking the code and translating it back into the original plain text message.

Consider the simplest of ciphers — a mono-alphabetic substitution; that is, the substitution of one letter for another. A cipher such as this is easy to crack if one has enough text to work with. A disadvantage of substitution ciphers is that they preserve the frequencies of individual letters, making it relatively easy to break the code by statistical methods. The percentage frequency of each letter in English is:

A	B	C	D	E	F	G	H	I	J	K	L	M
8.2	1.5	2.8	4.3	12.7	2.2	2.0	6.1	7.0	0.2	0.8	4.0	2.4
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
6.7	7.5	1.9	0.1	6.0	6.3	9.1	2.8	1.0	2.3	0.1	2.0	0.1

Naturally, coded texts vary in content, and the frequency of letters in a code will not follow these general proportions exactly. However, they do provide a starting point for someone attempting to decipher a text. The appearance of apostrophes and frequent letter associations in common words such as ‘the’ may also provide clues.

Mono-alphabetic substitution

The following message has been encoded using a mono-alphabetic substitution. Can you decipher it?

BTJPXRMLXPCUVAMLXICVJPIBTWXVRCIMLMT'RPMTN,MTN
YVCJXCDXVMWMBTRJJPXAMTNGXRJBAHUQCTJPXQGMBRJXVCI
JPXYMGGCIJPXHBTW'RQMGMAX;MTNJPXHBTWRMYJPXQMVJCI
JPXPMTNJPMJYVCJX.JPXTJPXHBTW'RACUTJXTMTAXYMR
APMTWXN,MTNPBRJPCUWPJRJVCUFGXNPBL,RCJPMJJPXSCBTJRCI
PBRGCBTRYXVXGCCRXN,MTNPBRHTXXRRLCJXCTXMWMBTRJ
MTCJPXV.JPXHBTWAVBXNMGCUNJCFVBTWBTJPX
MRJVCGCWXVR,JPXAPMGNXMTR,MTNJPXRCCJPRMEXVR.MTNJPX
HBTWRQMHX,MTNRMBNJCJPXYBRXLXTCIFMFEGCT,YPCRCXDXV
RPMGGVXMNJPBRYVBJBTW,MTNRPCYLXJPXBTJXVQVXJMJBCT
JPXVXCI,RPMGGFXAGCJPXNYBJPRAMVGXJ,MTNPMDXMAPMBTCI
WCGNMFCUJPBRTXAH,MTNRPMGGFXJPXJPBVNVUGXVBTJPX
HBTWNCL.JPXTAMLXBTMGGJPXHBTW'RYBRXLXT;FUJJPXE
ACUGNTCJVXMNJPXYVBJBTW,TCVLMHXHTCYTJCJPXHBTWJPX
BTJXVQVXJMJBCTJPXVXCI.JPXTYMRHBTWFXGRPMOOMVWVXMJGE
JVCUFGXN,MTNPBRACUTJXTMTAXYMRAPMTWXNBTPBL,MTNPBR
GCVNRYXVXMRJCTBRPXN.TCYJPXKUXXT,FEVXMRCTCIJPX
YCVNRCIJPXHBTWMTNPBRGCVNR,AMLXBTJCJPXFMTKUXJ
PCURX;MTNJPXKUXXTRQMHXMTNRMBN,CHBTW,GBDXICVXDXV;
GXJTCJJPEJPCUWPJRJVCUFGXJPXX,TCVGXJJPEACUTJXTMTAXFX
APMTWXN;JPXVXBRMLMTBTJPEHBTWNCL,BTYPCLBRJPX
RQBVBJCIJPXPCGEWCNR;MTNBTJPXNMERCIJPEIMJPXVGBWPJ
MTNUTNXVRJMTNBTWMTNYBRNCL,GBHXJPXYBRNCLCIJPX
WCNR,YMRICUTNBTPBL;YPCLJPXHBTWTXFUAPMNTXOOMVJPE
IMJPXV,JPXHBTW,BRME,JPEIMJPXV,LMNXLMRJXVCIJPX
LMWBABMTR,MRJVCGCWXVR,APMGNXMTR,MTNRCCJPRMEXVR;
ICVMRLUAPMRMTXZAXGGXTJRQBVBJ,MTNHTCYGXNWX,MTN
UTNXVRJMTNBTW,BTJXVQVXJBTWCINVXMLR,MTNRPCYBTWCI
PMVNRXTJXTAXR,MTNNBRRCGDBTWCINCUFJR,YXVXICUTNBT
JPXRMLXNMTBXG,YPCLJPXHBTWTMLXNFXGJXRPMOOMV;TCYGXJ
NMTBXGFXAMGGXN,MTNPXYBGGRPCYJPXBTJXVQVXJMJBCT.

One way to overcome the problems associated with mono-alphabetic substitution is to divide the plain text into groups of letters and encipher the plain text group by group, rather than one letter at a time. A system of cryptography in which the plain text is divided into sets of n letters, each of which is replaced by a set of n cipher letters, is called a *poly-graphic system.*

Enciphering

In the discussion to follow, we assume that each plain-text and cipher-text letter, except *Z,* is assigned the numerical value that specifies its position in the standard alphabet.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by a procedure that involves modulo arithmetic. Modulo arithmetic plays an important role in cryptology.

Let us consider this first.

Recall that we say:

$45 \equiv 3$ modulo 7, because the remainder when 45 is divided by 7 is 3.

This statement is usually abbreviated to:

$$45 \equiv 3 \pmod 7.$$

When operating in mod 7 there are only 7 distinct numbers 0, 1, 2, 3, 4, 5 and 6. Every other number can be linked to one of these remainders.

As there are 26 letters in the English alphabet, modulo 26 is frequently used in cryptology.

For example, $87 = 9 \pmod{26}$. This means that the remainder when 87 is divided by 26 is 9. Another way of looking at this is $87 = 26 \cdot\ 3 + 9$.

For negative numbers, we still look for a positive remainder. For example, $-38 = 14$ (mod 26). That is, $-38 = 26 \cdot\ -2 + 14$.

For modulo 26, $Z_{26} = \{0, 1, 2, \quad , 25\}$.

If a is a number in Z_m, then a number, a^{-1}, in Z_m is called a *reciprocal* or *multiplicative inverse* of a modulo m if $aa^{-1} = a^{-1}a = 1 \pmod m$.

Not every element of Z_m will have a multiplicative inverse.

The results for multiplication modulo 26 are listed in the table below.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	2	4	6	8	10	12	14	16	18	20	22	24	0	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	1	4	7	10	13	16	19	22	25	2	5	8	11	14	17	20	23
4	4	8	12	16	20	24	2	6	10	14	18	22	0	4	8	12	16	20	24	2	6	10	14	18	22
5	5	10	15	20	25	4	9	14	19	24	3	8	13	18	23	2	7	12	17	22	1	6	11	16	21
6	6	12	18	24	4	10	16	22	2	8	14	20	0	6	12	18	24	4	10	16	22	2	8	14	20
7	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22	3	10	17	24	5	12	19
8	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
9	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
10	10	20	4	14	24	8	18	2	12	22	6	16	0	10	20	4	14	24	8	18	2	12	22	6	16
11	11	22	7	18	3	14	25	10	21	6	17	2	13	24	9	20	5	16	1	12	23	8	19	4	15
12	12	24	10	22	8	20	6	18	4	16	2	14	0	12	24	10	22	8	20	6	18	4	16	2	14
13	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13
14	14	2	16	4	18	6	20	8	22	10	24	12	0	14	2	16	4	18	6	20	8	22	10	24	12
15	15	4	19	8	23	12	1	16	5	20	9	24	13	2	17	6	21	10	25	14	3	18	7	22	11
16	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
17	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9
18	18	10	2	20	12	4	22	14	6	24	16	8	0	18	10	2	20	12	4	22	14	6	24	16	8
19	19	12	5	24	17	10	3	22	15	8	1	20	13	6	25	18	11	4	23	16	9	2	21	14	7
20	20	14	8	2	22	16	10	4	24	18	12	6	0	20	14	8	2	22	16	10	4	24	18	12	6
21	21	16	11	6	1	22	17	12	7	2	23	18	13	8	3	24	19	14	9	4	25	20	15	10	5
22	22	18	14	10	6	2	24	20	16	12	8	4	0	22	18	14	10	6	2	24	20	16	12	8	4
23	23	20	17	14	11	8	5	2	25	22	19	16	13	10	7	4	1	24	21	18	15	12	9	6	3
24	24	22	20	18	16	14	12	10	8	6	4	2	0	24	22	20	18	16	14	12	10	8	6	4	2
25	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

From the table we can see that 9 is the multiplicative inverse of 3 (mod 26) as $9 \cdot 3 = 1$ (mod 26). That is, 3^{-1} mod 26 = 9.

WORKED Example 15

Simplify each of the following.

a **200 mod 26** **b** **5^{-1} mod 26** **c** **–9 mod 26**

THINK

a Since mod 26 refers to the remainder when dividing by 26, divide 200 by 26 and find the remainder.

b Use the table to find the multiplicative inverse of 5. 5^{-1} means $5^{-1} \cdot 5 = 1$.

c Divide –9 by 26 and find the positive remainder. Use $-9 = -1 \cdot 26 + 17$ rather than $-9 = 0 \cdot 26 - 9$.

WRITE

a 200 | 26 = 7 remainder 18
200 mod 26 = 18

b From the table, $5 \cdot 21 = 1$ (mod 26).
Therefore 5^{-1} mod 26 = 21.

c –9 | 26 = –1 remainder 17
Therefore –9 mod 26 = 17.

Now we can use this to transform plain text to cipher text. In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by the following procedure

Step 1 Choose a $2 \cdot 2$ matrix with integer entries $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This matrix will be used to perform the encoding.

Step 2 Group successive plain-text letters into pairs, adding an arbitrary 'dummy' letter to fill out the last pair if the plain text has an odd number of letters, and replace each plain-text letter by its numerical value.

Step 3 Successively convert each plain-text pair, $p_1 p_2$, into a column matrix:

$$P = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix},$$

and form the product AP. We will call P a *plain-text* matrix and AP the corresponding *cipher-text* matrix.

Step 4 Convert each cipher-text matrix into its alphabetic equivalent using modulo 26 if the numbers are greater than 26.

WORKED Example 16

Use the matrix to encipher the message: THE GAME IS UP.

THINK

① Group the letters in pairs, doubling the last if there is an odd number.

② Convert each letter to a number from 0 to 25. Use the table on page 78.

WRITE

TH EG AM EI SU PP

20 8 5 7 1 13 5 9 19 21 16 16

THINK	**WRITE**
3 Premultiply the first pair of numbers by the given cipher matrix.	$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}20\\8\end{bmatrix}=\begin{bmatrix}48\\36\end{bmatrix}$
4 Convert each number in the resulting matrix to modulo 26; that is, find the remainder when each number is divided by 26. In this case, 48 = 22 (mod 26) and 36 = 10 (mod 26).	$=\begin{bmatrix}22\\10\end{bmatrix}$ mod 26
5 Repeat steps 3 and 4 for the remaining pairs.	$\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix}=\begin{bmatrix}17\\19\end{bmatrix}=\begin{bmatrix}17\\19\end{bmatrix}$ mod 26 $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\13\end{bmatrix}=\begin{bmatrix}15\\27\end{bmatrix}=\begin{bmatrix}15\\1\end{bmatrix}$ mod 26 $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}5\\9\end{bmatrix}=\begin{bmatrix}19\\23\end{bmatrix}=\begin{bmatrix}19\\23\end{bmatrix}$ mod 26 $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}19\\21\end{bmatrix}=\begin{bmatrix}59\\61\end{bmatrix}=\begin{bmatrix}7\\9\end{bmatrix}$ mod 26 $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}16\\16\end{bmatrix}=\begin{bmatrix}48\\48\end{bmatrix}=\begin{bmatrix}22\\22\end{bmatrix}$ mod 26 So 20 8 5 7 1 13 5 9 19 21 16 16 becomes 22 10 17 19 15 1 19 23 7 9 22 22
6 Convert each number to a letter of the alphabet using the table on page 78.	The enciphered message is V J Q S O A S W G I V V.

Because the plain text was grouped in pairs and enciphered by a 2 · 2 matrix, the cipher in the previous worked example is referred to as a *2-cipher*. It is possible to group the plain text in triples and encipher by a 3 · 3 matrix with integer entries; this is called a *3-cipher*. In general, for an *n-cipher*, plain text is grouped into sets of n letters and enciphered by an n · n matrix with integer entries.

Deciphering

To decipher a message that has been encrypted using matrix A, we use the inverse of A, A^{-1}. That is, for the enciphering 2 · 2 matrix, $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, the deciphering 2 · 2 matrix, A^{-1} is given by

$$A^{-1}=(ad-bc)^{-1}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}\text{ (mod 26)} \qquad [2]$$

where $(ad-bc)^{-1}$ is the multiplicative inverse of $(ad-bc)$ in modulo 26.

Note that not all numbers have an inverse in mod 26. The inverse of 3 is 9 because $9 \cdot 3 = 27 = 1$ (mod 26); but there is no inverse for 2 or 4 or 13. (See the table on page 79.)

Note also that the inverse of -3 is different from the inverse of 3. The inverse of 3 is 9 but $-3 \cdot 9 = -27 = 25$ (mod 26). The inverse of -3 is 17 as $-3 \cdot 17 = -51 = 1$ (mod 26).

WORKED Example 17

Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

DPOOOHZZVOQOVYYYVQQ

THINK	WRITE
① To find the inverse matrix, A^{-1}, first calculate the determinant of matrix A.	$\det A = 2 \cdot 2 - 1 \cdot 1$ $= 3$
② Find the multiplicative inverse (mod 26) of det A using the relationship that $(\det A)^{-1} \cdot (\det A) = 1$ for modulo 26. (Use the table on page 79.)	From the table, the multiplicative inverse of 3 is 9. $9 \cdot 3 = 27 = 1$ (mod 26)
③ Find A^{-1}.	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = 9 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $= \begin{bmatrix} 18 & -9 \\ -9 & 18 \end{bmatrix}$
④ Convert the elements in the resulting matrix to modulo 26.	$= \begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$
⑤ Convert the cipher text to a string of numbers. (Use the table on page 78.) Group the numbers in pairs.	DPOOOHZZVOQOVYYYVQQ gives 4 16 15 15 15 8 0 0 22 15 17 25 22 25 25 22 17 17
⑥ Multiply the first pair of numbers by the deciphering matrix $\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$.	$\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 344 \\ 356 \end{bmatrix}$
⑦ Convert the elements in the resulting matrix to modulo 26.	$= \begin{bmatrix} 6 \\ 18 \end{bmatrix}$ mod 26
⑧ Repeat steps 6 and 7 for all pairs of numbers. List the decoded numbers.	4 16 15 15 15 8 0 0 22 15 17 25 22 25 25 22 17 17 gives 6 18 5 5 16 9 0 0 1 20 15 13 15 18 18 15 23 23
⑨ Convert the decoded numbers to text.	This gives FREEPIZZATOMORROWW.

remember

1. A mono-alphabetic substitution cipher exchanges one letter for another.
2. A 2-cipher substitution exchanges a block of two letters for another two letters.
3. One method of enciphering is to use a 2 · 2 matrix with integer entries and follow these steps:
   (a) Convert plain text to numbers and group them in pairs.
   (b) Form 2 · 1 matrices from the number pairs and multiply each by the enciphering matrix.
   (c) Convert the numbers in the resulting matrix to modulo 26.
   (d) Convert these numbers to letters.
4. To decipher an encrypted message, follow the steps in 3 but use the inverse (mod 26) of the enciphering matrix.

EXERCISE 2D Cryptology

WORKED Example 15

1 Simplify each of the following.

a 300 mod 26 **b** 7^{-1} mod 26 **c** –14 mod 26

d 200 mod 26 **e** 8^{-1} mod 26 **f** –5 mod 26

g 431 mod 26 **h** 15^{-1} mod 26 **i** –15 mod 26

WORKED Example 16

2 Encode the message: LEAVE IMMEDIATELY

using the enciphering matrix $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$.

3 Encode the message: THE IDES OF MARCH

using the enciphering matrix $\begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$.

WORKED Example 17

4 The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ has been used to encrypt the following message:

O L Y P X F R K R M E L.

a What is A^{-1} mod 26? **b** Use this matrix to decipher the message.

5 The matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$ has been used to encrypt the following message:

I V W L A E D C O U E D V W K R Y R.

a What is A^{-1} mod 26? **b** Use this matrix to decipher the message.

6 Encode the message: ALAN TURING IS A LEGEND

using a 3 · 3 enciphering matrix $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.

summary

Matrix operations

- Addition: To add two matrices of the same size add the corresponding elements.
- Scalar multiplication: To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.
- Matrix multiplication: To multiply $A \cdot B$: The number of rows in B must be the same as the columns in A.

  $$A_{ij} = A_{i1} \cdot B_{1j} + A_{i2} \cdot B_{2j} +$$

  For example, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- If I is the identity matrix,
  (a) $IA = AI = A$.
  (b) $A^{-1} A = AA^{-1} = I$.

Markov chains

- A transition matrix, M, is used to calculate the likelihood of certain states given the states at a previous stage.
  The values of M are arranged as follows:

$$\begin{array}{ccc}
 & & \textbf{From} \\
 & & \begin{array}{cc} A & \quad B \end{array} \\
\textbf{To} & \begin{array}{c} A \\ \\ B \end{array} & \begin{bmatrix} \quad & \quad \\ & \\ \quad & \quad \end{bmatrix}
\end{array}$$

- If a system can be in one of r states, the $r \cdot 1$ column matrix (or vector) p gives the likelihood of each of the states.
- If a system is initially in a state p, after n stages the system is in a state given by $M^n p$.
- The sum of columns in a transition matrix is 1. The sum of probabilities in the state matrix is 1.
- The long-term behaviour of the system can be found by considering $M^{50}p$, first checking that M^{51} is the same as M^{50}.

Eigenvalues

- For a matrix, A, if there is a column matrix (vector), X, such that $AX = \lambda X$, then X is called an eigenvector of A and λ is called an eigenvalue of the matrix, A.

Leontief matrices

In an economy:

- p is an $n \cdot 1$ matrix whose elements are the products of a process.
- q is an $n \cdot 1$ matrix whose elements are the inputs needed to produce p.
- d is an $n \cdot 1$ matrix whose elements are the external demands in the economy.

- C is an $n \cdot\ n$ matrix with C_{ij} giving the amount of product i needed to make one unit of product j.
- $q = Cp$
- A production, p, to match a given demand, d, may be found using the equation: $p = (I - C)^{-1}d$.

Leslie matrices

- A Leslie matrix has fecundity rates across the top row, survival rates along the sub-diagonal and 0s elsewhere.

  That is, $L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$ where F_i is the fecundity rate (the number of female offspring per female per year) of the ith age group and S_i is the survival rate of the ith age group.
- Given a Leslie matrix, L, and an initial female population, N_1, then the female population after r years is: $N_r = L^{r-1}N_1$.

  N_i is an $n \cdot\ 1$ matrix which gives the age breakdown of the female population.

Cryptology

- A mono-alphabetic substitution cipher exchanges one letter for another.
- A 2-cipher substitution exchanges a block of two letters for another two letters.
- One method of enciphering is to use a $2 \cdot\ 2$ matrix with integer entries and follow these steps.

  Step 1 Convert plain text to numbers and group them in pairs.

  Step 2 Form $2 \cdot\ 1$ matrices from the number pairs and multiply each by the enciphering matrix.

  Step 3 Convert the numbers in the resulting matrix to modulo 26.

  Step 4 Convert these numbers to letters.
- To decipher an encrypted message, follow the steps above but use the inverse (mod 26) of the enciphering matrix.

CHAPTER review

2A **1** If $A = \begin{bmatrix} 4 & 5 \\ -2 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 8 \\ 6 & -1 \end{bmatrix}$:

a calculate $2A + B$

b calculate AB

c calculate BA

d calculate A^{-1}

e calculate B^{-1}

f show that $(AB)^{-1} = B^{-1} \cdot A^{-1}$

g show that det AB = det $A \cdot$ det B.

2A **2** If $M = \begin{bmatrix} 1 & 0 & 4 \\ 3 & -2 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ use your graphics calculator to find the:

a determinant of M

b inverse of M.

2B **3** The transition matrix for a system is $\begin{bmatrix} 0.3 & 0.30 & 0.55 \\ 0.3 & 0.25 & 0.25 \\ 0.4 & x & 0.20 \end{bmatrix}$.

a Calculate the value of x.

b If the state of the system is $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$, calculate the state of the system 5 stages later.

2B **4** Commuters travelling to the city can travel by car, bus or train. Each month the preferences of commuters change in the following way:

Of the car drivers:	Of the bus travellers:	Of the train travellers:
80% stay with cars 10% change to bus travel and 10% change to train travel.	65% stay with bus travel 25% change to train travel and 10% change to car travel.	90% stay with train travel 10% change to car travel.

If, at the beginning of January, 1800 people were using cars to commute to the city, 2200 people were using the bus and 3000 people were using the train to get into the city, calculate:

a the number of people using the train at the beginning of February

b the number of people using car travel at the beginning of June

c the long-term distribution of commuter travel.

2B **5** If $A = \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}$, find the:

a eigenvalues associated with the matrix

b eigenvectors associated with the matrix.

6 Given the consumption matrix $C = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc|c} G & H & I & \\ \end{array}$ $$C = \begin{array}{cccc} G & H & I & \\ \left[\begin{matrix} 0.3 \\ 0.2 \\ 0.3 \end{matrix}\right. & \begin{matrix} 0.5 \\ 0.4 \\ 0.1 \end{matrix} & \left.\begin{matrix} 0.2 \\ 0.3 \\ 0.2 \end{matrix}\right] & \begin{matrix} G \\ H \\ I \end{matrix} \end{array}$$, calculate the following:

a inputs needed to produce 1 unit of H

b inputs needed to produce 12 units of G, 15 units of H and 30 units of I

c production needed to meet the external demand: 40 units of G, 80 units of H and 56 units of I.

7 Consider the following three-industry economy:
The external demand for A, B and C is 20, 30 and 50 respectively.

The exchange matrix is $$M = \begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0.5 \\ 0.1 \\ 0.5 \end{matrix}\right. & \begin{matrix} 0.1 \\ 0.4 \\ 0.6 \end{matrix} & \left.\begin{matrix} 0.2 \\ 0.2 \\ 0.2 \end{matrix}\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

a What is the meaning of the entry 0.6 in the matrix?

b Calculate the required output for each of the three industries.

Modelling and problem solving

1 Kelly has developed a method for predicting whether or not the surf will be good on any particular day. If it is good today, there is a 70% chance it will be good tomorrow. If it is poor today, there is a 40% chance it will be poor tomorrow.

a Construct a transition matrix to represent this situation.

b If the surf is good on Saturday, what is the likelihood that the surf will be good on:

- **i** the next day, Sunday?
- **ii** two days later, Monday?
- **iii** the following Tuesday?

c What is the long-term likelihood of good surf?

2 The Wentworth Show is held in July, but organisers begin planning in January. They are concerned about the weather and know from past records that if a month is dry, there is a 60% chance that the following month will be wet. If the month is wet then there is an 80% chance that the following month will be dry.

a Construct the transition matrix.

b If January is a dry month, calculate the probability that:

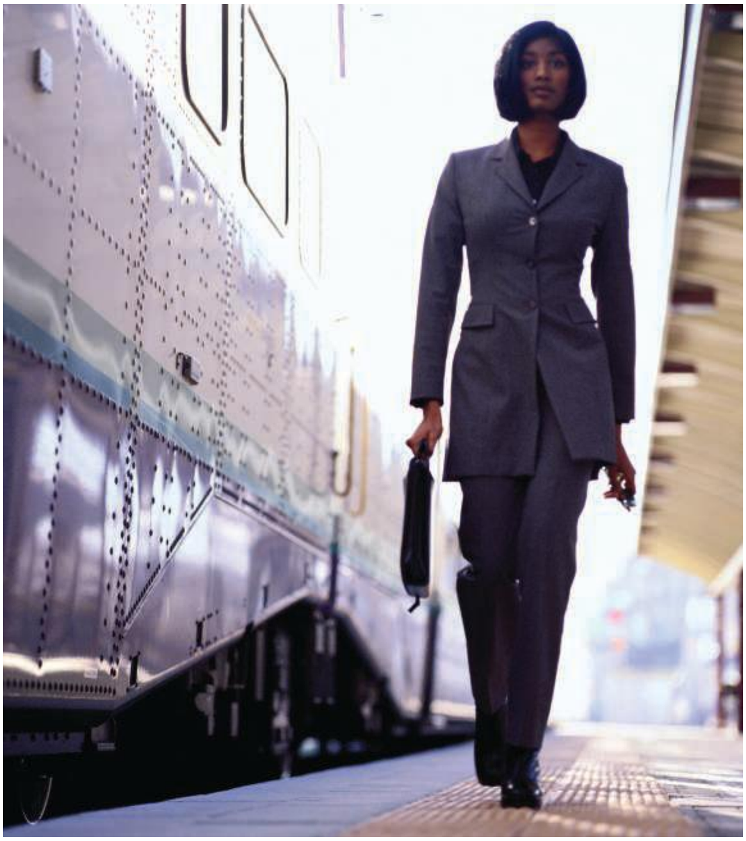
- **i** February will be a dry month
- **ii** July will also be a dry month.

3 Commuters travelling into the centre of Trenchtown use either bus or train. Research has shown that each month 20% of those using the bus switch to train travel and 30% of those using the train revert to bus travel.

If, at the beginning of January, 2500 people were using the bus and 3000 people were using the train to get into the city, calculate each of the following:

a the number of people using the train at the beginning of February

b the number of people using the bus and train in the steady-state situation.



4 Suppose the way in which the market changes its choice of soft drink each month follows this model.

	Croke	Pepola	Sprote	Funta
Croke	50%	20%	10%	20%
Pepola	5%	50%	15%	30%
Sprote	20%	20%	20%	40%
Funta	15%	15%	10%	60%

That is, for example, at the end of each month 20% of the people who use Croke will change to Pepola.

Previous experience indicates that unless a drink can command a 25% market share, it will not be profitable. Based on the above information, which companies will not be profitable in the long run?

5 Three companies are competing for the mobile phone market. At the end of January, market research revealed the following patterns in the subscriptions of mobile phone users.

Of the 400 Tellya customers who were interviewed:

- 320 staying with Tellya
- 50 changing to Oops
- 30 changing to Yodacall.

Of the 150 Oops customers who were interviewed:

- 130 staying with Oops
- 15 changing to Tellya
- 5 changing to Yodacall.

Of the 85 Yodacall customers who were interviewed:

- 80 staying with Yodacall
- 0 changing to Tellya
- 5 changing to Oops.

A company will fail to be viable if its market share falls below 25%. Which, if any, of these companies will not achieve this market share in the long run?

6 Consider the following data associated with sheep flocks.

Age (years)	Birth rate	Survival rate	Number (of females) in 2002
0	0	0.90	450
1	0.50	0.85	2200
2	0.45	0.50	860
3	0.40	0	55

a Construct a Leslie matrix for the sheep flock.

b Use the Leslie matrix to predict the sheep population in 5 years time.

7 On a farm, the following data are recorded for the sheep flock in a particular year.

Age	Number of females at start of year	Number of females at the end of the year	Number of females born to this group
0	330	280	0
1	880	870	350
2	150	0	12

Assume that there are no female sheep aged over 2 years.

a What is the survival rate of:
  - **i** the 0-year-old group?
  - **ii** the 1-year-old group?
  - **iii** the 2-year-old group?

b What is the birth rate for:
  - **i** the 0-year-old group?
  - **ii** the 1-year-old group?
  - **iii** the 2-year-old group?

c Construct a Leslie matrix for the sheep flock.

d Use the matrix to predict the sheep population in 3 years time.

8 The numbers of a particular species of lizard, all aged 1, 2 or 3 years, have been monitored over many years and the following Leslie matrix has been developed to model the population.

$$L = \begin{bmatrix} 0 & 5.30 & 3.50 \\ 0.20 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

a On average, how many new lizards do 1-year-old lizards produce?

b On average, how many new lizards do 2-year-old lizards produce?

c What proportion of 1-year-old lizards survive the first year?

d If the initial female population is $N_1 = \begin{bmatrix} 35 \\ 46 \\ 22 \end{bmatrix}$, calculate the female population in 3 years time.

9 Encode the message:

ET PLEASE COME HOME

using the enciphering matrix $\begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$.

10 The matrix, $A = \begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$ has been used to encrypt the following message.

B H A V Q F I Z L J O T

a What is A^{-1} mod 26?

b Use this matrix to decipher the message.

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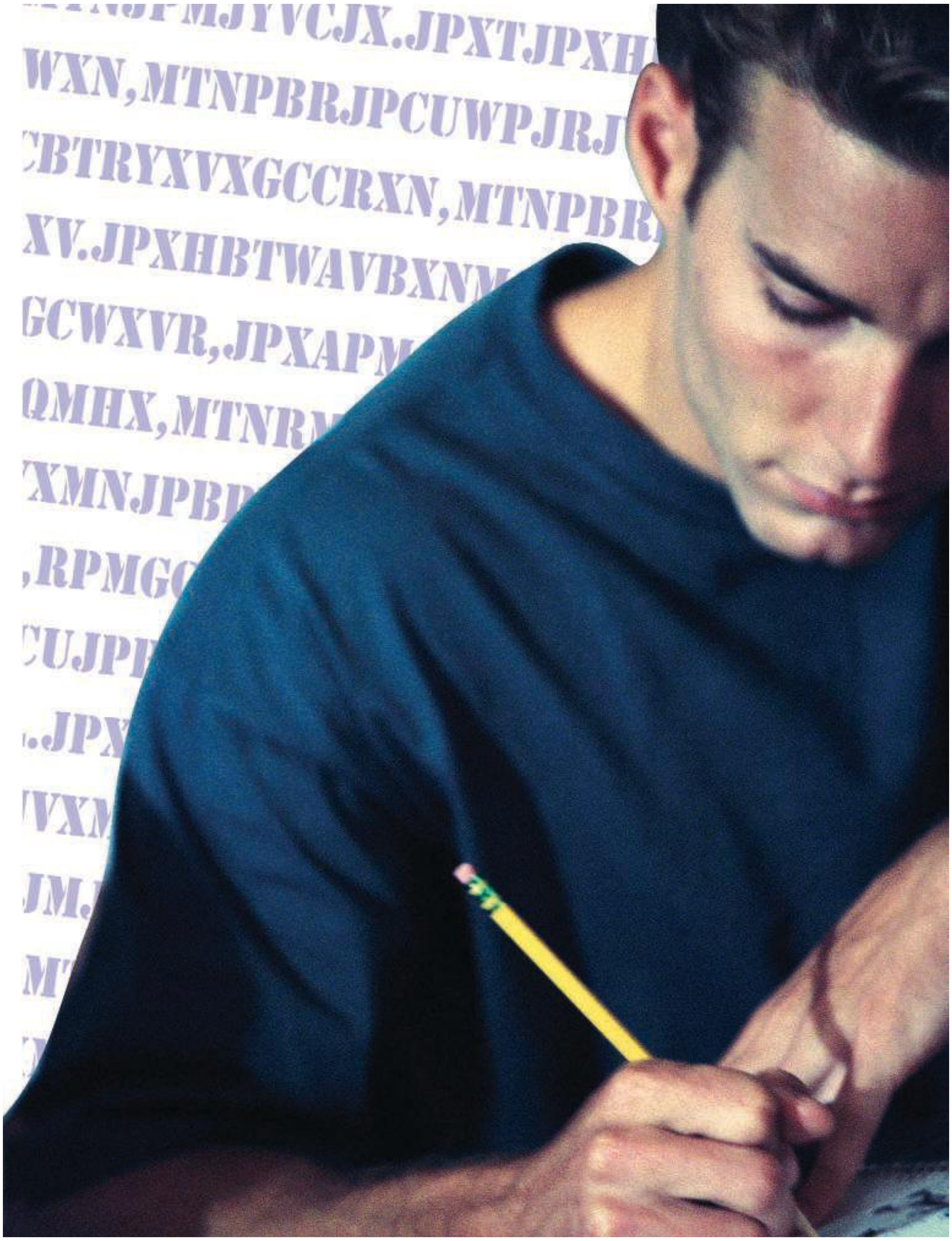
Test Yourself
Chapter 2



eBook*plus* ACTIVITIES

2A Review of matrix operations

Tutorials

- Int-0512: Watch a tutorial on matrix multiplication. (*page 47*)
- Int-0305: Watch a tutorial on using matrices to solve simultaneous equations. (*page 51*)

Digital doc

- WorkSHEET 2.1: Calculate determinants, inverses and unknown elements in matrices. (*page 55*)

2B Markov chains and eigenvalues

Interactivity

- Int-0270: Use the interactivity to investigate transition matrices. (*page 57*)

Tutorial

- Int-0515: Watch a tutorial on using matrices to represent Markov systems. (*page 61*)

Chapter review

Digital docs

- WorkSHEET 2.2: Use transition matrices to determine long term behaviour of a system, calculate eigenvalues and use Leslie matrices. (*page 77*)

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 91*)

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3

Vectors

syllabus reference

Core topic:
Vectors and applications

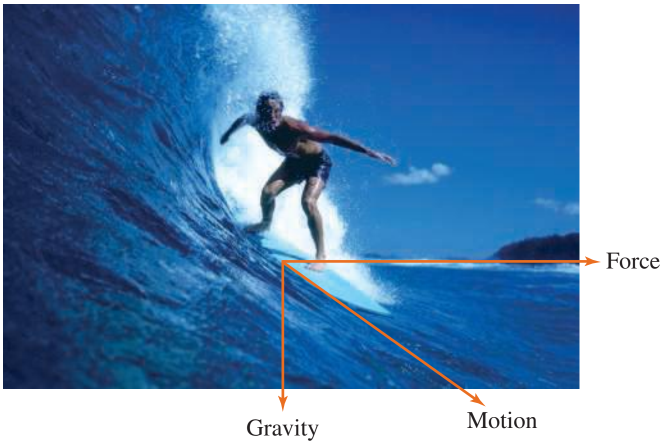
In this chapter

3A Review of vectors
3B Vector product
3C Scalar triple product
3D Applications of the vector product

Review of vectors

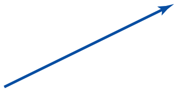
Physics is the subject that attempts to explain how energy and atoms and other particles interact to create the world as we know it. The physicist attempts to develop theories to explain why apples fall down and not up, why the sky is blue and why magnets attract each other.

In developing theories to explain these phenomena, physicists use the language of mathematics. Whenever forces and movement are considered, vectors are used.



We begin by reviewing a number of concepts related to vectors that were introduced in year 11.

A vector is a quantity that has both magnitude and direction.



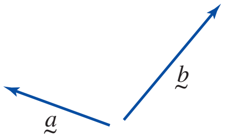
The vector may be represented by a directed line, or arrow. The arrow indicates the direction of the vector and the length of the line represents the magnitude of the vector. This makes a vector the ideal tool to investigate situations involving force, displacement, velocity and acceleration.

Vectors are often written as a single letter in bold, such as ***u***. However, this is difficult to write using pen and paper, so $\underset{\sim}{u}$ can also be used. The symbol (~) is called a tilde. Vectors with identified start and end points (for example A and B) may be written with an arrow: $\overrightarrow{AB}$. Vectors can be added, subtracted and multiplied by a scalar. Recall that a scalar is simply a number.

WORKED Example 1

For the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ shown at right, draw diagrams to represent each of the following.

a $3\underset{\sim}{a}$ **b** $\underset{\sim}{a} + \underset{\sim}{b}$ **c** $2\underset{\sim}{a} - \underset{\sim}{b}$

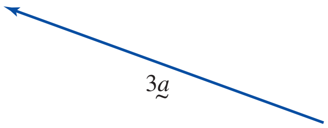


THINK

a Scalar multiplication changes the length of the vector. Draw the vector three times as long.

WRITE

a



THINK

b ① Add vectors by placing them head to tail.

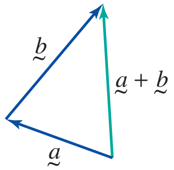
② Draw the new vector $\underset{\sim}{a} + \underset{\sim}{b}$.

c ① To add two vectors to obtain the vector $2\underset{\sim}{a} - \underset{\sim}{b}$, consider $2\underset{\sim}{a} - \underset{\sim}{b}$ as $2\underset{\sim}{a} + (-\underset{\sim}{b})$. Reverse the direction of vector $\underset{\sim}{b}$ to make it negative. The vector $2\underset{\sim}{a}$ is twice as long as vector $\underset{\sim}{a}$.

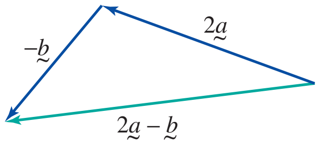
② Add vector $2\underset{\sim}{a}$ and the new vector $-\underset{\sim}{b}$ by placing them head to tail. Draw the vector $2\underset{\sim}{a} - \underset{\sim}{b}$.

WRITE

b



c



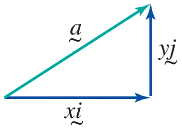
It is often convenient to represent a vector in terms of its components.

A *unit* vector is a vector of length 1. There are three special unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ that are parallel to the x-, y- and z-axes respectively.

Thus, in two dimensions, we may write any vector $\underset{\sim}{a}$ as the sum of its $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components.

$$\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

Notice that, as the triangle is right-angled, we have:



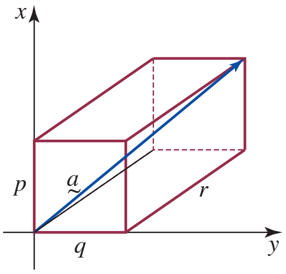
$$\cos\theta = \frac{x}{|\underset{\sim}{a}|} \quad \text{so } x = |\underset{\sim}{a}| \cos\theta \text{ and}$$

$$\sin\theta = \frac{y}{|\underset{\sim}{a}|} \quad \text{so } y = |\underset{\sim}{a}| \sin\theta$$

where $|\underset{\sim}{a}|$ is the magnitude or length of vector $\underset{\sim}{a}$.

So

$$\begin{aligned} \underset{\sim}{a} &= x\underset{\sim}{i} + y\underset{\sim}{j} \\ &= |\underset{\sim}{a}| \cos\theta\underset{\sim}{i} + |\underset{\sim}{a}| \sin\theta\underset{\sim}{j}. \end{aligned}$$



In three dimensions we may write any vector as the sum of its $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components.

The vector $\underset{\sim}{a}$ can be expressed in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.

$$\underset{\sim}{a} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$$

WORKED Example 2

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Tutorial:
Worked example 2
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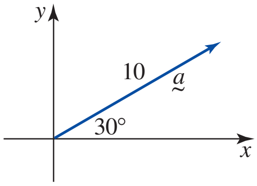
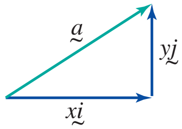
Express the vector, $\underset{\sim}{a}$, in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.



THINK	**WRITE**
1 Express $\underset{\sim}{a}$ in component form.	$\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$
2 Find the $\underset{\sim}{i}$ component by using the relationship $x = \lvert\underset{\sim}{a}\rvert \cos\theta$.	$\underset{\sim}{i}$ component: $x = \lvert\underset{\sim}{a}\rvert \cos\theta$ $= 10 \cos 30°$ $= 8.66$
3 Find the $\underset{\sim}{j}$ component by using the relationship $y = \lvert\underset{\sim}{a}\rvert \sin\theta$.	$\underset{\sim}{j}$ component: $y = \lvert\underset{\sim}{a}\rvert \sin\theta$ $= 10 \sin 30°$ $= 5$
4 Write the vector in $\underset{\sim}{i}$ and $\underset{\sim}{j}$ form.	$\underset{\sim}{a} = 8.66\underset{\sim}{i} + 5\underset{\sim}{j}$

If a vector is given in component form, say, $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$, we can find the magnitude and direction of the vector using a right-angled triangle as follows.



Magnitude: $\lvert\underset{\sim}{a}\rvert = \sqrt{x^2 + y^2}$

Direction: θ where $\tan\theta = \dfrac{y}{x}$

In three dimensions, the vector magnitude can be found by extending the formula:

$$\lvert\underset{\sim}{a}\rvert = \sqrt{x^2 + y^2 + z^2}$$

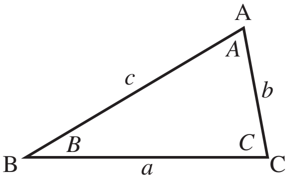
When adding two vectors written in component form, we add each corresponding component for $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$. For example, if $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ and $\underset{\sim}{b} = u\underset{\sim}{i} + v\underset{\sim}{j}$ then the resultant vector, $\underset{\sim}{r}$, for the sum of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is:

$$\begin{aligned}\underset{\sim}{r} &= \underset{\sim}{a} + \underset{\sim}{b}\\ &= x\underset{\sim}{i} + y\underset{\sim}{j} + u\underset{\sim}{i} + v\underset{\sim}{j}\\ &= (x + u)\underset{\sim}{i} + (y + v)\underset{\sim}{j}\end{aligned}$$

Alternatively, we can find the magnitude and direction of the resultant vector without using components, by using the following rules for a non-right-angled triangle.

Cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$

Sine rule $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

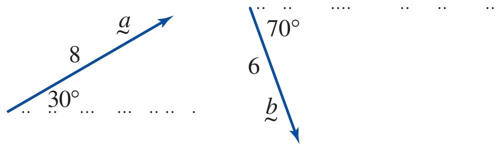


Note the convention where A, B and C are labels for points; A, B and C are labels for their respective angles; and a, b and c are labels for the opposite sides.

These two methods are shown in the following worked example.

WORKED Example 3

Find the resultant magnitude and direction of the vector sum $\underset{\sim}{a} + \underset{\sim}{b}$.

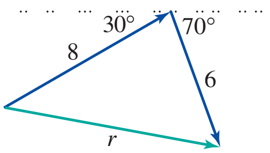


THINK **WRITE**

Method 1 Using components

1. Draw a diagram showing the resultant vector, $\underset{\sim}{r}$. The vector $\underset{\sim}{r}$ is found by placing the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ head to tail.

Let $\underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{b}$.



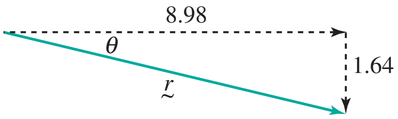
2. Convert each vector to component form.

$$\begin{aligned}\underset{\sim}{a} &= 8\cos 30°\underset{\sim}{i} + 8\sin 30°\underset{\sim}{j}\\ &= 6.93\underset{\sim}{i} + 4\underset{\sim}{j}\\ \underset{\sim}{b} &= 6\cos(-70°)\underset{\sim}{i} + 6\sin(-70°)\underset{\sim}{j}\\ &= 2.05\underset{\sim}{i} - 5.64\underset{\sim}{j}\end{aligned}$$

3. To obtain $\underset{\sim}{a} + \underset{\sim}{b}$ in component form, add the $\underset{\sim}{i}$ components of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and the $\underset{\sim}{j}$ components of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$\begin{aligned}\underset{\sim}{r} &= \underset{\sim}{a} + \underset{\sim}{b}\\ &= 6.93\underset{\sim}{i} + 4\underset{\sim}{j} + 2.05\underset{\sim}{i} - 5.64\underset{\sim}{j}\\ &= 8.98\underset{\sim}{i} - 1.64\underset{\sim}{j}\end{aligned}$$

4. Draw vector $\underset{\sim}{r}$, showing the horizontal and vertical components.



5. Calculate the magnitude of the resultant vector $\underset{\sim}{r}$.

$$\begin{aligned}|\underset{\sim}{r}| &= \sqrt{8.98^2 + (-1.64)^2}\\ &= 9.13\end{aligned}$$

6. Find the direction of $\underset{\sim}{r}$.

$$\begin{aligned}\tan\theta &= \frac{-1.64}{8.98}\\ \theta &= -10.3°\end{aligned}$$

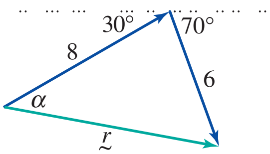
7. Write the answer.

The resultant vector has a magnitude of 9.13 in a direction 10.3° south of east.

Method 2 Using lengths and angles in a non-right-angled triangle

1. Draw a diagram showing the resultant vector, $\underset{\sim}{r}$. The vector $\underset{\sim}{r}$ is found by placing the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ head to tail.

Let $\underset{\sim}{r} = \underset{\sim}{a} + \underset{\sim}{b}$.



2. Use the cosine rule to find $|\underset{\sim}{r}|$.

$$\begin{aligned}|\underset{\sim}{r}|^2 &= 8^2 + 6^2 - 2\cdot 8\cdot 6\cos 80°\\ &= 83.33\\ |\underset{\sim}{r}| &= 9.13\end{aligned}$$

3. Use the sine rule to find α.

$$\begin{aligned}\frac{\sin\alpha}{6} &= \frac{\sin 80°}{9.13}\\ \sin\alpha &= 0.647\\ \alpha &= 40.3°\end{aligned}$$

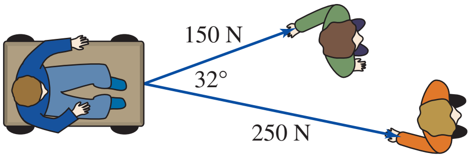
4. Write the answer.

The resultant vector has a magnitude of 9.13 in a direction 10.3° south of east.

WORKED Example 4

Two people are towing a third person in a 'billycart' by ropes attached to the cart. The forces exerted by them are given in the diagram at right.

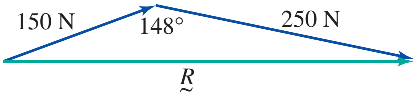
Calculate the magnitude of the resultant force on the 'billycart'.



THINK

WRITE

① The resultant force is the vector sum of the force.



② Draw the vectors head to tail.

③ Use the cosine rule to find the magnitude of the resultant force, $|\underset{\sim}{R}|$.

$$|\underset{\sim}{R}|^2 = 150^2 + 250^2 - 2 \cdot 150 \cdot 250 \cos 148°$$
$$= 148\ 603$$
$$|\underset{\sim}{R}| = 385.5 \text{ N}$$

④ Write the answer in a sentence.

The magnitude of the resultant force is 385.5 N.

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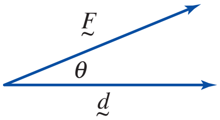
Scalar product

When a force, *F,* moves an object through a displacement, *d,* the object gains energy. If the force and the motion have the same direction then the amount of energy gained is $|F| \cdot |d|$. If the force and the direction of motion are not the same then only that component of force in the direction of motion increases the energy of the object.

The component of force in the direction of *d* is $|\underset{\sim}{F}| \cos \theta$.

Thus the increase in energy is now $|\underset{\sim}{F}| \cos \theta \cdot |\underset{\sim}{d}|$.

This gives rise to a useful concept in dealing with vectors: the scalar product sometimes called the *dot product*.



For two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ the scalar product is a scalar:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos \theta$$

where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Note that if the vectors are perpendicular then the value of the dot product, $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is 0, because $\cos 90° = 0$.

Returning to our discussion of force, $\underset{\sim}{F}$, and displacement, $\underset{\sim}{d}$: the energy, E, gained by an object when a force $\underset{\sim}{F}$ is applied through a displacement $\underset{\sim}{d}$ is:

$$E = \underset{\sim}{F} \cdot \underset{\sim}{d}$$

If a vector is given in terms of its $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components then the calculation of the dot product is relatively simple.

It is easy to see that $\underset{\sim}{i} \cdot \underset{\sim}{i} = |\underset{\sim}{i}||\underset{\sim}{i}| \cos 0° = 1 \cdot 1 \cdot 1 = 1$.

In the same way, $\underset{\sim}{j} \cdot \underset{\sim}{j} = 1$ and $\underset{\sim}{k} \cdot \underset{\sim}{k} = 1$.

Also, $\underset{\sim}{i} \cdot \underset{\sim}{j} = |\underset{\sim}{i}||\underset{\sim}{j}| \cos 90° = 1 \cdot 1 \cdot 0 = 0$.

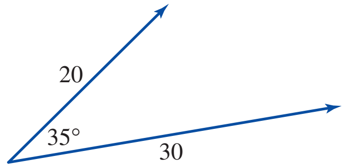
In the same way, $\underset{\sim}{i} \cdot \underset{\sim}{k} = 0$ and $\underset{\sim}{j} \cdot \underset{\sim}{k} = 0$.

Thus if $a = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $b = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$.

WORKED Example 5

Calculate the dot product for each of the following pairs of vectors.

a

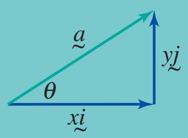


b $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

THINK	WRITE
a 1 Calculate the dot product (or scalar product) by using the appropriate relationship for the given information. In this case use $\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos \theta$.	**a** Let the two vectors be $\underset{\sim}{a}$ and $\underset{\sim}{b}$. $\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos \theta$
2 Substitute for $\|\underset{\sim}{a}\|$, $\|\underset{\sim}{b}\|$ and θ.	$= 20 \cdot 30 \cdot \cos 35°$
3 Evaluate. The answer is a scalar.	$= 491.5$
b 1 Write $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in terms of their components.	**b** $\underset{\sim}{a} \cdot \underset{\sim}{b} = (2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}) \cdot (-\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$
2 Use the Distributive Law, recognising that products of 'different' unit vectors are zero and products of the same unit vectors are 1. For example, $\underset{\sim}{i} \cdot \underset{\sim}{j} = 0$ and $\underset{\sim}{i} \cdot \underset{\sim}{i} = 1$.	$= 2\underset{\sim}{i} \cdot -\underset{\sim}{i} + -\underset{\sim}{j} \cdot 2\underset{\sim}{j} + 2\underset{\sim}{k} \cdot \underset{\sim}{k}$ $= -2 - 2 + 2$ $= -2$

remember

1. A vector can be represented by giving magnitude and direction or by giving $\underset{\sim}{i}, \underset{\sim}{j}$ and $\underset{\sim}{k}$ components.
2. Vectors can be added by placing them head to tail.
3. If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ we have



$\cos\,\theta = \dfrac{x}{|\underset{\sim}{a}|}$ so $x = |\underset{\sim}{a}|\,\cos\,\theta$ and

$\sin\,\theta = \dfrac{y}{|\underset{\sim}{a}|}$ so $y = |\underset{\sim}{a}|\,\sin\,\theta$

where $|\underset{\sim}{a}|$ is the magnitude or length of vector $\underset{\sim}{a}$.
So $\underset{\sim}{a} = |\underset{\sim}{a}|\,\cos\,\theta\underset{\sim}{i} + |\underset{\sim}{a}|\,\sin\,\theta\underset{\sim}{j}$.

4. The magnitude of $\underset{\sim}{a} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$ is $|\underset{\sim}{a}| = \sqrt{p^2 + q^2 + r^2}$.
5. The dot (or scalar) product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\,\cos\,\theta$$

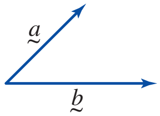
where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{j}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{j}$ then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$.

EXERCISE 3A

Review of vectors

WORKED Example 1

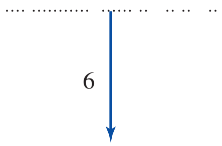
1 For the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ shown at right, draw diagrams to represent each of the following.



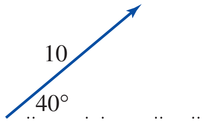
a $-2\underset{\sim}{a}$ **b** $\underset{\sim}{a} + 2\underset{\sim}{b}$ **c** $3\underset{\sim}{a} - 2\underset{\sim}{b}$
d $3\underset{\sim}{a}$ **e** $-3\underset{\sim}{b}$ **f** $-\underset{\sim}{a} - \underset{\sim}{b}$

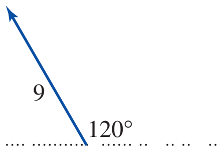
WORKED Example 2

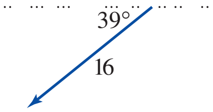
2 Express each of the following vectors in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

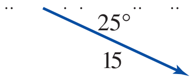
a


b

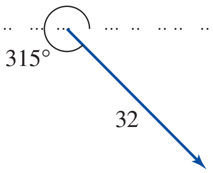

c


d


e


f


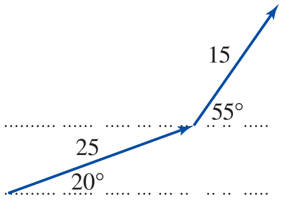
g


h


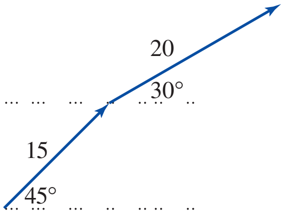
WORKED Example 3

3 Find the resultant magnitude and direction of each of the following vector sums.

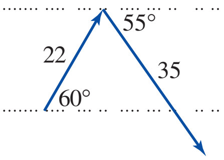
a



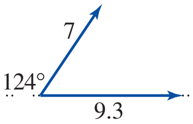
b



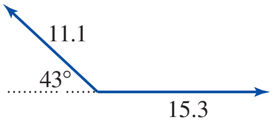
c



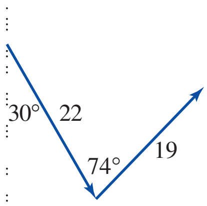
d



e



f



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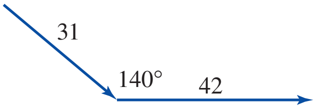
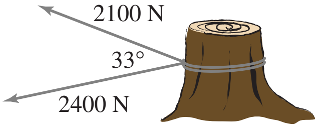


WORKED Example 4

4 In order to remove a tree stump, two wires are attached to it as shown at right.

Calculate the magnitude of the resultant force on the tree stump.



5 Find the magnitude of the vector sum $\underset{\sim}{a} + \underset{\sim}{b}$ for each of the following.

a $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$

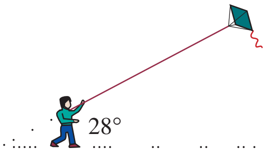
b $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

c $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$

d $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$

e $\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

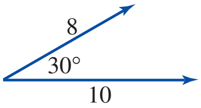
6 A boy holds a kite in a strong wind. The tension in the kite string is 400 N and the boy's weight is 600 N acting straight down. Calculate the magnitude of the resultant force on the boy.



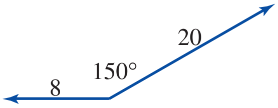
WORKED Example 5a

7 Calculate the dot product for each of the following pairs of vectors.

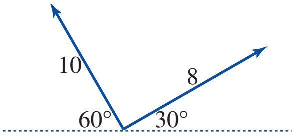
a



b



c



WORKED Example 5b

8 Calculate $\underset{\sim}{a} \cdot \underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

b $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$

c $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$.

9 Which of the pairs of vectors in question **8** are perpendicular?

10 Calculate $\underset{\sim}{a} \cdot \underset{\sim}{b}$ if:

a $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$

b $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}, \underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

c $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}, \underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

d $\underset{\sim}{a} = 2\underset{\sim}{j} + 2\underset{\sim}{k}, \underset{\sim}{b} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$

e $\underset{\sim}{a} = -4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}, \underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{k}$.

11 Which of the pairs of vectors in question **10** are perpendicular?

12 Calculate the value of n if the following vectors are perpendicular.

a $2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $n\underset{\sim}{i} + \underset{\sim}{j}$

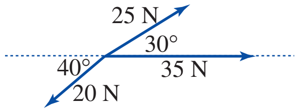
b $n\underset{\sim}{i} - 2\underset{\sim}{j}$ and $n\underset{\sim}{i} + 8\underset{\sim}{j}$

c $n\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $2\underset{\sim}{i} + 2\underset{\sim}{k}$

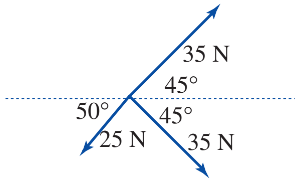
d $\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $3\underset{\sim}{i} + n\underset{\sim}{j} + 3\underset{\sim}{k}$

13 Three forces act on an object as shown. Calculate the magnitude and direction of the resulting force.

a

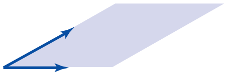


b

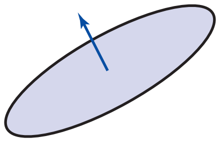


Vector product

Two vectors, provided that they are not parallel, define a plane. A plane is a flat surface extending indefinitely in all directions.



The orientation of a plane can be described, if a vector perpendicular to the plane is known.



Describing relationships between lines and planes in three dimensions is difficult. Plane geometry (geometry in two dimensions) can be written and analysed easily on paper. Extending to three dimensions on paper creates problems. However, three-dimensional vectors are extremely useful for describing relationships between lines and planes in three dimensions.

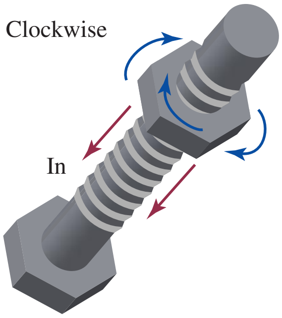
Given that a key element in the description of a plane is the normal vector, it seems natural to define an operation between two vectors that produces a vector normal to both. This leads to the definition of the *vector product*.

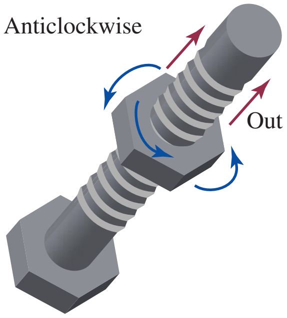
The *vector product* of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin \theta \hat{\underset{\sim}{n}}$$

where θ is the angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\hat{\underset{\sim}{n}}$ is a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the right-hand sense. (Recall that a unit vector has length 1.)

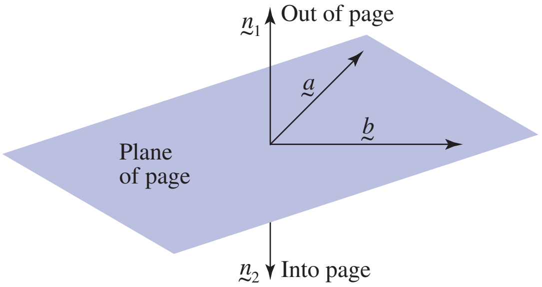
The phrase ‘in the right-hand sense’ needs some explanation. First let us agree on turning a nut (or a tap or a screw).





Turning the nut clockwise moves the nut inwards along the thread. Turning the nut anticlockwise moves the nut outwards. This phenomenon can be applied to the vector product.

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are any two vectors, and are not in the same direction, we can always find two unit vectors that are perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



Which of these two vectors $\hat{\underset{\sim}{n}}_1$ or $\hat{\underset{\sim}{n}}_2$ is used in the vector product? If the vector product is $\underset{\sim}{a} \cdot \underset{\sim}{b}$ then the motion of $\underset{\sim}{a}$ to $\underset{\sim}{b}$ represents a clockwise motion (the nut is screwed inwards) and the result is the vector that points in, $\hat{\underset{\sim}{n}}_2$.

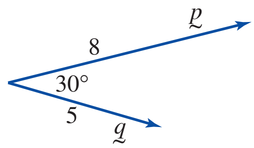
If the vector product is $\underset{\sim}{b} \cdot \underset{\sim}{a}$ then the motion of $\underset{\sim}{b}$ to $\underset{\sim}{a}$ is anticlockwise (the nut is screwed outwards) and the result is the vector that points out, $\hat{\underset{\sim}{n}}_1$.

Note that $\underset{\sim}{a} \cdot \underset{\sim}{b} = -\underset{\sim}{b} \cdot \underset{\sim}{a}$.

WORKED Example 6

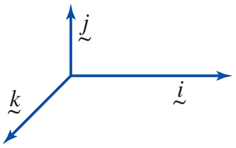
Vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ lie in a horizontal plane as shown. Calculate each of the following.

a $\underset{\sim}{p} \cdot \underset{\sim}{q}$ **b** $\underset{\sim}{q} \cdot \underset{\sim}{p}$



THINK	WRITE
a ① Recall the rule for the vector product.	**a** $\underset{\sim}{p} \cdot \underset{\sim}{q} = \lvert\underset{\sim}{p}\rvert\lvert\underset{\sim}{q}\rvert \sin\theta\, \hat{\underset{\sim}{n}}$
② Substitute the values for $\underset{\sim}{p}$, $\underset{\sim}{q}$ and θ.	$= 8 \cdot 5 \cdot \sin 30°\hat{\underset{\sim}{n}}$
③ Simplify.	$= 20\hat{\underset{\sim}{n}}$
④ Determine the direction of $\hat{\underset{\sim}{n}}$. Recall that if $\hat{\underset{\sim}{n}}$ is perpendicular to $\underset{\sim}{p}$ and $\underset{\sim}{q}$, it is either up or down. Moving $\underset{\sim}{p}$ to $\underset{\sim}{q}$ is a clockwise motion, so the result is down.	Therefore $\underset{\sim}{p} \cdot \underset{\sim}{q} = 20$ in the downward direction.
b Recall that $\underset{\sim}{a} \cdot \underset{\sim}{b} = -\underset{\sim}{b} \cdot \underset{\sim}{a}$.	**b** $\underset{\sim}{q} \cdot \underset{\sim}{p} = 20$ in the upward direction.

The vector product in $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ form



Recall that the vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are the mutually perpendicular, unit vectors that can be used to describe three-dimensional space.

The *x*, *y* and *z* directions cannot be arranged in just any order, even though they may be mutually perpendicular. The form is a right-hand triad. This means that the direction of *z* must be that obtained by 'screwing' *x* to *y*.

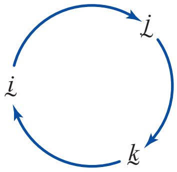
If no axes are specified, we can assume that *x* is horizontal positive to the right and *y* is vertical positive upwards. The *z*-axis then *must* be out of the page positive.

Consider some special vector products. Remember that $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin \theta \hat{\underset{\sim}{n}}$ where $\hat{\underset{\sim}{n}}$ is a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the right-hand sense.

$$\begin{aligned}
\underset{\sim}{i} \cdot \underset{\sim}{i} &= 1 \cdot 1 \cdot \sin 0^\circ \hat{\underset{\sim}{n}} \\
&= 0 \\
\underset{\sim}{i} \cdot \underset{\sim}{j} &= 1 \cdot 1 \cdot \sin 90^\circ \hat{\underset{\sim}{n}} \\
&= 1 \cdot 1 \cdot 1 \cdot \underset{\sim}{k} \\
&= \underset{\sim}{k}
\end{aligned}$$

Continuing in this way we can complete the multiplication table as follows. (Remember that the order of the vectors in the vector product is important.)

		Second vector		
	$\cdot$	$\underset{\sim}{i}$	$\underset{\sim}{j}$	$\underset{\sim}{k}$
First vector	$\underset{\sim}{i}$	0	$\underset{\sim}{k}$	$-\underset{\sim}{j}$
	$\underset{\sim}{j}$	$-\underset{\sim}{k}$	0	$\underset{\sim}{i}$
	$\underset{\sim}{k}$	$\underset{\sim}{j}$	$-\underset{\sim}{i}$	0



A useful way of summarising these results is shown in the figure at right.

A product following the direction indicated (clockwise) produces a positive value.

$$\underset{\sim}{j} \cdot \underset{\sim}{k} = \underset{\sim}{i}$$

A product in the anticlockwise direction produces a negative value.

$$\underset{\sim}{j} \cdot \underset{\sim}{i} = -\underset{\sim}{k}$$

To evaluate the vector product of two vectors that are expressed in $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ form we can use the Distributive Law. This is illustrated in the following worked example.

WORKED Example 7

If $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, use the Distributive Law of vectors to calculate the vector product $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

THINK	**WRITE**
① Write the product, remembering that the order of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is important.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) \cdot (\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

THINK	**WRITE**
② Expand the brackets. There are nine products we need to consider.	$= 2\underset{\sim}{i} \cdot \underset{\sim}{i} + 2\underset{\sim}{i} \cdot 2\underset{\sim}{j} + 2\underset{\sim}{i} \cdot 3\underset{\sim}{k}$ $-2\underset{\sim}{j} \cdot \underset{\sim}{i} - 2\underset{\sim}{j} \cdot 2\underset{\sim}{j} - 2\underset{\sim}{j} \cdot 3\underset{\sim}{k}$ $+ \underset{\sim}{k} \cdot \underset{\sim}{i} + \underset{\sim}{k} \cdot 2\underset{\sim}{j} + \underset{\sim}{k} \cdot 3\underset{\sim}{k}$
③ Evaluate each of these products individually using the results in the table on page 104.	$= 0 + 4\underset{\sim}{k} - 6\underset{\sim}{j} + 2\underset{\sim}{k} + 0 - 6\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{i} + 0$
④ Simplify.	$= -8\underset{\sim}{i} - 5\underset{\sim}{j} + 6\underset{\sim}{k}$

Clearly, this method of calculating the vector product is fairly complicated.

A simpler calculation relies on the use of a 3 · 3 determinant.

Recall that if $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then the determinant of A is $\begin{vmatrix} p & q \\ r & s \end{vmatrix} = ps - qr$.

The determinant of a 3 · 3 matrix is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{j}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{j}$

then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \underset{\sim}{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

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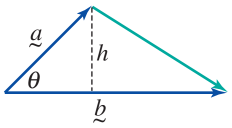
WORKED Example 8

If $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, calculate the vector product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ by the evaluation of determinants.

THINK	**WRITE**
① Write the vector product as a determinant.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$
② Evaluate the determinant.	$= \underset{\sim}{i}\begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}$ $= -8\underset{\sim}{i} - 5\underset{\sim}{j} + 6\underset{\sim}{k}$

Area of a triangle



The area of a triangle is given by $\frac{1}{2}\cdot$ base $\cdot$ perpendicular height.

For the triangle formed by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the base is $|\underset{\sim}{b}|$ and the height, h, can be calculated as follows:

$$\sin\theta = \frac{h}{|\underset{\sim}{a}|}$$

$$h = |\underset{\sim}{a}|\sin\theta.$$

That is, the area of the triangle, $A = \frac{1}{2}\cdot|\underset{\sim}{b}|\cdot h$

$$= \frac{1}{2}\cdot|\underset{\sim}{b}|\cdot|\underset{\sim}{a}|\sin\theta$$

$$= \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta.$$

This can be related to the vector product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$; that is, $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$. Therefore the area, A, of a triangle formed by vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given by:

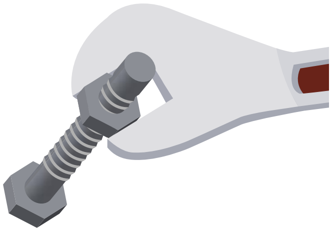
$$A = \tfrac{1}{2}|\underset{\sim}{a}\cdot\underset{\sim}{b}|.$$

WORKED Example 9

Calculate the area of triangle OAB if A and B have coordinates (2, 1, −1) and (3, 0, 2) respectively.

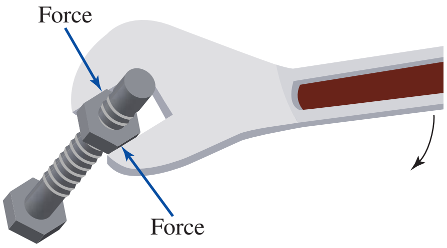
THINK	WRITE
1 The triangle is formed by the vectors OA and OB. Write each vector in component form.	OA $= 2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$ and OB $= 3\underset{\sim}{i}+2\underset{\sim}{k}$
2 State the area formula.	$A = \frac{1}{2}\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert$
3 Substitute for $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$A = \frac{1}{2}\lvert(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k})\cdot(3\underset{\sim}{i}+2\underset{\sim}{k})\rvert$
4 Use the determinant method to calculate the vector product.	$(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k})\cdot(3\underset{\sim}{i}+2\underset{\sim}{k})$ $= \begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & 1 & -1\\ 3 & 0 & 2\end{vmatrix}$ $= \underset{\sim}{i}\begin{vmatrix}1 & -1\\ 0 & 2\end{vmatrix} - \underset{\sim}{j}\begin{vmatrix}2 & -1\\ 3 & 2\end{vmatrix} + \underset{\sim}{k}\begin{vmatrix}2 & 1\\ 3 & 0\end{vmatrix}$ $= 2\underset{\sim}{i}-7\underset{\sim}{j}-3\underset{\sim}{k}$
5 Substitute the vector product back into the area formula.	Therefore $A = \frac{1}{2}\lvert 2\underset{\sim}{i}-7\underset{\sim}{j}-3\underset{\sim}{k}\rvert$
6 Calculate the magnitude of the vector and simplify to find the area.	$= \frac{1}{2}\sqrt{4+49+9}$ $= \frac{1}{2}\sqrt{62}$ $= 3.9$ square units.

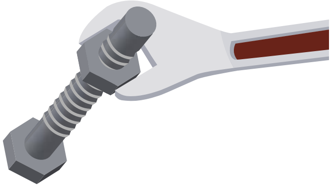
Torque

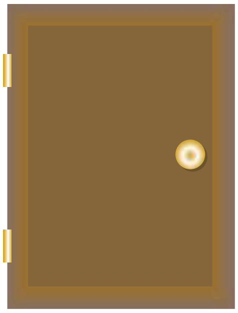


If you want to turn a nut on a thread and it is hard to move, do you use a small spanner or a large spanner?

Your intuition tells you, correctly, that the large spanner will turn the nut more easily. The turning effect of a force depends on the distance the force is from the point of application.







This principle is observed in many other applications such as:

1. Door handles are usually placed as far from the hinge as practicable.

2. When riding uphill on a geared bicycle a large cog is selected.

This turning force is called *torque*, and its magnitude is given by:

$$\text{Torque} = |\underset{\sim}{r} \cdot \underset{\sim}{F}|$$

where $\underset{\sim}{F}$ is the force and $\underset{\sim}{r}$ is the vector from the point of application of the force to the point, or axis, about which the turning is considered.

Note that, as this equation suggests, the size of the torque depends not only on the force and the distance the force is from the point of application, but also on the angle at which the force is applied. The torque is a maximum when the angle between $\underset{\sim}{F}$ and $\underset{\sim}{r}$ is 90°. If $\underset{\sim}{F}$ is measured in Newtons (N) and $\underset{\sim}{r}$ is measured in metres (m) then torque is measured in Newton metres (Nm).

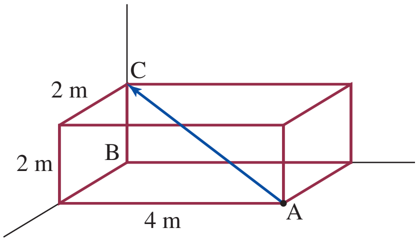
WORKED Example 10

A force, $\underset{\sim}{F}$, acts on an object at a point, P. Calculate the torque about a point O if $\underset{\sim}{F} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$ and the vector joining O to P is $2\underset{\sim}{i} + 5\underset{\sim}{k}$.

THINK	WRITE
① Write the formula for torque.	Torque $= \lvert \underset{\sim}{r} \cdot \underset{\sim}{F} \rvert$
② Substitute for $\underset{\sim}{r}$ and $\underset{\sim}{F}$.	Torque $= \lvert (2\underset{\sim}{i} + 5\underset{\sim}{k}) \cdot (2\underset{\sim}{i} - 3\underset{\sim}{j}) \rvert$
③ Use the determinant method to calculate the vector product.	$(2\underset{\sim}{i} + 5\underset{\sim}{k}) \cdot (2\underset{\sim}{i} - 3\underset{\sim}{j})$ $= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 0 & 5 \\ 2 & -3 & 0 \end{vmatrix}$ $= \underset{\sim}{i}\begin{vmatrix} 0 & 5 \\ -3 & 0 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 5 \\ 2 & 0 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & 0 \\ 2 & -3 \end{vmatrix}$ $= 15\underset{\sim}{i} + 10\underset{\sim}{j} - 6\underset{\sim}{k}.$
④ Substitute the simplified vector product into the relationship for torque.	So torque $= \lvert 15\underset{\sim}{i} + 10\underset{\sim}{j} - 6\underset{\sim}{k} \rvert$
⑤ Find the magnitude of the vector product.	$= \sqrt{15^2 + 10^2 + (-6)^2}$ $= \sqrt{361}$ $= 19$
⑥ Write the answer.	The torque is 19 Nm.

WORKED Example 11

A force of 40 N is applied to a rectangular crate at a point, A, as shown. Calculate the torque of the force about B.



THINK	WRITE
① First find the vector $\overrightarrow{BA}$. This will give us $\underset{\sim}{r}$ in the torque formula.	Take B as the origin with the *x*-, *y*- and *z*-axes taken as usual. $\overrightarrow{BA} = 4\underset{\sim}{i} + 2\underset{\sim}{k}$
② To find the vector of the force, first find the vector $\overrightarrow{AC}$.	$\overrightarrow{AC} = -4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$
③ Calculate the magnitude of this vector.	$\lvert \overrightarrow{AC} \rvert = \sqrt{(-4)^2 + 2^2 + (-2)^2}$ $= \sqrt{24}$ The magnitude of this vector is $\sqrt{24}$.

THINK	**WRITE**
④ Use the vector $\overrightarrow{\text{AC}}$ to express the force in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.	So $\underset{\sim}{F} = 40 \cdot \frac{1}{\sqrt{24}}(-4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$.
⑤ Use the formula for torque.	Torque $= \lvert \underset{\sim}{r} \cdot \underset{\sim}{F} \rvert$ $= \lvert \overrightarrow{\text{BA}} \cdot \underset{\sim}{F} \rvert$
⑥ Substitute for $\underset{\sim}{r}$ (or $\overrightarrow{\text{BA}}$ in this case) and $\underset{\sim}{F}$.	$= \lvert (4\underset{\sim}{i} + 2\underset{\sim}{k}) \cdot 40 \cdot \frac{1}{\sqrt{24}}(-4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) \rvert$ $= \frac{40}{\sqrt{24}} \lvert (4\underset{\sim}{i} + 2\underset{\sim}{k}) \cdot (-4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) \rvert$
⑦ Calculate the vector product.	$= \frac{40}{\sqrt{24}} \lvert -4\underset{\sim}{i} + 8\underset{\sim}{k} \rvert$
⑧ Evaluate the magnitude of the vector.	$= \frac{40}{\sqrt{24}} \cdot \sqrt{(-4)^2 + 8^2}$ $= \frac{40}{\sqrt{24}} \cdot \sqrt{80}$ $= 73$
⑨ Write the answer.	Therefore, the torque is 73 Nm.

remember

1. The *vector product* of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert \underset{\sim}{a} \rvert \lvert \underset{\sim}{b} \rvert \sin \theta \hat{\underset{\sim}{n}}$$

where θ is the angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and $\hat{\underset{\sim}{n}}$ is a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the right-hand sense.

2. If $\underset{\sim}{a} = p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$ and $\underset{\sim}{b} = s\underset{\sim}{i} + t\underset{\sim}{j} + u\underset{\sim}{k}$

then $$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ p & q & r \\ s & t & u \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} q & r \\ t & u \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} p & r \\ s & u \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} p & q \\ s & t \end{vmatrix}.$$

3. The area of a triangle formed by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is:

$$\text{Area} = \tfrac{1}{2} \lvert \underset{\sim}{a} \cdot \underset{\sim}{b} \rvert$$

4. The turning effect of a force is called *torque* and is calculated by:

$$\text{Torque} = \lvert \underset{\sim}{r} \cdot \underset{\sim}{F} \rvert$$

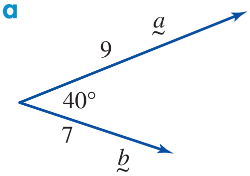
where $\underset{\sim}{F}$ is the force and $\underset{\sim}{r}$ is the vector from the point of application of the force to the point about which the turning is considered.

EXERCISE 3B Vector product

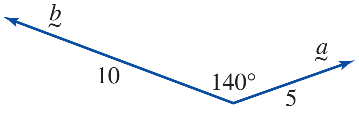
WORKED Example 6

1 Vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ lie in a horizontal plane as shown. Calculate **i** $\underset{\sim}{a} \cdot \underset{\sim}{b}$ and **ii** $\underset{\sim}{b} \cdot \underset{\sim}{a}$ for each of the following.

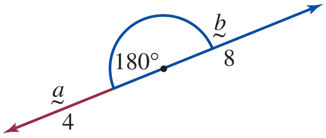
a



b



c



WORKED Example 7

2 Use the Distributive Law of vectors to calculate the following vector products.

a $\underset{\sim}{i} \cdot (2\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$ **b** $\underset{\sim}{j} \cdot (-2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k})$ **c** $(\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot (2\underset{\sim}{i} - 3\underset{\sim}{k})$

WORKED Example 8

3 Calculate the following vector products by the evaluation of determinants.

a $(2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}) \cdot (-3\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$

b $(-2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}) \cdot (-3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$

c $(3\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$

WORKED Example 9

4 Calculate the area of triangle OAB if the coordinates of A and B are:

a (2, 1, 2) and (0, 3, 0)

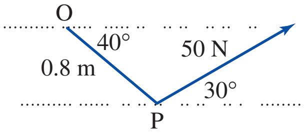
b (–2, 3, –1) and (2, 0, 2).

WORKED Example 10

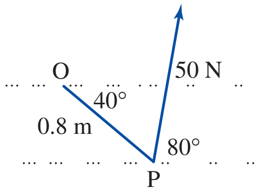
5 A force, $\underset{\sim}{F}$, acts on an object at a point, P. Calculate the torque about O if $\underset{\sim}{F} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and the vector joining O to P is $2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$.

6 Calculate the torque of a force, $\underset{\sim}{F}$, which acts at P, about a point, O, as shown.

a



b



7 In the table on page 104, the results of the vector products of the unit vectors, $\underset{\sim}{i} \cdot \underset{\sim}{j}$, $\underset{\sim}{k} \cdot \underset{\sim}{k}$, and so on, were listed. Use this table to help calculate the following vector products.

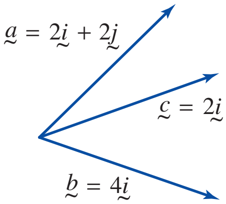
a $\underset{\sim}{i} \cdot (\underset{\sim}{i} \cdot \underset{\sim}{j})$ **b** $\underset{\sim}{j} \cdot (\underset{\sim}{i} \cdot \underset{\sim}{j})$

c $\underset{\sim}{i} \cdot (\underset{\sim}{j} \cdot \underset{\sim}{k})$ **d** $(\underset{\sim}{i} \cdot \underset{\sim}{j}) \cdot (\underset{\sim}{j} \cdot \underset{\sim}{k})$

8 Find a unit vector that is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ if:

a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{j} + \underset{\sim}{k}$ **b** $\underset{\sim}{a} = 2\underset{\sim}{i}$ and $\underset{\sim}{b} = \underset{\sim}{j} + \underset{\sim}{k}$.

9 Consider the three vectors, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, shown at right.



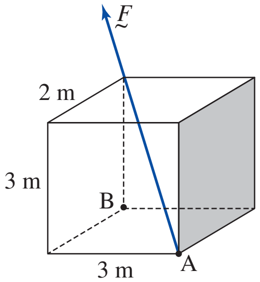
a Calculate the vector product $\underset{\sim}{a} \cdot (\underset{\sim}{b} \cdot \underset{\sim}{c})$.

b Show that the result of the vector product $\underset{\sim}{a} \cdot (\underset{\sim}{b} \cdot \underset{\sim}{c})$ lies in the plane of $\underset{\sim}{b}$ and $\underset{\sim}{c}$; that is, the result can be written in the form $n\underset{\sim}{b} + m\underset{\sim}{c}$ for some numbers n and m.

c By considering the orientation of $\underset{\sim}{b} \cdot \underset{\sim}{c}$ and $\underset{\sim}{a} \cdot (\underset{\sim}{b} \cdot \underset{\sim}{c})$, show that, in general, the vector $\underset{\sim}{a} \cdot (\underset{\sim}{b} \cdot \underset{\sim}{c})$ lies in the plane of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

WORKED Example 11

10 A force of 100 N is applied to a rectangular crate at a point, A, as shown. Calculate the torque of the force about B.



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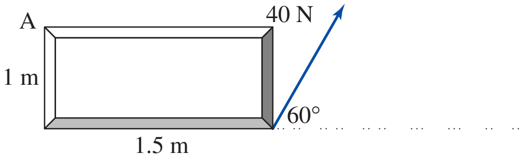
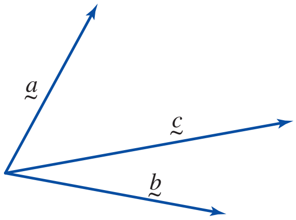
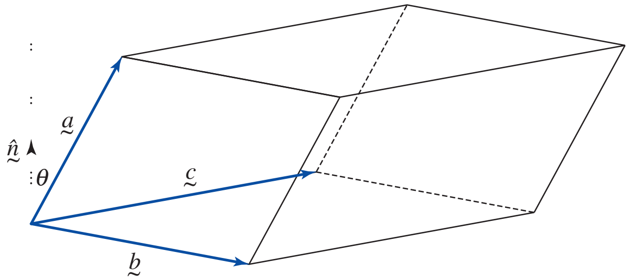
11 A force of 40 N acts on a picture frame as shown at left. What is the torque of this force about A?



Scalar triple product

Any three vectors that do not all lie in the same plane can form a *parallelepiped*. Note that this figure need not have any angles measuring 90°.



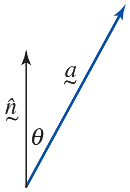


The volume of this figure can be found by multiplying the area of the base by the perpendicular height.

Recall that the area of a triangle formed by vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$, say, is $\frac{1}{2}|\underset{\sim}{p} \cdot \; \underset{\sim}{q}|$.

Thus, the area of the base is $2 \cdot \; \frac{1}{2}|\underset{\sim}{b} \cdot \; \underset{\sim}{c}| = |\underset{\sim}{b} \cdot \; \underset{\sim}{c}|$.

The perpendicular height of the parallelepiped is $|\underset{\sim}{a}| \cos \theta$.



The value $|\underset{\sim}{a}| \cos \theta$ can be obtained by calculating $\underset{\sim}{a} \bullet \hat{\underset{\sim}{n}}$.

Putting the two calculations together, we find that the volume, V, of the parallelepiped is:

$$V = |\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \; \underset{\sim}{c}|$$

This leads to a quantity called the *scalar triple product* of three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ which is defined as $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \; \underset{\sim}{c}$.

Notes

1. The result is a scalar.
2. $\underset{\sim}{b} \cdot \; \underset{\sim}{c}$ is a vector. This term must be calculated first. As $\underset{\sim}{a} \bullet \underset{\sim}{b}$ is a scalar it cannot form part of a vector product.

The scalar triple product of three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ can be evaluated using the determinant of a 3 · 3 matrix whose rows are the components of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

WORKED Example 12

Calculate $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c}$ if $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$.

THINK

WRITE/DISPLAY

① Use the determinant formula for the scalar triple product.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

② Substitute for $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

$$= \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

③ Evaluate the determinant.

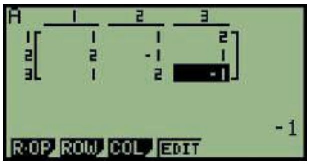
$$= 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$
$$= -1 + 3 + 10$$
$$= 12$$

Alternatively, a graphics calculator can be used.

For the CASIO *fx*-9860G AU

① To calculate the scalar triple product using matrices, press:
- MENU
- 1 (RUN)
- F1 (MAT)

For Mat A, enter 3 · 3 and substitute the values for vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ as shown.



```
A     1    2    3
1[    1    1    2]
2     2   -1    1
3[    1    2   -1]
                    -1
```

② Then press:
- EXIT
- EXIT
- OPTN
- F2 (MAT)
- F3 (Det)
- F1 (mat)

Complete the entry line as:

Det Mat A

then press EXE.

```
Det Mat A
                    12
```

THINK

WRITE/DISPLAY

For the TI-Nspire CAS

1. To calculate the scalar triple product using matrices on a Calculator page, press:
   - MENU (menu)
   - 1: Actions (1)
   - 1: Define (1)

   Complete the entry line as:

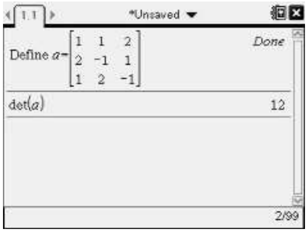
$$a = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

   then press ENTER (enter).

```
Define a=[1  1  2]                Done
         [2 -1  1]
         [1  2 -1]
|
                                  1/99
```

2. To find the determinant of a, press:
   - MENU (menu)
   - 7: Matrix & Vector (7)
   - 3: Determinant (3)

   Complete the entry line as:
   det(a)
   then press ENTER (enter).



```
Define a=[1  1  2]                Done
         [2 -1  1]
         [1  2 -1]
det(a)                              12
                                  2/99
```

WORKED Example 13

Find the volume of the parallelepiped formed by the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, where $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{j}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$.

THINK

WRITE/DISPLAY

1. State the formula for finding the volume of a parallelepiped.

Volume = $|\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \ \underset{\sim}{c}|$

2. Calculate the scalar triple product by using the determinant method. Substitute the components for $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

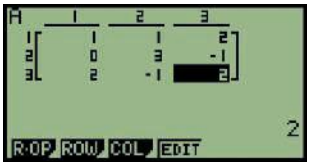
$$\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \ \underset{\sim}{c} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & -1 & 2 \end{vmatrix}$$

3. Evaluate the determinant using a graphics calculator.

For the CASIO *fx*-9860G AU

4. To calculate the scalar triple product using matrices, press:
   - MENU
   - 1 (RUN)
   - F1 (MAT)

   For Mat A, enter 3 · 3 and substitute the values for vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ as shown.



```
A     1     2     3
1[    1     1     2]
2[    0     3    -1]
3[    2    -1     2]
                  2
```

Continued over page

THINK | **WRITE/DISPLAY**

⑤ Then press:
- EXIT
- EXIT
- OPTN
- F2 (MAT)
- F3 (Det)
- F1 (Mat)

Complete the entry line as:

Det Mat A

then press EXE.

Det Mat A

-9

⑥ State the value obtained for the scalar triple product.

$\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c} = -9$

⑦ Use this value to calculate the required volume.

$$\begin{aligned}\text{Volume} &= |\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c}| \\ &= |-9| \\ &= 9 \text{ cubic units}\end{aligned}$$

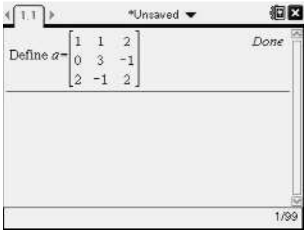
For the TI-Nspire CAS

④ To calculate the scalar triple product using matrices on a Calculator page, press:
- MENU (menu)
- 1: Actions (1)
- 1: Define (1)

Complete the entry line as:

$$a = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

then press ENTER (enter).



Define $a = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ Done

⑤ To find the determinant of *a*, press:
- MENU (menu)
- 7: Matrix & Vector (7)
- 3: Determinant (3)

Complete the entry line as:

det(*a*)

then press ENTER (enter).

Define $a = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ Done

det(*a*) -9

⑥ State the value obtained for the scalar triple product.

$\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c} = -9$

⑦ Use this value to calculate the required volume.

$$\begin{aligned}\text{Volume} &= |\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c}| \\ &= |-9| \\ &= 9 \text{ cubic units}\end{aligned}$$

remember

1. The scalar triple product of three vectors is $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c}$.
2. The scalar triple product of three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ can be evaluated using the determinant of a 3 · 3 matrix whose rows are the components of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.
$$\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
3. The volume of a parallelepiped formed by vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ is given by the formula:
$$\text{Volume} = |\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c}|.$$

EXERCISE 3C Scalar triple product

WORKED Example 12

1 Calculate $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c}$ for each of the following:

a $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$

b $\underset{\sim}{a} = \underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

c $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

d $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

e $\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

f $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{c} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$

g $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

WORKED Example 13

2 Find the volume of the parallelepiped formed by the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, where:

a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{j}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

b $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + 3\underset{\sim}{j}$

c $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$

d $\underset{\sim}{a} = -2\underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$.

3 If $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c} = 0$, calculate the value of n if:

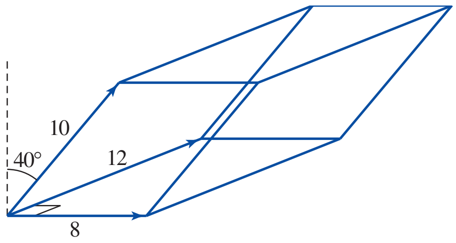
a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{k}$ and $\underset{\sim}{c} = n\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$

b $\underset{\sim}{a} = \underset{\sim}{i} + n\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

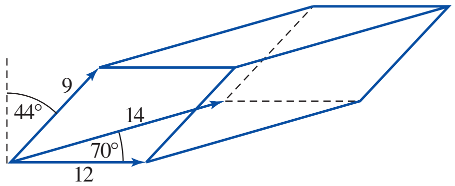
c $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + n\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - 4\underset{\sim}{k}$.

4 Calculate the volume of the parallelepiped shown in each diagram.

a



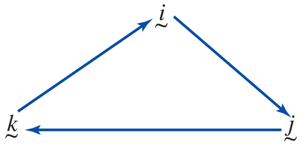
b



5 Calculate each of the following scalar triple products.

a $\underset{\sim}{i} \cdot \underset{\sim}{j} \cdot \underset{\sim}{k}$
b $\underset{\sim}{j} \cdot \underset{\sim}{j} \cdot \underset{\sim}{k}$
c $\underset{\sim}{j} \cdot \underset{\sim}{k} \cdot \underset{\sim}{i}$
d $\underset{\sim}{k} \cdot \underset{\sim}{j} \cdot \underset{\sim}{k}$
e $\underset{\sim}{i} \cdot \underset{\sim}{k} \cdot \underset{\sim}{j}$
f $\underset{\sim}{k} \cdot \underset{\sim}{i} \cdot \underset{\sim}{j}$

6 By considering the products in question **5**, develop an explanation to predict the outcome of the scalar product of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ without actually calculating the result.
Hint: The diagram at right might help.



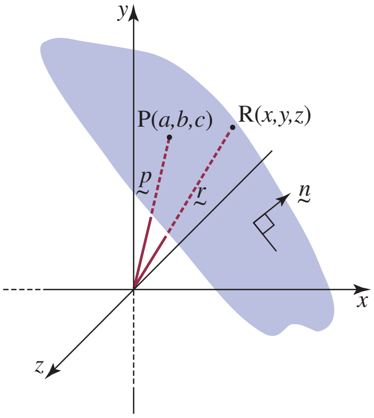
7 If the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are coplanar then $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \underset{\sim}{c} = 0$.
Explain this result by referring to:

a the volume of a parallelepiped
b angles between vectors.

Applications of the vector product

Planes in three dimensions

In two dimensions, the equation of a line in the x–y plane is given by the equation $ax + by = c$, where $-\frac{a}{b}$ is the gradient and $\frac{c}{b}$ is the y-intercept. If we extend this equation to three dimensions to get $ax + by + cz = d$, the result is the equation of the plane; that is, a ‘flat’ 2-dimensional surface extending indefinitely. In this next section we will see how the variables a, b, c and d relate to the type of plane obtained.



If a point, P, in the plane is known, and the vector, $\underset{\sim}{n}$, perpendicular to the plane is also known, then we can find the equation to the plane. Suppose that the coordinates of P are (a, b, c) and the components of $\underset{\sim}{n}$ are: $\underset{\sim}{n} = n_1\underset{\sim}{i} + n_2\underset{\sim}{j} + n_3\underset{\sim}{k}$.

Let R be any point in the plane with coordinates (x, y, z). The vectors $\underset{\sim}{p}$ and $\underset{\sim}{r}$ are shown in the diagram.

$$\begin{aligned} \overrightarrow{PR} &= \underset{\sim}{r} - \underset{\sim}{p} \\ &= (x-a)\underset{\sim}{i} + (y-b)\underset{\sim}{j} + (z-c)\underset{\sim}{k} \end{aligned}$$

As $\underset{\sim}{n}$ is perpendicular to $\overrightarrow{PR}$,

$$\underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{p}) = 0.$$

Writing the dot product in component form we have:

$$n_1(x-a) + n_2(y-b) + n_3(z-c) = 0.$$

This is the equation of the plane passing through (a, b, c) and perpendicular to $\underset{\sim}{n} = n_1\underset{\sim}{i} + n_2\underset{\sim}{j} + n_3\underset{\sim}{k}$.

WORKED Example 14

Find the equation of the plane containing the points P(1, 2, 1), Q(–1, 1, 0) and R(1, 0, 2).

THINK	**WRITE**
① Find the vector $\overrightarrow{PQ}$.	$\overrightarrow{PQ} = (-1-1)\underset{\sim}{i} + (1-2)\underset{\sim}{j} + (0-1)\underset{\sim}{k} = -2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$
② Find the vector $\overrightarrow{QR}$.	$\overrightarrow{QR} = (1 - -1)\underset{\sim}{i} + (0-1)\underset{\sim}{j} + (2-0)\underset{\sim}{k} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

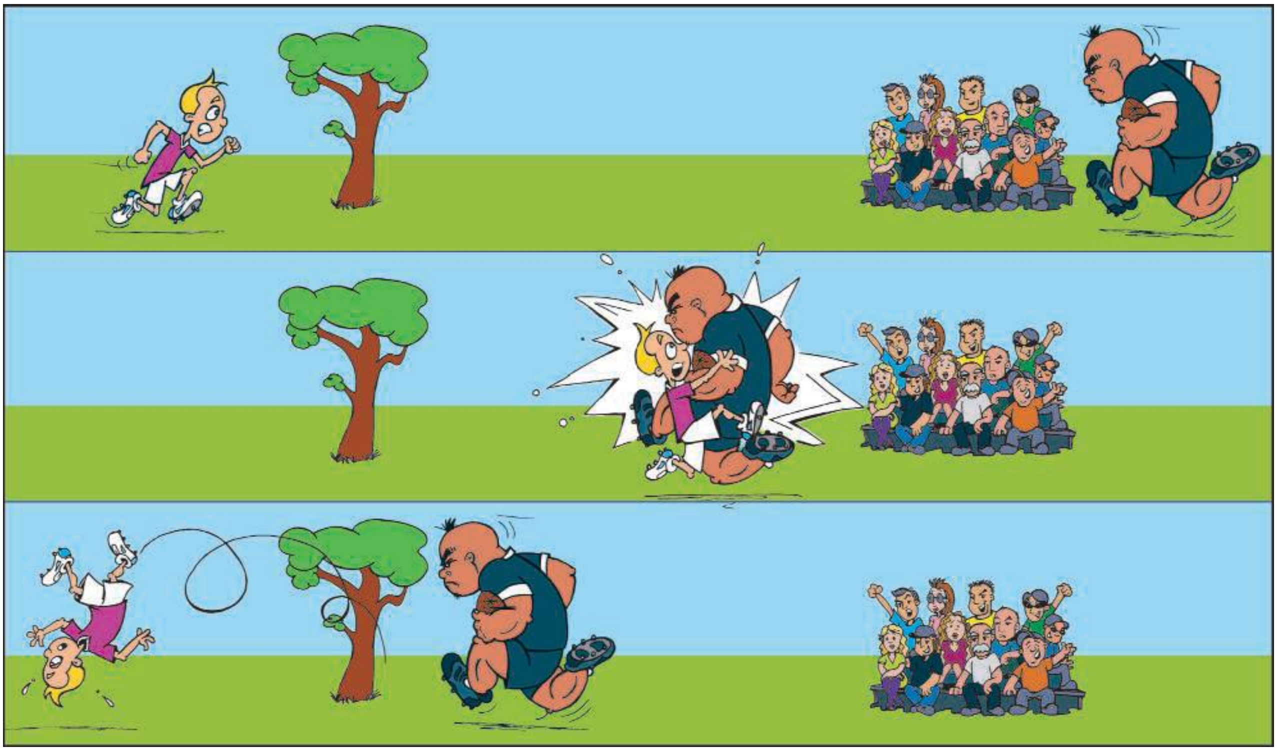
THINK	**WRITE**
③ Use the vector product of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ to find a vector, $\underset{\sim}{n}$, perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.	$\underset{\sim}{n} = \overrightarrow{PQ} \cdot \overrightarrow{QR}$ $= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & -1 & -1 \\ 2 & -1 & 2 \end{vmatrix}$ $= \underset{\sim}{i}\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & -1 \\ 2 & -1 \end{vmatrix}$ $= -3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$
④ Substitute the components of $\underset{\sim}{n}$ and the coordinates of the point (1, 2, 1) into the general relationship for the equation of a plane.	$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$ $-3(x - 1) + 2(y - 2) + 4(z - 1) = 0$ $-3x + 2y + 4z = 5$

Angular momentum

In year 11, we considered *momentum*, a property that is associated with a moving object. The momentum, $\underset{\sim}{p}$, of an object is the product of its mass and velocity and as such is a vector quantity.

$$\underset{\sim}{p} = m\underset{\sim}{v}$$

This quantity is useful in analysing the behaviour of systems of particles. In collisions between objects, for example, momentum is conserved. That is, the momentum of the system before the collision is equal to the momentum after the collision.



How do we measure the momentum of a spinning top?

Clearly it has momentum, but the formula $\underset{\sim}{p} = m\underset{\sim}{v}$ is not useful because the top has no velocity — it is not moving with linear motion. We need to consider the rotational motion or *angular momentum* of the top. The angular momentum, L, of an object of mass, m, moving with velocity, $\underset{\sim}{v}$, about a point O is:

$$\underset{\sim}{L} = \underset{\sim}{r} \cdot \ m\underset{\sim}{v},$$

where $\underset{\sim}{r}$ is the position vector of the object from O.

Because we are interested in only the general properties of angular momentum, no specific units will be used.

WORKED Example 15

An object of mass 1.5 kg is moving with a velocity $\underset{\sim}{v} = 2i - 2j$. Calculate the angular momentum of this object about O when its position is given by $r = i + 2j$.

THINK

1. Use the formula for angular momentum.
2. Substitute values into the formula.
3. Calculate the vector product.
4. Write the answer.

WRITE

$$\begin{aligned}\underset{\sim}{L} &= \underset{\sim}{r} \cdot \ m\underset{\sim}{v} \\ &= (\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot \ 1.5(2\underset{\sim}{i} - 2\underset{\sim}{j}) \\ &= (\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot \ (3\underset{\sim}{i} - 3\underset{\sim}{j}) \\ &= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & 0 \\ 3 & -3 & 0 \end{vmatrix} \\ &= \underset{\sim}{i}\begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} \\ &= 0 - 0 - 9\underset{\sim}{k} \\ &= -9\underset{\sim}{k}\end{aligned}$$

The angular momentum is $-9\underset{\sim}{k}$.

remember

1. If the vector $\underset{\sim}{n}$ is perpendicular to a plane and $\underset{\sim}{p}$ is the position vector of a point in the plane, then any point in the plane with position vector $\underset{\sim}{r}$ will satisfy the equation:
$$\underset{\sim}{n} \cdot (\underset{\sim}{r} - \underset{\sim}{p}) = 0.$$
2. In component form:
$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$
is the equation of the plane passing through (a, b, c) and perpendicular to $\underset{\sim}{n} = n_1 i + n_2 j + n_3 k$.
3. The angular momentum, $\underset{\sim}{L}$, of an object of mass, m, moving with velocity, $\underset{\sim}{v}$, about a point, O, is:
$$\underset{\sim}{L} = \underset{\sim}{r} \cdot \ m\underset{\sim}{v},$$
where $\underset{\sim}{r}$ is the position vector of the object from O.

EXERCISE 3D Applications of the vector product

WORKED Example 14

1 Find the equation of the plane containing each of the following sets of points.

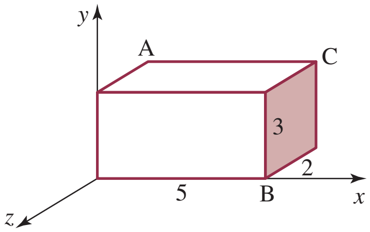
- **a** P(2, 1, 1), Q(1, 0, 1) and R(–3, 2, 0)
- **b** P(1, 0, 0), Q(0, 1, 0) and R(0, 0, 1)
- **c** P(3, 1, 1), Q(1, 1, 1) and R(–3, 2, 1)
- **d** P(1, –1, 1), Q(4, –2, 0) and R(–3, 1, 3)
- **e** P(2, 0, 1), Q(3, 3, 1) and R(3, 2, 2)
- **f** P(3, 2, 1), Q(2, 2, 2) and R(–1, 3, 1)

2 For each of the following, find the equation of the plane:

- **a** perpendicular to $\underset{\sim}{n} = \underset{\sim}{i} + 2\underset{\sim}{k}$ that contains the point (2, 1, 1)
- **b** perpendicular to $\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ that contains the point (0, 1, 2)
- **c** perpendicular to $\underset{\sim}{n} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ that contains the point (–2, 1, 0)
- **d** perpendicular to $\underset{\sim}{n} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ that contains the point (1, 3, 1).

3 Find the equation of the plane perpendicular to $\underset{\sim}{n} = 2\underset{\sim}{j} - \underset{\sim}{k}$ passing through the origin.

4 Examine the figure below.



- **a** Express vector $\overrightarrow{AB}$ in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.
- **b** Express vector $\overrightarrow{AC}$ in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.
- **c** Find a vector perpendicular to the plane containing the points A, B and C.
- **d** Find the equation of the plane passing through the points A, B and C.

5 Find the equation of the plane passing through the point (2, 1, 3) and which is perpendicular to the y-axis.

6 **a** Calculate the coordinates of points P and Q which are the x- and y-intercepts of the line with equation $2x + 3y = 6$.

- **b** Find the equation of the plane that passes through the point with coordinates (0, 0, 1) and contains the line $2x + 3y = 6$.

7 A plane has equation $3x + 4y + 12z = 6$. Find a unit vector perpendicular to this plane.

8 A plane has equation $z = 3x + 2y$. Find a unit vector perpendicular to this plane.

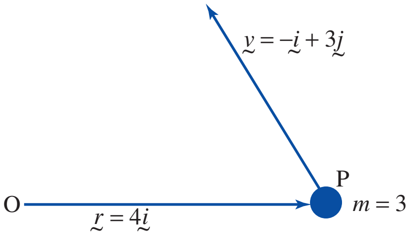
WORKED Example 15

9 An object of mass 2 kg is moving with a velocity $\underset{\sim}{v} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$. Calculate the angular momentum of this object about O when its position is given by $\underset{\sim}{r} = 4\underset{\sim}{i} + 2\underset{\sim}{j}$.

10 Examine the figure below.

a Calculate the momentum of the object at P.

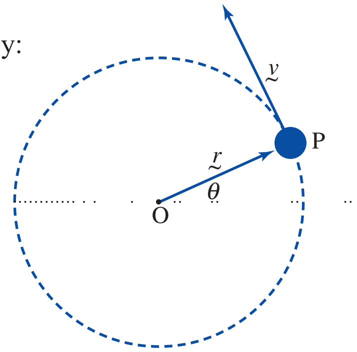
b Calculate the angular momentum of the object at P moving about O.



11 For an object of mass m, moving in a circle with speed v, the position, $\underset{\sim}{r}$, and the velocity, $\underset{\sim}{v}$, are given by:

$$\underset{\sim}{r} = r\cos\theta\underset{\sim}{i} + r\sin\theta\underset{\sim}{j}$$

$$\underset{\sim}{v} = -v\sin\theta\underset{\sim}{i} + v\cos\theta\underset{\sim}{j}.$$



a Show that $\underset{\sim}{v}$ and $\underset{\sim}{r}$ are perpendicular.

b Show that, in this case, the magnitude of the angular momentum is rmv.

The conservation of angular momentum means that if a change is made in r (or v) then a corresponding change will occur in v (or r) to keep the angular momentum constant.

c If r is doubled what happens to v?

d If r is halved, what happens to v?

e To create a rapid spin, the ballerina begins with arms outstretched. As she pulls her arms to her side her rate of spin increases. Use the conservation of angular momentum to explain this phenomenon.

12 A plane is perpendicular to the vector $\underset{\sim}{n}$ and contains the point P.

a Show that the shortest distance from the plane to the origin is given by $\dfrac{\underset{\sim}{n} \cdot \overrightarrow{\mathrm{OP}}}{|\underset{\sim}{n}|}$.

b Calculate the shortest distance from the plane $2x + 3y - z = 12$ to the origin.

13 a When two non-parallel planes intersect, a straight line is formed. Show that the direction of the straight line formed by the intersection of two planes is given by $n_1 \cdot \ n_2$ where n_1 and n_2 are the perpendiculars to the two planes.

b What is the direction of the line formed by the intersection of the planes $2x - 3y + z = 0$ and $3x = y + z$?

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summary

Review of vectors

- A vector is a quantity that has magnitude and direction.
- Vectors can be added, subtracted or multiplied by a scalar.
- Vectors can be represented using unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$.
- The magnitude of a vector $v = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is $|\underset{\sim}{v}| = \sqrt{x^2 + y^2 + z^2}$.
- For two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, the dot product is a scalar:

  $$\underset{\sim}{a} \bullet \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos \theta.$$

  If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ then $\underset{\sim}{a} \bullet \underset{\sim}{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Vector product

- The vector product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined as:

  $$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \sin \theta\hat{\underset{\sim}{n}}$$

  where θ is the angle between vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and $\hat{\underset{\sim}{n}}$ is a unit vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the right-hand sense.
- In component form:

  If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$ then $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- The area, A, of a triangle formed by vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given by:

  $$A = \tfrac{1}{2}|\underset{\sim}{a} \cdot \underset{\sim}{b}|.$$
- Torque $= |\underset{\sim}{r} \cdot \underset{\sim}{F}|$ where $\underset{\sim}{F}$ is the force and $\underset{\sim}{r}$ is the vector from the point of application of the force to the point about which the turning is considered.

Scalar triple product

- The scalar triple product of three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ is $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c}$.

  Notes

  1. The result is a scalar.
  2. As $\underset{\sim}{b} \cdot \underset{\sim}{c}$ is a vector, this term must be calculated first. Also, $\underset{\sim}{a} \bullet \underset{\sim}{b}$ is a scalar so it cannot form part of a vector product.
- The scalar triple product of three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ can be evaluated using the determinant of a 3 · 3 matrix whose rows are the components of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

  $$\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \underset{\sim}{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Applications of the vector product

- The equation of the plane passing through (a, b, c) and perpendicular to $\underset{\sim}{n} = n_1\underset{\sim}{i} + n_2\underset{\sim}{j} + n_3\underset{\sim}{k}$ is:

  $$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0.$$
- The angular momentum of a moving object about a point, O, is:

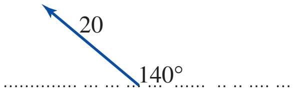
  $$\underset{\sim}{L} = \underset{\sim}{r} \cdot m\underset{\sim}{v},$$

  where m is the mass of the object, $\underset{\sim}{v}$ is its velocity and $\underset{\sim}{r}$ is the position vector of the object from O.

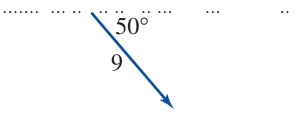
CHAPTER review

1 Write the following vectors in $\underset{\sim}{i}$ and $\underset{\sim}{j}$ form. **3A**

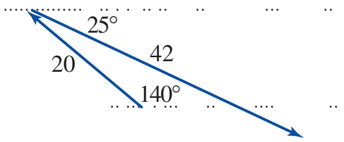
a



b



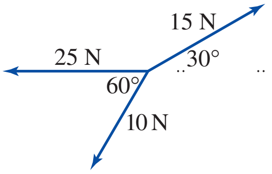
2 Find the resultant magnitude and direction of the vector sum at right. **3A**



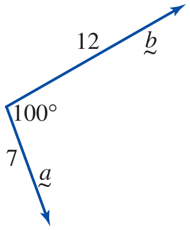
3 Calculate $\underset{\sim}{a} \cdot \underset{\sim}{b}$ if $\underset{\sim}{a} = \underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$. **3A**

4 Calculate the value of n if the vectors $3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ and $n\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ are perpendicular. **3A**

5 Three forces act on an object as shown. Calculate the magnitude and direction of the resulting force. **3A**



6 Calculate the vector product, $\underset{\sim}{a} \cdot \underset{\sim}{b}$, of the following pair of vectors. **3B**



7 Calculate the following vector products by the evaluation of determinants. **3B**

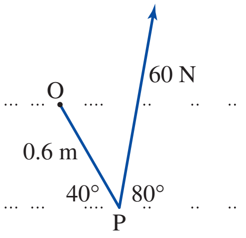
a $(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 4\underset{\sim}{j} - 3\underset{\sim}{k})$

b $(2\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}) \cdot (3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$

8 Calculate the area of triangle ACB if the coordinates of the points are (2, 1, 0), (3, 2, 1) and (4, –2, 2) respectively. **3B**

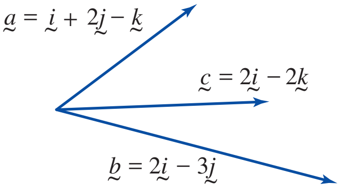
9 A force, $\underset{\sim}{F}$, acts on an object at a point, P. Calculate the torque about O if $\underset{\sim}{F} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and the vector joining O to P is $2\underset{\sim}{i} + 2\underset{\sim}{j}$. **3B**

10 Calculate the torque of a force, $\underset{\sim}{F}$, which acts at P about a point, O, as shown. **3B**



3B **11** Find a unit vector that is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ if $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

3B **12** For the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, show that $\underset{\sim}{a} \cdot \ (\underset{\sim}{b} \cdot \ \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{c}\, \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b}\, \underset{\sim}{c}$. (Note that $\underset{\sim}{a} \cdot \underset{\sim}{c}\, \underset{\sim}{b}$ represents the vector $\underset{\sim}{b}$ multiplied by the scalar $\underset{\sim}{a} \cdot \underset{\sim}{c}$.)



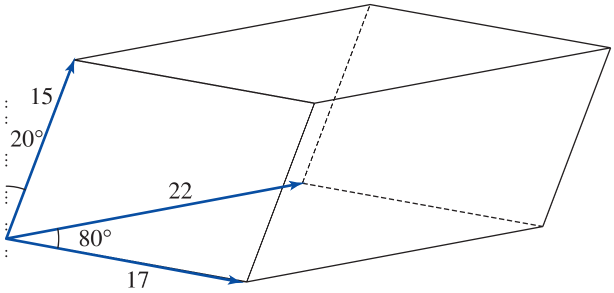
3C **13** Calculate $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \ \underset{\sim}{c}$ for the following:

a $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = \underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$

b $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

3C **14** If $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} + n\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \ \underset{\sim}{c} = 0$, calculate the value of n.

3C **15** Calculate the volume of the parallelepiped shown in the diagram.



3C **16** If $\underset{\sim}{a} \cdot \underset{\sim}{b} \cdot \ \underset{\sim}{c} = 0$, what can be said about the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$?

3D **17** The coordinates of P, Q and R are (2, 0, 2), (−3, −1, 3) and (2, 4, 0) respectively. Find the equation of the plane containing these three points.

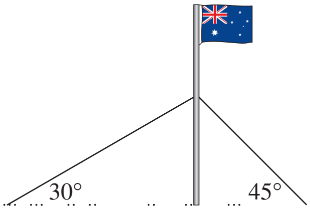
3D **18** Find the equation of the plane perpendicular to $\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ that contains the point (0, 3, 2).

3D **19** A plane has equation $x - y + 2z = 6$. Find a unit vector perpendicular to this plane.

Modelling and problem solving

1 Two supporting wires keep a flagpole in its vertical position.

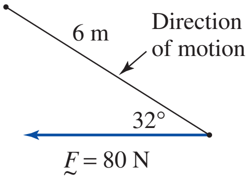
The horizontal components of the tensions in the supporting wires must be equal and opposite, otherwise the pole will move laterally. The vertical components may be different.



a Calculate the ratio of the tension in the shorter wire to the tension in the longer wire.

b If the tension in the shorter wire is 600 N, calculate the magnitude of the tension in the longer wire.

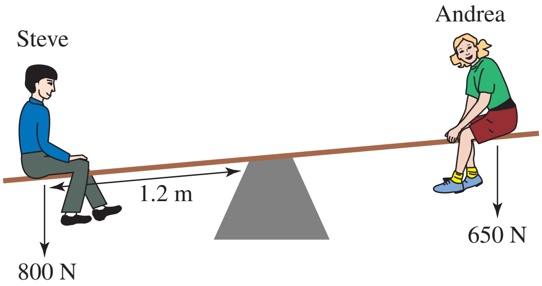
2 An ice skater exerts a force on his partner as shown. Calculate the work done (increase in energy) by the skater in moving his partner.



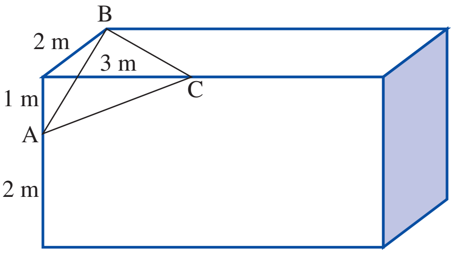
3 Steve and Andrea are sitting on a seesaw. Steve weighs 800 N and Andrea weighs 650 N.

a If Steve is sitting 1.2 m from the centre, calculate the torque that Steve's weight exerts about the centre point.

b Where should Andrea sit, on the opposite side, if she is to provide a torque equal (and opposite to) Steve's torque?

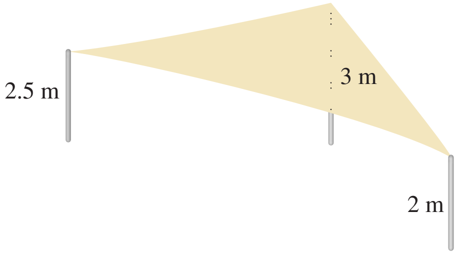


4



A corner is to be cut from a right prism as shown above. Calculate the area of the face, ABC. *Hint:* Find expressions for the vectors AB and AC.

5 A triangular shade cloth is to be suspended by three poles of different height as shown. The feet of the poles are positioned 10 metres from each other. The heights of each pole are shown in the diagram. Calculate the area of shade cloth needed.



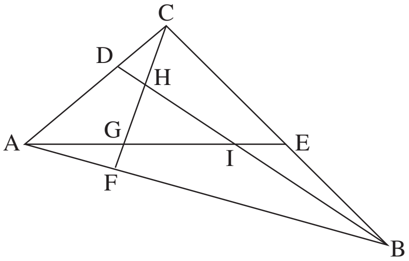
6 **a** If $\underset{\sim}{a} \cdot \ \underset{\sim}{b} = \underset{\sim}{p}$, give the value of $\underset{\sim}{b} \cdot \ \underset{\sim}{a}$.

b If $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \ \underset{\sim}{c} = 4$, give the value of $\underset{\sim}{a} \bullet \underset{\sim}{c} \cdot \ \underset{\sim}{b}$.

c Describe a general relationship between $\underset{\sim}{a} \bullet \underset{\sim}{b} \cdot \ \underset{\sim}{c}$ and its permutations $\underset{\sim}{a} \bullet \underset{\sim}{c} \cdot \ \underset{\sim}{b}$, $\underset{\sim}{c} \bullet \underset{\sim}{a} \cdot \ \underset{\sim}{b}$, $\underset{\sim}{c} \bullet \underset{\sim}{b} \cdot \ \underset{\sim}{a}$, $\underset{\sim}{b} \bullet \underset{\sim}{a} \cdot \ \underset{\sim}{c}$ and $\underset{\sim}{b} \bullet \underset{\sim}{c} \cdot \ \underset{\sim}{a}$.

7 An object of mass 3 kg is moving with velocity $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j}$. Calculate the angular momentum of this object about O when its position is given by $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$.

8



In the triangle ABC, D divides AC in the ratio 2:1, E divides CB in the ratio 2:1 and F divides BA in the ratio 2:1. Show that the area of triangle GHI is $\frac{1}{7}$ times the area of triangle ABC.

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Test Yourself
Chapter 3

eBook*plus* ACTIVITIES

3A Review of vectors

Tutorial

- Int-0406: Watch a tutorial on how to express direction and distance as a vector. (*page 96*)

Digital doc

- SkillSHEET 3.1: Practise using the sine and cosine rules. (*page 98*)

3B Vector product

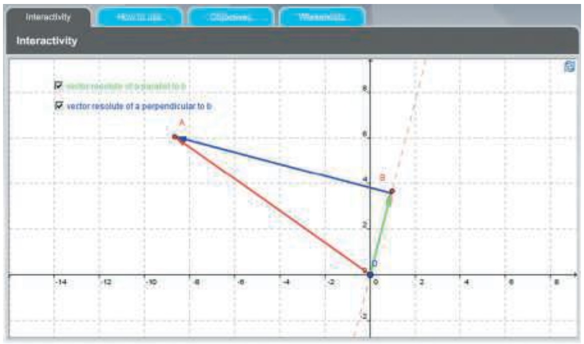
Digital docs

- SkillSHEET 3.2: Practise calculating the determinant of a 3 · 3 matrix. (*page 105*)
- WorkSHEET 3.1: Calculate resultant magnitude and direction of vectors, determine whether two vectors are perpendicular and calculate cross products between vectors. (*page 111*)

3D Applications of the vector product

Digital doc

- WorkSHEET 3.2: Calculate dot and cross products, find the equations of planes, angular momentum, volume and area. (*page 121*)



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 125*)

To access eBookPLUS activities, log on to

www.jacplus.com.au

Integral calculus and numerical methods

4

syllabus reference

Core topic:
Calculus

In this chapter

4A Approximation using the derivative
4B Substitution where the derivative is present in the integrand
4C Linear substitution
4D Antiderivatives involving trigonometric identities
4E Antidifferentiation using partial fractions
4F Integration by parts
4G Definite integrals
4H Volumes of solids of revolution
4I Approximation using Simpson’s rule

Numerical methods in calculus

Algebra, a system that uses letters to stand for numbers, is a powerful symbolic language. To illustrate this, consider life without algebra. Consider, for example, the problems that investors would face trying to compare variables (principal, interest rate and term) to find the best combination for an investment. Each situation would have to be laboriously worked out, the investor making dozens, or even hundreds of separate additions, subtractions, multiplications and divisions to find an answer.

Clever investors would keep the results of all this working out, and eventually produce tables of values — pages and pages of tables — giving results for a range of variables. They couldn't work them all out, of course, but if someone wanted a combination that didn't exactly fit the tables, they could always guess from the figures that were closest.

Now consider the alternative: using algebra, we can use a compound interest formula:

$$A = P\left(1 + \frac{r}{100}\right)^n$$

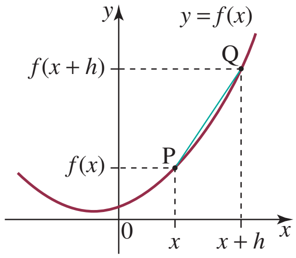
This formula calculates the amount, A, by which an investment grows after an amount, P has been invested for n periods at a rate of growth of r% per period.

This formula works for whatever values of P, r and n are chosen. It works, for example, when the interest rate is 10% per annum and the number of years is 7; or when the interest rate is 22% and the number of years is 1.5. This simple formula (with a calculator) replaces pages and pages of tables giving A for different values of P, r and n.

Calculus is essentially algebraic in character. The derivative of the function $f(x) = x^2$ is given by the equation $f'(x) = 2x$. This algebraic relationship gives the gradient on the curve $y = f(x)$ for any value of x. At times, however, it is either unfeasible or impossible to perform with algebra the demands made by the situation. Consider the integral $\int_0^{\pi} \frac{\sin x}{x}\,dx$. It appears innocuous, but it cannot be calculated using algebraic methods. The value of this integral can be calculated to any desired level of accuracy, not using algebraic techniques, but by using numerical techniques where the area under the curve is approximated by rectangles or trapeziums, as we saw in Mathematics B.

Much of the calculus that we examine in Mathematics B and C emphasises the algebraic aspect of the topic. In this section, and another later in the chapter, we consider numerical aspects of calculus.

Approximation using the derivative



Recall that the derivative of a function $f(x)$ is defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For small values of h it is clear that:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Rearranging this equation, we find:

$$hf'(x) \approx f(x + h) - f(x)$$

and for some x_0,

$$f(x_0 + h) \approx f(x_0) + hf'(x_0).$$

WORKED Example 1

a **For small values of t, show that $\sin t \approx t$.**

b **Use this approximation to give an approximate value of $\sin \frac{\pi}{8}$.**

c **What is the percentage error in the approximation?**

THINK	**WRITE**
a ① State the formula for the approximation using the derivative. ② Define $f(t)$ and $f'(t)$. ③ Identify values for x_0 and h. ④ Substitute these values into the formula. ⑤ Simplify.	**a** $f(x_0 + h) \approx f(x_0) + hf'(x_0)$ Let $f(t) = \sin t$. So $f'(t) = \cos t$. Let $x_0 = 0$ and $h = t$. $f(t) \approx f(0) + tf'(0)$ $\sin t \approx \sin 0 + t \cdot \cos 0$ $\sin t \approx 0 + t \cdot 1$ $\sin t \approx t$
b Find the approximate value of $\sin \frac{\pi}{8}$ by substituting $t = \frac{\pi}{8}$ into the approximation equation.	**b** Thus $\sin \frac{\pi}{8} \approx \frac{\pi}{8}$. $\sin \frac{\pi}{8} \approx 0.39$
c Calculate the percentage error. Include all decimal figures in calculations and then round down.	**c** Percentage error $= \dfrac{\left\lvert \sin \frac{\pi}{8} - \frac{\pi}{8} \right\rvert}{\sin \frac{\pi}{8}} \cdot 100\%$ $= 2.6\%$

WORKED Example 2

Use the method of approximation using the derivative to estimate the value of $\sqrt{38}$.

THINK	WRITE
1 Write the formula for approximation using the derivative.	$f(x_0 + h) \approx f(x_0) + hf'(x_0)$
2 Define $f(x)$ and $f'(x)$.	Let $f(x) = \sqrt{x} = x^{\frac{1}{2}}$. So $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$.
3 Identify values for x_0 and h. As $\sqrt{36}$ can be calculated easily, this is our choice for x_0.	Let $x_0 = 36$ and $h = 2$.
4 Substitute these values into the formula.	$f(36 + 2) \approx f(36) + 2 \cdot f'(36)$
5 Simplify.	Thus $\sqrt{38} \approx \sqrt{36} + 2 \cdot \dfrac{1}{2\sqrt{36}}$. $\sqrt{38} \approx 6 + \frac{1}{6}$ $\sqrt{38} \approx 6\frac{1}{6}$

remember

1. The formula for approximation using the derivative is $f(x_0 + h) \approx f(x_0) + hf'(x_0)$.
2. Choose x_0 so that it is easily computed. For example, choose $x_0 = \sqrt{25}$ when approximating $\sqrt{27}$.

EXERCISE 4A Approximation using the derivative

WORKED Example 1

1 a For small values of x show that $3 \sin x \approx 3x$.

b Use this approximation to give an approximate value of $3 \sin \dfrac{\pi}{6}$.

c What is the percentage error in the approximation?

2 a Find an approximation for $f(x) = x \cos x$ for small values of x.

b Use this approximation to estimate $\dfrac{\pi}{8} \cos \dfrac{\pi}{8}$.

3 a Find an approximation for $f(x) = (x + 1)^3$ for small values of x.

b Use this approximation to approximate 1.09^3.

WORKED Example 2

4 Use the method for approximation using the derivative to estimate the value of $\sqrt{84}$.

5 Find approximate values for the following:

a $\sqrt{96}$ **b** $\dfrac{1}{0.98}$ **c** $\sqrt{26}$ **d** $\sqrt[3]{28}$.

6 The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$.

a Show that a formula for the approximate change in the volume due to a small change in the radius, Δr is:

$$\Delta V = 4\pi r^2 \Delta r$$

b Thus, find the approximate change in the volume of a sphere when the radius changes from 8 cm to 8.2 cm.

7 If the volume of a sphere increases by 3%, find the approximate corresponding percentage increase in the radius.

8 If the surface area of a sphere increases by 3%, find the corresponding percentage increase in the radius.

Polynomial expansions

Recall that a polynomial is a function of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots$$

We have already met a number of functions that are not polynomials, such as $\dfrac{1}{x}$, sin x, ln x and e^x.

Provided that these functions can be differentiated repeatedly, it is possible to find polynomials that match these functions as closely as needed.

The following result produces what is known as the *Taylor series* for a polynomial and is stated without proof.

$$f(x) = f(x_0) + f'(x_0)x + f^{(2)}(x_0)\frac{x^2}{2!} + f^{(3)}(x_0)\frac{x^3}{3!} + \qquad (1)$$

Note that the series continues indefinitely; that is, the terms keep going. In practice, the number of terms one calculates depends upon the accuracy needed.

1 By letting $x_0 = 0$ in equation (1), show that the first 4 non-zero terms of the Taylor polynomial for $f(x) = \sin x$ are:

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

2 Find the Taylor expansion for each of the following.

a $f(x) = \cos x$

b $f(x) = e^x$

c $f(x) = \ln(1 + x)$

3 Use the Taylor expansion to find approximate values for each of the following to the number of decimal places indicated.

a sin 2 (to 2 decimal places)

b cos 4.5 (to 3 decimal places)

c e^3 (to 4 decimal places)

d ln 1.5 (to 3 decimal places)

Integration techniques and applications

You will have seen in your Mathematics B course and elsewhere that some functions can be antidifferentiated (integrated) using standard rules. These common results are shown in the table below where the function $f(x)$ has an antiderivative $F(x)$.

$f(x)$	$F(x)$
ax^n	$\frac{ax^{n+1}}{n+1}+c$
$\frac{1}{x}$	$\log_e kx + c$
e^{kx}	$\frac{e^{kx}}{k}+c$
$\sin kx$	$\frac{-\cos kx}{k}+c$
$\cos kx$	$\frac{\sin kx}{k}+c$
$\sec^2 kx$	$\frac{\tan kx}{k}+c$
$\frac{1}{\sqrt{a^2-x^2}}, x \in (-a, a)$	$\text{Sin}^{-1}\frac{x}{a}+c$
$\frac{-1}{\sqrt{a^2-x^2}}, x \in (-a, a)$	$\text{Cos}^{-1}\frac{x}{a}+c$
$\frac{a}{a^2+x^2}$	$\text{Tan}^{-1}\frac{x}{a}+c$

In this chapter you will learn how to find antiderivatives of more complex functions using various techniques.

Technique 1: Substitution where the derivative is present in the integrand

Since $\frac{d[f(x)]^{n+1}}{dx} = (n+1)f'(x)[f(x)]^n, n \neq -1$, as an application of the chain rule,

then it follows that: $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1.$

Since $\frac{d[\log_e f(x)]}{dx} = \frac{f'(x)}{f(x)}; f(x) \neq 0$

then it follows that $\int \frac{f'(x)}{f(x)} \, dx = \log_e f(x) + c$.

The method relies on the derivative, or multiples of the derivative, being present and recognisable. Then, the appropriate substitutions may be made according to the above rules.

WORKED Example 3

Find the antiderivative of the following expressions (i) by hand and (ii) by using the TI-Nspire CAS calculator.

a $(x + 3)^7$ **b** $4x(2x^2 + 1)^4$ **c** $\frac{3x^2 + 1}{\sqrt{x^3 + x}}$

THINK	WRITE
By hand	
a ① Recognise that the derivative of $x + 3$ is 1. Let $u = x + 3$.	**a** Let $u = x + 3$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 1$
③ Make dx the subject.	or $dx = du$
④ Substitute for $x + 3$ and dx.	So $\int (x + 3)^7\, dx = \int u^7\, du$
⑤ Antidifferentiate with respect to u.	$= \frac{u^8}{8} + c$
⑥ Replace u with $x + 3$ and state the answer in terms of x.	$= \frac{(x + 3)^8}{8} + c$
b ① Recognise that $4x$ is the derivative of $2x^2 + 1$. Let $u = 2x^2 + 1$.	**b** Let $u = 2x^2 + 1$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 4x$
③ Make dx the subject.	or $dx = \frac{du}{4x}$
④ Substitute u for $2x^2 + 1$ and $\frac{du}{4x}$ for dx.	So $\int 4x(2x^2 + 1)^4\, dx$ $= \int 4x\, u^4\, \frac{du}{4x}$
⑤ Simplify the integrand by cancelling out the $4x$.	$= \int u^4\, du$
⑥ Antidifferentiate with respect to u.	$= \frac{u^5}{5} + c$
⑦ Replace u with $2x^2 + 1$.	$= \frac{(2x^2 + 1)^5}{5} + c$
c ① Recognise that $3x^2 + 1$ is the derivative of $x^3 + x$. Let $u = x^3 + x$.	**c** Let $u = x^3 + x$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 3x^2 + 1$

Continued over page →

THINK	**WRITE**
3 Make dx the subject.	or $\mathrm{d}x = \dfrac{\mathrm{d}u}{3x^2+1}$
4 Substitute u for $x^3 + x$ and $\dfrac{\mathrm{d}u}{3x^2+1}$ for dx.	So $\displaystyle\int \frac{3x^2+1}{\sqrt{x^3+x}}\mathrm{d}x$ $= \displaystyle\int \frac{3x^2+1}{\sqrt{u}} \cdot \frac{\mathrm{d}u}{3x^2+1}$
5 Cancel out $3x^2 + 1$.	$= \displaystyle\int \frac{\mathrm{d}u}{\sqrt{u}}$
6 Express the integrand in index form.	$= \displaystyle\int u^{-\frac{1}{2}}\mathrm{d}u$
7 Antidifferentiate with respect to u.	$= 2u^{\frac{1}{2}} + c$
8 Replace u with $x^3 + x$.	$= 2(x^3 + x)^{\frac{1}{2}} + c$
9 Express in root notation.	$= 2\sqrt{x^3 + x} + c$

Using the TI-Nspire CAS calculator

a 1 Press:

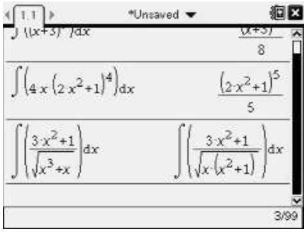
- Menu menu
- 4: Calculus 4
- 3: Integral 3

Leave the limits of integration blank by using tab twice.

Complete the entry line as:

$$\int (x+3)^7 \mathrm{d}x$$

Press Enter enter.



$\int ((x+3)^7)\mathrm{d}x \quad \dfrac{(x+3)^8}{8}$

$\int \left(4x(2x^2+1)^4\right)\mathrm{d}x \quad \dfrac{(2\cdot x^2+1)^5}{5}$

$\int \left(\dfrac{3\cdot x^2+1}{\sqrt{x^3+x}}\right)\mathrm{d}x \quad \int \left(\dfrac{3\cdot x^2+1}{\sqrt{x\cdot(x^2+1)}}\right)\mathrm{d}x$

b Repeat these steps for part **b**.

c Repeat these steps for part **c**.

Notes

1. The calculator cannot determine the solution for part **c**. It simply returns the input you entered. You will have to do this problem by working manually.
2. Also, it does not include the constant for any of the antiderivatives; you will have to do that yourself.

THINK	**WRITE**
2 Write your solutions, remembering to include the constant of integration.	$\displaystyle\int (x+3)^7 \mathrm{d}x = \frac{(x+3)^8}{8} + c$ $\displaystyle\int 4x(2x^2+1)^4 \mathrm{d}x = \frac{(2x^2+1)^5}{5} + c$

WORKED Example 4

Antidifferentiate the following functions with respect to x.

a $f(x) = \frac{x+3}{(x^2+6x)^3}$ **b** $f(x) = (x^2-1)\cos(3x-x^3)$

c **Repeat part a using a TI-Nspire calculator.**

THINK	WRITE
Working manually	
a ① Express in integral notation.	**a** $\int \frac{x+3}{(x^2+6x)^3}dx$
② Recognise that $x + 3$ is half of the derivative of $x^2 + 6x$.	
③ Let $u = x^2 + 6x$.	Let $u = x^2 + 6x$.
④ Find $\frac{du}{dx}$.	$\frac{du}{dx} = 2x+6$
⑤ Make dx the subject.	or $dx = \frac{du}{2x+6}$
⑥ Substitute u for $x^2 + 6x$ and $\frac{du}{2x+6}$ for dx.	So $\int \frac{x+3}{(x^2+6x)^3}dx = \int \frac{x+3}{u^3} \cdot \frac{du}{2x+6}$
⑦ Factorise $2x + 6$.	$= \int \frac{x+3}{u^3} \cdot \frac{du}{2(x+3)}$
⑧ Cancel out $x + 3$ and express u in index form on the numerator.	$= \int \frac{1}{2}u^{-3}\,du$
⑨ Antidifferentiate with respect to u.	$= -\frac{u^{-2}}{4} + c$
⑩ Replace u with $x^2 + 6x$.	$= -\frac{(x^2+6x)^{-2}}{4} + c$
⑪ Express the answer with a positive index number. (Optional.)	$= -\frac{1}{4(x^2+x)^2} + c$
b ① Express in integral notation.	**b** $\int (x^2-1)\cos(3x-x^3)\,dx$
② Recognise that $x^2 - 1$ is a multiple of the derivative of $3x - x^3$.	
③ Let $u = 3x - x^3$.	Let $u = 3x - x^3$.
④ Find $\frac{du}{dx}$.	$\frac{du}{dx} = 3-3x^2$
⑤ Make dx the subject.	or $dx = \frac{du}{3-3x^2}$

Continued over page ➔

THINK	**WRITE**
6 Substitute u for $3x - x^3$ and $\dfrac{du}{3 - 3x^2}$ for dx.	So $\int (x^2 - 1)\cos(3x - x^2)\,dx$ $= \int (x^2 - 1)\cos u \cdot \dfrac{du}{3 - 3x^2}$
7 Factorise $3 - 3x^2$.	$= \int (x^2 - 1)\cos u \cdot \dfrac{du}{3(1 - x^2)}$ $= \int (x^2 - 1)\cos u \cdot \dfrac{du}{-3(x^2 - 1)}$
8 Cancel out $x^2 - 1$.	$= \int \dfrac{-\cos u}{3}\,du$
9 Antidifferentiate with respect to u.	$= \dfrac{-\sin u}{3} + c$
10 Replace u with $3x - x^3$.	$= \dfrac{-\sin(3x - x^3)}{3} + c$

Using the TI-Nspire CAS calculator

c 1 Press:

- Menu (menu)
- 4: Calculus (4)
- 3: Integral (3)

Leave the limits of integration blank by using (tab) twice.

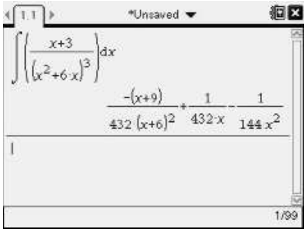
Complete the entry line as:

$$\int \frac{x+3}{(x^2+6x)^3}\,dx$$

Press Enter (enter).

Note: The answer is in an equivalent but more complex form than the solution found by hand.

c



$\int \left(\dfrac{x+3}{(x^2+6\cdot x)^3}\right)dx$

$\dfrac{-(x+9)}{432\cdot(x+6)^2}+\dfrac{1}{432\cdot x}-\dfrac{1}{144\cdot x^2}$

You can collect the terms in factorised form over a common denominator as follows:

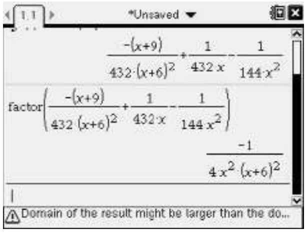
2 Press:

- Menu (menu)
- 3: Algebra (3)
- 2: Factor (2)
- Press the ‘up’ arrow (▲) on the NavPad. This will select the previous answer.
- Press Enter (enter)

This will complete the entry line as:

$$\text{factor}\left(\frac{-(x+9)}{432(x+6)^2}+\frac{1}{432x}-\frac{1}{144x^2}\right)$$

Press Enter (enter) again to get the answer.



$\dfrac{-(x+9)}{432\cdot(x+6)^2}+\dfrac{1}{432\cdot x}-\dfrac{1}{144\cdot x^2}$

$\text{factor}\left(\dfrac{-(x+9)}{432\cdot(x+6)^2}+\dfrac{1}{432\cdot x}-\dfrac{1}{144\cdot x^2}\right)$

$\dfrac{-1}{4\cdot x^2\cdot(x+6)^2}$

Domain of the result might be larger than the do...

3 Write your solution, remembering to include the constant of integration.

$$\int \frac{x+3}{(x^2+6x)^3}\,dx = -\frac{1}{4x^2(x+6)^2}+c$$

WORKED Example 5

Evaluate the following indefinite integrals.

a $\int \cos x \sin^4 x \, dx$ **b** $\int \frac{\text{Tan}^{-1}\frac{x}{2}}{4+x^2} dx$ **c** $\int \frac{\log_e 4x}{x} dx$ **d** $\int \sin^2 x \cos^3 x \, dx$

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SkillSHEET 4.1

THINK	WRITE
a 1 Recognise that cos x is the derivative of sin x.	**a**
2 Let u = sin x.	Let $u = \sin x$.
3 Find $\frac{du}{dx}$.	$\frac{du}{dx} = \cos x$
4 Make dx the subject.	or $dx = \frac{du}{\cos x}$
5 Substitute u for sin x and $\frac{du}{\cos x}$ for dx.	So $\int \cos x \sin^4 x \, dx = \int (\cos x) u^4 \frac{du}{\cos x}$
6 Cancel out cos x.	$= \int u^4 \, du$
7 Antidifferentiate with respect to u.	$= \frac{1}{5} u^5 + c$
8 Replace u with sin x.	$= \frac{1}{5} \sin^5 x + c$
b 1 Recognise that $\frac{1}{4+x^2}$ is half of the derivative of $\text{Tan}^{-1}\frac{x}{2}$.	**b**
2 Let $u = \text{Tan}^{-1}\frac{x}{2}$.	Let $u = \text{Tan}^{-1}\frac{x}{2}$.
3 Find $\frac{du}{dx}$.	$\frac{du}{dx} = \frac{2}{4+x^2}$
4 Make dx the subject.	or $dx = \frac{(4+x^2)du}{2}$
5 Substitute u for $\text{Tan}^{-1}\frac{x}{2}$ and $\frac{(4+x^2)du}{2}$ for dx.	So $\int \frac{\text{Tan}^{-1}\frac{x}{2}}{4+x^2} dx$ $= \int \frac{u}{4+x^2} \cdot \frac{(4+x^2)du}{2}$
6 Cancel out $4 + x^2$.	$= \int \frac{u}{2} du$
7 Antidifferentiate with respect to u.	$= \frac{u^2}{4} + c$
8 Replace u with $\text{Tan}^{-1}\frac{x}{2}$.	$= \frac{\left(\text{Tan}^{-1}\frac{x}{2}\right)^2}{4} + c$

Continued over page →

THINK	**WRITE**
c ① Recognise that $\frac{1}{x}$ is the derivative of $\log_e 4x$.	**c**
② Let $u = \log_e 4x$.	Let $u = \log_e 4x$.
③ Find $\frac{du}{dx}$.	$\frac{du}{dx} = \frac{1}{x}$
④ Make dx the subject.	or $dx = x\,du$
⑤ Substitute u for $\log_e 4x$ and x du for dx in the integral.	So $\int \frac{\log_e 4x}{x} dx$ $= \int \frac{u}{x} \cdot\ x\,du$
⑥ Cancel out x.	$= \int u\,du$
⑦ Antidifferentiate with respect to u.	$= \frac{1}{2}u^2 + c$
⑧ Replace u by $\log_e 4x$.	$= \frac{1}{2}(\log_e 4x)^2 + c$
d ① Express $\cos^3 x$ as $\cos x \cos^2 x$.	**d** $\int \sin^2 x \cos^3 x\,dx$ $= \int \sin^2 x \cos x \cos^2 x\,dx$
② Express $\cos x \cos^2 x$ as $\cos x\,(1 - \sin^2 x)$ (using the identity $\sin^2 x + \cos^2 x = 1$).	$= \int \sin^2 x \cos x\,(1 - \sin^2 x)\,dx$
③ Let $u = \sin x$ as its derivative is a factor of the new form of the function.	Let $u = \sin x$.
④ Find $\frac{du}{dx}$.	$\frac{du}{dx} = \cos x$
⑤ Make dx the subject.	or $dx = \frac{du}{\cos x}$
⑥ Substitute u for $\sin x$ and $\frac{du}{\cos x}$ for dx.	So $\int \sin^2 x \cos^3 x\,dx$ $= \int u^2 \cos x\,(1 - u^2)\,\frac{du}{\cos x}$
⑦ Cancel out $\cos x$.	$= \int u^2(1 - u^2)\,du$
⑧ Expand the integrand.	$= \int (u^2 - u^4)\,du$
⑨ Antidifferentiate with respect to u.	$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c$
⑩ Replace u by $\sin x$.	$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

WORKED Example 6

If $f'(x) = 4xe^{x^2}$ and $f(0) = 5$, find $f(x)$.

THINK	WRITE
1 Express $f(x)$ in integral notation.	$f(x) = \int 4xe^{x^2}\,dx$
2 Recognise that $4x$ is twice the derivative of x^2.	
3 Let $u = x^2$.	Let $u = x^2$.
4 Find $\frac{du}{dx}$.	$\frac{du}{dx} = 2x$
5 Make dx the subject.	or $dx = \frac{du}{2x}$
6 Substitute u for x^2 and $\frac{du}{2x}$ for dx.	So $f(x) = \int 4xe^u \frac{du}{2x}$
7 Cancel out $2x$.	$= \int 2e^u\,du$
8 Antidifferentiate with respect to u.	$= 2e^u + c$
9 Replace u by x^2.	$f(x) = 2e^{x^2} + c$
10 Substitute $x = 0$ and $f(0) = 5$.	$f(0) = 2e^0 + c = 5$
11 Solve for c.	$2 + c = 5$ $c = 3$
12 State the function $f(x)$.	Therefore $f(x) = 2e^{x^2} + 3$.

remember

1. Since $\frac{d[f(x)]^{n+1}}{dx} = (n+1)f'(x)[f(x)]^n,\ n \neq -1$

   then $\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$.

2. Since $\frac{d\log_e f(x)}{dx} = \frac{f'(x)}{f(x)}$

   then $\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + c$.

EXERCISE 4B

Substitution where the derivative is present in the integrand

WORKED Example 3 & 4

1 Find the antiderivative for each of the following expressions.

a $2x(x^2 + 3)^4$

b $2x(6 - x^2)^{-3}$

c $3x^2(x^3 - 2)^5$

d $2(x + 2)(x^2 + 4x)^{-3}$

e $(2x+5)\sqrt{x^2+5x}$

f $\dfrac{2x-3}{(x^2-3x)^4}$

g $3x^2(x^3 - 5)^2$

h $\dfrac{3x^2+4x}{\sqrt{x^3+2x^2}}$

i $4x^3e^{x^4}$

j $(2x + 3)\sin(x^2 + 3x - 2)$

k $(3x^2 + 5)\cos(x^3 + 5x)$

l $\cos x \sin^3 x$

m $-\sin^4 x \cos x$

n $\dfrac{\log_e x}{x}$

o $\sec^2 x \tan^3 x$

p $\dfrac{(\text{Sin}^{-1}\, x)^2}{\sqrt{1-x^2}}$

2 multiple choice

Given that the derivative of $(x^2 + 5x)^4$ is $4(2x + 5)(x^2 + 5x)^3$, then the antiderivative of $8(2x + 5)(x^2 + 5x)^3$ is:

A $2(x^2 + 5x)^4 + c$

B $\frac{1}{2}(x^2 + 5x)^4 + c$

C $4(x^2 + 5x)^4 + c$

D $2(x^2 + 5x)^2 + c$

E $\frac{1}{2}(x^2 + 5x)^2 + c$

3 multiple choice

a The integral $\displaystyle\int \frac{x}{\sqrt{x^2+3}}\,dx$ can be found by making the substitution ‘u’ equal to:

A x^2

B x

C $\sqrt{x}$

D $x^2 + 3$

E $2x$

b After the appropriate substitution the integral becomes:

A $\displaystyle\int u^{\frac{1}{2}}\,du$

B $\displaystyle\frac{1}{2}\int u^{-\frac{1}{2}}\,du$

C $\displaystyle\frac{1}{2}\int (u+3)^{-\frac{1}{2}}\,du$

D $\displaystyle\int (u+3)^{\frac{1}{2}}\,du$

E $\displaystyle 2\int u^{-\frac{1}{2}}\,du$

c Hence, the antiderivative of $\dfrac{x}{\sqrt{x^2+3}}$ is:

A $\frac{2}{3}(x^2+3)^{\frac{3}{2}}+c$

B $4(x^2+3)^{\frac{1}{2}}+c$

C $\frac{2}{3}(x^2+6)^{\frac{3}{2}}+c$

D $(x^2+6)^{\frac{1}{2}}+c$

E $(x^2+3)^{\frac{1}{2}}+c$

WORKED Example 4

4 Antidifferentiate each of the following expressions with respect to x.

a $6x^2(x^3 - 2)^5$

b $x(4 - x^2)^3$

c $x^2(x^3 - 1)^7$

d $(x + 3)(x^2 + 6x - 2)^4$

e $(x + 1)(x^2 + 2x + 3)^{-4}$

f $\dfrac{4x + 6}{\sqrt{x^2 + 3x}}$

g $\dfrac{2x - 5}{(x^2 - 5x + 2)^6}$

h $(x^2 - 1)\sqrt{4 - 3x + x^3}$

i $(6x - 3)e^{x^2 - x + 3}$

j $x^2e^{x^3 + 2}$

k $(x + 1)\sin(x^2 + 2x - 3)$

l $(x^2 - 2)\cos(6x - x^3)$

m $\sin 2x \cos^4 2x$

n $\cos 3x \sin^2 3x$

o $\dfrac{\log_e 3x}{2x}$

p $\dfrac{(4x - 2)\log_e(x^2 - x)}{x^2 - x}$

WORKED Example 5

5 Evaluate the following indefinite integrals.

a $\int x(x^2 + 1)^{\frac{5}{2}}\,dx$

b $\int x\sqrt{1 - x^2}\,dx$

c $\int e^x(3 + 2e^x)^4\,dx$

d $\int \dfrac{\sin x}{\cos^3 x}\,dx$

e $\int x^2 \sin x^3\,dx$

f $\int \sin x\, e^{\cos x}\,dx$

g $\int \dfrac{\cos x \log_e(\sin x)}{\sin x}\,dx$

h $\int e^{3x}(1 - e^{3x})^2\,dx$

i $\int \dfrac{-2\,\mathrm{Cos}^{-1}\frac{x}{3}}{\sqrt{9 - x^2}}\,dx$

j $\int (2x + 1)\sqrt{x + x^2 - 3}\,dx$

k $\int (x + 2)\cos(x^2 + 4x)\,dx$

l $\int \dfrac{e^{\sqrt{x+1}}}{\sqrt{x + 1}}\,dx$

m $\int \dfrac{\mathrm{Sin}^{-1} 4x}{\sqrt{1 - 16x^2}}\,dx$

n $\int \dfrac{\mathrm{Tan}^{-1} x}{1 + x^2}\,dx$

o $\int \dfrac{x}{1 - 4x^2}\,dx$

6 Find the antiderivative for each of the following expressions.

a $\dfrac{\cos x}{\sqrt{1 + 3\sin x}}$

b $\sec^2 x\sqrt{2 + \tan x}$

c $\sin x \sec^3 x$

d $\dfrac{e^{2x}}{(e^{2x} - 3)^2}$

e $\dfrac{\sec^2 x}{(5 - \tan x)^3}$

f $\dfrac{4}{x\log_e x}$

g $\dfrac{(\log_e x)^3}{x}$

h $\dfrac{e^{\tan x}}{\cos^2 x}$

i $\dfrac{e^x - e^{-x}}{\sqrt{e^x + e^{-x}}}$

j $\dfrac{\sin x - \cos x}{\sin x + \cos x}$

k $\sin^3 x \cos^2 x$

l $\cos^3 x \sin^4 x$

m $\dfrac{\log_e(\tan x)}{\sin x \cos x}$

WORKED Example 6

7 If $f'(x) = \dfrac{x}{\sqrt{x^2+5}}$ and $f(2) = 1$ find $f(x)$.

8 If $f'(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ and $f(0) = 3$ find $f(x)$.

9 If $g(1) = -2$ and $g'(x) = \dfrac{4 \log_e x^2}{x}$ then find $g(x)$.

10 If $g\left(\dfrac{\pi}{4}\right) = 0$ and $g'(x) = 16 \sin x \cos^3 x$ then find $g(x)$.

Technique 2: Linear substitution

For antiderivatives of the form $\int f(x)[g(x)]^n \, dx, n \neq 0$ where $g(x)$ is a linear function; that is, of the type $mx + c$, and $f(x)$ is not the derivative of $g(x)$, the substitution $u = g(x)$ is often successful in finding the integral. Examples of this type of integral are:

1. $\int \sqrt{4x+1} \, dx$. In this example $f(x) = 1$ and $g(x) = 4x + 1$ with $n = \frac{1}{2}$. By letting $u = 4x + 1$, and consequently $dx = \frac{1}{4} du$, the integral becomes $\frac{1}{4}\int \sqrt{u} \, du$ which can be readily antidifferentiated.
2. $\int 4x(x-3)^4 \, dx$. In this example $f(x) = 4x$ and $g(x) = x - 3$ with $n = 4$. By letting $u = x - 3$, the function $f(x)$ can be written in terms of u; that is, $u = x - 3$, thus $4x = 4(u + 3)$ and further, $dx = du$. The integral becomes $\int 4(u+3) \cdot u \, du$ which can be readily antidifferentiated.

The following worked examples illustrate how the use of the substitution $u = g(x)$ simplifies integrals of the type $\int f(x)[g(x)]^n \, dx$.

WORKED Example 7

i **Using the appropriate substitution, express the following integrals in terms of u only.**
ii **Evaluate the integrals as functions of x.**

a $\int x(x-2)^{\frac{5}{2}}\,dx$ **b** $\int \frac{x^2}{\sqrt{x+1}}\,dx$

THINK	WRITE
a **i** ① Let $u = x - 2$.	**a** **i** Let $u = x - 2$ and $x = u + 2$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 1$
③ Make dx the subject.	$dx = du$
④ Substitute u for $x - 2$, $u + 2$ for x and du for dx.	So $\int x(x-2)^{\frac{5}{2}}\,dx$ $= \int (u+2)u^{\frac{5}{2}}\,du$
⑤ Expand the integrand.	$= \int \left(u^{\frac{7}{2}} + 2u^{\frac{5}{2}}\right) du$
ii ① Antidifferentiate with respect to u.	**ii** $= \frac{2}{9}u^{\frac{9}{2}} + \frac{4}{7}u^{\frac{7}{2}} + c$
② Replace u with $x - 2$.	$= \frac{2}{9}(x-2)^{\frac{9}{2}} + \frac{4}{7}(x-2)^{\frac{7}{2}} + c$
③ Take out the factor of $2(x-2)^{\frac{7}{2}}$.	$= 2(x-2)^{\frac{7}{2}}\left(\frac{x-2}{9} + \frac{4}{7}\right) + c$
④ Simplify the other factor.	$= 2(x-2)^{\frac{7}{2}}\left(\frac{7x - 14 + 36}{63}\right) + c$ $= 2(x-2)^{\frac{7}{2}}\left(\frac{7x + 22}{63}\right) + c$
b **i** ① Express $\sqrt{x+1}$ in index form.	**b** **i** $\int \frac{x^2}{\sqrt{x+1}}\,dx$ $= \int x^2(x+1)^{-\frac{1}{2}}\,dx$
② Let $u = x + 1$.	Let $u = x + 1$.

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THINK	WRITE
③ Find $\frac{du}{dx}$.	$\frac{du}{dx} = 1$
④ Make dx the subject.	or $dx = du$
⑤ Express x in terms of u.	$x = u - 1$
⑥ Hence express x^2 in terms of u.	$x^2 = u^2 - 2u + 1$
⑦ Substitute u for $x + 1$, $u^2 - 2u + 1$ for x^2 and du for dx.	So $\int x^2(x+1)^{-\frac{1}{2}}\,dx$ $= \int (u^2 - 2u + 1)u^{-\frac{1}{2}}\,du$
⑧ Expand the integrand.	$= \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du$
ii ① Antidifferentiate with respect to u.	**ii** $= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c$
② Replace u with $x + 1$.	$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + c$
③ Take $2(x+1)^{\frac{1}{2}}$ out as a factor.	$= 2(x+1)^{\frac{1}{2}}\left[\frac{(x+1)^2}{5} - \frac{2(x+1)}{3} + 1\right] + c$
④ Simplify the other factor.	$= 2(x+1)^{\frac{1}{2}}\left[\frac{(x^2+2x+1)}{5} + \frac{(-2x-2)}{3} + 1\right] + c$ $= 2(x+1)^{\frac{1}{2}}\left[\frac{3x^2+6x+3-10x-10+15}{15}\right] + c$ $= 2(x+1)^{\frac{1}{2}}\left[\frac{3x^2-4x+8}{15}\right] + c$

WORKED Example 8

a **Find the antiderivative of $\frac{e^{2x}}{e^x + 1}$.**

b **State the domain of the antiderivative.**

THINK	WRITE
a ① Since $e^{2x} = (e^x)^2$, it can be antidifferentiated the same as a linear function by letting $u = e^x + 1$.	**a** Let $u = e^x + 1$.

THINK	**WRITE**
② Find $\frac{\mathrm{d}u}{\mathrm{d}x}$.	$\frac{\mathrm{d}u}{\mathrm{d}x} = e^x$
③ Make dx the subject.	or $\mathrm{d}x = \frac{\mathrm{d}u}{e^x}$
④ Express e^x in terms of u.	and $e^x = u - 1$
⑤ Substitute u for $e^x + 1$ and $\frac{\mathrm{d}u}{e^x}$ for dx.	So $\int \frac{e^{2x}}{e^x + 1}\,\mathrm{d}x$ $= \int \frac{e^x e^x}{u} \cdot \frac{\mathrm{d}u}{e^x}$
⑥ Cancel out e^x.	$= \int \frac{e^x}{u}\,\mathrm{d}u$
⑦ Substitute $u - 1$ for the remaining e^x.	$= \int \frac{u - 1}{u}\,\mathrm{d}u$
⑧ Simplify the rational expression.	$= \int \left(1 - \frac{1}{u}\right)\mathrm{d}u$
⑨ Antidifferentiate with respect to u.	$= u - \log_e u + c$
⑩ Replace u with $e^x + 1$.	$= e^x + 1 - \log_e(e^x + 1) + c$
b ① $e^x + 1 > 0$ for all values of x as $e^x > 0$ for all x. The function $\log_e f(x)$ exists wherever $f(x) > 0$.	**b** For $\log_e(e^x + 1)$ to exist $e^x + 1 > 0$, which it is for all x.
② State the domain.	Therefore the domain of the integral is R.

Note: Recall that the logarithm of a negative number cannot be found.

remember

For antiderivatives of the form $\int f(x)[g(x)]^n\,\mathrm{d}x$, $n \neq 0$, make the substitution $u = g(x)$ and so $[g(x)]^n\,\mathrm{d}x$, $n \neq 0$ becomes $g'(x)\,u^n\,\mathrm{d}u$, $n \neq 0$. This technique can be used for the specific case where $g = mx + c$ since $g'(x) = m$. The function $f(x)$ needs to be transformed in terms of the variable u as well.

EXERCISE 4C Linear substitution

WORKED Example 7

1 By making the appropriate substitution for u:

i express the following integrals in terms of u

ii evaluate the integrals as functions of x.

a $\int \frac{4}{x-3}\,dx$ **b** $\int \frac{2}{3x+5}\,dx$

c $\int \sqrt{4x+1}\,dx$ **d** $\int \sqrt{3-2x}\,dx$

e $\int x(x+1)^3\,dx$ **f** $\int 4x(x-3)^4\,dx$

g $\int 2x(2x+1)^4\,dx$ **h** $\int 3x(1-3x)^5\,dx$

i $\int 6x(3x-2)^{\frac{3}{4}}\,dx$ **j** $\int x(2x+7)^{\frac{1}{3}}\,dx$

k $\int x\sqrt{x+3}\,dx$ **l** $\int x\sqrt{3x-4}\,dx$

m $\int (x+2)(x-4)^{\frac{3}{2}}\,dx$ **n** $\int (x-3)(2x+1)^{\frac{5}{2}}\,dx$

o $\int \frac{2x}{\sqrt{x-6}}\,dx$ **p** $\int \frac{3x}{\sqrt{8-x}}\,dx$

2 **multiple choice**

a The integral $\int 4x\sqrt{x+2}\,dx$ can be found by letting u equal:

A $\sqrt{x+2}$ **B** x **C** $x+2$ **D** $4x$ **E** $2x$

b The integral then becomes:

A $\int u^{\frac{5}{2}}\,du$ **B** $\int \left(2u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right)du$ **C** $\int 2u^{\frac{1}{2}}\,du$

D $\int \left(4u^{\frac{1}{2}} - 2u^{\frac{3}{2}}\right)du$ **E** $\int \left(4u^{\frac{3}{2}} - 8u^{\frac{1}{2}}\right)du$

3 **multiple choice**

a Using the appropriate substitution, $\int \frac{x^2}{\sqrt{x-1}}\,dx$ becomes:

A $\int u\,du$ **B** $\int \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$ **C** $\int \left(u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right)du$

D $\int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$ **E** $\int u^{\frac{3}{2}}\,du$

b The result of the integration is:

A $\frac{2}{3}(x-1)^{\frac{3}{2}}+c$

B $\frac{2}{3}(x-1)^{\frac{3}{2}}+4(x-1)^{\frac{1}{2}}+c$

C $\frac{2}{5}(x-1)^{\frac{5}{2}}+c$

D $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{4}{3}(x-1)^{\frac{3}{2}}+2(x-1)^{\frac{1}{2}}+c$

E $\frac{2}{7}(x-1)^{\frac{7}{2}}+\frac{4}{5}(x-1)^{\frac{5}{2}}+\frac{4}{3}(x-1)^{\frac{3}{2}}+c$

WORKED Example 8a

4 Find the antiderivative of each of the following expressions.

a $x^2(x-4)^4$

b $x^2(5-x)^3$

c $x^2\sqrt{x-1}$

d $x^2\sqrt{3-x}$

e $x^2(x+2)^{\frac{4}{3}}$

f $x^2(1-x)^{\frac{3}{4}}$

g $(x+1)^2\sqrt{x-2}$

h $(x-3)^2\sqrt{x+1}$

i $\dfrac{e^x}{e^x+1}$

j $\dfrac{x^2}{\sqrt{x+1}}$

k $\dfrac{2x^2}{\sqrt{3-x}}$

l $x^3\sqrt{x-1}$

m $\dfrac{x^3}{\sqrt{x+4}}$

n $\dfrac{2x^3}{\sqrt{1-x}}$

o $\dfrac{x+3}{(x-2)^2}$

p $\dfrac{2x-1}{(x+1)^3}$

q $\dfrac{4x}{(x+2)^2}$

r $\dfrac{x^2}{(x-1)^2}$

s $\dfrac{(x+3)^2}{\sqrt{x+2}}$

t $\dfrac{(x-2)^2}{\sqrt{2-x}}$

u $\dfrac{e^{2x}}{e^x+2}$

v $\dfrac{e^{3x}}{e^x-1}$

WORKED Example 8b

5 a If $f'(x) = -(5-x)^{\frac{1}{2}}+10(5-x)^{-\frac{1}{2}}$ and $f(1) = -2$, find $f(x)$.

b State the domain of $f(x)$.

6 a If $f'(x) = \dfrac{5(x+1)^{\frac{3}{2}}}{2}-3(x+1)^{\frac{1}{2}}+\dfrac{(x+1)^{-\frac{1}{2}}}{2}$ and $f(0) = 1$, find $f(x)$.

b State the domain of $f(x)$.

7 a Given that $g'(x) = \dfrac{2x+1}{(x-1)^2}$ and $g(2) = 0$, find $g(x)$.

b State the domain of $g(x)$.

8 a Given that $g(0) = 2-\log_e 2$ and $g'(x) = \dfrac{e^{2x}}{e^x+1}$, find $g(x)$.

b State the domain of $g(x)$.

Technique 3: Antiderivatives involving trigonometric identities

Different trigonometric identities can be used to antidifferentiate $\sin^n x$ and $\cos^n x$; $n \in N$ depending on whether n is even or odd. Functions involving $\tan^2 ax$ are also discussed.

Even powers of sin x or cos x

The double-angle trigonometric identities can be used to antidifferentiate even powers of sin x or cos x. The first identity is:

$$\begin{aligned}\cos 2x &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1.\end{aligned}$$

Therefore $\quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

or $\quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

The second identity is: $\quad \sin 2x = 2\sin x \cos x$

or $\quad \sin x \cos x = \frac{1}{2}\sin 2x$.

These may be expressed in the following general forms:

$\boldsymbol{\sin^2 ax = \frac{1}{2}(1 - \cos 2ax)}$ **Identity 1**

$\boldsymbol{\cos^2 ax = \frac{1}{2}(1 + \cos 2ax)}$ **Identity 2**

$\boldsymbol{\sin ax \cos ax = \frac{1}{2}\sin 2ax}$ **Identity 3**

WORKED Example 9

Find the antiderivative of the following expressions.

a $\sin^2\frac{x}{2}$ **b** $2\cos^2\frac{x}{4}$ **c** **Repeat part a using a TI-Nspire CAS calculator.**

THINK	WRITE
Working manually	
a 1 Express in integral notation.	**a** $\int \sin^2\frac{x}{2}\,dx$
2 Use identity 1 to change $\sin^2\frac{x}{2}$.	$= \int \frac{1}{2}(1 - \cos x)\,dx$
3 Take the factor of $\frac{1}{2}$ to the front of the integral.	$= \frac{1}{2}\int (1 - \cos x)\,dx$
4 Antidifferentiate by rule.	$= \frac{1}{2}(x - \sin x) + c$
5 Simplify the answer.	$= \frac{x}{2} - \frac{1}{2}\sin x + c$

THINK

b ① Express in integral notation.

② Use identity 2 to change $\cos^2\frac{x}{4}$.

③ Simplify the integral.

④ Antidifferentiate by rule.

WRITE

b $\int 2\cos^2\frac{x}{4}\,dx$

$= \int 2\left(\tfrac{1}{2}\right)\left(1+\cos\frac{x}{2}\right)dx$

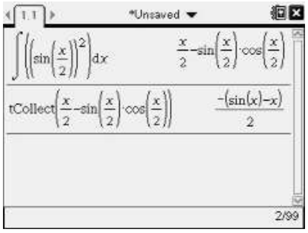
$= \int \left(1+\cos\frac{x}{2}\right)dx$

$= x + 2\sin\frac{x}{2} + c$

Using the TI-Nspire CAS calculator

Note: If you use a calculator for problems such as these, you may find the answer expressed in a form that is different from the ones above.

a The screen dump shows the result, in the first line, of part **a** done by calculator. The compact form shown in the second line can be obtained as follows:



$\int \left(\left(\sin\left(\frac{x}{2}\right)\right)^2\right)dx \qquad \frac{x}{2}-\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)$

$\text{tCollect}\left(\frac{x}{2}-\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)\right) \qquad \frac{-(\sin(x)-x)}{2}$

① Press:
- Menu (menu)
- 3: Algebra (3)
- A: Trigonometry (A)
- 2: Collect (2)

Use the NavPad to select the expression in the first line and copy it to complete the entry line as:

$\text{tCollect}\left(\frac{x}{2}-\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)$

Press Enter (enter).

② Write your solution, remembering to include the constant of integration.

$\int \sin^2\left(\frac{x}{2}\right)dx = \frac{x-\sin x}{2} + c$

WORKED Example 10

Evaluate the following indefinite integrals as functions of *x*.

a $\int \sin x \cos x\, dx$ **b** $\int 4\sin^2\frac{x}{2}\cos^2\frac{x}{2}\,dx$

THINK

a ① Use identity 3 to change $\sin x \cos x$.
Note: The integral could be antidifferentiated using technique 1 since the derivative of $\sin x$ is $\cos x$.

② Antidifferentiate by rule.

WRITE

a $\int \sin x \cos x\, dx$

$= \int \tfrac{1}{2}\sin 2x\, dx$

$= -\tfrac{1}{4}\cos 2x + c$

Continued over page →

THINK	**WRITE**
b ① Express $\sin^2\frac{x}{2}\ \cos^2\frac{x}{2}$ as a perfect square.	**b** $\int 4\ \sin^2\frac{x}{2}\ \cos^2\frac{x}{2}\ dx$ $= \int 4\left(\sin\frac{x}{2}\ \cos\frac{x}{2}\right)^2\ dx$
② Use identity 3 to change $\sin\frac{x}{2}\ \cos\frac{x}{2}$.	$= \int 4(\frac{1}{2}\sin x)^2\ dx$
③ Square the identity.	$= \int 4(\frac{1}{4}\sin^2 x)\ dx$
④ Simplify the integral.	$= \int \sin^2 x\ dx$
⑤ Use identity 1 to change $\sin^2 x$.	$= \int \frac{1}{2}(1 - \cos 2x)\ dx$
⑥ Antidifferentiate by rule.	$= \frac{1}{2}(x - \frac{1}{2}\sin 2x) + c$
⑦ Simplify the answer.	$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$

Odd powers of sin x or cos x

For integrals involving odd powers of sin x or cos x the identity:

$$\sin^2 x + \cos^2 x = 1$$

can be used so that the 'derivative method' of substitution then becomes applicable. The following worked example illustrates the use of this identity whenever there is an odd-powered trigonometric function in the integrand.

WORKED Example 11

Find the antiderivative of the following expressions.

a $\cos^3 x$ **b** $\cos x \sin 2x$ **c** $\cos^4 2x\ \sin^3 2x$

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Tutorial:
Worked example 11
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THINK	**WRITE**
a ① Express in integral notation.	**a** $\int \cos^3 x\ dx$
② Factorise $\cos^3 x$ as $\cos x\ \cos^2 x$.	$= \int \cos x\ \cos^2 x\ dx$
③ Use the identity: $(1 - \sin^2 x)$ for $\cos^2 x$.	$= \int \cos x(1 - \sin^2 x)\ dx$

THINK	WRITE
4 Let $u = \sin x$ so the derivative method can be applied.	Let $u = \sin x$.
5 Find $\frac{du}{dx}$.	$\frac{du}{dx} = \cos x$
6 Make dx the subject.	or $dx = \frac{du}{\cos x}$
7 Substitute u for $\sin x$ and $\frac{du}{\cos x}$ for dx.	So $\int \cos x(1 - \sin^2 x)\, dx$ $= \int \cos x(1 - u^2)\frac{du}{\cos x}$
8 Cancel out $\cos x$.	$= \int (1 - u^2)\, du$
9 Antidifferentiate with respect to u.	$= u - \frac{1}{3}u^3 + c$
10 Replace u with $\sin x$.	$= \sin x - \frac{1}{3}\sin^3 x + c$
b 1 Express in integral notation.	**b** $\int \cos x \sin 2x\, dx$
2 Use identity 3 in reverse to express $\sin 2x$ as $2 \sin x \cos x$.	$= \int \cos x\ (2 \sin x \cos x)\, dx$
3 Simplify the integrand.	$= \int 2 \sin x \cos^2 x\, dx$
4 Let $u = \cos x$ so that the derivative method can be applied.	Let $u = \cos x$.
5 Find $\frac{du}{dx}$.	$\frac{du}{dx} = -\sin x$
6 Make dx the subject.	or $dx = \frac{du}{-\sin x}$
7 Substitute u for $\cos x$ and $\frac{du}{-\sin x}$ for dx.	So $\int 2 \sin x \cos^2 x\, dx$ $= \int 2 \sin x\ (u^2)\frac{du}{-\sin x}$
8 Cancel out $\sin x$.	$= \int -2u^2\, du$

Continued over page →

THINK	**WRITE**
⑨ Antidifferentiate with respect to u.	$= -\frac{2}{3}u^3 + c$
⑩ Replace u with $\cos x$.	$= -\frac{2}{3}\cos^3 x + c$
c ① Express in integral notation.	**c** $\int \cos^4 2x \sin^3 2x \, dx$
② Factorise $\sin^3 2x$ as $\sin 2x \sin^2 2x$.	$= \int \cos^4 2x \sin 2x \sin^2 2x \, dx$
③ Use the identity $1 - \cos^2 2x$ for $\sin^2 2x$.	$= \int \cos^4 2x \sin 2x(1 - \cos^2 2x) \, dx$
④ Let $u = \cos 2x$ so that the derivative method can be applied.	Let $u = \cos 2x$.
⑤ Find $\frac{du}{dx}$.	$\frac{du}{dx} = -2\sin 2x$
⑥ Make dx the subject.	or $dx = \frac{du}{-2\sin 2x}$
⑦ Substitute u for $\cos 2x$ and $\frac{du}{-2\sin 2x}$ for dx.	So $\int u^4 \sin 2x(1 - u^2)\frac{du}{-2\sin 2x}$
⑧ Cancel out $\sin 2x$.	$= \int -\frac{1}{2}u^4(1 - u^2) \, du$
⑨ Expand the integrand.	$= \int \frac{1}{2}(u^6 - u^4) \, du$
⑩ Antidifferentiate with respect to u.	$= \frac{1}{2}(\frac{1}{7}u^7 - \frac{1}{5}u^5) + c$
⑪ Simplify the result.	$= \frac{1}{14}u^7 - \frac{1}{10}u^5 + c$
⑫ Replace u with $\cos 2x$.	$= \frac{1}{14}\cos^7 2x - \frac{1}{10}\cos^5 2x + c$

Using the identity $\sec^2 x = 1 + \tan^2 x$

The identity $\sec^2 ax = 1 + \tan^2 ax$ is used to antidifferentiate expressions involving $\tan^2 ax + c$ where c is a constant since the antiderivative of $\sec^2 x$ is $\tan x$.

Otherwise, expressions of the form $\tan^n x \sec^2 x$ can be antidifferentiated using the ‘derivative method’ of exercise 4B (p. 132).

WORKED Example 12

Find an antiderivative for each of the following expressions.

a $\int (2 + \tan^2 x)\, dx$ **b** $\int 3 \tan^2 3x \sec^2 3x\, dx$

THINK	WRITE
a ① Express $2 + \tan^2 x$ as $1 + \sec^2 x$ using the identity.	**a** $\int (2 + \tan^2 x)\, dx$ $= \int (1 + \sec^2 x)\, dx$
② Antidifferentiate by rule. There is no need to add c as one antiderivative only is required.	$= x + \tan x$
b ① Let $u = \tan 3x$ so that the derivative method can be applied.	**b** Let $u = \tan 3x$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 3 \sec^2 3x$
③ Make dx the subject.	or $dx = \frac{du}{3 \sec^2 3x}$
④ Substitute u for $\tan 3x$ and $\frac{du}{3 \sec^2 3x}$ for dx.	So $\int 3 \tan^2 3x \sec^2 3x\, dx$ $= \int 3\, u^2 \sec^2 3x \frac{du}{3 \sec^2 3x}$
⑤ Cancel out $3 \sec^2 3x$.	$= \int u^2\, du$
⑥ Antidifferentiate with respect to u.	$= \frac{1}{3} u^3$
⑦ Replace u with $\tan 3x$.	$= \frac{1}{3} \tan^3 3x$

remember

1. Trigonometric identities can be used to antidifferentiate odd and even powers of $\sin x$ and $\cos x$. These identities are:

$$\sin^2 ax = \tfrac{1}{2}(1 - \cos 2ax)$$

$$\cos^2 ax = \tfrac{1}{2}(1 + \cos 2ax)$$

$$\sin ax \cos ax = \tfrac{1}{2} \sin 2ax$$

2. The identity $\sec^2 ax = 1 + \tan^2 ax$ is used to antidifferentiate expressions involving $\tan^2 ax + c$ where c is a constant.

EXERCISE 4D Antiderivatives involving trigonometric identities

WORKED Example 9

1 Antidifferentiate each of the following expressions with respect to x.

a $\cos^2 x$ **b** $\sin^2 2x$ **c** $2\cos^2 4x$ **d** $4\sin^2 3x$

e $\cos^2 5x$ **f** $\sin^2 6x$ **g** $\cos^2 \frac{x}{2}$ **h** $\sin^2 \frac{x}{3}$

i $3\cos^2 \frac{x}{6}$ **j** $2\sin^2 \frac{x}{4}$ **k** $\cos^2 \frac{2x}{3}$ **l** $\sin^2 \frac{3x}{2}$

WORKED Example 10

2 Evaluate the following indefinite integrals as functions of x.

a $\int 2\sin x\cos\ dx$ **b** $\int 4\sin 2x\cos 2x\,dx$

c $\int \sin 3x\cos 3x\,dx$ **d** $\int -2\sin 4x\cos 4x\,dx$

e $\int \sin^2 x\cos^2 x\,dx$ **f** $\int \sin^2 2x\cos^2 2x\,dx$

g $\int 2\sin^2 4x\cos^2 4x\,dx$ **h** $\int 2\sin^2 3x\cos^2 3x\,dx$

i $\int 6\sin^2 \frac{x}{2}\cos^2 \frac{x}{2}\,dx$ **j** $\int 4\sin^2 \frac{x}{3}\cos^2 \frac{x}{3}\,dx$

k $\int \sin^2 \frac{5x}{2}\cos^2 \frac{5x}{2}\,dx$ **l** $\int -2\sin^2 \frac{4x}{3}\cos^2 \frac{4x}{3}\,dx$

3 **multiple choice**

If a is a constant, then,

a $\int \sin^2 ax\,dx$ is equal to:

A $2x - \sin 2ax + c$ **B** $x - 2a\sin 2ax + c$ **C** $\frac{x}{2} - \frac{\sin 2ax}{4a} + c$

D $x - \frac{1}{2}\sin\frac{ax}{2} + c$ **E** $\frac{x}{2} - \frac{1}{a}\sin\frac{ax}{2} + c$

b $\int \sin^2 ax\cos^2 ax\,dx$ is equal to:

A $\frac{x}{8} - \frac{\sin 4ax}{32a} + c$ **B** $\frac{x}{2} - \frac{\sin ax}{4a} + c$ **C** $\frac{x}{4} - \frac{\cos ax}{8a} + c$

D $x - \frac{\sin 4ax}{16a} + c$ **E** $\frac{x}{8} + \frac{\cos 4ax}{16a} + c$

c $\int \cos^3 ax\,dx$ is equal to:

A $a\cos ax - 3a\cos^3 ax + c$ **B** $a\sin ax - 3\cos^3 ax + c$

C $\frac{\cos^4 ax}{4a} + c$ **D** $\frac{\sin^4 ax}{4a} + c$

E $\frac{1}{3a}(3\sin ax - \sin^3 ax) + c$

WORKED Example 11a

4 Find an antiderivative of each of the following expressions.

a $\sin^3 x$ **b** $\cos^3 2x$ **c** $6\sin^3 4x$ **d** $4\cos^3 3x$

e $\sin^3 7x$ **f** $\cos^3 6x$ **g** $3\sin^3\frac{x}{2}$ **h** $2\cos^3\frac{x}{3}$

i $\sin^3\frac{3x}{2}$ **j** $\cos^3\frac{5x}{2}$ **k** $\sin^3\frac{3x}{4}$ **l** $\cos^3\frac{4x}{3}$

WORKED Example 11b

5 Use the appropriate identities to antidifferentiate the following expressions.

a $\sin x\cos 2x$ **b** $\cos 2x\cos 4x$ **c** $\sin 3x\cos 6x$

d $\cos 4x\cos 8x$ **e** $\sin\frac{x}{2}\cos x$ **f** $\cos\frac{x}{3}\cos\frac{2x}{3}$

6 Antidifferentiate each of the following expressions with respect to x.

a $\sin x\cos^4 x$ **b** $\sin 2x\cos^3 2x$ **c** $\sin\frac{x}{2}\cos^5\frac{x}{2}$

d $\cos 3x\sin^4 3x$ **e** $\cos\frac{x}{5}\sin^6\frac{x}{5}$ **f** $\cos\frac{2x}{3}\sin^7\frac{2x}{3}$

WORKED Example 11c

7 Find the following integrals.

a $\int\cos^2 x\sin^3 x\,dx$ **b** $\int\sin^2 x\cos^3 x\,dx$

c $\int\cos^2 2x\sin^3 2x\,dx$ **d** $\int\sin^2 3x\cos^3 3x\,dx$

e $\int\cos^2\frac{x}{2}\sin^3\frac{x}{2}\,dx$ **f** $\int\sin^2\frac{3x}{2}\cos^3\frac{3x}{2}\,dx$

g $\int 4\cos^2\frac{x}{3}\sin^3\frac{x}{3}\,dx$ **h** $\int -6\sin^2\frac{5x}{4}\cos^3\frac{5x}{4}\,dx$

i $\int\sin^3 x\cos^4 x\,dx$ **j** $\int\cos^3 2x\sin^4 2x\,dx$

k $\int 2\sin^3 2x\cos^5 2x\,dx$ **l** $\int -2\cos^3 3x\sin^6 3x\,dx$

m $\int 4\sin^3\frac{x}{2}\cos^6\frac{x}{2}\,dx$ **n** $\int\cos^3\frac{3x}{2}\sin^7\frac{3x}{2}\,dx$

WORKED Example 12

8 Find an antiderivative for each of the following expressions.

a $1+\tan^2 2x$ **b** $1+\tan^2\frac{x}{3}$ **c** $\tan^2 x\sec^2 x$

d $\tan^3 x\sec^2 x$ **e** $4\tan^5 2x\sec^2 2x$ **f** $8\tan^4\frac{x}{2}\sec^2\frac{x}{2}$

g $\tan^2 x\sec^4 x$ **h** $6\tan^2 2x\sec^4 2x$ **i** $2\tan^2\frac{x}{2}\sec^4\frac{x}{2}$

j $3\tan^3 3x\sec^4 3x$ **k** $\tan^4\frac{x}{5}\sec^4\frac{x}{5}$ **l** $12\tan^5 6x\sec^6 6x$

9 Find the following integrals.

a $\int\sin x\cos^n x\,dx$ **b** $\int\cos x\sin^n x\,dx$ **c** $\int\sec^2 x\tan^n x\,dx$

d $\int\sin^3 x\cos^n x\,dx$ **e** $\int\cos^3 x\sin^n x\,dx$

10 If $f'(x) = 6 \sin x \cos^2 x$ and $f\left(\frac{\pi}{3}\right) = 0$, find $f(x)$.

11 If $f'(x) = 4 \sin^2 2x \cos^2 2x$ and $f\left(\frac{\pi}{4}\right) = \pi$, find $f(x)$.

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12 Find $g(x)$ if $g'(x) = \sin^3 \frac{x}{2} \cos^4 \frac{x}{2}$ and $g(0) = -\frac{4}{35}$.

The graph of a function and the graphs of its antiderivatives

Note: This investigation relates to the TI-Nspire CAS calculator only. The Casio graphics calculator does not have the capability to graph an integral function.

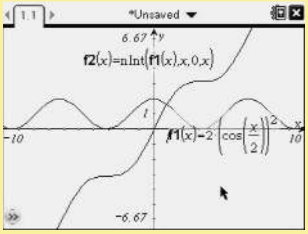
Given that $f(x) = 2\cos^2 \frac{x}{2}$, what do the graphs of its antiderivatives look like?

On a Graphs page, complete the entry line as:

$$f1(x) = 2 \cdot \left(\cos\left(\frac{x}{2}\right)\right)^2$$

$$f2(x) = \text{nInt}(f1(x), x, 0, x)$$

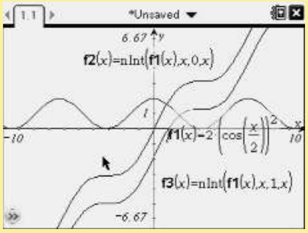
Press ENTER after each line.



The antiderivative graph cuts 0 at $x = 0$, since the integral from 0 to 0 of any function is 0.

To see another antiderivative graph, complete the entry line as:

$$f3(x) = \text{nInt}(f1(x), x, 1, x)$$



1 Generate another two antiderivative graphs on your calculator. Sketch the function and the four antiderivative graphs. Describe any relationships you can find.

2 Choose another function and investigate the realtionship between the graph of the function and the graphs of its antiderivatives.

Technique 4: Antidifferentiation using partial fractions

Rational expressions, in particular those with denominators that can be expressed with linear factors, can be transformed into partial fractions. A summary of two common transformations is shown in the table below. These transformations are useful when the degree of the numerator is less than the degree of the denominator; otherwise long division is generally required before antidifferentiation can be performed.

Rational expression	Equivalent partial fraction
$\dfrac{f(x)}{(ax+b)(cx+d)}$ where $f(x)$ is a linear function	$\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)}$
$\dfrac{f(x)}{(ax+b)^2}$ where $f(x)$ is a linear function	$\dfrac{A}{(ax+b)^2}+\dfrac{B}{(ax+b)}$

This procedure can be used to simplify the sketching of graphs of rational functions. Similarly, expressing rational functions as partial fractions enables them to be antidifferentiated quite easily. However, it is preferable to use a substitution method, if it is applicable, as the partial-fraction technique can be tedious.

WORKED Example 13

Find a, b and c if $ax(x-2)+bx(x+1)+c(x+1)(x-2)=2x-4$.

THINK	WRITE
1 Let $x=0$ so that c can be evaluated.	Let $x=0$, $-2c=-4$
2 Solve the equation for c.	$c=2$.
3 Let $x=2$ so that b can be evaluated.	Let $x=2$, $6b=0$
4 Solve the equation for b.	$b=0$.
5 Let $x=-1$ so that a can be evaluated.	Let $x=-1$, $3a=-6$
6 Solve the equation for a.	$a=-2$.
7 State the solution.	Therefore $a=-2$, $b=0$ and $c=2$.

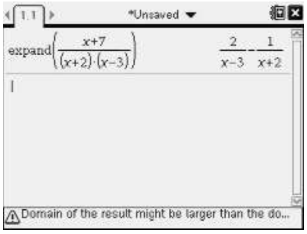
WORKED Example 14

For each of the following rational expressions:

i express as partial fractions **ii antidifferentiate the result.**

a $\dfrac{x+7}{(x+2)(x-3)}$ **b** $\dfrac{2x-3}{x^2-3x-4}$

Perform parts a and b manually, but use a CAS calculator to check the partial fractions for part a.

THINK	WRITE
a i ① Express the rational expression as two separate fractions with denominators $(x + 2)$ and $(x - 3)$ respectively.	**a i** $\dfrac{x+7}{(x+2)(x-3)} = \dfrac{a}{(x+2)} + \dfrac{b}{(x-3)}$
② Express the partial fractions with the original common denominator.	$= \dfrac{a(x-3)+b(x+2)}{(x+2)(x-3)}$
③ Equate the numerator on the left-hand side with the right-hand side.	So $x + 7 = a(x - 3) + b(x + 2)$
④ Let $x = -2$ so that a can be evaluated.	Let $x = -2$, and thus $5 = -5a$
⑤ Solve for a.	$a = -1.$
⑥ Let $x = 3$ so that b can be evaluated.	Let $x = 3$, and thus $10 = 5b$
⑦ Solve for b.	$b = 2.$
⑧ Rewrite the rational expression as partial fractions.	Therefore $\dfrac{x+7}{(x+2)(x-3)} = \dfrac{-1}{x+2} + \dfrac{2}{x-3}$
⑨ Use a CAS calculator to express **a** in partial fraction form as follows: Press: • Menu (menu) • 3: Algebra (3) • 3: Expand (3) Complete the entry line as: expand $\left(\dfrac{x+7}{(x+2)\cdot\ (x-3)}\right)$. Press Enter (enter).	 expand$\left(\dfrac{x+7}{(x+2)\cdot(x-3)}\right)$ $\dfrac{2}{x-3}-\dfrac{1}{x+2}$ Domain of the result might be larger than the do...
⑩ Write your answer.	$\dfrac{x+7}{(x+2)(x-3)} = \dfrac{2}{x-3} - \dfrac{3}{x+2}$
ii ① Express the integral in partial fraction form.	**ii** $\displaystyle\int \dfrac{x+7}{(x+2)(x-3)}\,dx$ $= \displaystyle\int\left(\dfrac{-1}{x+2} + \dfrac{2}{x-3}\right)dx$
② Antidifferentiate by rule.	$= -\log_e(x+2) + 2\log_e(x-3) + c \quad (x > 3)$
③ Simplify using log laws.	$= \log_e\left[\dfrac{(x-3)^2}{x+2}\right] + c$

THINK	**WRITE**
b i ① Factorise the denominator.	**b i** $\frac{2x-3}{x^2-3x-4} = \frac{2x-3}{(x-4)(x+1)}$
② Express the partial fractions with denominators $(x - 4)$ and $(x + 1)$ respectively.	$= \frac{a}{x-4} + \frac{b}{x+1}$
③ Express the right-hand side with the original common denominator.	$= \frac{a(x+1)+b(x-4)}{(x-4)(x+1)}$
④ Equate the numerators.	So $2x - 3 = a(x + 1) + b(x - 4)$
⑤ Let $x = 4$ to evaluate a.	Let $x = 4$, $5 = 5a$
⑥ Solve for a.	$a = 1$.
⑦ Let $x = -1$ to evaluate b.	Let $x = -1$, $-5 = -5b$
⑧ Solve for b.	$b = 1$.
⑨ Rewrite the rational expression as partial fractions.	Therefore $\frac{2x-3}{x^2-3x-4} = \frac{1}{x-4} + \frac{1}{x+1}$
ii ① Express the integral in its partial fraction form.	**ii** $\int \frac{2x-3}{x^2-3x-4}\,dx$ $= \int \left(\frac{1}{x-4} + \frac{1}{x+1}\right) dx$
② Antidifferentiate by rule.	$= \log_e(x-4) + \log_e(x+1) + c \quad (x > 4)$
③ Simplify using log laws.	$= \log_e[(x-4)(x+1)] + c \quad (x > 4)$ or $\log_e(x^2 - 3x - 4) + c \quad (x > 4)$

WORKED Example 15

Find the following integrals.

a $\int \frac{2}{1-x^2}\,dx$ **b** $\int \frac{x^2+6x-1}{(x+4)(x+1)}\,dx$

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Tutorial:
Worked example 15
int-0388

THINK	**WRITE**
a ① Factorise the denominator of the integrand.	**a** $\frac{2}{1-x^2} = \frac{2}{(1-x)(1+x)}$
② Express into partial fractions with denominators $(1 - x)$ and $(1 + x)$.	$= \frac{a}{1-x} + \frac{b}{1+x}$
③ Express the partial fractions with the original common denominator.	$= \frac{a(1+x)+b(1-x)}{1-x^2}$.
④ Equate the numerators.	So $2 = a(1 + x) + b(1 - x)$

Continued over page →

THINK	**WRITE**
⑤ Let $x = 1$ to find a.	Let $x = 1, \quad 2 = 2a$
⑥ Solve for a.	$a = 1.$
⑦ Let $x = -1$ to find b.	Let $x = -1, 2 = 2b$
⑧ Solve for b.	$b = 1.$
⑨ Express the integrand in its partial fraction form.	Therefore $\int \frac{2}{1 - x^2}\,dx$ $= \int \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right) dx$
⑩ Antidifferentiate by rule.	$= -\log_e(1 - x) + \log_e(1 + x) + c,$ $(-1 < x < 1)$
⑪ Simplify using log laws.	$= \log_e \left(\frac{1 + x}{1 - x} \right) + c \quad (-1 < x < 1)$
b ① Expand the denominator.	**b** $\frac{x^2 + 6x - 1}{(x + 4)(x + 1)} = \frac{x^2 + 6x - 1}{x^2 + 5x + 4}$
② The degree of the numerator is the same as the degree of the denominator and hence we need to rewrite the fraction.	$= \frac{x^2 + 5x + 4 + x - 5}{x^2 + 5x + 4}$
③ Break the fraction into two so that one of the resulting fractions cancels to a whole number.	$= \frac{x^2 + 5x + 4}{x^2 + 5x + 4} + \frac{x - 5}{x^2 + 5x + 4}$ Therefore $\frac{x^2 + 6x - 1}{(x + 4)(x + 1)} = 1 + \frac{x - 5}{(x + 4)(x + 1)}$
④ Express $\frac{x - 5}{(x + 4)(x + 1)}$ as partial fractions with denominators $(x + 4)$ and $(x + 1)$.	Now $\frac{x - 5}{(x + 4)(x + 1)} = \frac{a}{x + 4} + \frac{b}{x + 1}$
⑤ Rewrite the partial fractions with the original common denominator.	$= \frac{a(x + 4) + b(x + 1)}{(x + 4)(x + 1)}$
⑥ Equate the numerators.	and thus $x - 5 = a(x + 4) + b(x + 1).$
⑦ Let $x = -1$ to find a.	Let $x = -1, \quad -6 = 3a$
⑧ Solve for a.	$a = -2.$
⑨ Let $x = -4$ to find b.	Let $x = -4, \quad -9 = -3b$
⑩ Solve for b.	$b = 3.$
⑪ Express the original integrand in its partial fraction form.	Therefore, $\int \frac{x^2 + 6x - 1}{(x + 4)(x + 1)}\,dx$ $= \int \left(1 + \frac{-2}{x + 4} + \frac{3}{x + 1} \right) dx$
⑫ Antidifferentiate by rule.	$= x - 2\log_e(x + 4) + 3\log_e(x + 1) + c,$ $(x > -1).$

remember

Rational polynomials can be antidifferentiated by rewriting the expressions as partial fractions Two common partial fraction transformations are shown below.

Rational expression	Equivalent partial fraction
$\dfrac{f(x)}{(ax+b)(cx+d)}$ where $f(x)$ is a linear function	$\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)}$
$\dfrac{f(x)}{(ax+b)^2}$ where $f(x)$ is a linear function	$\dfrac{A}{(ax+b)^2}+\dfrac{B}{(ax+b)}$

EXERCISE 4E Antidifferentiation using partial fractions

WORKED Example 13

1 Find the values of a, b and c in the following identities.

- **a** $ax + b(x - 1) = 3x - 2$
- **b** $a(x + 2) + b(x - 3) = x - 8$
- **c** $a(x - 4) + b = 3x - 2$
- **d** $a(3x + 1) + b(x - 2) = 5x + 4$
- **e** $a(2 - 3x) + b(x + 5) = 9x + 11$
- **f** $a(x + 2) + bx = 2x - 10$
- **g** $a + b(x + 2) + c(x + 2)(x + 3) = x^2 + 4x - 2$
- **h** $a(x + 2)(x - 3) + bx(x - 3) + cx(x + 2) = 3x^2 - x + 6$

WORKED Example 14i

2 Express each of the following rational expressions as partial fractions.

- **a** $\dfrac{1}{(x+1)(x+2)}$
- **b** $\dfrac{12}{(x-2)(x+2)}$
- **c** $\dfrac{6x}{(x+3)(x-1)}$
- **d** $\dfrac{3x}{(x-2)(x+1)}$
- **e** $\dfrac{x+3}{(x+2)(x+3)}$
- **f** $\dfrac{x+20}{(x-4)(x+4)}$
- **g** $\dfrac{4x+5}{(x+2)^2}$
- **h** $\dfrac{5x-26}{(x-5)^2}$
- **i** $\dfrac{x+4}{x(x-2)}$
- **j** $\dfrac{7x-4}{(x-2)(x+3)}$
- **k** $\dfrac{8x-10}{(2x+1)(x-3)}$
- **l** $\dfrac{9x-11}{(3x-2)(x+1)}$
- **m** $\dfrac{11-3x}{(2-x)(x+3)}$
- **n** $\dfrac{12-2x}{(1-x)(3-x)}$

3 Find the antiderivative of each rational expression in question **2**.

WORKED Example 14ii

4 **multiple choice**

a If $\dfrac{5x+10}{24-2x-x^2} = \dfrac{a}{x+6} + \dfrac{b}{4-x}$, then:

A $a = 2, b = 3$ **B** $a = -2, b = -3$ **C** $a = 3, b = 2$

D $a = -2, b = 3$ **E** $a = 1, b = -1$

b Hence $\displaystyle\int \frac{5x+10}{24-2x-x^2}\,dx$ is equal to:

A $2\log_e(x+6) - 3\log_e(4-x) + c$ **B** $-2\log_e(x+6) - 3\log_e(4-x) + c$

C $3\log_e(x+6) + 2\log_e(4-x) + c$ **D** $3\log_e(x+6) - 2\log_e(4-x) + c$

E $\dfrac{\log_e(x+6)}{4-x}$

5 **multiple choice**

The antiderivative of $-\dfrac{10}{x^2+x-6}$ is equal to:

A $2\log_e(x+3) - \log_e(x-2) + c$ **B** $2\log_e\dfrac{x+1}{x-6} + c$

C $2\log_e\dfrac{x+3}{x-2} + c$ **D** $\log_e(x+3) - 2\log_e(x-2) + c$

E $\log_e(x+1) - 2\log_e(x-6) + c$

WORKED Example 15a

6 Antidifferentiate each of the following rational polynomials by first expressing them as partial fractions.

a $\dfrac{3x+10}{x^2+2x}$ **b** $\dfrac{5x-4}{x^2-x-2}$

c $\dfrac{x+3}{x^2+3x+2}$ **d** $\dfrac{6x-1}{x^2-5x-6}$

e $\dfrac{5x-7}{x^2-4x+3}$ **f** $\dfrac{x+16}{x^2+7x+6}$

g $\dfrac{7x+9}{x^2-9}$ **h** $\dfrac{7x+1}{x^2-1}$

i $\dfrac{5x}{2x^2-3x-2}$ **j** $\dfrac{16-2x}{3x^2+7x-6}$

k $\dfrac{x+4}{2x^2-5x+2}$ **l** $\dfrac{4}{4-x^2}$

m $\dfrac{3x-4}{16-x^2}$ **n** $\dfrac{x+13}{5+4x-x^2}$

WORKED Example 15b

7 By first simplifying the rational expression, find the antiderivative of each of the following expressions.

a $\dfrac{x-1}{x+5}$ **b** $\dfrac{x+3}{x-2}$

c $\dfrac{x^2-1}{x^2+3x}$ **d** $\dfrac{x^2+2x+4}{x^2-4x}$

e $\dfrac{x^2 - x}{(x+3)(x+1)}$ **f** $\dfrac{x^2 + x + 4}{x^2 - 2x - 3}$

g $\dfrac{x^2 + 3x - 2}{x^2 - 4}$ **h** $\dfrac{x^3 + 4x^2 - x}{(x+2)(x+1)}$

i $\dfrac{x^3 + 4x - 13}{x^2 - 4x - 5}$ **j** $\dfrac{2x^3 + x^2 - 5}{x^2 - 1}$

k $\dfrac{x^2 - x + 2}{x^2 + 2x + 1}$ **l** $\dfrac{2x^2 - 9x + 7}{x^2 - 6x + 9}$

8 Evaluate the following integrals in terms of x.

a $\displaystyle\int \frac{4 - x}{x(x+2)}\,dx$ **b** $\displaystyle\int \frac{9x + 8}{(x-3)(x+4)}\,dx$ **c** $\displaystyle\int \frac{5(x+1)}{x^2 - 25}\,dx$

d $\displaystyle\int \frac{x^2 + 3}{x^2 - 9}\,dx$ **e** $\displaystyle\int \frac{x^2 + 3x - 4}{(x-4)(x+2)}\,dx$ **f** $\displaystyle\int \frac{x^2 + 4x + 1}{x^2 + 6x - 7}\,dx$

g $\displaystyle\int \frac{x^3 + x^2 - 4x}{x^2 - 4x + 4}\,dx$ **h** $\displaystyle\int \frac{4x^2 + 6x - 4}{2x^2 - x - 6}\,dx$ **i** $\displaystyle\int \frac{x + 1}{x^2 + 4}\,dx$

j $\displaystyle\int \frac{4x - 2}{x^2 + 9}\,dx$

Challenge

k $\displaystyle\int \frac{5x^2 + 2x + 17}{(x-1)(x+2)(x-3)}\,dx$ **l** $\displaystyle\int \frac{x^2 + 18x + 5}{(x+1)(x-2)(x+3)}\,dx$

m $\displaystyle\int \frac{x^2 + 8x + 9}{(x-1)(x+2)^2}\,dx$ **n** $\displaystyle\int \frac{x^2 + 5x + 1}{(x^2+1)(2-x)}\,dx$

9 **a** If $f'(x) = \dfrac{6}{x^2 - 1}$ and $f(2) = 3\log_e 2$, find $f(x)$.

b State the domain of $f(x)$.

10 **a** Find $g(x)$ if $g'(x) = \dfrac{x^2 + 1}{x^2 - 2x - 3}$ and $g(4) = 4 - \log_e 5$.

b State the domain of $g(x)$.

Technique 5: Integration by parts

Consider the product rule for differentiation.

If $y = uv$ then $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Rearranging this equation gives a useful result for integration:

$$u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}$$

$$\int u\frac{dv}{dx}\,dx = \int \frac{d(uv)}{dx}\,dx - \int v\frac{du}{dx}\,dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

This result is called *integration by parts*.

WORKED Example 16

Find $\int x \sin x \, dx$.

THINK	WRITE
① If the integrand is a product, try integration by parts.	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
② Identify u and $\frac{dv}{dx}$.	Let $u = x$ and $\frac{dv}{dx} = \sin x$.
③ Calculate $\frac{du}{dx}$ and v.	So $\frac{du}{dx} = 1$ and $v = -\cos x$
④ Substitute into the formula.	$\int x \sin x\,dx = -x\cos x - \int(-\cos x)dx$
⑤ Simplify.	$= -x\cos x + \sin x + c$

At times the technique of integrating by parts may need to be repeated to obtain the desired result.

WORKED Example 17

Find $\int x^2 e^x \, dx$.

THINK	WRITE
① Write the formula for integration by parts.	$\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}\,dx$
② Identify u and $\frac{dv}{dx}$.	Let $u = x^2$ and $\frac{dv}{dx} = e^x$.
③ Calculate $\frac{du}{dx}$ and v.	So $\frac{du}{dx} = 2x$ and $v = e^x$
④ Substitute into the formula.	$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx.$
⑤ Repeat the process with the integrand $2xe^x$.	Let $u = 2x$ and $\frac{dv}{dx} = e^x$. So $\frac{du}{dx} = 2$ and $v = e^x$ $\int x^2e^x\,dx = x^2e^x - (2xe^x - \int 2e^x\,dx)$ $= x^2e^x - 2xe^x + 2e^x + c$

Sometimes some ingenuity is needed when applying this rule.

WORKED Example 18

Find $\int \log_e x \, dx$.

THINK	**WRITE**
① Use integration by parts.	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
② Identify u and $\frac{dv}{dx}$. Note that $\log_e x$ can be written as $\log_e x \cdot 1$.	Let $u = \log_e x$ and $\frac{dv}{dx} = 1$.
③ Calculate $\frac{du}{dx}$ and v.	So $\frac{du}{dx} = \frac{1}{x}$ and $v = x$
④ Substitute into the formula and simplify.	$\int \log_e x\,dx = x\log_e x - \int\left(x \cdot \frac{1}{x}\right)dx$ $= x\log_e x - x + c$

At times considerable ingenuity is required.

WORKED Example 19

Find $\int e^x \sin x \, dx$.

THINK	**WRITE**
① Use integration by parts.	$\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}\,dx.$
② Identify u and $\frac{dv}{dx}$.	Let $u = \sin x$ and $\frac{dv}{dx} = e^x$.
③ Calculate $\frac{du}{dx}$ and v.	So $\frac{du}{dx} = \cos x$ and $v = e^x$
④ Substitute into the formula.	$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$
⑤ Repeat the process with $\int e^x \cos x\,dx$.	Let $u = \cos x$ and $\frac{dv}{dx} = e^x$.
Identify u and $\frac{dv}{dx}$ and calculate $\frac{du}{dx}$ and v.	So $\frac{du}{dx} = -\sin x$ and $v = e^x$
⑥ Substitute into the formula to replace $\int e^x \cos x\,dx$ and simplify.	$\int e^x \sin x\,dx = e^x \sin x - [e^x \cos x - \int e^x (-\sin x)dx]$ $\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$
⑦ Group the $\int e^x \sin x\,dx$ terms together by adding $\int e^x \sin x\,dx$ to both sides of the equation.	$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x$
⑧ Divide both sides of the equation by 2 and include a constant in your final answer.	$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c$

remember

1. The formula for the technique of integration by parts is $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.
2. Choose u as a function that can be easily differentiated and $\frac{dv}{dx}$ as a function that can be easily integrated.

EXERCISE 4F Integration by parts

WORKED Example 16

1 Find each of the following using integration by parts.

a $\int x\cos x\,dx$ (Let $u = x$.) **b** $\int xe^x\,dx$ (Let $u = x$.)

c $\int x\log_e x\,dx$ (Let $u = \log_e x$.) **d** $\int x\sin 2x\,dx$ (Let $u = x$.)

WORKED Example 17

2 Find each of the following.

a $\int x^2\cos x\,dx$ **b** $\int x^2\sin x\,dx$

c $\int x^2e^{2x}\,dx$ **d** $\int x^2(x+1)^5\,dx$

WORKED Example 18

3 Find each of the following.

a $\int \log_e 2x\,dx$ **b** $\int \text{Sin}^{-1}x\,dx$

WORKED Example 19

4 Find each of the following.

a $\int e^x\cos x\,dx$ **b** $\int e^{2x}\cos x\,dx$ **c** $\int e^x\sin 2x\,dx$

5 Integrate the following expressions.

a $(x-1)\sin x$ **b** $(x+2)e^x$ **c** $x^3\log_e x$

d $(x+1)\log_e x$ **e** $x^2\sin^{-1}x$ **f** $(x+1)^2e^x$

g x^3e^x **h** $\frac{x}{e^x}$ **i** $x^n\log_e x$

6 a Show that $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$.

b Use this result to calculate $\int x^5e^x\,dx$.

7 a Prove that $\int \cos^n x\,dx = \sin x\cos^{n-1}x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx$.

b Use this result to calculate $\int \cos^3 x\,dx$.

8 Calculate $\int \sin^6\frac{x}{2}\,dx$.

Definite integrals

The quantity $\int f(x)\,dx$ is called the 'indefinite integral of the function $f(x)$'. However, $\int_a^b f(x)\,dx$ is called the 'definite integral of the function $f(x)$' and is evaluated using the result that:

$$\begin{aligned}\int_a^b f(x)\,dx &= [F(x)]_a^b \\ &= F(b) - F(a)\end{aligned}$$

where $F(x)$ is an antiderivative of $f(x)$.

The definite integral $\int_a^b f(x)\,dx$ can be found only if the integrand, $f(x)$, exists for all values of x in the interval $[a, b]$; that is, $a \le x \le b$.

WORKED Example 20

For each of the following integrals, state:

i the domain of the integrand **ii whether the integral exists.**

a $\displaystyle\int_{-2}^{2} \frac{1}{\sqrt{9-x^2}}\,dx$ **b** $\displaystyle\int_{0}^{4} \frac{2}{(x-1)(x+3)}\,dx$

THINK	WRITE
a i ① For the integrand to exist, $\sqrt{9-x^2}$ must be greater than 0.	**a i** The integrand exists if $\sqrt{9-x^2} > 0$. That is, $9 - x^2 > 0$.
② Solve the inequation for x.	$x^2 < 9$ $-3 < x < 3$
③ State the domain.	The domain is $(-3, 3)$.
ii The integral exists for all values of x between the terminals −2 and 2.	**ii** The integral exists.
b i ① The integrand does not exist for $x = -3$ and 1, as these values make the denominator equal to zero.	**b i** $x \neq -3, 1$
② State the domain.	Domain is $R\backslash\{-3, 1\}$.
ii The integral does not exist for all values of x between the terminals 0 and 4 (as 1 lies in the interval).	**ii** The integral does not exist.

When using substitution to evaluate definite integrals there is no need to return to an expression in terms of x providing the terminals are expressed in terms of u. In fact it is mathematically incorrect to show the integral in terms of u but with terminals in terms of x. Therefore when using a substitution, $u = f(x)$, the terminals should also be adjusted in terms of u.

WORKED Example 21

Use an appropriate substitution to express each of the following definite integrals in terms of u, with the terminals of the integral correctly adjusted.

a $\int_2^3 \frac{x}{x^2 - 1}\,dx$ **b** $\int_3^6 \frac{x}{\sqrt{x - 2}}\,dx$

THINK	WRITE
a ① Antidifferentiate the integrand by letting $u = x^2 - 1$ so the derivative method can be applied.	**a** Let $u = x^2 - 1$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 2x$
③ Express dx in terms of du.	or $dx = \frac{du}{2x}$
④ Adjust the terminals by finding u when $x = 2$ and $x = 3$.	When $x = 2$, $u = 2^2 - 1$ $= 3$ When $x = 3$, $u = 3^2 - 1$ $= 8$
⑤ Rewrite the integral.	Therefore the integral is $\int_3^8 \frac{x}{u} \cdot \frac{du}{2x}$
⑥ Simplify the integrand.	$= \int_3^8 \frac{1}{2u}\,du$
b ① Antidifferentiate the integrand by using the linear substitution $u = x - 2$.	**b** Let $u = x - 2$.
② Find $\frac{du}{dx}$.	$\frac{du}{dx} = 1$
③ Express dx in terms of du.	or $dx = du$
④ Express x in terms of u.	$x = u + 2$
⑤ Adjust the terminals by finding u when $x = 3$ and $x = 6$.	When $x = 3$, $u = 3 - 2$ $= 1$ When $x = 6$, $u = 6 - 2$ $= 4$
⑥ Rewrite the integral.	Therefore the integral is $\int_1^4 \frac{u + 2}{u^{\frac{1}{2}}}\,du$
⑦ Simplify the integrand.	$= \int_1^4 \left(u^{\frac{1}{2}} + 2u^{-\frac{1}{2}}\right) du$

WORKED Example 22

eBook*plus*

Tutorial:
Worked example 22
int-0389

Evaluate the following definite integrals manually. Also evaluate part a using a TI-Nspire CAS calculator, or a Casio *fx*-9860G AU.

a $\int_0^2 \frac{x-2}{x^2+5x+4}\,dx$ **b** $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin x}\,dx$

THINK	WRITE
a 1 Write the integral.	**a** $\int_0^2 \frac{x-2}{x^2+5x+4}\,dx$
2 Factorise the denominator of the integrand.	Consider: $\frac{x-2}{x^2+5x+4} = \frac{x-2}{(x+1)(x+4)}$
3 Express in partial fraction form with denominators $x + 1$ and $x + 4$.	$= \frac{a}{x+1} + \frac{b}{x+4}$
4 Express the partial fractions with the original common denominator.	$= \frac{a(x+4)+b(x+1)}{x^2+5x+4}$
5 Equate the numerators.	$x-2 = a(x+4) + b(x+1)$
6 Let $x = -1$ to find a.	Let $x = -1$, $-3 = 3a$ $a = -1$
7 Let $x = -4$ to find b.	Let $x = -4$, $-6 = -3b$ $b = 2$
8 Rewrite the integral in partial fraction form.	So $\int_0^2 \frac{x-2}{x^2+5x+4}\,dx$ $= \int_0^2 \frac{-1}{x+1} + \frac{2}{x+4}\,dx$
9 Antidifferentiate the integrand.	$= [-\log_e(x+1) + 2\log_e(x+4)]_0^2$
10 Evaluate the integral.	$= [-\log_e 3 + 2\log_e 6] - [-\log_e 1 + 2\log_e 4]$ $= -\log_e 3 + 2\log_e 6 - 2\log_e 4$
11 Simplify using log laws.	$= 2\log_e 1.5 - \log_e 3$ $= \log_e 2.25 - \log_e 3$ $= \log_e 0.75$ (or approx. -0.2877)

Continued over page →

THINK **WRITE**

Using the TI-Nspire CAS calculator

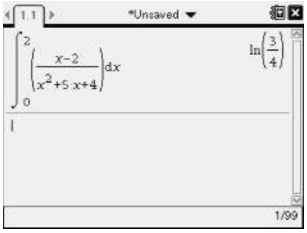
12 Press:

- Menu menu
- 4: Calculus 4
- 3: Integral 3

Insert the limits of integration using tab to move to the next entry point.
Complete the entry line as:

$$\int_0^2 \left(\frac{x-2}{x^2+5x+4}\right) \mathrm{d}x$$

Press Enter enter.



$\int_0^2 \left(\frac{x-2}{x^2+5\cdot x+4}\right) dx$ $\ln\left(\frac{3}{4}\right)$

13 Write your solution.

$$\int_0^2 \frac{x-2}{x^2+5x+4}\,\mathrm{d}x = \ln\left(\frac{3}{4}\right)$$

Using the Casio *fx*-9860G AU

12 Press:

- MENU
- 1 (RUN)
- OPTN
- F4 (CALC)
- F4 ($\int$dx)

Complete the entry line as:

$$\int((x-2) \mid (x^2+5x+4), 0, 2)$$

Press EXE.

```
∫((X-2)÷(X²+5X+4),0,2
)
                -0.2876820725
```

13 Write your solution.

$$\int_0^2 \frac{x-2}{x^2+5x+4}\,\mathrm{d}x = -0.2877$$

b **1** Write the integral.

b $\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin x}\,\mathrm{d}x$

2 Let $u = 1 + \sin x$ to antidifferentiate.

Let $u = 1 + \sin x$.

3 Find $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$

4 Express dx in terms of du.

$$\text{or } \mathrm{d}x = \frac{\mathrm{d}u}{\cos x}$$

THINK	WRITE
5 Change terminals by finding u when $x = 0$ and $x = \frac{\pi}{2}$.	When $x = 0$, $u = 1 + \sin 0$ $= 1$ When $x = \frac{\pi}{2}$, $u = 1 + \sin \frac{\pi}{2}$ $= 1 + 1$ $= 2$
6 Simplify the integrand.	So $\int_0^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin x}\, \mathrm{d}x$ $= \int_1^2 (\cos x) u^{\frac{1}{2}} \frac{\mathrm{d}u}{\cos x}$ $= \int_1^2 u^{\frac{1}{2}}\, \mathrm{d}u$
7 Antidifferentiate the integrand.	$= \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^2$
8 Evaluate the integral.	$= \frac{2}{3} \cdot 2^{\frac{3}{2}} - \frac{2}{3} \cdot 1^{\frac{3}{2}}$ $= \frac{4\sqrt{2}}{3} - \frac{2}{3}$ or $\frac{4\sqrt{2} - 2}{3}$

WORKED Example 23

By using the substitution $x = \sin \theta$, evaluate $\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, \mathrm{d}x$.

THINK	WRITE
1 Let $x = \sin \theta$.	Let $x = \sin \theta$.
2 Find $\frac{\mathrm{d}x}{\mathrm{d}\theta}$.	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos \theta$
3 Make $\mathrm{d}x$ the subject.	or $\mathrm{d}x = \cos \theta\, \mathrm{d}\theta$
4 Change the terminals by finding θ when $x = \frac{1}{2}$ and $x = 0$.	When $x = \frac{1}{2}$, $\frac{1}{2} = \sin \theta$ $\theta = \frac{\pi}{6}$ When $x = 0$, $0 = \sin \theta$ $\theta = 0$

Continued over page →

THINK	WRITE
⑤ Simplify the integrand. *Note:* With the substitution $x = \sin\theta$, $\sqrt{1-x^2}$ becomes $\sqrt{1-\sin^2\theta}$ which simplifies to $\cos\theta$.	$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$ $= \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2\theta}\cos\theta\,d\theta$ $= \int_0^{\frac{\pi}{6}} \cos\theta\cos\theta\,d\theta$ $= \int_0^{\frac{\pi}{6}} \cos^2\theta\,d\theta$
⑥ Replace $\cos^2\theta$ by its identity $\frac{1}{2}(1+\cos 2\theta)$.	$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1+\cos 2\theta)\,d\theta$ $= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1+\cos 2\theta)\,d\theta$
⑦ Antidifferentiate the integrand.	$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$
⑧ Evaluate the integral.	$= \frac{1}{2}\left[\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$ $= \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]$ $= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$

remember

1. $\int_a^b f(x)\,dx = [F(x)]_a^b$

   $= F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$.

2. The definite integral $\int_a^b f(x)\,dx$ can be found only if the integrand, $f(x)$, exists for all values of x in the interval $[a, b]$; that is, $a \le x \le b$.

EXERCISE 4G Definite integrals

WORKED Example 20

1 For each of the following definite integrals $\int_a^b f(x)\,dx$, state **i** the maximal domain of the integrand $f(x)$ and **ii** whether the integral exists.

a $\displaystyle\int_1^2 \frac{1}{9-x^2}\,dx$ **b** $\displaystyle\int_0^1 \frac{-1}{\sqrt{4-x^2}}\,dx$ **c** $\displaystyle\int_3^5 \frac{dx}{\sqrt{16-x^2}}$

d $\displaystyle\int_{-1}^2 \frac{dx}{1+x^2}$ **e** $\displaystyle\int_1^4 \frac{2}{x}\,dx$ **f** $\displaystyle\int_1^2 \frac{dx}{x(x+1)}$

g $\displaystyle\int_{-1}^1 \frac{4x+10}{x^2+5x+6}\,dx$ **h** $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$ **i** $\displaystyle\int_1^3 \frac{3x+2}{x^2-8x+12}\,dx$

j $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{4x^2+9}$ **k** $\displaystyle\int_{-1}^0 \frac{dx}{\sqrt{1-9x^2}}$ **l** $\displaystyle\int_{\frac{1}{2}}^2 (2x-1)^{\frac{3}{2}}\,dx$

m $\displaystyle\int_0^2 \left(x+\frac{1}{x-2}\right)dx$ **n** $\displaystyle\int_0^3 (e^x+e^{-x})^2\,dx$

2 Evaluate the integrals in question **1** provided that the integrand, $f(x)$, exists for all values within the domain of the integral.

3 **multiple choice**

The definite integral $\int_0^2 2x^2\sqrt{x^3+1}\,dx$ can be evaluated after substituting $u = x^3 + 1$.

a The integral will then be equal to:

A $\displaystyle\int_0^2 \frac{2\sqrt{u}}{3}\,du$ **B** $\displaystyle\int_0^9 \frac{3\sqrt{u+1}}{2}\,du$ **C** $\displaystyle\int_1^9 \frac{2\sqrt{u}}{3}\,du$

D $\displaystyle\int_0^2 2x^2\sqrt{u}\,du$ **E** $\displaystyle\int_0^7 \frac{4u^{\frac{3}{2}}}{9}\,du$

b The value of the integral is:

A $11\frac{5}{9}$ **B** $\frac{8\sqrt{2}}{9}$ **C** 9 **D** $12\frac{4}{9}$ **E** $10\sqrt{10}-1$

4 **multiple choice**

a $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin x}}\,dx$ can be evaluated by first making the substitution:

A $u = \sin x$ **B** $u = \cos x$ **C** $u = \sqrt{1+\sin x}$

D $u = \cot x$ **E** $u = 1 + \sin x$

b The integral will then be equal to:

A $\displaystyle\int_0^1 u^{-\frac{1}{2}}du$ **B** $\displaystyle\int_1^0 u^{\frac{1}{2}}\,du$ **C** $\displaystyle\int_0^2 u^{-\frac{1}{2}}\,du$ **D** $\displaystyle\int_1^2 u^{-\frac{1}{2}}\,du$ **E** $\displaystyle\int_0^1 \sqrt{1+u}\,du$

c When evaluated, the integral is equal to:

A 2 **B** $2\sqrt{2}-2$ **C** -2 **D** $2\sqrt{2}$ **E** $\frac{2}{3}$

WORKED Example 21

5 By choosing an appropriate substitution for u, express the following integrals in terms of u. (Do not forget to change the terminals.)

a $\int_0^2 x^2(2+x^3)^3\,dx$ **b** $\int_1^{\sqrt{2}} \frac{4x}{(x^2-3)^2}\,dx$ **c** $\int_0^1 x\sqrt{x^2+1}\,dx$

d $\int_2^4 (x-1)\sqrt{x^2-2x}\,dx$ **e** $\int_1^2 x\sqrt{x-1}\,dx$ **f** $\int_0^3 \frac{x^2}{\sqrt{x+1}}\,dx$

g $\int_1^3 \frac{\log_e x}{x}\,dx$ **h** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\, e^{\cos x}\,dx$ **i** $\int_0^1 x(1-x)^{10}\,dx$

j $\int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin x}\,dx$ **k** $\int_0^{\frac{\pi}{4}} \tan^3 x\sec^2 x\,dx$ **l** $\int_0^{\frac{\pi}{2}} x\sin x^2\,dx$

m $\int_{\frac{\pi}{2}}^{\pi} \cos^3 x\,dx$ **n** $\int_0^1 \frac{e^x}{\sqrt{e^x+1}}\,dx$

6 Evaluate each of the integrals in question **5**.

WORKED Example 22

7 Evaluate the following definite integrals.

a $\int_{-2}^{0} 4xe^{x^2}\,dx$ **b** $\int_0^1 (4x+7)\sqrt{2x^2+7x}\,dx$

c $\int_{-3}^{-2} 2\sqrt{x+3}\,dx$ **d** $\int_{-1}^{0} \frac{x+1}{\sqrt{1-x}}\,dx$

e $\int_0^{\frac{\pi}{3}} \sin x\cos^4 x\,dx$ **f** $\int_0^2 \frac{x+5}{x^2+4x+3}\,dx$

g $\int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$ **h** $\int_0^3 \frac{1}{(x-3)^2+1}\,dx$

i $\int_{-1}^{1} \frac{-1}{\sqrt{4-(x-1)^2}}\,dx$ **j** $\int_0^{\pi} \sin^3 x\cos^2 x\,dx$

k $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$ **l** $\int_5^6 \frac{3x-10}{x^2-7x+12}\,dx$

m $\int_{-1}^{1} \frac{2x^2}{x^2+1}\,dx$ **n** $\int_0^{\frac{\pi}{2}} 2\sin 2x\cos x\,dx$

o $\int_0^1 (2x+1)e^{x^2+x}\,dx$ **p** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (2+\tan^2 x)\,dx$

q $\int_2^5 \frac{x^2}{\sqrt{x-1}}\,dx$ **r** $\int_3^4 \frac{2x^3+x^2-2x-4}{x^2-4}\,dx$

WORKED Example 23

8 By substituting $x = \sin\theta$, evaluate $\int_0^1 \sqrt{1-x^2}\,dx$.

9 By substituting $x = 2\sin\theta$, evaluate $\int_0^{\sqrt{3}} \sqrt{4-x^2}\,dx$.

10 By making the substitution $x = \tan\,\theta$, evaluate $\displaystyle\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}$.

11 If $\displaystyle\int_0^a \frac{4}{1+x^2}\,\mathrm{d}x = \pi$, find the value of a.

12 If $\displaystyle\int_0^a \frac{4}{4-x^2}\,\mathrm{d}x = -\log_e 3$, find a.

13 If $\displaystyle\int_{-1}^a 3\sqrt{x+1}\,\mathrm{d}x = 6\sqrt{3}$, find a.

14 If $\displaystyle\int_{-a}^a \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \frac{\pi}{2}$, find a.

Volumes of solids of revolution

If part of a curve is rotated about the x-axis, or y-axis, a figure called a *solid of revolution* is formed. For example, a solid of revolution is obtained if the shaded region in figure 1 is rotated about the x-axis.

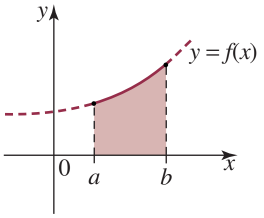


Figure 1

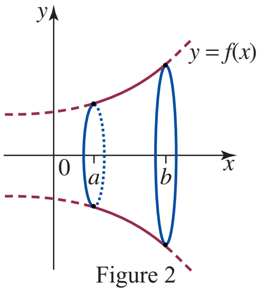


Figure 2

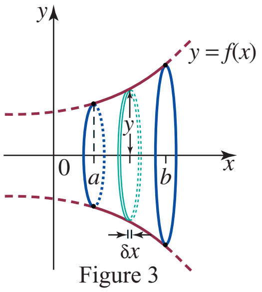


Figure 3

The solid generated (figure 2) is symmetrical about the x-axis and any vertical cross-section is circular, with a radius equal to the value of y at that point. For example, the radius at $x = a$ is $f(a)$.

Any thin vertical slice may be considered to be cylindrical, with radius y and height δx (figure 3).

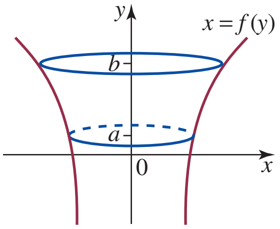
The volume of the solid of revolution generated between $x = a$ and $x = b$ is found by allowing the height of each cylinder, δx, to be as small as possible and adding the volumes of all of the cylinders formed between $x = a$ and $x = b$. That is, the volume of a typical strip is equal to $\pi y^2\,\delta x$.

Therefore the volume of the solid contained from $x = a$ to $x = b$ is the sum of all the infinitesimal volumes:

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\ \delta x$$

$$= \int_a^b \pi y^2\,\mathrm{d}x$$

The value of y must be expressed in terms of x so that the integral can be evaluated. From the figure above $y = f(x)$ and thus the volume of revolution of a curve $f(x)$ from $x = a$ to $x = b$ is $V = \pi\int_a^b [f(x)]^2\,dx$.



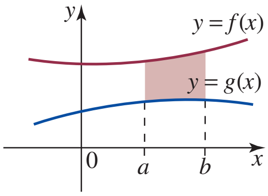
Similarly if a curve is rotated about the y-axis, the solid of revolution shown in the figure at right is produced.

The volume of the solid of revolution is likewise

$$V = \pi\int_a^b [f(y)]^2\,dy$$

For regions between two curves that are rotated about the x-axis:

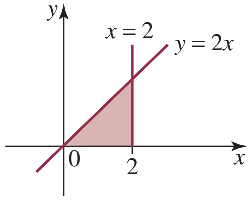
$$V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$$



WORKED Example 24

a **Sketch the graph of $y = 2x$ and show the region bounded by the graph, the x-axis and the line $x = 2$.**

b **Find the volume of the solid of revolution when the region is rotated about the x-axis.**

THINK	WRITE
a ① Sketch the graph. ② Shade the region required.	**a** 
b ① State the integral that gives the volume. (The volume generated is bounded by $x = 0$ and $x = 2$.)	**b** $V = \pi\int_0^2 (2x)^2\,dx$
② Simplify the integrand.	$= \pi\int_0^2 4x^2\,dx$
③ Antidifferentiate by rule.	$= \pi\left[\frac{4}{3}x^3\right]_0^2$
④ Evaluate the integral.	$= \pi\left[\frac{32}{3} - 0\right]$ $= \frac{32\pi}{3}$
⑤ State the volume.	The exact volume generated is $\frac{32\pi}{3}$ cubic units.

Graphics Calculator tip! Finding the volume of a solid of revolution

To find the volume of the solid of revolution formed in Worked example 24.

For the CASIO *fx*-9860G AU

1. Press:
   - MENU
   - 1 (RUN)
   - OPTN
   - F4 (CALC)
   - F4 ($\int dx$)

   Complete the entry line as:

   $\pi \cdot \int((2x)^2, 0, 2)$

```
π×∫((2X)²,0,2)
                33.51032165
```

2. Write the answer.

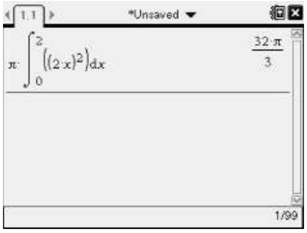
The volume of the solid of revolution formed when the region bounded by the graph of $y = 2x$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated about the x-axis is 33.5 square units.

For the TI-Nspire CAS

1. On a Calculator page, complete the entry line as:

   $\pi \cdot \int_0^2 (2x)^2 dx$

   Press ENTER.



$\pi \cdot \int_0^2 ((2\cdot x)^2)dx \quad \frac{32\cdot\pi}{3}$

2. Write the answer.

The volume of the solid of revolution formed when the region bounded by the graph of $y = 2x$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated about the x-axis is $\frac{32\pi}{3}$ square units.

WORKED Example 25

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Tutorial:
Worked example 25
int-0390

a **Sketch the region bounded by the curve $y = \log_e x$, the x-axis, the y-axis and the line $y = 2$.**

b **Calculate the volume of the solid generated if the region is rotated about the y-axis.**

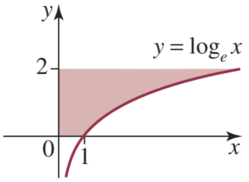
THINK

a ① Sketch the graph. (Use a graphics calculator if necessary.)

② Shade the region required.

WRITE

a



Continued over page

THINK	**WRITE**
b ① Write the rule $y = \log_e x$.	**b** $y = \log_e x$
② Take the exponent of both sides to get y as a function of x.	$e^y = e^{\log x}$
③ State the function.	$e^y = x$ or $x = e^y$
④ Express the volume in integral notation between $y = 0$ and $y = 2$.	So $V = \pi\int_0^2 (e^y)^2\,dy$
⑤ Simplify the integrand.	$= \pi\int_0^2 e^{2y}\,dy$
⑥ Antidifferentiate by rule.	$= \pi\left[\frac{1}{2}e^{2y}\right]_0^2$
⑦ Evaluate the integral.	$= \pi\left[\frac{1}{2}e^4 - \frac{1}{2}e^0\right]$ $= \frac{\pi}{2}(e^4 - 1)$
⑧ State the volume.	The volume is exactly $\frac{\pi}{2}(e^4 - 1)$ cubic units (or approximately 84.19 cubic units).

remember

1. To find the volume of revolution about the x-axis for the function $f(x)$ from $x = a$ to $x = b$, evaluate the integral:

$$V = \pi\int_a^b [f(x)]^2\,dx$$

2. To find the volume of revolution about the y-axis for the function $f(y)$ from $y = a$ to $y = b$, evaluate the integral:

$$V = \pi\int_a^b [f(y)]^2\,dy$$

3. To find the volume of revolution about the x-axis for the region between $f(x)$ and $g(x)$ where $f(x) \geq g(x)$ from $x = a$ to $x = b$, evaluate the integral:

$$V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$$

EXERCISE 4H Volumes of solids of revolution

Give exact answers where possible; otherwise use an appropriate number of decimal places when giving approximate answers. (Use a graphics calculator to check any graphing.)

WORKED Example 24

1 a Sketch the graph of the region bounded by the x-axis, the curve $y = 3x$ and the line $x = 2$.

b Calculate the volume generated by rotating this region about the x-axis.

c Verify this result by using the standard volume formula for the solid generated.

2 The region bounded by the graph of $y = \sqrt{16 - x^2}$ and the x-axis is rotated about the x-axis.

a Calculate the volume of the solid of revolution generated.

b Verify this answer using the standard volume formula.

eBook plus

Interactivity:
Volumes of solids of revolution
int-0347

WORKED Example 25

3 a Sketch the region bounded by the curve $y = \sqrt{x - 1}$, the y-axis and the lines $y = 0$ and $y = 2$.

b Calculate the volume generated when this region is rotated about the y-axis.

4 Find the volume generated when the area bounded by $y = x^2 - 1$ and the x-axis is rotated about:

a the x-axis

b the y-axis.

5 For the regions bounded by the x-axis, the following curves, and the given lines:

i sketch a graph shading the region

ii find the volume generated when the region is rotated about the x-axis.

a $y = x + 1$; $x = 0$ and $x = 2$

b $y = \sqrt{x}$; $x = 1$ and $x = 4$

c $y = x^2$; $x = 0$ and $x = 2$

d $y^2 = 2x + 1$; $x = 0$ and $x = 3$

e $x^2 + y^2 = 4$; $x = -1$ and $x = 1$

f $y = \dfrac{2}{x}$; $x = 1$ and $x = 3$

g $y = \cos x$; $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$

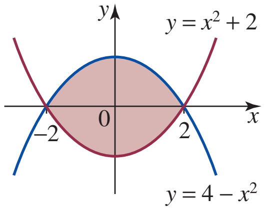
h $y = e^x + 1$; $x = -2$ and $x = -1$

6 For each region defined in question **5** (**a** to **f** only) find the volume generated by rotating it about the y-axis.

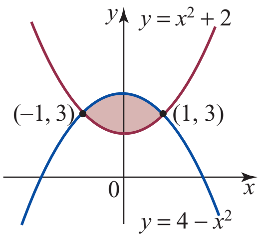
7 **multiple choice**

a The region bounded by the curves $y = x^2 + 2$ and $y = 4 - x^2$ is represented by the graph:

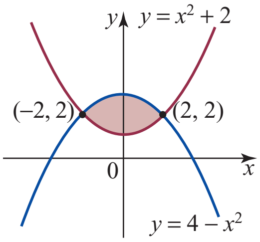
A



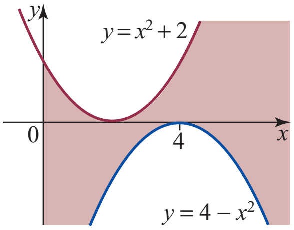
B



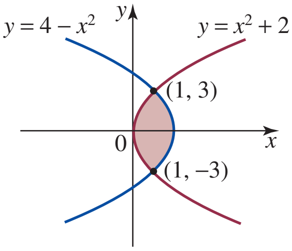
C



D



E



b The volume generated when the region is rotated about the *x*-axis is equal to:

A $\pi\int_0^2 (2 - 2x^2)^2\, dx$

B $\pi\int_{-1}^1 (2 - 2y)^2\, dy$

C $\pi\int_0^1 (2 - 2x^2)^2\, dx$

D $\pi\int_{-1}^1 (12 - 12x^2)\, dx$

E $\pi\int_{-1}^1 (6 - 2x^2)^2\, dx$

c The volume generated when the region is rotated about the *y*-axis is equal to:

A $\pi\int_3^4 (4 - y)\, dy + \pi\int_2^3 (y - 2)\, dy$

B $\pi\int_3^4 (y - 2)\, dy + \pi\int_2^3 (4 - y)\, dy$

C $\pi\int_2^4 (2 - 2y)\, dy$

D $\pi\int_2^4 (2y - 2)\, dy$

E $\pi\int_2^4 (2 - 2x^2)\, dx$

8 Find the volume generated when the region bounded by the curves $y = x^2$ and $y = -x$ is rotated about:

a the *x*-axis

b the *y*-axis.

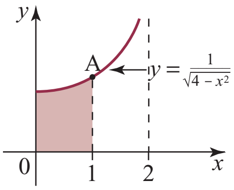
9 Find the volume generated when the area bounded by the curve $y = \sec x$, the line $x = \frac{\pi}{4}$ and the *x*- and *y*-axes is rotated about the *x*-axis.

10 Find the volume generated by rotating the area bounded by $y = e^{2x}$, the *y*-axis and the line $y = 2$ about the *x*-axis.

11 The area bounded by the curve $y = \text{Tan}^{-1}x$, the *x*-axis and the line $x = 1$ is rotated about the *y*-axis. Find the volume of the solid generated.

12 A model for a container is formed by rotating the area under the curve of $y = 2 - \frac{x^2}{6}$ between $x = -1$ and $x = 1$ about the *x*-axis. Find the volume of the container.

13 For the graph shown at right:



a find the coordinate of A

b find the volume generated when the shaded region is rotated about the x-axis

c find the volume generated when the shaded region is rotated about the y-axis.

14 What is the volume generated by rotating the ellipse with equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$ about:

a the x-axis?

b the y-axis?

15 Find the volume generated when the region bounded by $y = x^2$ and $y = \sqrt{8x}$ is rotated about:

a the x-axis

b the y-axis.

16 Find the volume generated by the rotation of the area bounded by the curves $y = x^3$ and $y = x^2$ about:

a the x-axis

b the y-axis.

Approximation using Simpson's rule

When we are calculating definite integrals or areas that involve integrands which cannot be antidifferentiated using techniques discussed in this chapter, an approximation method can be used.

Consider the following situation. The next time that you take a drive on a large highway observe how, as often as not, the road does not go over or around a hill but straight through it.

When the M1 highway from Brisbane to the Gold Coast was constructed, for months, all traffic on the road was halted at various times during the day. This was to allow blasting at Nerang so that a huge cutting could be constructed. Each day, after tonnes of rock had been blasted loose, earth moving equipment would remove the debris and the site would again be prepared for another blasting.

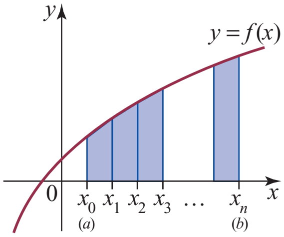
Engineers who work on such projects need to be able to estimate the time needed to clear the cutting. Numerous factors, such as the type of rock and a suitable site to relocate the debris, would influence the length of time needed for the project. Clearly though, the key element is the volume of rock to be removed from the cutting.

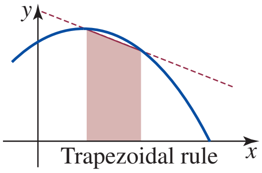
In this section, building on techniques developed in Mathematics B, we shall develop estimates for the area enclosed by a curve. Techniques of this sort can be applied to problems such as estimating the volume of rock in a road cutting.

From Mathematics B, you will recall that the area enclosed by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$ can be approximated by the trapezoidal rule.

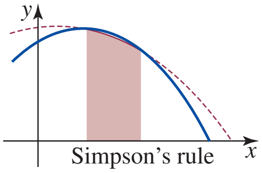
The idea behind this method is that the curve between two adjacent points is approximated by a straight line, and the area of the resulting figure is obtained. Each of the individual areas is added to give an estimate for the area under the curve.

It is obvious that a closer estimate could be achieved by approximating the curve $y = f(x)$ between two points, using a parabola. Simpson's rule is a method that estimates the area bounded by a curve, by approximating the curve with a parabola.





Trapezoidal rule



Simpson's rule

Simpson's rule

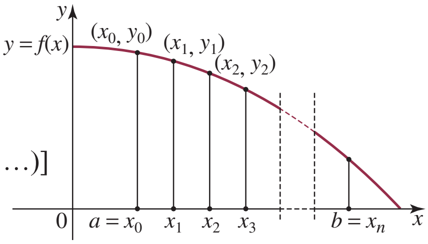
Recall that the area bounded by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.

Simpson's rule gives the estimate of this area by dividing the interval on the x-axis from a (or x_0) to b (or x_n) into an *even* number of intervals, n.

$$\text{Area} \approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

where w is the width of an interval given by

$$w = \frac{b-a}{n}.$$

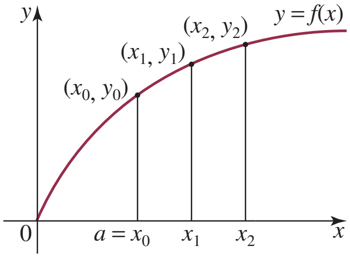


The derivation of this rule is an interesting exercise in algebra.

The interval from $x = a$ to $x = b$ has been divided into n equal segments of width, w, where:

$$w = \frac{b-a}{n}.$$

Only the first two strips are shown here.



A parabola is drawn through the points with coordinates (x_0, y_0), (x_1, y_1), (x_2, y_2). Let us suppose that this curve has the equation $y = Ax^2 + Bx + C$; then we can calculate the area under the curve between x_0 and x_2.

$$\begin{aligned}\text{Area} &= \int_{x_0}^{x_2} (Ax^2 + Bx + C)\mathrm{d}x \\ &= \left[\frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx\right]_{x_0}^{x_2} \\ &= (\tfrac{1}{3}Ax_2^{\ 3} + \tfrac{1}{2}Bx_2^{\ 2} + Cx_2) - (\tfrac{1}{3}Ax_0^{\ 3} + \tfrac{1}{2}Bx_0^{\ 2} + Cx_0) \\ &= \tfrac{1}{3}A(x_2^{\ 3} - x_0^{\ 3}) + \tfrac{1}{2}B(x_2^{\ 2} - x_0^{\ 2}) + C(x_2 - x_0) \\ &= \tfrac{1}{6}[2A(x_2^{\ 3} - x_0^{\ 3}) + 3B(x_2^{\ 2} - x_0^{\ 2}) + 6C(x_2 - x_0)]\end{aligned}$$

Now, recognising that:

$$x_2^{\ 3} - x_0^{\ 3} = (x_2 - x_0)(x_2^{\ 2} + x_0x_2 + x_0^{\ 2})$$

$$x_2^{\ 2} - x_0^{\ 2} = (x_2 - x_0)(x_2 + x_0)$$

and $$x_2 - x_0 = 2w$$

we see that the area becomes:

$$\begin{aligned}\text{Area} &= \frac{1}{6}[2A(2w)(x_2^{\ 2} + x_2x_0 + x_0^{\ 2}) + 3B(2w)(x_2 + x_0) + 6C(2w)] \\ &= \frac{w}{3}[2A(x_2^{\ 2} + x_2x_0 + x_0^{\ 2}) + 3B(x_2 + x_0) + 6C] \\ &= \frac{w}{3}[Ax_2^{\ 2} + Bx_2 + C + Ax_0^{\ 2} + Bx_0 + C + A(x_2^{\ 2} + 2x_2x_0 + x_0^{\ 2}) + 2B(x_2 + x_0) + 4C] \\ &= \frac{w}{3}[y_2 + y_0 + A(x_2 + x_0)^2 + 2B(x_2 + x_0) + 4C].\end{aligned}$$

Now x_1 is halfway between x_0 and x_2. That is, $x_1 = \dfrac{x_2 + x_0}{2}$ or $x_2 + x_0 = 2x_1$.

Thus the area can be written as:

$$\begin{aligned}\text{Area} &= \frac{w}{3}[y_2 + y_0 + A(2x_1)^2 + 2B(2x_1) + 4C] \\ &= \frac{w}{3}[y_2 + y_0 + 4(Ax_1^{\ 2} + Bx_1 + C)] \\ &= \frac{w}{3}(y_2 + y_0 + 4y_1)\end{aligned}$$

In a similar way, we can show that the area for the next two strips, that is, the strips between x_2 and x_4 will be:

$$\frac{w}{3}(y_4 + y_2 + 4y_3)$$

In total, the area of all strips is:

$$\begin{aligned}\text{Area} &= \frac{w}{3}(y_0 + y_2 + 4y_1) + \frac{w}{3}(y_2 + y_4 + 4y_3) + \ldots + \frac{w}{3}(y_{n-2} + y_n + 4y_{n-1}) \\ &= \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)].\end{aligned}$$

WORKED Example 26

Use Simpson's rule with four strips to approximate the value of the definite integral $\int_0^{\frac{\pi}{2}} \sin\sqrt{x}\, dx$.

THINK

WRITE

1. Calculate *w*, the width of each strip.

$$w = \frac{b-a}{n} \text{ where } a = 0,\ b = \frac{\pi}{2} \text{ and } n = 4$$

$$\text{so } w = \frac{\frac{\pi}{2} - 0}{4}$$

$$= \frac{\pi}{8}$$

2. Construct a table giving values of *x* and *y*.

n	x_n	y_n
0	0	0
1	$\frac{\pi}{8}$	0.586
2	$\frac{2\pi}{8}$	0.775
3	$\frac{3\pi}{8}$	0.884
4	$\frac{4\pi}{8}$	0.950

3. State Simpson's rule.

By Simpson's rule,

$$\text{Area} \approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

4. Write the formula to approximate the value of the required definite integral.

$$\text{So } \int_0^{\frac{\pi}{2}} \sin\sqrt{x}\, dx \approx \frac{w}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)]$$

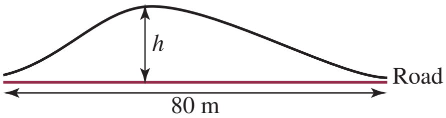
5. Substitute into this formula.

$$= \frac{\pi}{24}[0 + 0.950 + 4(0.586 + 0.884) + 2(0.775)]$$

$$= 1.097$$

WORKED Example 27

A road is to be built, cutting through a hillside as shown.



The width of the cutting is 80 m and it has a depth of 50 m. A measurement of the height of the cutting, *h*, is taken every 10 metres.

The results are recorded in the table below.

Measurement	Horizontal distance	Height
1	0	0
2	10	2
3	20	4.5
4	30	6
5	40	5
6	50	4
7	60	3
8	70	2.5
9	80	0

Calculate the volume, in cubic metres, of rock that is to be removed to build the cutting.

THINK	WRITE
① Consider the shape of the cutting to be a prism, and state the appropriate formula for volume.	Volume = area of cross-section · width
② Estimate the area of the cross-section; use Simpson's rule.	For the area of the cross-section: $\text{Area} \approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$
③ Calculate the width, *w*, of each strip by identifying the number of 'strips', *n*, and values for *a* and *b*.	$w = \frac{b - a}{n}$ where $n = 8$, $a = 0$ and $b = 80$ m so $w = \frac{80 - 0}{8}$ $= 10$ m

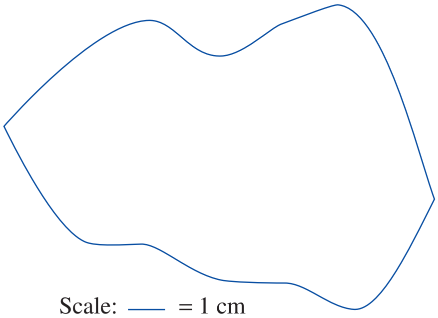
Continued over page →

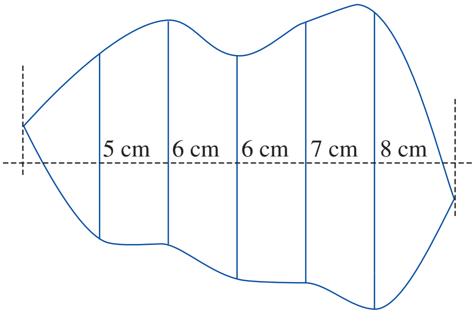
THINK	WRITE
4 Substitute into Simpson's rule to calculate the approximate area of the cross-section. Note that care needs to be taken with the numbering of the strips. Measurement 1 gives x_0 and y_0.	Area of cross-section $\approx \frac{w}{3}[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$ $= \frac{10}{3}[0 + 0 + 4(2 + 6 + 4 + 2.5) + 2(4.5 + 5 + 3)]$ $= 276.7 \text{ m}^2$
5 Substitute the estimate for the area into the volume formula.	Volume = $276.7 \cdot\ 50$ $= 13\ 835 \text{ m}^3$
6 Answer the question.	The volume of rock to be removed to build the cutting is 13 835 m^3.

Although Simpson's rule has been demonstrated to work when a curve is bounded by straight lines, it is easy to show that it can be applied to any convex figure.

WORKED Example 28

Use Simpson's rule with six strips to estimate the area of the following figure.


Scale: —— = 1 cm

THINK	WRITE/DRAW
1 Draw a horizontal line through the figure and take measurements at six equal intervals along this line.	

THINK | **WRITE/DRAW**

2 Use these measurements to complete a table of values for y.

n	y_n
0	0
1	5
2	6
3	6
4	7
5	8
6	0

3 State the width, w, of each of the six strips.

$w = 2$ cm

4 Calculate an estimate of the area using Simpson's rule.

$$\text{Area} \approx \frac{w}{3}[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$
$$= \tfrac{2}{3}[0 + 0 + 4(5 + 6 + 8) + 2(6 + 7)]$$
$$= 68$$

5 Answer the question.

The area of the figure is approximately 68 cm^2.

remember

1. Simpson's rule is an approximation method for calculating definite integrals or areas.
2. If the interval from $x = a$ to $x = b$ is divided into an even number of strips, n, of width w then

$$\text{Area} \approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

where y_0 and y_n are the lengths of the strips at the end points of the interval and $w = \dfrac{b - a}{n}$.

EXERCISE 4I

Approximation using Simpson's rule

WORKED Example 26

1 Use Simpson's rule to estimate the value of the definite integral $\int_0^{\frac{\pi}{2}} \sin x^2 \, dx$ using:

a 4 strips **b** 6 strips.

2 Use Simpson's rule to estimate the value of the definite integral $\int_0^1 2^{-x^2} \, dx$ using:

a 4 strips **b** 6 strips.

3 Use Simpson's rule to estimate the value of the definite integral $\int_0^2 \frac{1}{1+x^2}\,dx$ using:

a 4 strips **b** 6 strips.

4 **a** Find the exact value of the definite integral of $\int_0^2 \frac{1}{1+x^2}\,dx$ using an integration technique.

b Compare your answer to that obtained in question **3**.

5 **a** Find the area of a semicircle, of a circle of radius 3, represented by $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ using:

- **i** an exact method
- **ii** Simpson's rule with six strips.

b Compare the values obtained for the area using these two methods.

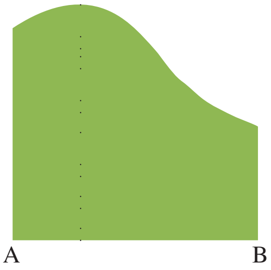
WORKED Example 27

6 A road is to be cut through a hill and engineers plan to calculate the volume of rock to be removed by finding the area of cross-section of the cut, and multiplying this by the width. The cut is 60 metres wide and 35 metres deep. Measurements are taken every ten metres along the cross-section. These results appear in the table below.

Measurement	Height of cross-section (m)
1	1
2	1.5
3	3
4	4
5	5.5
6	3
7	2

Use Simpson's rule to estimate the volume of rock to be removed to make this cutting.

7 Lot 22, Paradise Place, is pictured at right. The distance from A to B is 30 metres. Every 5 metres, a surveyor measures the distance from AB to the opposite boundary. The results are recorded below.

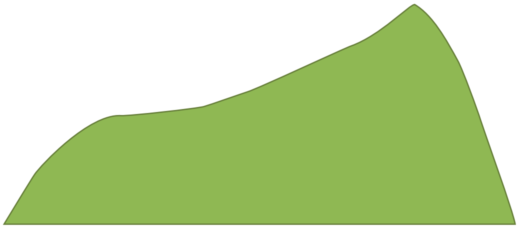


Distance from A (m)	0	5	10	15	20	25	30
Distance to boundary (m)	28	33	30	27	23	21	18

Use these measurements to approximate the area of the block.

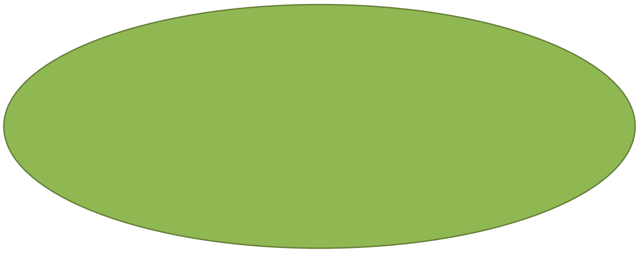
WORKED Example 28

8 Use Simpson's rule with four strips to approximate the area of the figure shown.



Scale: ——— = 1 cm

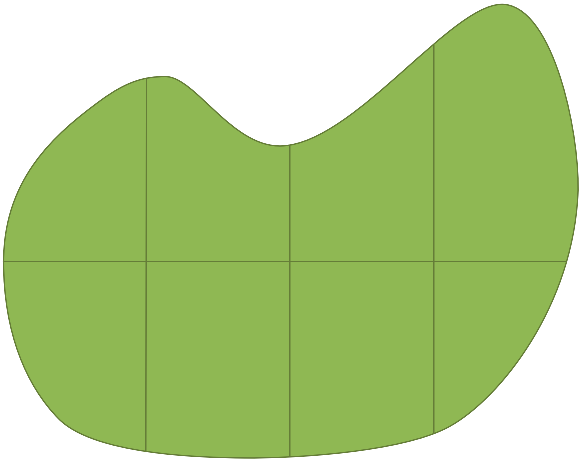
9 **a** Use Simpson's rule with six strips to approximate the area of the following figure.



Scale: ——— = 1 cm

b The formula for computing the area of an ellipse is πab where a is half the length and b is half the width of the ellipse. Does the result in part **a** agree with this formula?

10 A scale drawing of Lake Waipanini is shown below.



Scale: ——— = 10 m

Use Simpson's rule with four strips to estimate the area of the lake.

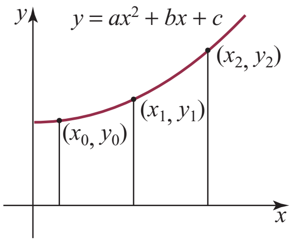
11 The area bounded by the curve $y = x^2$ between $x = a$ and $x = b$ and the x-axis is given by $\int_a^b x^2\,dx$.

Because Simpson's rule uses a parabola to approximate the curve, this method should give an exact value for $\int_a^b x^2\,dx$.

Show that if $\int_a^b x^2\,dx$ is approximated by Simpson's rule using two strips then it gives the exact value.

12 If $x_2 - x_1 = x_1 - x_0 = w$, show that

$$\int_{x_0}^{x_2} (ax^2 + bx + c)\,dx = \frac{w}{3}(y_2 + y_0 + 4y_1).$$



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Tutorial: WorkSHEET 4.2

Simpson's rule and cubic functions

Simpson's rule uses a series of parabolas to estimate the area bounded by a curve. As a result, it is clear that it will provide an exact result when used to calculate the area bounded by the graph of a quadratic function. A result that is not as clear (and, in fact, is quite unexpected) is that Simpson's rule will provide an exact value for the area bounded by the graph of a cubic function.

Show that:

$$\int_a^b x^3\,dx = \frac{b-a}{4}(b+a)(b^2+a^2)$$

Estimate the value of $\int_a^b x^3\,dx$ using Simpson's rule with two strips.

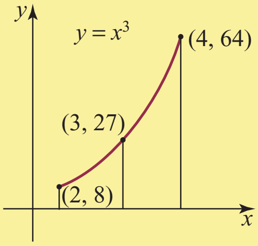
Thus, deduce that Simpson's rule produces an exact value for the area bounded by $y = x^3$.

Let us try to see what is happening here.

Consider the integral $\int_2^4 x^3\,dx$.

Simpson's rule, with two strips to estimate this area, uses a parabola that passes through the three points (2, 8), (3, 27) and (4, 64).

Find the parabola that passes through these points. (You should get $y = 9x^2 - 26x + 24$. One method for finding this equation is to use a quadratic regression on your graphics calculator.)



On your graphics calculator, graph both $y = x^3$ and $y = 9x^2 - 26x + 24$. Are they identical? Can you see why they produce the same area?

summary

Approximation using the derivative

- The formula for approximating using small changes is $f(x_0 + h) \approx f(x_0) + hf'(x_0)$.

Common antiderivatives

- The table below lists common antiderivatives.

$f(x)$	$F(x)$
Ax^n	$\frac{ax^{n+1}}{n+1} + c$
$\frac{1}{x}$	$\log_e kx + c$
e^{kx}	$\frac{e^{kx}}{k} + c$
$\sin kx$	$\frac{-\cos kx}{k} + c$
$\cos kx$	$\frac{\sin kx}{k} + c$
$\sec^2 kx$	$\frac{\tan kx}{k} + c$
$\frac{1}{\sqrt{a^2 - x^2}}, x \in (-a, a)$	$\text{Sin}^{-1}\frac{x}{a} + c$
$\frac{-1}{\sqrt{a^2 - x^2}}, x \in (-a, a)$	$\text{Cos}^{-1}\frac{x}{a} + c$
$\frac{a}{a^2 + x^2}$	$\text{Tan}^{-1}\frac{x}{a} + c$

Substitution where the derivative is present in the integrand

- $\int f'(x)\,[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{(n+1)} + c$
- $\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + c$

Linear substitution

The integral $\int f(x)\,[g(x)]^n\,dx, n \neq 0$ may be successfully antidifferentiated using the substitution $u = g(x)$, provided that $g(x)$ is linear. The function $f(x)$ must be written in terms of y also.

Useful trigonometric identities

- Trigonometric identities are used to integrate even and odd powered trigonometric functions:

$$\sin^2 ax = \tfrac{1}{2}(1 - \cos 2ax)$$

$$\cos^2 ax = \tfrac{1}{2}(1 + \cos 2ax)$$

$$\sin ax \cos ax = \tfrac{1}{2}\sin 2ax$$

Antidifferentiation using partial fractions

Many rational expressions can be antidifferentiated by transforming the expressions into partial fractions. Two common types are shown below.

Rational expression	Equivalent partial fraction
$\dfrac{f(x)}{(ax+b)(cx+d)}$ where $f(x)$ is a linear function	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}$
$\dfrac{f(x)}{(ax+b)^2}$ where $f(x)$ is a linear function	$\dfrac{A}{(ax+b)^2}+\dfrac{B}{ax+b}$

Integration by parts

- $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
- Choose u as a function that can be easily differentiated and $\dfrac{dv}{dx}$ as a function that can be easily integrated.

Definite integrals

- $$\int_a^b f(x)\,dx = [F(x)]_a^b$$
  $$= F(b) - F(a), \text{ where } F(x) \text{ is an antiderivative of } f(x).$$
- The definite integral $\displaystyle\int_a^b f(x)\,dx$ can be found only if the integrand $f(x)$ exists for all values of x in the interval $[a, b]$; that is, $a \le x \le b$.

Volumes of solids of revolution

- About x-axis: $\displaystyle V = \pi\int_a^b [f(x)]^2\,dx$
- About y-axis: $\displaystyle V = \pi\int_a^b [f(y)]^2\,dy$
- Between two functions $f(x)$ and $g(x)$ where $f(x) \ge g(x)$:

  $$V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$$

Approximation using Simpson’s rule

- Simpson’s rule is an approximation method for calculating definite integrals or areas.
- If the interval from $x = a$ to $x = b$ is divided into an even number of strips, n, of width, w, then:

  $$\text{Area} \approx \frac{w}{3}[y_0 + y_n + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots)]$$

  where y_0 and y_n are the lengths of the strips at the end points of the interval and $w = \dfrac{b-a}{n}$.

CHAPTER review

1 **a** Find an approximation for $f(x) = (2 + x)^4$ for small values of x. **4A**

b Use this approximation to give an approximate value of 2.03^4.

c What is the percentage error in the approximation?

2 Use the method of approximation using the derivative to estimate the value of $\sqrt{50}$. **4A**

3 Find an approximate value for $\sqrt[3]{67}$. **4A**

4 **multiple choice** **4B**

The expression $\int (x-1)(x^2-2x)^5 \, dx$ is equal to:

A $\int u^5 \, du$ **B** $\frac{1}{2}\int u^5 \, du$ **C** $2\int u^5 \, du$ **D** $\int 5u^4 \, du$ **E** $\int u^4 \, du$

5 Find an antiderivative of $\dfrac{\sin x}{\cos^3 x}$. **4B**

6 **multiple choice** **4B**

The expression $\int 6 \sec^2 3x \tan^4 3x \, dx$ is equal to:

A $2\int u^4 \, du$ **B** $\int u^4 \, du$ **C** $\frac{1}{2}\int u^4 \, du$ **D** $\int u^2 \, du$ **E** $2\int u^2 \, du$

7 Find the antiderivative of: **4B**

a $(\cos x)\, e^{\sin x}$ **b** $\dfrac{(\log_e x)^2}{x}$.

8 Find the antiderivative of $x(x + 2)^{10}$. **4C**

9 Find $\int x\sqrt{2-x} \, dx$. **4C**

10 **multiple choice** **4C**

Using an appropriate substitution, $\int e^{2x}\sqrt{e^x - 1} \, dx$ is equal to:

A $\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$ **B** $\int \left(2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$ **C** $\int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

D $\int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}}\right) du$ **E** $\int \left(u^{\frac{5}{2}} + u^{\frac{1}{2}}\right) du$

4C

11 Find the indefinite integral $\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x$.

4D

12 **multiple choice**

Using an appropriate substitution, $\int \cos^3 x \sin^2 x\,\mathrm{d}x$ is equal to:

A $\int (u^4 - u^2)\,\mathrm{d}u$ **B** $\int u^2 \cos x\,\mathrm{d}x$ **C** $\int (u^5 - u^3)\,\mathrm{d}u$

D $\int (u^3 - u^5)\,\mathrm{d}u$ **E** $\int (u^2 - u^4)\,\mathrm{d}x$

4D

13 **multiple choice**

If $f'(x) = 4\sin^2 x$ and $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, then $f(x)$ is equal to:

A $4\cos^2 x + \frac{\pi}{2} - 2$ **B** $2x - 1 + \sin 2x$ **C** $2x + \cos 2x$

D $2x - \cos 2x$ **E** $2x - \sin 2x$

4D

14 **multiple choice**

Using the appropriate substitution, $\int \sin^5 x\,\mathrm{d}x$ is equal to:

A $\int (u^2 - u^4)\,\mathrm{d}u$ **B** $\int (u^4 - 2u^2)\,\mathrm{d}u$ **C** $\int (2u^2 - 1 - u^4)\,\mathrm{d}u$

D $\int (-u^4 - 1)\,\mathrm{d}u$ **E** $\int (u^4 - u^2 + 1)\,\mathrm{d}u$

4D

15 **multiple choice**

The expression $\int (2 + \tan^2 x)\,\mathrm{d}x$ is equal to:

A $x + \sec^2 x + c$ **B** $2x + \sec^2 x + c$ **C** $\tan x + c$

D $x + \tan x + c$ **E** $x\tan x + c$

4D

16 Find:

a $\int \cos^2 2x\,\mathrm{d}x$ **b** $\int \sin^2\frac{x}{4}\cos^2\frac{x}{4}\,\mathrm{d}x$.

4D

17 Find $f(x)$ if $f'(x) = \sin 2x \cos x$ and $f(\pi) = 1$.

4D

18 If $f'(x) = 2\sqrt{1 - x^2}$ and $f(0) = -3$, find $f(x)$.
(*Hint:* Use the substitution $x = \sin\theta$ to antidifferentiate.)

19 **multiple choice** 4E

Given that $\dfrac{1}{x^2-9x+20} = \dfrac{1}{x-5} - \dfrac{1}{x-4}$, $x > 5$, an antiderivative of $\dfrac{1}{x^2-9x+20}$ is:

A $\log_e\left(\dfrac{x-5}{x-4}\right)$ **B** $\log_e\left(\dfrac{x-4}{x-5}\right)$ **C** $\log_e(x^2-9x+20)$

D $\log_e(x-5) - \dfrac{1}{(x-4)^2}$ **E** $\dfrac{1}{(x-5)^2} - \dfrac{1}{(x-4)^2}$

20 Find $\displaystyle\int \frac{2x+3}{(x+1)^2}\,dx$. 4E

21 Find an antiderivative of $f(x)$ where $f(x) = \dfrac{x^2-2x-12}{x^2-7x-8}$. 4E

22 Use integration by parts to evaluate $\displaystyle\int x^2e^x\,dx$. 4F

23 Find $\displaystyle\int e^{2x}\sin x\,dx$. 4F

24 **multiple choice** 4G

The integral $\displaystyle\int_a^b \frac{1}{\sqrt{9-x^2}}\,dx$ can be evaluated over the largest domain of:

A (–9, 9) **B** [–3, 3] **C** (–3, 0)

D P **E** (–3, 3)

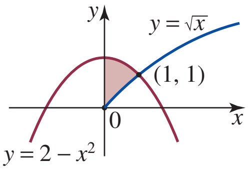
25 Evaluate: 4G

a $\displaystyle\int_{-1}^{1} \frac{x^2}{x^3+1}\,dx$ **b** $\displaystyle\int_0^{\pi} 2x\cos x^2\,dx$.

26 Evaluate: 4G

a $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$ **b** $\displaystyle\int_{-2}^{1} \frac{x}{\sqrt{2-x}}\,dx$.

Questions **27** and **28** refer to the shaded area in the figure at right.



27 **multiple choice** 4H

The volume generated when the region is rotated about the x-axis is equal to:

A $\pi\displaystyle\int_0^1 (4-2x^2+x^4-x)\,dx$ **B** $\pi\displaystyle\int_0^1 (4+x^4-x)\,dx$

C $\pi\displaystyle\int_0^1 (4-3x^2+x^4)\,dx$ **D** $\pi\displaystyle\int_0^1 (4-4x^2+x^4-x)\,dx$

E $\pi\displaystyle\int_0^2 (2-x^2-\sqrt{x})\,dx$

4H **28** **multiple choice**

The volume generated when the region is rotated about the y-axis is equal to:

A $\pi\int_0^2 (2-y)\,dy$

B $\pi\int_1^2 (2-y)\,dy + \pi\int_0^1 y^4\,dy$

C $\pi\int_0^2 y^2\,dy$

D $\pi\int_1^2 y^2\,dy + \pi\int_0^1 (2-y)\,dy$

E $\pi\int_0^2 (2-y-y^2)\,dy$

4H **29** Find the volume generated when the area under the graph of $y = e^x$, between $x = -1$ and $x = 0$, is rotated about the x-axis.

4H **30** Find the volume of water in a hemispherical bowl of radius 8 cm if the depth is 3 cm.

4I **31** Use Simpson's rule with four strips to approximate the value of $\int_0^{\pi} \frac{\sin x}{x}\,dx$.

4H **32** **a** Find the area of a quadrant of a circle of radius 6 represented by $\int_0^6 \sqrt{36-x^2}\,dx$ using:

- **i** an exact method
- **ii** Simpson's rule with six strips.

b Compare the values obtained for the area using these two methods.

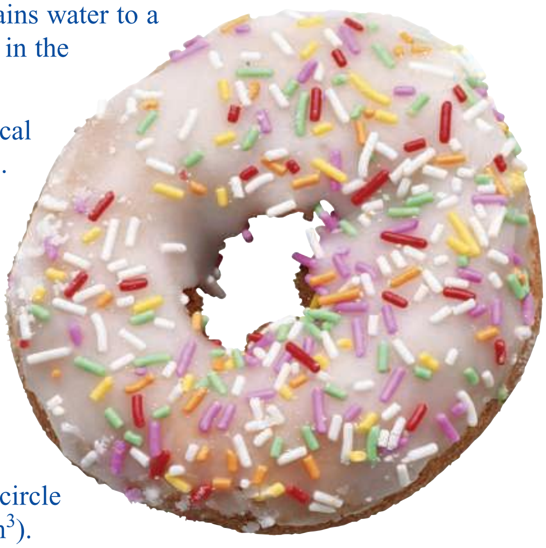
Modelling and problem solving

1 The power, P watts, that is generated when an electric current, I amps, flows through a resistor of resistance R ohms is given by the formula $P = I^2R$. If the resistance is 600 ohms, derive an expression for the change in power, P, due to a small change in current, ΔI.

2 A hemispherical bowl of radius 10 cm contains water to a depth of 5 cm. What is the volume of water in the bowl?

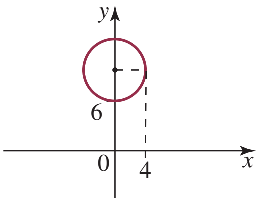
3 A solid sphere of radius 6 cm has a cylindrical hole of radius 1 cm bored through its centre. What is the volume of the remainder of the sphere?

4 Find the volume of a truncated cone of height 10 cm, a base radius of 5 cm and a top radius of 2 cm.

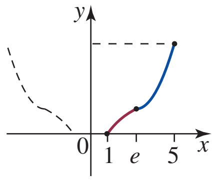


5 **a** Find the equation of the circle sketched below.

b Find the volume of a torus (doughnut-shaped figure) generated by rotating this circle about the x-axis (give your answer in cm^3).



6 The side view of the right side of a wine glass vessel can be modelled by two curves which join at $x = e$:



$y = 2\log_e x$, $0 < x \le e$ (red curve)
$y = x^2 - 2ex + e^2 + c$, $e \le x \le 5$ (blue curve)
(All measurements are in centimetres.)

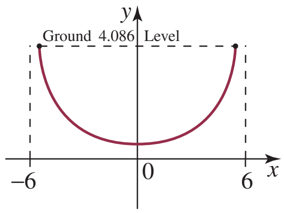
a Show that the value of c is 2 and find the height of the vessel correct to 2 decimal places.
The vessel is formed when the region between the curves and the y-axis is rotated about the y-axis.

b Find the volume of wine in the glass when the depth is 2 cm.

c What is the maximum volume of wine that the glass can hold (using maximum height to the nearest mm)?

7 A below-ground skating ramp is to be modelled by the curve

$$y = \frac{2}{\sqrt{36 - x^2}}, \ -5.98 \le x \le 5.98.$$



This is shown above, where the line $y = 4.086$ represents ground level. (All measurements are in metres.) (Give all answers correct to 2 decimal places.)

a Find the maximum depth of the ramp.

b Find the area under the curve.

c Find the volume generated if this area is rotated about the x-axis.

d If the ramp is 20 metres long, what is the volume of dirt which must be removed?

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Chapter 4

eBook*plus* ACTIVITIES

4A Approximation using the derivative

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- SkillSHEET 4.1: Practise finding derivatives and integrals of inverse trigonometric functions. (*page 137*)

4D Antiderivatives involving trigonometric identities

Tutorial

- int-0387: Watch a tutorial on antiderivatives of products of trigonometric functions using substitution. (*page 150*)

Digital doc

- WorkSHEET 4.1: Approximate numerical values, determine antiderivaties using the substitution method and trigonometric identities. (*page 156*)

4E Antidifferentiation using partial fractions

Tutorial

- int-0388: Watch a tutorial on how to antidifferentiate rational functions by first expressing them as partial fractions. (*page 159*)

4G Definite integrals

Tutorial

- int-0389: Watch a tutorial on how to evaluate definite integrals. (*page 169*)

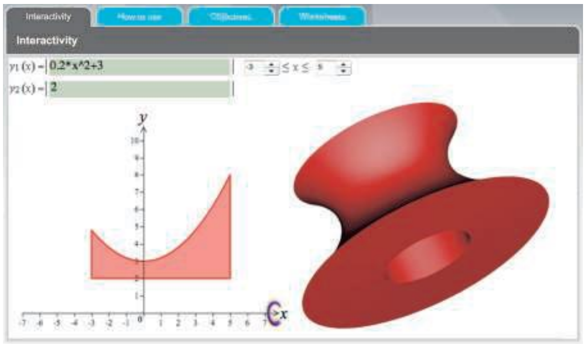
4H Volumes of solids of revolution

Tutorial

- int-0390: Watch a tutorial on how to evaluate the volume of revolution of a region rotated about an axis. (*page 177*)

Interactivity

- int-0347: Consolidate your understanding of calculating volumes of solids of revolution. (*page 179*)



$y_1(x) =$ 0.2*x^2+3

$y_2(x) =$ 2

4I Approximation using Simpson's rule

Tutorial

- WorkSHEET 4.2: Determine antiderivatives of rational functions and calculate volumes of solids of revolution. (*page 189*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 197*)

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5

Differential equations

syllabus reference

Core topic:
Calculus

In this chapter

5A Differential equations: related rates

5B Differential equations of the form $\frac{dy}{dx} = f(x)$

5C Differential equations of the form $\frac{dy}{dx} = g(y)$

5D Setting up and solving differential equations

Differential equations

A differential equation, as the name implies, is an equation that contains at least one derivative.

Differential equations can be used to model systems that are in the process of changing. Examples are:

- the depth of water in a tank which is filling up or being drained
- the velocity of a parachutist as she falls towards the earth
- the concentration of a chemical in solution
- the cost of producing a piece of equipment.

They are most useful when quantities can be measured by rates of change.

The *order* of a differential equation is defined as the order of the highest derivative that occurs in the equation.

1. First order differential equations involve first derivatives only, that is,

$$\frac{dy}{dx}, f'(x), y'$$

2. Second order differential equations involve second derivatives, but can also have first derivatives, that is,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right), \frac{d^2y}{dx^2}\ f''(x), y'' \text{ with } \frac{dy}{dx}, f'(x), y'$$

Related rates

Some problems requiring a rate of change between two variables cannot be solved directly. However, it is possible that a rate of change for each of the variables in terms of a third variable may be found. When this is the case, the chain rule can then be applied to find the rate of change between the original two variables.

The chain rule is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ (if } y \text{ is a function of } u\text{)}$$

WORKED Example 1

Use the chain rule to find the derivative indicated in brackets for each of the following.

a $\frac{dV}{dr} = 5r, \frac{dr}{dt} = 2 \quad \left(\frac{dV}{dt}\right)$ **b** $\frac{dP}{dn} = 4n, \frac{dP}{dt} = 6 \quad \left(\frac{dn}{dt}\right)$

THINK	WRITE
a 1 Use the chain rule $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$.	**a** $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$
2 Substitute $\frac{dV}{dr} = 5r$ and $\frac{dr}{dt} = 2$ into the rule.	$= 5r \cdot 2$
3 Simplify.	$= 10r$

THINK	WRITE
b ① Use the chain rule $\frac{dn}{dt} = \frac{dP}{dt} \cdot \frac{dn}{dP}$.	**b** $\frac{dn}{dt} = \frac{dn}{dP} \cdot \frac{dP}{dt}$
② Find $\frac{dn}{dP}$ by inverting $\frac{dP}{dn}$ and substituting $\frac{dP}{dn} = 4n$.	$\frac{dn}{dP} = \frac{1}{\left(\frac{dP}{dn}\right)} = \frac{1}{4n}$
③ Multiply 6 by $\frac{1}{4}n$.	$\frac{dn}{dt} = \frac{1}{4n} \cdot 6$
④ Simplify by removing the common factor of 2.	$= \frac{3}{2n}$ or $\frac{3}{2}n^{-1}$

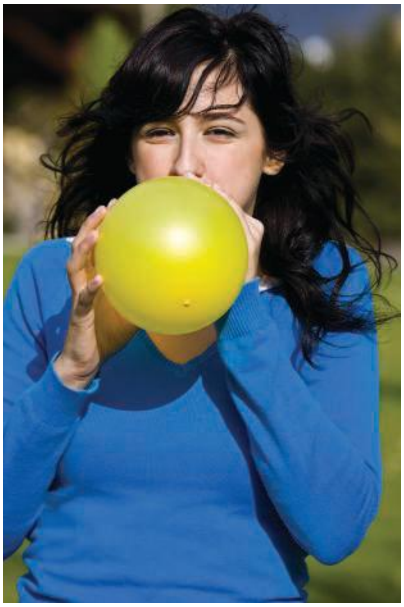
WORKED Example 2

If A = $h^2 - 5h$ find $\frac{dA}{dh}$ when $h = 2$.

THINK	WRITE
① Write the equation.	$A = h^2 - 5h$
② Differentiate A with respect to h.	$\frac{dA}{dh} = 2h - 5$
③ Evaluate $\frac{dA}{dh}$ when $h = 2$.	When $h = 2$, $\frac{dA}{dh} = 2(2) - 5$ $= 4 - 5$ $= -1$

Notes

1. Rates of change are given with respect to *time* unless otherwise stated.



2. An *increasing* rate is *positive*.

The volume of the balloon is increasing. The rate of change of volume with respect to time is positive.

$$\frac{dV}{dt} > 0$$

3. A *decreasing* rate is *negative*.

The mass of sand in the tip-truck is decreasing. The rate of mass change with respect to time is negative.

$$\frac{dm}{dt} < 0$$

WORKED Example 3

A balloon is being filled with air at the rate of 150 cm^3/s. Assuming that the balloon remains spherical, at what rate is the radius changing when the radius reaches 3 cm?

	THINK	WRITE
1	The given rate is $\frac{dV}{dt} = 150$ cm^3/s.	Given rate $\frac{dV}{dt} = 150$
2	The required rate is $\frac{dr}{dt}$.	Required rate $\frac{dr}{dt}$
3	The rate which will be found is $\frac{dr}{dV}$ as $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$.	Need to find $\frac{dr}{dV}$
4	The volume of a sphere is $V = \frac{4}{3}\pi r^3$.	$V = \frac{4}{3}\pi r^3$
5	Differentiate V with respect to r to obtain $\frac{dV}{dr}$.	$\frac{dV}{dr} = 4\pi r^2$
6	Invert $\frac{dV}{dr}$ to find $\frac{dr}{dV}$.	$\frac{dr}{dV} = \frac{1}{4\pi r^2}$
7	Use the chain rule to find $\frac{dr}{dt}$.	$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ (using the chain rule)
8	Simplify $\frac{dr}{dt}$.	$= \frac{150}{4\pi r^2}$
9	Substitute $r = 3$ into $\frac{dr}{dt}$ and evaluate.	When $r = 3$, $\frac{dr}{dt} = \frac{150}{4\pi(3)^2}$ $= \frac{150}{36\pi}$ $= \frac{25}{6\pi}$
10	State the answer.	The radius is increasing at $\frac{25}{6\pi}$ cm/s (or 1.326 cm/s) when the radius is 3 cm.

WORKED Example 4

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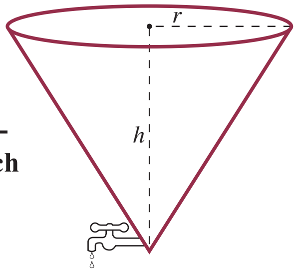
Tutorial:
Worked example 4
Int-0391

A conical tank full of water loses water at a rate of 1.2 m^3/min when a tap at the bottom is turned on. If the surface radius of water in the tank is always equal to two-thirds of the depth of the water, then find the rate at which the depth of water is changing when the depth is 2 m.



THINK	**WRITE**
① The given rate is $\frac{\mathrm{d}V}{\mathrm{d}t} = -1.2$ as the rate is decreasing.	Given rate $\frac{\mathrm{d}V}{\mathrm{d}t} = -1.2 \text{ m}^3\text{/min}$
② The required rate is $\frac{\mathrm{d}h}{\mathrm{d}t}$.	Required rate is $\frac{\mathrm{d}h}{\mathrm{d}t}$
③ The rate which has to be found is $\frac{\mathrm{d}h}{\mathrm{d}V}$ as $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$.	Need to find $\frac{\mathrm{d}V}{\mathrm{d}h}$
④ The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.	$V = \frac{1}{3}\pi r^2 h$
⑤ Substitute $r = \frac{2}{3}h$ in the rule to obtain V in terms of h only.	But $r = \frac{2}{3}h$ (given)
⑥ Simplify V.	$V = \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h$ $= \frac{\pi}{3}\left(\frac{4h^2}{9}\right)h$ $= \frac{4}{27}\pi h^3$
⑦ Differentiate V with respect to h to obtain $\frac{\mathrm{d}V}{\mathrm{d}h}$.	$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{12}{27}\pi h^2$ $= \frac{4}{9}\pi h^2$
⑧ Invert to obtain $\frac{\mathrm{d}h}{\mathrm{d}V}$.	$\frac{\mathrm{d}h}{\mathrm{d}V} = \frac{9}{4\pi h^2}$
⑨ Find $\frac{\mathrm{d}h}{\mathrm{d}t}$ using the chain rule.	$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t}$ $= \frac{9}{4\pi h^2} \cdot -1.2$ $= -\frac{2.7}{\pi h^2}$
⑩ Substitute $h = 2$ into $\frac{\mathrm{d}h}{\mathrm{d}t}$ and evaluate.	When $h = 2$, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{-2.7}{\pi\, 2^2}$ $= \frac{-0.675}{\pi}$
⑪ State the answer.	The height is changing at a rate of $\frac{-0.675}{\pi}$ m/min (or −0.215 m/min) when the depth is 2 m. or The height is decreasing by 0.215 m/min when the depth is 2 m.

remember

1. A rate of change is a derivative.
2. Chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
3. Reciprocal rule: $\dfrac{dx}{dy} = \dfrac{1}{\left(\dfrac{dy}{dx}\right)}$
4. *Increasing rates* are *positive*.
5. *Decreasing rates* are *negative*.
6. Solid geometry formulas:

   Total surface area of a cone = $\pi r(s + r)$
   Total surface area of a sphere = $4\pi r^2$
   Volume of a cone = $\dfrac{1}{3}\pi r^2 h$
   Volume of a sphere = $\dfrac{4}{3}\pi r^3$

EXERCISE 5A Differential equations: related rates

1 Express each of the following phrases as derivatives.

- **a** The rate of change of area (A) with respect to length (L)
- **b** The rate of change of volume (V) with respect to radius (r)
- **c** The rate of change of pressure (P) with respect to height (h)
- **d** The rate of change of acceleration (a) with respect to time (t)

2 Find the derivative indicated in the brackets for each of the following rules.

- **a** $V = \dfrac{4\pi r^3}{3} \quad \left(\dfrac{dV}{dr}\right)$
- **b** $M = 0.6\sqrt{A^3} \quad \left(\dfrac{dM}{dA}\right)$
- **c** $P = 2e^{0.01t^2} \quad \left(\dfrac{dP}{dt}\right)$
- **d** $5xy = 2 \quad \left(\dfrac{dy}{dx}\right)$
- **e** $S = 2\pi r(r + 10) \quad \left(\dfrac{dS}{dr}\right)$

WORKED Example 1

3 Use the chain rule to find the derivative indicated in brackets for each of the following.

- **a** $\dfrac{dA}{dr} = 2r,\ \dfrac{dr}{dt} = 3 \quad \left(\dfrac{dA}{dt}\right)$
- **b** $\dfrac{dV}{dL} = 3L^2,\ \dfrac{dL}{dt} = 1.5 \quad \left(\dfrac{dV}{dt}\right)$
- **c** $\dfrac{dV}{dh} = 8\pi,\ \dfrac{dh}{dt} = 0.4 \quad \left(\dfrac{dV}{dt}\right)$
- **d** $\dfrac{dS}{dr} = 3 + 2r,\ \dfrac{dr}{dt} = 4 \quad \left(\dfrac{dS}{dt}\right)$
- **e** $\dfrac{dP}{dh} = 2\sqrt{h},\ \dfrac{dh}{dt} = 0.5 \quad \left(\dfrac{dP}{dt}\right)$
- **f** $\dfrac{dV}{dh} = 6,\ \dfrac{dV}{dt} = 1.2 \quad \left(\dfrac{dh}{dt}\right)$
- **g** $\dfrac{dA}{dr} = 2\pi r,\ \dfrac{dA}{dt} = 4 \quad \left(\dfrac{dr}{dt}\right)$
- **h** $\dfrac{dV}{dL} = 8L,\ \dfrac{dV}{dt} = 0.5 \quad \left(\dfrac{dL}{dt}\right)$

WORKED Example 2

4 **a** If $M = 2n^3$, find $\frac{dM}{dn}$ when $n = 2$.

b If $V = 6r^2$, find $\frac{dV}{dr}$ when $r = 1.2$.

c Find $\frac{dP}{dt}$ when $t = 4$ given that $P = 0.5e^{1+\sqrt{t}}$.

d Find $\frac{dy}{dx}$ when $x = 1$ given that $3xy = 8$.

5 **multiple choice**

The side lengths of a square (L) are increasing at a constant rate of 2 cm/min. If the area of the square is A,

a then $\frac{dA}{dL}$ is equal to:

A L^2 **B** $2L$ **C** 4 **D** L **E** 2

b the rate of change of the area, $\frac{dA}{dt}$, when $L = 5$ cm is:

A 10 cm^2/min **B** 0.1 cm^2/min **C** 0.5 cm^2/min
D 5 cm^2/min **E** 20 cm^2/min

6 **multiple choice**

A spherical balloon is deflating so that its radius is decreasing at a constant rate of 0.8 cm/s. The rate of change of volume when the radius is 10 cm is:

A 80π cm^3/s **B** -400π cm^3/s **C** -320π cm^3/s
D 500π cm^3/s **E** -80π cm^3/s

WORKED Example 3

7 The radius of a circular oil slick is increasing at a rate of 10 m/h. If the slick remains circular, find the rate at which the area is increasing when the radius is 5 m.

8 The radius of an inflated spherical balloon is increasing uniformly by 2 cm/min. At what rate is the surface area increasing when the radius is 5 cm?

9 A large block of ice in the shape of a cube is melting uniformly at the rate of 1.2 cm^3/s. Find the rate of decrease of its side length when the side length is 2 cm.

WORKED Example 4

10 The volume of water in a glass is $V = 2x^{\frac{5}{3}}$ where x cm is the depth of water in the glass. If water is poured into the glass at a constant rate of 12 cm^3/s, find the rate at which the depth is changing when $x = 8$ cm.

11 The pulse rate (P beats per minute) and the speed travelled by an athlete (v m/s) are related by the rule $P = 60 + 15v$, where $0 \le v \le 8$. If the runner increases speed by a constant rate of 0.5 m/s per second, find the rate of change of pulse when the speed reaches 6 m/s.

12 A solid cylindrical container of fixed length 90 cm is being pressure tested and the radius is increasing at a constant rate of 0.01 cm/min. When the radius is 15 cm, find the rate of change of:

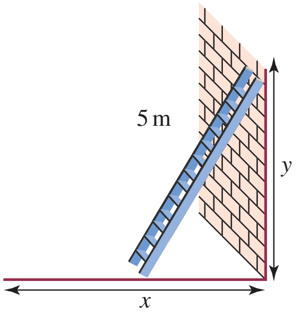
a the total surface area (S)
b the volume (V).

13 On a particular day the temperature (T °C) at a height (h m) above sea level is: $T = 12e^{-0.001h}$.
If a balloon is rising at 0.1 m/s, what is the rate of change of temperature on the balloon when it reaches a height of 1000 m above sea level?

14 In a particular circuit the current (I amps) in a resistor with resistance (R ohms) satisfies the rule $IR = 200$.
The resistance is increased at a constant rate of 100 ohms/s. Find the rate at which the current changes when the resistance is 400 ohms.

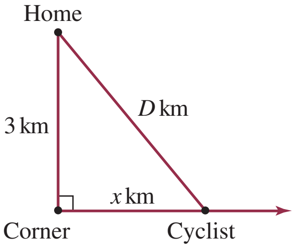
15 Water is poured at a constant rate of 20 cm^3/s into a container in the shape of an inverted cone whose semi-vertical angle is 45°. What is the rate at which the level of water is rising when the depth is 50 cm?

16 A 5-metre ladder leaning against a vertical wall starts to slip. The upper end is sliding at a constant rate of 0.1 m/s. Find the rate at which the lower end is sliding when the lower end of the ladder is 3 metres from the wall.



17 A cyclist is travelling from the corner along a straight road at a constant speed of 20 km/h.
Find:

a the distance (D) from home when the cyclist is x km from the corner
b the rate of change of distance from home when the cyclist is 4 km from the corner.



Differential equations of the form $\frac{dy}{dx} = f(x)$

The process of antidifferentiation can solve some differential equations. *General solutions* can be found by including the constant of antidifferentiation. If conditions are given which enable the constants of antidifferentiation to be found then *particular solutions* can be found. These conditions are known as *initial conditions* or *boundary conditions*.

The simplest type of first order differential equation is of the form: $\frac{dy}{dx} = f(x)$

This differential equation is solved by simply finding the antiderivative of $f(x)$.

$$\textbf{If } \frac{dy}{dx} = f(x)\textbf{, then } y = \int f(x)\,dx$$
$$= F(x) + c,$$
where $F(x)$ is the antiderivative of $f(x)$.

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WORKED Example 5

Find the general solution to $\frac{dy}{dx} = 2x + \frac{1}{\sqrt{1-x^2}}$.

THINK	WRITE
① Express y as an antiderivative of $\frac{dy}{dx}$.	$\frac{dy}{dx} = 2x + \frac{1}{\sqrt{1-x^2}}$ $y = \int\left(2x + \frac{1}{\sqrt{1-x^2}}\right)dx$
② Antidifferentiate $\frac{dy}{dx}$ by rule to obtain y and add the constant of antidifferentiation since no initial conditions are given.	$y = x^2 + \text{Sin}^{-1}x + c$

WORKED Example 6

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Tutorial:
Worked examaple 6
int-0393

Find the particular solution to $h'(t) = \frac{t}{\sqrt{t^2+9}}$ where $h(4) = 1$.

THINK	WRITE
① Express $h(t)$ in integral notation.	$h'(t) = \frac{t}{\sqrt{t^2+9}}$ $h(t) = \int \frac{t}{\sqrt{t^2+9}}\,dt$
② Antidifferentiate by substituting $u = t^2 + 9$, as the derivative is present.	Let $u = t^2 + 9$
③ Find $\frac{du}{dt}$.	$\frac{du}{dt} = 2t$
④ Make dt the subject.	or $dt = \frac{du}{2t}$
⑤ Substitute u and dt into the integrand.	$h(t) = \int \frac{t}{u^{\frac{1}{2}}} \cdot \frac{du}{2t}$
⑥ Simplify the integrand.	$= \int \frac{1}{2}u^{-\frac{1}{2}}du$
⑦ Antidifferentiate.	$= u^{\frac{1}{2}} + c$
⑧ Replace u by $t^2 + 9$.	$= \sqrt{t^2 + 9} + c$

Continued over page →

THINK	**WRITE**
⑨ Substitute the initial conditions of $t = 4$ and $h = 1$ into the equation.	Since $h(4) = 1$ $1 = \sqrt{25} + c$ $1 = 5 + c$
⑩ Solve for the constant, c.	$c = -4$
⑪ State the particular solution.	Therefore $h(t) = \sqrt{t^2 + 9} - 4$.

WORKED Example 7

The rate of change of shaded area covered by a shrub is modelled by the differential equation $\frac{dA}{dt} = 0.02t + 0.5\sin\frac{\pi t}{6}$, where A m^2 is the area covered by the shrub t months after it is planted.

If the shrub covers an area of 0.3 m^2 when planted, what area does it cover after 9 months?

THINK	**WRITE**
① Express A in integral notation.	$\frac{dA}{dt} = 0.02t + 0.5\sin\frac{\pi t}{6}$ $A = \int\left(0.02t + 0.5\sin\frac{\pi t}{6}\right)dt$
② Antidifferentiate the integrand by rule.	$= 0.01t^2 - \frac{3}{\pi}\cos\frac{\pi t}{6} + c$
③ State the general solution $A(t)$.	$A(t) = 0.01t^2 - 0.955\cos\frac{\pi t}{6} + c$ (approximating $\frac{3}{\pi}$ with 0.955)
④ Substitute 0.3 for A and 0 for t, these being the initial conditions.	Since $A(0) = 0.3$ $0.3 = -0.955 + c$
⑤ Solve the equation for the constant, c.	$c = 1.255$
⑥ State the particular solution $A(t)$.	$A = 0.01t^2 - 0.955\cos\frac{\pi}{6} + 1.255$
⑦ Evaluate $A(9)$, the shaded area at time $t = 9$ months.	$A(9) = 0.01(9)^2 - 0.955\cos\frac{3\pi}{2} + 1.255$ $= 0.81 + 1.255$ $= 2.065$
⑧ State the solution.	After 9 months the shrub covers an area of approximately 2.065 m^2.

Graphics Calculator tip! Solving a first order differential equation

The Casio graphics calculator is unable to solve a first order differential equation. However, the TI-Nspire CAS is able to do so using the nInt function. This function can be found by typing nInt into the calculator. The command nInt(f, x, a, b) finds the integral of function f with respect to x between the values a and b. If we integrate between 0 and x, the resulting function will have a y-intercept of 0. As we know the y-intercept, adding the intercept will result in the desired function.

To provide a general example, consider the equation in Worked example 7 using the variables x and y to stand for t and A.

This means that we are solving $\frac{dy}{dx} = 0.02x + 0.5 \sin\frac{\pi x}{6}$ where $y = 0.3$ when $x = 0$.

We want to find the value of y when $x = 9$. The solution can be found by using a graph or table.

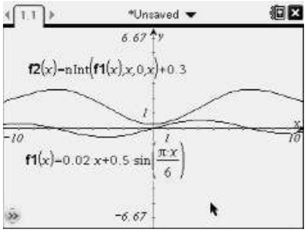
Graph

1. On a Graphs page, complete the entry lines as:

   $f1(x) = 0.02x + 0.5 \sin\frac{\pi x}{6}$

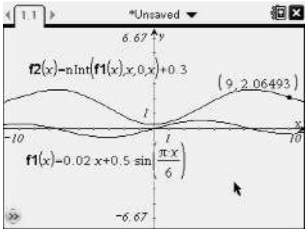
   $f2(x) = \text{nInt}(f1(x), x, 0, x) + 0.3$

   Press ENTER enter after each line.



2. To find the value of y when $x = 9$, press:
   - MENU menu
   - 5: Trace 5
   - 1: Graph Trace 1

   Trace along the curve of the solution, press ENTER enter to lock in a coordinate point, then move the pointer ↖ until it hovers over the point, press the Click button twice and change the x-value to 9, then press ENTER enter. This gives a y-value of 2.06.

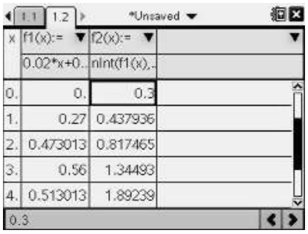


Table

1. On a Graphs page, repeat step 1 above to enter $f1(x)$ and $f2(x)$.
2. To display the data in a table, open a Lists & Spreadsheet page.

   Press:
   - MENU menu
   - 5: Table 5
   - 1: Switch to Table 1

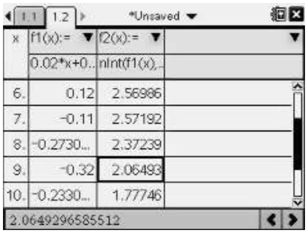
   Set the second column to be f1 and the third column to be f2.



x	f1(x):=	f2(x):=
	0.02*x+0...	nInt(f1(x),...
0.	0.	0.3
1.	0.27	0.437936
2.	0.473013	0.817465
3.	0.56	1.34493
4.	0.513013	1.89239

0.3

3. Scroll down to find the solution when $x = 9$ (so $y = 2.06493$).



x	f1(x):=	f2(x):=
	0.02*x+0...	nInt(f1(x),...
6.	0.12	2.56986
7.	−0.11	2.57192
8.	−0.2730...	2.37239
9.	−0.32	2.06493
10.	−0.2330...	1.77746

2.0649296585512

remember

If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\,dx$

$= F(x) + c$

When initial conditions are given then c can be evaluated giving a particular solution from the general solution.

EXERCISE 5B Differential equations of the form $\frac{dy}{dx} = f(x)$

WORKED Example 5

1 Find the general solution to the following differential equations.

a $\frac{dy}{dx} = 2x + 1$

b $\frac{dy}{dx} = x^3 - 4x$

c $y'(x) = (x + 3)(x - 2)$

d $y'(x) = 2e^{3x}$

e $\frac{dy}{dx} = x^2 + \frac{3}{x}$

f $y'(x) = x\sqrt{x^2 - 2}$

g $y'(x) = 4\sec^2 2x$

h $f'(x) = \frac{1}{x^3} + \sqrt{x}$

i $f'(x) = \frac{(2x + 1)(x - 1)}{x + 3}$

j $y'(x) = \frac{-1}{\sqrt{9 - x^2}}$

k $y'(x) = \frac{1}{4 + x^2}$

l $\frac{dy}{dx} = \sin 4x - 2\cos x$

m $h'(t) = 2(5 - 2t)^4$

n $\frac{dx}{d\sigma} = \sin^3\sigma \cos \sigma$

2 **multiple choice**

If $\frac{dy}{dx} = (2 - x)^5$ and $y = 0$ when $x = 1$, then the solution is:

A $y = 5 - 5(2 - x)^4$

B $y = (2 - x)^6$

C $y = \frac{-(2 - x)^6}{6}$

D $y = \frac{1 - (2 - x)^6}{6}$

E $y = (x - 2)^6$

3 **multiple choice**

Given that $\frac{dV}{dt} = \frac{2}{\sqrt{1 - t^2}}$ and $V = 2\pi$ when $t = 1$, then $V(t)$ is equal to:

A $2\,\mathrm{Cos}^{-1}t$

B $\frac{3\pi}{2} + \mathrm{Sin}^{-1}t$

C $\pi + 2\,\mathrm{Sin}^{-1}t$

D $2\pi + 1 - t^2$

E $\frac{7\pi}{4} + \mathrm{Tan}^{-1}t$

4 **multiple choice**

If $P'(u) = u\cos(u^2 - 3)$ and $P = 1$ when $u = \sqrt{3}$, then the solution is:

A $P = 1 + \frac{1}{2}\sin(u^2 - 3)$

B $P = \frac{1}{2} + \frac{1}{2}\cos(u^2 - 3)$

C $P = \sin(u^3 - 3u)$

D $P = u\sin(u^2 - 3)$

E $P = \cos(u^2 - 3)$

WORKED Example 6

5 Find the particular solution for each of the following differential equations.

a $y'(x) = 3x^2 + 1$, where $y = 3$ when $x = 0$

b $\frac{dy}{dx} = e^{-2x}$, where $y = 0$ when $x = 0$

c $f'(x) = 6\cos 3x$ and $f\left(\frac{\pi}{3}\right) = 1$

d $g'(x) = \sqrt{2x-1}$ and $g(5) = 4$

e $\dfrac{dh}{dt} = \dfrac{1}{\sqrt{16-t^2}}$, where $h = 3$ when $t = 0$

f $h'(t) = \dfrac{t-1}{t^2+4t+3}$, where $h(0) = \log_e 9$

g $\dfrac{dP}{dn} = \dfrac{n}{\sqrt{n^2-7}}$, where $P = 5$ when $n = 4$

h $\dfrac{dL}{d\sigma} = 4\sin^2\sigma$, where $L = \pi$ when $\sigma = \pi$

i $\dfrac{dM}{d\sigma} = 2 + \tan^2\sigma$, where $M = -3$ when $\sigma = 0$

j $\dfrac{du}{dx} = \dfrac{x^2}{x^2+4}$, where $u = 1$ when $x = 2$

k $\dfrac{dV}{dt} = 4te^{t^2+2}$, where $V = 3e^2$ when $t = 0$

l $(x^2+1)\dfrac{dy}{dx} = 4x + 1$, where $y = 7$ when $x = 0$

m $\dfrac{dL}{d\sigma} = 3\cos\sigma\sqrt{1+\sin\sigma}$, where $L = 0$ when $\sigma = 0$

n $(2-e^x)\dfrac{dy}{dx} = e^x$, where $y = 3$ when $x = 0$

6 Find the equation of the curve whose gradient at any point (x, y) is $4x$ and the curve passes through $(-1, 3)$.

7 A curve passes through the point $(-\frac{1}{2}, \pi)$ and its gradient at any point (x, y) is equal to $\dfrac{-3}{\sqrt{1-x^2}}$. What is the equation of the curve?

WORKED Example 7

8 The rate of change of height, h metres, of water at a port entrance is $\dfrac{dh}{dt} = 7.5 - 1.5\cos 0.5t$, where t is the time in hours after low tide.
Express the height, h, as a function of time, t, if the height at low tide (that is, $t = 0$) is 6 metres.

9 The rate $\dfrac{dV}{dt}$ at which V kilolitres of sewage is treated at a sewage plant t hours after daily operation begins is $\dfrac{dV}{dt} = 1000e^{0.1t}$ $0 \le t \le 15$.
Find:

a the total volume treated, $V(t)$, after time, t
b the total volume treated after 5 hours
c the average volume treated per hour if the plant operates for 15 hours per day.

10 The acceleration, a, of a particular car from rest is given by the expression $a = 3t\sqrt{t^2+4}$, $0 \le t \le 5$, where a is in km/h^2 and t is in seconds.

a Find the velocity at any time, t, in units of km/h.
b Find the time, to the nearest hundredth of a second, when the car reaches 56 km/h.

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WorkSHEET 5.1

Differential equations of the form $\frac{dy}{dx} = g(y)$

Differential equations of the type $\frac{dy}{dx} = g(y)$ can be solved by inverting the equation and then antidifferentiating. This can be done since $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, where both $\frac{dy}{dx}$ and $\frac{dx}{dy} \neq 0$.

$\frac{dy}{dx} = g(y)$ becomes $\frac{dx}{dy} = \frac{1}{g(y)}$

and therefore $x = \int \frac{1}{g(y)}\,dy + c$.

This gives x as a function of y. If it is possible, the solution is rearranged to express y as a function of x. However, not all equations can readily be transformed to make y the subject.

WORKED Example 8

Find general solutions for each of the following differential equations.

a $\frac{dy}{dx} = y$ **b** $y'(x) = \tan y$

THINK	WRITE
a ① Invert both sides of the differential equation.	**a** $\frac{dy}{dx} = y$ $\frac{dx}{dy} = \frac{1}{y}$
② Antidifferentiate the RHS with respect to y to obtain $x(y)$.	$x = \int \frac{1}{y}\,dy$ $x = \log_e y + c,\ y > 0$
③ Subtract c from both sides.	$x - c = \log_e y$
④ Take the exponent e to the power of both sides to convert to exponential form and make y the subject.	$e^{x-c} = y$ $y = e^{x-c}$
⑤ State the solution and express it more simply.	Now $e^{x-c} = e^x e^{-c} = Ae^x$, where $A = e^{-c}$. Therefore, the general solution is $y = Ae^x$, where $A = e^{-c}$.
b ① Express the derivative as $\frac{dy}{dx}$.	**b** $y'(x) = \tan y$ $\frac{dy}{dx} = \tan y$
② Invert both sides of the differential equation.	$\frac{dx}{dy} = \frac{1}{\tan y},\ y \neq 0$
③ Simplify $\frac{1}{\tan y}$.	$= \frac{\cos y}{\sin y}$
④ Let $u = \sin y$ to antidifferentiate using the ‘derivative present’ method.	$x = \int \frac{\cos y}{\sin y}\,dy$ Let $u = \sin y$
⑤ Find $\frac{du}{dy}$.	$\frac{du}{dy} = \cos y$

THINK	WRITE
6 Make d*y* the subject.	$dy = \dfrac{du}{\cos y}$
7 Substitute *u* and d*y* into the integral.	$x = \int \dfrac{\cos y}{u}\dfrac{du}{\cos y}$
8 Simplify the integrand.	$= \int \dfrac{du}{u}$
9 Antidifferentiate by rule.	$= \log_e u + c$
10 Replace *u* by sin *y*.	$x = \log_e(\sin y) + c,\ \sin y > 0$
11 Subtract *c* from both sides.	$x - c = \log_e(\sin y)$
12 Express the equation in exponential form.	$e^{x-c} = \sin y$
13 Take Sin^{-1} of both sides to make *y* the subject.	$y = \text{Sin}^{-1}(e^{x-c}),\ x < c$
14 State the solution.	Therefore the general solution is $y = \text{Sin}^{-1}\,(Ae^{x})$, where $A = e^{-c}$.

WORKED Example 9

a **Find the particular solution of $f'(x) = \sqrt{1-y^2}$ if $f(0) = \frac{1}{2}$.**

b **State the largest domain for which the solution applies.**

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Tutorial:
Worked example 9
int-0395

THINK	WRITE
a 1 Express the differential equation in the form $\dfrac{dy}{dx}$.	**a** $\dfrac{dy}{dx} = \sqrt{1-y^2}$
2 Invert both sides of the differential equation.	$\dfrac{dx}{dy} = \dfrac{1}{\sqrt{1-y^2}},\ -1 < y < 1$
3 Antidifferentiate by rule.	$x = \int \dfrac{1}{\sqrt{1-y^2}}\,dy$ $x = \text{Sin}^{-1}y + c,\ -1 < y < 1$
4 Substitute $y = \frac{1}{2}$ and $x = 0$.	Substituting $(0, \frac{1}{2})$ into the equation, $0 = \text{Sin}^{-1}\frac{1}{2} + c$
5 Solve the equation for *c*.	$0 = \dfrac{\pi}{6} + c$ $c = -\dfrac{\pi}{6}$
6 Rewrite the equation.	$x = \text{Sin}^{-1}y - \dfrac{\pi}{6}$
7 Add $\dfrac{\pi}{6}$ to both sides.	$x + \dfrac{\pi}{6} = \text{Sin}^{-1}y$
8 Take the sine of both sides to make *y* the subject.	$\sin\left(x + \dfrac{\pi}{6}\right) = y$

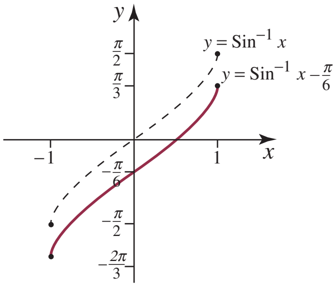
Continued over page

THINK	**WRITE**
⑨ State the solution.	Therefore $y = \sin\left(x + \frac{\pi}{6}\right)$ is the particular solution.
b ① From part **a**, $\frac{dy}{dx} = \frac{1}{\sqrt{1 - y^2}}$ exists for $-1 < y < 1$.	**b** From part **a**, $-1 < y < 1$
② We need to consider $x + \frac{\pi}{6} = \text{Sin}^{-1} y$, where $-1 < y < 1$ (see part **a**).	Since $\left(x + \frac{\pi}{6}\right) = \text{Sin}^{-1} y$
③ $\text{Sin}^{-1} y$ is defined as having a range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.	$-\frac{\pi}{2} < x + \frac{\pi}{6} < \frac{\pi}{2}$
④ Solve this inequation for x.	$-\frac{2\pi}{3} < x < \frac{\pi}{3}$
⑤ State the largest possible set of values for x; that is, the domain of $y = \sin\left(x + \frac{\pi}{6}\right)$.	Therefore, the largest possible domain for the solution of the differential equation in part **a** is $\left(-\frac{2\pi}{3}, \frac{\pi}{3}\right)$.

Notes

1. The graph of $y = \text{Sin}^{-1} x - \frac{\pi}{6}$, which has x and y interchanged in Worked example 9, is shown in the figure at right.

   This clearly shows that $-\frac{2\pi}{3} < y < \frac{\pi}{3}$, which corresponds to the domain, x, in the previous example.



2. As sin x, cos x and tan x are cyclic functions, for their inverses to be functions, the ranges of the inverse functions need to be restricted as follows:

   $y = \text{Sin}^{-1}x$, Domain $-1 \le x \le 1$, Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

   $y = \text{Cos}^{-1}x$, Domain $-1 \le x \le 1$, Range $0 \le y \le \pi$

   $y = \text{Tan}^{-1}x$, Domain $-\infty \le x \le \infty$, Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

WORKED Example 10

A kite is rising at a rate which is proportional to its height at any time; that is, $\frac{dh}{dt} = kh$, $0 < h < 3000$, where h is in metres and t is in minutes. (k is a constant.)
If the initial height is 2 metres and the height is 20 metres after 1 minute, find:

a **the height at any time, t**
b **the height to the nearest metre after 3 minutes.**

THINK	**WRITE**
a ① Invert both sides of the differential equation.	**a** $\frac{dh}{dt} = kh$ $\frac{dt}{dh} = \frac{1}{kh}$

THINK	WRITE
② Express t in integral notation.	$t = \int \frac{1}{kh}\,\mathrm{d}h$
③ Antidifferentiate $\frac{\mathrm{d}t}{\mathrm{d}h}$ with respect to h.	$t = \frac{1}{k}\log_e h + c$
④ Isolate the log.	$t - c = \frac{1}{k}\log_e h$ $k(t - c) = \log_e h$
⑤ Take the exponent e to the power of both sides to convert the exponential form and make h the subject.	$e^{k(t-c)} = h$
⑥ The expression can be simplified by letting $A = e^{-ck}$.	$h = Ae^{kt}$
⑦ Substitute $h = 2$ when $t = 0$.	When $t = 0$, $h = 2$, $2 = Ae^0$
⑧ Solve for A.	$A = 2$, since $e^0 = 1$
⑨ Substitute A into the equation.	$h = 2e^{kt}$
⑩ Substitute $h = 20$ and $t = 1$ into the equation.	When $t = 1$, $h = 20$ $20 = 2e^k$
⑪ Solve the equation for k.	$10 = e^k$ $k = \log_e 10$ $k = 2.303$
⑫ Substitute k into the equation.	$h = 2e^{2.303t}$
⑬ State the solution.	Therefore, $h(t) = 2e^{2.303t}$ is the height at any time t.
b ① Substitute $t = 3$ into the rule $h(t)$.	**b** When $t = 3$, $h(3) = 2e^{2.303 \cdot 3}$
② Evaluate $h(3)$.	$= 2e^{6.909}$
③ Round the height to the nearest metre.	≈ 2002
④ State the approximate height.	Therefore, after 3 minutes the kite reaches a height of approximately 2002 metres.

remember

1. If $\frac{\mathrm{d}y}{\mathrm{d}x} = g(y)$, then $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{g(y)}$ and $x = \int \frac{1}{g(y)}\,\mathrm{d}y + c$.
2. $y = \mathrm{Sin}^{-1}x$, Domain $-1 \le x \le 1$, Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
3. $y = \mathrm{Cos}^{-1}x$, Domain $-1 \le x \le 1$, Range $0 \le y \le \pi$.
4. $y = \mathrm{Tan}^{-1}x$, Domain $-\infty \le x \le \infty$, Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

EXERCISE 5C Differential equations of the form $\dfrac{dy}{dx} = g(y)$

WORKED Example 8

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Interactivity: Slope fields int-0348

1 Find the general solutions to the following differential equations.

a $\dfrac{dy}{dx} = \dfrac{1}{2y}$

b $\dfrac{dy}{dx} = \dfrac{y}{3}$

c $\dfrac{dy}{dx} = y + 3$

d $\dfrac{dy}{dx} = 3y - 5$

e $y'(x) = \dfrac{1}{2y + 7}$

f $y'(x) = \dfrac{2}{7 - 4y}$

g $y'(x) = 8 - \dfrac{y}{5}$

h $2y' - y^2 = 0$

i $\dfrac{dy}{dx} = \cos^2 y$

j $\dfrac{dy}{dx} = \dfrac{1}{\tan y}$

2 **multiple choice**

If $\dfrac{dh}{dt} = 0.2h$, where $h = \frac{1}{2}$ when $t = 0$, then the solution is:

A $h = e^{0.2t}$ **B** $h = \frac{1}{2}e^{0.2t}$ **C** $h = e^{5t}$

D $h = e^{5t} + \frac{1}{2}$ **E** $h = \frac{1}{2}e^{5t}$

3 **multiple choice**

Given that $\dfrac{dy}{dx} = \sqrt{4 - y^2}$ and $y = \sqrt{3}$ when $x = \dfrac{\pi}{6}$ then,

a the solution is:

A $y = 2\sin\left(x - \dfrac{\pi}{6}\right)$ **B** $y = 2\cos\left(x - \dfrac{\pi}{6}\right)$ **C** $y = \sqrt{4 - x^2}$

D $y = 2\sin\left(x + \dfrac{\pi}{6}\right)$ **E** $y = \cos\left(x + \dfrac{\pi}{6}\right)$

b the domain of the function is:

A $R\backslash\{-2, 2\}$ **B** $R\backslash\{2\}$ **C** $\left(-\dfrac{2\pi}{3}, \dfrac{\pi}{3}\right)$

D $[-2, 2]$ **E** $\left(-\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right)$

WORKED Example 9

4 For each of the following differential equations:

i find the particular solution

ii state the largest domain for which the solution applies.

a $(y + 3)^3\dfrac{dy}{dx} = 4$ and $y = -3$ when $x = 0$.

b $\dfrac{dy}{dx} = \frac{1}{4} + y^2$ and $y = \frac{1}{2}$ when $x = \dfrac{\pi}{4}$.

c $\dfrac{dy}{dx} = \dfrac{1}{y - 3}$, $y > 3$ and $y = 5$ when $x = 2$.

d $(y + 1)^2\dfrac{dy}{dx} = 1$ and $y = 2$ when $x = 4$.

e $y'(x) = -\sqrt{2 - y^2}$ and $y = 1$ when $x = 0$.

f $\dfrac{dh}{dt} = 2h(3 - h)$ where $h = 1$ when $t = 0$.

g $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-2v}$ where $v = 0$ when $x = 1$.

h $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^y + 1}{e^y}$ where $y = 0$ when $x = 0$.

i $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - y$ and $y = 2$ when $x = 0$.

j $\frac{\mathrm{d}u}{\mathrm{d}x} = 4 - u^2$ and $u = 1$ when $x = 0$.

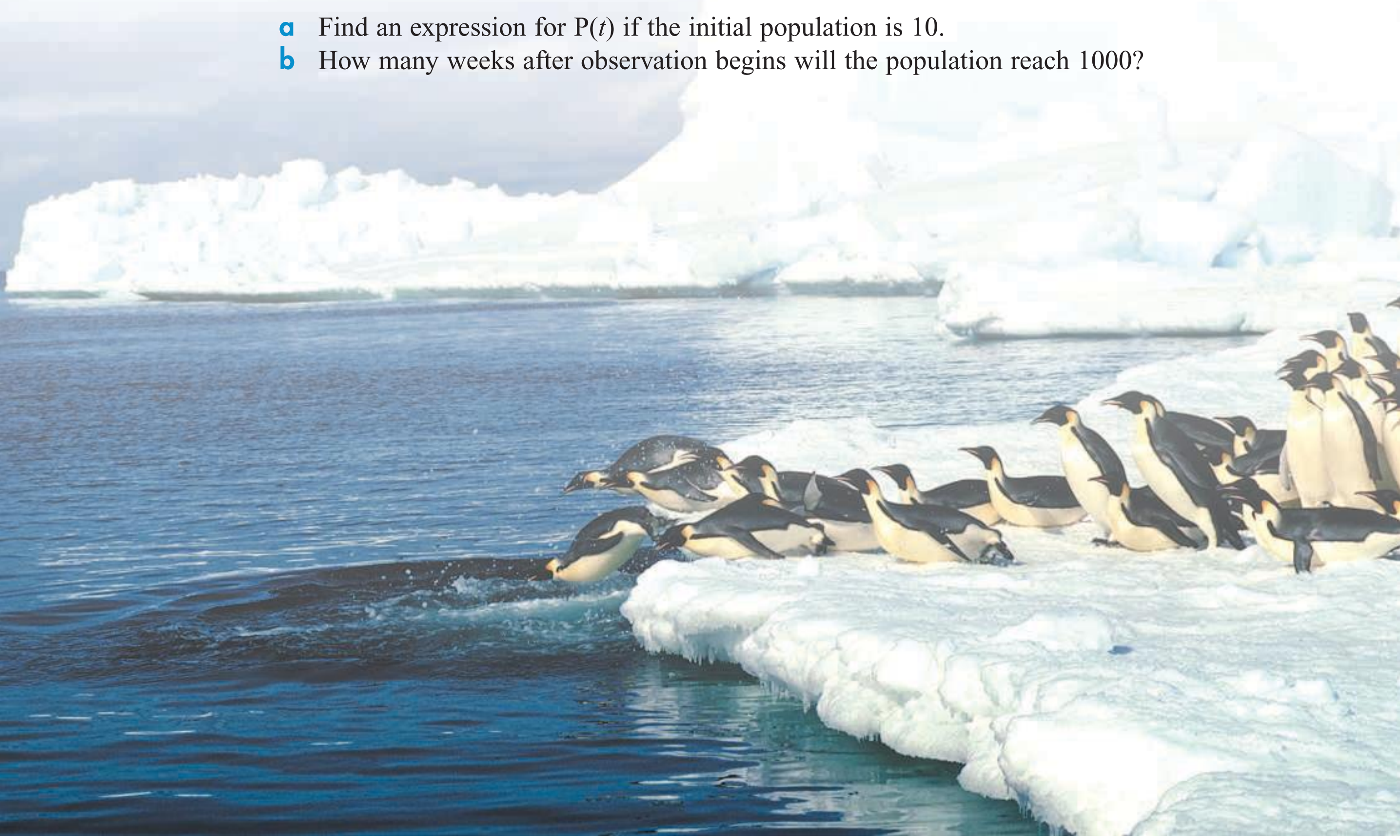
5 The gradient of a curve at any point (x, y) is equal to $(y + 1)^2$. If the curve passes through the point (4, 2), what is the equation of the curve?

WORKED Example 10

6 During the breeding season, the rate of growth of a particular colony of penguins is $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.1P$, where P is the number of penguins t weeks after observation is started.

a Find an expression for P(t) if the initial population is 10.

b How many weeks after observation begins will the population reach 1000?



7 A balloon is expanding at a rate that is inversely proportional to its volume (V cm^3) at any time t seconds. That is, $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{k}{V}$, where k is a constant.

If the initial volume is 10 cm^3 and after 5 seconds it is 40 cm^3, find:

a V at any time t

b the volume after 8 seconds.

8 The rate of radioactive decay of a particular substance is proportional to the amount present, Q, at any time, t. The differential equation which describes this rate is $\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ$, where k is a constant.

Initially $Q = 100$ g and when $t = 20$ years, $Q = 50$ g. Find the time taken for 80% of the initial amount to have decayed.

Setting up and solving differential equations

Many situations arise in economics, botany, biology, physics, chemistry and so on which require a differential equation to model data involving rates of change. We shall now consider problems that involve setting up and solving differential equations from given information.

It is useful to recall that:

1. y is proportional to (or varies directly with) x,
   abbreviates to $y \propto x$
   which means that $y = kx$, where k is a constant.

2. y varies inversely x
   abbreviates to $y \propto \frac{1}{x}$
   which means that $y = \frac{k}{x}$, where k is a constant.

3. Rates are with respect to time unless stated otherwise.

WORKED Example 11

Set up differential equations for the following situations.

a **The volume (V) of water in a tank is decreasing at a rate which is proportional to its current volume.**

b **The gradient of a curve varies inversely with the square of its x-coordinate.**

THINK	WRITE
a ① The rate of change of volume, $\frac{dV}{dt}$, is negative as the rate is decreasing.	**a**
② The current volume is V.	
③ Write the proportionality statement.	$\frac{dV}{dt} \propto V$
④ Write the differential equation with constant $-k$ (decreasing rate).	$\frac{dV}{dt} = -kV$
b ① The gradient is $\frac{dy}{dx}$.	**b**
② The square of the x-coordinate is x^2.	
③ Write the variation statement.	$\frac{dy}{dx} \propto \frac{1}{x^2}$
④ Write the differential equation with constant k.	$\frac{dy}{dx} = \frac{k}{x^2}$

Exponential growth and decay

We have seen that differential equations of the general form $\frac{dy}{dx} = ky$ have a general solution of the form:

$$y = Ae^{kx} \text{ where } y = A \text{ when } x = 0.$$

Since the solution to this differential equation is a function that is exponential, it represents exponential growth if $k > 0$ and exponential decay if $k < 0$.

Note: When antidifferentiating differential equations of the type above, it is sometimes more convenient to express the antiderivative in the form $x = \frac{1}{k}\log_e Ay$, where A is the constant of antidifferentiation. In this case, if the initial condition is $y(0)$, then $Ay(0) = 1$, as $\log_e 1 = 0$. Thus the constant A is the reciprocal of the initial condition $y(0)$.

Either method can be used, but it is usually easier to leave the antiderivative as a logarithm when the values are given as proportions of the initial number or if you are solving for time. This will be clearly demonstrated in Worked examples 13 and 14.

WORKED Example 12

The rate of growth of a town's population (N) is 5% of its population at any time t years. If its population is currently 100 000 people (at $t = 0$) find:

a **the population at any time t**

b **the population in 3 years time.**

THINK	WRITE
a ① The differential equation is $\frac{dN}{dt} = 0.05\,N$. This is because the rate of the population, $\frac{dN}{dt}$, is 5% of the current population; that is, $0.05 \cdot N$.	**a** $\frac{dN}{dt} = 0.05\,N$
② Invert both sides of the differential equation.	$\frac{dt}{dN} = \frac{1}{0.05N}$
③ Simplify the RHS.	$= \frac{20}{N}$
④ Express t in integral notation.	$t = \int \frac{20}{N}\,dN$
⑤ Antidifferentiate the integrand by rule.	$= 20\log_e N + c$
⑥ Isolate the logarithm.	$\frac{t-c}{20} = \log_e N$

Continued over page →

THINK	**WRITE**
⑦ Take the exponent e to the power of both sides to convert the exponential form and make N the subject.	$N = e^{\frac{t-c}{20}}$ $= Ae^{\frac{t}{20}}$ $\left[\textit{Note: } A = e^{\frac{-c}{20}}\right]$
⑧ Substitute $t = 0$ and $N = 100\,000$ into the equation.	When $t = 0$, $N = 100\,000$ $100\,000 = Ae^{0}$
⑨ Solve for A.	$100\,000 = A$
⑩ Replace the value of A in the equation.	$N = 100\,000\,e^{\frac{t}{20}}$
⑪ State the solution.	Therefore, the population at any time is $N = 100\,000e^{0.05\,t}$.
b ① Substitute $t = 3$ into the population equation.	**b** When $t = 3$, $N = 100\,000e^{0.05\,\cdot\,3}$
② Evaluate N.	$N = 100\,000\,e^{0.15}$ $= 116\,183.4$
③ Round down to the nearest whole number (population reached) and state the answer.	Therefore in 3 years the population will be 116 183 people.

WORKED Example 13

A sample of a radioactive isotope of radium loses half of its radioactive nuclei each 1200 years (that is, its half-life is 1200 years). If the rate of decay is proportional to the number of atoms present, find, to the nearest year, the time taken for the isotope to decay to 30% of its original amount.

THINK	**WRITE**
① Let N be the number of atoms present at any time, t years.	Let N = the number of nuclei present at any time, t years.
② Let N_o be the original number of nuclei present.	Let N_o = the original number of nuclei.
③ The differential equation is $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ because the rate of decay, $\frac{\mathrm{d}N}{\mathrm{d}t}$, is proportional to the amount N present. ($k < 0$, as will show in calculations.)	$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$
④ Invert the differential equation.	$\frac{\mathrm{d}t}{\mathrm{d}N} = \frac{1}{kN}$
⑤ Express t in integral form.	$t = \frac{1}{k}\int\frac{\mathrm{d}N}{N}$

THINK	**WRITE**
6 Antidifferentiate the integrand by rule.	$t = \frac{1}{k}\log_e N + c$ $t = \frac{1}{k}\log_e AN$ [*Note:* $c = \log_e A$]
7 Substitute $t = 0$ and $N = N_o$.	When $t = 0$, $N = N_o$
8 Solve for A in terms of N_o.	$AN_o = 1$ $A = \frac{1}{N_o}$
9 Substitute $A = \frac{1}{N_o}$ into the equation.	$t = \frac{1}{k}\log_e \frac{N}{N_o}$
10 Substitute $t = 1200$ and $N = 0.5N$ into the equation (since N is 50% of its original value).	When $t = 1200$, $N = 0.5N_o$ $1200 = \frac{1}{k}\log_e\left(\frac{0.5N_o}{N_o}\right)$
11 Solve for $\frac{1}{k}$.	$1200 = \frac{1}{k}\log_e 0.5$ $\frac{1}{k} = -1731.234$
12 Substitute the value of $\frac{1}{k}$ into the equation.	$t = -1731.234\log_e \frac{N}{N_o}$
13 Leave t the subject of the equation.	
14 Substitute $N = 0.3\ N_o$ into the equation (30% of its original value).	When $N = 0.3N_o$, $t = -1731.234\log_e\left(\frac{0.3N_o}{N_o}\right)$
15 Evaluate t, the time interval for which the number of radioactive nuclei falls to 30% of its initial value.	$= -1731.234\log_e 0.3$ $= 2084.359$
16 Round the value to the nearest year and state the solution.	Therefore it takes approximately 2084 years for the number of radioactive nuclei to be reduced to 30% of its original number.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate at which a body cools is proportional to the excess of its temperature above that of its surroundings.

If T is the temperature of the body and T_s is the temperature of the surroundings,

$$\text{then } \frac{dT}{dt} \propto (T - T_s)$$

$$\text{and } \frac{dT}{dt} = k(T - T_s).$$

That is, the rate that the temperature changes for an object is proportional to the temperature difference between the object and its local environment.

WORKED Example 14

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Tutorial:
Worked example 14
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A loaf of bread at a temperature of 20 °C is placed in a freezer whose temperature is a constant −15 °C. If it takes 18 minutes for the temperature of the bread to drop to 8 °C, how much longer, to the nearest minute, will it take for the bread to reach 0 °C? You may assume that Newton's Law of Cooling applies.

THINK	WRITE
1 $T_s = -15$ as this is the environment temperature.	$\frac{dT}{dt} = k(T - T_s)$ $T_s = -15$
2 Use Newton's Law of Cooling equation with $T_s = -15$.	$\frac{dT}{dt} = k(T + 15)$
3 Invert the differential equation.	$\frac{dt}{dT} = \frac{1}{k(T + 15)}$
4 Express t in integral notation.	$t = \int \frac{dT}{k(T + 15)}$ $= \frac{1}{k} \log_e (T + 15) + c$
5 Antidifferentiate by rule to obtain the general equation for t.	$= \frac{1}{k} \log_e A(T + 15)$ [*Note:* $c = \frac{1}{k} \log_e A$]
6 Substitute $t = 0$ and $T = 20$ into the equation.	When $t = 0$, $T = 20$ $0 = \frac{1}{k} \log_e 35A$
7 Solve the equation for A.	$35A = 1$ $A = \frac{1}{35}$
8 Replace $A = \frac{1}{35}$ back into the equation in step 5.	$t = \frac{1}{k} \log_e \left(\frac{T + 15}{35}\right)$
9 Substitute $t = 18$ and $T = 8$ into the equation.	When $t = 18$, $T = 8$
10 Solve the equation for $\frac{1}{k}$.	$18 = \frac{1}{k} \log_e \left(\frac{23}{35}\right)$ $\frac{1}{k} = -42.872$
11 Replace $\frac{1}{k} = -42.872$ back into the equation.	$t = -42.872 \log_e \left(\frac{T + 15}{35}\right)$
12 Leave t as the subject of the equation and substitute $T = 0$ into it.	When $T = 0$, $t = -42.872 \log_e \left(\frac{15}{35}\right)$
13 Evaluate t.	$= 36.325$
14 To calculate the further time after 18 minutes for the bread to reach 0 °C, subtract 18 from 36.325.	The extra time after 18 minutes is $36.325 - 18 = 18.325$.
15 Round the time down to the nearest minute and state the answer.	Therefore it takes approximately a further 18 minutes for the temperature of the bread to drop from 8 °C to 0 °C.

remember

1. If $y \propto x$, then $y = kx$.
2. If $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$.
3. Differential equations of the form $\frac{dy}{dx} = ky$ have solutions of the form $y(x) = Ae^{kx}$ where $y(0) = A$ and consequently the value of A gives the initial value.
4. Newton's Law of Cooling states that the rate of temperature change $\frac{dT}{dt}$ of an object is proportional to the difference in temperature of the object T and its surroundings T_s. This can be expressed as $\frac{dT}{dt} = k(T - T_s)$.

EXERCISE 5D Setting up and solving differential equations

WORKED Example 11

1 Set up differential equations that describe the following situations. (Do not solve.)

a A curve $y = f(x)$ whose gradient at any point is the square of the x-coordinate.

b The gradient of a curve $y = f(x)$ is proportional to the cube root of the y-coordinate.

c The population (P) of a town is decreasing at a rate that is proportional to its population at any time.

d The rate of a chemical reaction (v) is inversely proportional to the total amount of reactants present (x).

e The rate at which a balloon rises $\frac{dh}{dt}$ varies directly with the height, h, of the balloon.

f The rate of change of the speed of a car after the brakes are applied is proportional to the square root of time, t, cubed (where t is the interval of time that the brakes have been applied).

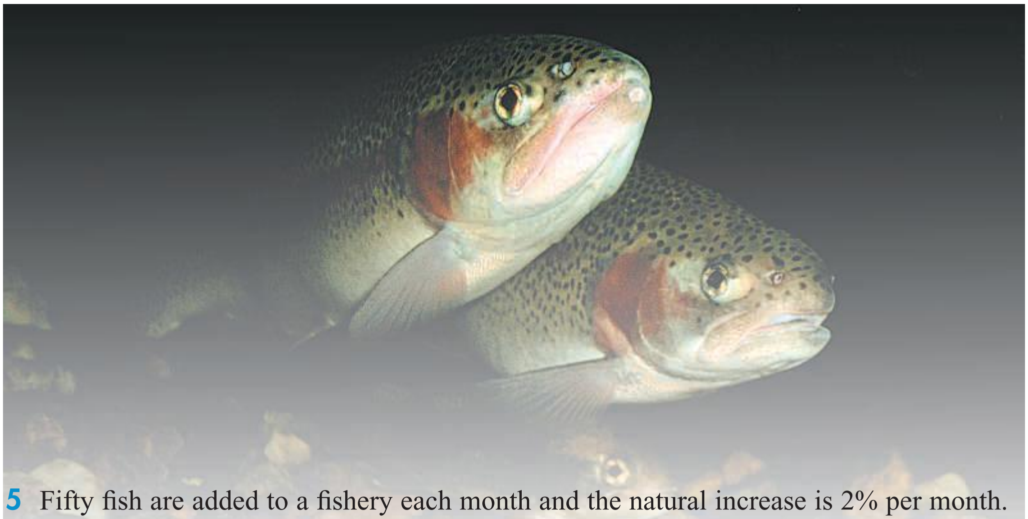
WORKED Example 12

2 A city's population is growing at a rate proportional to its current size. If the population was 50 000 at the beginning of 1998 and 60 000 at the beginning of 2008, predict the population at the beginning of 2013. [*Hint:* Take $t = 0$ years to be the beginning of 1998.]

WORKED Example 13

3 A substance is decaying radioactively at a rate proportional to its mass, m kg, such that at any time t years $\frac{dm}{dt} = -m$. If the initial mass is m_o, find the time taken for the mass of the substance to halve to $\frac{m_o}{2}$.

4 The number of bacteria in a food sample, t hours after being placed in the refrigerator, decreases at a rate proportional to its present number. If the number of bacteria halves after one hour, find the time taken for the number of bacteria to reach 10% of its initial value.

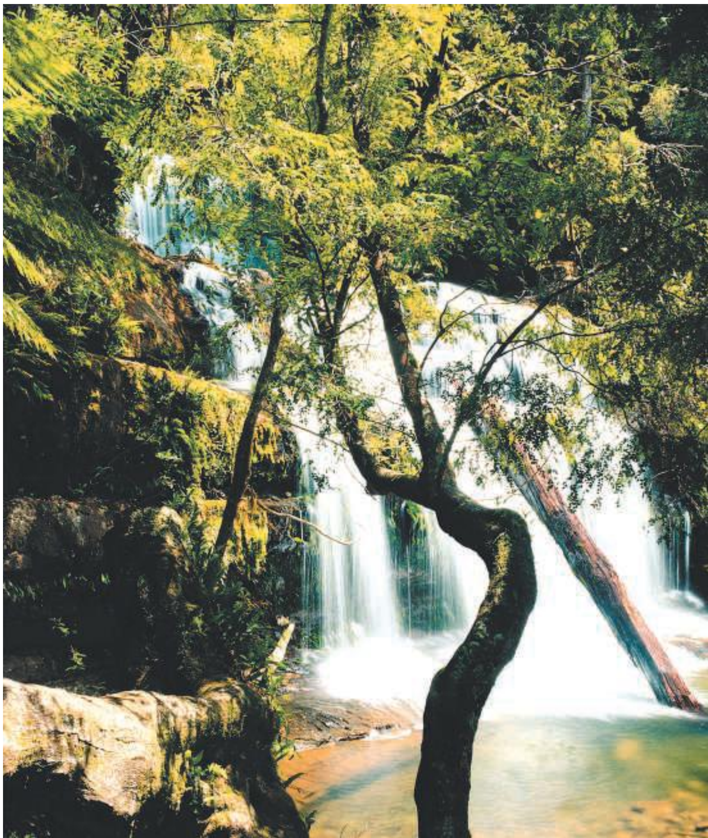


5 Fifty fish are added to a fishery each month and the natural increase is 2% per month.

a Find an expression for $\frac{dP}{dt}$, the growth rate of the fish population.

b If the initial number of fish is 1000, predict the number after one year.

6 To help protect the environment in national parks, a research group has been set up to develop environmentally friendly containers for foods commonly used by park visitors. One type of container material decomposes at a rate proportional to the amount of material present, Q. If it takes 2 weeks to be reduced by 10%, how long will it take to be reduced by 60%?



WORKED Example 14

7 A rod that has been heated to 110 °C is placed into a large tank of water whose temperature is a constant 10 °C. After 2 minutes the temperature of the rod is 70 °C. Assuming Newton's Law of Cooling applies, find the temperature of the rod after a further 2 minutes.

8 The temperature of a cup of coffee drops from 95 °C to 75 °C in 5 minutes in a room where the temperature remains at 15 °C. If Newton's Law of Cooling applies, find:

a the time taken for the temperature of the coffee to reach 50 °C

b the temperature of the coffee at any time, t, and hence after 2 minutes.

9 The temperature of a freezer is a constant −6 °C. A tray of water that is 12 °C is placed in the freezer, and its temperature drops by 4 °C in one minute. Find:

a the time taken for the tray of water to reach freezing point, 0 °C (to the nearest second)

b the temperature of the tray at any time, t, and hence after 6 minutes (to the nearest degree).

10 A mother boils a cup of milk to 100 °C for her young child. The temperature of the milk drops to 90 °C in 2 minutes in a room where the temperature is a constant 20 °C. The child will drink the milk only if its temperature is between 55 °C and 70 °C. If Newton's Law of Cooling applies, how long does the child have to drink all of the milk?

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WorkSHEET 5.2

Double decay

Consider two radioactive elements, A and B. Each second, 4% of A changes into B. B also decays and each second, 2% of B changes into C. Initially there are 800 g of A and 100 g of B.

Construct a spreadsheet showing the amount of material B over time.

Try to obtain the values obtained in your spreadsheet by solving the differential equations:

$$\frac{da}{dt} = -0.04a$$

$$\frac{db}{dt} = 0.04a - 0.02b$$

where a is the amount of A
b is the amount of B
c is the amount of C

Hint: $\frac{db}{dt} + 0.02b = \frac{1}{e^{0.02t}} \frac{d}{dt}(e^{0.02t}\, b)$.

History of mathematics

MARIA AGNESI (16 May 1718 – 9 January 1799)

During her lifetime ...

Easter Island is discovered with its giant statues.

Vivaldi writes The Four Seasons.

Benjamin Franklin establishes the first police force.

Celsius develops a scale for measuring temperature.

Maria Agnesi was an Italian mathematician. She was born in Milan and her father was a professor of mathematics at the University of Bologna. Her large family was wealthy and well-educated and their home was a meeting place for intellectuals. Agnesi, a child prodigy, was the eldest of 21 children. By the age of 9 she could read and speak Latin, Greek, Hebrew, Italian, French, German and Spanish. During her teens she studied mathematics, tutored the younger children in the family and served as hostess at the discussions held at her father's house.

Until the 20th century very few women in Europe or elsewhere received more than a very basic education. As a result, women were unable to work or carry out research in academic disciplines. Agnesi was an exception. When she was 20 years old she published a book of essays called *Propositiones Philosophicae*. For the next ten years she worked on her most famous book, *Analytic Institutions for the Use of Italian Youth*. It was one of the first and most complete works on calculus and was widely used as a textbook. In it she organised and analysed the work of many mathematicians, together with her own original interpretations. When the book was published in 1748 it caused a sensation and was praised for its clarity.

After the success of her book, Agnesi accepted a position at the University of Bologna and was eventually elected to the Bologna Academy of Sciences. One of her solutions for a differential equation has become known as 'the witch of Agnesi'. The solution to the equation follows a curve. Agnesi referred to this curve as a 'versiera' which in Italian means 'turn' or 'bend'. When translated into English *versiera* became confused with the Italian word *avversiera* which means 'witch'.

When her father died, Maria devoted the rest of her life to helping the poor and the sick, especially women. She never married, and died at the age of 81.

Questions

1. Which field of mathematics was the subject of Agnesi's most famous book?
2. Why did very few women work or carry out research in academic disciplines until the 20th century?
3. Which university offered a position to Agnesi?
4. Which curve is named after her?

Research

Find the equation for 'the witch of Agnesi'. Use a graphics calculator to investigate the curve for various values of a.

summary

Related rates

- Differential equations may be first or second order. First order differential equations contain (at most) first derivatives. Second order differential equations contain (at most) second derivatives and may also have first derivatives.
- 1st derivatives are $\frac{dy}{dx}$, $f'(x)$ or y'.
- 2nd derivatives are $\frac{d^2y}{dx^2}$, $f''(x)$ or y''.
- A rate of change is a derivative.
- Chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. This rule is often used when $\frac{dy}{du}$ is known and $\frac{du}{dx}$ is also known. This allows for $\frac{dy}{dx}$ to be determined.
- Reciprocal rule: $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$
- Increasing rates are positive.
- Decreasing rates are negative.
- Solid geometry formulas:
  Total surface area of a cone $= \pi r(s + r)$
  Total surface area of a sphere $A = 4\pi r^2$, thus $\frac{dA}{dr} = 8\pi r$.
  Volume of a cone $= \frac{1}{3}\pi r^2 h$, $\frac{dV}{dr}$ or $\frac{dV}{dh}$ can be found.
  Volume of a sphere $= \frac{4}{3}\pi r^3$, thus $\frac{dV}{dr} = 4\pi r^2$

Solving differential equations of the type $\frac{dy}{dx} = f(x)$

- If $\frac{dy}{dx} = f(x)$,

  then $y = \int f(x)\, dx$

  $= F(x) + c$ where $F(x)$ is the antiderivative of $f(x)$ and c is a constant of integration.

  When initial conditions are given then c can be evaluated giving a particular solution from the general solution.

Solving differential equations of the type $\frac{dy}{dx} = g(y)$

- If $\frac{dy}{dx} = g(y)$,

  then $\frac{dx}{dy} = \frac{1}{g(y)}$

  and $x = \int \frac{1}{g(y)}\,dy + c$.

  The equation can often be rearranged to make y the subject. The value for c can be found from the initial conditions.

Setting up and solving differential equations

- If $y \propto x$, then $y = kx$.
- If $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$.
- Differential equations of the form $\frac{dy}{dx} = ky$ have solutions of the form $y(x) = Ae^{kx}$ where $y(0) = A$ and consequently the value of A gives the the initial value.
- Newton's Law of Cooling states that the rate of temperature change $\frac{dT}{dt}$ of an object is proportional to the difference in temperature of the object, T, and its surroundings, T_s. This can be expressed as $\frac{dT}{dt} = k(T - T_s)$.

CHAPTER review

1 **multiple choice** 5A

If $\frac{dV}{dr} = 2r$ and $\frac{dr}{dt} = 5$, then when $r = 4$, $\frac{dV}{dt}$ is equal to:

A $10r$ **B** 8 **C** 20 **D** 1.6 **E** 40

2 **multiple choice** 5A

The sides of a square are increasing at a constant rate of 0.4 cm/min. At what rate is the area increasing when the side length is 8 cm?

A 0.1 cm^2/min **B** 6.4 cm^2/min **C** 10 cm^2/min **D** 1.6 cm^2/min **E** 25.6 cm^2/min

3 **multiple choice** 5A

A spherical balloon of radius r metres is leaking at a constant rate of 2 m^3/h. If the balloon retains its shape, the rate at which the radius is decreasing when the radius is 3 metres is equal to:

A $\frac{1}{18\pi}$ m/h **B** 36π m/h **C** $\frac{1}{36\pi}$ m/h **D** 72π m/h **E** $-\frac{1}{6\pi}$ m/h

4 An ice cube is melting so that its side length is decreasing at a constant rate of 0.2 cm/s. Find the rate at which the volume is decreasing when the side length is 3 cm, assuming the ice cube retains its shape. 5A

5 **multiple choice** 5B

The general solution to $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$ is:

A $y = -\sqrt{1-x^2} + c$ **B** $y = \text{Sin}^{-1}x + c$ **C** $y = x\,\text{Sin}^{-1}x + c$

D $y = \sqrt{1-x^2} + c$ **E** $y = -(1-x^2) + c$

6 **multiple choice** 5B

If $f'(x) = \frac{2}{x^2-1}$ then $f(x)$ is equal to:

A $\log_e\left(\frac{x+1}{x-1}\right) + c$ **B** $\log_e(x^2-1) + c$ **C** $2\log_e(x^2-1) + c$

D $2\,\text{Tan}^{-1}x + c$ **E** $\log_e\left(\frac{x-1}{x+1}\right) + c$

5B

7 **multiple choice**

If $g'(x) = \dfrac{4}{x^2 + 1}$ and $g(0) = 3$ then $g(x)$ is equal to:

A $4 \log_e (x^2 + 1) + 3$
B $4 \text{Tan}^{-1} x + 3$
C $3 + 4 \log_e 2x$
D $3 + \frac{1}{4} \text{Tan}^{-1} x$
E $-1 + 4 \text{Tan}^{-1} x$

5B

8 The gradient of a curve at any point (x, y) is equal to $4x$ and the curve passes through the point $(-1, 1)$. What is the equation of the curve?

5C

9 **multiple choice**

The general solution to $\dfrac{dy}{dx} = 2y + 3$ is of the form:

A $y = e^{2x} - 3$
B $y = e^{2x + c} - 3$
C $y = \frac{1}{2} e^{2x + c} - 3$
D $y = \frac{1}{2} e^{2x + c} + 3$
E $y = \frac{1}{2} (e^{2x + c} - 3)$

5C

10 **multiple choice**

Given that $\dfrac{dP}{dn} = e^{-2P}$ and $P = 0$ when $n = 1$, then the solution is:

A $P = \frac{1}{2} \log_e (2n - 1)$
B $P = \log_e (2n - 1)$
C $P = -\log_e (2n + 1)$
D $P = \frac{1}{2} \log_e (1 - 2n)$
E $P = -\frac{1}{2} \log_e (2n - 1)$

5C

11 **multiple choice**

The rate of increase of the volume of a container is inversely proportional to the square of the volume, V, of the container at any time, t. The differential equation which describes this situation (where k is a constant) is:

A $\dfrac{dV}{dt} = kV^2$
B $\dfrac{dV}{dt} = \dfrac{k}{V^2}$
C $\dfrac{dV}{dt} = k\sqrt{V}$
D $\dfrac{dV}{dt} = \dfrac{k}{\sqrt{V}}$
E $\dfrac{dV}{dt} = \dfrac{k}{V}$

5C

12 **multiple choice**

The rate of increase in a town's population varies directly with the population, P, at any time, t. The differential equation for this situation (where k is a constant) is:

A $\dfrac{dP}{dt} = \dfrac{k}{P}$
B $\dfrac{dP}{dt} = \dfrac{P}{k}$
C $\dfrac{dP}{dt} = kP^2$
D $\dfrac{dP}{dt} = kP$
E $\dfrac{dt}{dP} = kP$

5C

13 **a** Find the solution to the differential equation $\dfrac{dy}{dx} = \tan y$, given that $y = \dfrac{\pi}{2}$ when $x = 2$.

b State the domain for which the solution exists.

The following information applies to questions **14** to **16**.
An object whose temperature is 50 °C is placed in a room of constant temperature 12 °C. Newton's Law of Cooling applies and the object cools to 30 °C in 20 minutes.

14 **multiple choice** 5D

The differential equation which describes this situation is:

A $\frac{dT}{dt} = T - 12$ **B** $\frac{dT}{dt} = k(T - 30)$ **C** $\frac{dT}{dt} = k(T - 12)$

D $\frac{dT}{dt} = 50k(T - 12)$ **E** $\frac{dT}{dt} = 12(T - 50)$

15 **multiple choice** 5D

The time, t, expressed as a function of the temperature, T, is approximately:

A $t = 26.77 \log_e\left(\frac{T - 12}{38}\right)$ **B** $t = 0.0374 \log_e\left(\frac{T + 30}{20}\right)$ **C** $t = -0.0374 \log_e\left(\frac{T + 12}{38}\right)$

D $t = 7.89 \log_e\left(\frac{T - 12}{62}\right)$ **E** $t = -26.77 \log_e\left(\frac{T - 12}{38}\right)$

16 **multiple choice** 5D

The time taken for the object to reach 15 °C is, to the nearest minute:

A 68 min **B** 9 min **C** 50 min **D** 22 min **E** 105 min

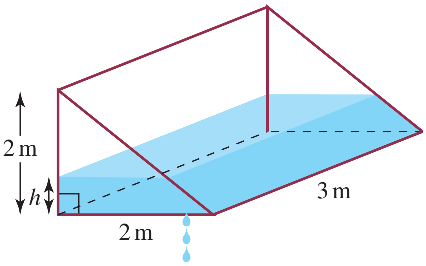
17 A mountain climber stops for a rest when the temperature is a constant –2 °C and pours a hot cup of soup which is initially 90 °C. After 2 minutes the temperature of the soup is 60 °C. What will be the temperature of the soup 5 minutes after it was poured, assuming Newton's Law of Cooling? 5D

18 The number of ants in a colony is increasing at a rate proportional to the number present on any day. If the number of ants increases by 50% in 100 days, how much longer will it be until the population is double the initial number? (Answer to the nearest day.) 5D

Modelling and problem solving

1 Water is leaking from the tank shown below at a constant rate of 0.03 m^3/min.

- **a** Find an expression for the volume, V, in terms of the depth, h.
- **b** Find the rate of change of depth, to the nearest mm/min, when:
  - **i** the depth is 1 metre
  - **ii** the tank is half full.



2 The number of ice-creams sold per day (N) at an ice-cream stand varies with the number of degrees Celsius (T) that the daily maximum temperature is above 15 °C:

$$\frac{dN}{dT} = 64\,420.4N^{-\frac{3}{2}},\ T \geq 0$$

If no ice-creams are sold when the daily maximum is 15 °C:

a express N as a function of T

b find the number of ice-creams sold when the maximum temperature is 30 °C

c find the maximum temperature if 280 ice-creams are sold one day

d on a particular day the kiosk can make only 440 ice-creams. What is the maximum temperature at which the kiosk will not sell out of ice-creams?

3 A sailor, whose core body temperature is 37 °C, falls from a ship into icy waters of temperature 1 °C. The temperature of the sailor drops to 36 °C in 5 minutes and he will not survive if his temperature drops below 35 °C.

a Assuming Newton's Law of Cooling is applicable, for how long can the sailor survive in the water after he first falls in?

Two minutes after the sailor falls in, the ship is 200 metres away and a rescue boat is despatched. Due to the seas and visibility the boat travels to the sailor according to the differential equation $\frac{dx}{dt} = \frac{5}{\sqrt{t+1}}$, where x is the distance travelled by the boat towards the sailor in metres and t is the time in seconds.

b Express x as a function of t.

c How long has the sailor been in the water when the rescue boat arrives?

d Hence, decide whether the sailor is still alive or not.

4 The rate at which a particular drug is absorbed by the body is proportional to the amount of the drug present (D) at any time t.

If 50 mL is initially administered to a patient and 50% remains after 4 hours, find:

a D as a function of t

b the amount present after 6 hours.

It is dangerous for the patient if the level of the drug in the body is below 10 mL.

c How long is it, after the initial dose was given, until another dose is required?

d If the patient is given a second dose of 50 mL at this time, how long will it take for the danger level to be reached again? (Assume that the second dose is absorbed separately but at the same rate as the first dose, and the amount present is the sum of the two doses.)

5 Matthew knows that a particular species of bacterium will grow at a rate proportional to the number of bacteria present. He sets up the experiment and leaves it for a week. From past experience, he expects his sample to have tripled in this time.

Holly wants to test a solution that will double the growth rate. She takes a quarter of Matthew's original sample. Once she's tested her solution, she plans to keep half and return the other half to Matthew's sample.

Assuming that the growth rate is only doubled while in Holly's solution, when should she return the bacteria if Matthew is to find the correct number of bacteria when he returns?

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Digital doc:
Test Yourself
Chapter 5

eBookplus ACTIVITIES

5A Differential equations: related rates

Tutorial

- int-0391: Watch a tutorial on how to use related rates to calculate the rate of change of the depth of water in a conical container (*page 202*)

5B Differential equations of the form dy/dx = *f*(*x*)

Tutorial

- int-0393 Watch how to find a particular solution to a differential equation. (*page 207*)

Digital docs

- WorkSHEET 5.1: Find particular solutions to differential equations and solve related rates problems. (*page 211*)
- SkillSHEET 5.1: Practise finding derivatives of expressions other than polynomials. (*page 207*)

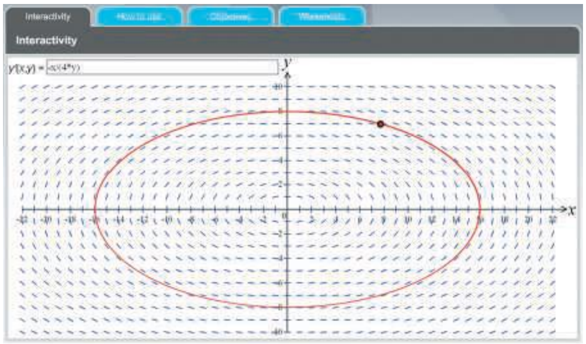
5C Differential equations of the form dy/dx = *g*(*y*)

Interactivity

- int-0348: Investigate solutions to differential equations using slope fields. (*page 216*)

Tutorial

- int-0395: Watch a tutorial on how to find a particular solution to a differential equation, and state the largest possible domain for which the solution applies. (*page 213*)



5D Setting up and solving differential equations

Tutorial

- int-0396: Watch how to apply Newton's law of cooling to solve differential equation. (*page 222*)

Digital doc

- WorkSHEET 5.2: Find general and particular solutions to differential equations. (*page 225*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 233*)

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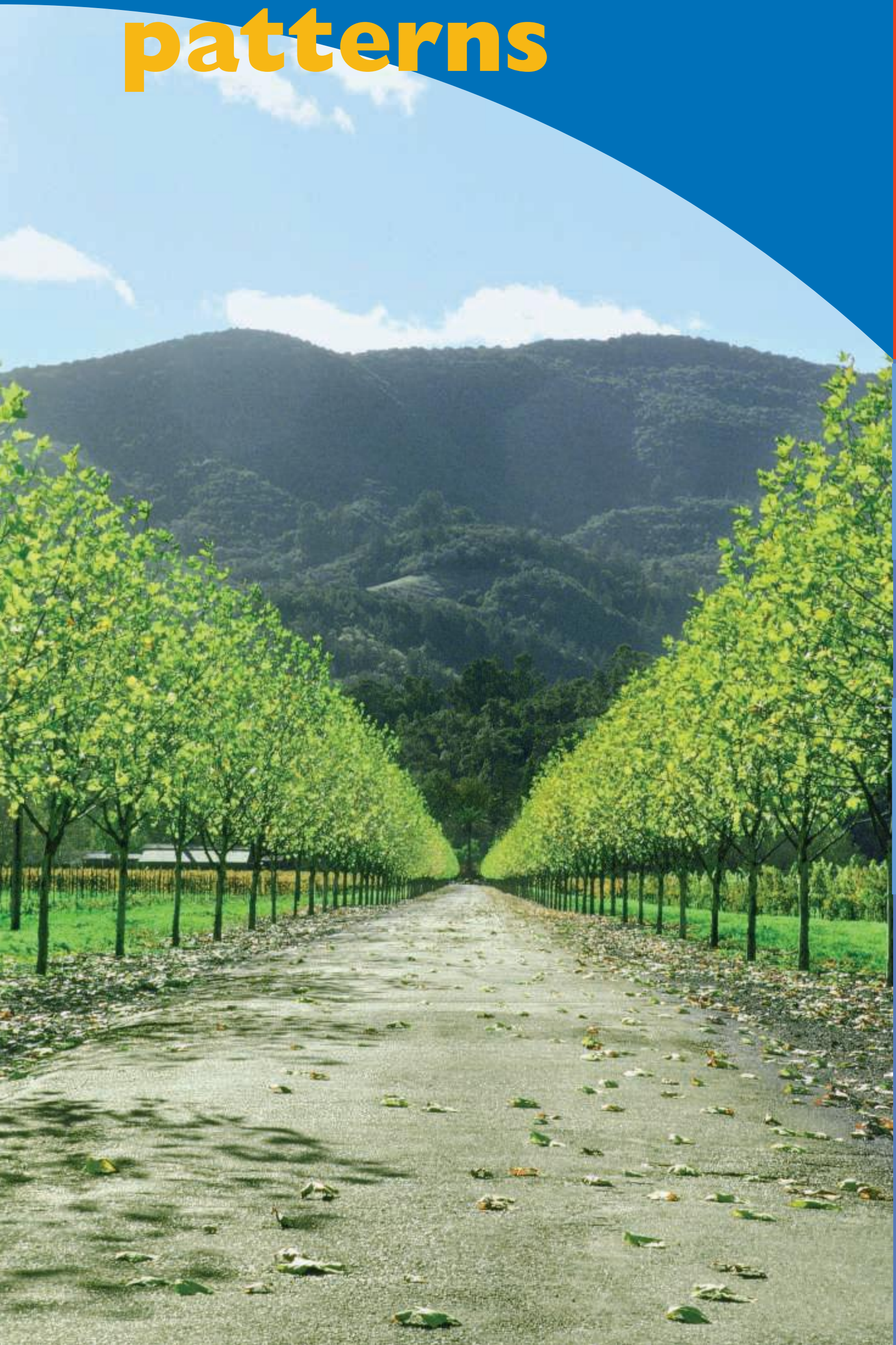
6

Structures and patterns



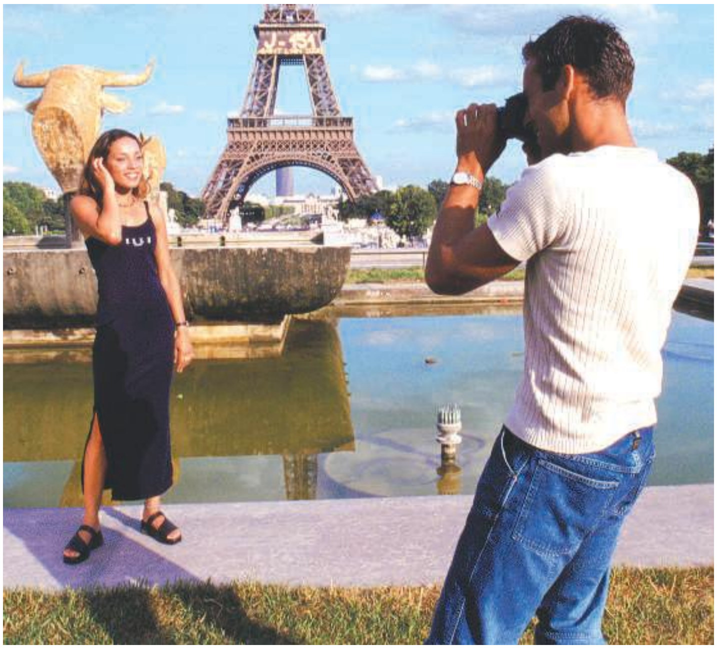
syllabus reference

Core topic:
Structures and patterns

In this chapter

6A Sequences and series
6B Mathematical induction
6C Finite differences

Structures and patterns



PARIS IN THE
THE SPRING.

Over time, human beings have evolved many powerful thinking strategies. To illustrate one of these thinking strategies please read, again, the text beside the photograph above.

If you are like most people you would have read the words ‘Paris in the spring’ not noticing that the word *THE* appears twice. The reason for this is that you are an efficient reader. You do not read each word individually but rather your eyes and brain scan the text, quickly imposing meaning on it. If you were not able to do this, you would find that reading was a slow and laborious task.

The skill of looking at a set of numbers and imposing order on them is also a useful thinking strategy.

Consider the set of numbers:

3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63 …

If you needed to communicate this set of numbers to another person you might give the entire list, or instead you might give the rule: 'Start at 3 and go up by 4 each time'. Clearly the second method is much more efficient. This method of communication is used by software that exchanges images between computers. An image can be exchanged by giving a complete digital description of each part of the image or it can be described using a rule to generate the image. The latter technique has allowed us to greatly reduce the amount of data transmitted on the Internet.

In this chapter we shall look at formal methods for identifying and communicating rules that connect sets of numbers. We begin with a revision of arithmetic and geometric sequences and series.

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Interactivity:
First order difference equations
int-0187

Sequences and series

Two common types of sequence are *arithmetic* and *geometric*. It is important to note that not all sequences fall into these two categories. In this section we look at arithmetic and geometric sequences in detail.

Arithmetic sequences and series

A sequence such as:

$$3, 7, 11, 15, 19, \ldots$$

is called an *arithmetic sequence* because there is a constant difference, in this case 4, between successive terms.

In general terms, an arithmetic sequence is represented by:

$$a, a + d, a + 2d, a + 3d, \ldots$$

Here, a is the first term, T_1, $(a + d)$ is the second term, T_2, and each successive term increases by the constant difference, d.

For an arithmetic sequence the general term, the nth term, is given by:

$$T_n = a + (n - 1)d.$$

The sum of an arithmetic sequence is called an *arithmetic series*, S_n,

$$S_n = a + (a + d) + (a + 2d) + \ldots [a + (n - 1)d].$$

(Notice that this series contains n terms.)

The formula for the sum of this series is

$$S_n = \frac{n}{2}[2a + (n - 1)d].$$

WORKED Example 1

Consider the sequence 3, 4.5, 6, 7.5, 9, 10.5, 12, …

a Write a formula for the *n*th term, T_n.

b What is the 100th term?

c What is the sum of the first 100 terms?

THINK		WRITE
a ①	Recall the formula for the general term of an arithmetic sequence.	a $T_n = a + (n - 1)d$
②	Identify *a* (the first term) and *d* (the constant difference).	$a = 3, d = 1.5$
③	Substitute the values for *a* and *d* into the formula and simplify.	$T_n = 3 + (n - 1)1.5$ $= 1.5n + 1.5$
b ①	Use the formula from part **a** and substitute for *n*.	b $T_n = 1.5n + 1.5$ $T_{100} = 1.5 \cdot\ 100 + 1.5$
②	Simplify.	$= 151.5$
③	Write the answer.	The 100th term is 151.5.
c ①	Write the general formula for the sum of an arithmetic series.	c $S_n = \frac{n}{2}[2a + (n - 1)d]$
②	Substitute for *n*, *a* and *d* and simplify.	$S_{100} = \frac{100}{2}(2 \cdot\ 3 + 99 \cdot\ 1.5)$ $= 7725$
③	Write the answer.	The sum of the first 100 terms is 7725.

Geometric sequences and series

A sequence such as:

$$3, 6, 12, 24, 48, \ldots$$

is called a *geometric sequence* because there is a constant ratio, in this case 2, between successive terms. An example would be a population of bacteria that doubled every hour.

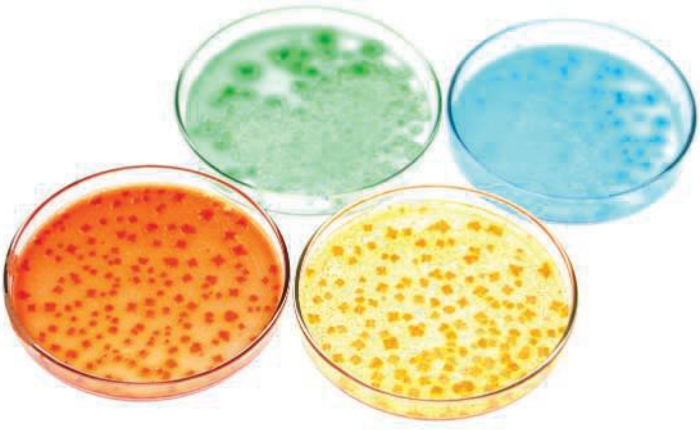
In general terms a geometric sequence is represented by:

$$a, ar, ar^2, ar^3, \ldots$$

Here, *a* is the first term, T_1, *ar* is the second term, T_2, and each successive term changes by the constant ratio *r*.

For a geometric sequence the general term, the *n*th term, is given by:

$$T_n = ar^{n-1}.$$



The sum of a geometric sequence is called a *geometric series* and is represented by the symbol S_n.

$$S_n = a + ar + ar^2 + ar^3 + \ldots ar^{n-1}$$

The formula for the sum of this series is $S_n = \dfrac{a(r^n - 1)}{r - 1}$.

Note: This formula may be written as $S_n = \dfrac{a(1 - r^n)}{1 - r}$ and when $|r| < 1$ some people prefer to use this version.

WORKED Example 2

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Tutorial:
Worked example 2
int-0449

Consider the sequence:

3, 6, 12, 24, 48, …

a **Write a formula for the *n*th term, T_n.**

b **What is the 10th term?**

c **What is the sum of the first 10 terms?**

THINK	WRITE
a ① Recall the formula for the general term of a geometric sequence.	**a** $T_n = ar^{n-1}$
② Identify *a* (the first term) and *r* (the constant ratio).	$a = 3, r = 2$
③ Substitute the values for *a* and *r*.	$T_n = 3 \cdot 2^{n-1}$
b ① Use the formula from part **a** and substitute for *n*.	**b** $T_n = 3 \cdot 2^{n-1}$ $T_{10} = 3 \cdot 2^9$
② Simplify.	$= 1536$
③ Write the answer.	The 10th term is 1536.
c ① Write the general formula for the sum of a geometric series.	**c** $S_n = \dfrac{a(r^n - 1)}{r - 1}$
② Substitute for *n*, *a* and *r* and simplify.	$S_{10} = \dfrac{3(2^{10} - 1)}{2 - 1}$ $= 3069$
③ Write the answer.	The sum of the first 10 terms is 3069.

WORKED Example 3

A psychologist conducts an experiment in which a subject is shown twenty words. The subject is then tested to determine the number of words that the subject can recall over subsequent days. The results are recorded below:

Day of test	1	2	3	4
Number of words recalled	16	12	9	6.75

If the sequence continues in this way, how many words will be recalled on the 7th day?

THINK	WRITE
1 Determine whether the sequence is arithmetic or geometric. If it is arithmetic there will be a constant difference between terms. Test for this first.	$T_2 - T_1 = 12 - 16 = -4$ $T_3 - T_2 = 9 - 12 = -3$ The sequence is not arithmetic.
2 Test for the geometric sequence.	$\frac{T_2}{T_1} = \frac{12}{16} = 0.75$ $\frac{T_3}{T_2} = \frac{9}{12} = 0.75$ $\frac{T_4}{T_3} = \frac{6.75}{9} = 0.75$ There is a constant ratio between terms. The sequence is geometric.
3 Recall the general formula for a term in a geometric sequence.	$T_n = ar^{n-1}$
4 Identify a and r.	$a = 16$ and $r = 0.75$
5 Substitute a and r into the formula.	So $T_n = 16 \cdot (0.75)^{n-1}$
6 Find the 7th term by substituting 7 for n.	$T_7 = 16 \cdot (0.75)^{7-1}$ $= 2.85$
7 Answer the question.	Seven days after the test, 2.85 words would be recalled.

Recursive definition

In the previous discussion of arithmetic and geometric sequences, an explicit formula was given for T_n in each case.

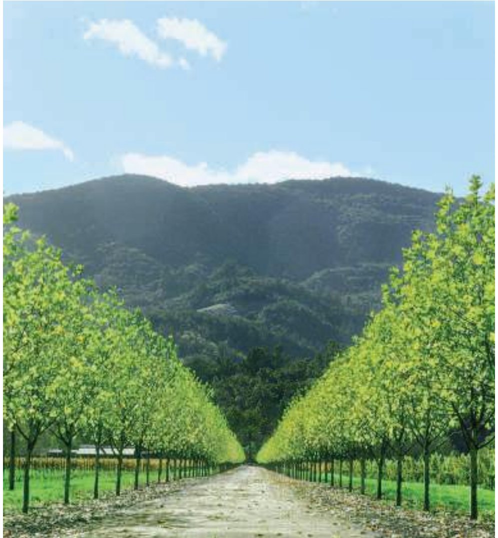
For the arithmetic sequence: $T_n = a + (n - 1)d$.

Note that if you are given a, d and n you can compute T_n directly.

Another common method used to describe a sequence of numbers is the *recursive definition*.

Consider the sequence:

$$2, 5, 11, 23, 47, \ldots$$



Can you see the pattern? The sequence is not arithmetic as there is no constant difference and the sequence is not geometric as the ratio between terms is not constant. However, the pattern can be articulated reasonably easily. The sequence is constructed by the rule:

Start at 2 and double and add 1 at each stage.

This can be written:

$$T_{n+1} = 2 \cdot T_n + 1 \text{ and } T_1 = 2.$$

A definition where the next term is generated using earlier terms is called a *recursive definition*.

WORKED Example 4

a List the first 5 terms of the sequence defined by:

$$\boldsymbol{T_{n+1} = 2 \cdot (T_n - 1),\ T_1 = 3.}$$

b For the sequence 3, 5, 7, 9, …, give a recursive formula for T_n.

THINK	WRITE
a 1 State the first term as this is given.	**a** $T_1 = 3$
2 Find the second term by substituting $n = 2$ into the recursive definition. Use the value for the first term.	$T_2 = 2(T_1 - 1)$ $= 2(3 - 1)$ $= 4$
3 Find the next three terms in the same way. Each term will be generated from the value of the previous term.	$T_3 = 2(T_2 - 1)$ $= 6$ $T_4 = 2(T_3 - 1)$ $= 10$ $T_5 = 2(T_4 - 1)$ $= 18$
4 Write the answer.	The first 5 terms of the sequences are 3, 4, 6, 10 and 18.
b Find a recursive formula for T_n. This will involve T_{n-1} and a value for the first term. Each term is 2 more than the previous term.	**b** $T_n = T_{n-1} + 2$ with $T_1 = 3$

Another mathematical device that is widely used in sequences and series is called *sigma notation*. The Greek letter, Σ (capital sigma), is used to indicate the sum of a sequence. For example:

$$\sum_{n=1}^{20} T_n \text{ stands for } T_1 + T_2 + T_3 + \ldots + T_{20}.$$

The limits of the sum (the numbers on the bottom and top of the Σ) indicate the terms that are to be included in the sum. When there is no chance of misinterpretation, the lower limit, ‘$n = 1$’, may be abbreviated to ‘1’.

WORKED Example 5

Write in detail the sum expressed by $\sum_{3}^{5} r^2$.

THINK

Write $\sum_{3}^{5} r^2$ in full by listing the values of r^2 to be added as r ranges from 3 to 5.

WRITE

$\sum_{3}^{5} r^2 = 3^2 + 4^2 + 5^2$

A graphics calculator can produce a sequence if the rule is known and then calculate the sum of the terms of that sequence. The following worked example illustrates some of the features of the calculator that can be used in this context.

WORKED Example 6

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Tutorial:
Worked example 6
int-1054

a **For the sequence 3, 5.5, 8, …**

i **find the 100th term**

ii **calculate the sum of the first 30 terms.**

b **For the sequence defined by $T_{n+1} = 2T_n - 1$, $T_0 = 3$,**

i **find the 10th term**

ii **calculate the sum of the first 20 terms.**

THINK

For the CASIO *fx*-9860G AU

a **i** ① First, identify whether the sequence is arithmetic or geometric, and list a and d or r as appropriate.

② Write the formula for the nth term.

③ Use the graphics calculator to find the 100th term.

(a) To store the values $a = 3$ and $d = 2.5$, press:

- MENU
- 1 (RUN)

Complete the entry lines as:
3→A
2.5→D
Press EXE after each line.

WRITE/DISPLAY

a **i** The sequence is arithmetic with $a = 3$ and $d = 2.5$.

Therefore, $T_n = 3 + (n - 1) \cdot\ 2.5$.

```
3→A
                3
2.5→D
              2.5
```

THINK | **WRITE/DISPLAY**

(b) To create a table of terms, press:
- MENU
- 8 (RECUR)
- F3 (TYPE)
- F1 (a_n)

Complete the entry line as:
$a_n = A + (n - 1) \cdot D$

```
Recursion
an=A+(n-1)×D        [—]
bn:                 [—]
cn:                 [—]
```

(c) To enter the range, press F5 (SET) and enter the table range as shown.

```
Table Setting    n
 Start:1
 End  :100
```

(d) To display the table, press:
- EXE
- F6 (TABL)

Scroll down to find the 100th term.

```
      n      an
     97     243
     98   245.5
     99     248
    100   250.5
                  100
```

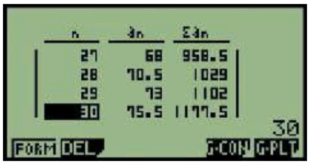
④ Write the answer.

$T_{100} = 250.5$

ii ① To find the sum of the first 30 terms, press:
- SHIFT
- MENU (SET UP)

Turn the Σ display on, by pressing F1 (On)
Then press EXE and return to Table.
Scroll down to $n = 30$.

ii



```
      n      an     Σan
     27      68   958.5
     28    70.5    1029
     29      73    1102
     30    75.5  1177.5
                        30
```

② Write the answer.

The sum of the first 30 terms is 1177.5.

b **i** ① Write the question

b **i** $T_{n+1} = 2T_n - 1, T_0 = 3$

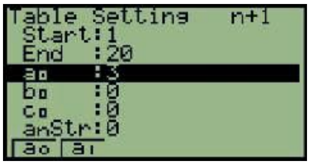
② To find the 10th term:

(a) first check the SET up to ensure the Σ display is on, then press:
- MENU
- 8 (RECUR)
- F3 (TYPE)
- F2 (a_{n+1})

Complete the entry line as:
$a_{n+1} = 2 \cdot a_n - 1$
Then press EXE.

```
Recursion
an+1=2×an-1         [—]
bn+1:               [—]
cn+1:               [—]
```

(b) Press F5 (SET) and enter the values as shown.
Note: The 10th term occurs at $n = 9$ because the sequence begins at a_0.



```
Table Setting    n+1
 Start:1
 End  :20
 a0   :3
 b0   :0
 c0   :0
 anStr:0
```

Continued over page →

THINK **WRITE/DISPLAY**

(c) Press:
- EXE
- F6 (TABL)

Scroll down to $n = 9$ to find the 10th term.

```
     n+1    an+1   Σan+1
       6     129     261
       7     257     518
       8     513    1031
       9    1025    2056
                       9
```

③ Write the answer.

$T_{10} = 1025$

ii ① To find the sum of the first 20 terms, continue to scroll down to $n = 19$ (the 20th term).
Note: The number is large and displayed as a decimal in the table. Place the cursor over this number and the full number will be displayed in the bottom right-hand corner of the screen.

ii

```
an+1=2Xan-1
     n+1    an+1   Σan+1
      16  131073  262159
      17  262145  524304
      18  524289  1.04E6
      19  1.04E6  2.09E6
                 2097170
```

② Write the answer.

The sum of the first 20 terms is 2 097 170.

For the TI-Nspire CAS

a i ① First, identify whether the sequence is arithmetic or geometric, and list a and d or r as appropriate.

a i The sequence is arithmetic with $a = 3$ and $d = 2.5$

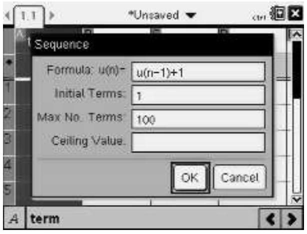
② Write the formula for the nth term.

Therefore $T_n = 3 + (n - 1) \cdot\ 2.5$.

③ Use the graphics calculator to find the 100th term.

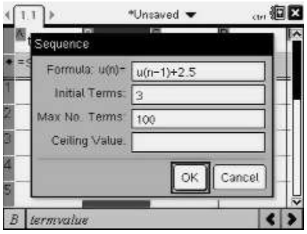
(a) On a Lists & Spreadsheet page, label Column A as 'term'. Highlight the grey header cell, press:
- MENU menu
- 3: Data 3
- 1: Generate Sequence 1

Complete the fields as shown and then select OK.
Note: Press Tab to move to the next field.



Sequence
Formula: u(n)= u(n-1)+1
Initial Terms: 1
Max No. Terms: 100
Ceiling Value:

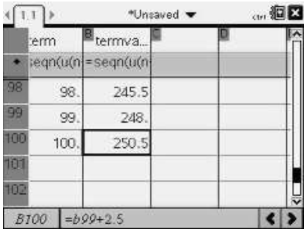
(b) Move the cursor to Column B and label Column B as 'termvalue'. Repeat the above instructions for Column B, complete the fields as shown and then select OK.



Sequence
Formula: u(n)= u(n-1)+2.5
Initial Terms: 3
Max No. Terms: 100
Ceiling Value:

THINK	**WRITE/DISPLAY**

(c) Scroll down to find the 100th term.



	term	termva…		
◆	seqn(u(n	=seqn(u(n		
98	98.	245.5		
99	99.	248.		
100	100.	250.5		
101				
102				

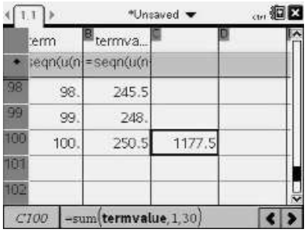
B100 =b99+2.5

④ Write the answer.

$T_{100} = 250.5$

ii ① To calculate the sum of the first 30 terms, place the cursor in an empty cell and complete the entry line as:
= sum(*termvalue*,1,30)

ii



	term	termva…		
◆	seqn(u(n	=seqn(u(n		
98	98.	245.5		
99	99.	248.		
100	100.	250.5	1177.5	
101				
102				

C100 =sum(**termvalue**,1,30)

② Write the answer.

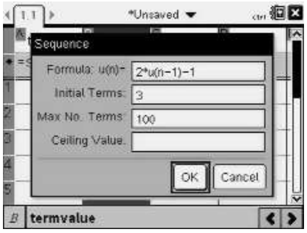
The sum of the first 30 terms is 1177.5.

b i ① To find the 10th term:

(a) on a Lists & Spreadsheet page, leave Column A labeled as ‘term’; and Column B as ‘termvalue’; highlight the grey header cell for Column B and press:
- MENU (menu)
- 3: Data ⟨3⟩
- 1: Generate Sequence ⟨1⟩

Complete the fields as shown and then select OK.

b i



Sequence

Formula: u(n)= 2*u(n−1)−1

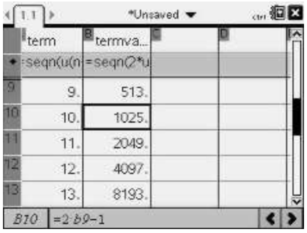
Initial Terms: 3

Max No. Terms: 100

Ceiling Value:

B termvalue

(b) Scroll down to the 10th term.



	term	termva…		
◆	seqn(u(n	=seqn(2*u		
9	9.	513.		
10	10.	1025.		
11	11.	2049.		
12	12.	4097.		
13	13.	8193.		

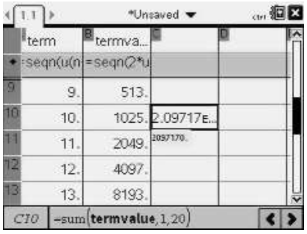
B10 =2·b9−1

② Write the answer.

$T_{10} = 1025$

ii ① To calculate the sum of the first 20 terms, place the cursor in an empty cell and complete the entry line as:
= sum(*termvalue*,1,20)

ii



	term	termva…		
◆	seqn(u(n	=seqn(2*u		
9	9.	513.		
10	10.	1025.	2.09717E…	
11	11.	2049.	2097170.	
12	12.	4097.		
13	13.	8193.		

C10 =sum(**termvalue**,1,20)

② Write the answer.

The sum of the first 20 terms is 2 097 170.

remember

1. For an arithmetic sequence:
   - (a) the difference between successive terms, d, is constant
   - (b) $T_n = a + (n - 1)d$ where a is the first term
   - (c) the sum of n terms, $S_n = \frac{n}{2}[2a + (n - 1)d]$.
2. For a geometric sequence:
   - (a) the ratio between successive terms, r, is constant
   - (b) $T_n = ar^{n-1}$ where a is the first term
   - (c) the sum of n terms, $S_n = \dfrac{a(r^n - 1)}{r - 1}$.
3. A sequence may be defined recursively; for example, $T_{n+1} = bT_n + c$ with $T_1 = a$.
4. The symbol Σ indicates a sum of terms.
5. Graphics calculators can be used to investigate sequences.

EXERCISE 6A Sequences and series

WORKED Example 1a, b

1 For each of the arithmetic sequences below:

i write a formula for the nth term, T_n

ii find the 50th term.

a 2, 5, 8, …

b 3, 5, 7, …

c 2, –4, –10, …

d 16, 13, 10, …

e 1, 1.75, 2.5, …

f 8, 6.2, 4.4, …

g 1000, 930, 860, …

h 220, 240, 260, …

i $100, 100 - d, 100 - 2d, \ldots$

j $N, N + 2.5, N + 5, \ldots$

WORKED Example 1c

2 For each of the sequences in question **1**, calculate or give an expression for the sum of the first 20 terms.

WORKED Example 2a, b

3 For each of the geometric sequences below:

i write a formula for the nth term, T_n

ii find the 20th term.

a 3, 6, 12, …

b 2, 6, 18, …

c 2, –4, 8, …

d –4, –2, –1, …

e 3, 1.5, 0.75, …

f 256, –128, 64, …

g 100, 110, 121, …

h 240, –180, 135, …

i $N, \frac{3}{4}N, (\frac{3}{4})^2N, \ldots$

j $100, -\dfrac{100}{d}, \dfrac{100}{d^2}, \ldots$

WORKED Example 2c

4 For each of the sequences in question **3**, give an expression for the sum of the first n terms.

5 For each of the following sequences, give an expression for the:

- **i** nth term
- **ii** 20th term
- **iii** sum of the first 20 terms.

a 2, 1, 0.5, …
b 2, 1, 0, …
c 4, 4, 4, …
d 5, 2, −1, …
e 0.25, 0.5, 1, …
f 17.1, 5.7, 1.9, …
g 54.1, 86.3, 118.5, …
h 3, 3, 5, 5, 7, 7, …
i 3, 3, 6, 6, 12, 12 , …
j $4a$, $2a$, 0, …

WORKED Example 3

6 An accountant has been working with the same company for 15 years. She commenced on a salary of \$28 000 and has received a \$2500 increase each year.

a How much did she earn in her 15th year of employment?
b How much has she earned from the company altogether?

7 A chemist has been working with the same company for 15 years. She commenced on a salary of \$28 000 and has received a 4% increase each year.

a How much did she earn in her 15th year of employment?
b How much has she earned from the company altogether?

WORKED Example 4a

8 List the first 5 terms of each of the sequences defined below.

a $T_{n+1} = 2T_n$, $T_1 = 2$
b $T_{n+1} = T_n - 2$, $T_1 = 0$
c $T_{n+1} = -2T_n$, $T_1 = 1$
d $T_{n+1} = 3 - 2T_n$, $T_1 = 2$
e $T_{n+1} = 2T_n + 3$, $T_1 = 2$

WORKED Example 4b

9 For each of the following sequences, give a recursive formula for T_n.

a 3, 5, 7, …
b 4, 5, 6.25, …
c 8, 5, 2, …
d 5, 7.5, 11.25, …
e 4, −2, 1, …

WORKED Example 5

10 Write each of the following sums in full detail.

a $\sum_{1}^{5} 2r$ **b** $\sum_{2}^{5} (r^2 + 2)$ **c** $\sum_{1}^{4} \frac{2}{r}$

d $\sum_{1}^{5} (1 + r)^2$ **e** $\sum_{1}^{5} \left(\frac{1}{r} - \frac{1}{r+1}\right)$

Use your graphics calculator to answer questions **11** to **15**.

WORKED Example 6a

11 For the sequence:

2, 5, 8, …

a calculate the 100th term **b** calculate the sum of the first 100 terms.

12 For the sequence:

2, 2.2, 2.42, …

a calculate the 30th term **b** calculate the sum of the first 30 terms.

WORKED Example 6b

13 For the sequence defined by:

$T_{n+1} = 1.5T_n + 1, T_0 = 2$

a calculate the 30th term **b** calculate the sum of the first 30 terms.

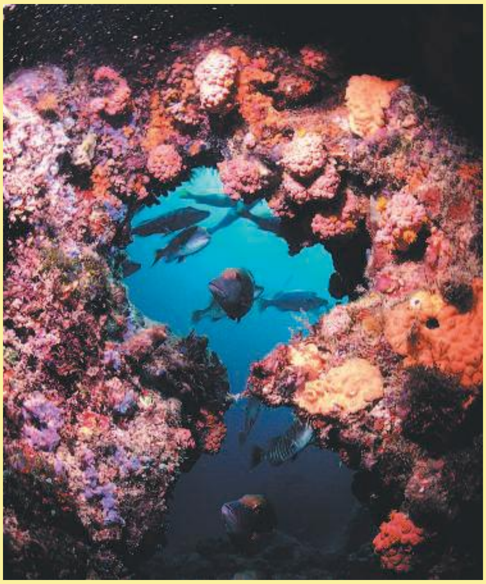
14 A regular payment of \$8000 is made into an account at the beginning of the year for 25 years. If the interest on the account is 9%, paid annually into the account, calculate the value of this investment at the end of the 25-year period.

15 Consider the geometric progression:

$Q, rQ, r^2Q, \ldots$

If $|r| < 1$ show that the sum to infinity of this sequence is $\frac{Q}{1-r}$.

Population equation



The population of an organism in a habitat is often modelled using the population equation:

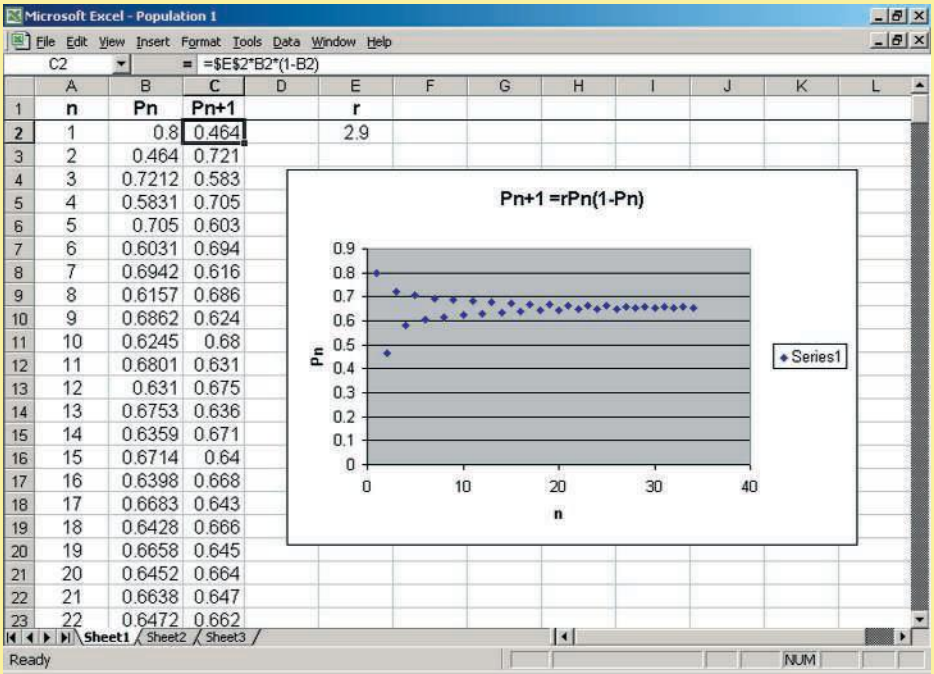
$$P_{n+1} = rP_n(1 - P_n)$$

where P_n is the population at the start of period n. In this equation P_n has been scaled so that it lies between 0 and 1.

The parameter r may depend on a variety of environmental factors such as food supply and fertility. Note that the equation ensures that P_n does not grow too large, for if it did, then the factor $(1 - P_n)$ would be small and the subsequent population would also be small.

To investigate this you can use your graphics calculator or a spreadsheet. The sequence formed by successive populations will be a set of numbers between 0 and 1. The *equilibrium* value of P_n is that value for which $P_{n+1} = P_n$.

From the spreadsheet, with $r = 2.9$ and $P_0 = 0.8$, we find that P_n approaches the equilibrium value of 0.6552. It does not matter what starting value of P_0 is used; the same equilibrium value is reached.



C2 = =$E$2*B2*(1-B2)

n	Pn	Pn+1	r
1	0.8	0.464	2.9
2	0.464	0.721	
3	0.7212	0.583	
4	0.5831	0.705	
5	0.705	0.603	
6	0.6031	0.694	
7	0.6942	0.616	
8	0.6157	0.686	
9	0.6862	0.624	
10	0.6245	0.68	
11	0.6801	0.631	
12	0.631	0.675	
13	0.6753	0.636	
14	0.6359	0.671	
15	0.6714	0.64	
16	0.6398	0.668	
17	0.6683	0.643	
18	0.6428	0.666	
19	0.6658	0.645	
20	0.6452	0.664	
21	0.6638	0.647	
22	0.6472	0.662	

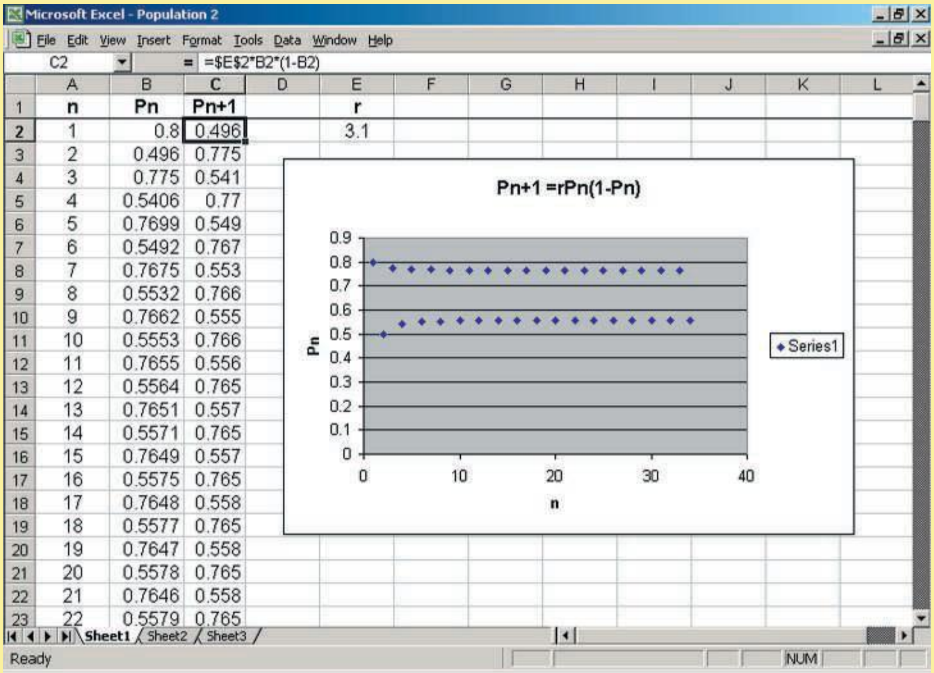
The equilibrium value could have been calculated algebraically. For the equilibrium value $P_{n+1} = P_n$, let:

$$P_{n+1} = P_n = x.$$

Then substituting into the population equation:

$$x = 2.9x(1 - x)$$
$$1 = 2.9(1 - x) \text{ provided } x \neq 0$$
$$x = 1 - \frac{1}{2.9}$$
$$= 0.6552$$

Sometimes, the population equation will have more than one equilibrium solution. Consider, for example, when the value of r is 3.1.



C2 = =$E$2*B2*(1-B2)

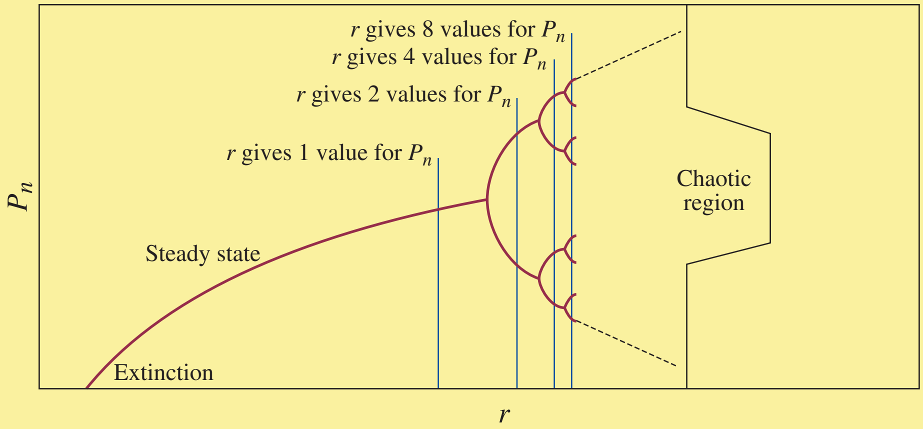
	A	B	C	D	E
1	n	Pn	Pn+1		r
2	1	0.8	0.496		3.1
3	2	0.496	0.775		
4	3	0.775	0.541		
5	4	0.5406	0.77		
6	5	0.7699	0.549		
7	6	0.5492	0.767		
8	7	0.7675	0.553		
9	8	0.5532	0.766		
10	9	0.7662	0.555		
11	10	0.5553	0.766		
12	11	0.7655	0.556		
13	12	0.5564	0.765		
14	13	0.7651	0.557		
15	14	0.5571	0.765		
16	15	0.7649	0.557		
17	16	0.5575	0.765		
18	17	0.7648	0.558		
19	18	0.5577	0.765		
20	19	0.7647	0.558		
21	20	0.5578	0.765		
22	21	0.7646	0.558		
23	22	0.5579	0.765		

As can be seen from the spreadsheet, the sequence does not have one equilibrium. Instead, it oscillates between 0.7646 and 0.5580.

1 Can you calculate these solutions algebraically? (In this case $P_{n+1} = P_{n-1}$.)

2 What value of r represents the transition from one equilibrium solution to two equilibrium solutions?

We can graph the equilibrium values for P_n against r.



Note that as r increases from 2.5 to about 3 there is one equilibrium solution. The solutions then ‘split’ into pairs of values. The solutions split again to yield 4 equilibrium solutions.

3. For what value of r do the equilibrium solutions form a cycle of four values? As the value of r increases further still, the solutions approach what is termed *chaos* — that is, a region where there is no regularity or equilibrium solutions.
4. Can you find a value of r that will yield a cycle of 8 equilibrium solutions?
5. For further research: Let the value of r for which the equilibrium solutions change from one solution to two solutions be r_1 and let r_2 be the value of r for which the solutions change from 2 solutions to 4 solutions. The ratio of $\frac{r_2}{r_1}$ is called a *Feigenbaum number*. Who identified these numbers and why are they important?

Mathematical induction

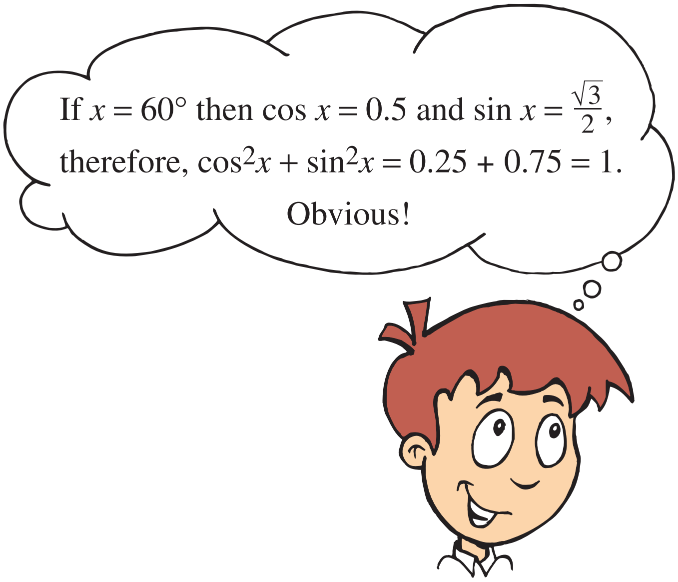
There is a philosophical difference between what constitutes a ‘fact’ in mathematics (such as the formula for the sum of a geometric progression) and a ‘fact’ in science (such as the law of gravity).

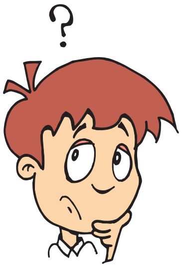
Mathematics is said to be *deductive*. That is, each new idea builds on earlier results — just as a tall building is built one floor on the other. If the foundations are strong the whole structure is strong. Science, by contrast, is said to be *inductive*. Because, many times in the past, the law of gravity has appeared to work we believe it will work again in the future. Usually such a method of proof would not be considered adequate in mathematics.

You may remember in earlier mathematics classes the frustration caused by questions such as the following.

$$\text{Prove that } \cos^2 x + \sin^2 x = 1.$$

Here is a common response to the problem.





However, even if we were to perform many such confirmations of the equation, they would still not constitute a proof. No number of examples can prove a general result. On the other hand, only one counter-example will suffice to show that a result is incorrect.

There is, however, a formal procedure called the *principle of mathematical induction* that can be used to prove a proposition under special circumstances. It is applied to propositions such as the following:

$$\sum_{1}^{n} r = \frac{n}{2}(n+1).$$

This equation we call P(n). That is, it is a proposition and it depends on n. For example, P(5) would be:

$$\sum_{1}^{5} r = \tfrac{5}{2}(5+1).$$

Now $\quad \displaystyle\sum_{1}^{5} r = 1 + 2 + 3 + 4 + 5$

$\qquad = 15$

and $\quad \tfrac{5}{2}(5+1) = \tfrac{5}{2} \cdot 6$

$\qquad = 15$

So the proposition is true for $n = 5$ or P(5) is true.

The principle of mathematical induction states:

A proposition, P(n), is true for all natural numbers if:
1. P(1) is true; and
2. whenever P(k) is true, P($k + 1$) is true.

Note that propositions must relate to natural numbers. The principle of mathematical induction can be likened to climbing a ladder. The nth rung of the ladder represents that the proposition is true for that value of n.

Step 1 says that you can get on the ladder.

Step 2 says that you can move from rung to rung.

Thus, there is a procedure for getting to any rung on the ladder no matter how far.

WORKED Example 7

Consider the proposition, P(n): $\displaystyle\sum_{1}^{n} \frac{1}{2^r} = \frac{2^n - 1}{2^n}$.

a **What is the statement P(1)?**

b **What is the statement P(k)?**

c **What is the statement P($k + 1$)?**

d **How can $\displaystyle\sum_{1}^{k} \frac{1}{2^r}$ be changed to $\displaystyle\sum_{1}^{k+1} \frac{1}{2^r}$?**

e **Use the principle of mathematical induction to prove $\displaystyle\sum_{1}^{n} \frac{1}{2^r} = \frac{2^n - 1}{2^n}$.**

THINK	**WRITE**
a ① State the proposition P(n).	**a** P(n): $\sum_{1}^{n} \frac{1}{2^r} = \frac{2^n - 1}{2^n}$
② Substitute $n = 1$ into P(n) to obtain the statement for P(1).	P(1): $\sum_{1}^{1} \frac{1}{2^r} = \frac{2^1 - 1}{2^1}$
b Substitute $n = k$ into P(n) to obtain the statement for P(k).	**b** P(k): $\sum_{1}^{k} \frac{1}{2^r} = \frac{2^k - 1}{2^k}$
c Substitute $n = k + 1$ into P(n) to obtain the statement for P(k + 1).	**c** P(k + 1): $\sum_{1}^{k+1} \frac{1}{2^r} = \frac{2^{k+1} - 1}{2^{k+1}}$
d Change $\sum_{1}^{k} \frac{1}{2^r}$ to $\sum_{1}^{k+1} \frac{1}{2^r}$ by adding the term where $r = k + 1$.	**d** $\sum_{1}^{k} \frac{1}{2^r} + \frac{1}{2^{k+1}} = \sum_{1}^{k+1} \frac{1}{2^r}$
e ① Show P(1) is true by simplifying each side of the equation.	**e** P(1): $\sum_{1}^{1} \frac{1}{2^r} = \frac{2^1 - 1}{2^1}$ $\sum_{1}^{1} \frac{1}{2^r} = \frac{1}{2}$ and $\frac{2^1 - 1}{2^1} = \frac{1}{2}$ so P(1) is true.
② Assume P(k) is true and try to use this to show P(k + 1) is true.	Assume P(k) is true: $\sum_{1}^{k} \frac{1}{2^r} = \frac{2^k - 1}{2^k}$
③ Try to arrange this expression into the form of the right-hand side of P(k + 1). Note that $\frac{1}{2^k} = \frac{2}{2 \cdot 2^k} = \frac{2}{2^{k+1}}$.	Then $\sum_{1}^{k} \frac{1}{2^r} + \frac{1}{2^{k+1}} = \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}}$ $= \frac{2(2^k - 1) + 1}{2^{k+1}}$ $= \frac{2 \cdot 2^k - 2 + 1}{2^{k+1}}$ $= \frac{2^{k+1} - 1}{2^{k+1}}$ Thus $\sum_{1}^{k+1} \frac{1}{2^r} = \frac{2^{k+1} - 1}{2^{k+1}}$ That is, P(k + 1) is true whenever P(k) is true.

Continued over page →

THINK

④ Write the answer.

WRITE

By the principle of mathematical induction, P(n) is true for all values of n.

That is, $\sum_{1}^{n} \frac{1}{2^r} = \frac{2^n - 1}{2^n}$ is true for all values of n.

De Moivre's Theorem

De Moivre's Theorem has already been referred to in Chapter 1.

It states that:

If $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all integer values of n.

We can prove de Moivre's Theorem by using mathematical induction.

P(n): $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$

1. P(1) is true, then both sides of the equation will have the same value.
2. Now we need to show that if P(k) is true, P(k + 1) is true.

$$\begin{aligned}
[\cos(x) + i\sin(x)]^{k+1} &= [\cos(x) + i\sin(x)]^k \cdot\ [\cos(x) + i\sin(x)] \\
&= [\cos(kx) + i\sin(kx)] \cdot\ [\cos(x) + i\sin(x)] \\
&\quad \text{(This is the induction hypothesis; P}(k)\text{ is true)} \\
&= [\cos(kx)\cos(x) - \sin(kx)\sin(x) + i[\sin(kx)\cos(x) \\
&\quad + \cos(kx)\sin(x)] \\
&\quad \text{(Expanding)}
\end{aligned}$$

Remembering the addition formulas:

$\cos(a + b) = \cos a \cos b - \sin a \sin b$

$\sin(a + b) = \sin a \cos b + \cos a \sin b$

$$\begin{aligned}
&= \cos(kx + x) + i\sin(kx + x) \\
&= \cos[(k + 1)x] + i\sin[(k + 1)x]
\end{aligned}$$

This proves that P(n + 1) is true whenever P(n) is true, so by the principle of mathematical induction, De Moivre's Theorem has been shown to be true.

remember

The principle of mathematical induction states:

A proposition, P(n), is true for all natural numbers if:

1. P(1) is true; and
2. whenever P(k) is true, P(k + 1) is true.

EXERCISE 6B Mathematical induction

For each of the proposition statements, P(n), in questions **1** to **17**, answer the following.

a Write the statement P(1).

b Verify that the statement is true for $n = 5$.

c Write the statement P(k).

d Write the statement P($k + 1$).

e What is needed to change the expression of the left-hand side of P(k) into the left-hand side of P($k + 1$)?

f Use the principle of mathematical induction to prove the statement P(n).

WORKED Example 7

1 $\sum_{1}^{n} r = \frac{n}{2}(n + 1)$

2 $\sum_{1}^{n} 2r = n(n + 1)$

3 $\sum_{1}^{n} 2r - 1 = n^2$

4 $\sum_{1}^{n} 2^r = 2^{n+1} - 2$

5 $\sum_{1}^{n} r^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

6 $\sum_{1}^{n} r^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

7 $a + (a + d) + (a + 2d) + \ldots(a + [n - 1]d) = \frac{n}{2}(2a + [n - 1]d)$

8 $a + ar + ar^2 + \ldots ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$

9 $3 + 6 + 9 + \ldots + 3n = \dfrac{3n(1 + n)}{2}$

10 $2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 2$

11 $-2 + 4 - 8 + \ldots + (-2)^n = \frac{2}{3}[(-2)^n - 1]$

12 $\sum_{1}^{n} \dfrac{1}{r(r + 1)} = 1 - \dfrac{1}{n + 1}$

13 $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1)$

14 $\sum_{1}^{n} r^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

15 $(n + 3)! > 2^n$

16 $3^{4n} - 1$ is divisible by 80.

17 $5^n + 2 \cdot 11^n$ is divisible by 3.

18 Fibonacci's sequence is given by the recursive relationship:

$$F_{n+1} = F_n + F_{n-1} \text{ with } F_0 = F_1 = 1.$$

A formula for the nth term of this sequence is:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n$$

Prove this result using the principle of induction.
Hint: In step 2 you will need to assume P(n) is true for k and $k - 1$ and then show it is true for $k + 1$.

19 Use mathematical induction to prove de Moivre's Theorem for all negative values of n.

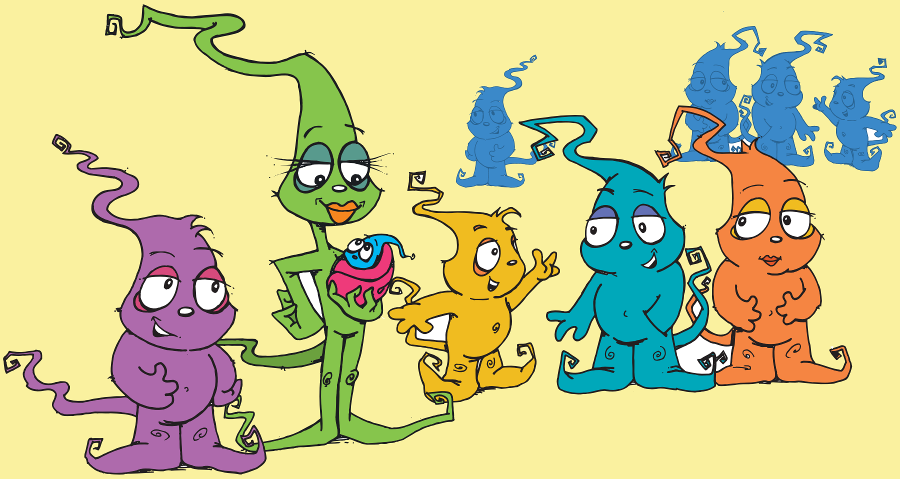
Fibonacci's sequence

Fibonacci's sequence is generated by $F_{n+1} = F_n + F_{n-1}$ with $F_0 = F_1 = 1$.

1 Use a spreadsheet to investigate the following properties of the Fibonacci sequence.

- **a** If a Fibonacci number, F_n, is prime then n is prime (except for $n = 4$). Give an example to show that the converse is not true.
- **b** The last digit of the sequence of Fibonacci numbers forms a cycle of 60. (As the sequence grows rapidly you may be able to observe the cycle in action for perhaps only 10 numbers after 60 before the spreadsheet starts rounding numbers.)
- **c** Only one Fibonacci number that is a square number has been identified. What is it?

2 The Lunx is a creature that reproduces according to the following rules.



1. All mature Lunx produce one baby each year.
2. A Lunx becomes mature at 2 years and will produce an offspring at the beginning of its third year.
3. Lunx die at the end of their 5th year.

At the end of year 1 the zoo has one Lunx aged 1 year. It will produce its first baby at the beginning of year 3. If the zoo population is left to grow uninhibited, what will the population be at the end of the 24th year?

Finite differences

A graphics calculator is a very powerful tool. However, it is not magical and needs to be used with caution. To illustrate this let us see what can happen when a set of data is modelled using the graphics calculator.

Consider, for example, the set of data:

x	1	1.25	1.5	1.75	2
y	1	1.5625	2.25	3.0625	4

Enter the data into your calculator.

For the CASIO *fx*-9860G AU

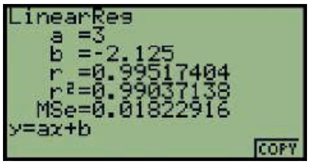
1. To generate a linear model for the data, press:
   - MENU
   - 2 (STAT)

   Enter the data into List 1 and List 2.

SUB	List 1	List 2	List 3	List 4
1	1	1		
2	1.25	1.5625		
3	1.5	2.25		
4	1.75	3.0625		

2. Press:
   - F2 (CALC)
   - F3 (REG)
   - F1 (X)

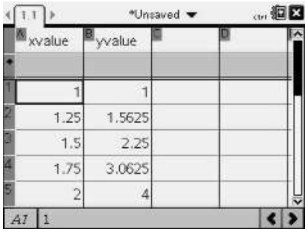
   The calculator then models the data using a linear model.



```
LinearReg
  a =3
  b =-2.125
  r =0.99517404
 r²=0.99037138
MSe=0.01822916
y=ax+b
```

For the TI-Nspire CAS

1. To generate a linear model for the data, on a Lists & Spreadsheet page, label Column A as ‘xvalue’ and Column B as ‘yvalue’ and then enter the data in Columns A and B.

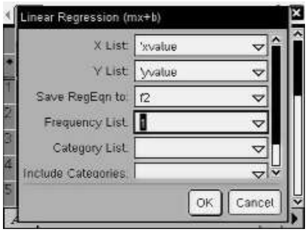


	A xvalue	B yvalue	C	D
1	1	1		
2	1.25	1.5625		
3	1.5	2.25		
4	1.75	3.0625		
5	2	4		

A1 1

2. Press:
   - MENU menu
   - 4: Statistics 4
   - 1: Stat Calculations 1
   - 3: Linear Regression 3

   Complete the fields as shown.

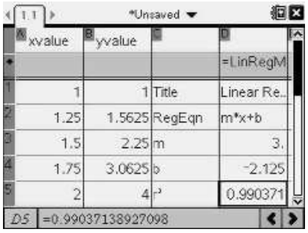


Linear Regression (mx+b)

X List: 'xvalue
Y List: 'yvalue
Save RegEqn to: f2
Frequency List: 1
Category List:
Include Categories:

3. Select OK.

   The calculator then models the data using a linear model.



	A xvalue	B yvalue	C	D
◆				=LinRegM
1	1	1	Title	Linear Re...
2	1.25	1.5625	RegEqn	m*x+b
3	1.5	2.25	m	3.
4	1.75	3.0625	b	-2.125
5	2	4	r²	0.990371

D5 =0.99037138927098

The calculator gives the linear model, $y = 3x - 2.125$ and suggests, with $r = 0.995$, that it is a very good match for the data. Yet this set of data comes from the equation $y = x^2$. The data fit this equation almost perfectly. The point of this example is to show that in some tasks an uncritical use of the calculator will yield an answer that looks correct but will be quite incorrect.

In developing models to fit data, a method called *finite differences* is used to identify whether a linear, quadratic or cubic model should be used to fit the data. In worked example 8 below, the method of finite differences is outlined. We change notation from x and y to n and t_n.

Suppose we have a relationship between t_n and n defined by:

$$t_n = 2n.$$

The first 6 terms can be placed in a table as shown.

n	t_n
1	2
2	4
3	6
4	8
5	10
6	12

The method of finite differences focuses on the difference between successive values of t_n and we include a column in the table to calculate these.

n	t_n	**First diff.**
1	2	
2	4	2
3	6	2
4	8	2
5	10	2
6	12	2

The 'First difference' column records the difference between successive terms. We note that the first difference is constant for a linear relationship such as $t_n = 2n$.

Now let us examine the differences for values of t_n generated by a quadratic equation, $t_n = n^2$.

n	t_n	**First diff.**
1	1	
2	4	3
3	9	5
4	16	7
5	25	9
6	36	11

We note that the first differences are not constant. An extra column is added to the table and second differences (the difference of the first differences) are calculated.

n	t_n	First diff.	Second diff.
1	1		
2	4	3	
3	9	5	2
4	16	7	2
5	25	9	2
6	36	11	2

For the equation $t_n = n^2$, and for any quadratic equation, the second differences are constant.

If the relationship is cubic, such as $t_n = n^3$, the first and second differences are not constant but the third differences are.

n	t_n	First diff.	Second diff.	Third diff.
1	1			
2	8	7		
3	27	19	12	
4	64	37	18	6
5	125	61	24	6
6	216	91	30	6

WORKED Example 8

For the set of data at right:

a use the method of finite differences to determine whether the relationship between t_n and n is linear, quadratic, cubic or none of these

b use a graphics calculator to find a formula relating t_n and n.

n	t_n
1	6
2	12
3	28
4	60
5	114
6	196

Continued over page

THINK

a ① Take the difference between successive values of t_n to construct a table of first, second and third differences.

② Continue to take further differences until the difference column is constant.

③ Determine whether the relationship is linear, quadratic, cubic or none of these.

(a) If the first difference column is constant the relationship is linear.

(b) If the second difference column is constant the relationship is quadratic.

(c) If the third difference column is constant the relationship is cubic.

④ Answer the question.

b ① Use a graphics calculator to find the cubic relationship between t_n and n.

For the Casio *fx*-9860G AU

(a) Press:

- MENU
- 2 (STAT)

Place the values of n and t_n into List 1 and List 2.

(b) Then press:

- F2 (CALC)
- F3 (REG)
- F4 (x^3)

For the TI-Nspire CAS

(a) On a Lists & Spreadsheet page, label Column A as ‘xvalue’ and Column B as ‘yvalue’ and then enter the data in Columns A and B.

(b) Then press:

- MENU menu
- 4: Statistics 4
- 1: Stat Calculations 1
- 7: Cubic Regression 7

Enter X List as xvalue and Y List as yvalue, then select OK.

② Write the answer in terms of t_n and n.

WRITE/DISPLAY

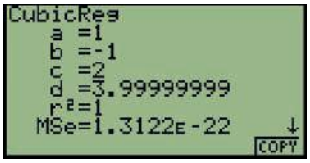
a

n	t_n	First diff.	Second diff.	Third diff.
1	6			
2	12	6		
3	28	16	10	
4	60	32	16	6
5	114	54	22	6
6	196	82	28	6

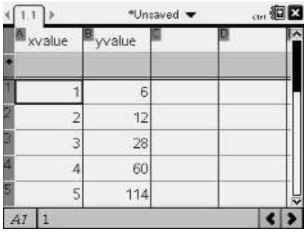
The relationship is cubic.

b The formula will be of the form $t_n = an^3 + bn^2 + cn + d$.

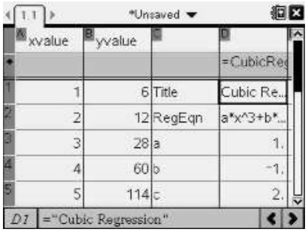
```
    List 1  List 2  List 3  List 4
SUB
 1       1       6
 2       2      12
 3       3      28
 4       4      60
                                 1
TOOL EDIT DEL DEL·A INS ▷
```



```
CubicReg
   a =1
   b =-1
   c =2
   d =3.99999999
   r²=1
  MSe=1.3122E-22           ↓
                        COPY
```



```
1.1          *Unsaved
A xvalue  B yvalue  C     D
1       1        6
2       2       12
3       3       28
4       4       60
5       5      114
A1 1
```



```
1.1          *Unsaved
A xvalue  B yvalue  C        D
                             =CubicReg
1       1        6  Title    Cubic Re...
2       2       12  RegEqn   a*x^3+b*...
3       3       28  a                1.
4       4       60  b               ⁻1.
5       5      114  c                2.
D1 ="Cubic Regression"
```

The formula is $t_n = n^3 - n^2 + 2n + 4$.

Essentially, the method of finite differences is used to decide the type of model that should be used to fit a set of data. By taking successive differences between terms, we can decide whether a linear, quadratic or cubic equation will fit. If no constant differences are found, then the relationship is not a simple polynomial equation.

Once the type of relationship has been established, we use the graphics calculator to find the equation. The equation it finds is not an approximation, but an exact fit for the data. In the next worked example we look at an interesting extension of the method of finite differences that actually calculates the coefficients of x^2 and x in the quadratic equation that fits the data.

WORKED Example 9

Use the method of finite differences to find the model for these data.

n	t_n
1	5
2	16
3	33
4	56
5	85

THINK

1. Add a column to calculate the first differences.
2. If the differences are not constant, continue to take further differences until the difference column is constant.

WRITE

n	t_n	First diff.	Second diff.
1	5		
2	16	11	
3	33	17	6
4	56	23	6
5	85	29	6

3. The second differences are constant so the model is of the form $y = ax^2 + bx + c$. Relate this to t_n and n.

The relationship is of the form $t_n = an^2 + bn + c$.

4. Add two more columns to the table. The first of these is to be a multiple of t_n. This multiple is determined by dividing 2 (quadratic relationship) by the constant second difference. That is, $\frac{2}{6}$ or $\frac{1}{3}$ in this case. Enter values in the fourth column for $\frac{1}{3}t_n$.
5. Complete the fifth column using $\frac{1}{3}t_n - n^2$. This has the effect of removing the quadratic part of t_n, leaving a linear relationship.

n	t_n	First diff.	Second diff.	$\frac{1}{3}t_n$	$\frac{1}{3}t_n - n^2$
1	5			$\frac{5}{3}$	$\frac{2}{3}$
2	16	11		$\frac{16}{3}$	$\frac{4}{3}$
3	33	17	6	$\frac{33}{3}$	$\frac{6}{3}$
4	56	23	6	$\frac{56}{3}$	$\frac{8}{3}$
5	85	29	6	$\frac{85}{3}$	$\frac{10}{3}$

6. Identify the relationship between $\frac{1}{3}t_n - n^2$ and n.

Comparing the first and last columns gives

$$\frac{1}{3}t_n - n^2 = \frac{2}{3}n$$

7. Solve the relationship for t_n in terms of n.

$$\frac{1}{3}t_n = n^2 + \frac{2}{3}n$$
$$t_n = 3n^2 + 2n$$

8. Answer the question.

The data can be modelled by the relationship $t_n = 3n^2 + 2n$.

remember

1. To investigate the relationship between t_n and n, calculate the differences between successive terms and then between successive differences.
2. If the first differences are constant, the relationship is linear.
3. If the second differences are constant, the relationship is quadratic.
4. If the third differences are constant, the relationship is cubic.
5. After the type of relationship has been identified, a graphics calculator can be used to find the equation relating t_n and n.

EXERCISE 6C Finite differences

WORKED Example 8

1 For each set of data:

- **i** use a table of finite differences to decide whether the relationship is linear, quadratic, cubic or none of these
- **ii** use a graphics calculator to determine the relationship between t_n and n.

a

n	t_n
1	1.5
2	6
3	13.5
4	24
5	37.5
6	54

b

n	t_n
1	5
2	16
3	33
4	56
5	85
6	120

c

n	t_n
1	−1
2	2
3	9
4	20
5	35
6	54

d

n	t_n
1	0.5
2	6
3	16.5
4	32
5	52.5
6	78

e

n	t_n
1	1
2	4
3	15
4	40
5	85
6	156

f

n	t_n
1	1
2	12
3	51
4	136
5	285
6	516

g

n	t_n
1	16
2	28
3	56
4	100
5	160
6	236

h

n	t_n
1	−1
2	−2
3	−1
4	4
5	17
6	46

i

n	t_n
1	0.5
2	4
3	25.5
4	80
5	182.5
6	348

j

n	t_n
1	3.5
2	12
3	10.5
4	−16
5	−82.5
6	−204

k

n	t_n
1	1.5
2	15
3	46.5
4	105
5	199.5
6	339

WORKED Example 9

2 The data below represent quadratic relationships between t_n and n. Use the method of finite differences to find the exact relationship.

a

n	t_n
1	3.5
2	11
3	22.5
4	38
5	57.5
6	81

b

n	t_n
1	6
2	8
3	6
4	0
5	–10
6	–24

c

n	t_n
1	2.5
2	13
3	26.5
4	43
5	62.5
6	85

3 For the equation $t_n = Cn + D$ where C and D are constants, show, by considering $t_{n+1} - t_n$, that the first difference is C.

a For the equation $t_n = Bn^2 + Cn + D$ where B, C and D are constants, show by considering $t_{n+1} - t_n$, that the first difference is $B + 2nB + C$. Thus show that the second difference is $2B$.

b For the equation $t_n = An^3 + Bn^2 + Cn + D$ where A, B, C and D are constants, show by considering $t_{n+1} - t_n$, that the third difference is $6A$.

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Digital doc:
WorkSHEET 6.1

Pascal's triangle

1 Row 0
1 1 Row 1
1 2 1 Row 2
1 3 3 1 Row 3
1 4 6 4 1 Row 4
1 5 10 10 5 1 Row 5
1 6 15 20 15 6 1 Row 6

In Year 11, Pascal's triangle was introduced in connection with combinations, ${}^nC_r$. You may recall that a number such as ${}^5C_2$ can be obtained from Pascal's triangle by looking at the 2nd position in the 5th row. Here we begin counting the rows and positions from 0. Thus ${}^5C_2 = 10$.

1 By referring to Pascal's triangle show that:

a ${}^nC_r = {}^nC_{n-r}$ **b** $\sum_{r=0}^{6} {}^6C_r = 2^6$ **c** $\sum_{r=0}^{n} {}^nC_r = 2^n$

2 In Pascal's triangle, find the triangular numbers.

3 If Pascal's triangle was continued for many rows and all the odd numbers were coloured black, a version of the Sierpinski triangle would result. Find an image of the Sierpinski triangle.

summary

Sequences and series

- For an arithmetic sequence:
  1. the difference between successive terms, d, is constant
  2. $T_n = a + (n - 1)d$ where a is the first term
  3. the sum of n terms, $S_n = \frac{n}{2}[2a + (n - 1)d]$.
- For a geometric sequence:
  1. the ratio between successive terms, r, is constant
  2. $T_n = ar^{n-1}$ where a is the first term
  3. the sum of n terms, $S_n = \frac{a(r^n - 1)}{r - 1}$.
- A sequence may be defined recursively, for example, $T_{n+1} = bT_n + c$ with $T_1 = a$.
- The symbol Σ indicates a sum of terms.
- A graphics calculator can be used to investigate sequences.

Mathematical induction

- The principle of mathematical induction states that a proposition, P(n), is true for all natural numbers if:
  1. P(1) is true; and
  2. whenever P(k) is true, P($k + 1$) is true.

Finite differences

- To investigate the relationship between t_n and n, calculate the differences between successive terms and then between successive differences.
  1. If the first differences are constant the relationship is linear.
  2. If the second differences are constant the relationship is quadratic.
  3. If the third differences are constant the relationship is cubic.
- After the type of relationship has been identified, a graphics calculator can be used to find the equation relating t_n and n.

CHAPTER review

1 For the sequence 20, 32, 44, … 6A

a calculate a formula for the *n*th term

b find the 100th term

c calculate the sum of the first 30 terms.

2 For the sequence 16, –8, 4, … 6A

a calculate a formula for the *n*th term

b find an expression for the 15th term

c find an expression for the sum of the first 30 terms.

3 For the expression $\sum_{r=3}^{7} 2r$ 6A

a write the sum in full detail

b calculate the value of the sum.

4 Calculate the value of $\sum_{r=1}^{100} 2r - 1$. 6A

5 Calculate the value of $\sum_{r=1}^{20} 3 \cdot (1.5)^r$. 6A

6 For the sequence defined by $T_{n+1} = 4 - \frac{1}{2} T_n$ with $T_1 = 100$: 6A

a list the first 5 terms in the sequence

b use a graphics calculator to find the sum of the first 25 terms.

7 For the sequence defined by $T_{n+1} = 4 + 0.1 \cdot T_n$ with $T_1 = 100$: 6A

a list the first 5 terms in the sequence

b use a graphics calculator to find the sum of the first 25 terms.

8 Prove each of the following statements using mathematical induction. 6B

a $\sum_{r=1}^{n} r + 1 = \frac{n}{2}(n + 3)$

b $\sum_{i=1}^{n} 1.5^{i-1} = 2 \cdot 1.5^n - 2$

c $\sum_{r=1}^{n} 3r - 2 = \frac{n}{2}(3n - 1)$

d $2 + 4 + 6 + \ldots + 2n = n(n + 1)$.

6B

9 Use induction to prove that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$, where F_i is the ith term in the Fibonacci sequence.

6C

10 Find a formula for T_n in terms of n for each of the following sets of data.

a

n	t_n
1	−3
2	−4
3	−3
4	0
5	5
6	12

b

n	t_n
1	−0.5
2	−1
3	4.5
4	19
5	45.5
6	87

Modelling and problem solving

1 A biologist is growing a tissue culture in a Petri dish. The initial mass of the culture was 20 milligrams. By the second day the culture had a mass of 28 milligrams.

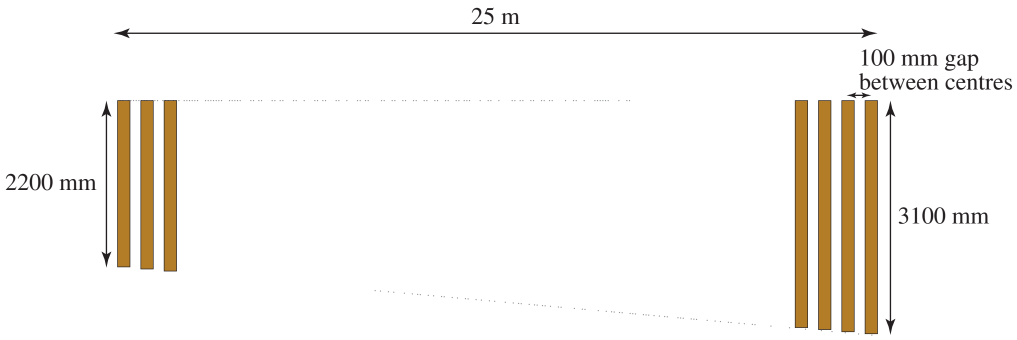
a Assuming that the daily growth is arithmetic, find the mass of the culture after the third, tenth and nth day.

b On what day will the culture mass first exceed 200 milligrams?

c Assuming that the daily growth is geometric, find the mass of the culture after the third, tenth and nth day.

d On what day will the culture mass first exceed 200 milligrams?

2 Battens (thin strips of timber) are to be used to enclose the area underneath the house shown in the diagram. The centres of the strips are 100 mm apart. The block slopes so that at one end the battens are 2200 mm in length while at the other end they are 3100 mm.



a Prove that the length of successive battens differs by a constant amount.

b If the first batten is 2200 mm long, give an expression for the length of the nth batten.

c How many battens are needed?

d Find the combined length of all battens.

3 An annuity consists of a payment made into an account on a regular basis. The amount already in the account attracts interest that is added to the amount invested. Consider an amount of \$5000 invested each year, for 30 years, in an account paying 10% per annum. (*Hint:* Although the funds are held in one account it is useful to think of the account as 30 separate accounts corresponding to the 30 payments.)

- a For the first \$5000 payment, calculate its value:
  - i after 1 year
  - ii after 5 years
  - iii after 30 years.
- b For the second \$5000 payment made at the start of year 2, calculate its value:
  - i at the end of year 2
  - ii at the end of 30 years.
- c Calculate the total amount in the account after 30 years by finding the sum of the value of 30 payments at the end of 30 years.

4 Consider the Fibonacci-type sequence defined by:

$$F_{n+1} = F_n + F_{n-1} \text{ with } F_0 = 3 \text{ and } F_1 = 4.$$

Use your graphics calculator to answer the following questions.

- a What is the value of F_{21}?
- b Calculate $\sum_{0}^{20} F_n$.
- c What value does the ratio $\frac{F_{n+1}}{F_n}$ tend to?

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Test Yourself
Chapter 6

eBook*plus* ACTIVITIES

6A Sequences and series

Interactivity

- Int-0187: Use the interactivity to investigate first order difference equations. (*page 237*)

Tutorial

- Int-0449: Watch a tutorial on finding the sum of geometric sequences. (*page 239*)
- Int-1054: Watch a tutorial on how to find a specific term in a geometric sequence. (*page 242*)

6C Finite differences

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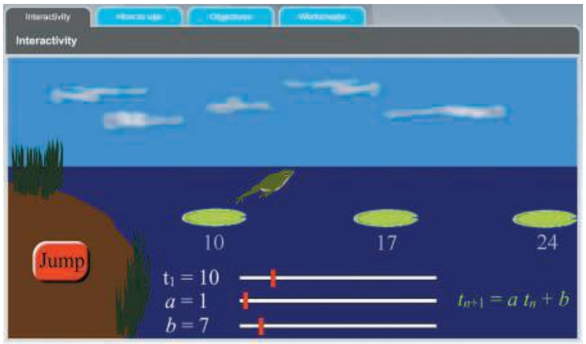
- WorkSHEET 6.1: Use mathematical induction for proofs and determine terms and term numbers for sequences and series. (*page 263*)



Chapter review

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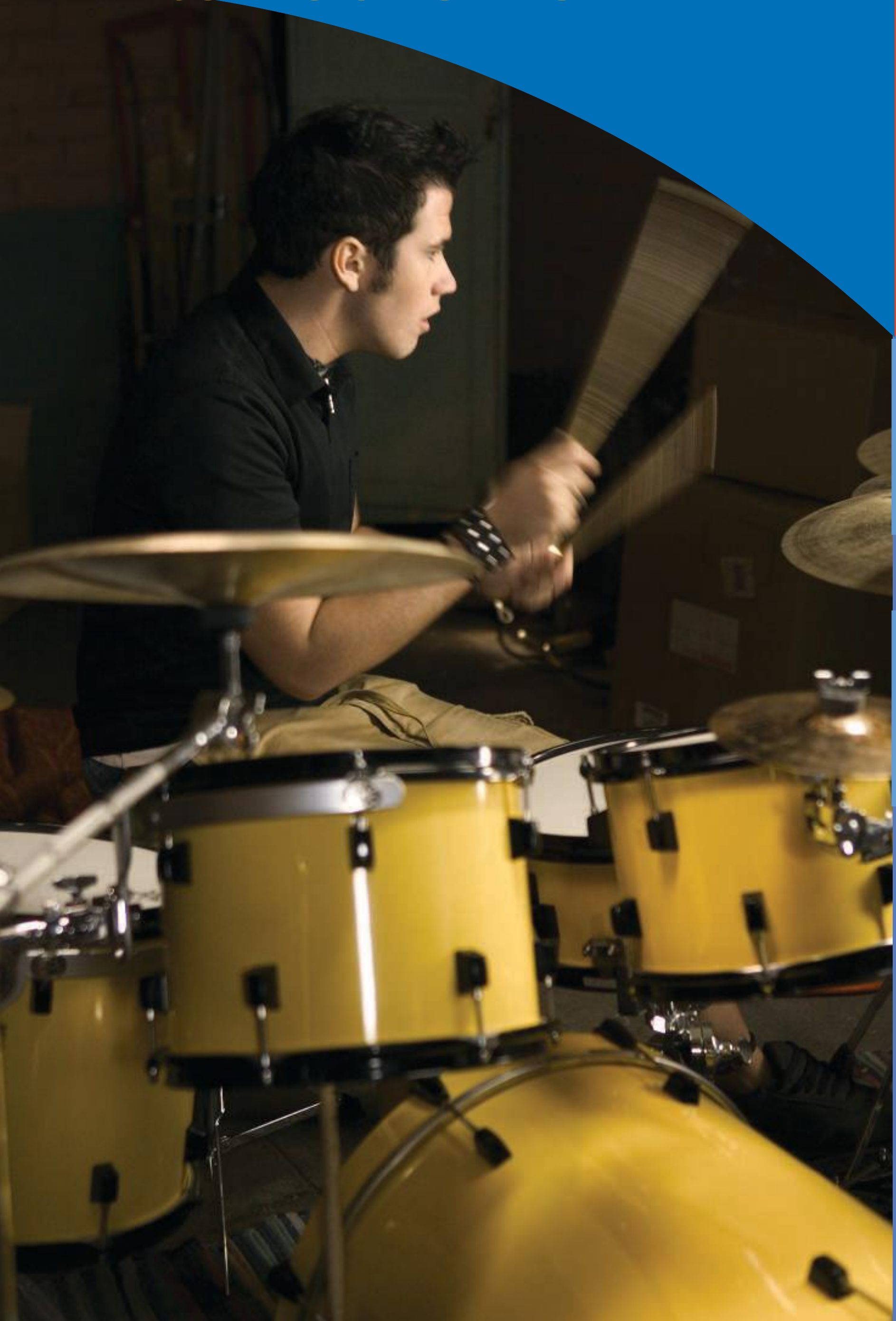
- Test Yourself: Take the end-of-chapter test to test your progress. (*page 267*)

To access eBookPLUS activities, log on to

www.jacplus.com.au

Advanced periodic functions

7



syllabus reference

Option topic:
Advanced periodic and exponential functions

In this chapter

7A Circular and reciprocal functions
7B Exact values and the Pythagorean relationships
7C Modelling soundwaves
7D Graphing the reciprocal trigonometric functions
7E Addition identities for $\sin(x \pm y)$ and $\cos(x \pm y)$
7F Addition identities for $\tan(x \pm y)$
7G The Werner or factorisation identities
7H The Simpson or half-sum/half-difference identities
7I Modelling and problem solving

Applications of periodic functions

In the physical world, everything vibrates. Vibrations follow cycles and tend to occur regularly. Mathematicians use the properties of periodic functions to create models to study phenomena like sound and the tides. Each musical instrument, for instance, has its own sound, or *sound signature*; the sound signature of a violin is different from that of, say, an oboe, a trumpet or a flute. By generating a sequence of sinusoidal waveforms, mathematicians can electronically generate the sound of a violin. This modelling of periodic phenomena can easily be extended to other musical instruments, to electronic and mechanical systems, and to nature itself.

In the study of advanced periodic exponential and logarithmic functions, we shall explore some of these periodic relationships and their applications in real life and in abstract mathematical contexts.

Circular and reciprocal functions

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Interactivity:
Sine and cosine graphs
int-0251

The functions that describe the horizontal and vertical positions of a point on a circle as a function of an angle are called *circular functions*. In Mathematics B, we gained some familiarity with the properties and graphs of the circular functions known as *sine* and *cosine*. We may say that for any angle measure, or real-valued variable θ, the tangent function is a relationship between sine and cosine such that for $\cos\theta \neq 0$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Further, by inverting $\cos\theta$, $\sin\theta$ and $\tan\theta$, we obtain the reciprocal functions. These functions are written respectively as *secant*, *cosecant*, and *cotangent* and are defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

The circular functions and their reciprocals are called *trigonometric functions* and their study is called *trigonometry*. Trigonometric relationships that may be familiar from Mathematics B include the following reduction formulas:

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\sin(\pi-\theta) = \sin\theta \qquad \cos(\pi-\theta) = -\cos\theta \qquad \tan(\pi-\theta) = -\tan\theta$$

$$\sin(\pi+\theta) = -\sin\theta \qquad \cos(\pi+\theta) = -\cos\theta \qquad \tan(\pi+\theta) = \tan\theta$$

$$\sin(2\pi-\theta) = -\sin\theta \qquad \cos(2\pi-\theta) = \cos\theta \qquad \tan(2\pi-\theta) = -\tan\theta$$

$$\sin(2\pi+\theta) = \sin\theta \qquad \cos(2\pi+\theta) = \cos\theta \qquad \tan(2\pi+\theta) = \tan\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

The following worked examples demonstrate how these trigonometric functions can be used.

WORKED Example 1

Show that tan $\left(\frac{\pi}{2}-\theta\right) = \cot\theta$.

THINK	**WRITE**
① Consider the left-hand side of the equation, and use the relationship $\tan\theta = \frac{\sin\theta}{\cos\theta}$.	$\tan\left(\frac{\pi}{2}-\theta\right) = \dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}$
② Simplify the numerator using $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$ and the denominator using $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$.	$= \dfrac{\cos\theta}{\sin\theta}$
③ Use the reciprocal function $\cot\theta = \frac{\cos\theta}{\sin\theta}$ to obtain the required right-hand side of the equation.	$= \cot\theta$ So $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$.

A relationship involving $\frac{\pi}{2} + \theta$ or $\frac{3\pi}{2} \pm \theta$ can also be simplified by:

1. identifying the quadrant in which the angle lies
2. determining the sign of the trigonometric ratio in that quadrant
3. changing to the complementary trigonometric function of θ.

These functions can also be simplified using the addition identities studied later in this chapter.

WORKED Example 2

Simplify:

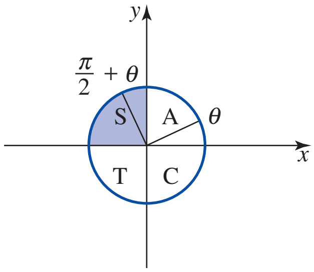
a $\cos\left(\frac{\pi}{2}+\theta\right)$ **b** $\sin\left(\frac{3\pi}{2}+\theta\right)$ **c** $\cos\left(\frac{3\pi}{2}-\theta\right)$.

THINK

WRITE/DISPLAY

a 1 Draw a diagram to find the quadrant in which $\left(\frac{\pi}{2}+\theta\right)$ lies.

a



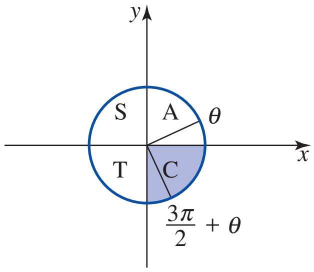
2 In quadrant 2, cos is negative. The complementary function is sin θ.

3 Answer the question.

$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$

b 1 Draw a diagram to find the quadrant in which $\frac{3\pi}{2}+\theta$ lies.

b



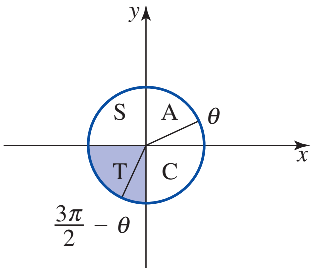
2 In quadrant 4, sin is negative. The complementary function is cos θ.

3 Answer the question.

$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$

c 1 Draw a diagram to find the quadrant in which $\frac{3\pi}{2}-\theta$ lies.

c



2 In quadrant 3, cos is negative. The complementary function is sin θ.

3 Answer the question.

$\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$

More complex expressions are handled as shown in the next worked example.

WORKED Example 3

Simplify $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sec(\pi+\theta)}$.

THINK	**WRITE**
① Consider the numerator and denominator separately. Use the appropriate reduction formula for the numerator.	$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$
② Rewrite the denominator using the reciprocal function $\sec\theta = \frac{1}{\cos\theta}$.	$\sec(\pi+\theta) = \frac{1}{\cos(\pi+\theta)}$
③ Simplify using the appropriate reduction formula.	$= \frac{1}{-\cos\theta}$
④ Substitute into the rational expression.	So $\dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sec(\pi+\theta)} = \dfrac{\cos\theta}{\frac{1}{-\cos\theta}}$
⑤ Simplify the rational expression.	$= \cos\theta \cdot -\cos\theta$ $= -\cos^2\theta$

remember

1. The tangent function is defined as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
2. The reciprocal functions are defined as:

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

EXERCISE 7A Circular and reciprocal functions

WORKED Example 1, 2

1 Simplify each of the following.

a $\sin\left(\frac{\pi}{2}+\theta\right)$ **b** $\sin\left(\frac{3\pi}{2}-\theta\right)$ **c** $\cos\left(\frac{3\pi}{2}+\theta\right)$

d $\tan\left(\frac{\pi}{2}+\theta\right)$ **e** $\cot\left(\frac{3\pi}{2}+\theta\right)$

2 Use trigonometric relationships to show each of the following.

a $\sec\left(\frac{\pi}{2}-\theta\right)=\operatorname{cosec}\theta$ **b** $\operatorname{cosec}\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$

c $\sec\left(\frac{3\pi}{2}-\theta\right)=-\operatorname{cosec}\theta$ **d** $\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta$

e $\dfrac{\cos\left(\frac{\pi}{2}+\theta\right)}{\sin\left(\theta-\frac{\pi}{2}\right)}=\tan\theta$

WORKED Example 3

3 Simplify each of the following.

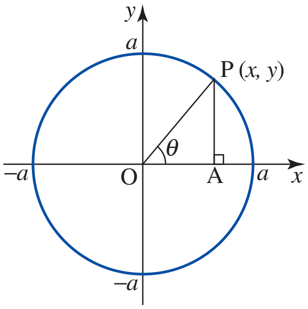
a $\cos\left(\frac{\pi}{2}-\theta\right)+\sin(-\theta)$ **b** $\sin\left(\frac{\pi}{2}+\theta\right)\cdot\sec(\pi+\theta)$

c $\operatorname{cosec}\left(\frac{3\pi}{2}-\theta\right)\cdot\cot(2\pi-\theta)$ **d** $\dfrac{\sin(\pi+\theta)}{\cos(2\pi+\theta)}+\dfrac{\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right)}{\sec\left(\frac{\pi}{2}+\theta\right)}$

e $\dfrac{\cot\left(\frac{\pi}{2}+\theta\right)\cdot\sec\left(\frac{3\pi}{2}-\theta\right)}{\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right)}$ **f** $\dfrac{\sin\left(\frac{\pi}{2}+\theta\right)-\cos\left(\frac{\pi}{2}+\theta\right)}{\sec(\pi-\theta)}$

Exact values and the Pythagorean relationships

The Pythagorean identity



In the illustration, the point P(x, y) is a variable point on the circle centred at the origin, O. Because the distance $\overline{\text{OP}}$ remains equal to the radius of the circle, say a, it follows that $x^2+y^2=a^2$ (by the distance formula). As the point A is the projection of P onto the x-axis, then we may say: $\overline{\text{OA}}=x=a\cos\theta$ and $\overline{\text{AP}}=a\sin\theta$.

Hence, as $x^2 + y^2 = a^2$, it follows that:

$$a^2 \cos^2 \theta + a^2 \sin^2 \theta = a^2$$
$$a^2(\cos^2 \theta + \sin^2 \theta) = a^2$$

so $$\cos^2 \theta + \sin^2 \theta = 1.$$

The Pythagorean identity is $\cos^2 \theta + \sin^2 \theta = 1$.

From this Pythagorean identity we can obtain two more useful formulas. In the first case, we divide the identity by $\cos^2 \theta$.

$$\cos^2 \theta + \sin^2 \theta = 1$$
$$\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} = 1$$

Hence, we may say that $$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

so $$1 + \tan^2 \theta = \sec^2 \theta.$$

In the second case, we divide by $\sin^2 \theta$.

$$\cos^2 \theta + \sin^2 \theta = 1$$
$$\frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta} = 1$$

And further, we have $$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

so $$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta.$$

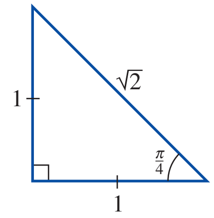
$$\mathbf{1 + \tan^2 \theta = \sec^2 \theta}$$
$$\mathbf{\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta}$$

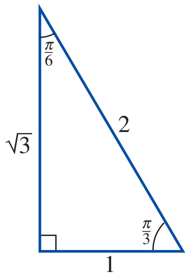
Some exact trigonometric values

We shall now examine the exact trigonometric values of the following.

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \tan \frac{\pi}{4} = 1$$

$$\left.\begin{aligned} \sin \frac{\pi}{3} &= \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \sin \frac{\pi}{6} &= \cos \frac{\pi}{3} = \frac{1}{2} \end{aligned}\right\} \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \tan \frac{\pi}{3} = \sqrt{3}$$





WORKED Example 4

eBookplus

Tutorial:
Worked example 4
int-0361

Find the exact value of cosec $\frac{2\pi}{3}$.

THINK	WRITE
① Use the reciprocal relationship to rewrite cosecant in terms of sine.	$\operatorname{cosec}\frac{2\pi}{3} = \frac{1}{\sin\frac{2\pi}{3}}$
② Express $\sin\frac{2\pi}{3}$ as $\sin\left(\pi - \frac{\pi}{3}\right)$ so that a reduction formula can be used.	$= \frac{1}{\sin\left(\pi - \frac{\pi}{3}\right)}$
③ Use the appropriate reduction formula so that the resulting angle is an angle in the first quadrant; that is, $\sin(\pi - \theta) = \sin\theta$.	$= \frac{1}{\sin\frac{\pi}{3}}$
④ Substitute the exact value for $\sin\frac{\pi}{3}$.	$= \frac{1}{\frac{\sqrt{3}}{2}}$
⑤ Simplify.	$= \frac{2}{\sqrt{3}}\ \left(\text{or } \frac{2\sqrt{3}}{3}\right)$

Why wouldn't a decimal answer be acceptable in this instance?

WORKED Example 5

Find the exact value of $\sec\frac{\pi}{6} \cdot \operatorname{cosec}\frac{5\pi}{4} + \cot\left(-\frac{2\pi}{3}\right)$.

THINK	WRITE
① Identify the reciprocal functions.	$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}}$, $\operatorname{cosec}\frac{5\pi}{4} = \frac{1}{\sin\frac{5\pi}{4}}$, $\cot\left(-\frac{2\pi}{3}\right) = \frac{1}{\tan\left(-\frac{2\pi}{3}\right)}$
② Consider each reciprocal function separately and reduce each to an exact value. Write the exact value for $\cos\frac{\pi}{6}$ and hence state the value for $\sec\frac{\pi}{6}$.	Now $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ so $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$
③ Rewrite $\sin\frac{5\pi}{4}$ so that a reduction formula can be used to obtain a first-quadrant angle.	$\sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right)$ $= -\sin\frac{\pi}{4}$

THINK	**WRITE**
④ Write the exact value for sin $\frac{\pi}{4}$ and hence state the exact value for the reciprocal function.	$= -\frac{1}{\sqrt{2}}$ so $\operatorname{cosec}\frac{5\pi}{4} = -\sqrt{2}$
⑤ Rewrite tan $\left(-\frac{2\pi}{3}\right)$ so that a reduction formula can be used to obtain a first quadrant angle.	$\tan\left(-\frac{2\pi}{3}\right) = -\tan\frac{2\pi}{3}$ $= -\tan\left(\pi - \frac{\pi}{3}\right)$ $= \tan\frac{\pi}{3}$
⑥ Write the exact value for tan $\frac{\pi}{3}$ and hence state the exact value for the reciprocal function.	$= \sqrt{3}$ so $\cot\left(-\frac{2\pi}{3}\right) = \frac{1}{\sqrt{3}}$
⑦ Substitute these exact values into the given expression and simplify.	Therefore $\sec\frac{\pi}{6} \cdot \operatorname{cosec}\frac{5\pi}{4} + \cot\left(-\frac{2\pi}{3}\right)$ $= \frac{2}{\sqrt{3}} \cdot -\sqrt{2} + \frac{1}{\sqrt{3}}$ $= \frac{-2\sqrt{2}}{\sqrt{3}} + \frac{1}{\sqrt{3}}$ $= \frac{1-2\sqrt{2}}{\sqrt{3}}$ or $\frac{\sqrt{3}(1-2\sqrt{2})}{3}$

WORKED Example 6

Given that sec $x = \sqrt{5}$ and $90° < x < 360°$, find the exact values of:

a **cot x**

b **cosec x.**

THINK	**WRITE**
a ① Use the given information to determine in which quadrant the angle is located.	**a** The value of sec x is positive in the 1st and 4th quadrants as cos x is positive in these quadrants. Also, $90° < x < 360°$, so x is located in the 4th quadrant.
② Since sec x is given, use the identity $1 + \tan^2 x = \sec^2 x$ to determine cot x. Remember that tan x is the reciprocal function for cot x. Substitute for sec x and solve to find tan x.	$1 + \tan^2 x = \sec^2 x$ $1 + \tan^2 x = 5$ $\tan^2 x = 4$ $\tan x = \pm 2$

Continued over page

THINK	**WRITE**
③ State the value of tan x in the 4th quadrant and use this to find the value of cot x.	As tangent is negative in the 4th quadrant, $\tan x = -2$ so $\cot x = -\frac{1}{2}$
b ① Since cot x is now known, use the identity $\cot^2 x + 1 = \text{cosec}^2 x$ to determine cosec x.	**b** $\cot^2 x + 1 = \text{cosec}^2 x$ $\frac{1}{4} + 1 = \text{cosec}^2 x$ $\text{cosec}^2 x = \frac{5}{4}$ $\text{cosec}\, x = \pm\frac{\sqrt{5}}{2}$
② State the value of cosec x in the 4th quadrant. As sine is negative in the 4th quadrant, cosecant is also negative in this quadrant.	As sine is negative in the 4th quadrant, $\text{cosec}\, x = -\frac{\sqrt{5}}{2}$

WORKED Example 7

Show that $\dfrac{\sin (90° + \alpha)}{1 - \sec \alpha} + \dfrac{\sin (90° - \alpha)}{1 + \sec \alpha} = -2 \cos \alpha \cot^2 \alpha.$

THINK	**WRITE**
① Consider the left-hand side of the equation.	$\frac{\sin (90° + \alpha)}{1 - \sec \alpha} + \frac{\sin (90° - \alpha)}{1 + \sec \alpha}$
② Simplify each numerator using the appropriate reduction formula. $\sin (90° + \alpha) = \sin [180° - (90° - \alpha)]$ $= \sin (90° - \alpha)$ $= \cos \alpha$ and $\sin (90° - \alpha) = \cos \alpha$.	$= \frac{\cos \alpha}{1 - \sec \alpha} + \frac{\cos \alpha}{1 + \sec \alpha}$
③ Write as a single fraction with a common denominator.	$= \frac{\cos \alpha(1 + \sec \alpha) + \cos \alpha(1 - \sec \alpha)}{(1 - \sec \alpha)(1 + \sec \alpha)}$
④ Expand and simplify both the numerator and the denominator.	$= \frac{2 \cos \alpha}{1 - \sec^2 \alpha}$
⑤ Use the identity $1 + \tan^2 \alpha = \sec^2 \alpha$ to simplify the denominator further.	$= \frac{2 \cos \alpha}{-\tan^2 \alpha}$
⑥ Rewrite the expression using the relationship $\cot \alpha = \frac{1}{\tan \alpha}$ to show that it equals the right-hand side of the equation.	$= -2 \cos \alpha \cot^2 \alpha$
⑦ Answer the question.	So $\frac{\sin (90° + \alpha)}{1 - \sec \alpha} + \frac{\sin (90° - \alpha)}{1 + \sec \alpha}$ $= -2 \cos \alpha \cot^2 \alpha$

remember

1. For any real-valued θ, the Pythagorean and associated relationships state:

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \text{cosec}^2\theta.$$

2. Useful exact values are:

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \tan\frac{\pi}{4} = 1$$

$$\left.\begin{aligned}\sin\frac{\pi}{3} &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\\ \sin\frac{\pi}{6} &= \cos\frac{\pi}{3} = \frac{1}{2}\end{aligned}\right\} \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \tan\frac{\pi}{3} = \sqrt{3}.$$

EXERCISE 7B

Exact values and the Pythagorean relationships

WORKED Example 4, 5

1 Find the exact value of:

a $\sin\frac{5\pi}{6}$ **b** $\text{cosec}\frac{5\pi}{6}$ **c** $\cos\frac{3\pi}{4}$

d $\sec\frac{3\pi}{4}$ **e** $\tan\frac{4\pi}{3}$ **f** $\cot\frac{4\pi}{3}$

g $\text{cosec}\left(-\frac{\pi}{3}\right)$ **h** $\cot\frac{3\pi}{4}\cdot\sec\frac{\pi}{6}$

i $\sec 315° + \text{cosec}(-240°)$ **j** $\text{cosec}^2(-330°) + \cot^2(-330°) + 1.$

WORKED Example 6

2 If $\sin x = \frac{3}{5}$ and $\cos x < 0$, find the exact value of:

a $\cot x$ **b** $\sec x$

c $(\text{cosec}\, x - \sec x)^2$ **d** $\frac{\cot x + \sec x}{\cos x}.$

3 An angle θ lies between $\frac{\pi}{2}$ and π, and $\sin\theta = \frac{5}{13}$. Find the exact values of:

a $\cot\theta - \tan\theta$ **b** $\text{cosec}\,\theta - \sec\theta$ **c** $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$

4 If θ is acute and $\cos\theta = \frac{1}{\sqrt{3}}$, find the exact value of $\frac{\cot\theta}{1 - \text{cosec}\,\theta}.$

5 Find the exact value of each of the following given that $\sin y = 0.8$ and y is acute.

a $\sec\left(\frac{\pi}{2} + y\right)$ **b** $\text{cosec}\left(\frac{\pi}{2} - y\right)$ **c** $\sin\left(\frac{3\pi}{2} + y\right)$

d $\cot\left(\frac{\pi}{2} - y\right)$ **e** $\sec(2\pi - y)$ **f** $\text{cosec}\left(\frac{3\pi}{2} - y\right)$

6 Simplify:

a $\cot^2 x - \operatorname{cosec}^2 x$

b $(1 - \sin^2 x)\sec^2 x$

c $\dfrac{1}{\sec\theta - 1} + \dfrac{1}{\sec\theta + 1}$

d $\dfrac{\cot\theta}{\operatorname{cosec}\theta - 1} + \dfrac{\cot\theta}{\operatorname{cosec}\theta + 1}$

e $\sqrt{(1-\cos^2 x)(1+\tan^2 x)}$

f $\dfrac{\tan x + \cot x}{\operatorname{cosec} x \sec x}$

g $\dfrac{\sin(-\theta)\tan\left(\frac{\pi}{2}-\theta\right)}{\cot(\pi+\theta)\sin\left(\frac{\pi}{2}+\theta\right)}$

h $\dfrac{\cos(-\theta)\cot\left(\frac{3\pi}{2}-\theta\right)}{\tan(\pi-\theta)\cos\left(\frac{\pi}{2}-\theta\right)}$.

WORKED Example 7

7 Show that:

a $\dfrac{1}{1+\sin x} + \dfrac{1}{1-\sin x} = 2\sec^2 x$

b $\dfrac{1+\cos x}{1-\cos x} = (\operatorname{cosec} x + \cot x)^2$

c $\dfrac{\tan x + \cot y}{\cot x + \tan y} = \dfrac{\tan x}{\tan y}$

d $\dfrac{1}{\sec y - \tan y} = \sec y + \tan y$.

8 If $\dfrac{\sin x}{1-\cos x} - \dfrac{\sin x}{1+\cos x} = y^2 \cot x$, find possible values for y if y is a real number.

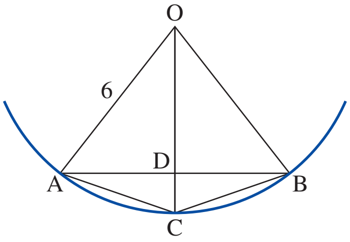
9 A sector OACB has a radius 6 cm and contains the equilateral triangle OAB. Given that OC is the perpendicular bisector of AB:

a find the lengths of:

i DA **ii** DC.

b Use your results from **a** above to deduce that $\tan(15°) = 2 - \sqrt{3}$.

c Hence, or otherwise, determine the exact value of $\tan(165°)$.



Modelling soundwaves

There are many terms we associate with sound; for example, pitch, frequency, loudness, tone and beat.

Pitch is an attribute of the sound frequency. The greater the number of vibrations per second (frequency), the higher the pitch. The range of frequencies of the human voice is from about 60 vibrations per second for a low bass to about 1300 vibrations per second for a soprano.

Loudness is a more complex attribute because it depends on physical factors such as distance. In general, if two sounds have the same frequency, the one with the greater amplitude will be perceived as the louder. On the other hand, if two sounds have the same amplitude, the one with the greater frequency will be perceived as being the louder.

The term *tone* has many connotations, but is usually used to describe a sound's quality. Partial tones are the separate components of a complex tone. The partial tone with the least frequency is called the *fundamental tone* and the others are called *overtones*.

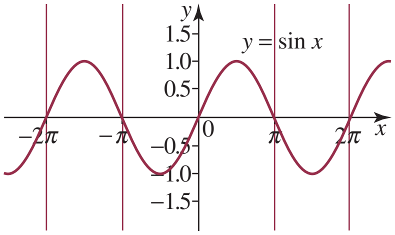
The term *beat* is also used in a variety of ways, all related to the rhythm of musical sounds. However, the *beat effect* is a special phenomenon: if two strings are in tune with each other, the sound is steady; if they are off by a number of vibrations per second, the resulting sound has a pulsation. Here, the number of beats per second is the difference in frequencies of the sounds.

A series of frequencies that have integer ratios are called *harmonics*. If the fundamental frequency is x, then the second harmonic will have a frequency of $2x$ and the third harmonic a frequency of $3x$, etc. Sounds that differ by an octave have frequencies that are in the ratio of 2:1.

The *timbre* of a sound is a manifestation of its tone quality. Electronic synthesisers make substantial use of timbre in producing sound. These synthesisers model sound using a branch of mathematics called *Fourier analysis*. Theoretically, using a synthesiser and Fourier modelling, it is possible to produce any sound.

Sound can be modelled using sinusoidal waveforms. Sinusoidal waveforms are a sum of sine waves of the form $a \sin(\omega x + b)$ where a is the amplitude, ω is the angular frequency, and $-\dfrac{b}{\omega}$ is the phase shift. Before we contemplate sinusoidal waveforms, it may be appropriate to review the significant features of the graphs of the main trigonometric functions.

$y = a \sin(\omega x + b)$



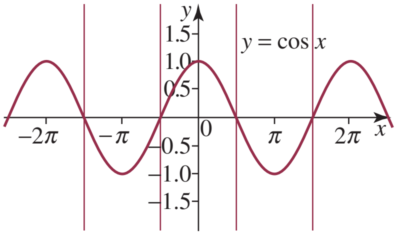
The period $= \dfrac{2\pi}{\omega}$

The natural period of the sine wave is 2π.
The amplitude is a where $a > 0$.

The phase shift is $-\dfrac{b}{\omega}$.

Also, $-\infty < x < \infty$ and $-a \le y \le a$.

$y = a \cos(\omega x + b)$



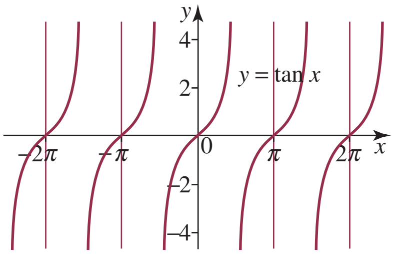
The period $= \dfrac{2\pi}{\omega}$

The natural period of the cosine wave is 2π.
The amplitude is a where $a > 0$.

The phase shift is $-\dfrac{b}{\omega}$.

Also, $-\infty < x < \infty$ and $-a \le y \le a$.

$y = a \tan(\omega x + b)$



The period $= \dfrac{2\pi}{\omega}$

The natural period of the tangent function is π.

The phase shift is $-\dfrac{b}{\omega}$.

Also, $-\infty < x < \infty$ and $-\infty \le y \le \infty$.

Remember that period is also equal to $\dfrac{1}{\text{frequency}}$.

WORKED Example 8

A radio that is not tuned properly produces an audible hum. This soundwave has a frequency of 60 vibrations per second and an amplitude of 0.2.

a **Find the angular frequency, ω, of this soundwave assuming a sinusoidal waveform.**

b **Write the equation of the waveform.**

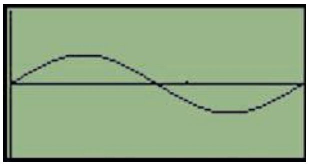
c **Model the waveform using a graphics calculator over the interval $0 \le t \le \frac{1}{60}$ seconds.**

THINK	WRITE/DISPLAY
a ① State the known information.	**a** Amplitude is 0.2. Frequency = 60 vibrations per second
② Calculate the period, since it relates to frequency and also gives a relationship involving ω for a sine function.	Period $= \frac{1}{\text{frequency}}$ $= \frac{1}{60}$ s Also period $= \frac{2\pi}{\omega}$ So $\frac{1}{60} = \frac{2\pi}{\omega}$ $\omega = 120\pi$
③ Write the answer.	The angular frequency is 120π vibrations per second.
b ① State the general form of a sine function.	**b** $y = a \sin(\omega t + b)$
② List the values for a and ω.	where $a = 0.2$, $\omega = 120\pi$
③ Assume that the phase shift is zero and hence state the value for b.	Assume the phase shift is zero, so $b = 0$.
④ Substitute for a, ω and b to write the equation of the waveform.	The equation of the waveform is $y = 0.2 \sin(120\pi t)$.
c Use a graphics or CAS calculator to draw a graph of the waveform from $t = 0$ to $t = \frac{1}{60}$ s.	**c**
For the Casio *fx*-9860G AU ① Ensure the calculator is in radian mode, then press: • MENU • 5 (GRAPH) Complete the entry line as: $Y1 = 0.2 \sin(120\pi x)$ Then press EXE. Adjust the View Window settings as shown.	View Window Xmin :0 max :0.01666666 scale:0.01 dot :1.3227E-04 Ymin :-0.5 max :0.5

THINK

2 Press:
- ENTER
- F6 (DRAW)

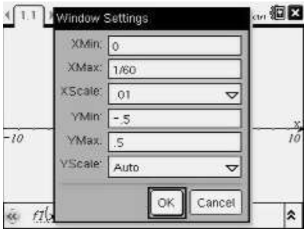
WRITE/DISPLAY



For the TI-Nspire CAS

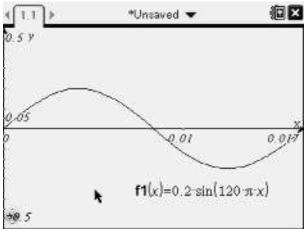
1 On a Graphs page, press:
- Menu menu
- 4: Window/Zoom 4
- 1: Window Settings 1

Enter the values in the fields as shown.
Note: Press tab tab to move between fields.
Select OK to return to the Graphs page.



XMin: 0
XMax: 1/60
XScale: .01
YMin: -.5
YMax: .5
YScale: Auto

2 Complete the entry line as:
$f1(x) = 0.2 \sin(120\pi x)$
then press ENTER enter.
Note: The calculator must be in radian mode.



f1(x)=0.2·sin(120·π·x)

WORKED Example 9

A soundwave from two musical instruments is made up of two partial waves both having an amplitude of 0.2. The period of the fundamental is 0.01 seconds and that of the overtone is 0.02 seconds.

a Find the angular frequency, ω, of both partial soundwaves.
b Write the equations of the separate and composite waveforms.
c Model the separate and composite waveforms using a graphics calculator over the interval $0 \le t \le 0.02$ seconds.
d How is the loudness related to the two separate instruments?

THINK	WRITE/DISPLAY
a 1 State the known information.	a First wave: period = 0.01, amplitude = 0.2 Second wave: period = 0.02, amplitude = 0.2
2 Write the relationship between period and angular frequency.	Period = $\frac{2\pi}{\omega}$

Continued over page

THINK | **WRITE/DISPLAY**

③ Calculate the angular frequency for each partial wave.

$$0.01 = \frac{2\pi}{\omega_1} \text{ so } \omega_1 = 200\pi$$

$$0.02 = \frac{2\pi}{\omega_2} \text{ so } \omega_2 = 100\pi$$

b ① Model using sinusoidal waveforms. Write the general equation using the sine function.

b $y = a \sin(\omega t + b)$

② Write the equations of the separate waveforms using appropriate values for a, ω and b.

$y_1 = 0.2 \sin(200\pi t)$

$y_2 = 0.2 \sin(100\pi t)$

③ Add each of the separate waveforms to obtain the equation for the composite.

$y = y_1 + y_2$

$\therefore\ y = 0.2 \sin(200\pi t) + 0.2 \sin(100\pi t)$

c Use a graphics calculator to draw the graphs of the waveforms from $t = 0$ to $t = 0.02$ s.

c

For the Casio *fx*-9860G AU

① Ensure the calculator is in radian mode then press:

- MENU
- 5 (GRAPH)

Complete the entry lines as:

Y1 = 0.2 sin(200πx)

Y2 = 0.2 sin(100πx)

Y3 = Y1 + Y2

Press EXE after each line.

Note: To input Y for Y1 and Y2 press:

- VARS
- F4 (GRPH)
- F1 (Y)

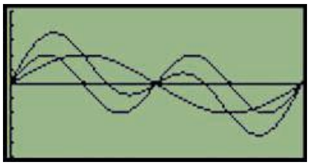
```
Graph Func  :Y=
Y1=.2sin (200πX)[—]
Y2=.2sin (100πX)[—]
Y3=Y1+Y2        [—]
Y4:             [—]
Y5:             [—]
Y6:             [—]
```

THINK | **WRITE/DISPLAY**

2 Enter appropriate V-window settings as shown then press EXE.

```
View Window
Xmin  :0
 max  :0.02
 scale:5E-03
 dot  :1.5873E-04
Ymin  :-0.5
 max  :0.5
```

3 Press F6 (DRAW) to draw the graph.



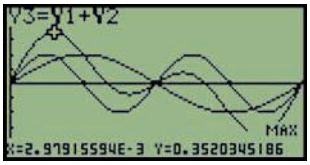
d Use a graphics calculator to check the amplitude of the composite wave.

d

1 To find the maximum value of the composite wave, press:

- SHIFT
- F5 (G-Solv)
- F2 (MAX)

Use the arrow keys to move to the graph of Y3 and press EXE. The coordinates of the maximum point are displayed at the bottom of the screen.



```
Y3=Y1+Y2
X=2.979155594E-3 Y=0.3520345186
```

2 Write the answer.

The amplitude of the composite function is 0.35. The loudness is almost doubled.

For the TI-Nspire CAS

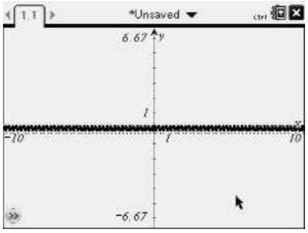
1 On a new Graphs page, complete the entry lines as:

$f1(x) = 0.2\ \sin(200\pi x)$

$f2(x) = 0.2\ \sin(100\pi x)$

$f3(x) = f1(x) + f2(x)$

Press ENTER after each line.



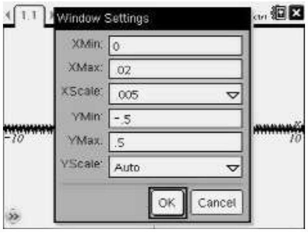
Continued over page

THINK **WRITE/DISPLAY**

2 To set the Window Settings, press:
- Menu (menu)
- 4: Window/Zoom (4)
- 1: Window Settings (1)

Enter the values in the fields as shown.

Select OK.



Window Settings

XMin: 0

XMax: .02

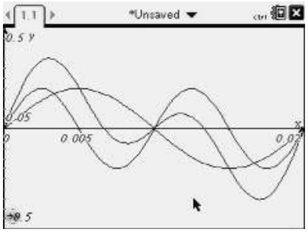
XScale: .005

YMin: -.5

YMax: .5

YScale: Auto

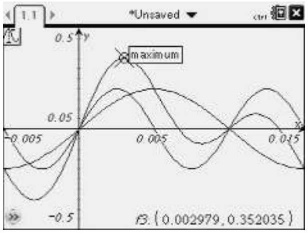
3 (a) Select OK to display the graph.



(b) Use the graphics calculator to check the amplitude of the composite wave.

4 To find the maximum of the composite wave, press:
- Menu (menu)
- 5: Trace (5)
- 1: Graph Trace (1)

When the maximum value is reached, the word 'maximum' appears in a text box.



f3: (0.002979, 0.352035)

5 Write the answer.

The amplitude of the composite function is 0.35. The loudness is almost doubled.

WORKED Example 10

Harmonic analysers may be used to generate square waves. These waves frequently appear in the sciences of electronics and engineering. In music, we may associate the square wave with sounds that are 'woody'. For example, the sound generated by a clarinet. The mathematical model used to generate a square wave is given by the formula:

$$y = \sin x + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + $$

Use a graphics calculator to generate an image of the square waveform using as many terms as possible over the domain $-2\pi \le x \le 2\pi$.

THINK **WRITE/DISPLAY**

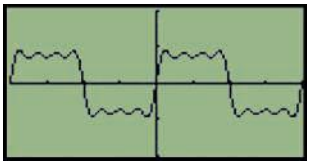
For the Casio *fx*-9860G AU

1. Ensure the calculator is in radian mode, then press:
   - MENU
   - 5 (GRAPH)

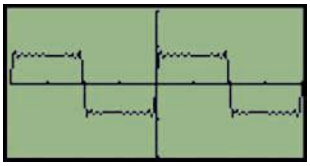
   Complete the entry line as:

   $Y1 = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7}$

   Set the domain in the View Window as $-2\pi \leq x \leq 2\pi$.



2. Edit the graph to add to the ‘squareness’ by adding more terms into the equation.

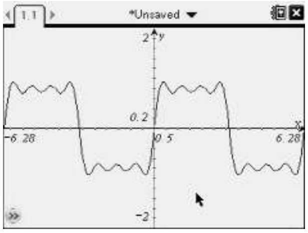


For the TI-Nspire CAS

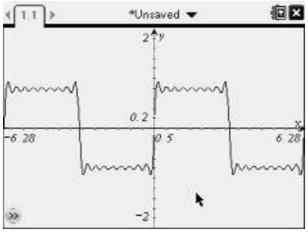
1. On a Graphs page, complete the entry line as:

   $f1(x) = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7}$

   Set the domain in the Window Settings as $-2\pi \leq x \leq 2\pi$.



2. Edit the graph to improve its ‘squareness’ by adding more terms.



remember

1. If two sounds have the same frequency, the one with the greater amplitude will be perceived as the louder. The partial tone with the least frequency is called the *fundamental tone* and the others are called *overtones*.
2. A series of frequencies that have integer ratios are called *harmonics*. Sounds that differ by an octave have frequencies that are in the ratio of 2:1.
3. A sinusoidal waveform has the general equation $y = a \sin(\omega t + b)$ where a is the amplitude, ω is the angular frequency and $-\frac{b}{\omega}$ is the phase shift.
4. The period of the waveform is given by $\frac{2\pi}{\omega}$ or $\frac{1}{\text{frequency}}$.

EXERCISE 7C Modelling soundwaves

In each of the following questions make use of appropriate software to generate each waveform.

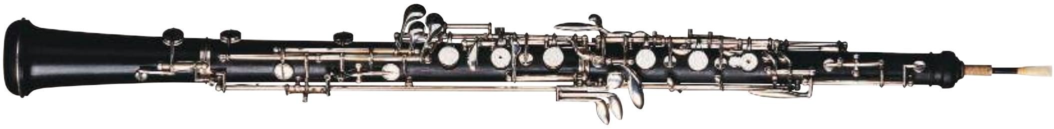
WORKED Example 8

1 A musical instrument generates a soundwave with a frequency of 100 vibrations per second and an amplitude of 0.2.

a Find the angular frequency, ω, of this soundwave, assuming a sinusoidal waveform.

b Write the equation of the waveform.

c Model the waveform using a graphics calculator over the interval $0 \le t \le 0.01$ seconds.



WORKED Example 9

2 A soundwave from two musical instruments is made up of two partial waves both having an amplitude of 0.2. The period of the fundamental is 0.1 seconds and that of the overtone is 0.2 seconds.

a Find the angular frequency, ω, of both soundwaves.

b Write the equations of the separate and composite waveforms.

c Model the separate and composite waveforms using a graphics calculator over the interval $0 \le t \le 0.2$ seconds.

d How is the loudness related to the two separate instruments?

3 For each of the following soundwaves state the amplitude, frequency and period. Graph each wave over an appropriate domain, t seconds, to produce one cycle.

a $y = 0.005 \sin (150\pi t)$

b $y = 0.04 \sin (400\pi t)$

c $y = 0.2 \sin (100\pi t) + 0.2 \sin (300\pi t)$

d $y = 0.4 \sin \left(200\pi t + \frac{\pi}{2}\right) + 0.25 \sin (200\pi t)$

e $y = 0.5 \sin (200\pi t) + 0.25 \sin (100\pi t)$

4 When two instrumentalists perform together, the vibrations may not be in phase.

a Graph each of the following functions and their composite $y = y_1 + y_2$:

$y_1 = 0.5 \sin (40\pi t)$

$y_2 = 0.5 \sin \left(40\pi t + \frac{\pi}{4}\right)$.

b Discuss how the frequency of the composite wave is related to the partials.

c How is the loudness related to that of the two separate instruments?

5 A soundwave is generated from two partials. The period of the fundamental is 0.01 seconds and that of the overtone 0.03 seconds. If both partials have an amplitude of 0.5:

a write the equation of the composite wave

b graph the waveform to show one complete cycle

c discuss the effects on both loudness and frequency in relation to the partials in connection with the composite soundwave.

6 Two oboists are playing together. If the soundwaves are modelled by the equations

$$y_1 = 4\sin(400\pi t) \text{ and } y_2 = 4\sin(400\pi t + \pi):$$

a What is the frequency of the sounds in cycles per second?

b Graph the composite waveform.

c Describe the nature of the sound.

7 Three stringed instruments in concert produce three harmonics. The fundamental frequency is 0.005 and they have the same amplitude of 2.

a Write an equation for the composite soundwave.

b Graph the composite wave and discuss what the outcomes are in relation to loudness and frequency.

WORKED Example 10

8 The sound generated by a recorder is modelled by the following equation:

$$y = \frac{4}{\pi}\left[\sin x - \frac{\sin(3x)}{3^2} + \frac{\sin(5x)}{5^2} - \frac{\sin(7x)}{7^2} + \quad\right].$$

Generate an image of the waveform using as many terms as possible over the domain $-2\pi \le x \le 2\pi$ to make an effective graph. Discuss the features of the graph. Compare and contrast the graph with that shown in Worked example 10.

Graphing the reciprocal trigonometric functions

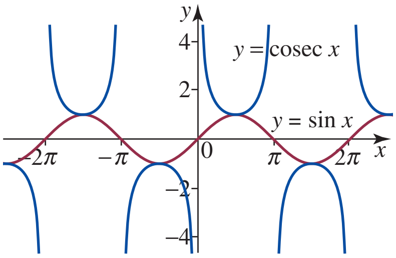
From the graphs of $y = \sin(ax + b)$ and $y = \cos(ax + b)$, we may deduce the respective forms of the reciprocal functions $y = \operatorname{cosec}(ax + b)$ and $y = \sec(ax + b)$. To do this we need to appreciate that the reciprocals of both the sine and cosine functions are undefined whenever the values of sine and cosine are zero. In the same fashion as for the cosec and sec graphs, the graph of $y = \cot x$ is related to $y = \tan x$. We shall now examine some of these features.

$y = \operatorname{cosec} x$

If $\sin x = 0$, then $x = n\pi$ for integral values of n.
As $-1 \le \sin x \le 1$, it follows that:

$$0 \le \sin x \le 1 \Rightarrow \operatorname{cosec} x \ge 1$$
$$-1 \le \sin x \le 0 \Rightarrow \operatorname{cosec} x \le -1$$

As $x \to n\pi$ for integral n, $\operatorname{cosec} x \to \pm\infty$.

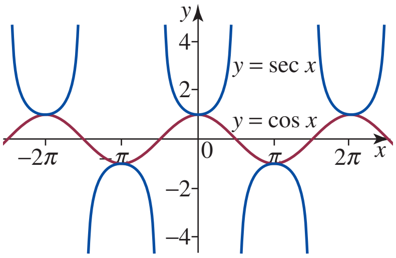


$y = \sec x$

If $\cos x = 0$, then $x = 2n\pi \pm \dfrac{\pi}{2}$ for integral values of n.
As $-1 \le \cos x \le 1$, it follows that:

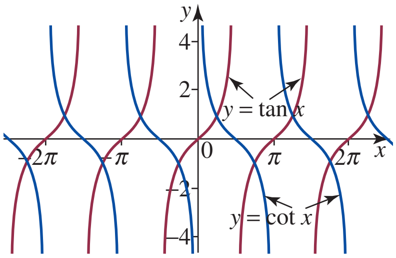
$$0 \le \cos x \le 1 \Rightarrow \sec x \ge 1$$
$$-1 \le \cos x \le 0 \Rightarrow \sec x \le -1.$$

As $x \to 2n\pi \pm \dfrac{\pi}{2}$ for integral n, then $\sec x \to \pm\infty$.



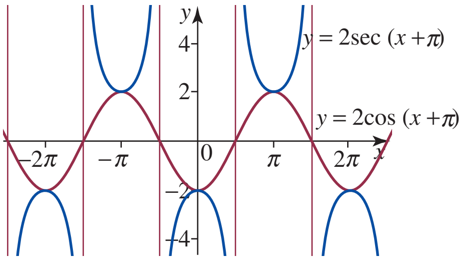
$y = \cot x$

If $\tan x = 0$, then $x = n\pi$ for integral n.
As $x \to n\pi$, $\cot x \to \pm\infty$.
As $\tan x \to \pm\infty$, $\cot x \to 0$ from above or below.



WORKED Example 11

Sketch $y = 2 \sec (x + \pi)$, $-2\pi \le x \le 2\pi$.

THINK	WRITE
① Consider the graph of the reciprocal function, $y = 2 \cos (x + \pi)$. State the features of the graph.	Reciprocal function is $y = 2 \cos (x + \pi)$. Amplitude is 2. Period is 2π. Phase shift is $-\pi$.
② Identify the discontinuities for $y = 2 \sec (x + \pi)$.	Vertical asymptotes occur where $x + \pi = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, etc. That is, where $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, etc.
③ Sketch $y = 2 \cos (x + \pi)$ over the required domain and mark in the vertical asymptotes for $y = 2 \sec (x + \pi)$. ④ Draw the sec graph in a different colour on the same set of axes. Label each function.	

You may like to check your graph by using a graphics or CAS calculator (see Worked example 13).

WORKED Example 12

Sketch $y = \frac{1}{2} \operatorname{cosec}\left(2x - \frac{\pi}{3}\right)$, $-2\pi \le x \le 2\pi$.

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Tutorial:
Worked example 12
int-0360

THINK	WRITE
① Consider the graph of the reciprocal function, $y = \frac{1}{2} \sin 2\left(x - \frac{\pi}{6}\right)$. State the features of the graph.	Reciprocal function is $y = \frac{1}{2} \sin \left(2x - \frac{\pi}{3}\right)$. Amplitude is $\frac{1}{2}$. Period is π. Phase shift is $\frac{\pi}{6}$.

THINK | **WRITE**

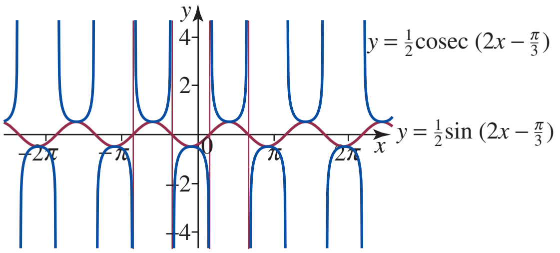
② Identify the discontinuities for $y = \frac{1}{2}\text{ cosec }\left(2x - \frac{\pi}{3}\right)$.

Vertical asymptotes for $y = \frac{1}{2}\text{ cosec }\left(2x - \frac{\pi}{3}\right)$ occur when

$$2\left(x - \frac{\pi}{6}\right) = \quad -2\pi, -\pi, 0, \pi, 2\pi$$

That is, when $x = \quad \frac{-5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$

③ Sketch $y = \frac{1}{2}\sin 2\left(x - \frac{\pi}{6}\right)$ over the required domain and mark in the vertical asymptotes for $y = \frac{1}{2}\text{ cosec }\left(2x - \frac{\pi}{3}\right)$.



④ Draw the cosec graph in a different colour on the same set of axes. Label each function.

WORKED Example 13

a **Sketch $y = -\sec(\pi x) + 1$, $0 \le x \le 2\pi$. Use a graphics calculator to verify your answer.**

b **Using a graphics calculator, find the smallest positive value of x for which $-\sec(\pi x) + 1 = 2.5$.**

THINK | **WRITE/DISPLAY**

a ① Consider the graph of the reciprocal function, $y = -\cos \pi x + 1$. State the features of the graph.

a Reciprocal function is $y = -\cos(\pi x) + 1$.
Amplitude is 1.
Period is 2 radians.
Reflect graph of $y = \cos \pi x$ in x-axis to obtain the graph of $y = -\cos \pi x$.
Vertical shift is +1.

② Identify the discontinuities for $y = -\sec(\pi x) + 1$.

Vertical asymptotes for $y = -\sec(\pi x) + 1$ occur when:

$$\pi x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \text{ etc.}$$

$$\Rightarrow x = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \text{ etc.}$$

That is, when $x = \quad , -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$

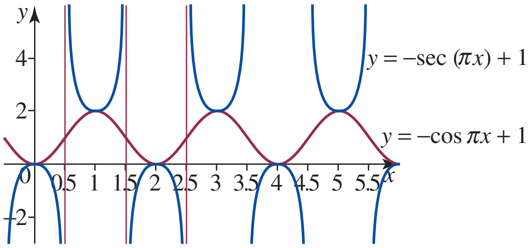
Continued over page →

THINK

WRITE/DISPLAY

③ Sketch $y = -\cos \pi x + 1$ over the required domain and mark in the vertical asymptotes for $y = -\sec(\pi x) + 1$.

④ Draw the sec graph in a different colour on the same set of axes. Label each function.



⑤ Verify your answer using a graphics calculator.

For the Casio *fx*-9860G AU

a ① Ensure the calculator is in radian mode and then press:

- MENU
- 5 (GRAPH)

Complete the entry lines as:

$Y1 = -\cos(\pi x) + 1$

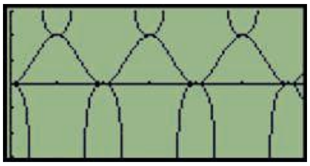
$$Y2 = \frac{-1}{\cos(\pi x)} + 1$$

Set the View Window as shown, then press EXE.

a

```
View Window
Xmin  :0
 max  :6.2831853
 scale:1
 dot  :0.04986655
Ymin  :-3
 max  :3
INIT TRIG STD STO RCL
```

② Press F6 (DRAW).



b ① To solve $-\sec(\pi x) + 1 = 2.5$ for the smallest value of x, complete the entry lines as:

$$Y1 = \frac{-1}{\cos(\pi x)} + 1$$

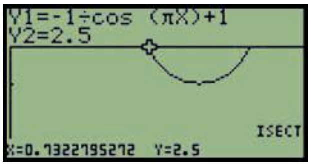
$Y2 = 2.5$

Press EXE after each line.
Then press:

- SHIFT
- F5 (G-Solv)
- F5 (ISCT)

The point of intersection will be displayed.

b



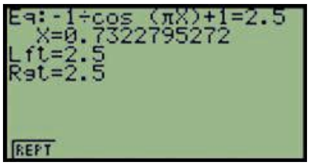
```
Y1=-1÷cos (πX)+1
Y2=2.5
ISECT
X=0.7322795272   Y=2.5
```

② Alternatively, press:

- MENU
- EQUA
- F3: Solver

Enter the equation $\frac{-1}{\cos(\pi x)} + 1 = 2.5$.

Then press EXE and the solution will be displayed.



```
Eq:-1÷cos (πX)+1=2.5
  X=0.7322795272
Lft=2.5
Rgt=2.5
REPT
```

THINK

WRITE/DISPLAY

For the TI-Nspire CAS

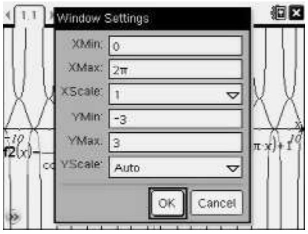
a ① On a Graphs screen, complete the entry lines as:

$f1(x) = -\cos(\pi x) + 1$

$f2(x) = \dfrac{-1}{\cos(\pi x)} + 1$

pressing ENTER (enter) after each line. Enter Window Settings as shown and select OK.

a



Window Settings

XMin: 0

XMax: 2π

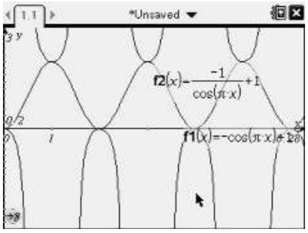
XScale: 1

YMin: -3

YMax: 3

YScale: Auto

② The graph will be displayed.



b ① On a new Graphs page, complete the entry lines as:

$f1(x) = \dfrac{-1}{\cos(\pi x)} + 1$

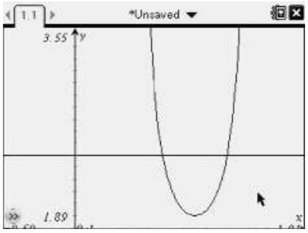
$f2(x) = 2.5$

pressing ENTER (enter) after each line. To improve the clarity of the graph, press:

- MENU (menu)
- 4: Window/Zoom (4)
- 3: Zoom In (3)

Move the centre close to the first point of intersection.

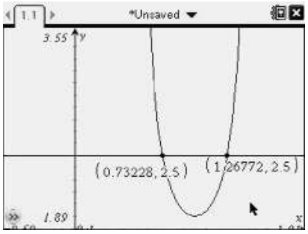
b



② To find the smallest value of x for which $-\sec(\pi x) + 1 = 2.5$, press:

- MENU (menu)
- 7: Points & Lines (7)
- 3: Intersection Point(s) (3)

Move the pointer ↖.
Click once on each graph.
It may be necessary to move the point labels for clarity.



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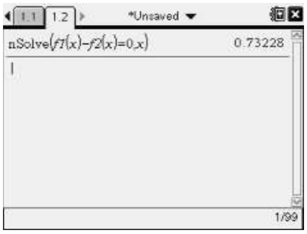
THINK

WRITE

③ Alternatively, on a Calculator screen, to obtain a numerical solution press:

- MENU (menu)
- 3: Algebra ⟨3⟩
- 5: Numerical Solve ⟨5⟩
- Complete the entry line as: nsolve($f1(x) - f2(x) = 0, x$)



```
nSolve(f1(x)-f2(x)=0,x)                    0.73228
```

remember

1. $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$
2. Both cosec x and sec x have a period of 2π but cot x has a period of π.
3. For integral values of n, as $x \to n\pi$, cosec $x \to \pm\infty$; as $x \to 2n\pi \pm \dfrac{\pi}{2}$, sec $x \to \pm\infty$; and as $x \to n\pi$, cot $x \to \pm\infty$.
4. For integral values of n, asymptotes exist for cosec x whenever $x = n\pi$; for sec x whenever $x = 2n\pi \pm \dfrac{\pi}{2}$; and for cot x whenever $x = n\pi$.

EXERCISE 7D Graphing the reciprocal trigonometric functions

WORKED Example 11, 12, 13a

1 Sketch each of the following trigonometric functions over the domain $-2\pi \le x \le 2\pi$ and check your graph against one generated using appropriate technology.

a $y = \sec\left(x + \dfrac{\pi}{4}\right)$ $\quad$ **b** $y = \text{cosec}\left(x - \dfrac{\pi}{3}\right)$

c $y = \cot\left(x + \dfrac{\pi}{2}\right)$ $\quad$ **d** $y = 2\sec 2\left(x - \dfrac{\pi}{4}\right)$

e $y = \frac{1}{2}\,\text{cosec}\left(3x + \dfrac{\pi}{2}\right)$ $\quad$ **f** $y = -\sec(4x + \pi)$

g $y = \cot(2x)$ $\quad$ **h** $y = \text{cosec}(\pi x) + 2$

i $y = 1 - \text{cosec}\left(\frac{1}{2}x + \pi\right)$ $\quad$ **j** $y = 2\sec\left(\frac{1}{2}\pi x\right) - 1$

WORKED Example 13b

2 Using a method that involves the use of appropriate technology, solve each of the following trigonometric equations as required.

a Find the smallest positive solution to the equation $\sec 3x = 2$ over the domain $0 \le x \le \pi$. How many possible solutions are there in this domain?

b Determine the number of roots of the equation $\frac{1}{2}$ cosec $2x = x$ in the domain $-\pi \le x \le \pi$. What is the least value of x in this domain?

c Find those values of x for which cot $2x$ = tan x in the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

d Determine the greatest value of x in the range 0° to 90° for which cosec $(3x°)$ = sec $(x°)$. For what values of $x°$ is cosec $(3x) \le$ sec (x) in the domain $0° \le x \le 45°$?

3 A minor arc, AB, of a circle of radius a units subtends an angle θ at the centre O. The area of the segment is one quarter of the whole circle.

a Show that $\sin\theta = \theta - \frac{\pi}{2}$.

b Use appropriate graphics calculator functions to find θ.

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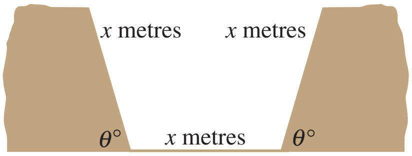
Digital docs:
SkillSHEET 7.1
WorkSHEET 7.1

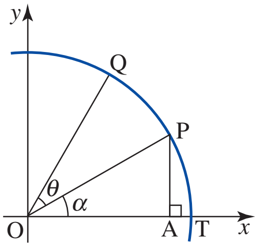
4 Mara is a civil engineer who is working on the construction of an irrigation channel like the one illustrated. This channel has a trapezoidal cross-section in which the bottom and sides are x metres in length. Mara requires $\theta°$ to be acute and for the sides to be equally inclined to the horizontal.



a Find an expression for the area of the cross-section of the channel as a function of θ.

b Using appropriate technology, assist Mara to determine the value of $\theta°$ for which the cross-sectional area is a maximum.

Addition identities for sin ($x \pm y$) and cos ($x \pm y$)



The illustration shows a point T(a, 0) rotated in an anticlockwise direction about the origin to the point P(x, y) through an angle α. This may be expressed mathematically as $\underset{\sim}{p} = \underset{\sim}{R}_\alpha \underset{\sim}{t}$ where $\underset{\sim}{p}$ and $\underset{\sim}{t}$ are the respective position vectors of the points P and T, and $\underset{\sim}{R}_\alpha$ is the associated rotation matrix defined as $\underset{\sim}{R}_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Hence,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos\alpha \\ a\sin\alpha \end{bmatrix}$$

$\therefore$ P(x, y) = P($a\cos\alpha$, $a\sin\alpha$).

In the same way, the point P is now rotated to Q(x, y) through angle θ producing the following matrix relationship:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos\alpha \\ a\sin\alpha \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos\theta\cos\alpha - a\sin\theta\sin\alpha \\ a\sin\theta\cos\alpha + a\cos\theta\sin\alpha \end{bmatrix}$$

$$\therefore \text{Q}(x, y) = \text{Q}(a\cos\theta\cos\alpha - a\sin\theta\sin\alpha,\ a\sin\theta\cos\alpha + a\cos\theta\sin\alpha).$$

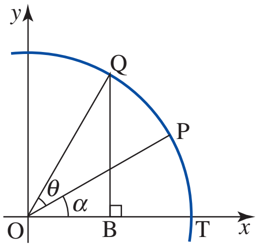
In $\triangle$OQB, B is the point of projection of Q onto the x-axis such that:

$$x = \overline{\text{OB}} = a\cos(\theta + \alpha) \text{ and } y = \overline{\text{QB}} = a\sin(\theta + \alpha).$$

In terms of the coordinates of the point Q, we may say that for any real values of θ and α:

$$a\cos(\theta + \alpha) = a\cos\theta\cos\alpha - a\sin\theta\sin\alpha$$

$$a\sin(\theta + \alpha) = a\sin\theta\cos\alpha + a\cos\theta\sin\alpha.$$



Hence we have:

$$\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$$

$$\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha.$$

We may observe from the above two results that by writing $-\alpha$ instead of α:

$$\cos(\theta - \alpha) = \cos\theta\cos(-\alpha) - \sin\theta\sin(-\alpha)$$

$$\sin(\theta - \alpha) = \sin\theta\cos(-\alpha) + \cos\theta\sin(-\alpha).$$

Now, utilising the identities $\cos(-A) = \cos(A)$ and $\sin(-A) = -\sin(A)$, we may say that:

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha.$$

$$\boldsymbol{\cos(x + y) = \cos x\cos y - \sin x\sin y}$$

$$\boldsymbol{\sin(x + y) = \sin x\cos y + \cos x\sin y}$$

$$\boldsymbol{\cos(x - y) = \cos x\cos y + \sin x\sin y}$$

$$\boldsymbol{\sin(x - y) = \sin x\cos y - \cos x\sin y}$$

WORKED Example 14

Using an appropriate addition identity, expand and simplify $\sin\left(x + \frac{\pi}{6}\right)$.

THINK	WRITE
1 Recognise that we require the sine of the sum of two angles. Write the appropriate identity.	$\sin(x + y) = \sin x\cos y + \cos x\sin y$
2 Replace y with $\frac{\pi}{6}$ and simplify.	$\sin\left(x + \frac{\pi}{6}\right) = \sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6}$ $= \sin x \cdot \frac{\sqrt{3}}{2} + \cos x \cdot \frac{1}{2}$ $\therefore \sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}(\sqrt{3}\sin x + \cos x)$

WORKED Example 15

Using appropriate identities, show that $\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$.

THINK	**WRITE**
1 Consider $\cos 2A$ as $\cos(A + A)$ and write the appropriate identity for the cosine of a sum.	$\cos(x + y) = \cos x \cos y - \sin x \sin y$
2 Replace both x and y with A in the general equation and simplify.	$\cos(A + A) = \cos A \cos A - \sin A \sin A$ $\therefore \cos(2A) = \cos^2 A - \sin^2 A$
3 To show that $\cos(2A) = 2\cos^2 A - 1$, we must eliminate the term in $\sin A$. Rearrange the Pythagorean identity to express sine in terms of cosine.	Now $\sin^2 A + \cos^2 A = 1$ so $\sin^2 A = 1 - \cos^2 A$
4 Substitute for $\sin^2 A$ in $\cos 2A = \cos^2 A - \sin^2 A$.	Substituting into $\cos(2A) = \cos^2 A - \sin^2 A$ gives $\cos(2A) = \cos^2 A - (1 - \cos^2 A)$ $\therefore \cos(2A) = 2\cos^2 A - 1$ Hence, $\cos(2A) = \cos^2 A - \sin^2 A$ $= 2\cos^2 A - 1.$

WORKED Example 16

Find the exact value of $\sin \frac{\pi}{12}$.

THINK	**WRITE**
1 To use a sine addition identity, find two angles that have a sum or difference of $\frac{\pi}{12}$ and for which exact sine and cosine values exist. Some experimentation may be required.	$\frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$
2 Write the appropriate identity involving the sine of the difference of these two angles.	$\sin \frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ $= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}$
3 Write the exact values for sine and cosine and simplify.	$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$ $= \frac{\sqrt{3} - 1}{2\sqrt{2}}$
4 A rational denominator can be obtained by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$. This is equivalent to multiplying by 1.	$= \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{1}{4}(\sqrt{6} - \sqrt{2})$

WORKED Example 17

Using an appropriate identity, find the exact value of sin $\frac{\pi}{8}$.

THINK	WRITE
① Try to find two angles whose sum or difference produces $\frac{\pi}{8}$ and where exact sine and cosine values are produced. There are none. Next try creating a multiple angle involving $\frac{\pi}{8}$ for which the value of the cosine or sine is an exact value.	$\frac{\pi}{4} = 2 \cdot \frac{\pi}{8}$
② Write the identity for the cosine of the sum of two angles. This will allow us to involve sin $\frac{\pi}{8}$ and cos $\frac{\pi}{4}$ in the equation. Replace y with x to obtain an equation for cos $2x$.	$\cos(x + y) = \cos x \cos y - \sin x \sin y$ If $y = x$, then $\cos(x + x) = \cos x \cos x - \sin x \sin x$ $\cos 2x = \cos^2 x - \sin^2 x$
③ Substitute for $\cos^2 x$ using the Pythagorean identity and simplify.	$= (1 - \sin^2 x) - \sin^2 x$ $= 1 - 2\sin^2 x$
④ Replace x with $\frac{\pi}{8}$ and simplify.	Let $x = \frac{\pi}{8}$,
⑤ Solve the quadratic for sin $\frac{\pi}{8}$.	so $\cos\left(2 \cdot \frac{\pi}{8}\right) = 1 - 2\sin^2\left(\frac{\pi}{8}\right)$ $\cos\frac{\pi}{4} = 1 - 2\sin^2\left(\frac{\pi}{8}\right)$ $\frac{1}{\sqrt{2}} = 1 - 2\sin^2\left(\frac{\pi}{8}\right)$ $\therefore 2\sin^2\left(\frac{\pi}{8}\right) = 1 - \frac{1}{\sqrt{2}}$ $\sin^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2} - 1}{2\sqrt{2}}$ $\sin^2\left(\frac{\pi}{8}\right) = \frac{2 - \sqrt{2}}{4}$ $\sin\frac{\pi}{8} = \pm\frac{\sqrt{2 - \sqrt{2}}}{2}$
⑥ Since $\frac{\pi}{8}$ is an acute measure, sin $\frac{\pi}{8}$ must be positive. Hence we reject the negative value.	As $\frac{\pi}{8}$ is an acute angle, sine is positive. $\therefore \sin\frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$.

remember

1. The addition identities for sine and cosine are:

$$\begin{aligned}\cos(x+y) &= \cos x \cos y - \sin x \sin y\\ \sin(x+y) &= \sin x \cos y + \cos x \sin y\\ \cos(x-y) &= \cos x \cos y + \sin x \sin y\\ \sin(x-y) &= \sin x \cos y - \cos x \sin y.\end{aligned}$$

2. These identities can be used both from left to right and right to left.
3. The results are true for both functions of angles and functions of real numbers.
4. The form of each expansion indicates the type. For instance, sine expansions involve forms of the type sin · cos and cosine expansions involve forms of the type cos · cos and sin · sin.

EXERCISE 7E Addition identities for sin $(x \pm y)$ and cos $(x \pm y)$

WORKED Example 14

1 Make use of an appropriate addition identity to expand and simplify the following.

a $\sin\left(\theta+\frac{\pi}{4}\right)$ **b** $\cos\left(x-\frac{\pi}{6}\right)$ **c** $2\sin\left(2x-\frac{\pi}{3}\right)$ **d** $\cos\left(\frac{\pi+x}{2}\right)$

2 Simplify each of the following by using an addition identity.

a $\sin\alpha\cos 2\beta - \sin 2\beta\cos\alpha$ **b** $\cos 25° \cos 30° + \sin 25° \sin 30°$

c $\cos\left(\frac{\pi}{4}+\theta\right)\cos\theta - \sin\left(\frac{\pi}{4}+\theta\right)\sin\theta$ **d** $\dfrac{\sin\left(\theta+\frac{\pi}{6}\right)}{\cos\left(\theta-\frac{\pi}{6}\right)}$

WORKED Example 15

3 Show that:

a $\cos 2x = 1 - 2\sin^2 x$

b $(\sin A + \cos A)(\sin B + \cos B) = \sin(A+B) + \cos(A-B)$

c $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A \sin B}$

d $2\sin\left(x+\frac{\pi}{4}\right)\sin\left(x-\frac{\pi}{4}\right) = \sin^2 x - \cos^2 x.$

WORKED Example 16

4 Make use of an appropriate addition identity to find the exact value of:

a $\sin\frac{7\pi}{12}$ **b** $\cos\frac{5\pi}{12}.$

WORKED Example 17

5 Make use of an appropriate identity to find the exact value of:

a $\cos\frac{\pi}{8}$ **b** $\sin\frac{3\pi}{8}.$

6 Prove each of the following identities.

a $\cos\left(\frac{\pi}{3}-x\right) + \sin\left(\frac{\pi}{6}-x\right) = \cos x$ **b** $\sin 3x = 3\sin x - 4\sin^3 x$

7 If $A + B + C = 180°$, then show that:

a $\sin A = \sin B \cos C + \cos B \sin C$ **b** $\cos B = \sin C \sin A - \cos C \cos A$.

8 **a** Given that $a \sin\left(x + \frac{\pi}{4}\right) = b \cos\left(x + \frac{\pi}{4}\right)$, show that $\tan x = \frac{b - a}{b + a}$.

b In part **a** above, $\tan x = \frac{1}{n}$ where $n > 0$.

- **i** Express the relationship between a and b in terms of n.
- **ii** Discuss the relationship between a and b as $n \to \infty$.

Addition identities for tan $(x \pm y)$

Given that $\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)}$ then we may write:

$$\tan(x + y) = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Now divide both the numerator and denominator by the product $\cos x \cos y$

$$\tan(x + y) = \frac{\dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}}$$

$$= \frac{\dfrac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} + \dfrac{\cos x \cancel{\sin y}}{\cancel{\cos x} \cos y}}{\dfrac{\cancel{\cos x \cos y}}{\cancel{\cos x \cos y}} - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$\tan(x + y) = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}}$$

$$\therefore \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

To find $\tan(x - y)$ we will replace y by $-y$ in the expression for $\tan(x + y)$

$$\tan(x + (-y)) = \frac{\tan x + \tan(-y)}{1 - \tan x \tan(-y)}$$

$$\therefore \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Note that $\tan(-\theta) = -\tan \theta$.

$$\boldsymbol{\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}}$$

$$\boldsymbol{\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}}$$

Note that the sign (±) between the terms in the numerator is the same as the sign between the variables and is opposite to the sign in the denominator.

WORKED Example 18

Expand tan 2θ.

THINK

1. Write the identity for tan of the sum of two angles.
2. Replace x and y with θ and simplify.

WRITE

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

$$\tan(\theta+\theta)=\frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$$

$$\therefore\ \tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$$

WORKED Example 19

Expand and simplify tan (A – 45°).

THINK

1. Write the identity for tan of the difference of two angles.
2. Substitute $x = A$ and $y = 45°$ into the identity and simplify.

WRITE

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

$$\tan(A-45°)=\frac{\tan A-\tan 45°}{1+\tan A\tan 45°}$$

$$\therefore\ \tan(A-45°)=\frac{\tan A-1}{1+\tan A}$$

WORKED Example 20

Find the exact value of tan (75°).

THINK

1. To use a tangent addition identity, find two angles that have a sum or difference of 75° and for which exact tangent values exist.
2. Write the appropriate identity for tan of the sum of two angles.

WRITE

$30° + 45° = 75°$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

Continued over page

THINK	WRITE
③ Substitute $x = 30°$ and $y = 45°$ into the identity.	$\begin{aligned} \tan(30° + 45°) &= \frac{\tan 30° + \tan 45°}{1 - \tan 30° \tan 45°} \\ \therefore \tan(75°) &= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}} \\ &= \frac{1 + \sqrt{3}}{\sqrt{3} - 1} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \cdot \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \\ &= \frac{(\sqrt{3} + 1)^2}{2} \\ &= \frac{3 + 2\sqrt{3} + 1}{2} \\ &= \frac{4 + 2\sqrt{3}}{2} \\ &= 2 + \sqrt{3} \end{aligned}$
④ Write the exact values for tan and simplify.	$\therefore \tan(75°) = 2 + \sqrt{3}$

WORKED Example 21

Show that $\frac{1}{1 - \tan x} - \frac{1}{1 + \tan x} = \tan 2x$.

THINK	WRITE
① Consider which side of the expression looks easier to deal with. Tackle the left-hand side and reduce it to a single fraction.	$\frac{1}{1 - \tan x} - \frac{1}{1 + \tan x} = \frac{(1 + \tan x) - (1 - \tan x)}{(1 - \tan x)(1 + \tan x)}$
② Simplify the new expression to see if it reduces to a recognisable tan expression.	$= \frac{\tan x + \tan x}{1 - \tan x \tan x}$
③ Recognise that the result is part of the addition identity for tan where $y = x$.	$= \tan(x + x)$
④ Simplify to obtain the required right-hand side.	$= \tan 2x$ So $\frac{1}{1 - \tan x} - \frac{1}{1 + \tan x} = \tan 2x$

remember

1. The addition identities for tangent are:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

2. The identities should be learnt both forwards and backwards to help with problem solving.
3. The sign (±) between the terms in the numerator is the same as the sign between the variables and is opposite to the sign in the denominator.

EXERCISE 7F

Addition identities for tan ($x \pm y$)

WORKED Example 18, 19

1 Expand and simplify where appropriate:

a $\tan(x + a)$ **b** $\tan(45° - x)$.

2 Simplify:

a $\dfrac{\tan 27° + \tan 18°}{1 - \tan 27° \tan 18°}$ **b** $\dfrac{\tan(A+3B) + \tan(B-A)}{1 - \tan(A+3B)\tan(B-A)}$.

3 **a** If $x = 45°$ and $y = 60°$ find the exact value of $\tan(x + y)$.

b If α and β are acute angles, and $\tan\alpha = \frac{1}{4}$, $\tan\beta = \frac{3}{5}$, find the value of $\tan(\alpha - \beta)$ by using an addition identity.

WORKED Example 20

4 **a** Prove that the exact value of $\tan\dfrac{5\pi}{12}$ is $2 + \sqrt{3}$.

b Find the exact value of $\tan\dfrac{\pi}{8}$.

WORKED Example 21

5 Show that:

a $\dfrac{1 - \tan y}{1 + \tan y} = \dfrac{\cos y - \sin y}{\cos y + \sin y}$ **b** $\dfrac{\sin(x+y)}{\cos(x-y)} = \dfrac{\tan x + \tan y}{1 + \tan x \tan y}$.

6 Show that if $\dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\sin(C-D)}{\sin(C+D)}$, then $\cot A \cot B \cot C = \cot D$.

7 Prove that $\cot(\theta + \alpha) = \dfrac{\cot\theta \cot\alpha - 1}{\cot\theta + \cot\alpha}$.

8 If $\alpha + \beta = \dfrac{\pi}{4}$, prove that $\cot\alpha = 1 + \cot\alpha \tan\beta + \tan\beta$.

9 **a** If $A + B + C = \pi$, show that $\dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$.

b From the above result, or otherwise, deduce that:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

10 For any triangle ABC, where a, b, and c respectively denote the sides opposite the angles A, B, C, prove that:

$$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

The Werner or factorisation identities

From our work with the addition identities we have:

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \end{aligned}$$

If we select the two sine identities and add them, we obtain:

$$\begin{array}{rcl} \sin(x+y) &=& \sin x \cos y + \cos x \sin y \quad + \\ \sin(x-y) &=& \sin x \cos y - \cos x \sin y \\ \hline \sin(x+y) + \sin(x-y) &=& 2 \sin x \cos y \\ \hline \end{array}$$

Now, if we subtract the same two sine identities, we have:

$$\begin{array}{rcl} \sin(x+y) &=& \sin x \cos y + \cos x \sin y \quad - \\ \sin(x-y) &=& \sin x \cos y - \cos x \sin y \\ \hline \sin(x+y) - \sin(x-y) &=& 2 \cos x \sin y \\ \hline \end{array}$$

Repeating this procedure for the two cosine identities we have, in the case of addition:

$$\begin{array}{rcl} \cos(x+y) &=& \cos x \cos y - \sin x \sin y \quad + \\ \cos(x-y) &=& \cos x \cos y + \sin x \sin y \\ \hline \cos(x+y) + \cos(x-y) &=& 2 \cos x \cos y \\ \hline \end{array}$$

And, in the case of subtraction:

$$\begin{array}{rcl} \cos(x+y) &=& \cos x \cos y - \sin x \sin y \quad - \\ \cos(x-y) &=& \cos x \cos y + \sin x \sin y \\ \hline \cos(x+y) - \cos(x-y) &=& -2 \sin x \sin y \\ \hline \end{array}$$

In summary, we have the following four results:

$$\begin{aligned} \sin(x+y) + \sin(x-y) &= 2 \sin x \cos y \\ \sin(x+y) - \sin(x-y) &= 2 \cos x \sin y \\ \cos(x+y) + \cos(x-y) &= 2 \cos x \cos y \\ \cos(x+y) - \cos(x-y) &= -2 \sin x \sin y \end{aligned}$$

These results enable us to express the sum or difference of two sines and the sum or difference of two cosines, as a product. By writing these results in reflexive form, we have the following factorisation identities.

$$\begin{aligned} 2 \sin x \cos y &= \sin(x+y) + \sin(x-y) \\ 2 \cos x \sin y &= \sin(x+y) - \sin(x-y) \\ 2 \cos x \cos y &= \cos(x+y) + \cos(x-y) \\ -2 \sin x \sin y &= \cos(x+y) - \cos(x-y) \end{aligned}$$

WORKED Example 22

Express 2 sin 5x cos 2x as a sum or difference.

THINK	WRITE
1 Note the product sin · cos. This indicates a sum of two sines. Write the appropriate identity.	$2 \sin x \cos y = \sin (x + y) + \sin (x - y)$
2 Replace x with 5x and y with 2x to write the equation for 2 sin 5x cos 2x.	$2 \sin 5x \cos 2x = \sin (5x + 2x) + \sin (5x - 2x)$
3 Simplify.	$= \sin 7x + \sin 3x$

WORKED Example 23

Express cos 3x cos x as a sum or difference.

THINK	WRITE
1 Note the product cos · cos. This indicates a sum of two cosines. Write the appropriate identity.	$2 \cos x \cos y = \cos (x + y) + \cos (x - y)$
2 Replace x with 3x and y with x to write an equation for 2 cos 3x cos x.	$2 \cos 3x \cos x = \cos (3x + x) + \cos (3x - x)$
3 Simplify.	$= \cos 4x + \cos 2x$
4 Multiply both sides of the equation by $\frac{1}{2}$ to obtain an expression for cos 3x cos x.	$\cos 3x \cos x = \frac{1}{2}(\cos 4x + \cos 2x)$

WORKED Example 24

Express 5 cos 4x sin 2x as a sum or difference.

THINK	WRITE
1 Note the product cos · sin. This indicates a difference of two sines. Write the appropriate identity.	$2 \cos x \sin y = \sin (x + y) - \sin (x - y)$
2 Replace x with 4x and y with 2x to write an equation for 2 cos 4x sin 2x.	$2 \cos 4x \sin 2x = \sin (4x + 2x) - \sin (4x - 2x)$
3 Simplify.	$= \sin 6x - \sin 2x$
4 Multiply both sides of the equation by $\frac{5}{2}$ to obtain an expression for 5 cos 4x sin 2x.	$5 \cos 4x \sin 2x = \frac{5}{2}(\sin 6x - \sin 2x)$

WORKED Example 25

Express sin 3x sin x as a sum or difference.

THINK	WRITE
1 Note the product sin · sin. This indicates a difference of two cosines. Write the appropriate identity.	$-2 \sin x \sin y = \cos(x + y) - \cos(x - y)$
2 Replace x with 3x and y with x to write an equation for −2 sin 3x sin x.	$-2 \sin 3x \sin x = \cos(3x + x) - \cos(3x - x)$
3 Simplify.	$= \cos 4x - \cos 2x$
4 Multiply both sides of the equation by $-\frac{1}{2}$ to obtain an expression for sin 3x sin x.	$\sin 3x \sin x = -\frac{1}{2}(\cos 4x - \cos 2x)$

remember

1. The Werner or factorisation identities are:

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2 \cos x \sin y = \sin(x + y) - \sin(x - y)$$
$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$
$$-2 \sin x \sin y = \cos(x + y) - \cos(x - y).$$

2. The identities should be learnt both forwards and backwards to help with problem solving.

EXERCISE 7G

The Werner or factorisation identities

Use the factorisation identities to simplify each of the following.

WORKED Example 22

1 2 sin 3x cos x

2 2 cos 4x sin x

WORKED Example 23

3 2 cos 2x cos x

4 −2 sin 3x sin 2x

WORKED Example 24

5 4 cos 4x sin 3x

6 cos (x + 30°) cos 30°

WORKED Example 25

7 −sin 5x sin 3x

8 $2 \sin (a + b) \cos (a - b)$

9 $a \cos 3b \cos 3b$

10 $5 \cos (2x - y) \sin (x + 2y)$

11 $\sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$

12 $\sin (x - y) \sin (x + y)$

The Simpson or half-sum/half-difference identities

The Simpson identities are a variation of the factorisation identities. To obtain the Simpson identities, we need to consider the form of the factorisation identities labelled [1] to [4] below and replace the a and b terms in them.

$$\sin (a + b) + \sin (a - b) = 2 \sin a \cos b \qquad [1]$$
$$\sin (a + b) - \sin (a - b) = 2 \cos a \sin b \qquad [2]$$
$$\cos (a + b) + \cos (a - b) = 2 \cos a \cos b \qquad [3]$$
$$\cos (a + b) - \cos (a - b) = -2 \sin a \sin b \qquad [4]$$

This is achieved by writing $x = a + b$ and $y = a - b$. Solving these equations simultaneously gives $a = \frac{1}{2}(x + y)$ and $b = \frac{1}{2}(x - y)$. (Try this for yourself.) If we now replace the variables in each of the above identities labelled [1] to [4] they become the Simpson formulas. These formulas are commonly known as the *half-sum* and *half-difference* formulas.

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$

$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$

$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$

WORKED Example 26

Express $\sin 3x + \sin x$ as a product.

THINK	WRITE
1 Consider the form of the expression. The sum of two sines indicates the product type sin · cos. Write the appropriate identity.	$\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)$
2 Replace x with $3x$ and y with x to write an equation for $\sin 3x + \sin x$.	$\sin 3x + \sin x = 2 \sin \frac{1}{2}(3x + x) \cos \frac{1}{2}(3x - x)$
3 Simplify by obtaining the half-sum and half-difference of the variables working from left to right in the order given.	$= 2 \sin 2x \cos x$

WORKED Example 27

Express cos $(2x + 30°)$ − cos 30° as a product.

THINK	WRITE
1 Consider the form of the expression. The difference of two cosines indicates the product type sin · sin. Write the appropriate identity.	$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$
2 Replace x with $2x + 30°$ and y with 30° to write an equation for cos $(2x + 30°)$ − cos 30°.	$\cos(2x + 30°) - \cos 30°$ $= -2 \sin \frac{1}{2}(2x + 30° + 30°) \sin \frac{1}{2}(2x + 30° - 30°)$
3 Simplify by obtaining the half-sum and half-difference of the variables working from left to right in the order given.	$= -2 \sin \frac{1}{2}(2x + 60°) \sin \frac{1}{2}(2x)$ $= -2 \sin(x + 30°) \sin x$

WORKED Example 28

Simplify $\dfrac{\sin 4x - \sin 2x}{\cos 4x - \cos 2x}$.

THINK	WRITE
1 Consider the numerator and denominator separately. The numerator will be of the form cos · sin because it involves the difference of two sines. Write the appropriate identity.	$\sin x - \sin y = 2 \cos \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$
2 Replace x with $4x$ and y with $2x$ to write an equation for sin $4x$ − sin $2x$ and simplify.	$\sin 4x - \sin 2x = 2 \cos \frac{1}{2}(4x + 2x) \sin \frac{1}{2}(4x - 2x)$ $= 2 \cos 3x \sin x$
3 The denominator will be of the type sin · sin because it involves the difference of two cosines. Write the appropriate identity.	$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)$
4 Replace x with $4x$ and y with $2x$ to write an equation for cos $4x$ − cos $2x$ and simplify.	$\cos 4x - \cos 2x = -2 \sin \frac{1}{2}(4x + 2x) \sin \frac{1}{2}(4x - 2x)$ $= -2 \sin 3x \sin x$
5 Write as a fraction and cancel the common factors.	$\dfrac{\sin 4x - \sin 2x}{\cos 4x - \cos 2x} = \dfrac{\cancel{2} \cos 3x \cancel{\sin x}}{-\cancel{2} \sin 3x \cancel{\sin x}}$
6 Simplify the resulting expression.	$= -\dfrac{\cos 3x}{\sin 3x}$ $= -\cot 3x$

WORKED Example 29

Solve $\cos \theta + \cos 2\theta + \cos 3\theta = 0$.

THINK	WRITE
1 Write the equation.	$\cos \theta + \cos 2\theta + \cos 3\theta = 0$
2 Select a pair of terms which, through the use of one of the identities, will reduce the given expression to a product containing the remaining cosine term. Try $\cos \theta + \cos 3\theta$.	$(\cos \theta + \cos 3\theta) + \cos 2\theta = 0$
3 Use the appropriate identity to write $\cos \theta + \cos 3\theta$ as a product.	$2 \cos \tfrac{1}{2}(\theta + 3\theta) \cos \tfrac{1}{2}(\theta - 3\theta) + \cos 2\theta = 0$
4 Simplify. Remember that $\cos(-\theta) = \cos \theta$.	$2 \cos 2\theta \cos(-\theta) + \cos 2\theta = 0$ $2 \cos 2\theta \cos \theta + \cos 2\theta = 0$
5 Factorise the left side of the equation and solve for θ.	$\cos 2\theta\,(2 \cos \theta + 1) = 0$ $\cos 2\theta = 0$ or $2 \cos \theta + 1 = 0$ $2\theta = 2n\pi \pm \dfrac{\pi}{2}$ or $\cos \theta = -\tfrac{1}{2}$ $\theta = n\pi \pm \dfrac{\pi}{4}$ or $\theta = 2n\pi \pm \dfrac{2\pi}{3}$ So $\theta = n\pi \pm \dfrac{\pi}{4}, 2n\pi \pm \dfrac{2\pi}{3}$ for integral values of n.

remember

1. The Simpson formulas are:

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$

$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$

$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y).$$

2. The identities should be learnt both forwards and backwards to help with problem solving.

EXERCISE 7H The Simpson or half-sum/half-difference identities

WORKED Example 26, 27

1 Write each sum or difference as a product.

a $\sin 4x + \sin 2x$ **b** $\sin 4x - \sin 2x$ **c** $\cos 3y + \cos y$

d $\cos 3\theta - \cos \theta$ **e** $\sin 5\alpha + \sin 3\alpha$ **f** $\sin 4\beta - \sin \beta$

g $\cos 3x + \cos 4x$ **h** $\cos 3x - \cos 4x$ **i** $\frac{1}{2}(\sin 3x + \sin 4x)$

j $\frac{1}{2}(\sin 2x - \sin 6x)$ **k** $-\frac{1}{2}[\cos (2\theta) - \cos (2\alpha)]$

l $4[\cos (x + y) + \cos (x - y)]$ **m** $\sin \alpha + \sin 2\alpha + \sin 3\alpha$

n $\cos y + \cos 3y + \cos 5y$

WORKED Example 28

2 Simplify:

a $\dfrac{\sin \theta - \sin \alpha}{\cos \theta - \cos \alpha}$ **b** $\dfrac{\cos x + \cos \alpha}{\sin x + \sin \alpha}$ **c** $\dfrac{\sin 2\theta - \sin 2\beta}{\cos 2\theta + \cos 2\beta}$

d $\dfrac{\sin 2x}{\cos x - \cos 5x}$ **e** $\dfrac{\cos 2x + \cos 4x}{\cos 3x \sin 2x}$

f $\dfrac{\sin y + \sin 2y + \sin 3y}{\cos y + \cos 2y + \cos 3y}$.

eBook plus

Digital docs:
SkillSHEET 7.2
WorkSHEET 7.2

3 Show that:

a $\sin (nx) = 2 \cos x \sin [(n - 1)x] - \sin [(n - 2)x]$

b $\cos (nx) = 2 \cos x \cos [(n - 1)x] - \cos [(n - 2)x]$.

4 Solve for x if $0 \le x \le 2\pi$.

a $\sin 2x + \sin x = 0$ **b** $\cos 2x + \cos x = 0$

WORKED Example 29

5 Find the general solution for x.

a $\sin x + \sin 2x + \sin 3x = 0$

b $\cos x + \cos 3x + \cos 5x = 0$

Modelling and problem solving

Graphing $y = a \sin x + b \cos x$

The function $y = a \sin x + b \cos x$ may be conveniently expressed in one of two forms:

$$y = A \sin (x + \alpha) \qquad \text{or} \qquad y = A \cos (x + \alpha)$$

given appropriate restrictions on α and by taking $A > 0$. To do this we consider the addition identities as follows:

$$\begin{aligned} A \sin (x + \alpha) &= A \sin x \cos \alpha + A \cos x \sin \alpha \\ &= A \cos \alpha \sin x + A \sin \alpha \cos x \end{aligned}$$

Comparing $y = a \sin x + b \cos x$ with $y = A \cos \alpha \sin x + A \sin \alpha \cos x$ gives:

$$A \cos \alpha = a \qquad \text{and} \qquad A \sin \alpha = b$$

So
$$A^2 \cos^2 \alpha + A^2 \sin^2 \alpha = a^2 + b^2$$
$$A^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2 + b^2$$

By applying the Pythagorean identity:

$$A^2 = a^2 + b^2$$

$$A = \sqrt{a^2 + b^2}, A > 0$$

Also:
$$\frac{A \sin \alpha}{A \cos \alpha} = \frac{b}{a}$$

$$\Rightarrow \tan \alpha = \frac{b}{a}$$

Therefore:
$$\alpha = \tan^{-1} \left(\frac{b}{a}\right)$$

WORKED Example 30

Express 2 cos x − 3 sin x in the form A cos (x − α) where $A > 0$ and $0° < \alpha < 360°$.

THINK	WRITE
① Write the appropriate addition identity.	$A \cos(x - \alpha) = A \cos x \cos \alpha + A \sin x \sin \alpha$
② Compare the given expression with the identity and assign appropriate values for the variables. Apply the Pythagorean identity to find A.	$A \cos \alpha = 2$ and $A \sin \alpha = -3$ $A^2 \cos^2 \alpha + A^2 \sin^2 \alpha = (-3)^2 + 2^2$ $\therefore A^2 = 13$ $\Rightarrow A = \sqrt{13}, A > 0$
③ Find α by obtaining a tan relationship. Decide in which quadrant α is located. Remember that $0° < \alpha < 360°$.	$\frac{A \sin \alpha}{A \cos \alpha} = \frac{-3}{2}$ $\Rightarrow \tan \alpha = \frac{-3}{2}$ $\therefore \alpha = \tan^{-1}\left(\frac{-3}{2}\right)$ As $\cos \alpha = \frac{2}{\sqrt{13}}$ and $\sin \alpha = \frac{-3}{\sqrt{13}}$ then α must lie in the fourth quadrant. Hence, $\alpha = \tan\left(\frac{-3}{2}\right)$ $\Rightarrow \alpha \approx 360° - 56°19'$ $\therefore \alpha \approx 303°41'$.
④ Write the answer.	So $2 \cos x - 3 \sin x = \sqrt{13} \cos(x - 303°41')$.

WORKED Example 31

a **Express $\sqrt{3}$ cos x + sin x in the form A sin (x + α) where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.**

b **Determine the extreme values of the given function.**

c **Using a graphics calculator, find those values of x for which $\sqrt{3}$ cos x + sin x = 1.5 and $0 < x < 2\pi$.**

THINK	WRITE/DISPLAY
a ① Write the appropriate addition identity.	**a** $A \sin(x + \alpha)$ $= A \sin x \cos \alpha + A \cos x \sin \alpha$
② Compare the given expression with the identity and assign appropriate values for the variables.	$A \sin \alpha = \sqrt{3}$ and $A \cos \alpha = 1$

Continued over page →

THINK | **WRITE/DISPLAY**

③ Determine the unknown quantities A and α.

$$A^2\cos^2\alpha + A^2\sin^2\alpha = (\sqrt{3})^2 + 1^2$$
$$\therefore\ A^2 = 4$$
$$\Rightarrow A = 2,\ A > 0.$$

$$\frac{A\sin\alpha}{A\cos\alpha} = \sqrt{3}$$

$$\Rightarrow \tan\alpha = \sqrt{3}$$

$$\therefore\ \alpha = \tan^{-1}(\sqrt{3})$$

Since α lies in the first quadrant, $\alpha = \frac{\pi}{3}$.

④ Write the answer.

$$\therefore\ \sqrt{3}\cos x + \sin x = 2\sin\left(x + \frac{\pi}{3}\right).$$

b The extreme values are the greatest and least values of the function.

b The extreme values follow from the fact that $2\sin\left(x + \frac{\pi}{3}\right)$ oscillates between ±2, since $\left|\sin\left(x + \frac{\pi}{3}\right)\right| \le 1$. The greatest value is 2 and the least value is –2.

c For the Casio *fx*-9860G AU

c

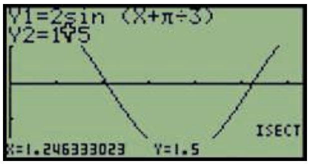
① To solve $\sqrt{3}\cos x + \sin x = 1.5$ for $0 < x < 2\pi$, press:

- MENU
- 5 (GRAPH)

Complete the entry lines as:

$$Y1 = 2\sin\left(x + \frac{\pi}{3}\right)$$
$$Y2 = 1.5$$

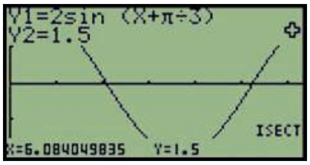
Press EXE after each line. Restrict View Window appropriately.



② To find the point of intersection, press:

- SHIFT
- F5 (G-Solv)
- F5 ISCT

One point of intersection will be displayed. To find the coordinates of the other point, press the right arrow.



③ Write the answer.

Solving $\sqrt{3}\cos x + \sin x = 1.5$ for $0 < x < 2\pi$ gives $x = 1.246$ and $x = 6.084$.

THINK **WRITE/DISPLAY**

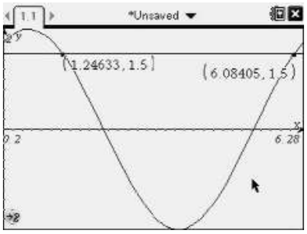
For the TI-Nspire CAS

1. To solve $\sqrt{3}\cos x + \sin x = 1.5$ for $0 < x < 2\pi$, on a Graphs page, complete the entry lines as:

   $f1(x) = 2\sin\left(x + \dfrac{\pi}{3}\right)$

   $f2(x) = 1.5$

   Press ENTER enter after each line. Restrict Window Settings appropriately.



2. To find the points of intersection, press:
   - MENU menu
   - 7: Points and Lines 7
   - 3: Intersection Point(s) 3

   Move the pointer ↖.
   Click ◉ once on each graph to display the coordinates of the points of intersection.

3. Write the answer.

Solving $\sqrt{3}\cos x + \sin x = 1.5$ for $0 < x < 2\pi$ gives $x = 1.25$ and $x = 6.08$.

Linking with calculus

These approaches can also be employed with calculus.

WORKED Example 32

Evaluate $\int \cos 2x \sin x \, dx$.

THINK **WRITE**

1. Use an appropriate identity to write the product as a sum or difference; that is, $2\cos x \sin y = \sin(x + y) - \sin(x - y)$.

2. Antidifferentiate each term.

$$\int \cos 2x \sin x \, dx$$

$$= \int \tfrac{1}{2}[\sin(2x + x) - \sin(2x - x)\, dx$$

$$= \int (\tfrac{1}{2}\sin 3x - \tfrac{1}{2}\sin x)\, dx$$

$$= -\tfrac{1}{6}\cos 3x + \tfrac{1}{2}\cos x + c$$

WORKED Example 33

a **Calculate the physical area between the curve $y = 5 \sin 4x \sin 2x$ and the x-axis from $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.**

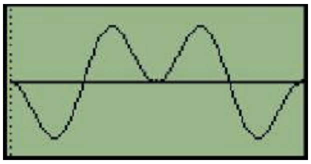
b **Verify your answer by using appropriate graphics calculator functions.**

THINK

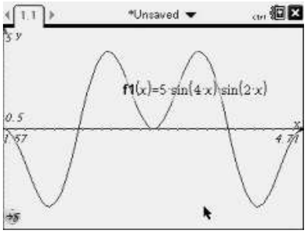
WRITE/DISPLAY

a ① Graph the given function using a graphics calculator, selecting appropriate (WINDOW) settings (for x, between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; for y, betweeen ±5).

a



② Sketch the graph showing an appropriate scale on the axes. Shade the required area.



The physical area is represented by four equal leaves (of equal area).

③ State the definite integral needed to find the area of one leaf and multiply the result by 4. Use the sketch to identify the upper and lower limits of the integral.

$$\text{Area} = 4\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 5 \sin 4x \sin 2x \, dx\right|$$

④ Express the integrand as a sum or difference using an appropriate addition identity.

$$\text{Area} = 4\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} -\frac{5}{2}(\cos 6x - \cos 2x) \, dx\right|$$

$$= 4\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \left(-\frac{5}{2} \cos 6x + \frac{5}{2} \cos 2x\right) dx\right|$$

⑤ Calculate the area and state your answer.

$$= 4\left|\left[-\frac{5}{12} \sin 6x + \frac{5}{4} \sin 2x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\right|$$

$$= 4\left|\left(-\frac{5}{12} - \frac{5}{4}\right) - (0 + 0)\right|$$

$$= 6\tfrac{2}{3}$$

The physical area is $6\frac{2}{3}$ square units.

THINK

WRITE/DISPLAY

b For the Casio *fx*-9860G AU

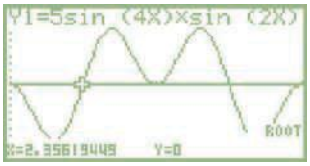
b

① To find the area between the curve $y = 5\sin(4x)\sin(2x)$ and the x-axis from $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, the x-intercept after $x = \frac{\pi}{2}$ needs to be identified.

To do this press:

- SHIFT
- F5 (G-Solv)
- F1 (ROOT)

Use the right arrow to move the cursor to the required x-intercept, $x = 2.356$.



```
Y1=5sin (4X)×sin (2X)
X=2.35619449      Y=0
```

② To determine the area, press:

- MENU
- 1 (RUN)

Complete the entry line as:

$4\cdot$ Abs($\int$(5 sin(4x) $\cdot$ sin(2x), x, π | 2, 2.356)

```
4×Abs (∫(5sin (4X)×si
n (2X),X,π÷2,2.356)
            6.666666668
```

③ Write the answer.

$$4\cdot\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 5\sin(4x)\cdot\sin(2x)dx\right| = 6.667$$

For the TI-Nspire CAS

① To find the area on a Calculator page, the first two x-intercepts are required. To find these two intercepts, complete the entry line as:

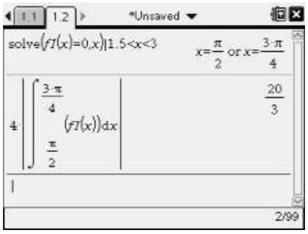
solve($f1(x) = 0$, x)

Then press ENTER.

To find the area of the four identical regions, complete the entry line as:

$$4\cdot\ \text{abs}\left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} f1(x)dx\right)$$

Then press ENTER.



$\text{solve}(f1(x)=0,x)|1.5<x<3 \qquad x=\frac{\pi}{2} \text{ or } x=\frac{3\cdot\pi}{4}$

$4\left|\int_{\frac{\pi}{2}}^{\frac{3\cdot\pi}{4}}(f1(x))dx\right| \qquad \frac{20}{3}$

② Write the answer.

$$4\cdot\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 5\sin(4x)\cdot\sin(2x)dx\right| = \frac{20}{3}$$

remember

1. The function $y = a \sin x + b \cos x$ may be conveniently expressed in one of two forms: $y = A \sin (x + \alpha)$ or $y = A \cos (x + \alpha)$ given appropriate restrictions on α and by taking $A > 0$. We do this by writing:

$$\begin{aligned} A \sin (x + \alpha) &= A \sin x \cos \alpha + A \cos x \sin \alpha \\ &= A \cos \alpha \sin x + A \sin \alpha \cos x \end{aligned}$$

   Comparing $y = a \sin x + b \cos x$ with $y = A \cos \alpha \sin x + A \sin \alpha \cos x$ yields $A \cos \alpha = a$ and $A \sin \alpha = b$. By applying the Pythagorean identity we have $A = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

2. When evaluating a physical area, work from a diagram to select the limits of integration. Be careful when assuming that the given function generates symmetric areas. This is not always the case.

EXERCISE 7I

Modelling and problem solving

WORKED Example 30

1 Express each sum or difference in the form requested.

a $\cos x - \sin x$ in the form $A \cos (x - \alpha)$ where $A > 0$ and $0° < \alpha < 360°$.

b $\sin x + \cos x$ in the form $A \sin (x + \alpha)$ where $A > 0$ and $0° < \alpha < 90°$.

c $4 \sin x - 3 \cos x$ in the form $A \sin (x - \alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

d $\cos x - 2 \sin x$ in the form $A \cos (x + \alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

e $\cos x - \sqrt{3} \sin x$ in the form $A \sin (x - \alpha)$ where $A > 0$ and $180° < \alpha < 270°$.

f $\sin x + \sqrt{2} \cos x$ in the form $A \cos (x - \alpha)$ where $A > 0$ and $0 < \alpha < 360°$.

WORKED Example 31

2 a Express $\sqrt{3} \cos x - 2 \sin x$ in the forms:

i $A \sin (x - \alpha)$ **ii** $B \sin (x + \beta)$ **iii** $C \cos (x - \theta)$ **iv** $D \cos (x + \phi)$

where A, B, C, D are positive; and $0 < \alpha, \beta, \theta, \phi < 2\pi$.

b Find the maximum and minimum values of $\sqrt{3} \cos x - 2 \sin x$.

c Use a graphics calculator to find the first values of x in the domain $0 \le x \le 2\pi$ where the maximums and minimums occur.

3 If $8 \cos x - 15 \sin x = r \cos (x + \alpha)$:

a find the values of r and α where $r > 0$ and $0° < \alpha < 360°$

b use a graphics calculator to determine the least value of x for which $8 \cos x - 15 \sin x = 9$ over $0° < x < 360°$.

WORKED Example 32

4 Evaluate each of the following integrals.

a $\int 2 \sin 2x \cos x \, dx$

b $\int 2 \cos 3x \sin x \, dx$

c $3\int \cos 4x \cos x \, dx$

d $-\int \sin 3x \sin x \, dx$

e $4\int \cos x \sin x \, dx$

f $-\frac{1}{2}\int \sin x \sin (x + \pi) \, dx$

g $-\int \sin 3x \cos 2x \, dx$

h $\int \cos (\frac{3}{2}\pi x) \cos (\frac{1}{2}\pi(x - 1)) \, dx$

5 **a** Evaluate each of the following integrals.

i $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 3 \cos 3x \sin x \, dx$ **ii** $\int_{\frac{\pi}{3}}^{\pi} \sin 4x \cos x \, dx$

iii $4\int_{\pi}^{2\pi} \sin 4x \cos 2x \, dx$ **iv** $-\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin 7x \sin x \, dx$

b Use a graphics calculator to verify your answers, writing each correct to three significant figures where appropriate.

6 **a** Show that $\cos(2x) = 2\cos^2 x - 1$.

b Hence, or otherwise, evaluate $\int_{\frac{\pi}{4}}^{\pi} \cos^2 x \, dx$.

7 **a** Show that $\cos(3x) = 4\cos^3 x - 3\cos x$.

b Hence, or otherwise, evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 x \, dx$.

WORKED Example 33

8 **a** Calculate the physical area between the curve $y = 3 \cos 3x \sin x$ and the x-axis from $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

b Verify your answer by using appropriate graphics calculator functions.

9 If $\sin 3x = 3 \sin x - 4 \sin^3 x$, then deduce that $8 \sin\left(\frac{\pi}{9}\right) \sin\left(\frac{2\pi}{9}\right) \sin\left(\frac{4\pi}{9}\right) = \sqrt{3}$.

10 **a** If m and n are positive integers such that $m > n$, evaluate:

i $\int_0^{\pi} \cos(mx)\cos(nx)\, dx$

ii $\int_0^{\pi} \cos(mx)\sin(nx)\, dx$, where m and n are both odd or both even

iii $\int_0^{\pi} \cos(mx)\sin(nx)\, dx$, where either m or n is odd.

b How would the above results change if $m = n$?

11 Using appropriate technology, graph each of the following functions. Discuss the features of both waveforms and for each, find the period, frequency and amplitude.

a $y = \cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2} + \ldots$

b $y = \sin x + \frac{\sin(2x)}{2^3} + \frac{\sin(3x)}{3^3} + \frac{\sin(4x)}{4^3} + \ldots$

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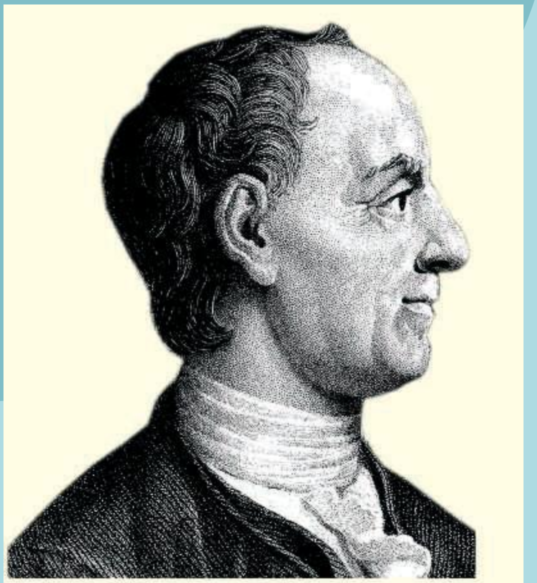
Digital doc: SkillSHEET 7.3

History of mathematics

LEONHARD EULER (1707–1783)



Leonhard Euler was born on 15 April 1707 in Bâle and was personally educated by John Bernoulli, with whom he formed a lifelong friendship.

In 1725, Euler went to Russia at the invitation of the Empress and, in 1733, he exchanged his position as a teacher of mathematics for the chair of mathematics vacated by another member of the famous Bernoulli family.

Euler's health problems began in 1735 when he had a severe fever and almost lost his life. The severity of the Russian climate affected his eyesight and, in 1735, he lost the use of one eye completely. In 1741, he moved to Berlin at the command of Frederick the Great. Here he stayed till 1766, when he returned to Russia and was succeeded at Berlin by Lagrange. Shortly after going back to St Petersburg, Leonhard Euler became blind. In spite of this — and although his house, together with many of his papers, were burnt in 1771 — he recast and improved most of his earlier works. A cataract operation in 1771 briefly restored his sight. Unfortunately for Leonhard Euler, he eventually succumbed to total blindness in 1773.

Because of his remarkable memory, Euler was able to continue with his work on optics, algebra, and lunar motion. Amazingly, after his return to St Petersburg, Euler produced almost half his total works in spite of his blindness. Euler's work in mathematics was, to say the least, profound. He made great progress in the study of modern analytic geometry and trigonometry, where he was the first to consider sin, cos etc. as functions, rather than as chords as Ptolemy had done.

We may sum up Euler's work by saying that he created much new analysis, and revised almost all the branches of pure mathematics that were then known, by filling in details, adding proofs, and arranging the whole in a consistent form. He wrote an immense number of memoirs on all kinds of mathematical subjects, including decisive and formative contributions to geometry, calculus, and number theory.

We owe to Euler the notation $f(x)$ for a function (1734), e for the base of natural logs (1727), i for the square root of -1 (1777), π for pi, Σ for summation (1755), Δy and $\Delta^2 y$ for the notation for finite differences, and many others. Work done by Euler on infinite series included the introduction in 1735 of his famous Euler's constant γ, which he showed to be the limit of $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \quad + \frac{1}{n - \log_e n}$ as n tends to infinity.

Euler gave modern mathematics $e^{ix} = \cos x + i \sin x$ from which he developed the formula $\ln(-1) = i$ in 1727. He published his full theory of logarithms of complex numbers in 1751. He also studied Fourier series and their application to music. In a letter to Goldbach in 1744 he presented the result:

$$\frac{\pi}{2} - \frac{x}{2} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} +$$

Leonhard Euler died of apoplexy in St Petersburg on 7 September 1783.

Questions

1. Where and in what year was Leonhard Euler born?
2. What significant disability afflicted Euler?
3. State a famous constant attributed to Euler.
4. In what year did Leonhard Euler die?
5. List three of Euler's notable achievements in mathematics.

summary

Circular and reciprocal functions

For any real-valued θ:

- The tangent function is defined as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
- The reciprocal functions are defined as:

$$\sec\theta = \frac{1}{\cos\theta} \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

Exact values and the Pythagorean relationships

- The Pythagorean and associated relationships state:

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta.$$

- Useful exact values are:

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \Rightarrow \tan\frac{\pi}{4} = 1$$

$$\left.\begin{aligned} \sin\frac{\pi}{3} &= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ \sin\frac{\pi}{6} &= \cos\frac{\pi}{3} = \frac{1}{2} \end{aligned}\right\} \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \tan\frac{\pi}{3} = \sqrt{3}.$$

Modelling soundwaves

- Of sounds having the same frequency, the one with the greatest amplitude will be perceived as the loudest. The partial tone with the least frequency is called the *fundamental tone* and the others are called *overtones*.
- A series of frequencies that have integer ratios are called *harmonics*. Sounds that differ by an octave have frequencies that are in the ratio of 2:1.

Graphing the reciprocal functions

- $\sec\theta = \dfrac{1}{\cos\theta} \quad \operatorname{cosec}\theta = \dfrac{1}{\sin\theta} \quad \cot\theta = \dfrac{1}{\tan\theta}$
- cosec x and sec x have periods 2π, but cot x has a period of π.
- For integer values of n, as $x \to n\pi$, $\operatorname{cosec} x \to \pm\infty$; as $x \to 2n\pi \pm \dfrac{\pi}{2}$, $\sec x \to \pm\infty$; and as $x \to n\pi$, $\cot x \to \pm\infty$.
- For integer values of n, asymptotes exist for cosec x when $x = n\pi$; for sec x when $x = 2n\pi \pm \dfrac{\pi}{2}$; and for cot x when $x = n\pi$.

Addition identities for sin ($x \pm y$) and cos ($x \pm y$)

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Addition identities for tan $(x \pm y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

The Werner or factorisation identities

$$2\sin x\cos y = \sin(x+y) + \sin(x-y)$$
$$2\cos x\sin y = \sin(x+y) - \sin(x-y)$$
$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$
$$-2\sin x\sin y = \cos(x+y) - \cos(x-y)$$

The Simpson or half-sum/half-difference identities

$$\sin x + \sin y = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\sin x - \sin y = 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

$$\cos x + \cos y = 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)$$

$$\cos x - \cos y = -2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

Modelling and problem solving

- The function $y = a\sin x + b\cos x$ may be conveniently expressed in one of two forms:
  $$y = A\sin(x+\alpha) \qquad \text{or} \qquad y = A\cos(x+\alpha)$$
  given appropriate restrictions on α and by taking $A > 0$. We do this by writing:
  $$\begin{aligned} A\sin(x+\alpha) &= A\sin x\cos\alpha + A\cos x\sin\alpha \\ &= A\cos\alpha\sin x + A\sin\alpha\cos x \end{aligned}$$
  Comparing $y = a\sin x + b\cos x$ with $y = A\cos\alpha\sin x + A\sin\alpha\cos x$ gives $A\cos\alpha = a$ and $A\sin\alpha = b$ and by applying the Pythagorean identity we have
  $A = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$.
- When evaluating a physical area, work from a diagram to select the limits of integration. Be careful before assuming that the given function generates symmetric areas. This is not always the case.

CHAPTER review

1 **multiple choice** 7A

sin (90° − x) in simplest form is:

A −sin x **B** cos x **C** −cos x **D** sin x **E** sec x

2 **multiple choice** 7A

The reciprocal of cos x is:

A sec x **B** cosec x **C** cot x **D** $\cos^{-1} x$ **E** sin x

3 Simplify cosec (90° − θ). 7A

4 Prove that $\dfrac{\cot(\pi-\theta)\ \text{cosec}\left(\frac{1}{2}\pi+\theta\right)}{\sin\theta} = -\text{cosec}^2\,\theta.$ 7A

5 **multiple choice** 7B

What is the value of $\cos^2 x + \sin^2 x$?

A x **B** cos (2x) **C** $\sec^2 x$ **D** 1 **E** −1

6 **multiple choice** 7B

The exact value of sec (30°) is:

A $\frac{1}{2}$ **B** $\frac{1}{\sqrt{3}}$ **C** $\frac{2}{\sqrt{3}}$ **D** $\sqrt{3}$ **E** 2

7 **multiple choice** 7B

The expression $\sqrt{1-\cos^2 x}$ is precisely:

A cos x **B** sin x **C** sec x **D** $|\sin x|$ **E** all of these

8 If $\cos\theta = -\frac{5}{13}$ and $0 < \theta < 180°$, determine the exact value of: 7B

a $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

b $\cot\theta + \tan\theta.$

9 Prove that $\dfrac{\cos x}{1-\sin x} + \dfrac{\cos x}{1+\sin x} = 2\sec x.$ 7B

10 **multiple choice** 7C

When modelling soundwaves, we may say that ‘pitch’ is an attribute of the sound’s:

A frequency **B** loudness **C** timbre **D** overtones **E** all of these

11 A soundwave represented by y is generated by the partials $y_1 = 0.6 \sin 50\pi t$ and $y_2 = 0.6 \sin\left(50\pi t + \frac{1}{3}\pi\right)$ such that $y = y_1 + y_2$. 7C

a Determine the amplitude, period and frequency of the composite waveform.

b Discuss the effects on loudness and frequency in relation to the partials.

12 **multiple choice** 7D

The graph of cosec x is related to the graph of which function?

A cos x **B** sin x **C** tan x **D** 2 sin x cos x **E** $\cos^2 x$

7D **13** **multiple choice**

If $y = \sqrt{2}\ \sec(\pi t)$, then which of the following is most correct?

A $-\sqrt{2} \le y \le \sqrt{2}$ **B** $y \le -\sqrt{2}$ **C** $y - \pi$ **D** $y \ge \sqrt{2}$ **E** both B and D

7D **14** Sketch $y = 2\sec 2(x + 45°)$ across the domain $-180° \le x \le 180°$.

7D **15** Solve $2\sec 2(x + 45°) = \cot(x - 45°)$ for $-45° < x < 45°$ using appropriate technology.

7E **16** **multiple choice**

$\cos\theta\cos\alpha - \sin\theta\sin\alpha =$

A $\cos(\theta + \alpha)$ **B** $\cos(\theta - \alpha)$ **C** $\sin(\theta - \alpha)$ **D** $\sin(\theta + \alpha)$ **E** $\cos\theta - \cos\alpha$

7E **17** **multiple choice**

$2\sin y\cos y =$

A $\sin y + \cos y$ **B** $\cos(2y)$ **C** $\sin y - \cos y$ **D** $2\sin(2y)$ **E** $\sin(2y)$

7E **18** **multiple choice**

If x is an acute angle such that $\sin x = t$, then $\sin 2x$ is:

A $2t$ **B** $2t\sqrt{1 - t^2}$ **C** $\dfrac{t}{\sqrt{1 - t^2}}$ **D** $\dfrac{2t}{\sqrt{1 - t^2}}$ **E** none of these

7E **19** Show that $\cot(\theta - 45°) = \dfrac{\cos\theta + \sin\theta}{\sin\theta - \cos\theta}$, and hence deduce that

$\cot(\theta - 45°) = -\sec 2\theta - \tan 2\theta$.

7E **20** **a** Prove that $\cos 2x = 2\cos^2 x - 1$.

b Find the exact values of: **i** $\cos\dfrac{\pi}{12}$ **ii** $\cos\dfrac{13\pi}{12}$.

7F **21** **multiple choice**

The expression $\dfrac{\tan x + \tan y}{1 - \tan x\tan y}$ is equivalent to:

A $\tan(2x)$ **B** $\tan\left(x + \dfrac{\pi}{4}\right)$ **C** $\tan(x + y)$ **D** $\tan(x - y)$ **E** none of these

7F **22** Show that $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$. Hence, or otherwise, find the exact value of:

a $\tan\dfrac{\pi}{12}$ **b** $\tan\dfrac{11\pi}{12}$.

7G **23** Express each of the following products as a sum or difference.

a $2\sin 4x\cos 2y$ **b** $a\sin^2\pi x\cos\pi x$

7H **24** **a** Simplify: **i** $\dfrac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x}$ **ii** $\dfrac{\cos 5x - \cos x}{\sin 5x + \sin x}$.

b Solve $\dfrac{\cos 5x - \cos x}{\sin 5x + \sin x} = 0$.

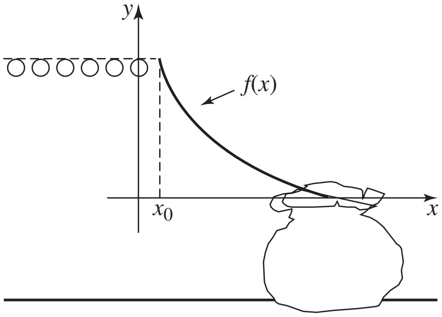
7I **25** **a** Evaluate: **i** $\int \cos 3x\sin x\,dx$ **ii** $\int_0^{\frac{\pi}{4}} \sin 4x\sin 2x\,dx$.

b Determine the physical area bounded by the curve $y = \sin 4x\sin 2x$, the x-axis and the abscissas at $x = 0$ and $x = \dfrac{\pi}{2}$.

c Verify your answer to **b** by utilising appropriate technology.

Modelling and problem solving

1 At the end of a conveyor belt that transports completed toys from Santa's factory is a chute that allows the toys to fall into a sack. The chute has the shape of a cot function of the form $f(x) = a \cot bx$, $x_0 \leq x \leq 3.5$, where x_0, a and b are real numbers. The end of the chute is 3.0 m horizontally away from the end of the conveyor belt, which is 2.0 m above the top of the sack. That is, $f(3.5) = 0$ and $f(0.50) = 2.0$.



- **a** Show that $x_0 = 0.50$.
- **b** Determine the value of b and hence a.
- **c** When the toys are a horizontal distance of 1.0 m from the sack, what is the value of x and how high above the sack are they?
- **d** Find the gradients of the chute at $x = x_0$ and $x = 3.5$.

2 **a** Evaluate each of the following integrals:

Hint: $\dfrac{d}{dx}\sin x° = \dfrac{\pi}{180}\cos x°$

- **i** $3\displaystyle\int_{45°}^{90°} \cos 4x° \cos 2x° \, dx$
- **ii** $-\displaystyle\int_{45°}^{180°} \sin 5x° \sin x° \, dx$.

b Use a graphics calculator to verify your answers, writing each correct to 3 significant figures where appropriate

3 **a** Calculate the physical area between the curve $y = \sin(\pi x)\sin(2\pi x)$ and the x-axis from $x = \dfrac{\pi}{3}$ and $x = \dfrac{2\pi}{3}$.

b Verify your answer by using appropriate graphics calculator functions.

4 A standing wave, at time t, is the result of the summation of two wave trains that have the same frequency, speed, and amplitude and are travelling in opposite directions along a taut string. These wave trains may be modelled as:

$$y_1 = a\sin(kx - \omega t) \quad \text{and} \quad y_2 = a\sin(kx + \omega t),\ a > 0.$$

- **a** Show that the standing wave has equation $y = [2a\sin(kx)]\cos(\omega t)$. We may assume that the bracketed expression represents the amplitude of the standing wave.
- **b** The *antinodes* of a standing wave are described as 'places of greatest amplitude'. If our wave trains have wavelength λ, where $\lambda = \dfrac{2\pi}{k}$, find the locations of the antinodes of our standing wave in terms of λ.

5 At a jetty in north Queensland, Ariel measures the depth of water at various times on 4 February 2009. Her observations yield the following results:

Time	6 am	7 am	8 am	9 am	10 am	11 am	12 noon	1 pm	2 pm
Depth (metres)	1.4	1.8	2.3	2.6	2.5	2.2	1.9	1.3	0.8
Time	3 pm	4 pm	5 pm	6 pm	7 pm	8 pm	9 pm	10 pm	11 pm
Depth (metres)	0.4	0.7	1.1	1.4	1.8	2.3	2.4	2.5	2.4

Ariel has information that suggests that boats which safely moor at this jetty can do so only when the depth of the water is above 1.8 metres.

a To predict future tidal behaviour, Ariel decides to model her observed data using appropriate functions of a graphics calculator. Assist Ariel to create a mathematical model that best reflects the given data.

b Use the model to determine if boats may safely moor at the end of the jetty on 7 February 2009 after 6 pm.

6 During the installation of 'O' rings into a launch vehicle, mechanical engineers need to solve the following configuration problem for both θ and α:
$\sin \theta + \sin (\theta - \alpha) + \sin (\theta + \alpha) = 0$.
It is known that $0 \leq \theta \leq 2\pi$ and $0 \leq \alpha \leq \pi$. The engineers seem to think that this problem does not have a unique solution for θ and α since the single equation has two unknowns. Determine if the engineers are correct by presenting a clear and convincing mathematical argument.

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Digital doc:
Test Yourself
Chapter 7

eBook*plus* ACTIVITIES

7A Circular and reciprocal functions

Interactivity

- Int-0251: Use the interactivity to review sine and cosine graphs. (*page 270*)

7B Exact values and the Pythagorean relationships

Tutorial

- int-0361: Watch a tutorial on finding exact values using trigonometric identities. (*page 276*)

7D Graphing the reciprocal trigonometric functions

Tutorial

- int-0360: Watch a tutorial on sketching the cosecant function over a given domain. (*page 290*)

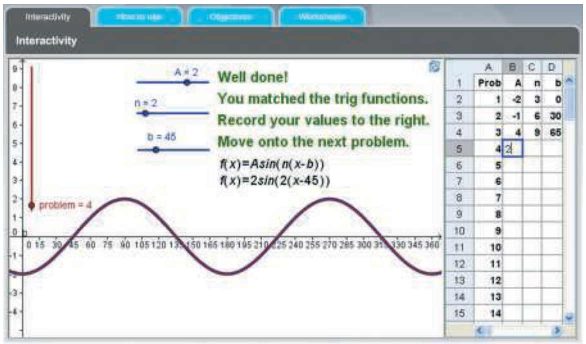
Digital docs

- SkillSHEET 7.1: Practise working with circular and reciprocal functions. (*page 295*)
- WorkSHEET 7.1: Simplify trigonometric expressions, sketch graphs of reciprocal trigonometric functions over a given domain and prove trigonometric identities. (*page 295*)

7H The Simpson or half-sum/half-difference identities

Digital docs

- SkillSHEET 7.2: Practise addition identities. (*page 310*)
- WorkSHEET 7.2: Simplify trigonometric expressions, prove trigonometric identities and solve trigonometric equations. (*page 310*)



Well done!
You matched the trig functions.
Record your values to the right.
Move onto the next problem.

$f(x)=A\sin(n(x-b))$

$f(x)=2\sin(2(x-45))$

	A	B	C	D
1	Prob	A	n	b
2	1	-2	3	0
3	2	-1	6	30
4	3	4	9	65
5	4	2		
6	5			
7	6			
8	7			
9	8			
10	9			
11	10			
12	11			
13	12			
14	13			
15	14			

7I Modelling and problem solving

Digital doc

- SkillSHEET 7.3: Practise using important formulas. (*page 317*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 325*)

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8

Advanced exponential functions

syllabus reference

Option topic:

Advanced periodic and exponential functions

In this chapter

8A Trigonometric links to complex numbers
8B Euler's formula and calculus applications
8C Applications of Euler's formula
8D Damped functions
8E Modelling real-life damped systems
8F Malthusian models and limited growth
8G Logistic model and population growth

Trigonometric links to complex numbers

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Interactivity:
Complex numbers in polar form
int-0343

Complex number theory and the associated trigonometric results are frequently applied to modelling real-life systems. Aeronautical engineering, signal theory, and viscous and electromagnetic flow are just a few of these systems that use complex number theory.

In our earlier studies involving complex numbers, we worked with de Moivre's theorem which states that for any complex number $z = r(\cos\theta + i\sin\theta)$, where z^n for real values of n is defined as:

$$z^n = r^n(\cos n\theta + i\sin n\theta) \qquad [1]$$

We can also write $z^n = r^n(\cos\theta + i\sin\theta)^n$ [2]

Equating [1] and [2] tells us that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Equating the real and imaginary components:

$$\text{Re}(\cos\theta + i\sin\theta)^n = \cos n\theta \text{ and}$$

$$\text{Im}(\cos\theta + i\sin\theta)^n = \sin n\theta$$

We can expand $(\cos\theta + i\sin\theta)^n$ by using the binomial theorem:

$$(\cos\theta + i\sin\theta)^n = \sum_{r=0}^{n} {}^nC_r \cos^{n-r}\theta(i\sin\theta)^r$$

This means that by using binomial expansion and de Moivre's theorem, $\cos n\theta$ and $\sin n\theta$ can be expressed in powers of sine and cosine.

$$\cos n\theta = \text{Re}(\cos\theta + i\sin\theta)^n$$
$$\sin n\theta = \text{Im}(\cos\theta + i\sin\theta)^n$$

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Digital docs:
SkillSHEET 8.1
SkillSHEET 8.2
SkillSHEET 8.3

WORKED Example 1

Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

THINK	WRITE
① Write the general formula for sin $n\theta$ and replace n with 3.	$\sin n\theta = \text{Im}(\cos\theta + i\sin\theta)^n$ so $\sin 3\theta = \text{Im}(\cos\theta + i\sin\theta)^3$
② Consider $(\cos\theta + i\sin\theta)^3$ and apply the binomial expansion using either Pascal's triangle or ${}^nC_r$ notation.	Now $(\cos\theta + i\sin\theta)^3$ $= {}^3C_0\cos^3\theta + {}^3C_1\cos^2\theta\,(i\sin\theta) +$ ${}^3C_2\cos\theta\,(i\sin\theta)^2 + {}^3C_3\,(i\sin\theta)^3$
③ Simplify, remembering that $i^2 = -1$ and $i^3 = i \cdot i^2 = -i$.	$= \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i^2\sin^2\theta) + i^3\sin^3\theta$ $= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$
④ Collect the real and imaginary parts.	$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\,(3\cos^2\theta\sin\theta - \sin^3\theta)$
⑤ Extract the imaginary part to write an expression for sin 3θ.	Since $\sin 3\theta = \text{Im}(\cos\theta + i\sin\theta)^3$, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
⑥ Use the Pythagorean identity to replace $\cos^2\theta$. That is, $\cos^2\theta = 1 - \sin^2\theta$.	$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$ $= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$
⑦ State your conclusion.	Hence, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

WORKED Example 2

Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

THINK	WRITE
① Write the general formula for cos $n\theta$ and replace n with 3.	$\cos n\theta = \text{Re}(\cos\theta + i\sin\theta)^n$ so $\cos 3\theta = \text{Re}(\cos\theta + i\sin\theta)^3$.
② Consider $(\cos\theta + i\sin\theta)^3$ and apply the binomial expansion using either Pascal's triangle or ${}^nC_r$ notation.	Now $(\cos\theta + i\sin\theta)^3$ $= {}^3C_0\cos^3\theta + {}^3C_1\cos^2\theta\,(i\sin\theta) +$ ${}^3C_2\cos\theta\,(i\sin\theta)^2 + {}^3C_3\,(i\sin\theta)^3$
③ Simplify, remembering that $i^2 = -1$ and $i^3 = i \cdot i^2 = -i$.	$= \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i^2\sin^2\theta) + i^3\sin^3\theta$ $= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$
④ Collect the real and imaginary parts.	$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\,(3\cos^2\theta\sin\theta - \sin^3\theta)$

Continued over page →

THINK	**WRITE**
⑤ Extract the real part to write an expression for cos 3θ.	Since $\cos 3\theta = \text{Re}\,(\cos\,\theta + i\sin\,\theta)^3$, $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
⑥ Use the Pythagorean identity to replace $\sin^2\theta$. That is, $\sin^2\theta = 1 - \cos^2\theta$.	$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$ $= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$
⑦ State your conclusion.	Hence, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Multiple-angle formulas

Consider a complex number $z = r(\cos\theta + i\sin\theta)$ where $|z| = 1$. This means that $z = \cos\theta + i\sin\theta$. Using de Moivre's theorem, if follows that:

$$z^n = \cos n\theta + i\sin n\theta \qquad \text{and} \qquad z^{-n} = \cos(-n\theta) + i\sin(-n\theta).$$

Since

$$\cos(-\theta) = \cos\theta \qquad \text{and} \qquad \sin(-\theta) = -\sin\theta$$

then

$$z^{-n} = \cos n\theta - i\sin n\theta.$$

Hence,

$$z^n + z^{-n} = 2\cos n\theta$$

and

$$z^n - z^{-n} = 2i\sin n\theta.$$

These multiple-angle formulas may be used more easily if written as follows.

$$\cos n\theta = \frac{z^n + z^{-n}}{2}$$

$$\sin n\theta = \frac{z^n - z^{-n}}{2i}$$

WORKED Example 3

Use the multiple-angle formulas to prove that $2\cos 3\theta\sin 2\theta = \sin 5\theta - \sin\theta$.

THINK	**WRITE**
① Consider the left-hand side of the equation. Use the multiple-angle formulas with appropriate values of n to replace cos 3θ and sin 2θ.	$2\cos 3\theta\sin 2\theta$ $= 2\cdot\dfrac{z^3 + z^{-3}}{2}\cdot\dfrac{z^2 - z^{-2}}{2i}$
② Simplify by cancelling a factor of 2 from numerator and denominator.	$= \dfrac{1}{2i}(z^3 + z^{-3})(z^2 - z^{-2})$
③ Expand the brackets.	$= \dfrac{1}{2i}(z^5 - z^1 + z^{-1} - z^{-5})$
④ Examine the right-hand side of the equation to be proved; that is, sin 5θ – sin θ. We require terms involving sine, so collect appropriate pairings of z to produce the necessary result.	$= \dfrac{1}{2i}[(z^5 - z^{-5}) - (z^1 - z^{-1})]$ $= \dfrac{z^5 - z^{-5}}{2i} - \dfrac{z^1 - z^{-1}}{2i}$ $= \sin 5\theta - \sin\theta$
⑤ State your conclusion.	$\therefore\ 2\cos 3\theta\sin 2\theta = \sin 5\theta - \sin\theta$

WORKED Example 4

Use the multiple-angle formulas and the binomial theorem to prove that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

eBook plus

Tutorial:
Worked example 4
int-0371

THINK	WRITE
1 Consider the left-hand side of the equation. Select an appropriate multiple-angle formula and set $n = 1$.	$\cos n\theta = \frac{z^n + z^{-n}}{2}$ so $\cos\theta = \frac{z^1 + z^{-1}}{2}$
2 Multiply both sides of the equation by 2 and write an equation involving $\cos^4\theta$.	$2\cos\theta = z^1 + z^{-1}$ $(2\cos\theta)^4 = (z^1 + z^{-1})^4$
3 Apply the binomial expansion to $(z^1 + z^{-1})^4$.	$16\cos^4\theta$ $= {}^4C_0\, z^4 + {}^4C_1\, z^3 \cdot z^{-1} + {}^4C_2\, z^2 \cdot z^{-2} + {}^4C_3\, z^1 \cdot z^{-3} + {}^4C_4\, z^{-4}$ $= z^4 + 4z^2 + 6z^0 + 4z^{-2} + z^{-4}$
4 We require terms including cosine, so collect appropriate pairings of z to produce the necessary result.	$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$
5 Express in a form so that a multiple-angle formula can be used.	$8\cos^4\theta = \frac{z^4 + z^{-4}}{2} + 4 \cdot \frac{z^2 + z^{-2}}{2} + \frac{6}{2}$ $= \cos 4\theta + 4\cos 2\theta + 3$
6 State your conclusion.	So $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

remember

1. De Moivre's theorem states that for any real values of n and complex number $z = r(\cos\theta + i\sin\theta)$:
   - (a) $z^n = r^n(\cos n\theta + i\sin n\theta)$
   - (b) Re $(\cos\theta + i\sin\theta)^n = \cos n\theta$ and Im $(\cos\theta + i\sin\theta)^n = \sin n\theta$
   - (c) if $i = \sqrt{-1}$, then $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.
2. The multiple-angle formulas are:

$$\cos n\theta = \frac{z^n + z^{-n}}{2} \text{ and } \sin n\theta = \frac{z^n - z^{-n}}{2i}.$$

EXERCISE 8A Trigonometric links to complex numbers

WORKED Example 1, 2

1 Using de Moivre's theorem and the binomial expansion, prove each of the following.

a $\sin 2\theta = 2 \sin \theta \cos \theta$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2 \cos^2\theta - 1$

c $\sin 4x = 4 \sin x \cos^3 x - 4 \cos x \sin^3 x$

d $\cos 4x = \cos^4 x - 6 \cos^2 x \sin^2 x + \sin^4 x = 8 \cos^4 x - 8 \cos^2 x + 1$

e $\sin 5\theta = 16 \sin^5\theta - 20 \sin^3\theta + 5 \sin \theta$

f $\cos 5\theta = 16 \cos^5\theta - 20 \cos^3\theta + 5 \cos \theta$

WORKED Example 3

2 Using the multiple-angle formulas, prove each of the following.

a $2 \sin 2x \cos x = \sin 3x + \sin x$

b $2 \cos 2\theta \sin \theta = \sin 3\theta - \sin \theta$

c $2 \sin 4x \cos 2x = \sin 6x + \sin 2x$

d $2 \cos 3\theta \cos 2\theta = \cos 5\theta + \cos \theta$

e $2 \sin 6x \sin 3x = \cos 3x - \cos 9x$

f $\sin 4\theta\,(1 + 2 \cos 2\theta) = \sin 2\theta + \sin 4\theta + \sin 6\theta$

WORKED Example 4

3 Prove each of the following.

a $\sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$

b $\cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x)$

c $\sin^4 x = \frac{1}{8}(\cos 4x - 4 \cos 2x + 3)$

d $\cos^5 x = \frac{1}{16}(\cos 5x + 5 \cos 3x + 10 \cos x)$

e $\sin^5 x = \frac{1}{16}(\sin 5x - 5 \sin 3x + 10 \sin x)$

4 If $z = \cos \theta + i \sin \theta$, then show that:

a $\tan \theta = \dfrac{z - z^{-1}}{i(z + z^{-1})}$

b $\tan^2\theta = \dfrac{2 - z^{-2} - z^2}{2 + z^{-2} + z^2}$

c $\cos^4\theta + \sin^4\theta \le 6 \cos^2\theta \sin^2\theta$ if $\dfrac{\pi}{8} \le \theta \le \dfrac{3\pi}{8}$

Euler's formula and calculus applications

A continuous function, $f(x)$, which may be repeatedly differentiated, can be expressed as a power series summation of the form:

$$f(x) = \sum_{r=0}^{\infty} \frac{f^{[n]}(0)x^n}{n!},$$

where $f^{[n]}(0)$ is the value of the nth derivative of $f(x)$ at 0.

Now, if we consider the derivatives of the functions e^x, sin x and cos x and build a table of values, we will be able to develop Euler's formula.

N	$f(x) = e^x$	$f^{[n]}$ (0)	$f(x) = \sin x$	$f^{[n]}$ (0)	$f(x) = \cos x$	$f^{[n\}}$ (0)
0	$f^{[0]} = e^x$	1	$f^{[0]} = \sin x$	0	$f^{[0]} = \cos x$	1
1	$f^{[1]} = e^x$	1	$f^{[1]} = \cos x$	1	$f^{[1]} = -\sin x$	0
2	$f^{[2]} = e^x$	1	$f^{[2]} = -\sin x$	0	$f^{[2]} = -\cos x$	−1
3	$f^{[3]} = e^x$	1	$f^{[3]} = -\cos x$	−1	$f^{[3]} = \sin x$	0
4	$f^{[4]} = e^x$	1	$f^{[4]} = \sin x$	0	$f^{[4]} = \cos x$	1
5	$f^{[5]} = e^x$	1	$f^{[5]} = \cos x$	1	$f^{[5]} = -\sin x$	0
6	$f^{[6]} = e^x$	1	$f^{[6]} = -\sin x$	0	$f^{[6]} = -\cos x$	−1
7	$f^{[7]} = e^x$	1	$f^{[7]} = -\cos x$	−1	$f^{[7]} = \sin x$	0
8	$f^{[8]} = e^x$	1	$f^{[8]} = \sin x$	0	$f^{[8]} = \cos x$	1

From the data in the table we may say:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \text{gives } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \qquad \text{gives } \sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \qquad \text{gives } \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Now, if we replace x with $i\theta$, we have the following.

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} +$$

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \quad + i\left\{\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \quad \right\}$$

Hence, we may say for any real values of θ,

$$e^{i\theta} = \cos\,\theta + i\,\sin\,\theta.$$

This is Euler's formula. It links the natural exponential function to trigonometry and also has application in calculus.

WORKED Example 5

a **Express $e^{i\pi}$ in standard form.**

b **Express both $z = -1 + i\sqrt{3}$ and $w = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ in Euler's form.**

c **Determine: i z^4w^5 ii $\frac{z^4}{w^5}$, leaving your answers in Euler's form.**

d **Write your answers to part c in standard form, correct to 4 decimal places.**

THINK	WRITE
a ① Apply Euler's formula with $\theta = \pi$.	**a** $e^{i\theta} = \cos\theta + i\sin\theta$ $e^{i\pi} = \cos\pi + i\sin\pi$
② Simplify. Note that the result is an entirely real number.	$= -1 + i\cdot 0$ $= -1$
③ Write the answer.	Hence, $e^{i\pi}$ in standard form is −1.
b ① Write z in polar (or modulus-argument) form.	**b** $z = -1 + i\sqrt{3}$ $\lvert z\rvert = \sqrt{(-1)^2 + (\sqrt{3})^2}$ $= 2$ $\text{Arg } z = \tan^{-1}\frac{\sqrt{3}}{-1}$ $= \frac{2\pi}{3}$ (in quadrant 2) So $z = 2\text{ cis }\frac{2\pi}{3}$ $= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
② Apply Euler's formula. Remember to express the exponent in radians and not in degrees.	$= 2e^{\frac{2\pi i}{3}}$
③ Repeat steps 1 and 2 for w.	$w = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ $\lvert w\rvert = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2}$ $= 1$ $\text{Arg } w = \tan^{-1} 1$ $= -\frac{3\pi}{4}$ (in quadrant 3) so $w = \text{cis}\left(-\frac{3\pi}{4}\right)$ $= \cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$ $= e^{-\frac{3\pi i}{4}}$

THINK	**WRITE**
c i ① Express z^4 in polar form.	**c i** $z^4 = 2^4 e^{\frac{4\cdot 2\pi i}{3}}$ $= 16e^{\frac{8\pi i}{3}}$
② Simplify the exponent by considering the equivalent trigonometric form of Euler's formula.	Now $\cos\frac{8\pi}{3} = \cos\left(2\pi+\frac{2\pi}{3}\right)$ $= \cos\frac{2\pi}{3}$ and $\sin\frac{8\pi}{3} = \sin\left(2\pi+\frac{2\pi}{3}\right)$ $= \sin\frac{2\pi}{3}$ so $z^4 = 16e^{\frac{2\pi i}{3}}$
③ Express w^5 in polar form.	$w^5 = e^{\frac{5\cdot -3\pi i}{4}}$ $= e^{-\frac{15\pi i}{4}}$
④ Simplify the exponent by considering the equivalent trigonometric form of Euler's formula.	Now $\cos\left(-\frac{15\pi}{4}\right) = \cos\left(-4\pi+\frac{\pi}{4}\right)$ $= \cos\frac{\pi}{4}$ and $\sin\left(-\frac{15\pi}{4}\right) = \sin\left(-4\pi+\frac{\pi}{4}\right)$ $= \sin\frac{\pi}{4}$ so $w^5 = e^{\frac{\pi i}{4}}$
⑤ Multiply z^4 and w^5 by multiplying the moduli and adding the arguments.	$z^4w^5 = 16e^{\frac{2\pi i}{3}} \cdot e^{\frac{\pi i}{4}}$ $= 16e^{\left(\frac{2\pi i}{3}+\frac{\pi i}{4}\right)}$ $= 16e^{\frac{11\pi i}{12}}$
ii Divide z^4 and w^5 by dividing the moduli and subtracting the arguments.	**ii** $\frac{z^4}{w^5} = \frac{16e^{\frac{2\pi i}{3}}}{e^{\frac{\pi i}{4}}}$ $= 16e^{\left(\frac{2\pi i}{3}-\frac{\pi i}{4}\right)}$ $= 16e^{\frac{5\pi i}{12}}$

Continued over page →

THINK	**WRITE**
d i ① Write z^4w^5 in Euler's form from part **c**.	**d i** $z^4w^5 = 16e^{\frac{11\pi i}{12}}$
② Use Euler's formula with $\theta = \dfrac{11\pi}{12}$ to write the result in standard form.	$= 16\left(\cos\dfrac{11\pi}{12} + i\sin\dfrac{11\pi}{12}\right)$
③ Simplify, writing the final answer to 4 decimal places.	$= 16(-0.965\,925\,8 + 0.258\,819i)$ $= -15.4548 + 4.1411i$ (to 4 decimal places)
ii ① Write $\dfrac{z^4}{w^5}$ in Euler's form from part **c**.	**ii** $\dfrac{z^4}{w^5} = 16e^{\frac{5\pi i}{12}}$
② Use Euler's formula with $\theta = \dfrac{5\pi}{12}$ to write the result in standard form.	$= 16\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$
③ Simplify, writing the final answer to 4 decimal places.	$= 16(0.258\,819 + 0.965\,925\,8i)$ $= 4.1411 + 15.4548i$ (to 4 decimal places)

Graphics Calculator tip! Using Euler's formula

The calculations in Worked example 5 can be performed on a graphics calculator.

For the CASIO *fx*-9860G AU

1. To ensure that the calculator is set in Complex Mode, press:
   - SHIFT
   - MENU (SET UP)

   Scroll down to Complex Mode and press F3 ($r\angle\theta$), then press EXE.

```
Mode          :Comp
Frac Result   :d/c
Func Type     :Y=
Draw Type     :Connect
Derivative    :Off
Angle         :Rad
Complex Mode:r∠θ
```

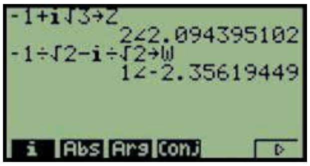
2. To access the various available complex number commands, press:
   - OPTN
   - F3 (CPLX)

   Using these commands, complete the entry lines as:

   $-1 + i\sqrt{3} \to z$

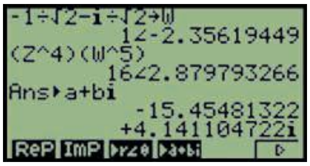
   $\dfrac{-1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}} \to w$

   Press EXE after each line.



```
-1+i√3→Z
            2∠2.094395102
-1÷√2-i÷√2→W
            1∠-2.35619449
```

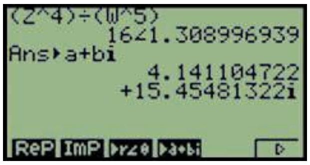
3. To calculate z^4w^5, complete the entry line as:
   (z^4)(w^5)
   Then press **EXE**.
   To express with rectangular coordinates, press:
   - **F6**
   - **F4** $(a + bi)$



```
-1÷√2-i÷√2→W
              1∠-2.35619449
(Z^4)(W^5)
             16∠2.879793266
Ans▸a+bi
               -15.45481322
               +4.141104722i
```

4. Therefore $z^4w^5 = 16e^{2.88i}$
   $= -15.45 + 4.14i.$

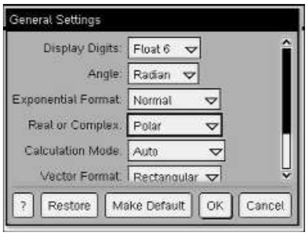
   Follow the steps above to find $\dfrac{z^4}{w^5}$.



```
(Z^4)÷(W^5)
             16∠1.308996939
Ans▸a+bi
                4.141104722
               +15.45481322i
```

For the TI-Nspire CAS

1. To ensure that the calculator is set in Complex Mode, go to General Settings.
   Use Tab **tab** to move between fields and set fields as shown.



General Settings

Display Digits: Float 6
Angle: Radian
Exponential Format: Normal
Real or Complex: Polar
Calculation Mode: Auto
Vector Format: Rectangular

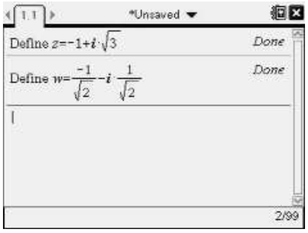
2. To define z and w, press:
   - MENU **menu**
   - 1: Actions **1**
   - 1: Define **1**

   Complete the entry lines as:

   Define $z = -1 + i\sqrt{3}$

   Define $w = \dfrac{-1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$

   Press ENTER **enter** after each line.

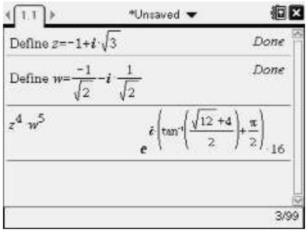


Define $z=^{-}1+i\cdot\sqrt{3}$ Done

Define $w=\frac{^{-}1}{\sqrt{2}}-i\cdot\frac{1}{\sqrt{2}}$ Done

2/99

3. To calculate z^4w^5, complete the entry line as:
   z^4w^5.
   Then press ENTER **enter**.
   The calculator defaults to an exact value of θ.



Define $z=^{-}1+i\cdot\sqrt{3}$ Done

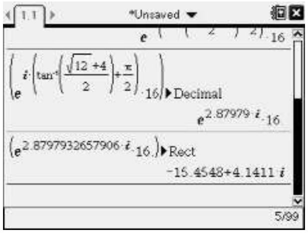
Define $w=\frac{^{-}1}{\sqrt{2}}-i\cdot\frac{1}{\sqrt{2}}$ Done

$z^4\cdot w^5$ $\quad e^{i\cdot\left(\tan^{-1}\left(\frac{\sqrt{12}+4}{2}\right)+\frac{\pi}{2}\right)}\cdot 16$

3/99

4. To find an approximate angle press:
   - MENU **menu**
   - 2: Number **2**
   - 1: Convert to Decimal **1**

   Ans ▸ Decimal will appear.
   Press ENTER **enter**.



$\left(e^{i\cdot\left(\tan^{-1}\left(\frac{\sqrt{12}+4}{2}\right)+\frac{\pi}{2}\right)}\cdot 16\right)$▸Decimal $\quad e^{2.87979\cdot i}\cdot 16$

$\left(e^{2.8797932657906\cdot i}\cdot 16.\right)$▸Rect $\quad ^{-}15.4548+4.1411\cdot i$

5/99

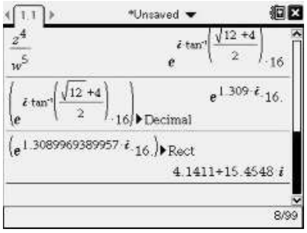
To express the result as rectangular coordinates, press:

- MENU menu
- 2: Number 2
- 9: Complex - Number Tools 9
- 7: Convert to Rectangular 7

This time Ans ▶ Rect will appear, press ENTER enter.

It can be seen that $z^4w^5 = 16e^{2.88i} = -15.45 + 4.14i$.

5. Follow the steps above to find, $\frac{z^4}{w^5}$.



$\frac{z^4}{w^5}$ $\quad e^{i\cdot\tan^{-1}\left(\frac{\sqrt{12}+4}{2}\right)}\cdot 16$

$\left(e^{i\cdot\tan^{-1}\left(\frac{\sqrt{12}+4}{2}\right)}\cdot 16\right)$▶Decimal $\quad e^{1.309\cdot i}\cdot 16.$

$\left(e^{1.3089969389957\cdot i}\cdot 16.\right)$▶Rect

$4.1411+15.4548\,i$

WORKED Example 6

Use Euler's formula to evaluate $\int e^x \cos x\, dx$.

THINK	WRITE
1 Use Euler's formula to express $\cos x$ as the real part of e^{ix}.	$e^{ix} = \cos x + i \sin x$ so $\cos x = \text{Re}\,(e^{ix})$ $\int e^x \cos x\, dx = \text{Re}\left(\int e^x e^{ix}\, dx\right)$
2 Find the integral of a new expression where $\cos x$ is replaced with e^{ix}.	$\int e^x e^{ix}\, dx = \int e^{x(1+i)}\, dx$ $= \frac{e^{x(1+i)}}{1+i}$
3 Multiply the numerator and denominator by the conjugate of $1 + i$ so that the denominator becomes a real number.	$= \frac{e^{x(1+i)}}{1+i} \cdot \frac{1-i}{1-i}$ $= \frac{(1-i)e^{x(1+i)}}{2}$
4 Simplify so that e^{ix} can be replaced with $(\cos x + i \sin x)$. This will make it easier to obtain the real part of the expression.	$= \frac{(1-i)e^x e^{ix}}{2}$ $= \frac{1}{2}e^x\,(1-i)(\cos x + i \sin x)$
5 Expand so that the real and imaginary parts can be seen clearly.	$= \frac{1}{2}e^x\,(\cos x + i \sin x - i \cos x - i^2 \sin x)$ $= \frac{1}{2}e^x\,(\cos x + i \sin x - i \cos x + \sin x)$
6 State the real part of the expression.	$\text{Re}\left(\int e^x e^{ix}\, dx\right) = \frac{1}{2}e^x\,(\cos x + \sin x)$
7 Write the answer. Remember to include the constant of integration.	So $\int e^x \cos x\, dx = \frac{1}{2}e^x\,(\cos x + \sin x) + c.$

WORKED Example 7

Use Euler's formula to find the exact value of $\int_0^{\frac{\pi}{2}} e^x \sin 2x\, dx$.

THINK	WRITE
① Use Euler's formula to express sin $2x$ as the imaginary part of e^{2ix}.	$e^{2ix} = \cos 2x + i \sin 2x$ so $\sin 2x = \text{Im}\,(e^{2ix})$ $\int e^x \sin 2x\, dx = \text{Im}\,(\int e^x e^{2ix}\, dx)$ $\int e^x e^{2ix}\, dx = \int e^{x(1+2i)}\, dx$
② Find the integral of a new expression where sin $2x$ is replaced with e^{2ix}.	$= \dfrac{e^{x(1+2i)}}{1+2i}$
③ Multiply the numerator and denominator by the conjugate of $1 + 2i$ so that the denominator becomes a real number.	$= \dfrac{e^{x(1+2i)}}{1+2i} \cdot \dfrac{1-2i}{1-2i}$ $= \dfrac{(1-2i)e^{x(1+2i)}}{5}$ $= \frac{1}{5}(1-2i)\, e^x e^{2ix}$
④ Simplify so that e^{2ix} can be replaced with (cos $2x$ + i sin $2x$). This will make it easier to obtain the imaginary part of the expression.	$= \frac{1}{5} e^x(1-2i)(\cos 2x + i \sin 2x)$
⑤ Expand so that the real and imaginary parts can be seen clearly.	$= \frac{1}{5} e^x(\cos 2x + i \sin 2x - 2i \cos 2x - 2i^2 \sin 2x)$ $= \frac{1}{5} e^x(\cos 2x + i \sin 2x - 2i \cos 2x + 2 \sin 2x)$
⑥ State the imaginary part of the expression.	$\text{Im}\left(\int e^x e^{2ix}\, dx\right) = \frac{1}{5} e^x(\sin 2x - 2 \cos 2x)$
⑦ Write the integral. Remember to include the constant of integration.	so $\int e^x \sin 2x\, dx = \frac{1}{5} e^x(\sin 2x - 2 \cos 2x) + c$
⑧ Find the definite integral by evaluating the limits.	Hence, $\int_0^{\frac{\pi}{2}} e^x \sin 2x\, dx$ $= \left[\frac{1}{5} e^x(\sin 2x - 2 \cos 2x)\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{5} e^{\frac{\pi}{2}}(\sin \pi - 2 \cos \pi) - \frac{1}{5} e^0(\sin 0 - 2 \cos 0)$ $= \frac{1}{5} e^{\frac{\pi}{2}}(0 - 2 \cdot -1) - \frac{1}{5}(0 - 2)$ $= \frac{2}{5}(e^{\frac{\pi}{2}} + 1)$

remember

1. For any real values of θ, Euler's formula states $e^{i\theta} = \cos\,\theta + i\,\sin\,\theta$.
2. θ is always a radian measure.
3. When working with Euler's formula we must remember that:
   (a) To *multiply*, we multiply ($\cdot$) the moduli and add (+) the arguments.
   (b) To *divide*, we divide (|) the moduli and subtract (–) the arguments.
4. $\int f(x)\cos ax\,dx = \text{Re}\left[\int e^{iax} f(x)\,dx\right]$
5. $\int f(x)\sin ax\,dx = \text{Im}\left[\int e^{iax} f(x)\,dx\right]$

EXERCISE 8B Euler's formula and calculus applications

Where appropriate, check your answers with a graphics calculator.

WORKED Example 5a

1 Express each of the following complex numbers in standard form.

a $e^{2\pi i}$
b $e^{-2\pi i}$
c $e^{\frac{\pi i}{2}}$
d $3e^{\frac{\pi i}{3}}$
e $2e^{\frac{2\pi i}{3}}$
f $5e^{\frac{-\pi i}{6}}$
g $\frac{1}{2}e^{-2.5i}$
h $4e^{-1.25i}$
i $3.25e^{1.04i}$
j $1.05e^{-0.75i}$
k $0.25e^{-0.45i}$
l $100e^{5.925i}$

WORKED Example 5b

2 Express each of the following complex numbers in Euler's form.

a $-1 + i$
b $-1 - i$
c $\sqrt{3} + i$
d $\sqrt{3} - i$
e $-1 - i\sqrt{3}$
f $\frac{\sqrt{3}}{2} - \frac{1}{2}i$
g $4 - 3i$
h $-3 - 4i$
i $-5 - 12i$
j $-5 + 12i$
k $24 - 7i$
l $2\sqrt{3} + 3\sqrt{5}\,i$

WORKED Example 5c, d

3 For each of the following, apply Euler's formula to find the given expression. Write your final answer in standard form correct to four decimal places where appropriate.

a If $z = 2 + 3i$ and $w = -1 + 2i$, find z^4w^5.
b If $z = -2 - 3i$ and $w = 1 - 2i$, find $\frac{z^4}{w^5}$.
c If $u = \sqrt{5} - i$ and $w = 1 - i$, find $\left(\frac{u}{w}\right)^3$.
d If $v = \sqrt{6} + i\sqrt{5}$ and $w = -3 + i$, find $4v^2w^3$.
e If $z = \sqrt{2} + i\sqrt{2}$ and $w = i$, find $(z^3w^4)^2$.
f If $z = \sqrt{8} - 3i$, find z^{-4}.
g If $v = -4 + 2i$ and $w = 5 + 4i$, find v^4w^{-3}.
h If $u = -3 - 2i$ and $v = 1 + i$, find $(u^2v^{-1})^4$.

WORKED Example 6 & 7

4 Using Euler's formula, determine each of the following integrals.

a $\int e^x \sin x \, dx$ **b** $\int e^{-x} \cos x \, dx$ **c** $\int e^{2x} \sin x \, dx$

d $\int e^{-x} \cos 2x \, dx$ **e** $\int e^{2x} \sin 2x \, dx$ **f** $\int_0^{\pi} e^x \cos x \, dx$

g $\int_0^1 e^x \sin \pi x \, dx$ **h** $\int_0^{\frac{\pi}{3}} e^{-x} \cos 3x \, dx$ **i** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2x} \sin 2x \, dx$

5 Given that $z = e^{ix}$ and $v = e^{iy}$, show that from the expressions zv and $\frac{z}{v}$, we may establish each of the following addition identities.

a $\cos (x + y) = \cos x \cos y - \sin x \sin y$ **b** $\cos (x - y) = \cos x \cos y + \sin x \sin y$

c $\sin (x + y) = \sin x \cos y + \sin y \cos x$ **d** $\sin (x - y) = \sin x \cos y - \sin y \cos x$

Applications of Euler's formula

Mathematicians use iterative formulas to deal with systems that involve repetitive calculations. These calculations frequently occur during the mathematical modelling of real-life situations; for instance, the repetitive multiplication of the Leslie matrix when we are calculating population changes in ecosystems. The iterative formula explored in this exercise is given by:

$$z_n = \sum_{k=1}^{n} \exp\left(\frac{2\pi i k^3}{M}\right), \; n = 1, 2, 3, \quad , M.$$

Note: $\exp\left(\frac{2\pi i k^3}{M}\right)$ means e to the power of $\frac{2\pi i k^3}{M}$.

This recurrence relation generates a sequence of complex numbers $\{z_n\}$, which may be connected in sequence to produce elegant patterns in the complex plane.

WORKED Example 8

Using the iterative formula $z_n = \sum_{k=1}^{n} \exp\left(\frac{2\pi i k^3}{M}\right)$, $n = 1, 2, 3, \ldots, M$, select $M = 4$, and generate the sequence of complex numbers $\{z_1, z_2, z_3, z_4\}$. Graph this sequence in the complex plane by linking in order: z_1 to z_2, z_2 to z_3, z_3 to z_4, and finally, z_4 back to z_1.

THINK / **WRITE**

1. Write the iterative formula with $M = 4$.

$$z_n = \sum_{k=1}^{n} \exp\left(\frac{\pi i k^3}{2}\right),\ n = 1, 2, 3, \quad , 4$$

2. Calculate the first term in the sequence; that is, $n = 1$. Use Euler's formula to simplify.

$$\begin{aligned} z_1 &= \sum_{k=1}^{1} \exp\left(\frac{\pi i k^3}{2}\right) \\ &= e^{\frac{\pi i}{2}} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i \cdot 1 \\ &= i \end{aligned}$$

3. Calculate the second term in the sequence; that is, $n = 2$. Notice that we can use the result for z_1.

$$\begin{aligned} z_2 &= \sum_{k=1}^{2} \exp\left(\frac{\pi i k^3}{2}\right) \\ &= e^{\frac{\pi i}{2}} + e^{4\pi i} \\ &= z_1 + (\cos 4\pi + i\sin 4\pi) \\ &= i + (1 + 0) \\ &= 1 + i \end{aligned}$$

4. Calculate the third and fourth terms in the same way.

$$\begin{aligned} z_3 &= \sum_{k=1}^{3} \exp\left(\frac{\pi i k^3}{2}\right) \\ &= z_2 + e^{\frac{27\pi i}{2}} \\ &= (1 + i) + \left(\cos\frac{27\pi}{2} + i\sin\frac{27\pi}{2}\right) \\ &= (1 + i) + (0 - i) \\ &= 1 \end{aligned}$$

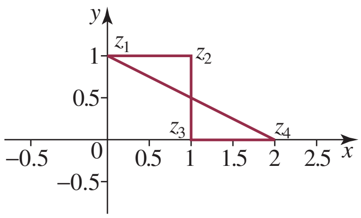
$$\begin{aligned} z_4 &= \sum_{k=1}^{4} \exp\left(\frac{\pi i k^3}{2}\right) \\ &= z_3 + e^{32\pi i} \\ &= 1 + (\cos 32\pi + i\sin 32\pi) \\ &= 1 + 1 + 0 \\ &= 2 \end{aligned}$$

5. List the sequence of terms.

So $\{z_1, z_2, z_3, z_4\} = \{i, 1 + i, 1, 2\}$.

THINK **WRITE**

6 Graphically represent the solution set $\{z_1, z_2, z_3, z_4\}$ and remember to join the last point to the first point.



Another useful result can be obtained by considering expressions for $e^{ik\theta}$ and $e^{-ik\theta}$. By using Euler's formula with each, we obtain the following:

$$e^{ik\theta} = \cos k\theta + i \sin k\theta \qquad [1]$$

and

$$e^{-ik\theta} = \cos k\theta - i \sin k\theta \qquad [2]$$

[as $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$].

Adding equations [1] and [2] together gives:

$$e^{ik\theta} + e^{-ik\theta} = 2\cos k\theta$$

so

$$\cos k\theta = \tfrac{1}{2}(e^{ik\theta} + e^{-ik\theta})$$

Subtracting equation [2] from [1] gives:

$$e^{ik\theta} - e^{-ik\theta} = 2i \sin k\theta$$

so

$$\sin k\theta = \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$$

In summary:

$$\boldsymbol{\cos k\theta = \tfrac{1}{2}(e^{ik\theta} + e^{-ik\theta})}$$

$$\boldsymbol{\sin k\theta = \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})}$$

These relationships are used in questions of the type shown in the following worked example.

WORKED Example 9

a **Show that** $\displaystyle\sum_{k=1}^{n} \sin k\theta = \text{Im}\,(e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \quad + e^{in\theta})$.

b **Hence, deduce that** $\displaystyle\sum_{k=1}^{n} \sin k\theta = \frac{\sin \tfrac{1}{2}(n+1)\theta \,.\, \sin \tfrac{1}{2}n\theta}{\sin \tfrac{1}{2}\theta}$.

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THINK	**WRITE**
a ① Consider the left-hand side and generate the sequence of terms.	**a** $\sum_{k=1}^{n} \sin k\theta = \sin \theta + \sin 2\theta + \sin 3\theta + \quad + \sin n\theta$
② Use the fact that Im $(e^{in\theta}) = \text{Im}\,(\cos n\theta + i \sin n\theta)$ to obtain an expression for $\sin n\theta$.	As $(e^{in\theta}) = (\cos n\theta + i \sin n\theta)$ then $\sin n\theta = \text{Im}\,(e^{in\theta})$
③ Substitute for each sine term and simplify.	So $\sum_{k=1}^{n} \sin k\theta$ $= \text{Im}\,(e^{i\theta}) + \text{Im}\,(e^{2i\theta}) + \text{Im}\,(e^{3i\theta}) + \quad + \text{Im}\,(e^{ni\theta})$ $= \text{Im}\,(e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \quad + e^{ni\theta})$
b ① Write the relationship from part **a**.	**b** $\therefore \sum_{k=1}^{n} \sin k\theta = \text{Im}\,(e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \quad + e^{in\theta})$
② Identify that there is the sum of a geometric progression on the right-hand side of the relationship.	Now $e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \quad + e^{ni\theta}$ is the sum of a geometric progression where the first term is $e^{i\theta}$ and the common ratio is $e^{i\theta}$.
③ Use the formula $\frac{a(r^n - 1)}{r - 1}$ where $a = e^{i\theta}$ and $r = e^{i\theta}$ to calculate the sum.	$\sum_{k=1}^{n} \sin k\theta = \text{Im}\left(\frac{e^{i\theta}(e^{in\theta} - 1)}{e^{i\theta} - 1}\right)$
④ Multiply both the numerator and the denominator by $e^{-\frac{1}{2}i\theta}$.	$\sum_{k=1}^{n} \sin k\theta = \text{Im}\left(\frac{e^{-\frac{1}{2}i\theta} \cdot e^{i\theta}(e^{in\theta} - 1)}{e^{-\frac{1}{2}i\theta}(e^{i\theta} - 1)}\right)$ $= \text{Im}\left(\frac{e^{\frac{1}{2}i\theta}(e^{in\theta} - 1)}{\left(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}\right)}\right)$
⑤ Consider Im $\left\{\frac{e^{\frac{1}{2}i\theta}(e^{in\theta} - 1)}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}\right\}$ in two parts:	

THINK **WRITE**

(a) Numerator

$$e^{in\theta} - 1 = e^{\frac{1}{2}n\theta i}\left(e^{\frac{1}{2}n\theta i} - e^{-\frac{1}{2}n\theta i}\right)$$

$$= e^{\frac{1}{2}n\theta i} \cdot 2i \sin\left(\tfrac{1}{2}n\theta\right)$$

$$\therefore\ e^{\frac{1}{2}i\theta}\left(e^{in\theta} - 1\right) = e^{\frac{1}{2}i\theta} \cdot e^{\frac{1}{2}n\theta i} \cdot 2i \sin\left(\tfrac{1}{2}n\theta\right)$$

$$= e^{\frac{1}{2}i\theta(n+1)} \cdot 2i \sin\left(\tfrac{1}{2}n\theta\right)$$

(b) Denominator

$$e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta} = 2i \sin\left(\tfrac{1}{2}\theta\right)$$

$$\text{Hence Im}\left\{\frac{e^{\frac{1}{2}i\theta}\left(e^{in\theta} - 1\right)}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}\right\}$$

$$= \text{Im}\left\{\frac{e^{\frac{1}{2}i\theta(n+1)} \sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}\right\}$$

$$= \frac{\sin \frac{1}{2}(n+1)\theta \cdot \sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta}$$

$$\therefore \sum_{k=1}^{n} \sin k\theta = \frac{\sin \frac{1}{2}(n+1)\theta \cdot \sin \frac{1}{2}n\theta}{\sin \frac{1}{2}\theta}$$

remember

1. The iterative formula, $z_n = \sum_{k=1}^{n} \exp\left(\frac{2\pi i k^3}{M}\right)$, $n = 1, 2, 3,\quad , M$, generates the complex sequence $\{z_n\}$.
2. As the value of M is increased the sequence $\{z_n\}$ becomes more intricate.
3. $\cos k\theta = \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})$ and $\sin k\theta = \frac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$.

EXERCISE 8C Applications of Euler’s formula

WORKED Example 8

1 a Generate the sequences of complex numbers for this iterative formula for each of the following values of M. $z_n = \sum_{k=1}^{n} \exp\left(\frac{2\pi i k^3}{M}\right),\ n = 1, 2, 3, \quad , M$

i $M = 4$ **ii** $M = 6$ **iii** $M = 8$ **iv** $M = 12$

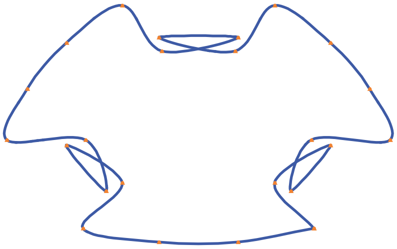
b Graph this sequence in the complex plane by linking in order z_1 to z_2, z_2 to z_3, z_3 to z_4, and so on, and finally, z_n back to z_1.

2 Consider the equation $z_n = \sum_{k=1}^{n} \exp\left(\frac{2\pi i k^3}{M}\right),\ n = 1, 2, 3, \quad , M$

a By making use of Euler’s formula, find expressions for the real and imaginary parts of z_n in terms of π, n, k and M in the iterative formula above.

b Derive a recurrence relationship connecting z_n to z_{n-1}.

c Prove your result.



3 Using an Excel spreadsheet package, calculate $\{z_n\}$ for $M = 24$ and generate its image as a closed connected sequence like the one at right. In generating $\{z_n\}$, use spreadsheet headings similar to the following:

	Real part		**Imaginary part**

WORKED Example 9

4 Prove that $\sum_{k=1}^{n} \cos k\theta = \frac{\cos\left[\frac{1}{2}(n+1)\theta\right] \cdot \sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}$.

5 A particular computer can execute 100 billion ($1 \cdot 10^{11}$) basic arithmetical operations per second. Estimate how long such a computer would take to calculate 100! basic arithmetical operations using Stirling’s formula, which states that:

$$n! \approx \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} e^{-n}$$

6 If $0 < \theta < \pi$, then solve $\frac{\sqrt{2}e^{-i\theta}}{i-2} = \frac{1+3i}{5}$ completely for θ.

eBook*plus*

Digital docs:
WorkSHEET 8.1
EXCEL Spreadsheet
$M = 29$
doc-0142

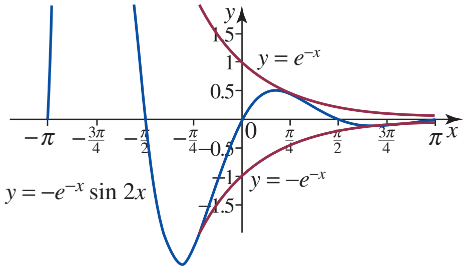
Damped functions

Heat generated by many mechanical and electrical systems inhibits motion or electron flow. This feature is called *damping*. In this section we shall explore functions that exhibit damping; in particular, $e^{ax}\sin bx$ and $e^{ax}\cos bx$ for various values of a and b.

As $\sin bx$ and $\cos bx$ are both functions that oscillate between ±1, then the multiplier function, e^{ax}, alters the amplitude of both according to the value of a. This enveloping effect constrains both $\sin bx$ and $\cos bx$ between $\pm e^{ax}$. We make use of this feature when sketching the functions $y = e^{ax}\sin bx$ and $y = e^{ax}\cos bx$. That is, we use the curves $y = e^{ax}$ and $y = -e^{ax}$ as the boundaries for the function.

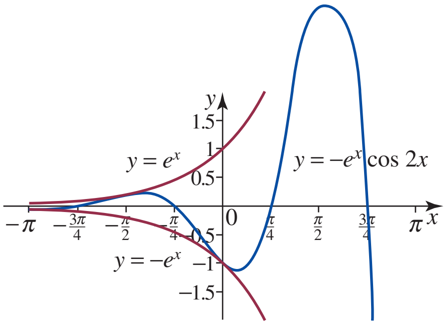
WORKED Example 10

Sketch $y = e^{-x}\sin 2x$ over the domain $-\pi \le x \le \pi$.

THINK	WRITE/DRAW
1 Identify the boundaries of the function.	For $y = e^{-x}\sin 2x$, the boundaries are $y = e^{-x}$ and $y = -e^{-x}$.
2 State the period of the trigonometric function $\sin 2x$.	The period of the function $y = \sin 2x$ is π.
3 As $\pm e^{-x}$ constrains $\sin 2x$, sketch these envelope functions first. That is, sketch the boundary functions $y = e^{-x}$ and $y = -e^{-x}$.	
4 Sketch the function between the drawn boundaries, following the periodic nature of $\sin 2x$.	

WORKED Example 11

Sketch $y = -e^{x}\cos 2x$ over the domain $-\pi \le x \le \pi$.

THINK	WRITE/DRAW
1 Identify the boundaries of the function.	For $y = -e^{x}\cos 2x$, the boundaries are $y = e^{x}$ and $y = -e^{x}$.
2 State the period of the trigonometric function $\cos 2x$.	The period of the function $y = -\cos 2x$ is π. This is a reflection of the graph $y = \cos 2x$ in the x-axis.
3 As $\pm e^{x}$ constrains $-\cos 2x$, sketch these envelope functions first. That is, sketch the boundary functions $y = e^{x}$ and $y = -e^{x}$.	
4 Sketch the function between the drawn boundaries, following the periodic nature of $-\cos 2x$.	

WORKED Example 12

Sketch $y = 2e^{-x} \cos \frac{1}{2}x$ over the domain $-\pi \le x \le \pi$.

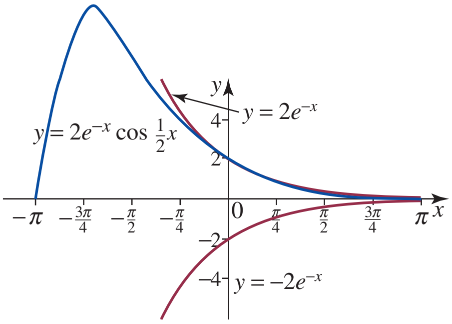
THINK

1. Identify the boundaries of the function.
2. State the period of the trigonometric function $\cos \frac{1}{2}x$.
3. As $\pm 2e^{-x}$ constrains $\cos \frac{1}{2}x$, sketch these envelope functions first. That is, sketch the boundary functions $y = 2e^{-x}$ and $y = -2e^{-x}$.
4. Sketch the function between the drawn boundaries, following the periodic nature of $\cos \frac{1}{2}x$.

WRITE/DRAW

For $y = 2e^{-x} \cos \frac{1}{2}x$, the boundaries are $y = 2e^{-x}$ and $y = -2e^{-x}$.

The period of the function $y = \cos \frac{1}{2}x$ is 4π.



WORKED Example 13

Using information associated with the graph of $y = e^{-x} \sin 2x$, determine:

a $\lim\limits_{x \to \infty} e^{-x} \sin 2x$

b $\int_0^c e^{-x} \sin 2x \, dx$

c $\lim\limits_{c \to \infty} \left(\int_0^c e^{-x} \sin 2x \, dx \right)$.

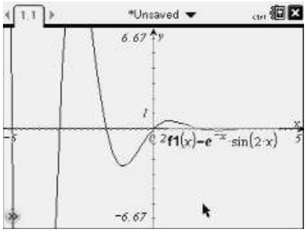
THINK

a 1. Draw a graph of the function. Use a graphics calculator (or in this case refer to the sketch produced in Worked example 10).

2. Examine the extremities of the graph $y = e^{-x} \sin 2x$.

WRITE/DISPLAY

a



f1(x)=e^{-x}·sin(2·x)

As x increases absolutely, the extremities of the envelopes ‘squeeze’ the function to zero.

So $\lim\limits_{x \to \infty} e^{-x} \sin 2x = 0$

THINK	**WRITE/DISPLAY**
b ① Use Euler's formula to express sin $2x$ as the imaginary part of e^{2ix} *Note:* As the expression is a product, the solution could be found by using integration by parts in a cyclic fashion.	**b** $e^{2ix} = \cos 2x + i \sin 2x$ so $\sin 2x = \text{Im}\,(e^{2ix})$ $\int e^{-x} \sin 2x \, dx = \text{Im}\left(\int e^{-x} e^{2ix} \, dx\right)$ $\int e^{-x} e^{2ix} \, dx = \int e^{x(2i-1)} \, dx$
② Find the integral of a new expression where sin $2x$ is replaced with e^{2ix}.	$= \dfrac{e^{x(2i-1)}}{2i-1}$
③ Multiply the numerator and denominator by the conjugate of $2i - 1$ so that the denominator becomes a real number.	$= \dfrac{e^{x(2i-1)}}{2i-1} \cdot \dfrac{2i+1}{2i+1}$ $= \dfrac{(2i+1)e^{x(2i-1)}}{-5}$
④ Simplify so that e^{2ix} can be replaced with $(\cos 2x + i \sin 2x)$. This will make it easier to obtain the imaginary part of the expression.	$= \dfrac{-1}{5}(2i+1)e^{-x}e^{2ix}$ $= \dfrac{-1}{5}e^{-x}(2i+1)(\cos 2x + i \sin 2x)$
⑤ Expand so that the real and imaginary parts can be seen clearly.	$= \dfrac{-1}{5}e^{-x}(2i\cos 2x + 2i^2 \sin 2x + \cos 2x + i \sin 2x)$ $= \dfrac{-1}{5}e^{-x}(-2\sin 2x + \cos 2x + 2i\cos 2x + i \sin 2x)$
⑥ State the imaginary part of the expression.	$\text{Im}\left(\int e^{-x}e^{2ix} \, dx\right) = \dfrac{-1}{5}e^{-x}(2\cos 2x + \sin 2x)$
⑦ Write the integral. Remember to include the constant of integration.	so $\int e^{x} \sin 2x \, dx = \dfrac{-1}{5}e^{-x}(2\cos 2x + \sin 2x) + c$
⑧ Evaluate the definite integral.	$\int_0^c e^{-x} \sin 2x \, dx = \left[-\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x)\right]_0^c$ $= -\frac{1}{5}e^{-c}(\sin 2c + 2\cos 2c) + \frac{1}{5}e^{-0}(\sin 0 + 2\cos 0)$ $= -\frac{1}{5}e^{-c}(\sin 2c + 2\cos 2c) + \frac{1}{5}(0 + 2)$ $= -\frac{1}{5}e^{-c}(\sin 2c + 2\cos 2c) + \frac{2}{5}$

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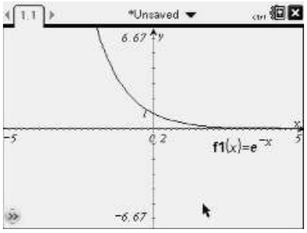
THINK	WRITE/DISPLAY
c ① Substitute for $\int_0^c e^{-x}\sin 2x\ dx$ using the result obtained in part **b**.	**c** $\lim\limits_{c\to\infty}\left(\int_0^c e^{-x}\sin 2x\ dx\right)$ $=\lim\limits_{c\to\infty}\left[-\frac{1}{5}e^{-c}(\sin 2c+2\cos 2c)+\frac{2}{5}\right]$ $=\lim\limits_{c\to\infty}\left[-\frac{1}{5}e^{-c}(\sin 2c+2\cos 2c)\right]+\lim\limits_{c\to\infty}\frac{2}{5}$
② Since e^{-x} approaches 0 as x approaches ∞, $-\frac{1}{5}e^{-c}(\sin 2c+2\cos 2c)$ also approaches 0 as c approaches ∞. This can be verified by viewing the graph of $y=e^{-x}$ or $y=-\frac{1}{5}e^{-c}(\sin 2c+2\cos 2c)$ on a graphics calculator or graphing program. The screen below shows $y=e^{-x}$.	$=0+\frac{2}{5}$ $=\frac{2}{5}$



remember

For damped functions $y=e^{ax}\sin bx$ and $y=e^{ax}\cos bx$:

1. $\sin bx$ and $\cos bx$ are functions which oscillate between ±1.
2. the multiplier function, e^{ax}, alters the amplitude according to the value of a constraining both $\sin bx$ and $\cos bx$ between $\pm e^{ax}$.
3. $\lim\limits_{x\to\infty} e^{ax}\sin bx=\begin{cases}0, & a<0\\ \pm\infty, & a>0\end{cases}$ and $\lim\limits_{x\to\infty} e^{ax}\cos bx=\begin{cases}0, & a<0\\ \pm\infty, & a>0\end{cases}$
4. $\lim\limits_{x\to-\infty} e^{ax}\sin bx=\begin{cases}\pm\infty, & a<0\\ 0, & a>0\end{cases}$ and $\lim\limits_{x\to-\infty} e^{ax}\cos bx=\begin{cases}\pm\infty, & a<0\\ 0, & a>0\end{cases}$
5. $\int e^{ax}\cos bx\ dx=\text{Re}\left(\int e^{x(a+bi)}\ dx\right)$
6. $\int e^{ax}\sin bx\ dx=\text{Im}\left(\int e^{x(a+bi)}\ dx\right)$

EXERCISE 8D Damped functions

WORKED Example 10,11,12

1 a Sketch each of the following graphs over the domain $-\pi \le x \le \pi$.

b Verify your sketches by using a graphing program such as 'Graphmatica' or a graphics calculator.

i $y = e^{x} \sin x$ **ii** $y = e^{x} \cos x$ **iii** $y = e^{-x} \sin x$

iv $y = e^{-x} \cos x$ **v** $y = 2e^{-2x} \sin x$ **vi** $y = e^{-\frac{1}{2}x} \cos x$

vii $y = -2e^{x} \cos 2x$ **viii** $y = e^{\frac{1}{2}x} \sin \frac{1}{2}x$ **ix** $y = \frac{1}{2}e^{-x} \cos \pi x$

WORKED Example 13a

2 a For each of the functions in question **1** above, determine $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.

b From the graphs generated for $y = e^{ax} \sin bx$ and $y = e^{ax} \cos bx$, describe the general features of each graph with respect to its frequency, period and amplitude in the cases that $a < 0$, $a = 0$ and $a > 0$.

WORKED Example 13b, c

3 For each of the following functions, determine:

a $\displaystyle\int_0^c f(x)\,\mathrm{d}x$

b $\displaystyle\lim_{c \to \infty} \left\{ \int_0^c f(x)\,\mathrm{d}x \right\}$.

i $f(x) = e^{-x} \sin x$ **ii** $f(x) = e^{-x} \cos x$ **iii** $f(x) = e^{-x} \sin 2x$
iv $f(x) = e^{-2x} \cos \pi x$ **v** $f(x) = e^{-2x} \sin x$ **vi** $f(x) = e^{-2x} \cos x$
vii $f(x) = e^{-x} \sin \pi x$ **viii** $f(x) = e^{-ax} \sin 2\pi x$ where $a > 0$

ix $f(x) = e^{-\frac{1}{2}x} \cos ax$ where $a > 0$

4 If $L = \displaystyle\int_0^x e^{-t} \sin 2t\,\mathrm{d}t$, show that $\dfrac{\mathrm{d}L}{\mathrm{d}x} = e^{-x} \sin 2x$.

5 $C = \displaystyle\int e^{ax} \cos bx\,\mathrm{d}x$ and $S = \displaystyle\int e^{ax} \sin bx\,\mathrm{d}x$.

a Show that:

i $aC - bS = e^{ax} \cos bx + c$

ii $aS + bC = e^{ax} \sin bx + c$.

b Use the equations from **a** to evaluate:

i $\displaystyle\int e^{-2t} \cos t\,\mathrm{d}t$

ii $\displaystyle\int e^{-2t} \sin t\,\mathrm{d}t$

6 Prove that for $a > 0$:

a $\displaystyle\int_0^\infty e^{-ax} \cos bx\,\mathrm{d}x = \frac{a}{a^2 + b^2}$

b $\displaystyle\int_0^\infty e^{-ax} \sin bx\,\mathrm{d}x = \frac{b}{a^2 + b^2}$.

7 a Use a graphics calculator or other suitable graphing software to graph $y = \frac{\sin x}{x}$ over the domain $-2\pi \le x \le 2\pi$.

b Using your graph, discuss the behaviour of $y = \frac{\sin x}{x}$ as:

i $x \to 0$
ii $x \to \pm\infty$

c It is possible to express sin x as a power series such that

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \quad \text{gives } \sin x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}.$$

i Show that $\int \frac{\sin x}{x}\, dx = \int \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) dx.$

ii Deduce that $\int_0^n \frac{\sin x}{x}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k n^{2k+1}}{(2k+1)(2k+1)!}.$

Approximating the sine integral

$Si(n) = \int_0^n \frac{\sin x}{x}\, dx$ is called the *sine integral*, and for some positive real values of n, it may be approximated using a spreadsheet incorporating the summation formulas from question **7 c** in exercise 8D.
For example, if we select $n = \pi$, then we may write:

$$Si(\pi) = \int_0^{\pi} \frac{\sin x}{x}\, dx \text{ gives } Si(\pi) = \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)(2k+1)!}$$

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1 Go to www.jacplus.com.au, and navigate to the Excel Spreadsheet ‘Summation’ and use it to find, correct to four decimal places, approximations for:
 a $Si(2)$
 b $Si(3)$
 c $Si(10)$
 d $Si(2\pi)$.

2 How many calculation rows are needed in the spreadsheet to give accuracy to four decimal places in each part of question **1**?

3 Use the graphics calculator to find approximations to question 1, correct to four decimal places.

4 Show that if $Si(x) = \int_0^x \frac{\sin t}{t}\, dt$, then $Si(-x) = -Si(x)$.

5 By generating a table of values for $Si(x)$, or by writing a computer program to do the same, graph **a** $Si(x)$, $0 < x \le 40$ and **b** $Si(x)$, $-40 \le x < 0$.

6 Deduce a value for $\lim_{x \to \infty} Si(x)$.

Modelling real-life damped systems

As we have seen, oscillations frequently occur in systems such as mechanical structures and electrical circuits. Usually, frictional forces such as air resistance and internal resistance in the form of heat act to damp, or inhibit, the amplitude of the system's oscillation over time.

In some situations, however, engineers must consider the cumulative effects of external forces that encourage a system to oscillate. In these situations, the amplitude of oscillation may continue to grow unless it can be physically dissipated. In mechanical structures such as bridges, for example, if the cumulative effect of amplitude growth cannot be dissipated, the bridge may disintegrate. This resonance effect caused the Tacoma Narrows Bridge in Washington, USA, to collapse four months after it was opened in 1940. Today, engineers give far greater consideration to the aerodynamics of external forces acting on their bridges.

For our purposes, we shall use the term *damped harmonic motion* for oscillatory motion damped by resistive forces, and we shall model its oscillatory behaviour by expressions of the forms $Ae^{at} \sin bt$ and $Be^{at} \cos bt$ or some combination of these.

For instance, the vertical motion of a motor vehicle's suspension system could be modelled by the expression $y = Ae^{at} \sin bt$, where y is the vertical displacement from the centre of oscillation at time t. By calculating the time-derivative of y, we may also determine the velocity of oscillation about the central point for some appropriate values of the constants.

$$y = Ae^{at} \sin bt$$

Noting that $\frac{dy}{dt} = \dot{y}$,

$$\dot{y} = Aae^{at} \sin bt + Abe^{at} \cos bt$$

or

$$\dot{y} = Ae^{at}(a \sin bt + b \cos bt).$$

In a similar way we may obtain an expression for the acceleration, $\ddot{y}$, of the system.

WORKED Example 14

The vertical motion of the suspension system of a motor vehicle is given by $y = 0.5e^{-t} \sin 4t$ where y is the displacement from the centre of oscillation at time t. For this system calculate, relative to the centre of oscillation, when $t = \pi$:

a **the displacement**
b **the velocity**
c **the acceleration of the system.**

THINK	WRITE
a ① Substitute $t = \pi$ into the equation for y to find the displacement.	**a** $y = 0.5e^{-t} \sin 4t$ When $t = \pi$, $y = 0.5e^{-\pi} \sin 4\pi$ $= 0.5e^{-\pi} \cdot 0$ $= 0$
② Answer the question.	When $t = \pi$, the displacement is 0 so the system is at the centre of oscillation.

Continued over page

THINK	WRITE
b ① Differentiate y to obtain the expression for the velocity, $\dot{y}$.	**b** $y = 0.5e^{-t} \sin 4t$ $\dot{y} = -0.5e^{-t} \sin 4t + 2e^{-t} \cos 4t$
② Substitute $t = \pi$ into the equation for $\dot{y}$ to find the velocity.	When $t = \pi$, $\dot{y} = -0.5e^{-\pi} \sin 4\pi + 2e^{-\pi} \cos 4\pi$ $= -0.5e^{-\pi} \cdot 0 + 2e^{-\pi} \cdot 1$ $= 2e^{-\pi}$
③ Answer the question.	When $t = \pi$, the velocity is $2e^{-\pi}$.
c ① Differentiate $\dot{y}$ to obtain the expression for the acceleration, $\ddot{y}$.	**c** $\dot{y} = -0.5e^{-t} \sin 4t + 2e^{-t} \cos 4t$ $\ddot{y} = 0.5e^{-t} \sin 4t - 2e^{-t} \cos 4t - 2e^{-t} \cos 4t - 8e^{-t} \sin 4t$ $= -7.5e^{-t} \sin 4t - 4e^{-t} \cos 4t$
② Substitute $t = \pi$ into the equation for $\ddot{y}$ to find the acceleration.	When $t = \pi$, $\ddot{y} = -7.5e^{-\pi} \sin 4\pi - 4e^{-\pi} \cos 4\pi$ $= -7.5e^{-\pi} \cdot 0 - 4e^{-\pi} \cdot 1$ $= -4e^{-\pi}$
③ Answer the question.	When $t = \pi$, the acceleration is $-4e^{-\pi}$.

In an electrical system, the oscillatory motion might involve the calculation of the amount of charge present in a circuit at time t, given that the current is modelled by:

$$I = Ae^{at} \cos bt.$$

The charge, q, in an electrical circuit is related to the current, I, by $I = \dfrac{dq}{dt}$.

Hence in our example, $I = \dot{q} = Ae^{at} \cos bt$.

We may determine q by integrating $\dot{q}(t)$ if we have an initial value for q.

WORKED Example 15

In a particular electrical circuit, at time t the current (measured in Amps) is given by $I = -2e^{-0.1t} \cos \pi t$. Initially, that is at time $t = 0$ s, the charge in the system is 10 coulombs.

a What is the current when $t = 5$ s?

b What is the charge in the system when $t = 5$ s?

c How does the charge change over time?

THINK	WRITE
a ① Write the expression for the current, I.	**a** $I = -2e^{-0.1t} \cos \pi t$
② Substitute $t = 5$ s into the given expression to determine the current at this time.	When $t = 5$, $I = -2e^{-0.1 \cdot 5} \cos 5\pi$ $= -2e^{-0.5} \cdot -1$ $= 2e^{-0.5}$ ≈ 1.21

THINK	**WRITE**
③ Answer the question.	When $t = 5$ s, the current is $2e^{-0.5}$ or approximately 1.21 Amps.
b ① As $\dot{q} = I$, write the expression for $\dot{q}$.	**b** $\dot{q} = -2e^{-0.1t} \cos \pi t$
② To find the charge on the plates, we must first obtain q by integrating the expression for I. Use Euler's formula to express $\cos \pi t$ as the real part of $e^{\pi i t}$. *Note:* As the expression is a product, the solution could be found by using integration by parts in a cyclic fashion.	$q = \int (-2e^{-0.1t} \cos \pi t)\, dt$ $e^{\pi i t} = \cos \pi t + i \sin \pi t$ so $\cos \pi t = \mathrm{Re}\,(e^{\pi i t})$ $\int -2e^{-0.1t} \cos \pi t \; dt = \mathrm{Re}\left(\int -2e^{-0.1t} e^{\pi i t} \; dt\right)$ $-2\int e^{-0.1t} e^{\pi i t}\, dx = -2\int e^{t(\pi i - 0.1)} dx$
③ Find the integral of a new expression where $\sin \pi t$ is replaced with $e^{\pi i t}$.	$= \dfrac{-2e^{t(\pi i - 0.1)}}{\pi i - 0.1}$
④ Multiply the numerator and denominator by the conjugate of $\pi i - 0.1$ so that the denominator becomes a real number.	$= \dfrac{-2e^{t(\pi i - 0.1)}}{\pi i - 0.1} \cdot \dfrac{\pi i + 0.1}{\pi i + 0.1}$ $= \dfrac{-2(\pi i + 0.1)e^{t(\pi i - 0.1)}}{-\pi^2 - 0.01}$
⑤ Multiply the numerator and denominator by 100 to clear the decimals.	$= \dfrac{-20(10\pi i + 1)e^{t(\pi i - 0.1)}}{-(100\pi^2 + 1)}$
⑥ Simplify so that $e^{\pi i t}$ can be replaced with $(\cos \pi t + \mathrm{i} \sin \pi t)$. This will make it easier to obtain the real part of the expression.	$= \dfrac{20}{(100\pi^2 + 1)} e^{-0.1t}(10\pi i + 1)(\cos \pi t + i \sin \pi t)$

Continued over page →

THINK	WRITE
⑦ Expand so that the real and imaginary parts can be seen clearly.	$= \frac{20}{(100\pi^2 + 1)}e^{-0.1t}(10\pi i\cos\pi t + 10i^2\pi\sin\pi t + \cos\pi t + i\sin\pi t)$ $= \frac{20}{(100\pi^2 + 1)}e^{-0.1t}(10\pi i\cos\pi t - 10\pi\sin\pi t + \cos\pi t + i\sin\pi t)$
⑧ State the real part of the expression.	$\text{Re}(-2\int e^{-0.1t}e^{\pi it}\,dt) = \frac{20}{(100\pi^2 + 1)}e^{-0.1t}(\cos\pi t - 10\pi\sin\pi t)$
⑨ Write the integral. Remember to include the constant of integration.	so $q = \int(-2e^{-0.1t}\cos\pi t)dt$ $= \frac{20}{100\pi^2 + 1}e^{-0.1t}(\cos\pi t - 10\pi\sin\pi t) + c$ so $q = \frac{20e^{-0.1t}(\cos\pi t - 10\pi\sin\pi t)}{1 + 100\pi^2} + c$
⑩ Determine c by substituting the given information into the equation.	When $t = 0$, $q = 10$, $10 = \frac{20e^0(\cos 0 - 10\pi\sin 0)}{1 + 100\pi^2} + c$ so $c = 10 - \frac{20}{1 + 100\pi^2}$
⑪ Substitute $t = 5$ to calculate the charge at this time.	When $t = 5$, $q = \frac{20e^{-0.5}(\cos 5\pi - 10\pi\sin 5\pi)}{1 + 100\pi^2} + 10 - \frac{20}{1 + 100\pi^2}$ $= \frac{20e^{-0.5}(-1 - 0) - 20}{1 + 100\pi^2} + 10$ $= 10 - \frac{20(1 + e^{-0.5})}{1 + 100\pi^2}$ ≈ 9.97
⑫ Answer the question.	When $t = 5$ s, the charge in the system is $10 - \frac{20(1 + e^{-0.5})}{1 + 100\pi^2}$ or approximately 9.97 coulombs.

THINK | **WRITE**

c State how the charge and current are related.

c Over time, both the current and charge decrease with the charge oscillating towards a limiting value of 9.98 coulombs. This may be observed mathematically from our solution to part **b**:

$$\lim_{t \to \infty} q = \lim_{t \to \infty} \left(\frac{20e^{-0.1t}(\cos \pi t - 10\pi \sin \pi t)}{1 + 100\pi^2} + 10 - \frac{20}{1 + 100\pi^2} \right)$$

$$= 10 - \frac{20}{1 + 100\pi^2} \text{ as } e^{-0.1t} \to 0 \text{ as } t \to \infty.$$

$$\approx 9.98$$

WORKED Example 16

The motion of a pendulum subject to damping has a solution of the form $x = Ae^{-0.2t}\cos\left(3t + \frac{\pi}{4}\right)$ where x cm is the displacement of the pendulum at time t seconds.

If $x = 4$ cm when $t = 0$ seconds, find:

a **the value of the constant A**

b **the velocity of the pendulum when it first passes through its central position**

c **the first value of t for which the pendulum has zero velocity.**

THINK | **WRITE**

a Determine the constant A by substituting $x = 4$ and $t = 0$ into $x(t)$.

a $x = Ae^{-0.2t}\cos\left(3t + \frac{\pi}{4}\right)$

When $t = 0$, $x = 4$,

$$4 = Ae^{0}\cos\left(0 + \frac{\pi}{4}\right)$$

$$= A \cdot 1 \cdot \frac{1}{\sqrt{2}}$$

$$A = 4\sqrt{2}$$

b ① Write the relationship for $x(t)$ with A replaced with $4\sqrt{2}$.

b $x(t) = 4\sqrt{2}\, e^{-0.2t} \cos\left(3t + \frac{\pi}{4}\right)$

② Find the velocity by differentiating $x(t)$ with respect to t.

$$\dot{x}(t) = -0.8\sqrt{2}\, e^{-0.2t} \cos\left(3t + \frac{\pi}{4}\right) - 12\sqrt{2}\, e^{-0.2t} \sin\left(3t + \frac{\pi}{4}\right)$$

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THINK	WRITE
③ Since the pendulum passes through its central position when $x(t) = 0$, write an equation and solve to find the first value of t.	The pendulum passes through its central position when $x(t) = 0$. $4\sqrt{2}\,e^{-0.2t}\cos\left(3t+\frac{\pi}{4}\right) = 0$ $\cos\left(3t+\frac{\pi}{4}\right) = 0$ $3t+\frac{\pi}{4} = \frac{\pi}{2}$ $3t = \frac{\pi}{4}$ $t = \frac{\pi}{12}$
④ Having found the first instance of the pendulum passing through its central position, substitute this value of t into the expression for the velocity.	When $t = \frac{\pi}{12}$, $\dot{x}(t) = -0.8\sqrt{2}\,e^{-\frac{\pi}{60}}\,e^{-0.2t}\cos\frac{\pi}{2} - 12\sqrt{2}\,e^{-\frac{\pi}{60}}\sin\frac{\pi}{2}$ $= -12\sqrt{2}\,e^{-\frac{\pi}{60}}$
⑤ Write the answer.	The velocity of the particle when it passes through its central position is given by $-12\sqrt{2}\,e^{-\frac{\pi}{60}}$ cm/s (or approximately -16.105 cm/s).
c ① The pendulum has zero velocity when $\dot{x}(t) = 0$. So we must find the first value of t for which $\dot{x}(t) = 0$.	**c** For $\dot{x}(t) = 0$ $-0.8\sqrt{2}\,e^{-0.2t}\cos\left(3t+\frac{\pi}{4}\right) - 12\sqrt{2}\,e^{-0.2t}\sin\left(3t+\frac{\pi}{4}\right) = 0$ $-0.8\sqrt{2}\,e^{-0.2t}\left[\cos\left(3t+\frac{\pi}{4}\right) + 15\sin\left(3t+\frac{\pi}{4}\right)\right] = 0$ $\cos\left(3t+\frac{\pi}{4}\right) + 15\sin\left(3t+\frac{\pi}{4}\right) = 0$

THINK	WRITE
② Divide both sides of the equation by $\cos\left(3t+\frac{\pi}{4}\right)$ to produce one term rather than two containing the variable t.	Dividing by $\cos\left(3t+\frac{\pi}{4}\right)$ gives $1 + 15\tan\left(3t+\frac{\pi}{4}\right) = 0$ $\tan\left(3t+\frac{\pi}{4}\right) = -\frac{1}{15}$ $3t + \frac{\pi}{4} = \tan^{-1}\left(-\frac{1}{15}\right)$
③ Note that $\tan^{-1}\left(-\frac{1}{15}\right)$ is an angle in the second or fourth quadrants. As we are seeking the first value of t, choose the second quadrant angle. Rewrite $\tan^{-1}\left(-\frac{1}{15}\right)$ as $\pi - \tan^{-1}\frac{1}{15}$; that is, π – (first quadrant angle).	$3t + \frac{\pi}{4} = \pi - \tan^{-1}\frac{1}{15}$ $3t = \frac{3\pi}{4} - \tan^{-1}\frac{1}{15}$ $t = \frac{\pi}{4} - \frac{1}{3}\tan^{-1}\frac{1}{15}$
④ Write the answer.	The first instance where the pendulum has zero velocity is at $t = \frac{\pi}{4} - \frac{1}{3}\tan^{-1}\frac{1}{15}$ (or approximately 0.763 s).

remember

Modelling damping

1. $y = Ae^{at}\sin bt$
   (a) By calculating the time-derivative of y, we may also determine the velocity of oscillation about the central point for some appropriate values of the constants.
   (b) Noting that $\frac{dy}{dt} = \dot{y}$.

$$\dot{y} = Aae^{at}\sin bt + Abe^{at}\cos bt$$
$$\therefore\ \dot{y} = Ae^{at}(a\sin bt + b\cos bt).$$

2. $I = Ae^{at}\cos bt$
   (a) In an electrical system the amount of charge present in a circuit at time t may be modelled by $I = Ae^{at}\cos bt$.
   (b) The charge, q, in an electrical circuit is related to the current, I, by $I = \frac{dq}{dt}$; that is, $\dot{q} = Ae^{at}\cos bt$ gives $q = \int Ae^{at}\cos bt$.
3. We may determine q by integrating $\dot{q}$ if we have an initial value for q.

EXERCISE 8E Modelling real-life damped systems

WORKED Example 14

1 The vertical motion of the suspension system of a motor vehicle may be represented by $y = 0.4e^{-0.2t} \sin 2\left(t - \frac{\pi}{8}\right)$, where y cm is the displacement from the centre of oscillation at time t seconds. When $t = \frac{\pi}{4}$ seconds, calculate:

- a the displacement
- b the velocity
- c the acceleration of the system.

2 The motion of a spring with a mass immersed in a viscous fluid has solution of the form $y = 5e^{kt} \sin\left(\pi t - \frac{\pi}{4}\right)$, where y cm is the displacement of the pendulum at time t seconds, relative to its central position. If $y = 1$ cm when $t = 1$ second, find each of the following:

- a the value of the constant k
- b the velocity and acceleration of the spring:
  - i at time $t = 1$
  - ii as it passes through its central position for the first time
- c the first value of t for which $y = 1.25$ cm.
  Hint: Use *Solver* in the TI graphics calculator.

3 In a low frequency oscillating circuit at time t, the charge is given by

$$q = -5e^{-0.1t} \sin\left(\frac{\pi t}{3} - \frac{\pi}{2}\right).$$

- a Find the value of the charge when $t = \frac{1}{2}$.
- b If the current in the circuit is given by $\dot{q}(t)$, determine its magnitude at time $t = 1$.
- c Determine how long it takes for the current to be reduced to zero.

WORKED Example 15

4 In an LCR circuit at time t, the current is given by $I = 20e^{-0.2t} \sin\left(\frac{\pi t}{4}\right)$. Initially the charge in the circuit is 100.

- a When $t = 3$, what is the value of the current?
- b What is the charge in the circuit when $t = 5$?
- c Determine $\lim\limits_{t \to \infty} q(t)$.
- d Calculate the long-term percentage change in the charge.

WORKED Example 16

5 The motion of a pendulum subject to damping has a solution of the form $x = Ae^{-0.2t} \cos\left(t + \frac{\pi}{4}\right)$ where x centimetres is the displacement of the pendulum at time t seconds. If $x = 5$ cm when $t = 0$ seconds, find:

- a the value of the constant A
- b the velocity of the pendulum at time $t = \frac{\pi}{4}$ seconds

c the first value of t for which:
  i the pendulum has zero velocity
  ii the pendulum is located at $x = 0$ cm

d the acceleration of the pendulum when $t = \dfrac{3\pi}{4}$ seconds.

6 The velocity at time t seconds of a damped spring oscillating in the vertical plane is given by $\dot{y} = Ae^{kt}\cos\left(t + \dfrac{\pi}{6}\right)$ where $\dot{y}(0) = 1$ cm/s and $\dot{y}\left(\dfrac{\pi}{2}\right) = -\sqrt{3}\,e^{-\frac{\pi}{4}}$ cm/s.

a Find the value of the constants A and k.

b Determine the acceleration when $t = \dfrac{\pi}{6}$ seconds.

c Find the displacement of the spring when $t = \dfrac{\pi}{2}$ seconds given that

$y\left(\dfrac{\pi}{6}\right) = 2e^{-\frac{\pi}{12}}$ cm.

d Using a graphics calculator, obtain the first value of t for which $y(t) = 1$ cm.

7 An oscillating damped circuit has a frequency of 15 cycles per second. The current in the circuit at time t seconds is given by $I(t) = -Ae^{kt}\sin\left(\omega t - \dfrac{3\pi}{4}\right)$.

a Show that the angular frequency of the system is given by $\omega = 30\pi$ cycles per second.

b Given that $I(0) = 10\sqrt{2}$ and $I(1) = 5.20$:
  i find the values of A and k
  ii find the current when $t = 3$ seconds.
  iii When is the current equal to 3 for the first time?

c If $q(t)$ represents the charge in the system at time t seconds, and $\dot{q}(t) = I$:
  i if $q(0) = 0$, show that

$$q(t) = \frac{20e^{-t}}{1 + 900\pi^2}\left[\sin\left(30\pi t - \frac{3\pi}{4}\right) + 30\pi\cos\left(30\pi t - \frac{3\pi}{4}\right)\right] + 0.1516$$

  ii using a graphics calculator, find t if $q(t) = 0.15$.

d Determine $\lim\limits_{t \to \infty} q(t)$.

8 A mass is attached to a spring system and is set oscillating in the horizontal plane. After time t seconds, the velocity of the system about its central position is given by:

$$\dot{x} = Ae^{kt}\cos 2\pi t \text{ cm/s.}$$

If $\dot{x}(0) = 4$ cm/s and $\dot{x}(2) = 4e^{-\frac{1}{2}}$ cm/s:

a find the value of the constants A and k

b find the first value of t for which $x(t) = 5.05$ cm given that $x(0) = 5$ cm

c find the velocity of the system when $x(t) = 5.05$

d show that $4\ddot{x} + \dot{x} + 32\pi e^{-\frac{1}{4}t}\sin(2\pi t) = 0$

e find $\lim\limits_{t \to \infty} x(t)$.

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WorkSHEET 8.2

Malthusian models and limited growth

Population changes over time may be modelled by a simple differential equation or by a system of such equations, given certain assumptions. These assumptions reflect the characteristics of the target population. For instance, a model may need to incorporate something about ecological and economic interaction, population density, resource scarcity and so on. It is difficult to build a model that gives appropriate consideration to all possible exigencies in all circumstances.

The British economist Thomas Malthus (1776–1834) observed that biological populations tended to increase at a rate proportional to the population size. If we consider a small time interval, say δt, in which the birth and death rates are both proportional to the population size, say $N(t)$, then the change in the population may be expressed, for some positive constants a and b, as $\delta N = aN\delta t - bN\delta t$. This may be expressed as the more familiar $\frac{\delta N}{\delta t} = (a - b)N$. If $\delta t \to \infty$, then we may say $\frac{dN}{dt} = kN$ where $k = a - b$. Irrespective of k being positive or negative, a solution for $N(t)$ may be found as follows:

$$\begin{aligned} \frac{dN}{dt} &= kN \\ \int \frac{dN}{dt} &= \int k\,dt \\ \ln N &= kt + c \\ N &= e^{kt+c} \\ &= e^{kt} \cdot e^{c}. \end{aligned}$$

If $N = N_0$ when $t = 0$ then $N_0 = e^c$

so $$N(t) = N_0\,e^{kt}.$$

In the Malthusian model, the rate of change of the population is proportional to the population at time t.

Apart from population growth, mathematicians use the Malthusian approach to model and predict other changes including learning efficiency and rumour spread. These models may be reliable predictors provided they involve small populations.

WORKED Example 17

In 1961, the Earth's population was estimated to be $3060 \cdot 10^6$ and during that decade, the population was increasing at a rate of 2% per year.

a Based on these statistics, what would be the Earth's population in the year 2020?

b If the total surface area of our planet is about $1.67 \cdot 10^5$ billion square metres, how many square metres will be available for each person in the year 2020?

THINK	WRITE
a ① Write a relationship to reflect that the rate of increase of the population is 2% (or 0.02) of the existing population per annum.	**a** Let $P(t)$ be the population at time, t. $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.02P$
② Integrate to find an expression for $P(t)$.	$\int \frac{\mathrm{d}P}{P} = \int 0.02\,\mathrm{d}t$ $\ln P = 0.02t + c$ $P = e^{0.02t + c}$ $= e^{0.02t} \cdot e^{c}$
③ Determine the constant c by considering that $P(0) = 3060 \cdot 10^6$.	Now $P(0) = 3060 \cdot 10^6$ so $3060 \cdot 10^6 = e^{0 + c}$ $e^c = 3060 \cdot 10^6$
④ Write the relationship for $P(t)$.	$P(t) = 3060 \cdot 10^6 \cdot e^{0.02t}$
⑤ Determine the population predicted in 2020 noting that $t = 59$.	For the year 2010, $t = 59$ $P(59) = 3060 \cdot 10^6 \cdot e^{0.02 \cdot 59}$ $\approx 9958 \cdot 10^6$
⑥ Answer the question.	In the year 2020, the Earth's population would be approximately $9958 \cdot 10^6$.
b ① Find the area per person by dividing the total area available by the population.	**b** Area per person $= \frac{\text{Total area}}{\text{Number of people}}$ $= \frac{1.67 \cdot 10^5 \text{ billion}}{9958 \cdot 10^6}$ $= \frac{1.67 \cdot 10^{14}}{9958 \cdot 10^6}$ $= 16\,770 \text{ m}^2$
② Answer the question.	Approximately $16\,770 \text{ m}^2$ per person will be available in 2020.

WORKED Example 18

An employee in a production plant is performing a job more efficiently each day in such a way that if y units are produced after t days, then the expression $\frac{dy}{dt} = k(80 - y)$ represents his increasing productivity for some positive constant k and $y \le 80$ for all t.

The employee produces 15 units on the first day and 40 units per day after being on the job for 20 days.

a **How many units per day will the person produce after being on the job 25 days?**
b **How long will it take for him to be working at 70% of his full potential?**

THINK	WRITE
a ① Separate the variables and integrate $\frac{dy}{dt} = k(80 - y)$.	**a** $\frac{dy}{dt} = k(80 - y)$ $\int \frac{dy}{(80 - y)} = k\,dt$ $-\ln(80 - y) = kt + c$, since $y < 80$ $80 - y = e^{-kt - c}$
② Use the given data to find the integration constant, c; that is, $y(0) = 15$.	When $t = 0$, $y = 15$ units $65 = e^{-c}$ That is, $80 - y = 65e^{-kt}$
③ Determine k using the second set of data; that is, $y(20) = 40$.	When $t = 20$, $y = 40$ $40 = 65\,e^{-20k}$ $e^{-20k} = \frac{40}{65}$ $-20k = \ln \frac{40}{65}$ $k = -\frac{1}{20} \ln \frac{8}{13}$ $\approx 0.024\,28$ $y(t) = 80 - 65e^{-0.024\,28t}$
④ To determine y after being on the job 25 days, we must calculate $y(25)$.	$y(25) = 80 - 65e^{-0.024\,28 \cdot 25}$ ≈ 44.57
⑤ Answer the question.	After 25 days the worker will be producing 44 units per day.
b ① Determine the number of units produced by the employee working at 70% of his full potential.	**b** Full potential is 80 units per day so 70% of full potential is 56 units per day. That is, $y(t) = 56$.
② Substitute $y = 56$ into the equation for $y(t)$ and solve for t.	$56 = 80 - 65e^{-0.024\,28t}$ $e^{-0.024\,28t} = \frac{24}{65}$ $-0.024\,28t = \ln \frac{24}{65}$ $t \approx 41.04$
③ Answer the question.	About 42 days will be required for the worker to achieve 70% proficiency.

WORKED Example 19

In a termite population, the rate of growth of the number of reproductive termites at time *t* days after 200 days is given by:

$$\dot{r}(t) = 0.05e^{0.02t}(200 - t),\ r(0) = 1000.$$

a **Show that $r(t) = 2.5e^{0.02t}(250 - t) + 375$.**

b **What is the number of reproductive termites when $t = 60$ days?**

c **Use a graphics calculator to find when the number of reproductive termites is:**

i **3000** **ii** **at least 3000** **iii** **the greatest.**

THINK	WRITE/DISPLAY
a ① Integrate $\dot{r}(t)$ to find $r(t)$ by applying the integration-by-parts formula.	**a** $\dot{r}(t) = 0.05e^{0.02t}(200 - t)$ $r(t) = \int 0.05e^{0.02t}(200 - t)\, dt$ $= uv - \int v\frac{du}{dt}\, dt$ where $u = 200 - t$, $\frac{dv}{dt} = 0.05e^{0.02t}$, $\frac{du}{dt} = -1$ and $v = 2.5e^{0.02t}$ $r(t) = (200 - t) \cdot 2.5e^{0.02t} - \int 2.5e^{0.02t}(-1)\, dt$ $= 2.5e^{0.02t}(200 - t) + \int 2.5e^{0.02t}\, dt$ $= 500e^{0.02t} - 2.5te^{0.02t} + 125e^{0.02t} + c$ $= 625e^{0.02t} - 2.5te^{0.02t} + c$ $= 2.5e^{0.02t}(250 - t) + c$
② Apply the boundary condition, $r(0) = 1000$, to determine the constant of integration, c.	Now $r(0) = 1000$ so $1000 = 2.5e^{0} \cdot 250 + c$ $c = 1000 - 625$ $= 375$
③ State the final relationship for $r(t)$.	$r(t) = 2.5e^{0.02t}(250 - t) + 375$
b ① Substitute $t = 60$ into the relationship for $r(t)$ and evaluate.	**b** $r(60) = 2.5e^{0.02 \cdot 60} \cdot 190 + 375$ $= 475e^{1.2} + 375$ ≈ 1952
② Answer the question.	There are about 1952 reproductive termites.
c ① To determine graphically when $r(t) = 3000$ with a graphics calculator, find the points of intersection of two graphs with equations $y_1 = 2.5e^{0.02t}(250 - t) + 375$ and $y_2 = 3000$. This will provide the t values for part **i**.	**c**

Continued over page →

THINK | **WRITE/DISPLAY**

For the CASIO *fx*-9860G AU

1. Go to setup and ensure that the calculator is in radians.

2. To sketch the graphs, press:
   - (MENU)
   - (5) (GRAPH)

   Complete the entry lines as:
   Y1 = $2.5\,e^{0.02x}(250 - x) + 375$
   Y2 = 3000
   Press (EXE) after each line.

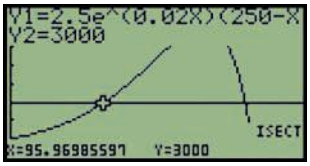
```
Graph Func  :Y=
Y1■2.5e^(0.02X)([—]
Y2=3000▮
Y3:            [—]
Y4:            [—]
Y5:            [—]
Y6:            [—]
```

3. Enter the V-Window settings shown.
   Note: The Y scale selected was 1000.

```
View Window
Xmin  :0
 max  :300
 scale:50
 dot  :2.38095238
Ymin  :0
 max  :8000
```

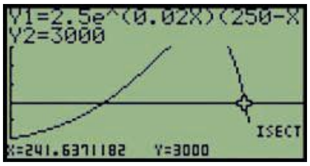
4. To find the first point of intersection, press:
   - (F5) (G-solv) followed by
   - (F5) (ISCT).

   The first point of intersection will be displayed.



```
Y1=2.5e^(0.02X)(250-X
Y2=3000
                 ISECT
X=95.96985597  Y=3000
```

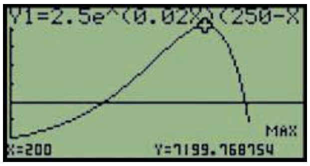
5. Press the right arrow to give the second point of intersection.



```
Y1=2.5e^(0.02X)(250-X
Y2=3000
                 ISECT
X=241.6371182  Y=3000
```

6. To locate the maximum point, press:
   - (F5) (G-solv)
   - (F2) (MAX)

   This gives the time taken for the number of termites to reach a maximum (required for part **iii**).



```
Y1=2.5e^(0.02X)(250-X
                   MAX
X=200      Y=7199.768754
```

7. Answer the questions.

i $r(t) = 3000$ when $t \approx 96$ and when $t \approx 241$.
There will be 3000 reproductive termites when $t = 96$ and 242 days.

ii For $r(t) \geq 3000$, $96 \leq t \leq 241$.
There will be at least 3000 reproductive termites for t values between and including 96 and 242 days.

iii Maximum $r(t) \approx 7200$ when $t = 200$.
There will be the maximum number of termites when $t = 200$ days.

THINK | **WRITE/DISPLAY**

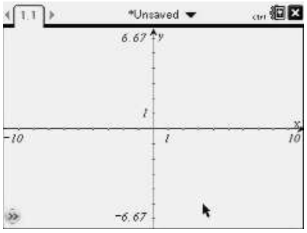
For the TI-Nspire CAS

① On a Graphs page complete the entry lines as:

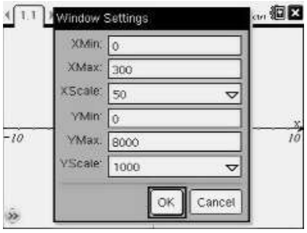
$f1(x) = 2.5\,e^{0.02x}(250 - x) + 375$

$f2(x) = 3000$

Press ENTER enter after each line.



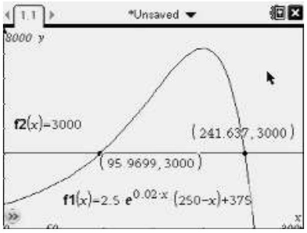
② Enter the Window Settings as shown. Then select OK.



Window Settings
XMin: 0
XMax: 300
XScale: 50
YMin: 0
YMax: 8000
YScale: 1000
OK Cancel

③ To find the points of intersection, press:
- MENU menu
- 7: Points & Lines 7
- 3: Intersection point(s) 3

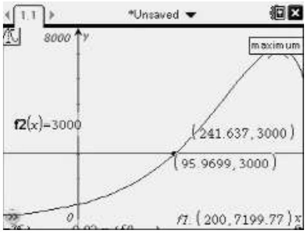
Select the two functions and the points of intersection will be displayed.



④ To find the maximum value, press:
- MENU menu
- 5: Trace 5
- 1: Graph Trace 1

As the graph is traced, 'maximum' will be highlighted when the maximum is reached.

This gives the time taken for the number of termites to reach its maximum (required for part **iii**).



⑤ Answer the questions.

i $r(t) = 3000$ when $t \approx 96$ and when $t \approx 241$. There will be 3000 reproductive termites when $t = 96$ and 242 days.

ii For $r(t) \geq 3000$, $96 \leq t \leq 241$.

There will be at least 3000 reproductive termites for t values between and including 96 and 242 days.

iii Maximum $r(t) \approx 7200$ when $t = 200$. There will be the maximum number of termites when $t = 200$ days.

remember

Malthusian population model

1. $\frac{dN}{dt} = kN$ and if $N = N_0$ when $t = 0$, then $N(t) = N_0e^{kt}$.
2. These models may be reliable predictors, provided that populations are small.

EXERCISE 8F Malthusian models and limited growth

WORKED Example 17

1 In 2009, the Earth's population was estimated to be $6.76 \cdot 10^9$. During the first decade of the 21st century, the population was increasing at a rate of 1.14% per year. If we assume that the rate of increase remains the same, what would be the projected population in the year 2014?

WORKED Example 18

2 A new worker on an assembly line can perform a particular task in such a way that if x units are completed in a day, then after t days on the assembly line $\frac{dx}{dt} = k(90 - x)$ for some positive constant k and $x \le 90$ for all t.
The worker produces 20 units on the first day and 40 units per day after being on the job 10 days.

a How many units per day will she produce after being on the job 25 days?
b How long will it take for her to be working at 70% of her full potential?

WORKED Example 19

3 In a fruit-fly population, the rate of growth of the number of reproductive flies at time t days after 100 days is given by $\dot{r}(t) = 0.03e^{0.01t}(100 - t)$, $r(0) = 250$.

a Show that $r(t) = 3e^{0.01t}(200 - t) - 350$.
b What is the number of reproductive fruit flies when $t = 30$ days?
c Use a graphics calculator to find when the number of reproductive flies is:
  i 400 **ii** at least 400 **iii** the greatest **iv** $300 \le r(t) \le 500$.

4 The rate of natural increase of a population of a particular city on the east coast is proportional to the population. If the population increases from 25 000 to 50 000 over 25 years, when will the population reach 100 000?

5 In a bacterial culture, where the rate of growth is proportional to the number present, scientists observe that after one hour the number of bacteria in the culture has *trebled*. If at the end of four hours the number of bacteria present is five million, how many bacteria were present initially?

6 A student has 4 hours to study for an examination and during that time she needs to memorise a set of 50 facts. According to education experts, the rate at which a person may memorise a set of facts is proportional to the number of facts remaining to be memorised. Thus if a student memorises m facts in t minutes,

$$\frac{dm}{dt} = k(50 - m), \text{ where } k > 0 \text{ and } m < 50 \text{ for all } t.$$

a If the student has initially memorised zero facts, but manages to memorise 15 facts in the first 20 minutes, how many facts will she have memorised after:
  i 1 hour? **ii** 4 hours?
b How long will it take her to memorise 90% of her facts?

7 In a wasp population the number of reproductive wasps after 300 days is given by $r(t) = 0.05e^{0.04t}(100 - t) + 10$ for $t \le 100$.
By making use of appropriate graphic calculator functions, find the range of values of t for which the rate of growth of reproductive wasps is greater than 0.25 wasps per day.

8 A telecommunications company is trying to advertise a new product through the internet. Modelling by the advertising executives suggests that the rate of exposure to new people is proportional to the number of people, $N(t)$, at time t, who have not seen the product out of L possible viewers. No one is aware of the product at the start of the advertising campaign and after 10 days 40% of L are aware of the product.

- **a** Show that $\frac{dN}{dt} = k(L - N)$, $N(0) = 0$, $N(10) = 0.4L$, $k > 0$.
- **b** Find the number of people who are aware of the product after t days of advertising.
- **c** How long will it take for 75% of L to have viewed the product?
- **d** Determine $\lim_{t \to \infty} N(t)$.

9 For a person learning to type using a new touch typing technique, it was discovered that the rate of improvement after t hours of practice is given by:

$\frac{dN}{dt} = 3 + 5e^{-0.2t}$ where $N(t)$ is the number of words typed per minute.

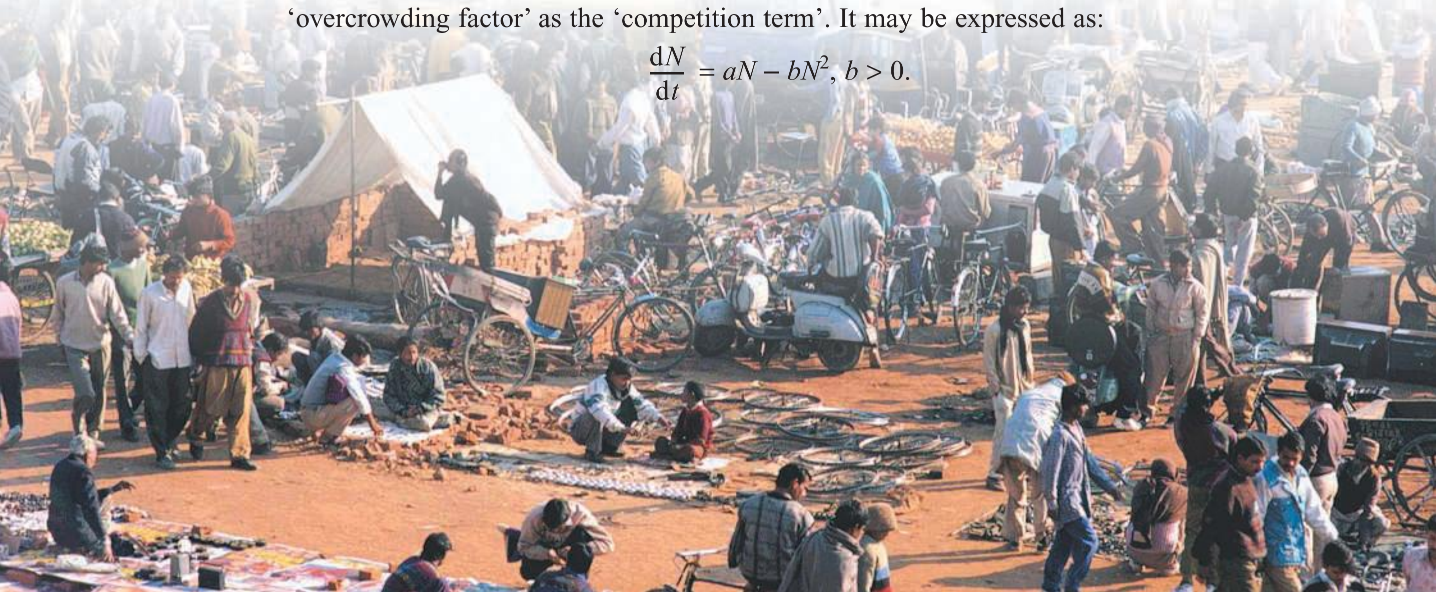
- **a** What is the rate of improvement after 12 hours of practice?
- **b** If the number of words typed per minute was initially 40, what is the percentage improvement in the typing speed after 12 hours of practice?

10 After a publicity campaign for a particular political party ended, the attendance at their subsequent meetings decreased at a rate proportional to each meeting's size. If the initial meeting attracted 3000 people, and the attendance at the fourth meeting was 1500, what is the expected attendance at the fifth meeting?

Logistic model and population growth

All populations have a limiting size determined by the carrying capacity of the population and the adaptability of the population to environmental conditions. To account for factors like these, demographers build in an 'overcrowding' factor into a Malthusian or exponential model. The simplest overcrowding factor is proportional to the square of the current population. P. F. Verhulst (1804–49), a Belgian biologist, referred to this 'overcrowding factor' as the 'competition term'. It may be expressed as:

$$\frac{dN}{dt} = aN - bN^2,\ b > 0.$$



This model is called the *Verhulst or Logistic model for population growth*. By varying the parameters a and b, we may observe changes in the limiting population size. Before we can do this we need to find the generalised solution to the above differential equation.

We will begin by stating that when $t = 0$, $N = N_0$ and writing that:

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= aN - bN^2 \\ &= N(a - bN).\end{aligned}$$

From this we may say:

$$\int \frac{\mathrm{d}N}{N(a-bN)} = \int \mathrm{d}t.$$

Consider breaking the expression $\frac{1}{N(a-bN)}$ into partial fractions:

$$\begin{aligned}\frac{1}{N(a-bN)} &= \frac{A}{N} + \frac{B}{a-bN} \\ &= \frac{A(a-bN)+BN}{N(a-bN)}.\end{aligned}$$

Considering the numerators:

$$1 = A(a - bN) + BN.$$

When $N = 0$,

$$\begin{aligned}1 &= Aa \\ A &= \frac{1}{a}\end{aligned}$$

When $N = \frac{a}{b}$,

$$\begin{aligned}1 &= A(a - a) + \frac{Ba}{b} \\ 1 &= \frac{Ba}{b} \\ B &= \frac{b}{a}\end{aligned}$$

so $$\frac{1}{N(a-bN)} = \left(\frac{\frac{1}{a}}{N} + \frac{\frac{b}{a}}{a-bN}\right)$$

and $$\begin{aligned}\int \frac{\mathrm{d}N}{N(a-bN)} &= \int\left(\frac{\frac{1}{a}}{N} + \frac{\frac{b}{a}}{a-bN}\right)\mathrm{d}N \\ &= \frac{1}{a}\ \ln N - \frac{1}{a}\ \ln (a - bN) + c \\ &= \frac{1}{a}\ \ln\left(\frac{N}{a-bN}\right) + c\end{aligned}$$

Now $$\int \frac{\mathrm{d}N}{N(a-bN)} = \int \mathrm{d}t \text{ becomes:}$$

$$\frac{1}{a}\ln\left(\frac{N}{a-bN}\right) + c = t$$

From this expression, we may obtain:

$$\frac{a-bN}{N} = ke^{-at}, \text{ for some constant, } k$$

From which we may see that:

$$N(t) = \frac{a}{b+ke^{-at}}$$

Using the initial conditions; that is, when $t = 0$, $N = N_0$

$$N_0 = \frac{a}{b+k}$$

so $$k = \frac{a-N_0b}{N_0}$$

With some algebraic manipulation, it follows that:

$$N(t) = \frac{\frac{a}{b}N_0}{N_0 + \left(\frac{a}{b} - N_0\right)e^{-at}}$$

Two significant features emerge from this solution:

Feature 1: $\lim\limits_{t\to\infty} N(t) = \frac{a}{b}$

This value is called *the limiting population.*

Feature 2: If $\frac{\mathrm{d}N}{\mathrm{d}t} = N(a - bN)$ then it follows, on implicitly differentiating this expression, that $\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = \frac{\mathrm{d}N}{\mathrm{d}t}(a - 2bN)$. Further, when $\frac{\mathrm{d}^2N}{\mathrm{d}t^2} = 0$ and provided that $\frac{\mathrm{d}N}{\mathrm{d}t} \neq 0$, then N will be maximal.

Hence, when N is maximal, $N_{\text{maximal}} = \frac{a}{2b}$. In other words, this *maximal value* is always half the limiting value of the population.

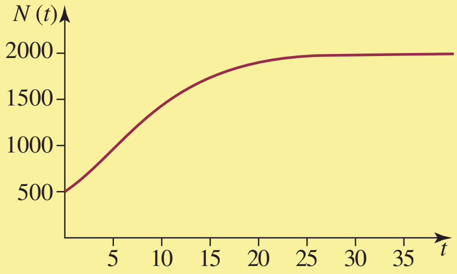
By considering the solution to the logistic equation,

$$N(t) = \frac{\frac{a}{b}N_0}{N_0 + \left(\frac{a}{b} - N_0\right)e^{-at}}$$

we can explore the carrying capacity of a population for different values of a and b. The constants a and b are called the *vital coefficients* of the population. Factors that influence these values include low energy, pollution and technological developments. Consequently, the values assigned to a and b need periodic appraisal to ensure the relevance of the model. Like the Malthusian model, the logistic law is not a great predictor of the world's population.

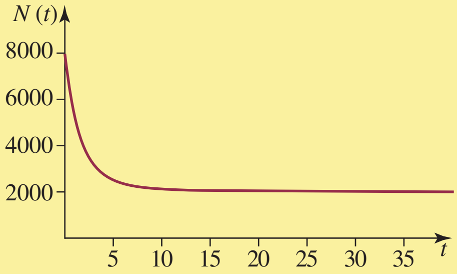
Logistic curves

To illustrate the general shape of the logistic curve we will consider a model in which $a = 0.2$, $b = 0.0001$, and $N_0 = 500$. Incorporating these values in the logistic equation and using suitable graphing software or a graphics calculator, produce the following graphs.



$N(t)$ with $N_0 < \frac{a}{b}$.

Now, for the same values of a and b, select $N_0 = 8000$ and observe a different relationship.



$N(t)$ with $N_0 > \frac{a}{b}$.

The growth curve of a typical logistic model demonstrates a slow establishment phase, a rapid expansion phase, and a long-term entrenchment phase while the population approaches its limiting size.

1. Record your observations.
2. What is the limiting population shown in each graph?
3. How does this value compare with the value of N_0 in each case?
4. Research: Explore the effects of changing a, b and c on the graph of a logistic model at http://orion.math.iastate.edu/algebra/sp/xcurrent/applets/logisticmodels.html.
5. Investigate mathematical careers involving modelling.

WORKED Example 20

A population exhibits logistic growth such that $\frac{dN}{dt} = N - 0.002N^2$, $N(0) = 100$.

- **a** **Solve for $N(t)$ by using the logistic equation directly.**
- **b** **Verify the solution by differentiation.**
- **c** **Solve for $N(t)$ by applying the integral calculus.**
- **d** **Using either a graphics calculator or the program 'Graphmatica', draw $N(t)$.**
- **e** **Using a graphics calculator, determine when the population is maximal.**

THINK	WRITE/DISPLAY
a 1 Compare the given equation with the generalised logistic equation and write the values for a and b.	**a** $\frac{dN}{dt} = N - 0.002N^2,\ N_0 = 100$ $= aN - bN^2$ where $a = 1,\ b = 0.002$
2 Write the general equation for $N(t)$ and substitute for a, b and N_0.	$N(t) = \dfrac{\frac{a}{b}N_0}{N_0 + \left(\frac{a}{b} - N_0\right)e^{-at}}$ $= \dfrac{50\,000}{100 + 400e^{-t}}$
3 Simplify.	$= \dfrac{500}{1 + 4e^{-t}}$
b 1 Use the relationship for N from part **a** and differentiate to obtain $\frac{dN}{dt}$.	**b** $N(t) = 500(1 + 4e^{-t})^{-1}$ $\frac{dN}{dt} = -1 \cdot -4e^{-t} \cdot 500\,(1 + 4e^{-t})^{-2}$ $= \dfrac{2000e^{-t}}{(1 + 4e^{-t})^2}$
2 Substitute for N in the expression $N - 0.002N^2$.	As $N(t) = \dfrac{500}{1 + 4e^{-t}}$, $N - 0.002N^2$ $= \dfrac{500}{1 + 4e^{-t}} - 0.002 \cdot \left(\dfrac{500}{1 + 4e^{-t}}\right)^2$
3 Simplify and show that this result is the same as that obtained for $\frac{dN}{dt}$ in step 1.	$= \dfrac{500(1 + 4e^{-t}) - 500}{(1 + 4e^{-t})^2}$ $= \dfrac{2000e^{-t}}{(1 + 4e^{-t})^2}$ Therefore $\frac{dN}{dt} = N - 0.002N^2$.

Continued over page →

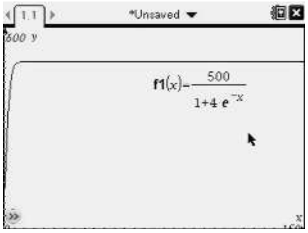
THINK	**WRITE/DISPLAY**
c ① To facilitate the integration using partial fractions, remove the decimals to make the arithmetic more manageable; that is, multiply the numerator and the denominator by 500.	**c** $\frac{\mathrm{d}N}{\mathrm{d}t} = N - 0.002N^2,\ N(0) = 100$ $\int \frac{\mathrm{d}N}{N(1 - 0.002N)} = \int \mathrm{d}t$ $\int \frac{500}{N(500 - N)}\ \mathrm{d}N = \int \mathrm{d}t$
② Consider the integrand on the left-hand side and break it into partial fractions.	Consider $\frac{500}{N(500 - N)}$ $\frac{500}{N(500 - N)} \equiv \frac{A}{N} + \frac{B}{500 - N}$ $= \frac{A(500 - N) + BN}{N(500 - N)}$ So $500 \equiv A(500 - N) + BN$
③ Determine the constants A and B.	When $N = 0$, $500 = 500A + 0$ so $A = 1$. When $N = 500$, $500 = 0 + 500B$ so $B = 1$.
④ Substitute the partial fractions into the integral statement.	Hence $\int \frac{\mathrm{d}N}{N(1 - 0.002N)} = \int \mathrm{d}t$ $\int \left(\frac{1}{N} + \frac{1}{500 - N}\right)\ \mathrm{d}N = \int \mathrm{d}t$
⑤ Perform the integration on both sides of the equation and include a combined constant of integration on the right-hand side only.	$\ln N - \ln(500 - N) = t + c$ $\ln\left(\frac{N}{500 - N}\right) = t + c$
⑥ Apply the boundary condition of $N(0) = 100$ to determine c.	When $t = 0$, $N = 100$ so $c = \ln \frac{1}{4}$.
⑦ Rearrange and simplify to make N the subject of the equation.	$\ln\left(\frac{N}{500 - N}\right) = t + \ln \frac{1}{4}$ $\ln\left(\frac{4N}{500 - N}\right) = t$ $\ln\left(\frac{500 - N}{4N}\right) = -t$ $e^{-t} = \frac{500 - N}{4N}$ $4Ne^{-t} = 500 - N$ $N(4e^{-t} + 1) = 500$ So $N(t) = \frac{500}{1 + 4e^{-t}}$

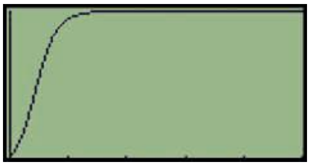
THINK

WRITE/DISPLAY

d Draw $N(t)$ using a graphics calculator.

d





e ① Find the maximal value of the population by finding half of the limiting value or calculating $\frac{a}{2b}$.

e

$$\text{Maximal } N = \frac{1}{2}\cdot \lim_{t\to\infty}\frac{500}{1+4e^{-t}}$$

$$= \frac{1}{2}\cdot 500$$

$$= 250$$

$$\text{or Maximal } N = \frac{a}{2b}$$

$$= \frac{1}{2\cdot 0.002}$$

$$= 250$$

Maximal population occurs when $N = 250$.

That is, when $\frac{500}{1+4e^{-t}} = 250$.

② To find when the maximal population is 250 we need to solve $\frac{500}{1+4e^{-t}} = 250$ to find t. To determine t for a maximal population, use a graphics calculator to graph the function for N and find the point of intersection with the line $N = 250$.

Using the Casio 9860G AU

① To draw the graphs, press:

- MENU
- 5 (GRAPH)

Complete the entry lines as:

$$Y1 = \frac{500}{1+4e^{-x}}$$

$$Y2 = 250$$

Press EXE after each line.

Adjust the V-Window and press F6 (DRAW).

Continued over page →

THINK

WRITE

② To find the point of intersection, press:
- SHIFT
- F5 (G-Solv)
- F5 (ISCT)

The point of intersection will be displayed.

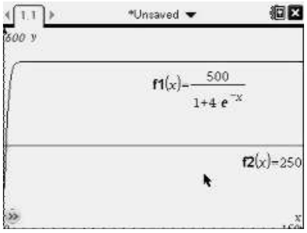
Using the TI-Nspire CAS

① To draw the graphs on a Graphs page, complete the entry lines as:

$$f1(x) = \frac{500}{1 + 4e^{-x}}$$

$f2(x) = 250$

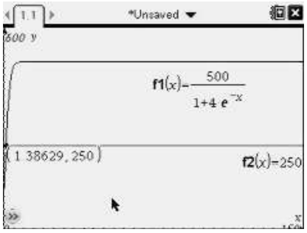
Press ENTER after each line.
Adjust the Window settings to find an appropriate scale.



② To find the point of intersection, press:
- MENU
- 7: Points & Lines 7
- 3: Intersection Point(s) 3

Move the pointer to $f1(x)$ and press ENTER; then move the pointer to $f2(x)$ and press ENTER.

The point of intersection will be displayed.



③ Answer the question.

The maximal value of the population is 250, and thus occurs when $t = 1.386$.

WORKED Example 21

An insect population N is subject to a birth rate α and a death rate $\beta + \gamma N$ per head of population, per unit time. Ecologists have modelled the rate of change of this population over time using the following logistic differential equation:

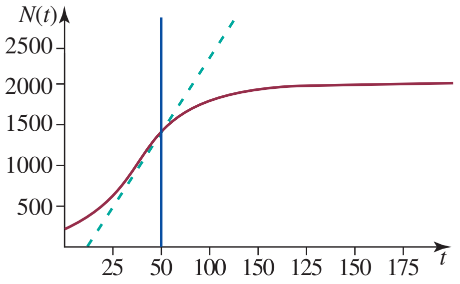
$$\frac{dN}{dt} = (\alpha - \beta)N - \gamma N^2.$$

It is known that $\alpha = 0.1$, $\beta = 0.05$, $\gamma = 2 \cdot 10^{-5}$, and the population is 200 when $t = 0$.

a **Determine each of the following correct to three significant figures.**
  - **i** $N(40)$
  - **ii** $N(60)$
  - **iii** **The average rate of change in $N(t)$ across the interval $40 \le t \le 60$**
  - **iv** **The instantaneous rate of change when $t = 50$**

b **Graph $N(t)$.**

c **Use algebra to determine when the population has doubled its initial size.**

THINK	**WRITE**
a Formulate the logistic differential equation and obtain (by rule) its solution.	**a** $\frac{\mathrm{d}N}{\mathrm{d}t} = aN - bN^2$ and $N(t) = \dfrac{\frac{a}{b}N_0}{N_0 + \left(\frac{a}{b} - N_0\right)e^{-at}}$ $a = \alpha - \beta = 0.05$, $b = \gamma = 2 \cdot 10^{-5}$, $N_0 = 200$ $\frac{\mathrm{d}N}{\mathrm{d}t} = 0.05N - 0.000\,02N^2$ and $N(t) = \dfrac{5000}{2 + 23e^{-0.05t}}$
i Substitute $t = 40$ to calculate $N(40)$.	**i** $N(40) = \dfrac{5000}{2 + 23e^{-2}}$ $= 978$
ii Substitute $t = 60$ to calculate $N(60)$.	**ii** $N(60) = \dfrac{5000}{2 + 23e^{-3}}$ $= 1590$
iii Calculate the average rate of change across the interval $40 \leq t \leq 60$.	**iii** Average rate $= \dfrac{N(60) - N(40)}{60 - 40}$ $= \dfrac{1590 - 978}{20}$ $= 30.6$ Therefore, the average rate is 31 head of population per unit time.
iv Calculate the required instantaneous rate of change by substituting $t = 50$ first into $N(t)$ then using this value for N in the expression for $\frac{\mathrm{d}N}{\mathrm{d}t}$.	**iv** $N(50) = \dfrac{5000}{2 + 23e^{-2.5}}$ $= 1286$ When $t = 50$, $\frac{\mathrm{d}N}{\mathrm{d}t} = 0.05N - 0.000\,02N^2$ $= 0.05 \cdot 1286 - 0.000\,02 \cdot 1286^2$ $= 31.22$ The instantaneous rate of change in N is 31.2 head of population per unit time.
b Use a graphics calculator or ‘Graphmatica’ to obtain a graph of $N(t)$.	**b** 

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THINK	**WRITE**
c ① Use the initial population to determine the size of the population when doubled.	**c** $N_0 = 200$; so for double the population, $N = 400$.
② Write an equation for this population.	When $N = 400$, $400 = \dfrac{5000}{2 + 23e^{-0.05t}}$
③ Solve for t.	$400(2 + 23e^{-0.05t}) = 5000$ $2 + 23e^{-0.05t} = 12.5$ $23e^{-0.05t} = 10.5$ $-0.05t = \ln\left(\dfrac{10.5}{23}\right)$ $t = \dfrac{-1}{0.05}\ \ln\left(\dfrac{10.5}{23}\right)$ $= 15.682$ The population has doubled its initial size when $t = 15.7$.

WORKED Example 22

A group of 400 caregivers is waiting at Brisbane airport for a student charter returning after a six-month exchange program in Europe. The plane's arrival has been significantly delayed and the caregivers are becoming anxious. A rumour starts that the plane is experiencing engine problems. Psychologists have established that a rumour tends to spread at a rate jointly proportional to the number who have heard it, say x, and the number who have not, $400 - x$. In this case we may write: $\dfrac{dx}{dt} = ax(400 - x)$.

If we assume initially that $x(0) = 1$, that is, when $t = 0$ minutes *one person* has started the rumour, and that the constant of proportionality is 0.000 02:

a **solve the differential equation using integral calculus, and use your result to find the number of people who have heard the rumour after $6\frac{1}{2}$ hours**

b **determine the time required for 200 people to have heard the rumour**

c **find $\lim\limits_{t \to \infty} x(t)$.**

THINK	**WRITE**
a ① Write the proportionality statement as an equation.	**a** $\dfrac{dx}{dt} \propto x(400 - x)$ $\dfrac{dx}{dt} = kx(400 - x)$
② Replace k, the constant of proportionality, with 0.000 02.	Hence, with $k = 0.000\,02$, $\dfrac{dx}{dt} = 0.000\,02x(400 - x)$
③ Write the integral statement to solve the differential equation.	$\displaystyle\int \frac{dx}{x(400 - x)} = \int 0.000\,02\ dt$

THINK	**WRITE**
④ Consider the integrand on the left-hand side and break it into partial fractions.	Consider $\frac{1}{x(400-x)}$. $\frac{1}{x(400-x)} \equiv \frac{A}{x} + \frac{B}{400-x}$ $= \frac{A(400-x)+Bx}{x(400-x)}$ So $1 \equiv A(400-x) + Bx$
⑤ Determine the constants A and B.	When $x = 0$, $1 = 400A + 0$ so $A = \frac{1}{400}$ When $x = 400$, $1 = 0 + 400B$ so $B = \frac{1}{400}$ $\frac{1}{x(400-x)} = \frac{\frac{1}{400}}{x} + \frac{\frac{1}{400}}{400-x}$
⑥ Substitute the partial fractions into the integral statement.	Hence $\int \frac{dx}{x(400-x)} = \int 0.000\,02\,dt$ becomes $\frac{1}{400}\int\left(\frac{1}{x} + \frac{1}{400-x}\right)dx = \int 0.000\,02\,dt$
⑦ Perform the integration on both sides of the equation and include a combined constant of integration on the right-hand side only.	$\frac{1}{400}[\ln x - \ln(400-x)] = 0.000\,02t + c$
⑧ Apply the boundary condition of $x(0) = 1$ to determine c.	When $t = 0$, $x = 1$ so $c = -\frac{1}{400}\ln 399$ $\frac{1}{400}[\ln x - \ln(400-x)] = 0.000\,02t - \frac{1}{400}\ln 399$
⑨ Rearrange and simplify to make x the subject of the equation.	$\frac{1}{400}\ln\left(\frac{399x}{400-x}\right) = 0.000\,02t$ $\frac{1}{400}\ln\left(\frac{400-x}{399x}\right) = -0.000\,02t$ $\ln\left(\frac{400-x}{399x}\right) = -0.008t$ $e^{-0.008t} = \frac{400-x}{399x}$ $399xe^{-0.008t} = 400 - x$ $x(399e^{-0.008t} + 1) = 400$ So $x(t) = \frac{400}{1+399e^{-0.008t}}$

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THINK	WRITE
⑩ The number of people who have heard the rumour after $6\frac{1}{2}$ hours is obtained by writing $t = 390$ minutes and calculating $x(390)$.	$6\frac{1}{2}$ hours = 390 minutes $x(390) = \dfrac{400}{1 + 399e^{-0.008 \cdot 390}}$ ≈ 21.484
⑪ Answer the question.	Therefore, 21 people will have heard the rumour after $6\frac{1}{2}$ hours.
b ① Find the time required for 200 people to have heard the rumour by finding t where $x(t) = 200$.	**b** When $x(t) = 200$, $\dfrac{400}{1 + 399e^{-0.008t}} = 200$ $400 = 200(1 + 399e^{-0.008t})$ $2 = 1 + 399e^{-0.008t}$ $\frac{1}{399} = e^{-0.008t}$ $\ln\left(\frac{1}{399}\right) = -0.008t$ $t = \dfrac{1}{-0.008}\ln\left(\frac{1}{399}\right)$ $= -125\ln\left(\frac{1}{399}\right)$ $= 125\ln 399$ ≈ 748.6
② Answer the question.	Therefore, the time taken for two hundred people to have heard the rumour is approximately 12 hours 29 minutes.
c ① Find $\lim\limits_{1 \to \infty} x(t)$. Remember that as $t \to \infty$, $e^{-0.008t} \to 0$.	**c** $\lim\limits_{t \to \infty} x(t) = \lim\limits_{t \to \infty} \dfrac{400}{1 + 399e^{-0.008t}}$ $= \dfrac{400}{1 + 399 \cdot 0}$ $= 400$
② State your conclusion.	If we wait long enough, eventually everyone will have heard the rumour.

remember

The Verhulst, or logistic, model for population growth involves the following.

1. The logistic differential equation, $\frac{dN}{dt} = aN - bN^2$ $b > 0$, with solution

$$N(t) = \frac{\frac{a}{b}N_0}{N_0 + \left(\frac{a}{b} - N_0\right)e^{-at}}$$ where the population at time $t = 0$ is N_0.

2. The limiting population is determined as $\lim\limits_{t \to \infty} N(t) = \frac{a}{b}$ and the maximal population by $N_{\max}(t) = \frac{a}{2b}$ at time t. $N_{\max}$ occurs at the point of oblique inflection.

EXERCISE 8G Logistic model and population growth

WORKED Example 20

1 A population, N, of mice at time t, exhibits logistic growth such that:

$$\frac{dN}{dt} = 0.2N - 0.0004N^2,\ N(0) = 100.$$

a Solve for $N(t)$ by using the logistic equation directly.
b Verify the solution by differentiation.
c Solve for $N(t)$ by applying integral calculus.
d Use either a graphics calculator or the program 'Graphmatica' to graph $N(t)$ and determine when the population is maximal.

2 A colony, P, of possums at time t, exhibits logistic growth such that:

$$\frac{dP}{dt} = 0.25P - 0.0005P^2,\ P(0) = 200.$$

a Solve for $P(t)$ by using the logistic equation directly.
b Verify the solution by differentiation.
c Solve for $P(t)$ by applying integral calculus.
d Use either a graphics calculator or the program 'Graphmatica' to graph $N(t)$ and determine when the population is maximal.

3 A population, F, of fire-ants at time t, exhibits logistic growth such that:

$$\frac{dF}{dt} = 0.5F - 0.0005F^2,\ F(0) = 50.$$

a Solve for $F(t)$ by using the logistic equation directly.
b Verify the solution by differentiation.
c Solve for $F(t)$ by applying integral calculus.
d Use either a graphics calculator or the program 'Graphmatica' to graph $N(t)$ and determine when the population is maximal.

WORKED Example 21

4 A marsupial population, N, is subject to a birth rate α and a death rate $\beta + \gamma N$ per head of population, per annum. Ecologists have modelled the rate of change of this population over time using the following logistic differential equation.

$$\frac{dN}{dt} = (\alpha - \beta)N - \gamma N^2$$

It is known that $\alpha = 0.2$, $\beta = 0.04$, $\gamma = 2 \cdot\ 10^{-4}$, and the population is 200 when $t = 0$ years.

a Determine each of the following, correct to four significant figures:
  i $N(4)$
  ii $N(8)$
  iii the average rate of change in $N(t)$ across the interval
  iv the instantaneous rate of change when $t = 7$ years.
b Graph $N(t)$.
c Determine algebraically when the population has doubled its initial size.

5 A butterfly population, B, is subject to a birth rate α and a death rate $\beta + \gamma B$ per head of population, per hour. Environmentalists have modelled the rate of change of this population over time using the following logistic differential equation.

$$\frac{dB}{dt} = (\alpha - \beta)B - \gamma B^2$$

It is known that $\alpha = 0.1$, $\beta = 0.08$, $\gamma = 4 \cdot 10^{-5}$, and the population is 150 when $t = 0$ hours.

a Determine each of the following, correct to three significant figures:

- **i** $B(30)$
- **ii** $B(50)$
- **iii** the average rate of change in $B(t)$ across the interval $30 \le t \le 50$
- **iv** the instantaneous rate of change when $t = 40$ hours.

b Graph $B(t)$.

c Determine algebraically when the population has trebled its initial size.

6 A colony of wasps with a population W at time t days is studied by a group of researchers. This colony has a birth rate α and a death rate $\beta + \gamma W$ per head of population, per day. The researchers have applied the following logistic model to predict the population rate of growth at time t days.

$$\frac{dW}{dt} = (\alpha - \beta)W - \gamma W^2$$

Data collected by the research group have revealed the vital coefficients of the colony as: $\alpha = 2 \cdot 10^{-2}$, $\beta = 5 \cdot 10^{-4}$, $\gamma = 3 \cdot 10^{-6}$, and the population is 100 when $t = 0$ days.

a Determine each of the following, correct to four significant figures:

- **i** $W(150)$
- **ii** $W(200)$
- **iii** the average rate of change in $W(150)$ across the interval $150 \le t \le 200$.
- **iv** the instantaneous rate of change when $t = 300$.

b Graph $W(t)$.

c Determine algebraically when the population is 4000 wasps.

WORKED Example 22

7 A group of 2000 supporters of the Japanese pop group *Big Sisters,* is waiting excitedly at Brisbane airport for their arrival. Rumour spreads that the group did not make their connecting flight from the USA. Sociologists have established that a rumour tends to spread at a rate jointly proportional to the number who have heard it, say x at time t minutes, and the number who have not, $2000 - x$. In this case we may write:

$$\frac{dx}{dt} = kx(2000 - x).$$

If we assume that initially, $x(0) = 1$; that is, when $t = 0$ minutes *one person* has started the rumour, and that 190 people per minute hear the rumour when $x(t) = 100$ people.

a determine the constant of proportionality, k

b solve the differential equation using integral calculus

c use your result to find the number of people who have heard the rumour after 5 minutes

d determine the time required for 800 people to have heard the rumour

e find $\lim_{t \to \infty} x(t)$.

8 In a small community of 1000 people, a person who has been exposed to influenza has contracted the virus. If we assume that all people in the community are susceptible to this virus, and if the epidemic spreads at a rate in direct proportion to the number who have it, N, and to the number who have not contracted it, $1000 - N$, then we may model the spread of the virus as follows:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = a\,N(1000 - N),\ N(0) = 1.$$

a If we select $a = 0.0004$, show that at time t days, the number of people who have contracted the virus is given by $N(t) = \dfrac{1000}{(1 + 999e^{-0.4t})}$.

b How many people will have contracted the virus after 14 days?

c How long will it take for 75% of the community to contract the virus?

9 In a community of 45 000 people, the rate of growth of an influenza epidemic is directly proportional to the number of people who have contracted the virus and the number of people who have not contracted it.

a If 200 people had the virus at the start of the epidemic and 2800 have it now after three weeks, find a mathematical model that describes the outbreak of the epidemic.

b Using your model, determine how many people will be infected after 8 weeks.

c If the epidemic continues indefinitely, how many people are likely to get this virus?

History of mathematics

CATHLEEN SYNGE MORAWETZ (1923–)

During her lifetime ...

World War II

The Vietnam War

First humans on the moon and in orbit

Cathleen Synge Morawetz was born on 5 May 1923 in Toronto, Canada. Her Irish parents were trained in mathematics. In 1943, Morawetz began her mathematical studies at Toronto University. She had married Herbert Morawetz, a chemist, while studying for her Masters Degree at the Massachusetts Institute of Technology. She completed her PhD from the New York University in 1951, with a thesis on the stability of a spherical implosion.

In 1978, she was appointed associate director of the Courant Institute of Mathematical Sciences, deputy director in 1981, and chair of the mathematics department from 1981–84. She was the first woman in the United States to head a mathematical institute when she was named director in 1984. She is now Professor Emeritus at New York University, Courant Institute.

A mother of four, Cathleen Morawetz has been able to balance her energies between research and her family. Morawetz's earliest published works were on the stability of steady viscous flows. During the 1950s, she turned to the mathematics of transonic flow, and showed that specially designed shockless airfoils develop shocks if they are altered by even a small amount. This theory is fundamental to aerofoil design in modern, supersonic aircraft. The discovery opened the problem of developing a theory for transonic flow with shocks.

Much of her research has focused on the wave equation. The classical problem of whether light should be treated as waves or as streams of particles can be answered, '*either will do*', if it can be shown that high frequency waves are, asymptotically, streams of weightless particles moving along the ray. In collaboration with another researcher, D. Ludwig, Morawetz showed that this is true for a medium with constant light speed outside a reflecting star-shaped object.

During her career, Cathleen Morawetz has been active in mathematical society and has received many honours. She is a Fellow of both the American Association for the Advancement of Science and the American Academy of Arts and Sciences. She is a member of the Advisory Committee for the Mathematical Sciences for the National Science Foundation and of the Mathematical Advisory Committee to the National Bureau of Standards. She is also the first woman to belong to the Applied Mathematics Section of the National Academy of Sciences. In 1997, she received the Krieger-Nelson Award given by the Canadian Mathematical Society to recognise outstanding research by a female mathematician. On December 8, 1998, Morawetz's thinking earned her the highest honour in US science, a National Medal of Science, awarded by President Clinton. She is the first woman to receive the medal for work in mathematics.

Questions

1. Cathleen Morawetz was the first women in the USA to head a mathematical institute. What was the name of this institute and in which year was she appointed as director?
2. What was the area of mathematics, which is associated with supersonic aircraft, that Morawetz worked on during the 1950s?
3. In which year was Cathleen Morawetz awarded the US National Medal of Science?

summary

de Moivre's theorem

- For any real-valued n and complex number $z = r(\cos\theta + i\sin\theta)$, de Moivre's theorem states:
  $z^n = r^n(\cos n\theta + i\sin n\theta)$.
- $\text{Re}(\cos\theta + i\sin\theta)^n = \cos n\theta$ and $\text{Im}(\cos\theta + i\sin\theta)^n = \sin n\theta$.
- If $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.

Multiple-angle formulas

- $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$ and $\sin n\theta = \dfrac{z^n - z^{-n}}{2i}$

Euler's formula

- For any real values of θ, Euler's formula states: $e^{i\theta} = \cos\theta + i\sin\theta$.
- θ is always a radian measure.
- When working with Euler's formula we must remember that:
  1. To *multiply*, we multiply $(\cdot\ )$ the moduli and add $(+)$ the arguments.
  2. To *divide*, we divide $(\ |)$ the moduli and subtract $(-)$ the arguments.
- $\displaystyle\int f(x)\cos ax\,\mathrm{d}x = \text{Re}\left(\int e^{iax} f(x)\,\mathrm{d}x\right)$
- $\displaystyle\int f(x)\sin ax\,\mathrm{d}x = \text{Im}\left(\int e^{iax} f(x)\,\mathrm{d}x\right)$

Applications of Euler's formula

- $z_n = \displaystyle\sum_{k=1}^{n} \exp\left(\frac{2\pi i k^3}{M}\right)$, $n = 1, 2, 3, \quad$, M generates the complex sequence $\{z_n\}$.
- As the value of M is increased, the sequence $\{z_n\}$ becomes more intricate.
- $\cos k\theta = \frac{1}{2}(e^{ik\theta} + e^{-ik\theta})$ and $\sin k\theta = \dfrac{1}{2i}(e^{ik\theta} - e^{-ik\theta})$

Damped functions $y = e^{ax}\sin bx$ and $y = e^{ax}\cos bx$

- $\sin bx$ and $\cos bx$ are functions which oscillate between ±1.
- the multiplier function e^{ax} alters the amplitude according to the value of a constraining both $\sin bx$ and $\cos bx$ between $\pm e^{ax}$.
- $\displaystyle\lim_{x\to\infty} e^{ax}\sin bx = \begin{cases} 0, & a < 0 \\ \pm\infty, & a > 0 \end{cases}$ and $\displaystyle\lim_{x\to\infty} e^{ax}\cos bx = \begin{cases} 0, & a < 0 \\ \pm\infty, & a > 0 \end{cases}$
- $\displaystyle\lim_{x\to-\infty} e^{ax}\sin bx = \begin{cases} \pm\infty, & a < 0 \\ 0, & a > 0 \end{cases}$ and $\displaystyle\lim_{x\to-\infty} e^{ax}\cos bx = \begin{cases} \pm\infty, & a < 0 \\ 0, & a > 0 \end{cases}$
- $\displaystyle\int e^{ax}\sin bx\,\mathrm{d}x = \text{Im}\left(\int e^{x(a+bi)}\mathrm{d}x\right)$
- $\displaystyle\int e^{ax}\cos bx\,\mathrm{d}x = \text{Re}\left(\int e^{x(a+bi)}\mathrm{d}x\right)$

Modelling damping

- $y = Ae^{at} \sin bt$
  1. By calculating the time-derivative of y, we may also determine the velocity of oscillation about the central point for some appropriate values of the constants.
  2. Noting that $\dfrac{dy}{dt} = \dot{y}$,

     $$\dot{y} = Aae^{at} \sin bt + Abe^{at} \cos bt$$

     $$\therefore\ \dot{y} = Ae^{at} (a \sin bt + b \cos bt)$$
- $I = Ae^{at} \cos bt$
  1. In an electrical system the amount of charge present in a circuit at time t may be modelled by $I = Ae^{at} \cos bt$.
  2. The charge, q, in an electrical circuit is related to the current, I, by $I = \dfrac{dq}{dt}$; that is, $\dot{q} = Ae^{at} \cos bt$ gives $q = \int Ae^{at} \cos bt \, dt$.

Malthusian population model

- $\dfrac{dN}{dt} = kN$ and if $N = N_0$, when $t = 0$, then $N(t) = N_0 e^{kt}$
- These models may be reliable predictors provided they are about small populations.

The Verhulst or logistic model for population growth

- The logistic differential equation, $\dfrac{dN}{dt} = aN - bN^2$, $b > 0$, has solution

  $$N(t) = \frac{\frac{a}{b}N_0}{\left[N_0 + \left(\frac{a}{b} - N_0\right)e^{-at}\right]}$$ where the population at time $t = 0$ is N_0.
- The limiting population is determined as $\lim_{t \to \infty} N(t) = \dfrac{a}{b}$ and the maximal population by $N_{\max}(t) = \dfrac{a}{2b}$ at time t. $N_{\max}$ occurs at the point of oblique inflection.

CHAPTER review

1 **multiple choice** 8A

Simplifying $\cos^2 x - \sin^2 x$ produces:

A 1 **B** -1 **C** $\sin 2x$

D $\cos 2x$ **E** None of these

2 Using de Moivre's theorem and the binomial expansion, prove that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

3 Using an appropriate multiple-angle formula, prove that $\sin^3(x) = \frac{1}{4}[3\sin(x) - \sin(3x)]$.

4 **multiple choice** 8B

Which of the following expresses $-1 + i$ in Euler's form?

A $2e^{-\frac{\pi}{4}i}$ **B** $\sqrt{2}e^{-\frac{\pi}{4}i}$ **C** $\sqrt{2}e^{-\frac{3\pi}{4}i}$ **D** $\sqrt{2}e^{\frac{\pi}{4}i}$ **E** $\sqrt{2}e^{\frac{3\pi}{4}i}$

5 **multiple choice** 8B

Which of the following expresses $2e^{-\frac{\pi}{6}i}$ in standard form?

A $-\sqrt{3} - i$ **B** $\sqrt{3} - i$ **C** $\sqrt{3} + i$ **D** $1 - i\sqrt{3}$ **E** $-1 - i\sqrt{3}$

6 **multiple choice** 8B

For the complex variable z, the expression $\frac{i}{2}(z^2 - z^{-2})$ is equivalent to:

A $-\sin 2\theta$ **B** $-\cos 2\theta$ **C** $2\,i\sin\theta$ **D** $2i\cos\theta$ **E** $i\sin 2\theta$

7 **a** Express $\sqrt{3} - i$ and $-1 + i$ in Euler's form. 8B

b Use your results from **a** above to evaluate $\dfrac{(\sqrt{3} - i)^5}{(-1 + i)^6}$.

8 Use Euler's formula to evaluate: 8B

a $\int e^x \cos x \, dx$

b $\int_0^{\frac{\pi}{3}} e^{-x} \sin x \, dx$.

9 Find all values of x and y for which $e^{\frac{i\pi}{3}} = \dfrac{x}{1 - i} + \dfrac{y}{2 + 3i}$. 8C

10 **multiple choice** 8D

If $x(t) = e^{2t}\sin t$ metres represents the displacement of a spring from its central position at time t seconds, then the initial *speed*, in metres/second, is:

A -2 m/s **B** -1 m/s **C** 1 m/s **D** 0 m/s **E** 2 m/s

8D **11** **multiple choice**

The value of $\lim\limits_{t \to \infty} e^{-2t} \cos 3t$ is:

A -1 **B** 1 **C** $\cos 3\pi$ **D** undefined **E** 0

8D **12** **multiple choice**

Given that $z_2 = \sum\limits_{k=1}^{2} \exp\left(\frac{\pi i k^3}{2}\right)$, what is z_2 in standard form?

A 0 **B** $1 + i$ **C** i **D** $1 - i$ **E** π

8D **13** **a** Sketch $y = e^{-x} \cos(2x)$ across the domain $-\frac{\pi}{4} \le x \le \pi$.

b Find $\int_0^c e^{-x} \cos(2x)\, dx$.

c Evaluate $\lim\limits_{c \to \infty} \int_0^c e^{-x} \cos(2x)\, dx$.

8E **14** **multiple choice**

The current in an LCR circuit is given by $I(t) = -2e^{kt} \sin\left(\omega t - \frac{3\pi}{4}\right)$. What is the initial current in this circuit?

A $\sqrt{2}$ **B** $\frac{1}{2}\sqrt{2}$ **C** $-\sqrt{2}$ **D** $\frac{1}{2\sqrt{2}}$ **E** 0

8E **15** **multiple choice**

Given that $\ddot{y}(t) = -3e^{-0.4t} \sin 2\pi t$, determine the value of $\dot{y}(1)$ if $\dot{y}(0) = \frac{112.5\pi}{3 + 75\pi^2}$ m/s.

A $\frac{e^{-0.4}}{3 + 75\pi^2}$ **B** $\frac{112.5\pi e^{0.4}}{3 + 75\pi^2}$ **C** none of these

D $\frac{112.5\pi e^{-0.4}}{3 + 75\pi^2}$ **E** $\frac{112.5\pi}{3 + 75\pi^2}$

8E **16** The motion of a pendulum subject to damping has a solution of the form $x = Ae^{-0.3t} \cos\left(2t + \frac{\pi}{3}\right)$, where x cm is the displacement of the pendulum at time t seconds.

If $x = 4$ cm when $t = 0$ seconds, find:

a the value of the constant A

b the velocity of the pendulum when it first passes through its central position

c the first value of t for which the pendulum has zero velocity.

8E **17** At time t, a particle moving in a straight line has an acceleration of $\ddot{x} = e^{-2t} \cos(t)$. When $t = \tan^{-1}(2)$, $\dot{x} = 0$, and when $t = \tan^{-1}\left(\frac{3}{4}\right)$, $x = 0$. Find $x(t)$.

8E **18** In an LCR circuit at time t, the current, I, is given by $I = 25e^{-0.2t} \sin\left(\frac{\pi t}{4}\right)$. Initially, the charge in the circuit is 0.

a When $t = 3$, what is the value of the current?

b Show that the charge, $q(t)$, is given by:

$$q(t) = \frac{500}{16 + 25\pi^2}\left[5\pi - 4e^{-0.2t}\left\{\sin\left(\frac{\pi t}{4}\right) + \frac{5\pi}{4}\cos\left(\frac{\pi t}{4}\right)\right\}\right]$$

c Graph $q(t)$.

d What is the charge in the circuit when $t = 5$?

e Determine $\lim\limits_{t \to \infty} q(t)$.

19 **multiple choice** 8F

The equation, $N(t) = \dfrac{500}{1 + 4e^{-t}}$ represents the number of rabbits at time t months in a colony.

What is the number of rabbits in the colony after 6 months?

A 505	**B** 250	**C** 100	**D** 500	**E** 495

20 In a fruit-fly population, the rate of growth of the number of reproductive flies at time t days after 100 days is given by: 8F

$$\dot{r}(\mathrm{t}) = 0.04e^{0.02t}(150 - t),\ r(0) = 300.$$

a Show that $r(t) = 2e^{0.02t}(200 - t) - 100$.

b What is the number of reproductives when $t = 30$ days?

c Use your graphics calculator to find when the number of reproductive flies is:

  i 1000 **ii** at least 1000 **iii** greatest **iv** $1200 \le r(t) \le 1500$.

21 A group of 800 devotees of the spiritual leader Sri Swami X is waiting at Brisbane airport for the arrival of their hero. The plane has been significantly delayed and the devotees are becoming anxious. A rumour starts that the plane has been hijacked to Rio. 8G

This rumour spreads at a rate jointly proportional to the number who have heard it, say x, and the number who have not, $800 - x$. It follows that:

$$\frac{\mathrm{d}x}{\mathrm{d}t} \propto x(800 - x).$$

Assume that (initially) $x(0) = 1$; that is, when $t = 0$ hours a single person started the rumour, and that the constant of proportionality is 0.004.

a Solve the differential equation using integral calculus and use your result to find the number of people who have heard the rumour after 2 hours.

b Determine the time required for 500 people to have heard the rumour.

c Find $\lim\limits_{t \to \infty} x(t)$.

22 Astrid and Mara are behavioural environmentalists studying a colony of wild dogs in a farming district in northern Queensland. The size of the dog colony was $W(t)$ at time t years. Their research suggests the following growth model: 8G

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 0.3W - 0.025W^2 \text{ where } W(0) = 3.$$

a What is the carrying capacity of this environment?

b The wild dog population will be culled when it reaches its maximum growth rate. When will the culling begin?

c How long will it take for the initial wild dog population to treble without any culling?

d What is the anticipated average rate of increase in the wild dog population across the period $5 \le t \le 10$?

Modelling and problem solving

1 A model of a *memorisation process* is given by the differential equation:

$\frac{dx}{dt} = a(m - x) - bx$, where the parameters associated with the process are defined as:

1. $x(t)$, which represents the amount of material memorised at time t
2. m, which represents the total amount of material to be memorised
3. a and b, which are positive constants associated with the memorising process.

Solve the above differential equation for $x(t)$ and use your result to discuss the efficiency of the memorisation process over time in the case that:

a $b = 0$

b $b \neq 0$.

2 A team of research scientists from the Department of Primary Industries has formulated a growth model for the fire-ant population in Brisbane. This model is represented by the differential equation $\frac{dN}{dt} = aN^2$, where $N(t)$ is the number of fire-ants present at time t, and a is a positive constant.

a If the fire-ant population is N_0 at time $t = 0$, show that $N = \frac{N_0}{1 - N_0 at}$.

b Determine how long it takes for the population to double, given that $a = 0.001$.

3 The struggle between two species with similar habits is commonly observed in natural ecosystems and is known as the *principle of competitive exclusion*. The following system of differential equations defines the simple model, which describes this phenomenon:

$$\frac{dx}{dt} = Ax - Bxy \text{ and } \frac{dy}{dt} = Cy - Dxy,$$

where $x(t)$ and $y(t)$ represent the populations of species x and species y at time t. If in this model, A, B, C and D are positive constants associated with the populations, then prove that the mathematical relationship between x and y is defined by $y^A e^{-By} = kx^C e^{-Dx}$, where k is an arbitrary constant.

4 In an isolated Queensland town which has a population x, 20% of the residents heard a radio announcement about a local political scandal at the recent elections. The rate of growth of the spread of information about the scandal was directly proportional to the number who had heard it and the number of people who had not. If, after *one hour,* 40% of the population heard about the scandal, how long would it take for 60% of the population to have heard it?

5 For the generalised logistic equation, obtain an algebraic expression for t in terms of a, b and N_0 when:

a the population is maximal

b the population has doubled.

6 By converting to Euler's form (or another method), find all possible values of m and n, if any, for which $\dfrac{(\sqrt{3}-i)^m}{(-1+i)^n} = -4$.

7 Leroy and Billie Brown are ecologists maintaining a colony of animals in a remote section of a national park. In 2001, Leroy and Billie began collecting data using a statistical sampling procedure. This procedure continued for the next 6 years. In the following table, the population at time t is $N(t)$, and the rate at which the population is growing at time t is $\dfrac{dN}{dt}$. In their calculations, Billie and Leroy Brown assigned the value 0 to the year 2001.

Year	2001	2002	2003	2004	2005	2006	2007
$N(t)$	10	15	25	38	56	78	102
$\dfrac{dN}{dt}$	4.800	7.050	11.250	16.112	21.728	26.832	30.192

a Using a quadratic regression, determine the positive constants a and b, correct to three decimal places, such that $\dfrac{dN}{dt} = aN - bN^2$.

b **i** Determine $N(t)$ and **ii** sketch your curve.

c Billie and Leroy know from experience that when the population approaches its limiting size, they must relocate part of the colony. When should this be done?

d Find the maximal rate of growth for this population and state when this occurs.

e What is the anticipated average rate of increase in the population for the period 2007–2012?

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Test Yourself
Chapter 8

eBook*plus* ACTIVITIES

8A Trigonometric links to complex numbers

Interactivity

- int-0343: Use the interactivity to make trigonometric links to complex numbers. (*page 328*)

Tutorial

- int-0371: Watch a tutorial on applying de Moivre's theorem to raising complex number to a power. (*page 331*)

Digital docs

- SkillSHEET 8.1: Practise working with Euler's formula. (*page 328*)
- SkillSHEET 8.2: Practise applications of damped functions. (*page 328*)
- SkillSHEET 8.3: Practise the logistic problem model. (*page 328*)

8C Applications of Euler's formula

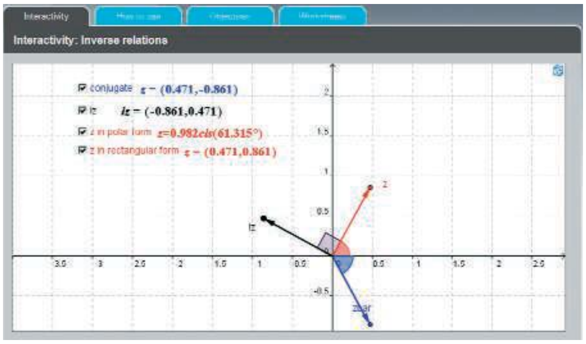
Digital docs

- WorkSHEET 8.1: Use Euler's formula and de Moivre's theorem to solve trigonometric equations, prove identities and evaluate expressions. (*page 346*)
- Spreadsheet doc-0142: Use a complex iterative formula to investigate M = 29. (*page 346*)

8D Damped functions

Digital doc

- Spreadsheet doc-0143: Investigate summation. (*page 352*)



Interactivity: Inverse relations

☑ conjugate z − (0.471,-0.861)

☑ iz iz − (-0.861,0.471)

☑ z in polar form z=0.982cis(61.315°)

☑ z in rectangular form z − (0.471,0.861)

8E Modelling real-life damped systems

Digital doc

- WorkSHEET 8.2: Use advanced period functions to make practical calculations given application questions. (*page 361*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 391*)

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9

Conics on the Cartesian and complex planes

syllabus reference

Option topic:
Conics

In this chapter

9A Cartesian form of the straight line and the circle
9B Cartesian form of the parabola
9C Cartesian form of the ellipse
9D Cartesian form of the hyperbola
9E Lines, circles and regions on the complex plane
9F The ellipse and hyperbola on the complex plane

Introduction to conics

Chapters 9 and 10 provide a study of planar curves called *conic sections* (that is, the curves we get at the junction where a plane cuts through a cone). In chapter 9, the Cartesian forms will be developed ahead of their equivalent forms in the complex plane. The parametric and polar forms of conic sections follow in chapter 10.

The general second-degree equation

For the Cartesian plane, we need to develop a broad familiarity with the types of planar curves that arise as graphs from equations of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad [1]$$

where A, B, C, D, E, F are all real numbers.

These curves occur as (a) graphs of boundaries for domains of functions in two variables, (b) vector-valued functions and (c) intersections of planes and surfaces in three dimensions. These graphs have important and useful geometric properties that we shall discuss further.

Equation [1] above is sometimes referred to as a *general second-degree* equation in two variables, x and y. When we consider particular values for A, B and C, some special geometric conditions are met, and these give rise to special curves that are called *conic sections*. The resulting curves can be classified as straight lines, parabolas, circles, ellipses and hyperbolas.

The table below gives a brief summary of the curves, associated with the conditions applied.

Curve	**Condition**	**Equation**
Straight line	$A = B = C = 0$	$Dx + Ey + F = 0$
Parabola (Axis of symmetry parallel to y-axis)	$B = C = 0$	$Ax^2 + Dx + Ey + F = 0$
Parabola (Axis of symmetry parallel to x-axis)	$A = B = 0$	$Cy^2 + Dx + Ey + F = 0$
Circle	$A = C > 0, B = 0$	$Ax^2 + Ay^2 + Dx + Ey + F = 0$
Ellipse (Major and minor axes parallel to corresponding coordinate axes)	$A, C > 0, B = 0$	$Ax^2 + Cy^2 + Dx + Ey + F = 0$
Hyperbola (Transversal and conjugate axes parallel to corresponding coordinate axes)	$A > 0, C < 0,$ or $A < 0, C > 0,$ $B = 0$	$Ax^2 - Cy^2 + Dx + Ey + F = 0$

These curves are called ‘conic sections’ because they are the result of the intersection of a plane and two infinite right circular cones. (See figures **a**, **b**, **c** and **d**.) Note that a straight line (table on page 394) is considered to be a *degenerate* conic section.

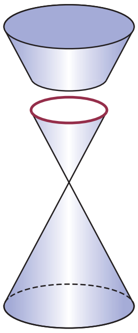


Figure **a**

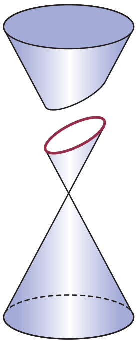


Figure **b**

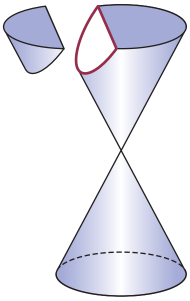


Figure **c**

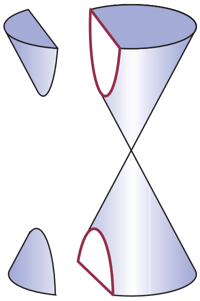


Figure **d**

Locus and equation

The *locus*, or equation in two variables, of a curve is the path traced out by a variable point while it obeys certain prescribed condition(s). These constraining conditions may be given in terms of distances, or ratios of distances from fixed points and/or fixed lines.

Since distances between two points, and the perpendicular distance of a point from a line, are essential in understanding how to derive the equations of conic sections, we shall now revise these formulas.

1. The distance, d, between two given points $A(x_1,y_1)$ and $B(x_2,y_2)$ is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. The distance, d, of a point $A(x_1, y_1)$ from a given line $Ax + By + C = 0$ is given by the formula $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

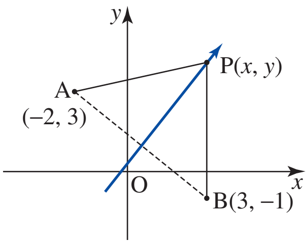
Cartesian form of the straight line

The straight line can be defined as the locus of a variable or moving point $P(x, y)$ such that it is equidistant from two given points $A(x_1, y_1)$ and $B(x_2, y_2)$.
Note: The straight line can also be described as the perpendicular bisector of a line segment.

The straight line on the Cartesian plane: $Dx + Ey + F = 0$

WORKED Example 1

Derive the equation of the locus of a point P(x, y) which moves in such a way that it is always equidistant from the points A(–2, 3) and B(3, –1).



THINK	WRITE
① To write a relationship, use the fact that the distance PA is equal to the distance PB.	PA = PB
② Substitute for PA and PB using the distance formula.	$\sqrt{(x+2)^2+(y-3)^2} = \sqrt{(x-3)^2+(y+1)^2}$
③ Square both sides of the equation.	$(x + 2)^2 + (y - 3)^2 = (x - 3)^2 + (y + 1)^2$
④ Expand both sides of the equation.	$x^2 + 4x + 4 + y^2 - 6y + 9 = x^2 - 6x + 9 + y^2 + 2y + 1$
⑤ Collect like terms and simplify.	$10x - 8y + 3 = 0$

Since the result is of the form $Dx + Ey + F = 0$, this equation represents a straight line.

Cartesian form of the circle

The circle can be defined as the locus of a point P(x, y) that is equidistant from a given point C(h, k).
Every equation of a circle can be reduced to the form $x^2 + y^2 + Dx + Ey + F = 0$.
If we rearrange and complete the squares we have:

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} = \frac{D^2}{4} + \frac{E^2}{4} - F$$

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}$$

The centre is at the point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$, and the radius $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

1. If $D^2 + E^2 - 4F > 0$ then the circle is real.
2. If $D^2 + E^2 - 4F < 0$ then the circle is imaginary.
3. If $D^2 + E^2 - 4F = 0$ then the radius is zero and the equation will represent the point $\left(-\frac{D}{2}, -\frac{E}{2}\right)$.

The circle on the Cartesian plane: $Ax^2 + Ay^2 + Dx + Ey + F = 0$

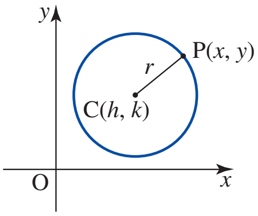
By definition, the distance PC = r where r is the radius of the circle.

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

Squaring both sides gives:

$$(x-h)^2+(y-k)^2=r^2.$$

This gives us the origin 'shift' form of the equation of a circle with centre C(h, k) and radius r.



WORKED Example 2

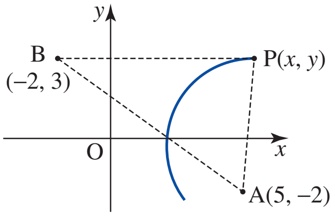
Find the equation of the circle with radius 5 units and centre at (–1, 5).

THINK	WRITE
① State the general form of the Cartesian equation of a circle.	$(x-h)^2+(y-k)^2=r^2$
② Substitute values for r, h and k.	For $r = 5$ and $(h, k) = (-1, 5)$ $(x+1)^2+(y-5)^2=5^2$
③ Expand & simplify.	$x^2+2x+1+y^2-10y+25=25$ $x^2+y^2+2x-10y+1=0$

Geometric conditions can describe distances in terms of ratios, as can be seen in Worked example 3. The circle formed is known as the *Circle of Apollonius*, and is named after the great Greek mathematician Apollonius of Perga (260–170 BC) because of his contributions to the study of conic sections. Apollonius' circle is the locus of a point that moves such that the ratio of its distances from two fixed points is a constant.

WORKED Example 3

Find the locus of the point P(x, y) that moves in the plane so that its distance from B(–2, 3) is twice its distance from A(5, –2).



THINK	WRITE
① State the relationship between the distance PA and the distance PB.	2PA = PB
② Substitute for PA and PB using the distance formula.	$2\sqrt{(x-5)^2+(y+2)^2} = \sqrt{(x+2)^2+(y-3)^2}$
③ Square both sides of the equation.	$4[(x-5)^2+(y+2)^2]=(x+2)^2+(y-3)^2$
④ Expand both sides of the equation.	$4[x^2-10x+25+y^2+4y+4]$ $=x^2+4x+4+y^2-6y+9$
⑤ Collect like terms and simplify.	$4x^2-40x+100+4y^2+16y+16$ $=x^2+4x+4+y^2-6y+9$ $3x^2+3y^2-44x+22y+103=0$

Since the result is of the form $Ax^2 + Ay^2 + Dx + Ey + F = 0$, this equation represents a circle.

WORKED Example 4

Determine the equation of the circle passing through the points A(5, 3), B(6, 2) and C(3, −1).

THINK	WRITE
1 State the expanded form of the general Cartesian equation for a circle.	$x^2 + y^2 + Dx + Ey + F = 0$
2 Since the circle must pass through all three points, substitute the coordinates of the points for x and y to obtain three equations in three unknowns.	For A(5, 3): $25 + 9 + 5D + 3E + F = 0$ For B(6, 2): $36 + 4 + 6D + 2E + F = 0$ For C(3, −1): $9 + 1 + 3D - E + F = 0$
3 Simplify the three equations and number them.	So $5D + 3E + F = -34$ [1] $6D + 2E + F = -40$ [2] $3D - E + F = -10$ [3]
4 Determine the coefficients D, E and F by solving the three equations simultaneously.	[1] − [2]: $-D + E = 6$ [4] [1] − [3]: $2D + 4E = -24$ [5] 2 × [4] + [5]: $6E = -12$ That is, $E = -2$. Substituting $E = -2$ into [4]: $-D - 2 = 6$ $D = -8$ Substituting E and D into [3]: $-24 + 2 + F = -10$ $F = 12$
5 Substitute the values for D, E and F to write the equation.	As $D = -8$, $E = -2$, $F = 12$, the equation of the circle is $x^2 + y^2 - 8x - 2y + 12 = 0$.

remember

1. The distance, d, between two given points A(x_1, y_1) and B(x_2, y_2) is given by the formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

2. The distance, d, of a point A(x_1, y_1) from a given line $Ax + By + C = 0$ is given by the formula:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

3. A straight line can be described as the locus of a moving point P(x, y) such that it is equidistant from two given points A(x_1, y_1) and B(x_2, y_2). The equation is of the form:

$$Dx + Ey + F = 0.$$

4. A circle can be described as the locus of a point P(x, y) that is equidistant from a given point C(h, k). The equation is of the form:

$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) are the coordinates of the centre and r is the radius.

EXERCISE 9A Cartesian form of the straight line and the circle

WORKED Example 1

1 Find the locus of the point P(x, y) which moves in the Cartesian plane such that it is equidistant from the points A and B if A and B are:

- **a** A(0, 3), B(0, 7)
- **b** A(2, 4), B(2, –6)
- **c** A(–2, 3), B(4, 5)
- **d** A(1, 7), B(4, –2)
- **e** A(5, 7), B(10, 7)
- **f** A(–4, 0), B(0, –4).

2 Find the equation of the locus of the point P(x, y) that is subject to the following conditions.

- **a** The distance of the point from the y-axis is three times the distance from the x-axis.
- **b** The distance from the line $y = -5$ is three quarters the distance from the line $x = 2$.

3 Find the locus of a point P(x, y) which moves such that $PA^2 - PB^2 = 6$ with A(1, 4) and B(–2, –1).

WORKED Example 2

4 Find the equation of each of the following circles.

- **a** Centre (–1, –4), radius 3 units
- **b** Centre (3, –3), radius 4 units
- **c** Centre (3, 2), passing through the point (5, –5)
- **d** Centre (–1, 4), passing through the origin

5 Write the coordinates of the centre, and the length of the radius, of each of the following circles.

- **a** $x^2 + y^2 - 5x + 3y - 1 = 0$
- **b** $(x + 3)^2 + y^2 = 3$
- **c** $2x^2 + 2y^2 - 8x + 5y + 3 = 0$
- **d** $x^2 + y^2 + 4x + 2y - 5 = 0$
- **e** $3x^2 + 3y^2 + 9x - 4y - 24 = 0$

WORKED Example 3

6 Find the locus of the point P(x, y) that moves in the plane so that its distance from A(3, –1) is twice its distance from B(–1, –1).

7 Find the equation of the circle that passes through (4, 0), (0, 4) and (0, 0).

WORKED Example 4

8 Determine the equation of the circle passing through the three points in each of the following.

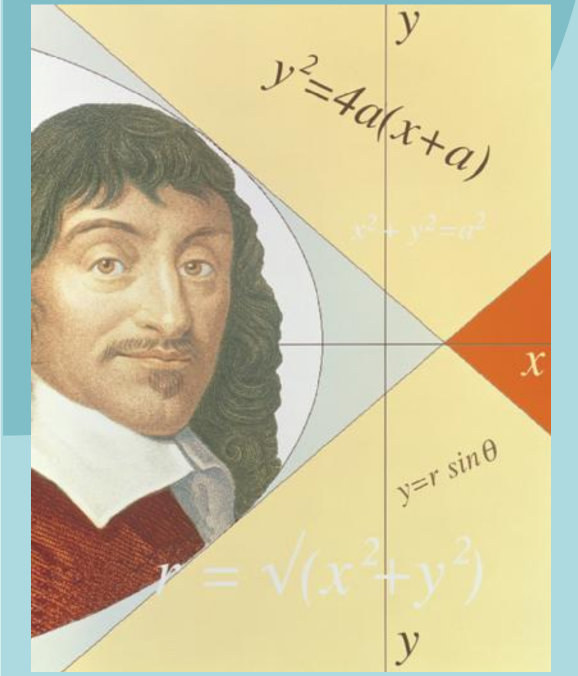
- **a** (4, 5), (3, –2) and (1, –4)
- **b** (8, –2), (6, 2) and (3, –7)
- **c** (1, 1), (1, 3) and (9, 2)
- **d** (–4, –3), (–1, –7) and (0, 0)

9 Show that (2, 1) lies on the circle $(x + 3)^2 + (y - 1)^2 = 25$ and find the coordinates of the other end of the diameter through (2, 1).

10 A diameter intersects a circle at (6, –4) and (–2, 6). Find the centre and radius of the circle.

History of mathematics

RENE DESCARTES (1596–1650)



Rene Descartes (*Reh-nay Day-cart*) (1596–1650) left school at the age of 16 and went to Paris, where he only occasionally studied mathematics. At 21 he became a soldier-for-pay, hiring out to the armies of various nations and minor political subdivisions. He alternated military service with study, and travelled throughout Europe, meeting a number of mathematicians. Although he was offered a commission as a lieutenant general, he declined. After nine years as a soldier his interests focused on mathematics and philosophy, which led him to settle in relative peace in the Netherlands. Here he studied and wrote for some twenty years.

During his slumber on 10 November 1619, Descartes had three profound dreams that affected his life. He claimed that in one of these dreams he was given the key to understanding nature. Although he never revealed exactly what this was, many believed that he was referring to the relationship between algebra and geometry. Whatever the truth of the story, this date may be said to be the founding date of analytic geometry. However, Descartes' works were not to be published for another 18 years.

In 1637 Descartes published *Discours de la methode* (or, more completely, *Discourse on the method of rightly conducting the reason and seeking truth in the sciences*). This was a program for conducting philosophical research. It was also an attempt to reach valid conclusions in many fields by systematic reasoning. The *Discours* included three appendices, each of which was a brilliant illustration of the application of the method to a different field. The first, *La dioptrique*, included the first published statement of the law of refraction, which was earlier discovered by the Dutch mathematician Willebrord Snell (1591–1626). The second was on meteorology, which included an explanation of the colours of the rainbow. The third, and most famous, was *La geometrie*.

La geometrie began with an ancient problem from the Greek mathematician Pappus (3rd century BC). Descartes applied algebra to geometry and geometry to algebra. Coordinate axes were implied but not explicitly used, and were thought of as oblique rather than the perpendicular coordinates we now call Cartesian. The notions of distance, angle between lines, and gradient/slope were not included, and no curves were plotted from an equation. Descartes did not consider negative numbers or negative coordinates, and only one equation was discussed in detail — the general second-degree equation that is discussed in this chapter. Descartes gave the conditions on the coefficients of the equation required for a curve to be an ellipse, parabola, or hyperbola. The details were left out, and the entire work was very challenging to read. Many readers found it difficult to make connections between the appendices and the body of the text, which related to philosophy. Regardless, the concepts of analytic geometry were introduced in this work. This impressive achievement was the only book on mathematics that Rene Descartes ever wrote.

Queen Christina of Sweden lured Descartes away from the Netherlands with an offer of a position at court, where there was an interest in philosophy and science. However, Descartes was unable to maintain good health. He contracted pneumonia from the cold, damp, early (5:00 am) rises that the Queen demanded for philosophy tutors, and he died at the age of 54.

Questions

1. When did Descartes leave school?
2. What did he start to do at the age of 21?
3. What did many believe that Descartes was referring to when he claimed that, in one of his dreams, he was given the key to understanding nature?
4. Descartes wrote the *Discours*, which included three appendices. What were they?
5. In *La geometrie*, Descartes discussed only one equation in detail. What was it?
6. Descartes gave the conditions on the coefficients of this equation. These conditions gave rise to what type of curves?

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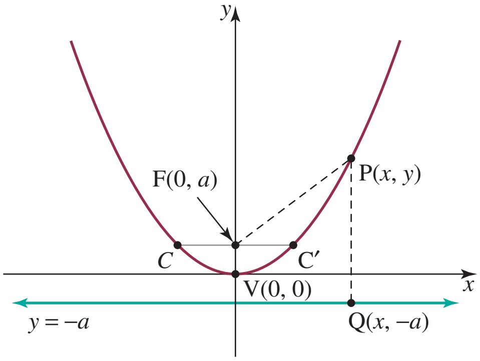
SkillSHEET 9.1

Cartesian form of the parabola

A parabola may be defined as the locus of a point P that moves so that the distance from P to a fixed point is equal to the distance from P to a fixed line. The fixed point is called the *focus* and the fixed line is called the *directrix* of the parabola.

We can obtain a general equation of a parabola in a simple form if we place the focus on the y-axis and the directrix parallel to the x-axis.

Let F(0, a) be a fixed point and D be the line $y = -a$. A point P(x, y) moves so that the distance PF is equal to the distance PQ from the line D.



From the definition of P:

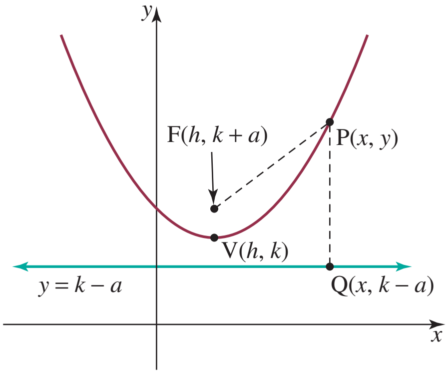
$$\begin{aligned} PF &= PQ \\ PF^2 &= PQ^2 \\ (x-0)^2 + (y-a)^2 &= (x-x)^2 + (y+a)^2 \\ x^2 + y^2 - 2ay + a^2 &= y^2 + 2ay + a^2 \\ x^2 &= 4ay \end{aligned}$$

Therefore, the equation of a parabola with the focus at F(0, a) and directrix $y = -a$ is $x^2 = 4ay$.

In this case, the *axis of symmetry* is the y-axis. The vertex V(0, 0) is at the origin and lies on the axis of symmetry midway between the focus and the directrix. The chord CC′ through the focus and perpendicular to the axis of symmetry is called the *latus rectum* and has length 4a units.

If the focus is moved to the x-axis at F(a, 0) and the directrix parallel to the y-axis has the equation is $x = -a$, then by following the same procedure, it can be shown that the equation of the parabola is $y^2 = 4ax$.

A more general form for the equation of a parabola can be found by shifting the first parabola so that the vertex is now (h, k). This would move the focus to the point F(h, $k + a$) and the directrix to the line $y = k - a$.



From the definition of P:

$$\begin{aligned} PF &= PQ \\ PF^2 &= PQ^2 \\ (x-h)^2 + [y-(k+a)]^2 &= (x-x)^2 + [y-(k-a)]^2 \end{aligned}$$

Expanding gives:

$$(x-h)^2 + y^2 - 2(k+a)y + (k+a)^2 = y^2 - 2(k-a)y + (k-a)^2$$

Expanding further, we have:

$$(x-h)^2 + y^2 - 2ky - 2ay + k^2 + 2ak + a^2 = y^2 - 2ky + 2ay + k^2 - 2ak + a^2$$

Collecting like terms and simplifying gives:

$$(x-h)^2 = 4ay - 4ak$$

$$\text{or}\quad (x-h)^2 = 4a(y-k)$$

Expansion of this equation would result in the form $Ax^2 + Dx + Ey + F = 0$ described in the table at the start of this chapter.

As before, if the focus is at F($h + a$, k) and the directrix parallel to the y-axis has the equation $x = h - a$, then it can be shown that the equation of the parabola would be $(y - k)^2 = 4a(x - h)$. This can be expanded to give $Cy^2 + Dx + Ey + F = 0$.

It is now possible to summarise the possible parabolas with the following table:

	Parabola	**Vertex**	**Focus**	**Directrix**
1.	$(x-h)^2 = 4a(y-k)$	(h, k)	$(h, k+a)$	$y = k - a$
2.	$(x-h)^2 = -4a(y-k)$	(h, k)	$(h, k-a)$	$y = k + a$
3.	$(y-k)^2 = 4a(x-h)$	(h, k)	$(h+a, k)$	$x = h - a$
4.	$(y-k)^2 = -4a(x-h)$	(h, k)	$(h-a, k)$	$x = h + a$

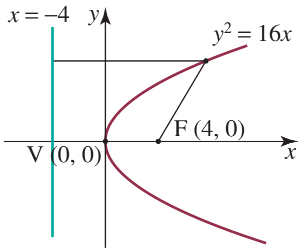
When we are solving problems with parabolas, it is useful to be aware of the orientation of the parabola we are dealing with; that is, decide if its type will be 1, 2, 3 or 4 (from the table above). Once we have established this, then (in most cases) a diagram may be used to consolidate our thinking and yield a solution very quickly.

The following worked examples will demonstrate the application of the properties discussed above and show how they can be used to derive information and equations.

WORKED Example 5

a **Find the coordinates of the vertex and the focus, and the equation of the directrix for the parabola with equation $y^2 = 16x$.**

b **Sketch the parabola $y^2 = 16x$.**

THINK	WRITE
a ① State the general form of the required parabola.	**a** $(y - k)^2 = 4a(x - h)$
② Compare this to $y^2 = 16x$ and list the values of h, k and a.	For $y^2 = 16x$, $h = 0$, $k = 0$ and $a = 4$.
③ Write the coordinates of the vertex using (h, k).	So the vertex is at (0, 0).
④ Write the coordinates of the focus using $(h + a, k)$.	The focus is at (4, 0).
⑤ Write the equation of the directrix using $x = h - a$.	The equation of the directrix is $x = -4$.
b ① On a Cartesian plane, mark in the vertex, focus and directrix of the parabola. ② Sketch the parabola in relation to these features.	**b** 

WORKED Example 6

Find the equation of the parabola whose focus is at (6, –2) and whose directrix is the line $x - 2 = 0$.

THINK	WRITE
① State the general form of the required parabola.	$(y - k)^2 = 4a(x - h)$
② State the general form of the focus and match it to the known coordinates of the focus.	The focus is at $(h + a, k)$ or $(6, -2)$ so $k = -2$ and $h + a = 6$. [1]
③ State the general form of the directrix equation and match it to the known equation.	The directrix has equation $x = h - a$ or $x - 2 = 0$ so $h - a = 2$. [2]
④ Determine h and a by solving the pair of simultaneous equations [1] and [2].	[1] + [2]: $2h = 8$ or $h = 4$ Substituting $h = 4$ into [1]: $4 + a = 6$ or $a = 2$.
⑤ Substitute the values for h, k and a to write the equation.	The equation of the parabola is: $(y + 2)^2 = 8(x - 4)$ or $y^2 + 4y - 8x + 36 = 0$.

In some cases, we need to use the technique of 'completing the square' to rewrite the equation into the general form so that h, k and a can be determined. This is demonstrated in the following worked example.

WORKED Example 7

Given the parabola whose equation is $y^2 + 8y - 6x + 4 = 0$, determine the coordinates of the vertex, the coordinates of the focus and the equation of the directrix.

THINK	WRITE
1 Write the equation.	$y^2 + 8y - 6x + 4 = 0$
2 Rearrange the equation so that the y terms are on the left-hand side.	$y^2 + 8y = 6x - 4$
3 Use the technique of completing the square on the left-hand side of the equation. (Remember to balance the equation by adding the same amount on the right-hand side.)	$y^2 + 8y + 16 = 6x - 4 + 16$ $y^2 + 8y + 16 = 6x + 12$
4 Factorise both sides of the equation.	$(y + 4)^2 = 6(x + 2)$
5 Compare this equation with the appropriate general form of a parabola and determine the values for h, k and a.	This is of the form $(y - k)^2 = 4a(x - h)$ where $h = -2$, $k = -4$ and $a = \frac{3}{2}$.
6 State the coordinates of the vertex using (h, k).	The vertex has the coordinates $(-2, -4)$.
7 State the coordinates of the focus using $(h + a, k)$.	The focus has the coordinates $(-\frac{1}{2}, -4)$.
8 State the equation of the directrix using $x = h - a$.	The equation of the directrix is $x = -3\frac{1}{2}$.

remember

1. A parabola can be defined as the locus of a point P(x, y) that moves so that its distance from a point F is equal to its distance from a line D.
2. For parabolas of the form $(x - h)^2 = \pm 4a(y - k)$ on the Cartesian plane:
   (a) the vertex is (h, k)
   (b) the focus is $(h, k \pm a)$
   (c) the directrix is $y = k \mp a$.
3. For parabolas of the form $(y - k)^2 = \pm 4a(x - h)$ on the Cartesian plane:
   (a) the vertex is (h, k)
   (b) the focus is $(h \pm a, k)$
   (c) the directrix is $x = h \mp$ a.

EXERCISE 9B

Cartesian form of the parabola

WORKED Example 5a

1 Find the coordinates of the vertex (V) and the focus (F), and the equation of the directrix (D) for each of the following parabolas.

a $y^2 = 6x$ **b** $x^2 = 8y$ **c** $3y^2 = -4x$

WORKED Example 5b

2 Sketch the following parabolas, clearly identifying the important features.

a $y = 2x^2$ **b** $y = -2x^2 + 2$ **c** $x = 3y^2 + 1$

d $8y + x^2 = 2$ **e** $(x - 2)^2 = 8(y - 1)$ **f** $(y - 2)^2 = 6(x + 3)$

WORKED Example 6

3 Determine the equation of each of the following parabolas.

a Focus at (3, 0) and the equation of the directrix is $x + 3 = 0$

b Focus at (0, 6) and the equation of the directrix is $y = 0$

c Vertex at the origin, axis of symmetry along the x-axis, and passes through the point (–3, 6)

d Derive the equation of the parabola with its vertex at (–2, 3) and its focus at (1, 3)

4 Derive the equation of the locus of a point that moves so that its distance from (–2, 3) equals its distance from the line $x + 6 = 0$.

WORKED Example 7

5 For each of the following parabolas, find the coordinates of the vertex, the coordinates of the focus and the equation of the directrix.

a $y^2 - 4y + 6x - 8 = 0$ **b** $3x^2 - 9x - 5y - 2 = 0$

c $y^2 - 4y - 6x + 13 = 0$

6 Find the equation of the parabola with its axis of symmetry parallel to the x-axis, and which passes through the points (3, 3), (6, 5) and (6, –3).

7 Find the equation of the parabola with a vertical axis of symmetry and which passes through the points (4, 5), (–2, 11) and (–4, 21).

Practical applications of parabolas

An important application of parabolic arcs follows from the physical law that when a ray (light or radio waves) strikes a reflecting surface, the angle of incidence equals the angle of reflection. The relationship between this law and parabolas lies in the fact that the line tangent to the graph of a parabola at a point P(x, y) makes equal angles with the ray from P to the focus of the parabola F and the line through P parallel to the axis of symmetry of the parabola.

Thus, rays parallel to the axis of a parabolic reflector will be reflected precisely to the focus of the parabola. This principle is used in reflecting telescopes, where the parallel rays of light from the stars are reflected off a parabolic mirror to the eyepiece located at the focus. The same principle applies to many radar, microwave and television antennas, which are designed to gather rays by means of their parabolic reflecting dishes.

The same principle, in reverse, explains the design of automobile headlamps and all types of spotlights. By placing the bulb at the focus of a parabolic mirror, designers ensure that the beams of light will leave the headlamp parallel to the axis of the parabolic mirror.

For a very readable account of the many roles played by parabolas in astronomy, radio communications, radar, wind tunnel photography, submarine tracking and bridge construction, see Lee Whit's *The standup conic presents: The parabola and applications.*[1]

The parabola is defined as the locus of a point P(x, y) that is equidistant from a fixed point F (focus) and a fixed line D (directrix).

Eccentricity

A parabola is defined as the locus of a point that moves so that the distance from a focus, F, is equal to the distance from the directrix, D. To describe this relationship in a slightly different fashion, if PF is the distance from the point P to the focus F, and PQ is the distance from the point P to the directrix D, then the ratio of PF:PQ = 1.

All conics can be described by using a similar ratio. We call the value of the ratio the *eccentricity*, e. As we've just shown, if the eccentricity is 1, the conic is a parabola. Changing the value of e results in the following conics:

1. If $0 < e < 1$, the conic section is an ellipse.
2. If $e = 1$, the conic section is a parabola
3. If $e > 1$, the conic section is a hyperbola
4. If $e = 0$, the conic section is a circle. (Note that this definition is not helpful in determining the equation of the circle.)

Cartesian form of the ellipse

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Interactivity:
Elliptical graph
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Ellipses are found in many practical situations and in nature. For example, they are used in the wings of aircraft and sometimes in gears designed for racing bicycles. Water pipes may be designed with elliptical cross-sections to allow the pipes to expand without breaking when the water freezes. Ellipses are also used in instruments employed to study aircraft noise in wind tunnels (the sound at one focus can be received at the other with relatively little noise interference from other sources). The triggering mechanisms in some lasers are elliptical, and stones on a beach become more and more elliptical as they are ground down by waves. Ellipses have been used in the study of fossils: the *ellipsolith*, once interpreted as a separate species, is now known to be merely an elliptically deformed nautilus.

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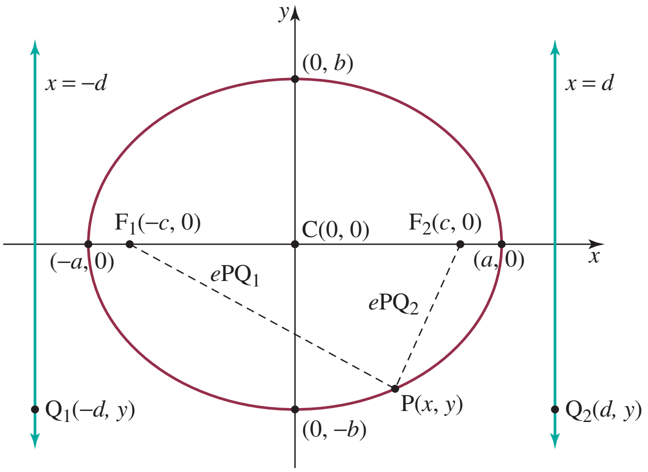
[1] L. Whit, 'The standup conic presents: The parabola and applications', *UMAP Journal*, vol. 3:3, 1982, pp. 285–313.



An ellipse has 2 foci and 2 directrices. We know that for an ellipse, $0 < e < 1$ where $e = \frac{PF}{PQ}$. (PF is the distance from the point P to the focus and PQ is the distance from P to the directrix.)

In the ellipse shown, the centre of the ellipse lies on the origin. The vertices of the ellipse are the points ($\pm a$, 0) and (0, $\pm b$). If one focus F_1 is the point ($-c$, 0), the other focus F_2 is the point (c, 0). If PQ_1 is the distance from P (x, y) to the directrix D_1, then $PF_1 = ePQ_1$; likewise $PF_2 = ePQ_2$.

Consider the sum of the distances from P to the foci.

$$\begin{aligned} PF_1 + PF_2 &= ePQ_1 + ePQ_2 \\ &= e(PQ_1 + PQ_2) \\ &= e(Q_1Q_2) \text{ (which will be a constant value)} \end{aligned}$$

As the point (a, 0) lies on the ellipse, it can be seen that $PF_1 + PF_2 = 2a$, where $PF_1 = a + c$, $PF_2 = a - c$ for P(a, 0).

This leads to an alternative definition of an ellipse. An ellipse can be defined as the locus of a point P(x, y) such that the sum of its distances from two given points F_1 and F_2 is constant.

To find the equation of an ellipse with vertices ($\pm a$, 0), (0, $\pm b$) and centre (0, 0) we use

$$PF_1 + PF_2 = 2a$$

which can be rearranged to give $PF_1 = 2a - PF_2$

Squaring gives
$$\begin{aligned} PF_1{}^2 &= (2a - PF_2)^2 \\ &= 4a^2 - 4aPF_2 + PF_2{}^2 \end{aligned}$$

Substituting for P, F_1 and F_2 gives:

$$(x + c)^2 + (y - 0)^2 = 4a^2 - 4aPF_2 + (x - c)^2 + (y - 0)^2$$

Expanding results in:

$$\begin{aligned} x^2 + 2cx + c^2 + y^2 &= 4a^2 - 4aPF_2 + x^2 - 2cx + c^2 + y^2 \\ 4cx &= 4a^2 - 4aPF_2 \\ cx &= a^2 - aPF_2 \\ cx - a^2 &= -aPF_2 \end{aligned}$$

Squaring again gives:

$$\begin{aligned} (cx - a^2)^2 &= a^2PF_2{}^2 \\ c^2x^2 - 2a^2cx + a^4 &= a^2[(x - c)^2 + (y - 0)^2] \\ &= a^2(x^2 - 2cx + c^2 + y^2) \\ &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 \\ a^4 - a^2c^2 &= a^2x^2 - c^2x^2 + a^2y^2 \\ a^2(a^2 - c^2) &= (a^2 - c^2)x^2 + a^2y^2 \end{aligned}$$

Dividing both sides by $a^2(a^2 - c^2)$ gives:

$$1 = \frac{(a^2 - c^2)x^2}{a^2(a^2 - c^2)} + \frac{a^2y^2}{a^2(a^2 - c^2)}$$

$$= \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2}$$

The point $(0, b)$ lies on the ellipse.

Therefore $$1 = \frac{b^2}{a^2 - c^2}$$

and $$a^2 - c^2 = b^2 \text{ [1]}.$$

The equation of the ellipse becomes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

We need to consider the foci and directrices in terms of a and e.

If the directrices have equations $x = \pm d$, then Q_1 is the point $(-d, y)$ and Q_2 is the point (d, y). This means that $Q_1Q_2 = 2d$. We noted earlier that $PF_1 + PF_2 = eQ_1Q_2$ and this can also be represented as $PF_1 + PF_2 = 2a$. We can now say that:

$$eQ_1Q_2 = 2a$$
$$e(2d) = 2a$$
$$ed = a$$
$$d = \frac{a}{e}$$

Therefore, the equations of the directrices are $x = \pm\frac{a}{e}$.

The foci have the form $(\pm c, 0)$. If we consider $PF_2 = ePQ_2$ for the point $P(a, 0)$, then Q_2 will be $\left(\frac{a}{e}, 0\right)$, PF_2 will equal $a - c$ and PQ_2 will equal $\frac{a}{e} - a$. If we substitute these values we get:

$$a - c = e\left(\frac{a}{e} - a\right)$$
$$a - c = a - ae$$
$$c = ae$$

Therefore the foci are $(\pm ae, 0)$.

We previously found that $a^2 - c^2 = b^2$ [1]

We know that $c = ae$ and therefore $c^2 = a^2e^2$.

This means that $$a^2 - a^2e^2 = b^2$$
$$a^2e^2 = a^2 - b^2$$
$$e^2 = \frac{a^2 - b^2}{a^2}$$
$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

Summary

We shall now summarise the characteristics of four forms of the ellipse — two forms of the ellipse whose major axis is parallel to the x-axis, and two forms whose major axis is parallel to the y-axis.

Form 1

For an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$:

1. The *centre* is at the origin.
2. The *vertices* are given by $(\pm a, 0)$ and $(0, \pm b)$.
3. The horizontal segment joining the vertices $(\pm a, 0)$ is called the *major axis* and has length $2a$.
4. The vertical segment joining the vertices $(0, \pm b)$ is called the *minor axis* and has length $2b$.
5. The *eccentricity* is $e = \frac{\sqrt{a^2 - b^2}}{a}$.
6. The *foci* are given by $(\pm ae, 0)$.
7. The equations of the *directrices* are $x = \pm\frac{a}{e}$.

Form 2

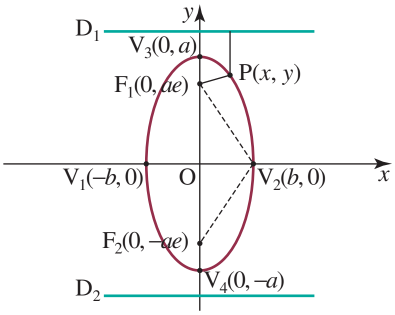
Now if we look at shifting this conic so that its centre moves from the origin to the point (h, k), we have the following.

For an ellipse of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where $a > b$:

1. The *centre* is at (h, k).
2. The *vertices* are given by $(h \pm a, k)$ and $(h, k \pm b)$.
3. The horizontal segment joining vertices $(h \pm a, k)$ is called the *major axis* and has length $2a$.
4. The vertical segment joining vertices $(h, k \pm b)$ is called the *minor axis* and has length $2b$.
5. The *eccentricity* is $e = \frac{\sqrt{a^2 - b^2}}{a}$.
6. The *foci* are given by $(h \pm ae, k)$.
7. The equations of the *directrices* are $x = h \pm \frac{a}{e}$.

Form 3

The other type of ellipse that we can consider is the ellipse whose major axis is parallel to the y-axis (see diagram).



For an ellipse of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ where $a > b$:

1. The *centre* is at the origin.
2. The *vertices* are given by $(\pm b, 0)$ and $(0, \pm a)$.
3. The vertical segment joining the vertices $(0, \pm a)$ is called the *major axis* and has length $2a$.
4. The horizontal segment joining the vertices $(\pm b, 0)$ is called the *minor axis* and has length $2b$.

5. The *eccentricity* is $e = \frac{\sqrt{a^2 - b^2}}{a}$.
6. The *foci* are given by $(0, \pm ae)$.
7. The equations of the *directrices* are $y = \pm\frac{a}{e}$.

Form 4

Now if we look at shifting this conic so that its centre moves from the origin to the point (h, k) we have the following:

For an ellipse of the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ where $a > b$:

1. The *centre* at (h, k).
2. The *vertices* are given by $(h, k \pm a)$ and $(h \pm b, k)$.
3. The vertical segment joining vertices $(h, k \pm a)$ is called the *major axis* and has length $2a$.
4. The horizontal segment joining vertices $(h \pm b, k)$ is called the *minor axis* and has length $2b$.
5. The *eccentricity* is $e = \frac{\sqrt{a^2 - b^2}}{a}$.
6. The *foci* are $(h, k \pm ae)$.
7. The equations of the *directrices* are: $y = k \pm \frac{a}{e}$.

You can see that if these equations are rearranged, they can be written as:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

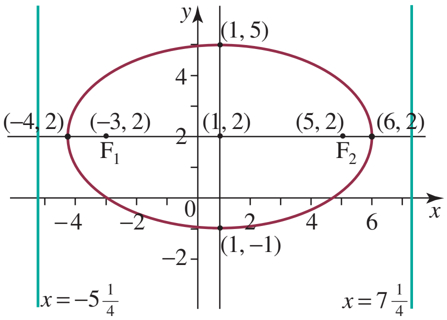
the form for an ellipse given at the start of this chapter.

WORKED Example 8

a For the ellipse $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$, find the eccentricity; the coordinates of the centre, the vertices and the foci; the equations of the directrices and the length of the major and minor axes.

b Sketch the ellipse $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$.

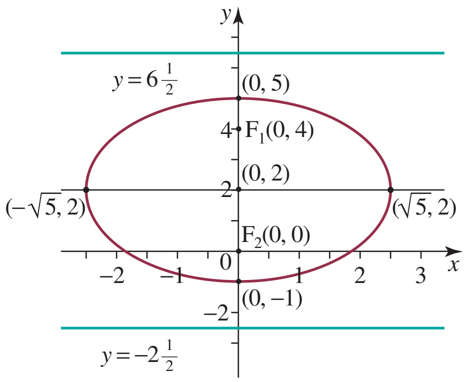
THINK	WRITE
a ① Identify whether the major axis is horizontal or vertical and state the corresponding general form of the ellipse.	**a** Horizontal major axis, so the ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$.
② Compare this to the given equation for the ellipse and identify h, k, a and b.	For $\frac{(x-1)^2}{25} + \frac{(y-2)^2}{9} = 1$, $h = 1$, $k = 2$, $a = 5$ and $b = 3$.

THINK	**WRITE**
③ Calculate the eccentricity using $e = \frac{\sqrt{a^2 - b^2}}{a}$. Note that $0 < e < 1$ which matches the fact that we are considering an ellipse.	Eccentricity: $e = \frac{\sqrt{a^2 - b^2}}{a}$ $= \frac{\sqrt{25 - 9}}{5}$ $= \frac{4}{5}$
④ Write the coordinates of the centre using (h, k).	Centre: (h, k) The centre is at (1, 2).
⑤ Write the coordinates of the vertices using $(h \pm a, k)$ and $(h, k \pm b)$.	Vertices: $(h \pm a, k)$ and $(h, k \pm b)$ The vertices are at (–4, 2), (6, 2), (1, –1) and (1, 5).
⑥ Write the coordinates of the foci using $(h \pm ae, k)$.	Foci: $(h \pm ae, k)$ The foci are at (–3, 2) and (5, 2).
⑦ Write the equations of the directrices using $x = h \pm \frac{a}{e}$.	Directrices: $x = h \pm \frac{a}{e}$ The equations of the directrices are $x = -5\frac{1}{4}$ and $x = 7\frac{1}{4}$.
⑧ Determine the length of the major axis using $2a$.	The length of the major axis is 10.
⑨ Determine the length of the minor axis using $2b$.	The length of the minor axis is 6.
b ① On a Cartesian plane, mark the centre, the vertices, the foci and the directrices of the ellipse. ② Sketch the ellipse in relation to these features.	**b** 

Another way of completing Worked example 8 is to consider the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and translate all features one unit to the right and two units up.

WORKED Example 9

Sketch the ellipse $9x^2 + 5(y - 2)^2 = 45$, identifying the important features.

THINK	WRITE
① To be able to compare the equation to a general form, divide both sides of the equation by 45.	$9x^2 + 5(y - 2)^2 = 45$ $\frac{x^2}{5} + \frac{(y-2)^2}{9} = 1$
② Identify whether the major axis is horizontal or vertical and state the corresponding general form of the ellipse.	Vertical major axis, so the ellipse is of the form $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,\ a > b$
③ Compare this to the given equation for the ellipse and identify h, k, a and b.	where $h = 0$, $k = 2$, $a = 3$ and $b = \sqrt{5}$.
④ Calculate the eccentricity using $e = \frac{\sqrt{a^2 - b^2}}{a}$.	Eccentricity: $e = \frac{\sqrt{a^2 - b^2}}{a}$ $= \frac{\sqrt{9-5}}{3}$ $= \frac{2}{3}$
⑤ Write the coordinates of the centre using (h, k).	Centre: (h, k) The centre is at (0, 2).
⑥ Write the coordinates of the vertices using $(h, k \pm a)$ and $(h \pm b, k)$.	Vertices: $(h, k \pm a)$ and $(h \pm b, k)$ The vertices are at $(0, -1)$, $(0, 5)$, $(-\sqrt{5}, 2)$ and $(\sqrt{5}, 2)$.
⑦ Write the coordinates of the foci using $(h, k \pm ae)$.	Foci: $(h, k \pm ae)$ The foci are at (0, 0) and (0, 4).
⑧ Write the equations of the directrices using $y = k \pm \frac{a}{e}$.	Directrices: $y = k \pm \frac{a}{e}$ The equations of the directrices are $y = -2\frac{1}{2}$ and $y = 6\frac{1}{2}$.
⑨ On a Cartesian plane, mark the centre, the vertices, the foci and the directrices of the ellipse. Sketch the ellipse in relation to these features.	

WORKED Example 10

Given the ellipse whose equation is $4x^2 + 9y^2 - 48x + 72y + 144 = 0$, find the eccentricity, the coordinates of the centre, the vertices, the foci and the equations of the directrices.

THINK	WRITE
① Use the technique of completing the square to write the equation in the general form of an ellipse.	$4x^2 + 9y^2 - 48x + 72y + 144 = 0$ $4(x^2 - 12x) + 9(y^2 + 8y) = -144$ $4(x^2 - 12x + 36) + 9(y^2 + 8y + 16) = -144 + 288$ $4(x - 6)^2 + 9(y + 4)^2 = 144$ $\frac{(x-6)^2}{36} + \frac{(y+4)^2}{16} = 1$
② Identify whether the major axis is horizontal or vertical and state the corresponding general form of the ellipse.	Horizontal major axis, so the ellipse is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$
③ Compare this to the given equation for the ellipse and identify h, k, a and b.	where $h = 6$, $k = -4$, $a = 6$ and $b = 4$.
④ Calculate the eccentricity using $e = \frac{\sqrt{a^2 - b^2}}{a}$.	Eccentricity: $e = \frac{\sqrt{a^2 - b^2}}{a}$ $= \frac{\sqrt{36 - 16}}{6}$ $= \frac{\sqrt{20}}{6}$ $= \frac{2\sqrt{5}}{6}$ $= \frac{\sqrt{5}}{3}$
⑤ Write the coordinates of the centre using (h, k).	Centre: (h, k) The centre is at $(6, -4)$.
⑥ Write the coordinates of the vertices using $(h \pm a, k)$ and $(h, k \pm b)$.	Vertices: $(h \pm a, k)$ and $(h, k \pm b)$ The vertices are at $(0, -4)$, $(12, -4)$, $(6, -8)$ and $(6, 0)$.
⑦ Write the coordinates of the foci using $(h \pm ae, k)$.	Foci: $(h \pm ae, k)$ The foci are at $(6 - 2\sqrt{5}, -4)$ and $(6 + 2\sqrt{5}, -4)$ or approximately $(1.5, -4)$ and $(10.5, -4)$.
⑧ Write the equations of the directrices using $x = h \pm \frac{a}{e}$.	Directrices: $x = h \pm \frac{a}{e}$ The equations of the directrices are $x = \frac{30 - 18\sqrt{5}}{5}$ and $x = \frac{30 + 18\sqrt{5}}{5}$ or approximately $x = -2.05$ and $x = 14.05$.

WORKED Example 11

Find the equation of the ellipse with its centre at (1, 2), focus at (6, 2) and which passes through the point (4, 6).

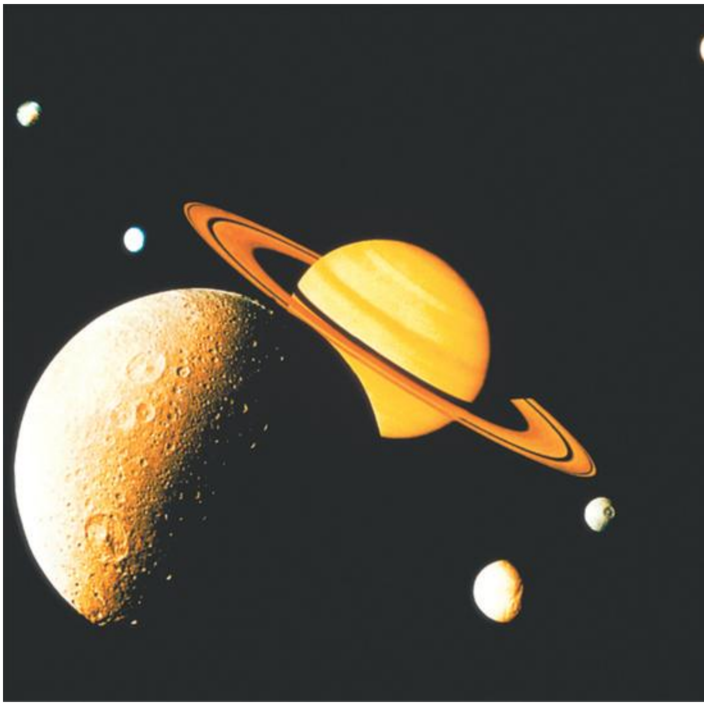
THINK	WRITE
① Identify whether the major axis is horizontal or vertical and state the corresponding general form of the ellipse.	As the foci and centre lie along the major axis, the major axis is horizontal, so the equation of the ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, a>b.$
② We need to use the given information to identify h, k, a^2 and b^2. First use the coordinates of the centre to identify h and k.	Since the centre is at (1, 2), $h = 1$ and $k = 2$. So $\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1$
③ Since (4, 6) lies on the curve, substitute $x = 4$ and $y = 6$ into the equation. Label this as equation [1].	For (4, 6): $\frac{(4-1)^2}{a^2}+\frac{(6-2)^2}{b^2}=1$ $\frac{9}{a^2}+\frac{16}{b^2}=1$ [1]
④ Since one of the foci is (6, 2), write a relationship for the x-coordinate and express e in terms of a. Note that the given focus is to the right of the centre. That is, consider $(h + ae, k)$.	Since one of the foci is (6, 2), $h+ae=6$ $1+ae=6$ $ae=5$ $e=\frac{5}{a}$
⑤ Replace e in the relationship $e=\frac{\sqrt{a^2-b^2}}{a}$ to obtain another relationship involving a and b only. Label this as equation [2].	Now $e=\frac{\sqrt{a^2-b^2}}{a}$ so $\frac{5}{a}=\frac{\sqrt{a^2-b^2}}{a}$ $5=\sqrt{a^2-b^2}$ $25=a^2-b^2$ $a^2=25+b^2$ [2]
⑥ Replace a^2 with $25 + b^2$ in equation [1].	Substituting [2] into [1]: $\frac{9}{25+b^2}+\frac{16}{b^2}=1$

THINK	**WRITE**
7 Simplify and solve for b^2.	$\frac{9b^2 + 16(25 + b^2)}{b^2(25 + b^2)} = 1$ $9b^2 + 400 + 16b^2 = 25b^2 + b^4$ $400 = b^4$ $b^2 = 20$
8 Find a^2 by substituting for b^2 in equation [2].	Substituting for b^2 in [2]: $a^2 = 25 + 20$ $= 45$
9 Write the equation for the given ellipse.	Equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1.$

Elliptical orbits

The planets in the solar system revolve around the sun in elliptical orbits with the sun at one focus. Most of the planets, including Earth, have orbits that are nearly circular. This is demonstrated in the table as their eccentricities are close to zero. Mercury is the exception, with an eccentricity of 0.21. Other members of the solar system have orbits that are more eccentric. The body Pluto, previously thought of as a planet, has an eccentricity of 0.25. Icarus, an asteroid about 1.6 km wide, revolves around the sun every 409 earth days and has an orbital eccentricity of 0.83.

Planet	Eccentricity
Earth	0.02
Jupiter	0.05
Mars	0.09
Mercury	0.21
Neptune	0.01
Saturn	0.06
Uranus	0.05
Venus	0.01



WORKED Example 12

The orbit of the comet Koboutek is about 44 astronomical units (AU) wide by 3600 astronomical units (AU) long. (One AU is half the major axis of the Earth's orbit; that is, about 1.48×10^8 km or 9.26×10^7 miles). Find the eccentricity of the orbit.

THINK	WRITE
① State the relationship for eccentricity.	$e = \frac{\sqrt{a^2 - b^2}}{a}$
② To calculate e, we need the values of a and b. Identify the values for a and b from the given information.	Length of the major axis is 3600 AU, so $a = 1800$. Length of the minor axis is 44 AU, so $b = 22$.
③ Substitute the values for a and b into the equation and calculate e.	$e = \frac{\sqrt{1800^2 - 22^2}}{1800}$ $e \approx 0.999\,925\,3$
④ State the answer.	The orbit has an eccentricity of approximately 0.9999.

remember

1. An ellipse can be defined as the locus of a point $P(x, y)$ that moves so that the sum of its distance from two given points F_1 and F_2 is a constant.
2. For and ellipse, $0 < e < 1$ where $e = \frac{\sqrt{a^2 - b^2}}{a}$.
3. The length of the major axis is $2a$.
4. The length of the minor axis is $2b$.
5. For an ellipse with a horizontal major axis the equation is of the form:

   $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$ with
   (a) centre (h, k)
   (b) vertices $(h \pm a, k)$ and $(h, k \pm b)$
   (c) foci $(h \pm ae, k)$
   (d) directrices $x = h \pm \frac{a}{e}$.
6. For an ellipse with a vertical major axis $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, $a > b$:
   (a) centre (h, k)
   (b) vertices $(h, k \pm a)$ and $(h \pm b, k)$
   (c) foci $(h, k \pm ae)$
   (d) directrices $y = k \pm \frac{a}{e}$.

EXERCISE 9C Cartesian form of the ellipse

WORKED Example 8

1 **a** For each of the following ellipses, find the eccentricity; the coordinates of the centre, the vertices and the foci; the equations of the directrices and the length of the major and minor axes.

b Sketch each ellipse.

i $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **ii** $\frac{x^2}{169} + \frac{y^2}{144} = 1$

iii $\frac{x^2}{64} + \frac{y^2}{81} = 1$ **iv** $x^2 + \frac{y^2}{25} = 1$

v $\frac{(x-4)^2}{25} + \frac{(y-3)^2}{16} = 1$ **vi** $\frac{(x-1)^2}{169} + \frac{(y+2)^2}{144} = 1$

vii $\frac{(x+3)^2}{9} + \frac{(y-2)^2}{25} = 1$ **viii** $\frac{(x+5)^2}{4} + \frac{(y+1)^2}{9} = 1$

WORKED Example 9

2 Sketch each of the following ellipses, identifying the important features.

a $225x^2 + 289y^2 = 65\,025$ **b** $9x^2 + 4y^2 = 36$

c $49(x - 2)^2 + 64y^2 = 3136$ **d** $36(x + 3)^2 + 25(y - 6)^2 = 900$

WORKED Example 10

3 Given the ellipse whose equation is $9x^2 + 16y^2 - 36x + 96y + 36 = 0$, find the eccentricity, the coordinates of the centre, the foci, the vertices, and the equations of the directrices.

4 State the equation of the following ellipses if the centre of each ellipse is at the origin.

a Foci $(\pm 4, 0)$, vertices $(\pm 5, 0)$

b Foci $(0, \pm 8)$, vertices $(0, \pm 17)$

c Foci $(0, \pm 6)$, length of minor axis = 16

d Foci $(\pm 5, 0)$, eccentricity = $\frac{5}{8}$

WORKED Example 11

5 Find the equation of the ellipse with:

a its centre at $(4, -1)$, focus at $(1, -1)$ and which passes through the point $(8, 0)$.

b its centre at $(3, 1)$, vertex on major axis at $(3, -2)$ and eccentricity = $\frac{1}{3}$

c focus at $(-1, -1)$, directrix is the y-axis and eccentricity = $\frac{\sqrt{2}}{2}$

d foci at $(\pm 8, 0)$ and which passes through the point $(8, \frac{18}{5})$.

6 Write the equation of the ellipse with its centre at the origin, its foci on the x-axis, and which passes through the points $(-3, 2\sqrt{3})$ and $\left(4, \frac{4\sqrt{5}}{3}\right)$.

7 A point P(x, y) moves so that the sum of its distances from two points $(2, -3)$ and $(2, 7)$ is 12. Derive the equation of its locus. What is the curve?

8 A point P(x, y) moves so that the sum of its distances from $(3, 1)$ and $(-5, 1)$ is 10. Derive the equation of its locus.

9 An arch is in the form of a semi-ellipse, with its major axis as the span of the arch. If the span is 80 m and the height is 30 m, find the height of the arch at a point 15 m from the minor axis.

WORKED Example 12

10 The orbit of a comet is about 220 astronomical units (AU) wide by 2800 astronomical units (AU) long. Find the eccentricity of the orbit.

11 Compare the orbit of a comet with eccentricity 0.87 to the orbit of the comet in question **10**.

Cartesian form of the hyperbola

eBook plus

Interactivities:
Conical hyperbolas I
int-0832
Conical hyperbolas II
int-0833

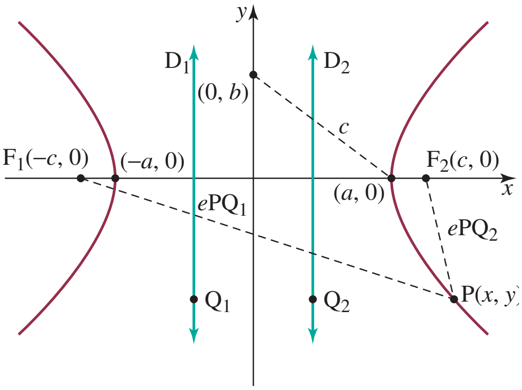
Hyperbolic curves are common in nature and in science. Hyperbolic paths occur in Einstein's theory of relativity. They form the basis for the LORAN (Long Range Navigation) radio navigation system. A comet that does not return to its sun follows a hyperbolic path (the probability of its being parabolic is zero). Many telescopes, from simple reflecting instruments to the space telescope used by NASA, employ small hyperbolic mirrors in conjunction with larger, parabolic ones.

Like an ellipse, a hyperbola has 2 foci and 2 directrices. If the line connecting the foci is horizontal, we say that the hyperbola has a horizontal transverse axis. If the line connecting the foci is vertical, we say that the hyperbola has a vertical transverse axis. We shall also consider hyperbolas where the foci lie on the line $y = x$ or $y = -x$. These hyperbolas can be written in the general conic form $Ax^2 - Cy^2 + DX + EY + F = 0$.

Hyperbolas with a horizontal transverse axis

We know that for a hyperbola, $e > 1$ where $e = \frac{\text{PF}}{\text{PQ}}$ (PF is the distance from the point P to the focus and PQ is the distance from P to the directrix).

In the hyperbola shown, the centre of the hyperbola lies on the origin. The vertices of the hyperbola are the points $(\pm a, 0)$. If one focus F_1 is the point $(-c, 0)$, the other focus F_2 is the point $(c, 0)$. If PQ_1 is the distance from $\text{P}(x, y)$ to the directrix D_1, then $\text{PF}_1 = e\text{PQ}_1$; likewise $\text{PF}_2 = e\text{PQ}_2$.



Consider the absolute value of the difference between the distances from the point P to the two foci.

$$\begin{aligned}|\text{PF}_1 - \text{PF}_2| &= |e\text{PQ}_1 - e\text{PQ}_2|\\ &= e|\text{PQ}_1 - \text{PQ}_2|\\ &= e|\text{Q}_1\text{Q}_2| \text{ (which will be a constant value).}\end{aligned}$$

As the point $(a, 0)$ lies on the hyperbola, it can be seen that $\text{PF}_1 = 2a + c$ and $\text{PF}_2 = c$. Therefore $|\text{PF}_1 - \text{PF}_2| = 2a$

This leads to an alternative definition of a hyperbola. A hyperbola can be defined as the locus of a point $\text{P}(x, y)$ such that the absolute value of the difference between its distances from two fixed points F_1 and F_2 is a constant.

To find the equation of a hyperbola with vertices at $(\pm a, 0)$ and centre $(0, 0)$ we use:

$$\text{PF}_1 - \text{PF}_2 = 2a$$

which can be rearranged to give $\quad \text{PF}_1 = 2a + \text{PF}_2$

Squaring both sides gives

$$\begin{aligned}\text{PF}_1{}^2 &= (2a + \text{PF}_2)^2\\ &= 4a^2 + 4a\text{PF}_2 + \text{PF}_2{}^2\end{aligned}$$

Substituting for P, F_1 and F_2 gives:

$$(x+c)^2+(y-0)^2=4a^2+4a\mathrm{PF}_2+(x-c)^2+(y-0)^2$$

Expanding results in $x^2+2cx+c^2+y^2=4a^2+4a\mathrm{PF}_2+x^2-2cx+c^2+y^2$

$$4cx=4a^2+4a\mathrm{PF}_2$$
$$cx=a^2+a\mathrm{PF}_2$$
$$cx-a^2=a\mathrm{PF}_2$$

Squaring again gives $(cx-a^2)^2=a^2\mathrm{PF}_2{}^2$

$$\begin{aligned}c^2x^2-2a^2cx+a^4&=a^2[(x-c)^2+(y-0)^2]\\&=a^2(x^2-2cx+c^2+y^2)\\&=a^2x^2-2a^2cx+a^2c^2+a^2y^2\end{aligned}$$

Rearranging gives $a^4-a^2c^2=a^2x^2-c^2x^2+a^2y^2$

$$a^2(a^2-c^2)=(a^2-c^2)x^2+a^2y^2$$

Dividing both sides by $a^2(a^2-c^2)$ gives

$$1=\frac{(a^2-c^2)x^2}{a^2(a^2-c^2)}+\frac{a^2y^2}{a^2(a^2-c^2)}$$

$$=\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}$$

Consider the point B(0, b) on the y-axis placed so that the distance from B to the vertex is c. This means that

$$c^2=a^2+b^2\ [2]$$

Rearranging gives $b^2=c^2-a^2$ or $a^2-c^2=-b^2$. The equation of the hyperbola becomes:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

We need to consider the foci and directrices in terms of a and e.

If the directrices have equations $x=\pm d$, then Q_1 is the point $(-d, y)$ and Q_2 is the point (d, y). This means that $Q_1Q_2=2d$. We noted earlier that $|\mathrm{PF}_1-\mathrm{PF}_2|=eQ_1Q_2$, and this can also be represented as $|\mathrm{PF}_1-\mathrm{PF}_2|=2a$. We can now say that:

$$e\,Q_1Q_2=2a$$
$$e\,(2d)=2a$$
$$ed=a$$
$$d=\frac{a}{e}$$

Therefore, the equations of the directrices are $x=\pm\frac{a}{e}$.

The foci have the form $(\pm c, 0)$. If we consider $\mathrm{PF}_2=e\mathrm{PQ}_2$ for the point P(a, 0), then Q_2 will be $\left(\frac{a}{e},0\right)$, PF_2 will equal $c-a$ and PQ_2 will equal $a-\frac{a}{e}$. If we substitute these values we get:

$$c-a=e\left(a-\frac{a}{e}\right)$$
$$c-a=ae-a$$
$$c=ae$$

Therefore the foci are $(\pm ae, 0)$.

We previously found that $c^2 = a^2 + b^2$ [1]
We know that $c = ae$ and therefore $c^2 = a^2e^2$. This means that $a^2e^2 = a^2 + b^2$. So:

$$e^2 = \frac{a^2 + b^2}{a^2}$$

$$e = \frac{\sqrt{a^2 + b^2}}{a}$$

Let us now consider the asymptotes. The general equation of a hyperbola is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = 1$$

$$b^2x^2 - a^2y^2 = a^2b^2$$

Rearranging gives $$a^2y^2 = b^2x^2 - a^2b^2$$ $$= b^2(x^2 - a^2)$$

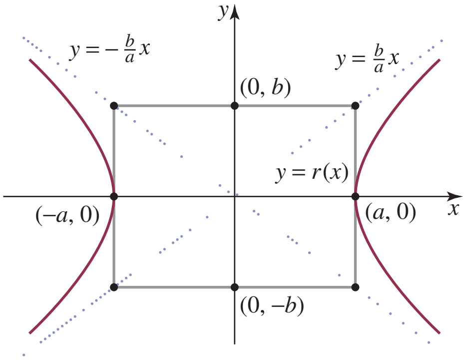
Dividing by a^2 results in $$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

As $x \to \infty$, x moves further away from a; this means that $x^2 - a^2$ will approach x^2. The asymptotes will be $y = \pm\frac{b}{a}x$.

A rectangle can be drawn passing through the vertices ($\pm a$, 0) with the corners on the asymptotes.

The axis of symmetry for the rectangle connecting the two foci is called the *transverse axis* and has length $2a$. The other axis of symmetry for the rectangle is called the *conjugate axis* and has length $2b$.



As we saw earlier with the ellipse, the conic can be shifted so that the centre moves from the origin to the point (h, k). The following table summarises both these cases.

Equation of the hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
Centre	(0, 0)	(h, k)
Vertices	$(\pm a, 0)$	$(h \pm a, k)$
Eccentricity	$e = \frac{\sqrt{a^2 + b^2}}{a}$	$e = \frac{\sqrt{a^2 + b^2}}{a}$
Foci	$(\pm ae, 0)$	$(h \pm ae, k)$
Directrices	$x = \pm\frac{a}{e}$	$x = h \pm \frac{a}{e}$
Asymptotes	$y = \pm\frac{b}{a}x$	$y - k = \pm\frac{b}{a}(x - h)$

WORKED Example 13

a **For the hyperbola with equation $\frac{(x-3)^2}{16}-\frac{(y-2)^2}{9}=1$, find the eccentricity; the coordinates of the centre, the vertices and the foci; the equations of the directrices and the equations of the asymptotes.**

b **Sketch the hyperbola $\frac{(x-3)^2}{16}-\frac{(y-2)^2}{9}=1$.**

THINK	WRITE
a ① Write the given equation for the hyperbola.	**a** $\frac{(x-3)^2}{16}-\frac{(y-2)^2}{9}=1$
② Compare this to the corresponding general form of the hyperbola and identify h, k, a and b.	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $h=3$, $k=2$, $a=4$ and $b=3$.
③ Calculate the eccentricity using $e=\frac{\sqrt{a^2+b^2}}{a}$. Note that $e>1$, which matches the fact that we are considering a hyperbola.	Eccentricity: $e=\frac{\sqrt{a^2+b^2}}{a}$ $=\frac{\sqrt{16+9}}{4}$ $=\frac{5}{4}$
④ Write the coordinates of the centre using (h, k).	Centre: (h, k) The centre is at (3, 2).
⑤ Write the coordinates of the vertices using $(h \pm a, k)$.	Vertices: $(h \pm a, k)$ The vertices are at $(-1, 2)$ and $(7, 2)$.
⑥ Write the coordinates of the foci using $(h \pm ae, k)$.	Foci: $(h \pm ae, k)$ The foci are at $(-2, 2)$ and $(8, 2)$.
⑦ Write the equations of the directrices using $x = h \pm \frac{a}{e}$.	Directrices: $x = h \pm \frac{a}{e}$ The equations of the directrices are $x=-\frac{1}{5}$ and $x=6\frac{1}{5}$.
⑧ Write the equations of the asymptotes using $y-k=\pm\frac{b}{a}(x-h)$.	Asymptotes: $y-k=\pm\frac{b}{a}(x-h)$ The equations of the asymptotes are $y-2=-\frac{3}{4}(x-3)$ and $y-2=\frac{3}{4}(x-3)$. That is, $y=-\frac{3}{4}x+\frac{17}{4}$ and $y=\frac{3}{4}x-\frac{1}{4}$.

Continued over page →

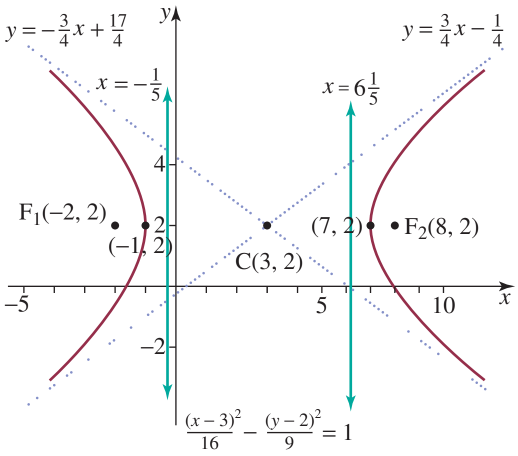
THINK

b ① On a Cartesian plane, mark the centre, the vertices, the foci, the directrices and the asymptotes of the hyperbola.

② Sketch the hyperbola in relation to these features.

WRITE

b

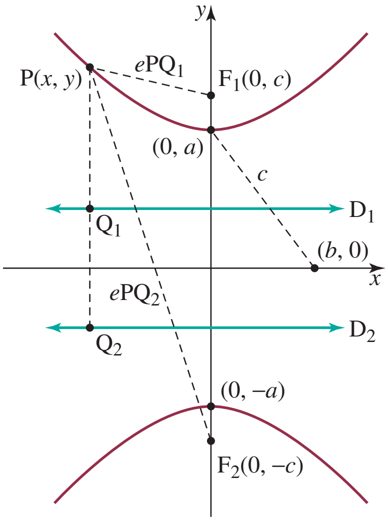


Another way of completing Worked example 13 is to consider the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and translate all the features three units to the right and two units up.

The hyperbola in Worked example 13 could also be written as $9(x - 3)^2 - 16(y - 2)^2 = 144$. If it were presented in this format, you would need to divide both sides by 144 to solve the problem.

Hyperbolas with a vertical transverse axis

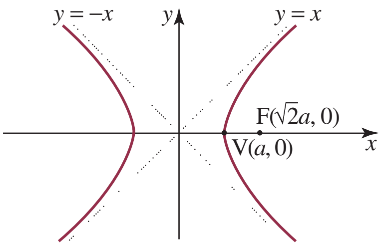
Consider what happens when the hyperbola we were looking at (see page 418) is rotated by 90° about its centre (see the diagram below).



The foci become the points $(0, \pm c)$ and now lie on a vertical transverse axis. This hyperbola will have the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. As before, this conic can be shifted so that its centre moves from the origin to the point (h, k). The following table summarises both cases.

Equation of the hyperbola	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Centre	$(0, 0)$	(h, k)
Vertices	$(0, \pm a)$	$(h, k \pm a)$
Eccentricity	$e = \frac{\sqrt{a^2 + b^2}}{a}$	$e = \frac{\sqrt{a^2 + b^2}}{a}$
Foci	$(0, \pm ae)$	$(h, k \pm ae)$
Directrices	$y = \pm\frac{a}{e}$	$y = k \pm \frac{a}{e}$
Asymptotes	$y = \pm\frac{a}{b}x$	$y - k = \pm\frac{a}{b}(x - h)$

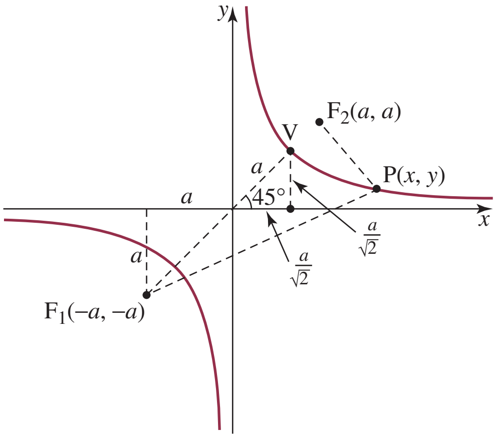
The rectangular hyperbola: $xy = c^2$



In the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, when the values of a and b are equal, the equation becomes $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$. This can be written as $x^2 - y^2 = a^2$. The asymptotes will be $y = \pm x$ and are perpendicular to each other. A hyperbola with perpendicular asymptotes is sometimes called a *rectangular* hyperbola.

The rectangular hyperbola above has the following features.

1. The centre is at the origin.
2. The vertices are given by $(\pm a, 0)$.
3. The eccentricity is $e = \frac{\sqrt{a^2 + a^2}}{a} = \sqrt{2}$.
4. The foci are given by $(\pm a\sqrt{2}, 0)$.
5. The asymptotes are $y = \pm x$.
6. The directrices are $x = \pm\frac{a}{\sqrt{2}}$.



Suppose that we wish to find the equation of this hyperbola rotated by 45° about the origin. The asymptotes are now the x- and y-axes. The vertices will have the coordinates $(\pm a \cos 45°, \pm a \sin 45°)$ or $\left(\pm\frac{a}{\sqrt{2}}, \pm\frac{a}{\sqrt{2}}\right)$. The foci will become $(\pm a\sqrt{2}\cos 45°, \pm a\sqrt{2}\sin 45°)$ or $(\pm a, \pm a)$.

If we think back to our earlier definition of a hyperbola, we have a point P(x, y) moving so that the absolute value of the difference between two fixed points F_1 and F_2 is a constant value of $2a$.

$$PF_1 - PF_2 = 2a$$

$$PF_1 = 2a + PF_2$$

Squaring both sides gives $$PF_1{}^2 = (2a + PF_2)^2$$

$$= 4a^2 + 4aPF_2 + PF_2{}^2$$

$$(x + a)^2 + (y + a)^2 = 4a^2 + 4aPF_2 + (x - a)^2 + (y - a)^2$$

$$x^2 + 2ax + a^2 + y^2 + 2ay + a^2 = 4a^2 + 4aPF_2 + x^2 - 2ax + a^2 + y^2 - 2ay + a^2$$

Simplifying: $$4ax + 4ay = 4a^2 + 4aPF_2$$

Dividing by $4a$ gives $$x + y = a + PF_2$$

$$x + y - a = PF_2$$

Squaring again gives $$(x + y - a)^2 = PF_2{}^2$$

$$(x + y)^2 - 2a(x + y) + a^2 = (x - a)^2 + (y - a)^2$$

Expanding: $$x^2 + 2xy + y^2 - 2ax - 2ay + a^2 = x^2 - 2ax + a^2 + y^2 - 2ay + a^2$$

$$2xy = a^2$$

$$xy = \frac{a^2}{2} \text{ or } xy = c^2 \text{ where } c^2 = \frac{a^2}{2}.$$

In summary, a rectangular hyperbola of the form $xy = c^2$ has the following features.

1. The centre is at the origin.
2. The vertices are given by $\left(\pm\frac{a}{\sqrt{2}}, \pm\frac{a}{\sqrt{2}}\right)$ or ($\pm c$, $\pm c$).
3. The oblique segment joining vertices ($-c$, $-c$) and (c, c) is called the *transverse axis* and has length $2\sqrt{2}\,c$.
4. The oblique segment joining points ($-c$, c) and (c, $-c$) is called the *conjugate axis* and has length $2\sqrt{2}\,c$.
5. The eccentricity is $\sqrt{2}$.
6. The foci are given by $(\pm c\sqrt{2}, \pm c\sqrt{2})$.
7. The equations of the directrices are $x + y = \pm c\sqrt{2}$.
8. The equations of the asymptotes are the coordinate axes.

WORKED Example 14

Find the locus of a point P(x, y) that moves so that the difference of its distances from the points A(–2, –2) and B(2, 2) is 4.

THINK	WRITE
1 State the relationship described in the question.	$PA - PB = 4$
2 Isolate PA.	$PA = 4 + PB$
3 Square both sides and expand.	$PA^2 = (4 + PB)^2$ $PA^2 = 16 + 8PB + PB^2$
4 Substitute for PA and PB using the distance formula. Expand and simplify by collecting like terms.	$(x + 2)^2 + (y + 2)^2 = 16 + 8PB + (x - 2)^2 + (x - 2)^2$ $x^2 + 4x + 4 + y^2 + 4y + 4 = 16 + 8PB + x^2 - 4x + 4$ $+ y^2 - 4y + 4$ $8x + 8y = 16 + 8PB$

THINK	**WRITE**
⑤ Simplify by dividing both sides by 8. Subtract 2 from both sides.	$x + y = 2 + \text{PB}$ $x + y - 2 = \text{PB}$
⑥ Square both sides again.	$(x + y - 2)^2 = \text{PB}^2$
⑦ Substitute for PB using the distance formula. Expand and simplify.	$(x + y)^2 - 4(x + y) + 4 = (x - 2)^2 + (x - 2)^2$ $x^2 + 2xy + y^2 - 4x - 4y + 4 = x^2 - 4x + 4 + y^2 - 4y + 4$ $2xy = 4$
⑧ Divide both sides by 2 and state the equation of the locus.	$xy = 2$ The equation of the required locus is $xy = 2$.

Conjugate hyperbolas

Two hyperbolas are called *conjugate hyperbolas* if the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other.

If the equation of a hyperbola is written in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equation of the conjugate hyperbola is found by changing the signs of the coefficients of x^2 and y^2 giving us:

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

WORKED Example 15

Write the equation of the hyperbola conjugate to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Find the coordinates of the foci for each hyperbola and write the equations of the asymptotes.

THINK	**WRITE**
① Write the equation of the given hyperbola.	$\frac{x^2}{9} - \frac{y^2}{16} = 1$
② Write the equation for the conjugate hyperbola by changing the sign in front of both the x^2 and y^2 terms of the original equation.	The conjugate is $-\frac{x^2}{9} + \frac{y^2}{16} = 1$ or $\frac{y^2}{16} - \frac{x^2}{9} = 1$
③ Use the relationship for eccentricity to write an expression for *ae*.	Now $e = \frac{\sqrt{a^2 + b^2}}{a}$ $ae = \sqrt{a^2 + b^2}$
④ Substitute the values for *a* and *b* to calculate the value of *ae*.	$= \sqrt{9 + 16}$ $= 5$
⑤ Use this value for *ae* to write the coordinates of the foci for each hyperbola.	The original hyperbola has foci at (±5, 0). The conjugate hyperbola has foci at (0, ±5).
⑥ Write the equations of the asymptotes for each hyperbola.	Both hyperbolas have asymptotes with equations $y = \pm\frac{4}{3}x$.

History of mathematics

PIERRE DE FERMAT (1601–1665)

During his life . . . Galileo Galilei develops the telescope, Johannes Kepler formulates the three laws of planetary motion, William Shakespeare writes some of his plays, and Peter Paul Rubens paints some of his masterpieces.

Pierre de Fermat (*Pee-air day Fair-mah*) (1601–1665) was a lawyer, but loved mathematics and pursued a variety of mathematical topics as a hobby. His work in analytical geometry was produced several years before that of Descartes, but it was not published until 1670, fourteen years after Fermat's death and fifty years after its completion. Descartes wrote in notation that was more modern than Fermat's, but Fermat's ideas more closely resemble those of modern mathematics.

In this rather short work, *Isogoge ad locus planos et solidos* (or *Introduction to plane and solid loci*), Fermat presented, in quite archaic terminology, an astonishing amount of material that is familiar to us today. His linear equation, written *D in A aequitur B in E*, we would write as $Dx = By$, and Fermat sketched it as a line (or rather as a half-line, since he did not accept negative coordinates). He dealt with hyperbolas, parabolas, circles and ellipses. He even extended his analytical geometry to space, a generalisation that had not occurred to Descartes.

Fermat considered the family of equations of the form $y = x^n$, where n is a positive or negative integer. In an attempt to discover the maximums and minimums of the associated curves, he found the derivatives of these special types of function. He also used what was essentially the differentiation process (but without the idea of the limit) for finding tangents to these same curves. Later, Fermat rectified the functions (that is, found the area of the region between such a curve and a line, which corresponds to the integral of x^n). It is curious that he did not seem to have recognised the inverse nature of these two processes.

Although Fermat is sometimes called a discoverer of calculus, he worked with only a few polynomial functions, he lacked the concept of a limit, and he failed to note the fundamental theorem of calculus. Thus, he cannot be said to have reached the intellectual summit that might have allowed him to be ranked with Newton and Leibniz.

Fermat's predominant interests lay in topics such as prime numbers, perfect numbers and magic squares; he made notable discoveries in these subjects, which fall under the heading of theory of numbers. He was also one of the earliest to develop the theory of probability. However, although his work was often more significant than that of many professional mathematicians, he published virtually nothing during his lifetime, and his brilliance went unrecognised until long after his death.

Questions

1. In one topic, Fermat made a generalisation (or extension) that had not occurred to Descartes. In what subject was Fermat's extension?
2. One of Fermat's linear equations was written as *D in A aequitur B in E*. How would we write it today?
3. In what form did Fermat sketch it?
4. What were the limitations imposed on this sketch?
5. What family of equations did Fermat consider?
6. To determine the *extrema* of the associated curves, Fermat found the derivatives of these functions for finding tangents to these same curves (but he did not use an idea that we use today). What was this idea?
7. What was Fermat's chief interest?
8. Of what else was Fermat one of the earliest developers?

remember

1. A hyperbola is defined as the locus of a point P (x, y) such that the absolute value of the difference between its distances from 2 given points is a constant.
2. For a hyperbola, $e > 1$.
3. For hyperbolas of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ on the Cartesian plane:
    (a) The transverse axis is parallel to the x-axis (horizontal); length is $2a$.
    (b) The conjugate axis is parallel to the y-axis (vertical); length is $2b$.
    (c) The centre is (h, k).
    (d) The vertices are $(h \pm a, k)$.
    (e) The eccentricity $e = \frac{\sqrt{a^2 + b^2}}{a}$.
    (f) The foci are $(h \pm ae, k)$.
    (g) The directrices are $x = h \pm \frac{a}{e}$.
    (h) The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.
4. For hyperbolas of the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ on the Cartesian plane:
    (a) The transverse axis is parallel to the y-axis (vertical); length is $2a$.
    (b) The conjugate axis is parallel to the x-axis (horizontal); length is $2b$.
    (c) The centre is (h, k).
    (d) The vertices are $(h, k \pm a)$.
    (e) The eccentricity $e = \frac{\sqrt{a^2 + b^2}}{a}$.
    (f) The foci are $(h, k \pm ae)$.
    (g) The directrices are $y = k \pm \frac{a}{e}$.
    (h) The asymptotes are $y - k = \pm\frac{a}{b}(x - h)$.
5. For rectangular hyperbolas of the form $(x - h)(y - k) = c^2$:
    (a) The transverse axis has the equation $y - k = x - h$; length is $2a$.
    (b) The conjugate axis has the equation $y - k = -(x - h)$; length is $2b$.
    (c) The centre is (h, k).
    (d) The vertices are $(h \pm c, k \pm c)$.
    (e) The foci are $(h \pm \sqrt{2}c, k \pm \sqrt{2}c)$.
    (f) The eccentricity is $\sqrt{2}$.
    (g) The directrices are $(x - h) + (y - k) = \pm\sqrt{2}c$.
    (h) The asymptotes are $x = h, y = k$.
6. Two hyperbolas are called *conjugate hyperbolas* if the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other.

EXERCISE 9D Cartesian form of the hyperbola

WORKED Example 13

1 **a** For the following hyperbolas, find the eccentricity; the coordinates of the centre, the vertices and the foci; the equations of the directrices and the equations of the asymptotes.

b Sketch each hyperbola.

i $\frac{x^2}{9} - \frac{y^2}{4} = 1$

ii $\frac{(x+2)^2}{16} - \frac{(y-1)^2}{4} = 1$

iii $4x^2 - 45y^2 = 180$

iv $\frac{y^2}{9} - \frac{x^2}{16} = 1$

v $\frac{(y+1)^2}{16} - \frac{(x-3)^2}{25} = 1$

vi $49y^2 - 16x^2 = 784$

vii $\frac{x^2}{9} - \frac{y^2}{9} = 1$

viii $x^2 - y^2 = 25$

2 Write the equations of the hyperbolas for which the following conditions are given.

a The transverse axis is 8; the foci are at (±5, 0)

b The centre is at the origin; a focus is at (8, 0); a vertex is at (6, 0)

c The conjugate axis is 24; the foci are at (0, ±13)

3 Derive the equation of the hyperbola whose centre is at the origin, whose axes are on the coordinate axes, and which passes through (3, 1) and (9, 5).

4 What is the equation of the hyperbola whose vertices are (±6, 0) and whose asymptotes are the lines $y = \pm\frac{7}{6}x$.

5 Find the coordinates of the centre, the foci, the vertices, and the equations of the asymptotes of the hyperbola $9x^2 - 16y^2 - 36x - 32y = 124$.

WORKED Example 14

6 Write the equation of the hyperbola conjugate to the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$. Find the coordinates of the foci for each hyperbola and write the equations of the asymptotes.

7 Find the equation of the hyperbola that has its centre at the origin, a vertex at (3, 0) and the equation of one asymptote $y = \frac{2}{3}x$. Write the equation of its conjugate hyperbola.

WORKED Example 15

8 Find the locus of a variable point P(x, y) in the Cartesian plane which moves such that the difference between its distances from the points A(4, 4) and B(–4, –4) is 8.

9 A point moves such that the difference between its distances from (0, 3) and (0, –3) is 5 units long. Find the equation of its locus.

10 Find the equation of the locus of a point that moves such that the difference between its distances from the points (–6, –4) and (2, –4) is 6.

11 A point moves such that the product of the gradients of the lines connecting it with (–2, 1) and (3, 2) is 4. Find the equation of its locus.

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Lines, circles and regions on the complex plane

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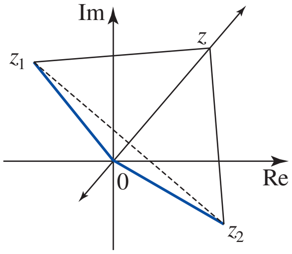
Interactivity: **Combination graphs and regions on the complex plane**
int-0345

In this section, we shall examine the behaviour of lines, circles and regions on the complex plane.

The line on the complex plane

The locus of a point, z, that moves in such a way as to be always equidistant from two points z_1 and z_2 on the complex plane will form a line.
The equation of the line can be written as $|z - z_1| = |z - z_2|$.
Note that the complex plane is also known as the Argand plane.



WORKED Example 16

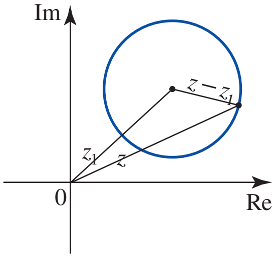
Find the Cartesian equation of the locus of a point on the complex plane, $z = x + iy$, which moves in such a way that it is always equidistant from the points $z_1 = -2 + 3i$ and $z_2 = 3 - i$.

THINK	WRITE
1 State the relationship described in the question; that is, the distance from z to z_1 is equal to the distance from z to z_2.	$\lvert z - z_1\rvert = \lvert z - z_2\rvert$
2 Substitute values for z_1 and z_2.	$\lvert z - (-2 + 3i)\rvert = \lvert z - (3 - i)\rvert$
3 Replace z with $x + yi$.	$\lvert(x + yi) - (-2 + 3i)\rvert = \lvert(x + yi) - (3 - i)\rvert$
4 Simplify by grouping real and imaginary parts.	$\lvert(x + 2) + i(y - 3)\rvert = \lvert(x - 3) + i(y + 1)\rvert$
5 Evaluate the moduli.	$\sqrt{(x+2)^2 + (y-3)^2} = \sqrt{(x-3)^2 + (y+1)^2}$
6 Square both sides of the equation.	$(x + 2)^2 + (y - 3)^2 = (x - 3)^2 + (y + 1)^2$
7 Expand both sides of the equation.	$x^2 + 4x + 4 + y^2 - 6y + 9 = x^2 - 6x + 9 + y^2 + 2y + 1$
8 Collect like terms and simplify.	$10x - 8y + 3 = 0$
9 State the answer. Note that the result is of the form $Dx + Ey + F = 0$, which represents a straight line.	The Cartesian equation of the required locus is $10x - 8y + 3 = 0$.

The circle on the complex plane

The locus of a point, z, that moves in such a way as to be always equidistant from another point, z_1, on the complex plane will form a circle.

For a circle on the complex plane with centre at z_1 and radius r, the equation can be written as $|z - z_1| = r$.



WORKED Example 17

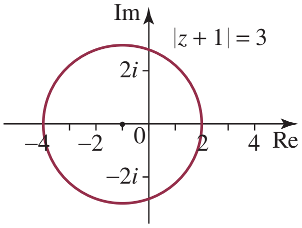
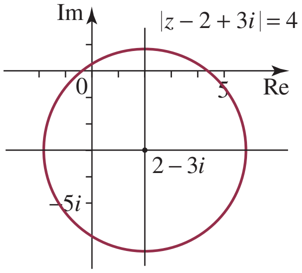
Consider the following equations.

a $|z + 1| = 3$

b $|z - 2 + 3i| = 4$

i **Sketch each of the following on a separate complex plane.**

ii **Determine the Cartesian equation of each.**

THINK	**WRITE**
a **i** ① Write the equation.	**a** **i** $|z + 1| = 3$
② Recognise that this equation looks similar to the general equation of a circle $|z - z_1| = r$.	This is of the form $|z - z_1| = r$
③ Identify the centre of the circle, z_1, and the radius, r.	where $z_1 = -1$ and $r = 3$. That is, the centre is at $-1 + 0i$ and the radius is 3 units.
④ On a complex plane, mark the centre and draw a circle with radius 3 units.	
ii ① Replace z with $x + yi$ in the given equation.	**ii** For $z = x + yi$, $|x + yi + 1| = 3$
② Simplify by grouping real and imaginary parts.	$|(x + 1) + yi| = 3$
③ Evaluate the modulus.	$\sqrt{(x+1)^2 + y^2} = 3$
④ Square both sides of the equation.	$(x + 1)^2 + y^2 = 9$
⑤ State the answer. Check that the features match the circle drawn on the complex plane.	The Cartesian equation is $(x + 1)^2 + y^2 = 9$. This is a circle with centre at $(-1, 0)$ and radius 3 units.
b **i** ① Write the equation.	**b** **i** $|z - 2 + 3i| = 4$
② Recognise that this equation looks similar to the general equation of a circle $|z - z_1| = r$.	This is of the form $|z - z_1| = r$
③ Identify the centre of the circle, z_1, and the radius, r.	where $z_1 = 2 - 3i$ and $r = 4$. That is, the centre is at $2 - 3i$ and the radius is 4 units.
④ On a complex plane, mark the centre and draw a circle with radius 4 units.	

THINK	WRITE
ii ① Replace z with $x + yi$ in the given equation.	**ii** For $z = x + yi$, $\lvert x + yi - 2 + 3i\rvert = 4$
② Simplify by grouping real and imaginary parts.	$\lvert(x - 2) + i(y + 3)\rvert = 4$
③ Evaluate the modulus.	$\sqrt{(x-2)^2 + (y+3)^2} = 4$
④ Square both sides of the equation.	$(x - 2)^2 + (y + 3)^2 = 16$
⑤ State the answer. Check that the features match the circle drawn on the complex plane.	The Cartesian equation is $(x - 2)^2 + (y + 3)^2 = 16$. This is a circle with centre at $(2, -3)$ and radius 4 units.

Regions on the complex plane

It is a simple matter to represent regions on the complex plane by making use of the definitions of the conic sections and the concept of representation of inequations on the coordinate axes. These ideas combine nicely so that we can look at single or combined relations represented as regions on the complex plane. We proceed as follows.

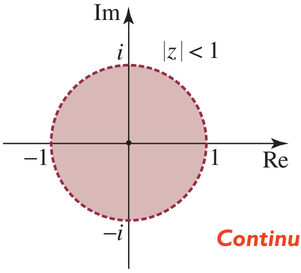
1. Regions are described as complex inequations such as $\lvert z - (1 + i)\rvert \le 5$ or $-\frac{\pi}{2} < \arg(z - 1) \le 0$.
2. For the boundaries, a solid line is used to represent $\le$ and $\ge$, and a broken line is used to represent $<$ and $>$.
3. The region required is then shaded.

We can further combine regions and look at the interaction between them. This is usually represented by the intersection (use the symbol $\cap$) of the regions. For example, for $\{\lvert z - (1 + i)\rvert \le 5\} \cap \{\lvert z - 2\rvert < 10\}$, we would consider each relation separately and then draw the region where they intersect. In other words, we could superimpose the graph of each relation on the same complex plane and shade the region that overlaps.

WORKED Example 18

Illustrate the following on an Argand diagram.

a $\lvert z\rvert < 1$ **b** $\lvert z - (2 + i)\rvert \le 4$ **c** $0 < \arg(z) \le \frac{\pi}{4}$

THINK	WRITE
a ① State the equation and features of the boundary of the relation.	**a** The boundary equation is $\lvert z\rvert = 1$. This is a circle centred at $0 + 0i$ with a radius of 1 unit.
② On a complex plane, draw a circle with centre at $0 + 0i$ and a radius of 1 unit. Use a broken line to indicate that the boundary is not included. ③ Shade the region inside the circle.	

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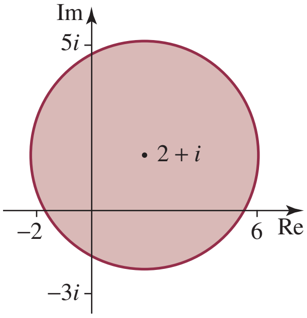
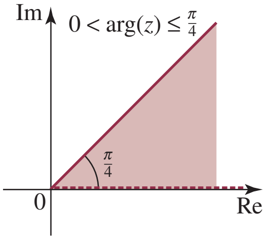
THINK	**WRITE**
b ① State the equation and features of the boundary of the relation. ② On a complex plane, draw a circle with centre at $2 + i$ and a radius of 4 units. Use a solid line to indicate that the boundary is included. ③ Shade the region inside the circle.	**b** The boundary equation is $\lvert z - (2 + i)\rvert = 4$. This is a circle centred at $2 + i$ with a radius of 4 units. 
c ① Consider the two relations separately and interpret the region required. ② On a complex plane, first draw a line starting at $0 + 0i$ with an angle of 0 from the positive Real axis. Use a broken line to indicate that the boundary is not included. ③ On the same complex plane, draw a line starting from $0 + 0i$ with an angle of $\frac{\pi}{4}$ (or 45°) from the positive Real axis. ④ Shade the region between the two lines.	**c** For the inequation $0 < \arg(z) \leq \frac{\pi}{4}$, consider the intersection of $\arg(z) > 0$ and $\arg(z) \leq \frac{\pi}{4}$. That is, the required region lies between the lines $\arg(z) = 0$ and $\arg(z) = \frac{\pi}{4}$. 

remember

1. On the complex plane, $\lvert z - z_1\rvert = \lvert z - z_2\rvert$ represents the locus of a point, z, which moves so that it is equidistant from two other points, z_1 and z_2.
2. On the complex plane, $\lvert z - z_1\rvert = r$ represents the locus of the points of a circle centred at z_1 with radius r.
3. An inequation can be represented on the complex plane by a region, where
   - (a) the boundary is represented by a solid line for inequations involving ≤ or ≥
   - (b) the boundary is represented by a broken line for inequations involving < or >.
4. The intersection of two or more regions can be represented by shading where the regions overlap.

EXERCISE 9E Lines, circles and regions on the complex plane

WORKED Example 16

1 Find the Cartesian equation of the locus of a point, z, on the complex plane, where $z = x + yi$ and moves in such a way that it is always equidistant from two points z_1 and z_2 where:

a $z_1 = 2 + 3i,\ z_2 = 5 - i$ **b** $z_1 = -2 + 3i,\ z_2 = 5 - i$ **c** $z_1 = 2 + i,\ z_2 = -5 + i$.

WORKED Example 17

2 **i** Sketch each of the following on a separate complex plane.
ii Determine the Cartesian equation of each.

a $|z - 3| = 3$ **b** $|z + 2| = 2$ **c** $|z - 4i| = 3$
d $|z + i| = 4$ **e** $|z - 2 - i| = 3$ **f** $|z + 3 - 2i| = 1$
g $|z + 4 + 3i| = 5$ **h** $|z - 5 + 5i| = 2$ **i** $|z - 2 + i| = |z + 1 - 2i|$
j $|z - 2| = |z - 3 - i|$ **k** $|z - 3i| = \overline{|z + 5 - 3i|}$

WORKED Example 18a, b

3 Illustrate each of the following on an Argand diagram.

a $|z| \le 2$ **b** $|z| > 5$ **c** $|z - 2| < 3$
d $|z + 3| \ge 1$ **e** $|z - 3 + 2i| \le 4$ **f** $|z + 5 - i| > 3$

WORKED Example 18c

4 Represent the following regions on the complex plane.

a $0 < \arg(z) \le \frac{\pi}{6}$ **b** $-\frac{2\pi}{3} < \arg(z) \le \frac{\pi}{2}$ **c** $-\frac{\pi}{2} \le \arg(z) < \pi$

d $0 \le \arg(z + 1) \le \frac{2\pi}{3}$ **e** $-\frac{3\pi}{4} \le \arg(z - 2i) \le -\frac{\pi}{2}$

f $\frac{\pi}{6} < \arg(z - 3 + i) < \frac{5\pi}{6}$

5 Sketch the graph of each of the following.

a $1 < |z| \le 3$ **b** $3 \le |z| < 4$ **c** $2 < |z - 1| \le 5$
d $1 < |z - 2i| < 2$ **e** $3 \le |z - 2 + 3i| \le 4$ **f** $4 \le |z + 3 - 4i| < 6$

6 Sketch each of the following.

a $\left\{-\frac{\pi}{2} < \arg(z) \le \frac{3\pi}{4}\right\} \cap \{|z| \le 2\}$

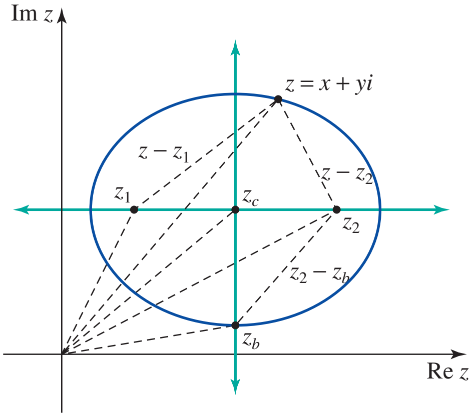
b $\{|z + 3| > 2\} \cap \left\{\frac{\pi}{6} < \arg(z + 2) \le \frac{2\pi}{3}\right\}$

The ellipse and hyperbola on the complex plane

The ellipse on the complex plane

Consider two complex numbers z_1 and z_2 on the complex plane as shown at right.

Knowing that an ellipse can be defined as the locus of a point moving so that the sum of its distance from two given points is constant, we have $|z - z_1| + |z - z_2| = k$. From our earlier work with ellipses, we know that $k = 2a$ (the length of the major axis).



It follows that the equation of an ellipse can be written as:

$$|z - z_1| + |z - z_2| = 2a \text{ if and only if } |z_1 - z_2| < 2a$$

or

$$|z - z_1| + |z - z_2| = k \text{ if and only if } |z_1 - z_2| < k.$$

The centre of the ellipse is the midpoint of z_1 and z_2 and is given by $z_c = \frac{z_1 + z_2}{2}$. The foci are located at z_1 and z_2.

To find the length of the minor axis, consider the triangle formed by z_c, z_2 and z_b. In the section on ellipses, we defined $|z_2 - z_b| = a = \frac{k}{2}$, $|z_2 - z_c| = \frac{1}{2}\,|z_2 - z_1| = c$ and $|z_b - z_c| = b$. Using Pythagorus' theorem we know that $b^2 = a^2 - c^2$.

By substituting for a and c we get $b^2 = \left(\frac{k}{2}\right)^2 - \left(\frac{|z_2 - z_1|}{2}\right)^2$.

Multiplying both sides by 4 results in $(2b)^2 = k^2 - |z_2 - z_1|^2$.

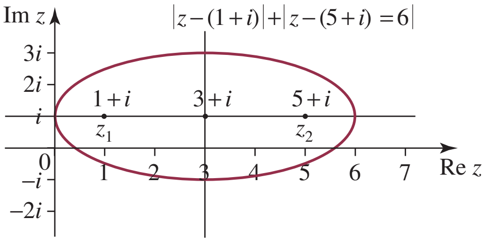
Taking the square root of both sides gives us the length of the minor axis, that is:

$$2b = \sqrt{k^2 - |z_2 - z_1|^2}.$$

WORKED Example 19

Sketch the ellipse $|z - (1 + i)| + |z - (5 + i)| = 6$. Identify the important features.

THINK	WRITE
1 Check whether the equation matches the general form of an ellipse. Identify z_1, z_2 and k and check whether $\lvert z_1 - z_2\rvert < k$.	For the ellipse to be of the form $\lvert z - z_1\rvert + \lvert z - z_2\rvert = k$ where the foci are z_1 and z_2, $\lvert z_1 - z_2\rvert < k$. If $z_1 = 1 + i$, $z_2 = 5 + i$ and $k = 6$, $\lvert z_1 - z_2\rvert = \lvert 1 + i - 5 - i\rvert$ $= \lvert -4\rvert$ $= 4$ Since $k = 6$, $\lvert z_1 - z_2\rvert < k$. Therefore, the locus is that of an ellipse with foci z_1 and z_2.
2 State the length of the major axis using k.	Length of major axis is 6 units.
3 State the length of the minor axis using $\sqrt{k^2 - \lvert z_2 - z_1\rvert^2}$.	Length of minor axis: $\sqrt{k^2 - \lvert z_2 - z_1\rvert^2} = \sqrt{36 - \lvert 4\rvert^2}$ $= \sqrt{20}$ $= 2\sqrt{5}$ units
4 Find the location of the centre and foci.	Centre of ellipse: $z_c = \frac{z_1 + z_2}{2}$ $= \frac{1 + i + 5 + i}{2}$ $= \frac{6 + 2i}{2}$ $= 3 + i$ Foci: z_1 and z_2 The foci are located at $1 + i$ and $5 + i$.

THINK	**WRITE**
⑤ On a complex plane, mark the centre and foci. Use the lengths of the major and minor axes to locate the vertices. ⑥ Sketch the ellipse in relation to the features.	

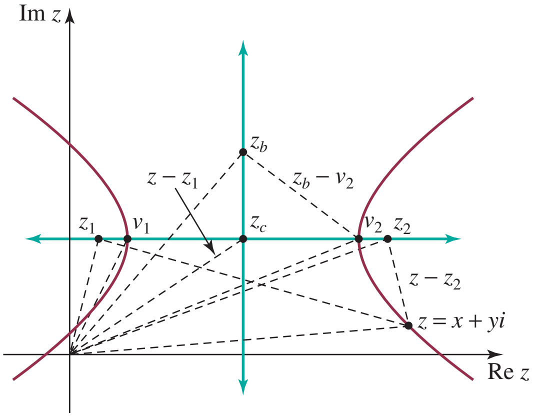
WORKED Example 20

Convert the equation of an ellipse on the complex plane given by $|z-(1+3i)|+|z-(1+7i)|=6$ into Cartesian form.

THINK	**WRITE**
① Write the equation and replace z with $x + yi$.	$\lvert z-(1+3i)\rvert+\lvert z-(1+7i)\rvert=6$ $\lvert x+yi-1-3i\rvert+\lvert x+yi-1-7i\rvert=6$
② Group the real and imaginary parts together.	$\lvert (x-1)+i(y-3)\rvert+\lvert (x-1)+i(y-7)\rvert=6$
③ Evaluate the moduli.	$\sqrt{(x-1)^2+(y-3)^2}+\sqrt{(x-1)^2+(y-7)^2}=6$
④ Rearrange the equation.	$\sqrt{(x-1)^2+(y-3)^2}=6-\sqrt{(x-1)^2+(y-7)^2}$
⑤ Square both sides of the equation.	$(x-1)^2+(y-3)^2=36-12\sqrt{(x-1)^2+(y-7)^2}$ $+(x-1)^2+(y-7)^2$
⑥ Cancel the $(x-1)^2$ term on both sides of the equation. Expand and collect like terms.	y^2-6y+9 $=36-12\sqrt{(x-1)^2+(y-7)^2}+y^2-14y+49$ $12\sqrt{(x-1)^2+(y-7)^2}=76-8y$
⑦ Simplify by dividing both sides of the equation by 4.	$3\sqrt{(x-1)^2+(y-7)^2}=19-2y$
⑧ Again square both sides of the equation to remove the radical term.	$9[(x-1)^2+(y-7)^2]=(19-2y)^2$ $9(x-1)^2+9y^2-126y+441=4y^2-76y+361$
⑨ Collect the y terms and simplify using the completing-the-square technique.	$9(x-1)^2+5y^2-50y+80=0$ $9(x-1)^2+5(y-5)^2=45$
⑩ Divide both sides by 45 to write the equation in the general form of an ellipse.	$\frac{(x-1)^2}{5}+\frac{(y-5)^2}{9}=1$

The hyperbola on the complex plane

Using the same approach that we adopted for the ellipse, we consider the locus of a point that moves so that the absolute value of the difference between its distances from two given points z_1 and z_2 is a constant.



We can write $|z - z_1| - |z - z_2| = \pm k$. From our previous work with hyperbolas, we know that k is the length of the transverse axis; that is $k = 2a$.

The equation of the hyperbola can be written as:

$$|z - z_1| - |z - z_2| = \pm 2a\text{, if and only if } |z_1 - z_2| > 2a$$

$$|z - z_1| - |z - z_2| = \pm k\text{, if and only if } |z_1 - z_2| > k.$$

The centre of this hyperbola is the midpoint of z_1 and z_2 and is found using $z_c = \dfrac{z_1 + z_2}{2}$, where z_1 and z_2 are the foci of the hyperbola.

To find the length of the conjugate axis, consider the triangle formed with vertices at z_c, z_b and v_2. From the previous section on hyperbolas we know that:

$$|z_c - v_2| = a = \frac{k}{2}, \qquad |z_b - v_2| = \frac{1}{2}|z_2 - z_1| = c \qquad \text{and} \qquad |z_b - z_c| = b.$$

Using Pythagoras' theorem we can obtain a value for b by substituting for a and c into $b^2 = c^2 - a^2$.

Therefore, we have $b^2 = \left(\dfrac{|z_2 - z_1|}{2}\right)^2 - \left(\dfrac{k}{2}\right)^2$.

Multiplying both sides by 4 we get $(2b)^2 = |z_2 - z_1|^2 - k^2$.

Taking the square root of both sides gives us the length of the conjugate axis; that is:

$$2b = \sqrt{|z_2 - z_1|^2 - k^2}$$

Notes

1. The form $|z - z_1| - |z - z_2| = -k$ represents the branch of the hyperbola near z_1 only. This is because points on that arm are closer to z_1 than to z_2 so the difference in distances will be negative.
2. Similarly, the form $|z - z_1| - |z - z_2| = k$ represents the branch of the hyperbola near z_2.
3. For both branches of the hyperbola, it is necessary to consider $\pm k$.

WORKED Example 21

Sketch $|z - (1 + i)| - |z - (5 + i)| = -3$ on the complex plane. Identify the important features.

THINK	WRITE
① Check whether the equation matches the general form of a hyperbola. Identify z_1, z_2 and k and check whether $\lvert z_1 - z_2\rvert > k$.	For the hyperbola to be of the form $\lvert z - z_1\rvert - \lvert z - z_2\rvert = \pm k$ where the foci are z_1 and z_2 and $\lvert z_1 - z_2\rvert > k$. If $z_1 = 1 + i$, $z_2 = 5 + i$ and $k = 3$ $\lvert z_1 - z_2\rvert = \lvert 1 + i - 5 - i\rvert$ $= \lvert -4\rvert$ $= 4$ $> k$ So $\lvert z_1 - z_2\rvert > k$.
② Identify whether we have the full hyperbola or just a branch. In this case, we have $-k$ and not $\pm k$, so we are considering the branch near z_1.	As the locus has an equation of the form $\lvert z - z_1\rvert - \lvert z - z_2\rvert = -k$, the locus is that around z_1.
③ State the length of the transverse axis using k.	The length of the transverse axis is 3 units.
④ Find the location of the centre and foci.	Centre of hyperbola: $z_c = \dfrac{z_1 + z_2}{2}$ $= \dfrac{1 + i + 5 + i}{2}$ $= \dfrac{6 + 2i}{2}$ $= 3 + i$ Foci: z_1 and z_2 The foci are located at $1 + i$ and $5 + i$.
⑤ Find the length of the conjugate axis.	Length of conjugate axis: $2b = \sqrt{\lvert z_2 - z_1\rvert^2 - k^2}$ $= \sqrt{4^2 - 3^2}$ $= \sqrt{7}$ The conjugate axis is $\sqrt{7}$ units long.
⑥ On a complex plane, mark the centre and foci. Use the length of the transverse axis to locate the vertices. Use the conjugate and transverse axes to locate the asymptotes.	

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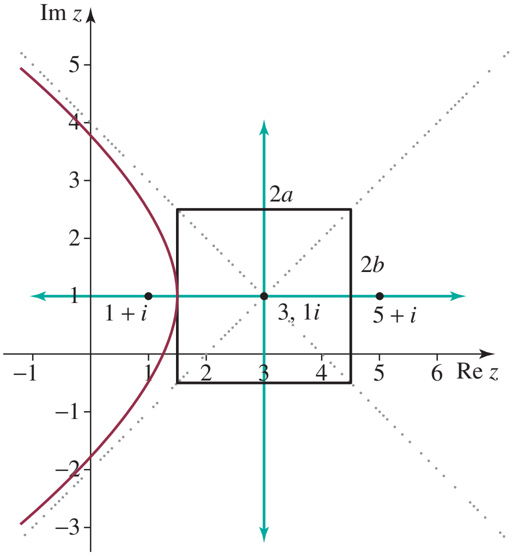
THINK	**WRITE**
7 Sketch the hyperbola in relation to the features. Remember to draw the branch around $1 + i$ only.	

remember

1. An ellipse can be written in the form $|z - z_1| + |z - z_2| = k$ if and only if $|z_1 - z_2| < k$.
2. A hyperbola can be written in the form $|z - z_1| - |z - z_2| = \pm k$ if and only if $|z_1 - z_2| > k$.
3. Both the ellipse and the hyperbola have a centre at $z_c = \dfrac{z_1 + z_2}{2}$ and foci at z_1 and z_2.
4. The ellipse has a major axis of length k and a minor axis of length $\sqrt{k^2 - |z_2 - z_1|^2}$.
5. The hyperbola has a transverse axis of length k and a conjugate axis of length $\sqrt{|z_2 - z_1|^2 - k^2}$.

EXERCISE 9F The ellipse and hyperbola on the complex plane

WORKED Example 19

1 Sketch each of the following ellipses in the complex plane. Identify the important features.

a $|z - (2 + i)| + |z - (8 + i)| = 10$ **b** $|z - (3 - i)| + |z - (3 + 5i)| = 8$

c $|z + 3 + 2i| + |z - 3 + 2i| = 8$ **d** $|z - 5 + i| + |z - 5 + 9i| = 12$

WORKED Example 20

2 Express each of the following ellipses in Cartesian form.

a $|z - i| + |z - 7i| = 10$ **b** $|z + 3| + |z - 7| = 14$

c $|z - (3 + 4i)| + |z - (11 + 4i)| = 16$ **d** $|z + 1 - 7i| + |z - 11 - 7i| = 14$

WORKED Example 21

3 Sketch each of the following hyperbolas in the complex plane. Identify the important features.

a $|z - i| - |z - 7i| = \pm 4$ **b** $|z + 3| - |z - 7| = \pm 8$

c $|z - (3 + 4i)| - |z - (11 + 4i)| = \pm 6$ **d** $|z + 1 - 7i| - |z - 11 - 7i| = -10$

4 Express each of the following hyperbolas in Cartesian form.

a $|z - (2 + i)| - |z - (8 + i)| = \pm 5$ **b** $|z - (3 - i)| - |z - (3 + 5i)| = \pm 5$

c $|z + 3 + 2i| - |z - 3 + 2i| = \pm 4$ **d** $|z - 5 + i| - |z - 5 + 9i| = \pm 7$

5 Determine the region in the complex plane represented by:

a $\{|z - 1 - 2i| \le 4\} \cap \{|z + 2| + |z - 5| < 10\}$

b $\{|z + 2| + |z - 4| = 8\} \cap \{|z + 4| - |z - 2| = 2\}$

c $\{|z + 3 - 5i| = 4\} \cap \{|z - 6i| - |z + 2i| = 3\}$.

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History of mathematics

HALLEY'S COMET

Edmund Halley (1656–1742), the British biologist, geologist, sea captain, Antarctic voyager, astronomer, adviser on fortifications, company founder and director, and classicist, was also the mathematician who financed the publication of Newton's *Principia* and used Newton's theory to calculate the orbit of the great comet of 1682. He wrote:

> . . . wherefore if according to what we have already said [the comet] should return again about the year 1758, candid posterity will not refuse to acknowledge that this was first discovered by an Englishman.

Indeed, ever since the comet's return in 1758 it has been known as Halley's comet.

The comet was last seen during the latter part of 1985 and early part of 1986 and is due to return in the year 2062. Records of its passing go back 30 orbit cycles to 240 BC. Recent studies have indicated that the comet has made about 2000 cycles so far with about the same number to go before the sun erodes it away completely. The orbit of the comet is an ellipse 35.88 AU long by 9.12 AU wide. Its eccentricity is:

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{17.94^2 - 4.56^2}}{17.94} \approx 0.967.$$

summary

Introduction to conics

- The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are all real numbers, is sometimes referred to as a *general second-degree equation* in two variables x and y. When we consider particular values for A, B and C, some special geometric conditions are met and what arises are some special curves that are called *conic sections*.
- A straight line is considered to be a conic section, and is sometimes referred to as a *degenerate conic section*.

Curve	Condition	Equation
Straight line	$A = B = C = 0$	$Dx + Ey + F = 0$
Parabola (Axis of symmetry parallel to y-axis)	$B = C = 0$	$Ax^2 + Dx + Ey + F = 0$
Parabola (Axis of symmetry parallel to x-axis)	$A = B = 0$	$Cy^2 + Dx + Ey + F = 0$
Circle	$A = C > 0, B = 0$	$Ax^2 + Ay^2 + Dx + Ey + F = 0$
Ellipse (Major and minor axes parallel to corresponding coordinate axes)	$A, C > 0, B = 0$	$Ax^2 + Cy^2 + Dx + Ey + F = 0$
Hyperbola (Transversal and conjugate axes parallel to corresponding coordinate axes)	$A > 0, C < 0,$ or $A < 0, C > 0,$ $B = 0$	$Ax^2 - Cy^2 + Dx + Ey + F = 0$

- The locus, or equation in two variables of a curve, is the path of a moving point given some specific geometric conditions. These conditions usually describe constraints that are imposed in terms of distances, or ratio of distances from given points (foci) and/or given lines (directrices).
- The eccentricity, e, is the ratio of the distance from a point P to the focus and the distance from that point to the directrix.
- The distance, d, between two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The distance, d, of a point $A(x_1, y_1)$ from a given line $Ax + By + C = 0$ is given by the formula $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

Cartesian form of the straight line and the circle

- A straight line can be described as the locus of a variable or moving point P(x, y) such that it is equidistant from two given points.
- A circle can be described as the locus of a point P(x, y) that is equidistant from a given point C(h, k). The equation is of the form $(x - h)^2 + (y - k)^2 = r^2$ where (h, k) are the coordinates of the centre and r is the radius.

Cartesian form of the parabola

- A parabola can be described as the locus of a point P(x, y) that is equidistant from a given point F (focus) and a given line D (directrix).

Form of parabola	$(x - h)^2 = \pm 4a(y - k)$	$(y - k)^2 = \pm 4a(x - h)$
Orientation	Vertical axis of symmetry	Horizontal axis of symmetry
Vertex	(h, k)	(h, k)
Focus	$(h, k \pm a)$	$(h \pm a, k)$
Directrix	$y = k \mp a$	$x = h \mp a$

- For a parabola, $e = 1$.

Cartesian form of the ellipse

- An ellipse can be described as the locus of a point P(x, y) such that the sum of its distances from two given points F_1 and F_2 (the foci) is constant, $2a$.

Form of ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, a > b$
Orientation	Horizontal major axis	Vertical major axis
Centre	(h, k)	(h, k)
Vertices	$(h \pm a, k)$ and $(h, k \pm b)$	$(h, k \pm a)$ and $(h \pm b, k)$
Foci	$(h \pm ae, k)$	$(h, k \pm ae)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{a}{e}$

- For an ellipse, $0 < e < 1$ where $\frac{\sqrt{a^2 - b^2}}{a}$.
- The length of the major axis is $2a$.
- The length of the minor axis is $2b$.

Cartesian form of the hyperbola

- A hyperbola can be described as the locus of a point P(x, y) such that the absolute value of the difference of its distances from two given points F_1 and F_2 (foci) is a constant, $2a$.

Form of hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Orientation	Horizontal transverse axis	Vertical transverse axis
Centre	(h, k)	(h, k)
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Foci	$(h \pm ae, k)$	$(h, k \pm ae)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{a}{e}$
Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{b}{a}(x - h)$

- For a hyperbola, $e > 1$ where $e = \frac{\sqrt{a^2 + b^2}}{a}$.
- The length of the transverse axis is $2a$.
- The length of the conjugate axis is $2b$.
- For a rectangular hyperbola, $e = \sqrt{2}$.

Form of rectangular hyperbola	$(x - h)(y - k) = c^2$
Centre	(h, k)
Vertices	$(h \pm c, k \pm c)$
Foci	$(h \pm \sqrt{2}c, k \pm \sqrt{2}c)$
Directrices	$(x - h) + (y - k) = \pm\sqrt{2}c$
Asymptotes	$x = h, y = k$
Transverse axis	$y - k = x - h$
Conjugate axis	$y - k = -(x - h)$

- Two hyperbolas are called *conjugate hyperbolas* if the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other.

Lines, circles and regions on the complex plane

- On the complex plane, $|z - z_1| = |z - z_2|$ represents the locus of a point, z, which moves so that it is equidistant from two points, z_1 and z_2.
- On the complex plane, $|z - z_1| = r$ represents the locus of the points of a circle centred at z_1 with radius r.
- An inequation can be represented on the complex plane by a region, where
  1. the boundary is represented by a solid line for inequations involving ≤ or ≥
  2. the boundary is represented by a broken line for inequations involving < or >.
- The intersection of two or more regions can be represented by shading where the regions overlap.

Ellipses and hyperbolas on the complex plane

- An ellipse can be written in the form $|z - z_1| + |z - z_2| = k$ — if and only if $|z_1 - z_2| < k$.
- A hyperbola can be written in the form $|z - z_1| - |z - z_2| = \pm k$ — if and only if $|z_1 - z_2| > k$.
- Both the ellipse and the hyperbola have a centre at $z_c = \dfrac{z_1 + z_2}{2}$ and foci at z_1 and z_2.
- The ellipse has a major axis of length k and a minor axis of length $\sqrt{k^2 - |z_2 - z_1|^2}$.
- The hyperbola has a transverse axis of length k and a conjugate axis of length $\sqrt{|z_2 - z_1|^2 - k^2}$.

CHAPTER review

9A

1 **multiple choice**

The locus of a point P(x, y) that is equidistant from a given point describes which conic section?

A A straight line
B A circle
C An ellipse
D An hyperbola

9A

2 **multiple choice**

The equation $(x-2)^2+(y+4)^2=16$ represents a circle with:

A Centre (−2, −4), radius 4
B Centre (−2, −4), radius 16
C Centre (2, −4), radius 8
D Centre (2, −4), radius 4

9A

3 **multiple choice**

Every equation of a circle can be reduced to $x^2+y^2+Dx+Ey+F=0$. To determine that the circle is real, the discriminant must satisfy which of the following conditions?

A $D^2+E^2-4F>0$
B $D^2+E^2-4F<0$
C $D^2+E^2-4F=0$
D $D^2+E^2-4F\neq 0$

9B

4 **multiple choice**

The equation $y^2=-8x$ represents a conic section called:

A A parabola
B A circle
C An ellipse
D An hyperbola

9B

5 Find the coordinates of the vertex (V) and the focus (F) and the equation of the directrix (D) for each of the following parabolas.

a $y=8x^2$
b $x+2y^2=8$
c $y^2-2y+3x=0$
d $3y^2+6y-4x+1=0$
e $2x^2+4x+6y=0$
f $4x^2-16x-5y+6=0$

9B

6 Determine the equation of each of the following parabolas:

a Focus at (2, 0) and the equation of the directrix is $x=-4$
b Focus at (0, 5) and the equation of the directrix is $y=1$
c Vertex is the origin, axis of symmetry along the y-axis and passes through the point (2, −12)

9C

7 **multiple choice**

The graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$ represents which conic section?

A A straight line
B A circle
C An ellipse
D An hyperbola

8 For each of the following ellipses, find the eccentricity, the coordinates of the centre, the vertices and the foci, the equations of the directrices and the length of the major and minor axes. **9C**

a $\frac{x^2}{81}+\frac{y^2}{64}=1$

b $\frac{x^2}{16}+\frac{y^2}{25}=1$

c $\frac{(x+1)^2}{9}+\frac{(y+5)^2}{4}=1$

d $\frac{(x+2)^2}{144}+\frac{(y-1)^2}{169}=1$

9 Given the ellipse whose equation is $16x^2+9y^2+96x-36y+36=0$, find the eccentricity, the coordinates of the centre, the foci, the vertices and the equations of the directrices. **9C**

10 State the equation of the following ellipses: **9C**

a Centre origin, foci $(\pm 8, 0)$, vertices $(\pm 17, 0)$

b Centre origin, foci $(0, \pm 4)$, vertices $(0, \pm 5)$

c Centre (6, 1), focus at (3, 1) and passing through the point (10, 2)

d Centre (5, 3), vertex on major axis at (5, 0) and eccentricity $= \frac{1}{3}$

11 **multiple choice** **9D**

The only conic section with asymptotes is:

A A parabola

B A circle

C An ellipse

D An hyperbola

12 **multiple choice** **9D**

The locus of a point, such that the difference between its distances from two given points is constant, is called:

A A parabola

B A circle

C An ellipse

D An hyperbola

Consider the hyperbola $\frac{(x-4)^2}{25}-\frac{(y+1)^2}{9}=1$ for questions **13** and **14**.

13 **multiple choice** **9D**

The hyperbola has vertices given by:

A $(-1, -1)$ and $(9, -1)$

B $(-1, -1)$ and $(-1, 7)$

C $(4, 4)$ and $(4, 2)$

D $(4, -1)$ and $(5, 3)$

E $(-6, -1)$ and $(4, -1)$

14 **multiple choice** **9D**

The graph of the hyperbola has asymptotes with equations:

A $y+1=\pm\frac{5}{3}(x-4)$

B $y-1=\pm\frac{9}{25}(x-4)$

C $y-1=\pm\frac{5}{3}(x-4)$

D $y+1=\pm\frac{3}{5}(x-4)$

E $y-1=\pm\frac{3}{5}(x+4)$

9D

15 Calculate the equation of the hyperbola given that:
- a The transverse axis is 8; the foci are $(0, \pm 5)$
- b The centre is at $(1, 1)$; a focus is at $(9, 1)$; a vertex is at $(7, 1)$
- c The conjugate axis is 24; the foci are at $(1, \pm 13)$
- d The vertices are $(0, \pm 6)$; asymptotes are $\pm\frac{6}{7}x$

9E

16 **multiple choice**

The complex equation $|z - 3 + 2i| = |z + 3|$ can be expressed in Cartesian form as:

A $y = -1$
B $y = x^2 + 1$
C $y = 3x - 1$
D $y = 3 - x$
E $y = 1 - 3x$

9E

17 Determine the Cartesian equation of:
- a $|z - 2 + 3i| = 3$
- b $|z + 2 - i| = |z + 5 - 4i|$

9F

18 Determine the Cartesian equation of the following ellipses.
- a $|z - 2 - i| + |z - 8 - i| = 10$
- b $|z - 5 + i| + |z - 5 + 9i| = 12$

9F

19 Determine the Cartesian equation of the following hyperbolas.
- a $|z - i| - |z - 7i| = \pm 4$
- b $|z - 3 - 4i| - |z - 11 - 4i| = \pm 6$

Modelling and problem solving

1 A ladder that is 6 m long rests with one end on the horizontal ground, and the other end against a vertical wall. Considering the ground to be the x-axis and the wall to be the y-axis, find the locus of the midpoint of the ladder.

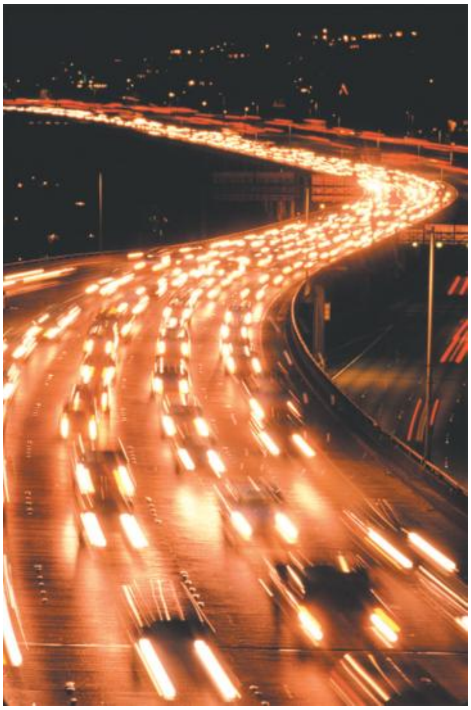
2 If A is a point where the circle $x^2 + y^2 = 16$ cuts the x-axis, find the locus of the midpoints of all the chords of this circle that contain A.

3 Find the equation of the circle that passes through the points $(2, 3)$ and $(-1, 1)$ and has its centre on the line $x - 3y - 11 = 0$.

4 A racing automobile travels around a circular track at a speed of 52.8 m/s and makes one revolution every 39.25 s. Taking the origin of the coordinate system at the centre of the track, determine an equation for the path of the automobile.

5 When a particle of mass m and charge q enters a magnetic field of induction B with a velocity v at right angles to B, it can be shown that the particle will travel in a circle of radius r, where $r = \frac{mv}{qB}$. (This is one of the basic operating principles of a mass spectrograph and an important consideration in the design of cyclotrons.) If a singly charged lithium ion enters a magnetic field of induction $B = 0.4$ webers/m^2 with speed of 1.17×10^5 m/s, determine an equation of its circular path. For this ion, $q = 1.6 \times 10^{-19}$ C and $m = 1.16 \times 10^{-26}$ kg.

6 Find the equation of the parabola with its vertex on the line $2y - 3x = 0$, its axis of symmetry parallel to the x-axis, and which passes through the two points $(3, 5)$ and $(6, -1)$.



7 If a light source is placed between the focus and vertex of a parabolic mirror, the rays of light diverge after reflection, whereas if the light source is placed outside the focus, the rays converge after reflection. This is how car headlights can give a broad beam of light (high beam) from one filament and a narrow beam of light (low beam) from a second filament. If the equation of a cross-sectional view of a parabolic mirror is $y^2 = 20x$, where x and y are in centimetres, what could be the distance between the vertex of the mirror and a filament if the light rays are to diverge after reflection from the mirror?

8 If the latus rectum of an ellipse is the line from one side to the other passing through the focus and perpendicular to the major axis, determine the equations of the following ellipses.

- **a** Centre is at the origin, vertices on the major axis are $(\pm 10, 0)$ and the length of the latus rectum is 5.
- **b** Centre is at the origin, the major axis is the y-axis and is 8 units long, the length of the latus rectum is 4.5 units.

9 A point moves so that its distance from the point (3, 2) is one half of its distance from the line $x = -2$. Derive the equation of its locus.

10 A point P(x, y) moves so that the product of the gradients of the two lines joining P to the two points (–2, 1) and (6, 5) is –4. Show that the locus is an ellipse and locate its centre.

11 A line segment AB, 18 units long, is moved so that A is always on the y-axis and B on the x-axis. Find the equation of the locus of a point P(x, y), where P is on the line and 6 units from B.

12 The orbit of Halley's comet has a semi-major axis of 17.94 AU and a eccentricity of 0.967.

- **a** Write an equation for the orbit of Halley's comet in a coordinate system in which the sun lies at the origin and the other focus lies on the positive x-axis, scaled in astronomical units.
- **b** Approximately how close does the comet come to the sun in astronomical units?
- **c** What is the farthest the comet goes from the sun in astronomical units?

13 What values of the constants a, b and c make the ellipse $4x^2 + y^2 + ax + by + c = 0$ tangent to the x-axis at the origin and pass through the point (–1, 2)?

14 The rectangular hyperbola with equation $xy = 1$ is transformed according to

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$. Find the equation of the image.

15 The hyperbola with the equation $x^2 - y^2 - 8x + 4y + 2 = 0$ is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$ and then further transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Find the equation of the final image.

16 Point P(x, y) is a variable point on the rectangular hyperbola $xy = 12$. Find the locus of the midpoint of the line segment OP where O is the origin.

17 Points P and Q lie on the x- and y-axis respectively, and R is the midpoint of PQ. If the triangle POQ has an area of 6 units, find the locus of R.

18 A circle whose equation is $x^2 + y^2 = 36$ is cut by the line whose equation is $x = 1$. Find the dimensions of the rectangle of largest area that will fit inside the smaller segment cut from the circle. Assume that one side of the rectangle lies along the line.

19 A parabolic searchlight is hinged at its vertex. It is aimed into the night sky at an angle of 45° to the horizontal. Its 1000-watt bulb is positioned at the focus (3, 3). The operator now tilts the searchlight so that the beam of light is horizontal. Assuming that the vertex is at the origin, find the Cartesian equation of the parabolic curve, in the vertical plane, when the beam is horizontal.

20 A line segment of length $a + b$ runs from the x-axis to the y-axis in the first quadrant. The point P on the line segment lies a units from one end and b units from the other end. Show that the point P traces an elliptical path as the ends of the segment slide along the axes.

21 A 'ripple tank' is being constructed. The wall of the tank is made by bending a strip of metal around the perimeter of an ellipse. The tank is completed by soldering a flat base onto the wall. A few centimetres of water is added to the tank and the experimenter pokes a finger into it, exactly at one of the foci of the ellipse. Ripples radiate outward through the water, reflect from the strip around the edge of the tank, and in a short time a drop of water shoots up at the second focus. Why?

22 A vehicular tunnel is to be constructed with a semi-elliptical cross-section. There are to be two roadways each 10 metres wide, with a gap between them of 4 metres. The vertical clearance at the outside extremity of each road is to be 6 metres. The eccentricity of the ellipse is to be 0.8. Give a mathematical description of the ellipse and state why a lower eccentricity would not be acceptable to the tunnel builders.

23 A triangle has two fixed vertices, $A(c, 0)$ and $B(-c, 0)$, and a third vertex, C, which moves in such a way that $\angle A = 2(\angle B)$. Find the path traced by C.

24 A circular tablecloth made of plain material has the complex equation $|z| = 4.5$. If a small textile flower is hand-stitched onto the tablecloth at all points $x + yi$ for integer values for x and y with $y > x$, how many flowers will decorate the tablecloth?

25 Rene is a designer of banners and flags. Currently she is designing a flag on which she is planning to sew a brilliant sequin (colourful button) at each point $a + bi$ such that a and b are integer values, and $a + bi$ belongs simultaneously to both regions $|z - 2 - 2i| > 4$ and $|z + 1 - 2i| + |z - 5 - 2i| \leq 12$. The flag is large and rectangular. Consider the centre of the flag to be the origin. How many sequins are required?

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Chapter 9

eBook*plus* ACTIVITIES

9B Cartesian form of the parabola

Digital doc

- SkillSHEET 9.1: Practise conics in the Cartesian plane. (*page 401*)

9C Cartesian form of the ellipse

Interactivity

- int-0338: Use the interactivity to investigate graphs of circles and ellipses. (*page 406*)

9D Cartesian form of the hyperbola

Interactivities

- int-0832: Use the interactivity to consolidate your understanding of graphs of conical hyperbolas I. (*page 418*)
- int-0833: Use the interactivity to consolidate your understanding of graphs of conical hyperbolas II. (*page 418*)

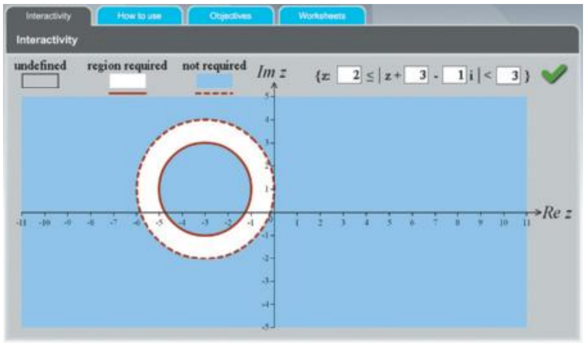
Digital doc

- WorkSHEET 9.1: Determine Cartesian equations of graphs of conics. (*page 428*)

9E Lines, circles and regions on the complex plane

Interactivity

- int-0345: Use the interactivity to investigate combination graphs and regions on the complex plane. (*page 429*)



$\{z: 2 \leq |z+3-1i| < 3\}$

9F The ellipse and hyperbola on the complex plane

Digital docs

- SkillSHEET 9.2: Practise conics in the complex plane. (*page 439*)
- WorkSHEET 9.2: Determine Cartesian equations of graphs of conics and sketch graphs in the complex plane. (*page 439*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 449*)

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Parametric and polar equations of conic sections

10

syllabus reference

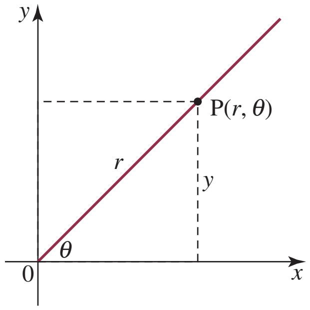
Option topic:
Conics

In this chapter

10A Parametric equations
10B Tangents and normals
10C Parametric equations of conics
10D Parametric equations of circles and ellipses
10E Parametric equations of hyperbolas
10F Applying technology to parametric equations
10G Polar equations
10H Conics in polar form
10I Some special curves and interesting problems

Introduction

In chapter 9, we saw how the equations of conic sections were expressed in Cartesian and complex number form. In this chapter, we show how they may be expressed in parametric and polar forms of conic sections.



Many plane curves can be more easily described in a coordinate system in which the coordinates of a point P(r, θ) measure its radial distance, r, along a ray from the origin that makes an angle of θ with some fixed reference ray — usually the x-axis. This is called the *polar coordinate system*. The origin is called the *pole* and the right-hand branch of the x-axis is called the *polar axis*.

We are already familiar with these concepts, and have dealt with them while using vector quantities.

Parametric equations

We use *parametric equations* as a particularly useful way of viewing certain ‘difficult’ curves in the plane. By using parametric equations, we can interpret these curves as graphs of *functions* where we have not previously been able to do so.

The power of this approach may be seen in how it handles a variety of troublesome situations. For example, the graph of an ellipse in the Cartesian plane has not (up until now) been regarded as corresponding to a function $y = f(x)$, since some values of x are associated with more than one value of y (a relation). The same difficulty can arise in polar coordinates since, for example, points on a spiral may be described conveniently by their polar coordinates, r and θ, but the spiral cannot be interpreted by a function $r = f(\theta)$ since many points correspond to a given value of θ.

This obstacle can be overcome by viewing the curve as the path traced out by the point P[$x(t)$, $y(t)$] as the independent variable, t, ranges through some interval. Here $x(t)$ and $y(t)$ are functions of the single variable, t.

Similarly, we can obtain the graph of a curve in polar coordinates using P[$r(t)$, $\theta(t)$]. These equations are called *parametric equations*; t is called the *parameter*. Note that one can easily use any letter to represent the parameter.

Simple algebraic techniques can be used to convert a set of parametric equations into Cartesian form (and vice versa).

The following worked example shows how to eliminate the parameter and form a single Cartesian equation involving only x and y.

WORKED Example 1

Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = t, \quad y = t + 2$ **b** $x = t + 1, \quad y = t^2$ **c** $x = t, \quad y = \frac{1}{t}$ **d** $x = \frac{2t}{1+t^2}, \quad y = \frac{1-t^2}{1+t^2}$

THINK	WRITE
a ① Write the parametric equations and number them.	**a** $x = t$ [1] $y = t + 2$ [2]
② Substitute equation [1] into equation [2]. That is, replace t with x in the second equation. This produces a Cartesian equation of a straight line.	Substituting [1] into [2] gives $y = x + 2$
b ① Write the parametric equations and number them.	**b** $x = t + 1$ [1] $y = t^2$ [2]
② Rearrange equation [1] to write t in terms of x.	Equation [1] becomes $t = x - 1$
③ Substitute this expression for t into equation [2]. This produces a Cartesian equation of a parabola.	Substituting $t = x - 1$ into [2] gives $y = (x - 1)^2$
c ① Write the parametric equations and number them.	**c** $x = t$ [1] $y = \frac{1}{t}$ [2]
② Substitute equation [1] into equation [2]. That is, replace t with x in the second equation. This produces a Cartesian equation of a rectangular hyperbola.	Substituting $t = x$ (from [1]) into [2] gives $y = \frac{1}{x}$ $xy = 1$
d ① Write the parametric equations and number them.	**d** $x = \frac{2t}{1+t^2}$ [1] $y = \frac{1-t^2}{1+t^2}$ [2]
② Square both equations and number them.	Squaring [1] gives $x^2 = \frac{(2t)^2}{(1+t^2)^2}$ Squaring [2] gives $y^2 = \frac{(1-t^2)^2}{(1+t^2)^2}$
③ Add equations [1] and [2].	$x^2 + y^2 = \frac{(2t)^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$

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THINK	**WRITE**
④ Simplify the right-hand side by adding the fractions.	$x^2 + y^2 = \dfrac{(2t)^2 + (1 - t^2)^2}{(1 + t^2)^2}$
⑤ Expand the numerator of the fraction and simplify further to eliminate *t*. This produces a Cartesian equation of a circle.	$x^2 + y^2 = \dfrac{4t^2 + t^4 - 2t^2 + 1}{(1 + t^2)^2}$ $x^2 + y^2 = \dfrac{(1 + t^2)^2}{(1 + t^2)^2}$ $x^2 + y^2 = 1$

Let us now look at some examples that involve another parameter. When parametric equations involve trigonometric functions, we often need to make use of standard trigonometric identities to eliminate the parameter. For example, let's consider Pythagorean identities, addition identities, double-angle identities and factorisation identities.

1. The Pythagorean identities
   $\sin^2 \theta + \cos^2 \theta = 1$
   $\sec^2 \theta = 1 + \tan^2 \theta$
   $\text{cosec}^2 \theta = 1 + \cot^2 \theta$

2. The addition identities
   $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
   $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
   $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \pm \tan x \tan y}$

3. The double-angle identities
   $\sin 2x = 2 \sin x \cos x$
   $\cos 2x = \cos^2 x - \sin^2 x$
   $\cos 2x = 2 \cos^2 x - 1$
   $\cos 2x = 1 - 2 \sin^2 x$
   $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

4. The factorisation identities
   $\sin A + \sin B = 2 \sin\left(\dfrac{A + B}{2}\right) \cos\left(\dfrac{A - B}{2}\right)$
   $\sin A - \sin B = 2 \cos\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{A - B}{2}\right)$
   $\cos A + \cos B = 2 \cos\left(\dfrac{A + B}{2}\right) \cos\left(\dfrac{A - B}{2}\right)$
   $\cos A - \cos B = -2 \sin\left(\dfrac{A + B}{2}\right) \sin\left(\dfrac{A - B}{2}\right)$

WORKED Example 2

Find the Cartesian equation of the curve whose parametric equations are given by:

a $x = \cos\theta, \quad y = \sin\theta$

b $x = 3\cos\theta, \quad y = 2\sin\theta$

c $x = 2\sec\theta, \quad y = \tan\theta$

d $x = \sin\theta, \quad y = \cos 2\theta$

e $x = t - \sin t, \quad y = 1 - \cos t$

THINK	WRITE
a ① Write the parametric equations and number them.	**a** $x = \cos\theta$ [1] $y = \sin\theta$ [2]
② Square both equations and number them.	Squaring [1] gives $x^2 = \cos^2\theta$ [3] Squaring [2] gives $y^2 = \sin^2\theta$ [4]
③ Add equations [3] and [4].	Adding [3] and [4] gives $x^2 + y^2 = \cos^2\theta + \sin^2\theta$
④ Use the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ to simplify. This produces a Cartesian equation of a circle.	$x^2 + y^2 = 1$
b ① Write the parametric equations and number them.	**b** $x = 3\cos\theta$ [1] $y = 2\sin\theta$ [2]
② Square equation [1] and then rearrange it to isolate the trigonometric function. Number the equation.	Squaring [1] gives $x^2 = 9\cos^2\theta$ or $\frac{x^2}{9} = \cos^2\theta$ [3]
③ Square equation [2] and then rearrange it to isolate the trigonometric function. Number the equation.	Squaring [2] gives $y^2 = 4\sin^2\theta$ or $\frac{y^2}{4} = \sin^2\theta$ [4]
④ Add equations [3] and [4].	Adding [3] and [4] gives $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2\theta + \sin^2\theta$
⑤ Use the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ to simplify. This produces a Cartesian equation of an ellipse.	$\frac{x^2}{9} + \frac{y^2}{4} = 1$
c ① Write the parametric equations and number them.	**c** $x = 2\sec\theta$ [1] $y = \tan\theta$ [2]
② Square equation [1] and then rearrange it to isolate the trigonometric function. Number the equation.	Squaring [1] gives $x^2 = 4\sec^2\theta$ or $\frac{x^2}{4} = \sec^2\theta$ [3]
③ Square equation [2] and number the equation.	Squaring [2] gives $y^2 = \tan^2\theta$ [4]

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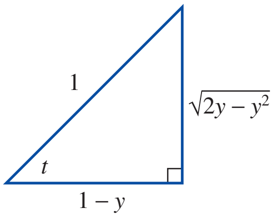
THINK	**WRITE**
④ Subtract equation [4] from equation [3].	Subtracting [4] from [3] gives $\frac{x^2}{4} - y^2 = \sec^2\theta - \tan^2\theta$
⑤ Use the Pythagorean identity $\sec^2\theta = 1 + \tan^2\theta$, which can be rearranged to $\sec^2\theta - \tan^2\theta = 1$ to simplify. This produces a Cartesian equation of a hyperbola.	$\frac{x^2}{4} - y^2 = 1$
d ① Write the parametric equations and number them.	**d** $x = \sin\theta$ [1] $y = \cos 2\theta$ [2]
② Use the double-angle identity $\cos 2\theta = 1 - 2\sin^2\theta$ to rewrite equation [2]. Number the equation.	Since $\cos 2\theta = 1 - \sin^2\theta$, [2] becomes $y = 1 - 2\sin^2\theta$ [3]
③ Substitute equation [1] into equation [3]. This produces a Cartesian equation of a parabola.	Substituting [1] into [3] gives $y = 1 - 2x^2$
e ① Write the parametric equations and number them.	**e** $x = t - \sin t$ [1] $y = 1 - \cos t$ [2]
② Rearrange equation [2] to isolate the parameter.	Rearranging [2] gives $\cos t = 1 - y$ or $t = \cos^{-1}(1 - y)$
③ Set up a right-angled triangle using the cosine value. Use Pythagoras' theorem to find the length of the side opposite the angle.	
④ Use this triangle to write the relationship for sin t.	From the triangle, $\sin t = \sqrt{2y - y^2}$
⑤ Substitute the values for t and sin t into equation [1]. This produces a Cartesian equation of a cycloid.	Substituting for t and $\sin t$ in [1] gives $x = \cos^{-1}(1 - y) - \sqrt{2y - y^2}$

The curve described by the Cartesian equation in part **e** of Worked example 2 is called a *cycloid* (a special case of trochoids) and is produced by tracing a point on the circumference of a circle that rolls along a straight line. The general parametric equation of this family of curves is $x = a(t - \sin t)$, $y = b(1 - \cos t)$. When $a = b$, we have a cycloid.

remember

1. To convert parametric equations into Cartesian equations, we solve the set of equations to eliminate the parameter.
2. If the parametric equations contain trigonometric functions, we can use trigonometric identities to assist in eliminating the parameter.

EXERCISE 10A Parametric equations

WORKED Example 1

1 For each of the following, find the Cartesian equation of the curve whose parametric equations are given.

a $x = t, \quad y = t - 7$

b $x = t + 1, \quad y = t^2 + 4$

c $x = t^3, \quad y = t^2$

d $x = 2t + 3, \quad y = 4t^2 - 9$

e $x = 2 + \frac{1}{t}, \quad y = 2 - t$

f $x = t^2 + t, \quad y = t^2 - t$

WORKED Example 2

2 For each of the following, find the Cartesian equation of the curve whose parametric equations are given.

a $x = \sin \theta, \quad y = \cos \theta$

b $x = 4 \cos \theta, \quad y = 2 \sin \theta$

c $x = -\sec \theta, \quad y = \tan \theta$

d $x = \cos 2\theta, \quad y = \sin \theta$

e $x = \operatorname{cosec} \theta, \quad y = \cot \theta$

f $x = t - \sin t, \quad y = 3 - 2 \cos t$

g $x = 4 \cos \theta, \quad y = 5 \sin \theta$

h $x = \cos \theta, \quad y = \sin 2\theta$

i $x = \sec^2\theta - 1, \quad y = \tan \theta$

j $x = \cos 2\theta, \quad y = \sin 2\theta$

k $x = 2 \cos^3 \theta, \quad y = 2 \sin^3 \theta$

Tangents and normals

One of the applications of derivatives is to find the equations of the tangent and normal at a given point on a curve. (You will have done similar work with functions and derivatives in Mathematics B.)

As you will recall, the gradient-point form of a straight line is $y - y_1 = m(x - x_1)$ where (x_1, y_1) is a point on the line and m is the gradient of the line. If the gradient of a line is m, then the gradient of a line perpendicular to this is $-\frac{1}{m}$.

Cartesian equations of conics do not define functions, but imply the existence of two functions. These equations are called *implicit functions* and can be expressed as a set of two or more equations of functions in terms of other variables (parametric functions). *Note*: The focus of this section will be on conic sections, but the techniques can apply to any implicit function.

Since we have two ways to represent these curves — that is, Cartesian equations and parametric equations — we have two techniques we can use to find the derivative.

Implicit differentiation

We shall now look at two techniques by which we can find the derivative of implicit functions. As mentioned above, implicit functions imply the existence of several functions such as a set of parametric equations.

Worked example 3 shows the first technique that applies directly to the Cartesian equation of the conic.

WORKED Example 3

Find the equations of the tangents and the normals to the circle $x^2 + y^2 = 25$ at $x = 3$.

THINK	WRITE
① Write the equation of the circle.	$x^2 + y^2 = 25$
② Find the derivative of each term in the equation with respect to x.	$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$
③ Use the chain rule to find the derivative of y^2.	$\frac{d}{dx}(x^2) + \frac{d}{dy}(y^2)\frac{dy}{dx} = \frac{d}{dx}(25)$
④ Simplify to find $\frac{dy}{dx}$.	$2x + 2y\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{x}{y}$
⑤ Find the value of y when $x = 3$.	When $x = 3$, $x^2 + y^2 = 25$ becomes $9 + y^2 = 25$ $y^2 = 16$ $y = \pm 4$
⑥ State the coordinates of the required points on the circle. The equations of the tangent and normal to each of these points are to be found.	The required points on the circle are (3, 4) and (3, –4).
⑦ Find the value of the gradient at (3, 4). That is, substitute $x = 3$ and $y = 4$ into $\frac{dy}{dx} = -\frac{x}{y}$.	At (3, 4), $\frac{dy}{dx} = -\frac{3}{4}$
⑧ Use the gradient-point form of a straight line, $y - y_1 = m(x - x_1)$, where $(x_1, y_1) = (3, 4)$ and $m = \frac{dy}{dx} = -\frac{3}{4}$, to find the equation of the tangent to the circle at the point (3, 4).	The equation of the tangent at (3, 4) is $y - 4 = -\frac{3}{4}(x - 3)$ $3x + 4y - 25 = 0$
⑨ Use $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (3, 4)$ and $m = \frac{4}{3}$, to find the equation of the normal to the circle at the point (3, 4).	The equation of the normal at (3, 4) is $y - 4 = \frac{4}{3}(x - 3)$ $4x - 3y = 0$
⑩ Find the value of $\frac{dy}{dx}$ at (3, –4). That is, substitute $x = 3$ and $y = -4$ into $-\frac{x}{y}$.	At (3, –4), $\frac{dy}{dx} = \frac{3}{4}$
⑪ Repeat step 8 with the appropriate values for (x_1, y_1) and m to find the equation of the tangent to the circle at the point (3, –4).	The equation at the tangent at (3, –4) is $y + 4 = \frac{3}{4}(x - 3)$ $3x - 4y - 25 = 0$
⑫ Repeat step 9 with the appropriate values for (x_1, y_1) and m to find the equation of the normal to the circle at the point (3, –4).	The equation of the normal at (3, –4) is $y + 4 = -\frac{4}{3}(x - 3)$ $4x + 3y = 0$

Worked examples 4 and 5 demonstrate the second technique with parametric equations. Firstly, let us consider an equation that uses the parameter t.

WORKED Example 4

Find the equations of the tangent and normal to the parabola with parametric equations $x = t$ and $y = t^2$ at $x = 1$.

THINK	WRITE
① Write the parametric equations.	$x = t$ and $y = t^2$
② Find the derivative of each equation with respect to the variable t.	$\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2t$
③ Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ $= 2t \cdot 1$ $= 2t$
④ Find the value of y when $x = 1$ by using the fact that when $x = 1$, $t = 1$.	When $x = 1$, $t = 1$ so $y = 1^2$ $= 1$
⑤ State the coordinates of the required point on the parabola.	The required point on the parabola is (1, 1).
⑥ Find the value of the gradient at the point (1, 1) by substituting $t = 1$ into $\frac{dy}{dx} = 2t$.	At the point (1, 1), $t = 1$ so $\frac{dy}{dx} = 2$.
⑦ Use $y - y_1 = m(x - x_1)$, where $(x_1, y_1) = (1, 1)$ and $m = 2$, to find the equation of the tangent to the parabola at the point (1, 1).	The equation of the tangent at (1, 1) is $y - 1 = 2(x - 1)$ $2x - y - 1 = 0$
⑧ Use the $y - y_1 = m(x - x_1)$, where $(x_1, y_1) = (1, 1)$ and $m = -\frac{1}{2}$, to find the equation of the normal to the parabola at the point (1, 1).	The equation of the normal at (1, 1) is $y - 1 = -\frac{1}{2}(x - 1)$ $x + 2y - 3 = 0$

Now consider a curve with parametric equations using the parameter θ.

WORKED Example 5

Find the equations of the tangents and the normals to the circle $x = 5\cos\theta$, $y = 5\sin\theta$ at $\theta = \frac{\pi}{3}$.

THINK	WRITE
① Write the parametric equations.	$x = 5\cos\theta$ and $y = 5\sin\theta$
② Find the derivative of each equation with respect to the variable θ.	$\frac{dx}{d\theta} = -5\sin\theta$ and $\frac{dy}{d\theta} = 5\cos\theta$

Continued over page →

THINK	**WRITE**
③ Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$ $=5\cos\theta\cdot-\frac{1}{5\sin\theta}$ $=-\frac{\cos\theta}{\sin\theta}$
④ Find the value of x and y when $\theta=\frac{\pi}{3}$.	When $\theta=\frac{\pi}{3}$ $x=5\cos\frac{\pi}{3}$ $=\frac{5}{2}$ and $y=5\sin\frac{\pi}{3}$ $=\frac{5\sqrt{3}}{2}$
⑤ State the coordinates of the required point on the parabola.	The required point on the parabola is $(\frac{5}{2},\frac{5\sqrt{3}}{2})$.
⑥ Find the value of the gradient at the point $(\frac{5}{2},\frac{5\sqrt{3}}{2})$ by substituting $\theta=\frac{\pi}{3}$ into $\frac{dy}{dx}=-\frac{\cos\theta}{\sin\theta}$.	At the point $(\frac{5}{2},\frac{5\sqrt{3}}{2})$, $\theta=\frac{\pi}{3}$ so $\frac{dy}{dx}=-\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}}$ $=-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}$ $=-\frac{1}{\sqrt{3}}$ $=-\frac{\sqrt{3}}{3}$
⑦ Use $y-y_1=m(x-x_1)$, where $(x_1,y_1)=(\frac{5}{2},\frac{5\sqrt{3}}{2})$ and $m=-\frac{\sqrt{3}}{3}$, to find the equation of the tangent to the parabola at the required point.	The equation of the tangent at $(\frac{5}{2},\frac{5\sqrt{3}}{2})$ is $y-\frac{5\sqrt{3}}{2}=-\frac{\sqrt{3}}{3}(x-\frac{5}{2})$ $\sqrt{3}x+3y-5\sqrt{3}=0$
⑧ Use $y-y_1=m(x-x_1)$, where $(x_1,y_1)=(\frac{5}{2},\frac{5\sqrt{3}}{2})$ and $m=\sqrt{3}$ to find the equation of the normal to the parabola at the required point.	The equation of the normal at $(\frac{5}{2},\frac{5\sqrt{3}}{2})$ is $y-\frac{5\sqrt{3}}{2}=\sqrt{3}\left(x-\frac{5}{2}\right)$ $\sqrt{3}x-y=0$

In Worked example 6, we shall look at a more general case for finding the equations of tangents and normals to conic sections.

WORKED Example 6

Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $P(x_1, y_1)$.

THINK	WRITE
1 Write the equation of the parabola.	$y^2 = 4ax$
2 Find the derivative of each term in the equation with respect to x.	$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$
3 Use the chain rule to find the derivative of y^2.	$\frac{d}{dy}(y^2)\frac{dy}{dx} = \frac{d}{dx}(4ax)$
4 Simplify to find $\frac{dy}{dx}$.	$2y\frac{dy}{dx} = 4a$ $\frac{dy}{dx} = \frac{2a}{y}$
5 Find the value of the gradient at (x_1, y_1). That is, substitute $y = y_1$ into $\frac{dy}{dx} = \frac{2a}{y}$.	At (x_1, y_1), $\frac{dy}{dx} = \frac{2a}{y_1}$
6 Use the gradient-point form of a straight line, $y - y_1 = m(x - x_1)$, where $m = \frac{2a}{y_1}$, to find the equation of the tangent to the parabola at the point (x_1, y_1).	The equation of the tangent at (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$ $yy_1 - y_1^2 = 2ax - 2ax_1$
7 Replace y_1^2 with $4ax_1$ and simplify.	$yy_1 - 4ax_1 = 2ax - 2ax_1$ $yy_1 = 2ax + 2ax_1$ $yy_1 = 2a(x + x_1)$
8 Use $y - y_1 = m(x - x_1)$ where $m = -\frac{y_1}{2a}$, to find the equation of the normal to the circle at the point (x_1, y_1).	The equation of the normal at (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ $2ay - 2ay_1 = -xy_1 + x_1y_1$ $xy_1 + 2ay = x_1y_1 + 2ay_1$

We may now take $y^2 = -4ax$ and show that the tangent and normal at $P(x_1, y_1)$ can be found using $yy_1 = -2a(x + x_1)$ and $xy_1 - 2ay = x_1y_1 - 2ay_1$ respectively. This is left up to the student to investigate.

remember

1. Tangents and normals are perpendicular to each other. If the gradient of a line is m, then the gradient of a line perpendicular to this is $-\frac{1}{m}$.
2. To find the equations of tangents and normals for implicit functions use
   (a) implicit differentiation to find $\frac{dy}{dx}$
   (b) the gradient-point form of a straight line to write the equation; that is, $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a given point and m is the gradient of the line. Use the relationship for $\frac{dy}{dx}$ to find m.

EXERCISE 10B Tangents and normals

WORKED Example 3

1 Write the equations of the tangent and normal to each of the following curves at the given points.

a $x^2 + y^2 = 25, (3, 4)$
b $2x^2 + 2y^2 - 3x + 5y - 2 = 0, (2, 0)$
c $x^2 + y^2 - 6x + 8y - 25 = 0, (-2, 1)$
d $2x^2 + 3y^2 - 30 = 0, (-3, 2)$
e $3x^2 + 4y^2 - 6x + 8y - 45 = 0, (-3, -2)$
f $x^2 - 4y = 0, (2, 1)$
g $6x^2 - 9y^2 - 8x + 3y + 16 = 0, (-1, 2)$
h $xy = 4, (2, 2)$
i $y^2 - 8x - 6y - 31 = 0, (-3, -1)$

WORKED Example 4, 5

2 Find the equations of the tangent and normal to the following curves at the given points.

a $x = t, y = \frac{t^2}{2} + 3, t = 2$
b $x = t^2 - 5, y = t + 3, t = 1$
c $x = \cos 2t, y = \sin t, t = \frac{\pi}{2}$
d $x = 4\cos\theta, y = 2\sin\theta, \theta = \frac{\pi}{4}$
e $x = 2\cos\theta, y = \sin\theta, \theta = \frac{5\pi}{6}$
f $x = 3\cos\theta + 1, y = 2\sin\theta - 1, \theta = \frac{\pi}{2}$

3 a Find the equations of the tangents to $5x^2 - 4y^2 = 4$ at those points where the curve is cut by $5x - 2y = 4$.

b Find the equations of the tangents to $xy = 2$ that are perpendicular to the line $x - 2y = 7$.

c Find the equations of the tangents to the curve $x^2 + 3y^2 = 4$ which are perpendicular to the line $2x + y - 1 = 12$.

d Find the equations of the tangents to $16x^2 - 9y^2 = 144$ that are parallel to the line $4x - y = 14$.

e A parabola with equation $y^2 = 4ax$ passes through the point $(-8, 4)$. Find the equation of the tangent to the parabola, parallel to the line $3x + 2y = 6$.

4 Find the points on $x^2 - 2y^2 = 8$ at which the tangents are perpendicular to the line $4x + 5y = 2$.

5 Find the equations of the tangent and normal to $x^2 = 5y$ at the point whose abscissa is 3.

WORKED Example 6

6 Show that for a parabola of the form $x^2 = \pm 4ay$, the equations of the tangent and the normal to the curve at the point $P(x_1, y_1)$ are given by $xx_1 = \pm 2a(y + y_1)$ and $x_1 y \pm 2ax = x_1 y_1 \pm 2ax_1$ respectively.

7 Show that the tangents to the parabola $y^2 = 4ax$ at the ends of the latus rectum are perpendicular.

8 Show that the equations of the tangents with gradient m for:

a the parabola $y^2 = 4ax$ are $y = mx + \dfrac{a}{m}$, $m \neq 0$

b the circle $x^2 + y^2 = a^2$ are $y = mx \pm a\sqrt{m^2 + 1}$

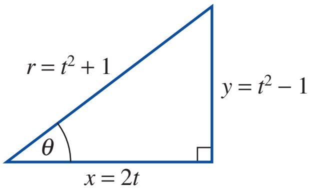
c the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2m^2 - b^2}$.

9 Find the value of b if the line $y = mx + b$ is tangent to the parabola $x^2 = 4ay$, and hence find the equation of the tangent to the parabola $x^2 = -2y$, which is parallel to $x - y = 4$.

10 Show that for a rectangular hyperbola of the form $xy = c^2$, the equations of the tangent and the normal to the curve at the point $P(x_1, y_1)$ are given by $y_1 x + x_1 y = 2x_1 y_1$ and $x_1 x - y_1 y = {x_1}^2 - {y_1}^2$ respectively.

Parametric equations of conics

It can be shown that for any right-angled triangle and for any integer value for t, the sides of the triangle can be of the form $2t$, $t^2 - 1$ and $t^2 + 1$, and this form will represent all the possible Pythagorean triplets. (Worked example 1d showed that a variation of these terms can still form parametric equations for a circle.) That is, for any value of t, the terms $2t$, $t^2 - 1$, $t^2 + 1$ form all the possible combinations of the Pythagorean triplets. From this triangle we have the following:



$$\cos\theta = \frac{2t}{t^2+1} = \frac{x}{r} \qquad \sin\theta = \frac{t^2-1}{t^2+1} = \frac{y}{r} \qquad \tan\theta = \frac{t^2-1}{2t} = \frac{y}{x}$$

$$\sec\theta = \frac{t^2+1}{2t} = \frac{r}{x} \qquad \operatorname{cosec}\theta = \frac{t^2+1}{t^2-1} = \frac{r}{y} \qquad \cot\theta = \frac{2t}{t^2-1} = \frac{x}{y}$$

We shall use combinations of these equations (both left-hand sides and right-hand sides) and the Pythagorean trigonometric identities to study the parametric equations of each of the conics: parabolas, circles, ellipses and hyperbolas.

The sections that follow will describe these parametric equations in terms of both types of parameters; that is, we shall look at parametric equations involving the rectangular parameter t and the circular (trigonometric) parameter θ.

As before, the parametric equations will be converted into the appropriate Cartesian forms by eliminating the parameter and simplifying.

Parametric equations of parabolas

The parametric equations that represent parabolas of the form $(x-h)^2 = \pm 4a(y-k)$ are:

$$x = \pm 2at + h \quad \text{and} \quad y = \pm at^2 + k$$

where a (in this case) is the focal length.

The parametric equations that represent the parabolas of the form $(y-k)^2 = \pm 4a(x-h)$ are:

$$y = \pm 2at + k \quad \text{and} \quad x = \pm at^2 + h.$$

This may be established using the technique of Worked examples 1 and 2, shown earlier. For example, let us consider the following set of parametric equations:

$$x = \pm 2at + h \qquad [1]$$
$$y = \pm at^2 + k \qquad [2]$$

Rearranging [1] to eliminate the parameter gives:

$$t = \pm\frac{(x-h)}{2a}.$$

Substituting this into [2], we have:

$$(y-k) = \pm a\left(\frac{x-h}{2a}\right)^2.$$

Simplifying and collecting terms we have:

$$(x-h)^2 = \pm 4a(y-k)$$

which is the required Cartesian equation.

A similar result will occur with the other set of parametric equations. Students are to investigate this on their own.

In the last section, we were able to find a general equation for the tangent and normal to the parabola $y^2 = \pm 4ax$ at the point $P(x_1, y_1)$. That is, at the point $P(x_1, y_1)$ on the parabola $y^2 = \pm 4ax$:

1. the equation of the tangent is $yy_1 = \pm 2a(x + x_1)$
2. the equation of the normal is $xy_1 \pm 2ay = \pm x_1y_1 + 2ay.$

Similarly, at the point $P(x_1, y_1)$ on the parabola $x^2 = \pm 4ay$:

1. the equation of the tangent is $xx_1 = \pm 2a(y + y_1)$
2. the equation of the normal is $x_1y \pm 2ax = x_1y_1 \pm 2ax_1.$

WORKED Example 7

Consider the parametric equations $x = 4t$, $y = 2t^2$.

a Find the Cartesian equation and state the vertex, focus and equation of the directrix.

b The normal at any point $P(4t, 2t^2)$ on the parabola cuts the y-axis at T. Find the Cartesian equation of the locus of the midpoint, M, of PT.

THINK	WRITE
a 1 Write the parametric equations and number them.	**a** $x = 4t$ [1] $y = 2t^2$ [2]
2 Rearrange equation [1] to write t in terms of x.	Rearranging [1] gives $t = \frac{x}{4}$

THINK	**WRITE**
③ Substitute this expression for t into equation [2]. This produces the Cartesian equation for the parabola.	Substituting $t = \frac{x}{4}$ into [2] gives $y = 2\left(\frac{x}{4}\right)^2$ $= \frac{x^2}{8}$ $x^2 = 8y$
④ Compare the equation to the general form for a parabola.	This is a parabola of the form $x^2 = 4ay$ where $a = 2$.
⑤ State the coordinates of the vertex.	The vertex is at (0, 0).
⑥ State the coordinates of the focus using $(0, a)$.	The focus is at (0, 2).
⑦ State the equation of the directrix using $y = -a$.	The equation of the directrix is $y = -2$.
b ① To find the equation of the normal at $P(4t, 2t^2)$ first find the gradient $\frac{dy}{dx}$ at this point. Alternatively, use the general equation $x_1y + 2ax = x_1y_1 + 2ax_1$, where $(x_1, y_1) = (4t, 2t^2)$.	**b** $\frac{d}{dx}(x^2) = \frac{d}{dy}(8y)\frac{dy}{dx}$ $2x = 8\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{x}{4}$ At $P(4t, 2t^2)$, $\frac{dy}{dx} = \frac{4t}{4}$ $= t$
② Find the gradient of the normal at P.	So the gradient of the normal at P is $-\frac{1}{t}$.
③ Write the equation of the normal at P and simplify.	The equation of the normal at P is $y - 2t^2 = -\frac{1}{t}(x - 4t)$ $-ty + 2t^3 = x - 4t$ $x + ty = 2t^3 + 4t$
④ Find the coordinates of the point T. This is where the normal cuts the y-axis; that is, when $x = 0$.	At the point T, $x = 0$ so $ty = 2t^3 + 4t$ $y = 2t^2 + 4$ The coordinates of T are $(0, 2t^2 + 4)$
⑤ Find the coordinates of the midpoint, M.	The midpoint of PT is $\left(\frac{4t + 0}{2}, \frac{2t^2 + 2t^2 + 4}{2}\right)$ so M has the coordinates $(2t, 2t^2 + 2)$.
⑥ Use the coordinates of M to write the parametric equations for the locus of M.	The parametric equations of the locus of M are $x = 2t$ [1] $y = 2t^2 + 2$ [2]

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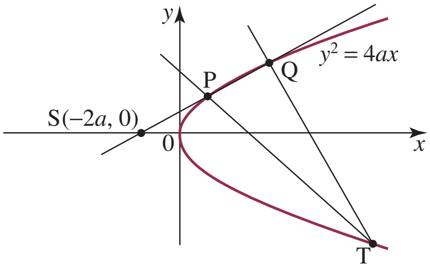
THINK	WRITE
⑦ Find the Cartesian equation for the locus of M by rearranging the first equation and substituting it into the second equation to eliminate the parameter.	Rearranging [1] gives $t = \frac{x}{2}$ Substituting $t = \frac{x}{2}$ into [2] gives $y = 2\left(\frac{x}{2}\right)^2 + 2$ $y = \frac{x^2}{2} + 2$ This is the Cartesian equation of a parabola with vertex at (0, 2), focus at $(0, 2\frac{1}{2})$ and directrix with equation $y = 1\frac{1}{2}$.

WORKED Example 8

Let P(ap^2, $2ap$) and Q(aq^2, $2aq$) be two points on the parabola $y^2 = 4ax$. The normals to the parabola at the points P and Q intersect at T. If the chord PQ varies such that, for all positions of P and Q, the chord PQ when produced passes through the fixed point S($-2a$, 0), show that T lies on the parabola.

THINK	WRITE
① Draw a diagram of the situation.	
② Use the fact that P, Q and S are collinear (lie on the same line) to find a relationship between p and q; that is, the gradient of PQ equals the gradient of QS.	As P, Q and S are collinear: gradient of PS = gradient of QS $\frac{2ap}{ap^2 + 2a} = \frac{2aq}{aq^2 + 2a}$
③ Cancel the common factor $\frac{2a}{a}$.	$\frac{p}{p^2 + 2} = \frac{q}{q^2 + 2}$
④ Cross-multiply by the denominators.	$p(q^2 + 2) = q(p^2 + 2)$
⑤ Create a quadratic in terms of p.	$qp^2 - (q^2 + 2)p + 2q = 0$
⑥ Factorise.	$(qp - 2)(p - q) = 0$
⑦ Solve the equation for p.	$p = q$ or $p = \frac{2}{q}$
⑧ Since P and Q are distinct points on the parabola, reject the solution $p = q$.	Since P and Q are distinct points, $p \neq q$ so $p = \frac{2}{q}$.

THINK	**WRITE**
⑨ To find the equation of the normal at P(ap^2, $2ap$) first find the gradient $\frac{dy}{dx}$ at this point. Alternatively, use the general equation $xy_1 + 2ay = x_1y_1 + 2ay$ where $(x_1, y_1) = (ap^2, 2ap)$.	$\frac{d}{dy}(y^2)\frac{dy}{dx} = \frac{d}{dx}(4ax)$ $2y\frac{dy}{dx} = 4a$ $\frac{dy}{dx} = \frac{2a}{y}$ At P(ap^2, $2ap$), $\frac{dy}{dx} = \frac{2a}{2ap}$ $= \frac{1}{p}$
⑩ Find the gradient of the normal at P.	so the gradient of the normal at P is $-p$.
⑪ Write the equation of the normal at P and simplify.	The equation of the normal at P is $y - 2ap = -p(x - ap^2)$ $y = 2ap - px + ap^3$ [1]
⑫ To find the equation of the normal at Q(aq^2, $2aq$), first find the gradient $\frac{dy}{dx}$ at this point. Alternatively, use the general equation $xy_1 + 2ay = x_1y_1 + 2ay$ where $(x_1, y_1) = (aq^2, 2aq)$.	At Q(aq^2, $2aq$), $\frac{dy}{dx} = \frac{2a}{2aq}$ $= \frac{1}{q}$
⑬ Find the gradient of the normal at Q.	so the gradient of the normal at Q is $-q$.
⑭ Write the equation of the normal at Q and simplify.	The equation of the normal at Q is $y - 2aq = -q(x - aq^2)$ $y = 2aq - qx + aq^3$ [2]
⑮ Since the normals at P and Q intersect at T, equate equations [1] and [2] to find T.	Equating [1] and [2] gives $2ap - px + ap^3 = 2aq - qx + aq^3$
⑯ Rearrange the resulting equation to solve for x.	$qx - px = 2aq + aq^3 - 2ap - ap^3$ $x(q - p) = 2a(q - p) + a(q^3 - p^3)$ $x = \frac{2a(q-p) + a(q-p)(q^2 + qp + p^2)}{q - p}$ $= 2a + a(q^2 + qp + p^2)$
⑰ Replace p with $\frac{2}{q}$ and simplify.	$= 2a + aq^2 + aq \cdot \frac{2}{q} + a\left(\frac{2}{q}\right)^2$ since $p = \frac{2}{q}$ $= 2a + aq^2 + 2a + \frac{4a}{q^2}$ $= \frac{2aq^2 + aq^4 + 2aq^2 + 4a}{q^2}$ $= \frac{a(q^4 + 4q^2 + 4)}{q^2}$ $= \frac{a(q^2 + 2)^2}{q^2}$

Continued over page →

THINK	WRITE
18 Substitute this x-value into equation [2] to find the y-value at T.	Substitute $x_T = \frac{a(q^2+2)^2}{q^2}$ into [2] to find y at T. $y_T = 2aq - \frac{qa(q^2+2)^2}{q^2} + aq^3$ $= \frac{2aq^2 - a(q^2+2)^2 + aq^4}{q}$ $= \frac{2aq^2 - aq^4 - 4aq^2 - 4a + aq^4}{q}$ $= \frac{-2aq^2 - 4a}{q}$ $= -\frac{2a(q^2+2)}{q}$
19 To show that T lies on the parabola, we need to show that the coordinates of T fit the relationship $y^2 = 4ax$. First square y_T to find an expression for $y_T{}^2$, then multiply x_T by $4a$ to find an expression for $4ax_T$.	Now $y_T{}^2 = \frac{4a^2(q^2+2)^2}{q^2}$ and $4ax_T = 4a \cdot \frac{a(q^2+2)^2}{q^2}$ $= \frac{4a^2(q^2+2)^2}{q^2}$ so $y^2 = 4ax_T$ and hence T lies on the parabola.

remember

1. The parametric equations that represent parabolas of the form $(y - k)^2 = \pm 4a(x - h)$ are:
$$y = \pm 2at + k \quad\text{and}\quad x = \pm at^2 + h.$$
2. The parametric equations that represent parabolas of the form $(x - h)^2 = \pm 4a(y - k)$ are:
$$x = \pm 2at + h \quad\text{and}\quad y = \pm at^2 + k$$
where a (in this case) is the focal length.
3. At the point $P(x_1, y_1)$ on the parabola $y^2 = \pm 4ax$
   (a) the equation of the tangent is $yy_1 = \pm 2a(x + x_1)$
   (b) the equation of the normal is $xy_1 \pm 2ay = \pm x_1y_1 + 2ay$.
4. At the point $P(x_1, y_1)$ on the parabola $x^2 = \pm 4ay$
   (a) the equation of the tangent is $xx_1 = \pm 2a(y + y_1)$
   (b) the equation of the normal is $x_1y \pm 2ax = x_1y_1 \pm 2ax_1$.

EXERCISE 10C Parametric equations of conics

WORKED Example 7

1 Consider the parametric equations $x = -2t^2$, $y = 4t$.
   - a Find the Cartesian equation and state the vertex, focus and equation of the directrix.
   - b The normal at any point, P, cuts the x-axis at Q. Find the Cartesian equation of the locus of the midpoint, M, of PQ.

2 Consider the parabola $y^2 = 8x$.
   - a P($2t^2$, $4t$) and Q($8t^2$, $8t$) are two variable points on the parabola. Tangents from P and Q intersect at R. Find the Cartesian equation of the locus of R.
   - b The normal at P($2t^2$, $4t$) intersects the x-axis at T. Find the coordinates of T and find the Cartesian equation of the locus of T.

3 If $A(2t, t^2)$ and B($4t$, $4t^2$) are two variable points on the parabola $x^2 = 4y$, and the tangents at A and B meet at C, find the equation of the locus of C.

WORKED Example 8

4 Let P(ap^2, $2ap$) and Q(aq^2, $2aq$) be two points on the parabola $y^2 = 4ax$ such that PQ is a focal chord. Let A(at^2, $2at$) and B(as^2, $2as$) be two other variable points on $y^2 = 4ax$.
   - a Show that $pq = -1$.
   - b If P is joined to the vertex, V, and PV is produced to meet the directrix at D, show that DQ is parallel to the axis of the parabola.
   - c The normal at P intersects the x-axis at R. The directrix intersects the x-axis at S, and F is the focus. The midpoint of RF is M. Show that $\text{SM}^2 - \text{MP}^2 = 4a^2$.
   - d The normals at A and B meet at H. Find H in terms of s and t. If $st = -2$, find the Cartesian equation of the locus of H.

5 P($2t^2$, $4t$) is any point on a certain parabola.
   - a Find the equation of the parabola.
   - b Find the equation of the tangent at P and where this tangent cuts the axes.
   - c If the tangent at P cuts the x-axis at T, show that ST = SP where S is the focus.

6 P is a variable point on the parabola $x^2 = 4ay$, which has a focus S and a vertex A. Q is the midpoint of SP, and R is the midpoint of AQ. Determine the locus of R.

7 PT is the tangent at P, a variable point on the parabola $x^2 = -16y$. If T lies on the directrix, find the locus of the midpoint of PT.

Parametric equations of circles and ellipses

We shall now examine the parametric equations that relate to circles and ellipses.

The circle

The parametric equations that represent a circle of the form $(x - h)^2 + (y - k)^2 = a^2$ are:

$$x = a\cos\theta + h, \quad y = a\sin\theta + k$$

where a (in this case) is the radius of the circle, and the centre is located at (h, k). This can be shown as follows. Consider the parametric equations:

$$x = a\cos\theta + h,\ \ y = a\sin\theta + k$$

Rearranging both equations to isolate the trigonometric functions gives:

$$\frac{(x-h)}{a} = \cos\theta \quad \text{and} \quad \frac{(y-k)}{a} = \sin\theta$$

Squaring both these equations and adding gives:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = \cos^2\theta + \sin^2\theta$$

Since $\cos^2\theta + \sin^2\theta = 1$, we obtain:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1.$$

A final rearranging produces:

$$(x-h)^2 + (y-k)^2 = a^2$$

which is a circle with centre (h, k) and radius of a units.

The ellipse

The parametric equations that represent an ellipse of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ are:

$$x = a\cos\theta + h,\ \ y = b\sin\theta + k$$

where a and b are the lengths of the semi-major and semi-minor axes respectively and the centre is located at (h, k).

In a manner similar to that employed with the circle, we may consider the parametric equations:

$$x = a\cos\theta + h \quad \text{and} \quad y = b\sin\theta + k.$$

Rearranging both equations to isolate the trigonometric functions gives:

$$\frac{(x-h)}{a} = \cos\theta \ \text{ and } \ \frac{(y-k)}{b} = \sin\theta.$$

Squaring both the equations and adding gives:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \cos^2\theta + \sin^2\theta.$$

Since $\cos^2\theta + \sin^2\theta = 1$, we obtain:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

which is the Cartesian equation of an ellipse.

As for the parabola, a general equation for the tangent and the normal can be developed for the ellipse.

At the point $P(x_1, y_1)$ on an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

1. the equation of the tangent is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
2. the equation of the normal is $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$.

WORKED Example 9

The points P(4 cos θ, 3 sin θ) and Q(–4 sin θ, 3 cos θ) lie on the ellipse $9x^2 + 16y^2 = 144$. Find the Cartesian equation of the locus of the point of intersection of the tangents at P and Q.

THINK	WRITE
① Write the equation of the ellipse and rearrange it into the form $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ so that a and b can be easily identified.	$9x^2 + 16y^2 = 144$ $\frac{x^2}{16}+\frac{y^2}{9} = 1$
② Use the general form to write the equation of the tangent to the ellipse at the point P.	At P(4 cos θ, 3 sin θ), the equation of the tangent to the ellipse is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$ where $a = 4$, $b = 3$, $x_1 = 4\cos\theta$ and $y_1 = 3\sin\theta$. $\frac{4x\cos\theta}{16}+\frac{3y\sin\theta}{9} = 1$ $\frac{x\cos\theta}{4}+\frac{y\sin\theta}{3} = 1$ [1]
③ Use the general form to write the equation of the tangent to the ellipse at the point Q.	At Q(–4 sin θ, 3 cos θ), the equation of the tangent to the ellipse is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$ where $a = 4$, $b = 3$, $x_1 = -4\sin\theta$ and $y_1 = 3\cos\theta$. $\frac{-4x\sin\theta}{16}+\frac{3y\cos\theta}{9} = 1$ $\frac{-x\sin\theta}{4}+\frac{y\cos\theta}{3} = 1$ [2]
④ Solve equations [1] and [2] to find the coordinates of the point of intersection of the tangents. First, arrange the equations to eliminate the terms involving y, then solve for x.	For the point of intersection, solve equations [1] and [2]. Multiplying [1] by cos θ gives $\frac{x\cos^2\theta}{4}+\frac{y\sin\theta\cos\theta}{3} = \cos\theta$ [3] Multiplying [2] by sin θ gives $\frac{-x\sin^2\theta}{4}+\frac{y\cos\theta\sin\theta}{3} = \sin\theta$ [4] Subtracting [4] from [3] gives $\frac{x}{4}(\cos^2\theta+\sin^2\theta) = \cos\theta - \sin\theta$
⑤ Simplify, using the identity $\cos^2\theta + \sin^2\theta = 1$.	$\frac{x}{4} = \cos\theta - \sin\theta$ [5]

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THINK	WRITE
⑥ Arrange the equations to eliminate the terms involving x, then solve for y.	Multiplying [1] by sin θ gives $\frac{x\cos\theta\sin\theta}{4}+\frac{y\sin^2\theta}{3} = \sin\theta$ [6] Multiplying [2] by cos θ gives $\frac{-x\sin\theta\cos\theta}{4}+\frac{y\cos^2\theta}{3} = \cos\theta$ [7] Adding [6] and [7] gives $\frac{y}{3}(\sin^2\theta+\cos^2\theta)=\sin\theta+\cos\theta$ $\frac{y}{3}=\sin\theta+\cos\theta$ [8]
⑦ State the coordinates of the point of intersection. This gives us the parametric equations for the locus of the point of intersection.	The point of intersection is located at $(4\cos\theta-4\sin\theta, 3\sin\theta+3\cos\theta)$.
⑧ Arrange to eliminate the parameter θ, to obtain the Cartesian equation of the required locus. Again use the identity $\cos^2\theta+\sin^2\theta=1$.	To obtain the required Cartesian equation, eliminate the parameter θ. Squaring both [5] and [8] and adding gives $\left(\frac{x}{4}\right)^2+\left(\frac{y}{3}\right)^2=(\cos\theta-\sin\theta)^2+(\sin\theta+\cos\theta)^2$ $\frac{x^2}{16}+\frac{y^2}{9}=\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta+\sin^2\theta$ $+2\sin\theta\cos\theta+\cos^2\theta$ $\frac{x^2}{16}+\frac{y^2}{9}=(\cos^2\theta+\sin^2\theta)+(\sin^2\theta+\cos^2\theta)$ $\frac{x^2}{16}+\frac{y^2}{9}=2$ The Cartesian equation for the locus of the point of intersection of the tangents at P and Q is $9x^2+16y^2=288$.

remember

1. The parametric equations that represent an ellipse of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} = 1$ are:

$$x = a\cos\theta+h,\ \ y = b\sin\theta+k$$

where a and b are the lengths of the semi-major and semi-minor axes respectively, and the centre is located at (h, k).

2. At the point $P(x_1, y_1)$ on an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

(a) the equation of the tangent is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$

(b) the equation of the normal is $a^2y_1x-b^2x_1y = (a^2-b^2)x_1y_1$.

EXERCISE 10D Parametric equations of circles and ellipses

WORKED Example 9

1 The points P(4 cos θ, 4 sin θ) and Q(4 sin θ, −4 cos θ) lie on the circle $x^2 + y^2 = 16$. Find the Cartesian equation of the locus of the point of intersection of the tangents at P and Q.

2 Show that the points P(3 cos θ, 2 sin θ) and Q(−3 sin θ, 2 cos θ) lie on the same ellipse and find the Cartesian equation of this ellipse. Find the Cartesian equation of the locus of the point of intersection of the tangents at P and Q.

3 The points A(x_A, y_A) and B(x_B, y_B) are variable points on two concentric (same centre) circles $x^2 + y^2 = r_1{}^2$ and $x^2 + y^2 = r_2{}^2$, respectively where $r_1 > r_2$. As the points move around the circles, the points O, A and B are collinear. Find the locus of the point C(x_A, y_B).

4 Tangents to the ellipse $x^2 + 4y^2 = 4$ at the points (2 cos θ, sin θ) and (2 cos ϕ, sin ϕ) are perpendicular. Show that $\tan\theta = -\frac{1}{4}\cot\phi$.

Parametric equations of hyperbolas

The parametric equations for a hyperbola of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ are:

$$x = a\sec\theta + h \quad \text{and} \quad y = b\tan\theta + k$$

where a and b are the lengths of the semi-transverse and semi-conjugate axes respectively, and the centre is located at (h, k).

The parametric equations for a hyperbola of the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ are:

$$y = a\,\text{cosec}\,\theta + k \quad \text{and} \quad x = b\cot\theta + h.$$

Consider the first set of parametric equations involving the trigonometric functions:

$$x = a\sec\theta + h \quad \text{and} \quad y = b\tan\theta + k.$$

Rearranging each equation to isolate the parameter gives:

$$\frac{(x-h)}{a} = \sec\theta \quad \text{and} \quad \frac{(y-k)}{b} = \tan\theta.$$

Squaring both sides of each equation then subtracting gives:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \sec^2\theta - \tan^2\theta.$$

Using the trigonometric identity $\sec^2\theta - \tan^2\theta = 1$ produces:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

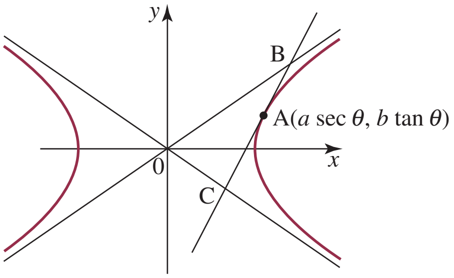
The second set of parametric equations can be verified in a similar fashion.

For the parabola and ellipse, a general equation for the tangent and the normal can be developed for the hyperbola. At the point P(x_1, y_1) on a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

1. the equation of the tangent is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
2. the equation of the normal is $a^2y_1x - b^2x_1y = (a^2 + b^2)x_1y_1$.

WORKED Example 10

The tangent at a point A(a sec θ, b tan θ) on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intersects the asymptotes at B and C. Show that A is the midpoint of BC.



THINK	WRITE
① Write the Cartesian equation of the hyperbola.	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
② Use the general form to write the equation of the tangent to the hyperbola at the point A.	At A($a$sec θ, b tan θ), the equation of the tangent to the hyperbola is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ where $x_1 = a \sec \theta$ and $x_2 = b \tan \theta$. $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$ [1]
③ Write the equations of the asymptotes of the hyperbola.	The equations of the asymptotes are $y = \frac{b}{a}x$ [2] and $y = -\frac{b}{a}x$ [3]
④ Find the point of intersection, B, by solving equations [1] and [2]. (a) First substitute equation [2] into equation [1] to eliminate y. (b) Solve for x.	For point B: substituting [2] into [1] gives $\frac{x \sec \theta}{a} - \frac{x \tan \theta}{a} = 1$ $x \sec \theta - x \tan \theta = a$ $x(\sec \theta - \tan \theta) = a$ $x\left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta}\right) = a$

THINK	**WRITE**
	$x\left(\frac{1-\sin\theta}{\cos\theta}\right) = a$ $x = \frac{a\cos\theta}{1-\sin\theta}$
5 Substitute this x-value into equation [2] to find the corresponding y-value for B.	Substituting $x = \frac{a\cos\theta}{1-\sin\theta}$ into $y = -\frac{b}{a}x$ gives $y = \frac{b\cos\theta}{1-\sin\theta}$
6 Find the point of intersection, C, by solving equations [1] and [3]. (a) First substitute equation [3] into equation [1] to eliminate y. (b) Solve for x.	For point C: substituting [3] into [1] gives $\frac{x\sec\theta}{a} + \frac{x\tan\theta}{a} = 1$ $x\sec\theta + x\tan\theta = a$ $x(\sec\theta + \tan\theta) = a$ $x\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right) = a$ $x\left(\frac{1+\sin\theta}{\cos\theta}\right) = a$ $x = \frac{a\cos\theta}{1+\sin\theta}$
7 Substitute this x-value into equation [3] to find the corresponding y-value for C.	Substituting $x = \frac{a\cos\theta}{1+\sin\theta}$ into $y = -\frac{b}{a}x$ gives $y = -\frac{b\cos\theta}{1+\sin\theta}.$
8 State the coordinates of the points B and C.	So the coordinates of B are $\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right)$ and the coordinates of C are $\left(\frac{a\cos\theta}{1+\sin\theta}, -\frac{b\cos\theta}{1+\sin\theta}\right).$
9 Find the coordinates of the midpoint of BC.	The x-coordinate of the midpoint of BC is $\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right)$ $= \frac{1}{2}\left(\frac{a\cos\theta(1+\sin\theta) + a\cos\theta(1-\sin\theta)}{1-\sin^2\theta}\right)$ $= \frac{1}{2}\left(\frac{2a\cos\theta}{\cos^2\theta}\right)$ $= a\sec\theta$

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THINK	**WRITE**
	The y-coordinate of the midpoint BC is $\frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta}-\frac{b\cos\theta}{1+\sin\theta}\right)$ $=\frac{1}{2}\left(\frac{b\cos\theta(1+\sin\theta)-b\cos\theta(1-\sin\theta)}{1-\sin^2\theta}\right)$ $=\frac{1}{2}\left(\frac{2b\sin\theta\cos\theta}{\cos^2\theta}\right)$ $=b\tan\theta$
⑩ Show that the result gives the coordinates of A.	So the coordinates of the midpoint of BC are ($a\sec\theta$, $b\tan\theta$) which are the same as the coordinates of A.

The rectangular hyperbola

The parametric equations for a rectangular hyperbola of the form $(x-h)(y-k)=c^2$ are

$$x=ct+h \text{ and } y=\frac{c}{t}+k \quad\text{or}\quad y=ct+k \text{ and } x=\frac{c}{t}+h$$

where the centre is located at (h, k).

The rectangular hyperbola is relatively simple to convert from parametric to Cartesian forms. Let us consider the first set of rectangular parametric equations:

$$x=ct+h \quad\text{and}\quad y=\frac{c}{t}+k.$$

Rearranging each equation to isolate the parameter gives:

$$x-h=ct \quad\text{and}\quad y-k=\frac{c}{t}.$$

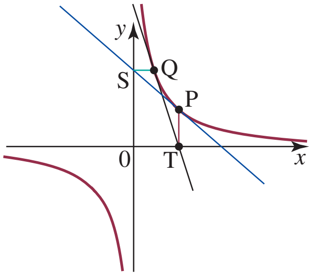
Multiplying the two equations together gives:

$$(x-h)(y-k)=ct\cdot\frac{c}{t} \quad\text{or}\quad (x-h)(y-k)=c^2.$$

The equation of the tangent is $x(y_1-k)+y(x_1-h)=2x_1y_1-kx_1-hy_1$.
The equation of the normal is $x(x_1-h)-y(y_1-k)=x_1{}^2-y_1{}^2-hx_1+ky_1$.

WORKED Example 11

The point P$\left(cp, \frac{c}{p}\right)$ is any point on the rectangular hyperbola $xy=c^2$. Point T is the foot of the perpendicular from P on to the x-axis. The tangent at P cuts the y-axis at S and the line through S that is parallel to the x-axis cuts the hyperbola at Q. Show that TQ is a tangent to the hyperbola.



THINK	**WRITE**
① Write the equation of the hyperbola.	$xy = c^2$
② State the coordinates of T using the fact that the x-coordinate is the same as that for P, and the y-coordinate is 0 (since T lies on the x-axis).	The coordinates of T are $(cp, 0)$.
③ Use the general form to write the equation of the tangent at P.	At P$\left(cp, \frac{c}{p}\right)$, the equation of the tangent to the hyperbola is $x_1y + xy_1 = 2x_1y_1$, where $x_1 = cp$ and $y_1 = \frac{c}{p}$ $cpy + \frac{cx}{p} = 2(cp)\left(\frac{c}{p}\right)$ $py + \frac{x}{p} = 2c$
④ As S lies on the y-axis and the tangent, find the coordinates of S by substituting $x = 0$ into the equation for the tangent to find y.	At S, $x = 0$ so $py + 0 = 2c$ $y = \frac{2c}{p}$ Hence the coordinates of S are $\left(0, \frac{2c}{p}\right)$.
⑤ Since Q has the same y-coordinate as S, and also lies on the hyperbola, find the coordinates of Q by substituting $y = \frac{2c}{p}$ into the equation of the hyperbola to find x.	At Q, $y = \frac{2c}{p}$ so $x\left(\frac{2c}{p}\right) = c^2$ $x = \frac{cp}{2}$ Hence the coordinates of Q are $\left(\frac{cp}{2}, \frac{2c}{p}\right)$.
⑥ Use implicit differentiation to find the gradient, $\frac{dy}{dx}$, of the hyperbola.	The gradient of the hyperbola is $x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = \frac{d}{dx}(c^2)$ $x\frac{dy}{dx} + y = 0$ $\frac{dy}{dx} = -\frac{y}{x}$
⑦ Substitute $x = \frac{cp}{2}$ and $y = \frac{2c}{p}$ into the expression for $\frac{dy}{dx}$ to find the gradient at Q.	At Q$\left(\frac{cp}{2}, \frac{2c}{p}\right)$, $\frac{dy}{dx} = -\frac{2c}{p} \cdot \frac{2}{cp} = -\frac{4}{p^2}$

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THINK	**WRITE**
⑧ Use the coordinates of T and Q to find the gradient of TQ.	For TQ, the gradient is $\dfrac{\left(\dfrac{2c}{p}-0\right)}{\left(\dfrac{cp}{2}-cp\right)}$ $= \dfrac{2c}{p} \cdot -\dfrac{2}{cp}$ $= -\dfrac{4}{p^2}$
⑨ Compare the gradient of TQ and the gradient of the hyperbola at Q and write a conclusion.	Since the gradient of the line TQ is the same as the gradient of the hyperbola at Q, TQ is a tangent to the hyperbola.

remember

1. The parametric equations for a hyperbola of the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ are:
$$x = a\sec\theta + h \quad \text{and} \quad y = b\tan\theta + k$$
where a and b are the lengths of the semi-transverse and semi-conjugate axes respectively, and the centre is located at (h, k).
2. The parametric equations for a hyperbola of the form $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ are:
$$y = a\operatorname{cosec}\theta + k \quad \text{and} \quad y = b\cot\theta + h.$$
3. The parametric equations for a rectangular hyperbola of the form $(x-h)(y-k) = c^2$ are:
$$x = ct + h \text{ and } y = \frac{c}{t} + k \quad \text{or} \quad y = ct + k \text{ and } x = \frac{c}{t} + h$$
where the centre is located at (h, k).
4. At the point $P(x_1, y_1)$ on a hyperbola of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
   (a) the equation of the tangent is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$
   (b) the equation of the normal is $a^2y_1x + b^2x_1y = (a^2 + b^2)x_1y_1$.
5. At the point $P(x_1, y_1)$ on a rectangular hyperbola of the form $(x-h)(y-k) = c^2$:
   (a) the equation of the tangent is $x(y_1 - k) + y(x_1 - h) = 2x_1y_1 - kx_1 - hy_1$
   (b) the equation of the normal is $x(x_1 - h) - y(y_1 - k) = {x_1}^2 - {y_1}^2 - hx_1 + ky_1$.

EXERCISE 10E Parametric equations of hyperbolas

1 Show that any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be represented by $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$ and find the equation of the tangent at this point.

WORKED Example 10

2 The tangent to the hyperbola $x^2 - 4y^2 = 4$ at the point $P(2\sec\theta, \tan\theta)$ cuts the asymptotes at S and T. Show that P is the midpoint of ST.

WORKED Example 11

3 Consider the rectangular hyperbola $xy = 1$.

a Let $P\left(p, \frac{1}{p}\right)$ be a point on the hyperbola. Let the tangent at P cut the asymptotes at A and B. Find the area of the triangle OAB, where O is the origin.

b Let $S\left(s, \frac{1}{s}\right)$ and $T\left(t, \frac{1}{t}\right)$ be two points on the branch of the hyperbola that lies in the first quadrant. Show that the gradient of the line that passes through S and T is $-\frac{1}{st}$.

c If the point (2, 2) lies on the line ST, show that:

i $t = \frac{s-2}{2s-1}$

ii $(p - 1)(q - 1)$ is a constant, where $M(p, q)$ is the midpoint of ST.

4 $P(x_1, y_1)$ is a point on the rectangular hyperbola $x^2 - y^2 = a^2$ whose foci are F_1 and F_2.

a Show that $PF_1 \cdot PF_2 = PO^2$, where O is the origin.

b If the normal at P cuts the axes at A and B, show that PA = PB = PO.

Applying technology to parametric equations

By using appropriate technology, we can quickly generate graphs of conic sections in parametric form. The following examples illustrate the procedure using a graphics or CAS calculator.

For the CASIO *fx*-9860G AU

1. To sketch a graph of a conic section, press:
   - MENU
   - 5 (GRAPH)

   To ensure the calculator is in Radian Mode, press:
   - SHIFT
   - MENU (SET UP)

   Scroll down to Angle and press F2 (Rad).
   Press EXIT to return to main graph screen.

```
Graph Func   :On
Dual Screen  :Off
Simul Graph  :Off
Derivative   :Off
Background   :None
Sketch Line  :Norm
Angle        :Rad
Deg Rad Gra
```

2. To enter the parametric equations, press:
   - F3 (TYPE)
   - F3 (Parm)

   Complete the entry lines as:

   $Xt1 = 2 \cdot (1 - T^2) \,|\, (1 + T^2)$

   $Yt1 = 6T \,|\, (1 + T^2)$

   Press EXE after each line.

Graph Func :Param
Xt1■2(1−T²)÷(1+T[—]
Yt1■6T÷(1+T²)
Xt2: [—]
Yt2:
Xt3: [—]
Yt3:

3. To change V-Window, press:
   - SHIFT
   - F3 (V-Window)

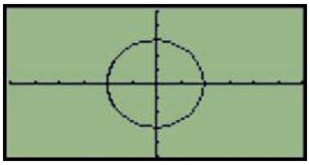
   Enter the values as shown.

View Window
Xmin :-6
max :6
scale:1
dot :0.09523809
Ymin :-5
max :5

4. Continue to scroll down to the T values and enter (T) min as -15π and (T) max as 15π and the pitch as $\frac{\pi}{16}$.

View Window
Ymin :-5
max :5
scale:0.5
Tθmin :-47.123889
max :47.1238898
ptch:π÷16

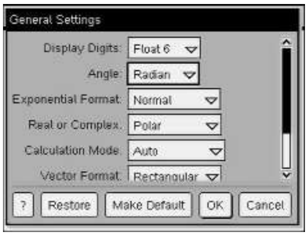
5. Press EXIT to return to the graph screen and press F6 (DRAW).
   The curve is an ellipse.



For the TI-Nspire CAS

1. To ensure that the calculator is in radian mode, press:
   - HOME
   - 5: Settings & Status 5
   - 1: General 1

   Change the Angle Mode to Radian.



General Settings
Display Digits: Float 6
Angle: Radian
Exponential Format: Normal
Real or Complex: Polar
Calculation Mode: Auto
Vector Format: Rectangular

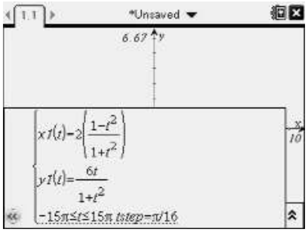
2. To graph the conic section using parametric equations, on a Graphs page, press:
   - MENU menu
   - 3: Graph Type 3
   - 2: Parametric 2

   Complete the entry lines as:

   $$x1(t) = 2\left(\frac{1-t^2}{1+t^2}\right)$$

   $$y1(t) = \frac{6t}{1+t^2}$$

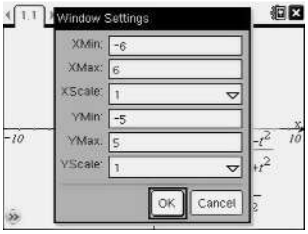
   Set appropriate values for t and then press ENTER enter.



$x1(t)=2\left(\frac{1-t^2}{1+t^2}\right)$
$y1(t)=\frac{6t}{1+t^2}$
$-15\pi \le t \le 15\pi$ tstep=$\pi/16$

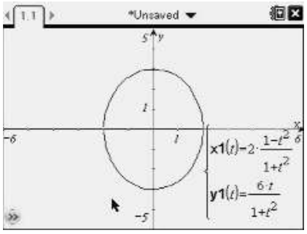
3. To change the Window Settings, press:
   - MENU menu
   - 4: Window 4
   - 1: Window settings 1

   Enter the window settings as shown.



Window Settings
XMin: -6
XMax: 6
XScale: 1
YMin: -5
YMax: 5
YScale: 1

4. The graph produced is an ellipse.



WORKED Example 12

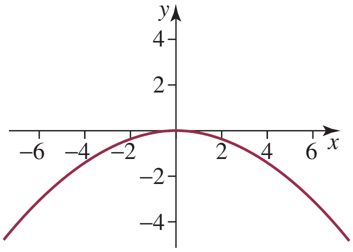
The length, L, of a continuous curve may be calculated as $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

For the curve defined parametrically as $x = 6t$, $y = -3t^2$ find L for $-3 \le x \le 3$.

THINK

WRITE/DISPLAY

① Use a graphics calculator or graphing program to produce a graph of the curve given by the parametric equations.



② Write the parametric equations and determine $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$x = 6t$ and $y = -3t^2$

$\frac{dx}{dt} = 6$ and $\frac{dy}{dt} = -6t$

③ Calculate the limits of the integral in terms of t.

When $x = -3$, $t = -\frac{1}{2}$.

When $x = 3$, $t = \frac{1}{2}$.

④ Calculate L by substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, t_1 and t_2.

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{6^2 + (-6t)^2}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{36(1 + t^2)}\, dt$$

$$= 6\int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 + t^2}\, dt$$

Continued over page ➔

THINK | **WRITE/DISPLAY**

5 Use the symmetry of the curve to simplify the calculation.

$$= 12\int_0^{\frac{1}{2}} \sqrt{1+t^2}\ dt$$

6 Use a graphics calculator to evaluate L.

Using the CASIO *fx*-9860G AU

To find the value of L:

(a) Press:

- MENU
- 1 (RUN)
- SHIFT SHIFT
- MENU (SET UP)

(b) Scroll down to Func Type and press F1 (Y=).

(c) Scroll down to Angle and press F2 (Rad).

Then press:

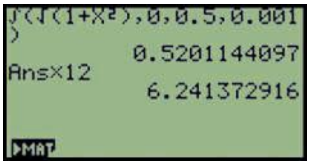
- EXE
- OPTN
- F4 (CALC)
- F4 ($\int dx$)

(d) Complete the entry line as:

$$\int(\sqrt{(1+x^2)}, 0, 0.5)$$

Then press EXE.

(e) Multiply by 12 to obtain the result for L.



```
∫(√(1+X²),0,0.5,0.001
)
                0.5201144097
Ans×12
                 6.241372916
```

(f) Write the answer.

The length, L, of the curve for $-3 \le x \le 3$ is 6.241 units.

For the TI-Nspire CAS

To find the value of L:

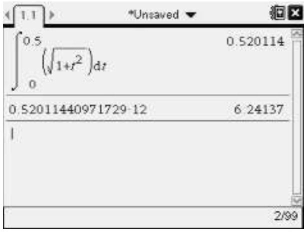
(a) On a Calculator page press:

- MENU menu
- 4: Calculus 4
- 3: Integral 3

Then complete the entry line as:

$$\int_0^{\frac{1}{2}}(\sqrt{1+x^2})dx$$

Press ENTER enter.



$\int_0^{0.5}(\sqrt{1+t^2})dt$ 0.520114

0.52011440971729·12 6.24137

(b) Multiply by 12 to obtain the result for L.

(c) Write the answer.

The length, L, of the curve for $-3 \le x \le 3$ is 6.241 units.

remember

Curves represented by parametric equations can be graphed easily using a graphics calculator or graphing software such as Graphmatica. Check that the operational mode is in radians and parametric form.

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Interactivity:
Parametric
int-0839

EXERCISE 10F Applying technology to parametric equations

1 Use appropriate technology to graph and identify the conic whose parametric equations are:

a $x = 2t^2, y = 4t$

b $x = \frac{2(1-t^2)}{1+t^2}, y = \frac{4t}{1+t^2}$

c $x = 6t, y = -3t^2$

d $x = 2\left(t + \frac{1}{t}\right), y = t - \frac{1}{t}$

e $x = -2t^2, y = 4t$

f $x = 2\left(t + \frac{1}{t}\right), y = 2\left(t - \frac{1}{t}\right)$

g $x = 2 \sec \theta, y = \tan \theta$

h $x = \frac{3(1-t^2)}{1+t^2}, y = \frac{2t}{1+t^2}$

i $x = 5 \cos \theta, y = 3 \sin \theta$

j $x = -2t, y = t^2$

k $x = \frac{-2}{t}, y = 2t$

l $x = 3 \cos \theta + 4, y = \sin \theta + 2.$

2 Use appropriate technology to graph each of the following curves, whose parametric equations are:

a $x = 2(t - \cos t)$
$y = 3(1 - \sin t)$

b $x = 3 \cos^3 (t)$
$y = 2 \sin^3 (t)$

c $x = \frac{3t}{1+t^3}$
$y = \frac{3t^2}{1+t^3}$

d $x = 3 \cos (t) - \cos (3t)$
$y = 3 \sin (t) - \sin (3t)$

e $x = 2 \cos (t) + \cos (2t)$
$y = 2 \sin (t) - \sin (2t)$

f $x = 3t$
$y = 3(1 - \cos t)$

g $x = 3t - 2 \sin (t)$
$y = 3 - 2 \cos (t)$

h $x = 3 \cos (t) - \cos (3t)$
$y = 3 \sin (t) - \sin (3t)$

i $x = 4 \cot (t)$
$y = 4 \sin^2 (t).$

WORKED Example 12

3 The length, L, of a continuous curve may be calculated as $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, where $x(t)$ and $y(t)$ are the parametric forms of the curve. For each curve, calculate L across the supplied domain.

a $x = 2t^2, y = 4t, 0 \le x \le 3$

b $x = 5\cos t, y = 3\sin t, -\pi \le t \le \pi$

c $\begin{aligned} x &= 3t - 2\sin(t) \\ y &= 3 - 2\cos(t) \end{aligned}$ for $-6 \le t \le 6$

d $x = 2\left(t + \frac{1}{t}\right), y = 3\left(t - \frac{1}{t}\right), 0 \le t \le 12$

e $x = \frac{-2}{t}, y = 2t, -2 \le x \le -1$

f $\begin{aligned} x &= 2\cos(t) + \cos(2t) \\ y &= 2\sin(t) - \sin(2t) \end{aligned}$ for $-\pi \le t \le \pi$

g $\begin{aligned} x &= 2(t - \sin t) \\ y &= 3(1 - \cos t) \end{aligned}$ for $0 \le t \le 2\pi$

h $\begin{aligned} x &= 3\cos^3(t) \\ y &= 2\sin^3(t) \end{aligned}$ for $(0 \le t \le \pi)$

4 By varying the value of the constant a and alternating the $\pm$ in the parametric equations $x = a(t \pm \sin t), y = a(1 - \cos t)$, describe the curves generated and state the effect produced by the changes to a and the $\pm$ sign.

5 Show that $x = \frac{3t}{1 + t^3}, y = \frac{3t^2}{1 + t^3}$ satisfies the equation called the *Folium of Descartes*, $x^3 + y^3 - 3xy = 0$. For this curve, show that $y = tx$ and explain what this means.

6 The parametric equations of the curve known as an *astroid* are given by $x = a\cos^3(t)$, $y = a\sin^3(t)$.

a Analyse the general shape of the astroid. Sketch this curve and mark the significant features.

b $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ represents the arc length of the astroid for $t_1 \le t \le t_2$.

Generate a table of values for L using $2 \le a \le 6$.

c From your results, or otherwise, deduce the value of L in terms of a.

d Determine the Cartesian equation of the astroid.

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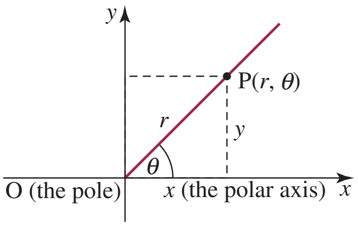
Digital doc: SkillSHEET 10.1

7 A particular cycloid is defined parametrically by the equations: $x = a(t - \sin t)$ and $y = a(1 - \cos t)$. The area underneath this cycloid for $0 \le x \le 2\pi a$ is given by $\int_0^{2\pi a} y\,dx$.

a Show that this area is equivalent to the integral $\int_0^{2\pi} a^2(1 - \cos t)^2 dt$.

b Deduce that this area is $3\pi a^2$.

Polar equations



As mentioned at the start of this chapter, many plane curves can be more easily described in a coordinate system in which the coordinates of a point P(r, θ) measure its radial distance, r, along a ray from the origin (pole) that makes an angle of θ radians with some fixed reference ray (polar axis) — usually the x-axis. This is called the *polar coordinate system*.

$$r = \sqrt{x^2 + y^2} \qquad [1]$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad [2]$$

$$x = r\cos\theta \qquad [3]$$

$$y = r\sin\theta \qquad [4]$$

The diagram above, and the associated equations, show the connection between polar and Cartesian coordinates. Mathematics C students have already encountered these relationships in several topics including vectors and their applications and complex numbers.

We shall now look at equations involving the radial variable, r, and the vectorial angle variable, θ, and discuss relationships of the form $r = \text{P}(\theta)$. Conversions from Cartesian to polar and vice versa are processes that will make use of the above equations [1] to [4].

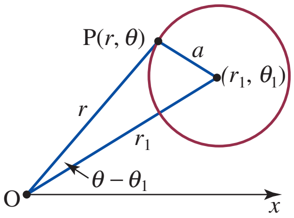
WORKED Example 13

a Express the relation (circle) $x^2 + y^2 + 2x - 2y = 0$ in polar form.

b Transform the polar relation $r = \dfrac{1}{1 - \cos\theta}$ into Cartesian form.

THINK	WRITE
a ① Write the Cartesian equation.	a $x^2 + y^2 + 2x - 2y = 0$
② Replace the x and y terms with r and θ terms and simplify. Use $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$.	$r^2 + 2r\cos\theta - 2r\sin\theta = 0$
③ Since $r \neq 0$, divide both sides of the equation by r.	$r + 2\cos\theta - 2\sin\theta = 0$
④ Simplify.	$r = 2(\sin\theta - \cos\theta)$
b ① Write the polar equation and rearrange.	b $r = \dfrac{1}{1 - \cos\theta}$ $r - r\cos\theta = 1$
② Replace r and θ terms with x and y terms. Substitute $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$.	$\sqrt{x^2 + y^2} - x = 1$ $\sqrt{x^2 + y^2} = x + 1$
③ Square both sides of the equation.	$x^2 + y^2 = (x + 1)^2$
④ Expand the right-hand side of the equation and simplify.	$x^2 + y^2 = x^2 + 2x + 1$ $y^2 = 2x + 1$

Polar equations of circles



Consider a circle on the polar plane with its centre at (r_1, θ_1) and radius a. Let P(r, θ) be any point on the circle.

Using the cosine rule, we have:

$$a^2 = r^2 + r_1{}^2 - 2rr_1 \cos(\theta - \theta_1).$$

Rearranging this equation gives the polar equation of a circle with its centre at (r_1, θ_1) and radius a:

$$r^2 - 2rr_1 \cos(\theta - \theta_1) + r_1{}^2 = a^2.$$

Consider a circle whose radius is a and whose centre is at the pole — that is, the origin (0, 0). We have the following:

$$r_1 = 0, \quad \theta_1 = 0$$

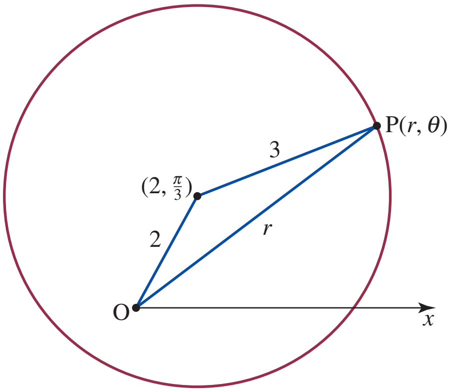
which gives $\quad r^2 = a^2$

or $\quad r = \pm a$.

Note: There are two different equations in polar form for this circle because the polar coordinates of a point are not unique.

WORKED Example 14

Find the polar equation of the circle whose radius is 3 and whose centre is at the point $\left(2, \frac{\pi}{3}\right)$.



THINK	WRITE
① Assume that P(r, θ) is any point on the circle. Write the general polar equation.	If P(r, θ) is any point on the circle, the polar equation is $r^2 - 2rr_1 \cos(\theta - \theta_1) + r_1{}^2 = a^2$ with centre at (r_1, θ_1) and radius a.
② Use the coordinates of the centre and the radius to identify r_1, θ_1 and a.	Now $r_1 = 2$, $\theta_1 = \frac{\pi}{3}$ and $a = 3$ so the polar equation is
③ Substitute r_1, θ_1 and a into the equation and simplify.	$r^2 - 4r\cos\left(\theta - \frac{\pi}{3}\right) + 4 = 9$ $r^2 - 4r\cos\left(\theta - \frac{\pi}{3}\right) = 5$

Let us now consider the process of working the opposite way; that is, if we have the polar equation of a circle, how can we find the centre and radius? Worked example 15 will demonstrate.

WORKED Example 15

Find the centre and radius of the circle whose polar equation is

$$r^2 + 4r\cos\theta - 4\sqrt{3}r\sin\theta = 20.$$

THINK	WRITE
① Write the polar equation.	$r^2 + 4r\cos\theta - 4\sqrt{3}r\sin\theta = 20$
② To express the polar equation in Cartesian form, replace r^2 with $x^2 + y^2$, $r\cos\theta$ with x, and $r\sin\theta$ with y.	$x^2 + y^2 + 4x - 4\sqrt{3}y = 20$
③ Complete the square with the x-terms and with the y-terms and simplify.	$(x+2)^2 - 4 + (y - 2\sqrt{3})^2 - 12 = 20$ $(x+2)^2 + (y - 2\sqrt{3})^2 = 36$
④ Identify the radius and the coordinates of the centre.	The radius is 6 and the centre is $(-2, 2\sqrt{3})$.
⑤ Express the coordinates of the centre in polar form, (r, θ), using $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ where $x = -2$ and $y = 2\sqrt{3}$.	To find the polar coordinates of the centre, consider $r = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ where $x = -2$ and $y = 2\sqrt{3}$. $r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$ $= \sqrt{16}$ $= 4$ $\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right)$ $= \tan^{-1}(-\sqrt{3})$ $= \frac{2\pi}{3}$
⑥ State the centre and radius of the circle.	So the circle has its centre at $\left(4, \frac{2\pi}{3}\right)$ and has a radius of 6 units.

remember

1. We can convert from Cartesian form to polar form and vice versa by using the following relationships:
   (a) $r = \sqrt{x^2 + y^2}$
   (b) $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
   (c) $x = r\cos\theta$
   (d) $y = r\sin\theta$
2. The polar equation of a circle with centre at (r_1, θ_1) and radius a is:

$$r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2 = a^2.$$

EXERCISE 10G Polar equations

WORKED Example 13a

1 Express the following Cartesian equations in polar form.

a $2x + 3y = 4$

b $3x - 4y = 0$

c $x = 4$

d $y = -3$

e $x^2 + y^2 = 16$

f $x^2 + y^2 - 2x - 2y = 0$

g $x^2 + y^2 - ax = 0$

h $y^2 = 2x$

i $4y = x^2$

j $\frac{x^2}{9} + \frac{y^2}{16} = 1$

k $\frac{x^2}{16} + \frac{y^2}{25} = 1$

l $\frac{x^2}{9} - \frac{y^2}{4} = 1$

m $xy = 2$

n $x^2 + y^2 - 2y = 0$

WORKED Example 13b

2 Express the following polar equations in Cartesian form.

a $r\cos\theta = 3$

b $\theta = \frac{\pi}{4}$

c $r = 3\cos\theta$

d $r = 3\operatorname{cosec}\theta$

e $r = 4$

f $r = \tan\theta\sec\theta$

g $r = \frac{2}{1 - \cos\theta}$

h $r = 4\sin\theta$

WORKED Example 14

3 Find the polar equations of the following circles, with the given conditions.

a Centre at the pole, radius 3

b Centre at the pole, radius 5

c Centre at (1, 0°), radius 1

d Centre at $\left(2, \frac{\pi}{2}\right)$, radius 2

e Centre at $\left(4, \frac{3\pi}{2}\right)$, radius 4

f Centre at $\left(3, \frac{\pi}{3}\right)$, radius 3

g Centre at $\left(5, \frac{2\pi}{3}\right)$, passing through the pole

h Centre at $\left(2, \frac{\pi}{2}\right)$, radius 3

i Centre at $\left(3, \frac{2\pi}{3}\right)$, radius 4

WORKED Example 15

4 Find the centre and radius of the following circles in polar form.

a $r^2 - 4\sqrt{3}r\cos\theta - 4r\sin\theta - 15 = 0$

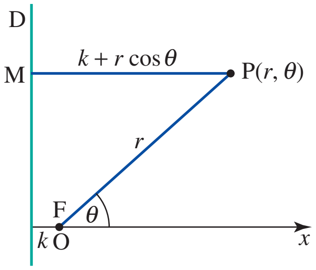
b $r = 5\cos\theta - 5\sqrt{3}\sin\theta$

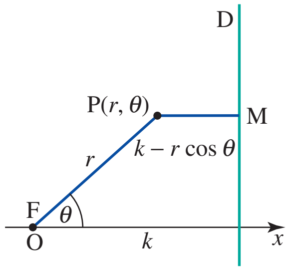
Conics in polar form

In chapter 9 we defined a conic as the locus of a point, P, such that the ratio of its distances from a fixed point F (the focus) to the fixed line D (the directrix) is a constant, e, which we called the *eccentricity*.

If $0 < e < 1$, the locus is an ellipse.
If $e = 1$, the locus is a parabola.
If $e > 1$, the locus is a hyperbola.

If we choose F (the focus) to be the pole and D (the directrix) to be perpendicular to the polar axis and k is the distance between the pole and directrix, we can derive the general polar equation of a conic.





From the above diagrams we see that from the definition, we get PF = ePM. This becomes:

$$r = e(k \pm r\cos\theta)$$

which takes into consideration both of the above situations.
Expanding the right-hand side, we get:

$$r = ek \pm er\cos\theta$$

Rearranging to isolate r gives

$$r \pm er\cos\theta = ek$$

$$r(1 \pm e\cos\theta) = ek$$

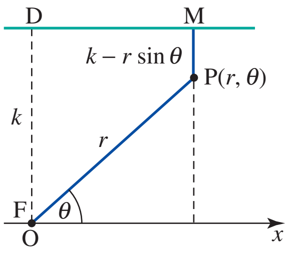
$$r = \frac{ek}{1 \pm e\sin\theta}$$

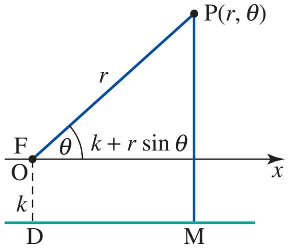
This gives us the general polar form of any conic with the directrix perpendicular to the polar axis. We use + in the equation if the directrix is to the right of the pole and – in the equation if the directrix is to the left of the pole.

Now if the directrix is parallel to the polar axis, the polar equation of a conic will be of the form:

$$r = \frac{ek}{1 \pm e\sin\theta}.$$

To establish this, consider the following diagrams.





From the diagrams we see that PF = ePM. This result becomes:

$$r = e(k \pm r \sin \theta)$$

which takes into consideration both of the above situations.

As before, after expanding and rearranging to isolate r we obtain:

$$r = \frac{ek}{1 \pm e \sin \theta}.$$

This gives us the general form of any conic with the directrix parallel to the polar axis. We use + in the equation if the directrix is above the pole and – in the equation if the directrix is below the pole.

WORKED Example 16

Determine the nature of the following conics in polar form.

a $r = \dfrac{3}{1 + \cos \theta}$ **b** $r = \dfrac{12}{4 - 3 \cos \theta}$ **c** $r = \dfrac{6}{1 + 2 \sin \theta}$

THINK	WRITE
a 1 Write the polar equation.	**a** $r = \dfrac{3}{1 + \cos \theta}$
2 Compare this to the appropriate form of the general polar equation of a conic and state the values of e and k.	This is of the form $r = \dfrac{ek}{1 + e \cos \theta}$ where $e = 1$ and $k = 3$.
3 Use the values of e and k to determine the nature of the conic.	The conic is a parabola where the directrix is perpendicular to the polar axis and 3 units to the right of the pole.
b 1 Write the polar equation.	**b** $r = \dfrac{12}{4 - 3 \cos \theta}$
2 Divide the numerator and denominator by 4 so that a comparison to the appropriate form of the general polar equation of a conic can be made. State the values of e and k.	$r = \dfrac{3}{1 - \frac{3}{4} \cos \theta}$ This is of the form $r = \dfrac{ek}{1 - e \cos \theta}$ where $e = \frac{3}{4}$ and $k = 4$.
3 Use the values of e and k to determine the nature of the conic.	The conic is an ellipse whose directrix is perpendicular to the polar axis and 4 units to the left of the pole.
c 1 Write the polar equation.	**c** $r = \dfrac{6}{1 + 2 \sin \theta}$
2 Compare this to the appropriate form of the general polar equation of a conic and state the values of e and k.	This is of the form $r = \dfrac{ek}{1 + e \sin \theta}$ where $e = 2$ and $k = 3$.
3 Use the values of e and k to determine the nature of the conic.	The conic is a hyperbola whose directrix is parallel to the polar axis and 3 units above the pole.

WORKED Example 17

State the polar equation of the conic with $e = 1.5$ and directrix 5 units to the right of the pole.

THINK / **WRITE**

1. List the known information. Since $e = 1.5$, the conic is a hyperbola.

$e = 1.5$ and $k = 5$ with the directrix to the right of the pole, so the polar equation is

2. From the position of the directrix in relation to the pole, write the appropriate form of the general polar equation of the conic.

$$r = \frac{ek}{1 + e\cos\theta}$$

3. Substitute for e and k to determine the polar equation of the hyperbola.

$$r = \frac{1.5 \cdot 5}{1 + 1.5\cos\theta}$$

4. Multiply the numerator and denominator by 2 to simplify the equation.

$$r = \frac{15}{2 + 3\cos\theta}$$

WORKED Example 18

Convert the following conics in polar form into Cartesian form.

a $r = \dfrac{3}{2 + 2\cos\theta}$ **b** $r = \dfrac{6}{3 - 2\cos\theta}$ **c** $r = \dfrac{6}{1 - 2\cos\theta}$

THINK / **WRITE**

a 1. Write the polar equation and rearrange.

$$r = \frac{3}{2 + 2\cos\theta}$$

$$2r + 2r\cos\theta = 3$$

2. Use $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$ to replace the terms involving r and θ.

$$2\sqrt{x^2 + y^2} + 2x = 3$$

$$2\sqrt{x^2 + y^2} = 3 - 2x$$

3. Square both sides of the equation and simplify. This conic is a parabola.

$$4(x^2 + y^2) = 9 - 12x + 4x^2$$

$$4y^2 = -12x + 9$$

$$y^2 = -3\left(x - \frac{3}{4}\right)$$

b 1. Write the polar equation and rearrange.

$$r = \frac{6}{3 - 2\cos\theta}$$

$$3r - 2r\cos\theta = 6$$

2. Use $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$ to replace the terms involving r and θ.

$$3\sqrt{x^2 + y^2} - 2x = 6$$

$$3\sqrt{x^2 + y^2} = 2x + 6$$

3. Square both sides of the equation and simplify.

$$9x^2 + 9y^2 = 4x^2 + 24x + 36$$

$$5x^2 - 24x + 9y^2 = 36$$

4. Complete the square with the x-terms.

$$5\left[x^2 - \frac{24}{5}x + \left(\frac{12}{5}\right)^2\right] - \frac{144}{5} + 9y^2 = 36$$

Continued over page →

THINK **WRITE**

$$25\left(x-\frac{12}{5}\right)^2-144+45y^2 = 180$$

⑤ Multiply both sides by 5 and simplify. This conic is an ellipse.

$$25\left(x-\frac{12}{5}\right)^2+45y^2 = 224$$

c ① Write the polar equation and rearrange.

c $$r = \frac{6}{1-2\cos\theta}$$

$$r-2r\cos\theta = 6$$

② Use $r = \sqrt{x^2+y^2}$ and $x = r\cos\theta$ to replace the terms involving r and θ.

$$\sqrt{x^2+y^2}-2x = 6$$

$$\sqrt{x^2+y^2} = 2x+6$$

③ Square both sides of the equation and simplify.

$$x^2+y^2 = 4x^2+24x+36$$

$$3x^2+24x+36-y^2 = 0$$

$$3(x^2+8x+4^2)-48+36-y^2 = 0$$

$$3(x+4)^2-y^2 = 12$$

$$\frac{(x+4)^2}{4}-\frac{y^2}{12} = 1$$

④ Complete the square with the x-terms. This conic is a hyperbola.

Applying technology to polar equations

Graphs of conic sections in polar form can be generated efficiently using appropriate technology. The following examples illustrate the procedure using Graphmatica and a graphics calculator.

WORKED Example 19

Graph and identify the curve that has $r = \dfrac{12}{6+\cos\theta}$ as its polar equation.

THINK **WRITE/DISPLAY**

For the CASIO *fx*-9860G AU

① Ensure that the calculator is in Radian Mode.
Press:

- **MENU**
- **5** (GRAPH)
- **F3** (TYPE)
- **F2** ($r =$)

Complete the entry line as:

$$r1 = \frac{12}{6+\cos\theta}$$

Then press **EXE**.

Graph Func :r=
r1=12÷(6+cos θ)
r2: [—]
r3: [—]
r4: [—]
r5: [—]
r6: [—]

THINK | **WRITE/DISPLAY**

2 To enter appropriate Window Settings, press:
- SHIFT
- F3 (V-Window)

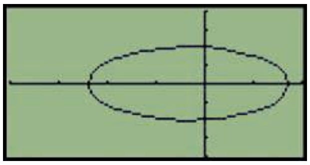
Enter the settings as shown.

```
View Window
Xmin  :-4
 max  :2
 scale:1
 dot  :0.04761904
Ymin  :-4
 max  :4
```

3 Scroll down to enter values for θ. Enter the values as shown.

```
View Window
Ymin  :-4
 max  :4
 scale:1
Tθmin :0
  max :6.2831853
  ptch:0.19634954
```

4 Press:
- EXIT
- F6 (DRAW)



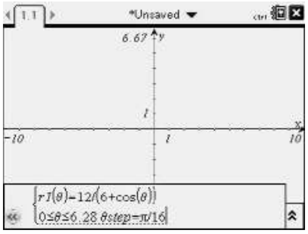
5 Write the answer.

The curve is an ellipse.

For the TI-Nspire CAS

1 On a Graphs page, press:
- MENU menu
- 3: Graph Type 3
- 3: Polar 3

Enter the polar equation and the appropriate values for θ as shown.

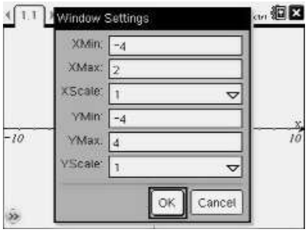


$r1(\theta)=12/(6+\cos(\theta))$

$0\le\theta\le6.28\ \theta step=\pi/16$

2 To change the Window Settings, press:
- MENU menu
- 4: Window/Zoom 4
- 1: Window settings 1

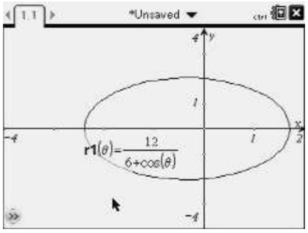
Enter the window settings as shown, then select OK.



Window Settings

XMin:	-4
XMax:	2
XScale:	1
YMin:	-4
YMax:	4
YScale:	1

3 The following graph is the result.



$r1(\theta)=\frac{12}{6+\cos(\theta)}$

4 Write the answer.

The curve is an ellipse.

WORKED Example 20

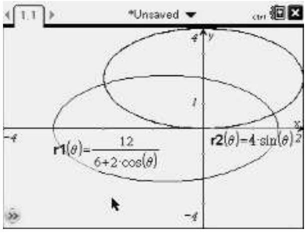
Using a graphics calculator, determine the Cartesian points of intersection of $r = \dfrac{12}{6 + 2\cos\theta}$ and $r = 4\sin\theta$.

The Casio graphics calculator can only find the intersection of graphs that are in the form $y =$.

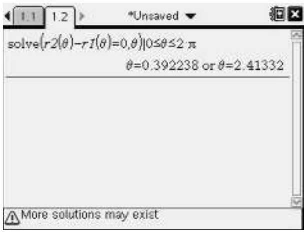
THINK | **WRITE/DISPLAY**

For the TI-Nspire CAS

1. On a Graphs page, complete the entry lines as:
   $r1(\theta) = \dfrac{12}{6 + 2\cos\theta}$
   $0 \le \theta \le 6.28 step = 0.13$
   $r2(\theta) = 4\sin\theta$
   $0 \le \theta \le 6.28 step = 0.13$
   Press ENTER enter after each line.



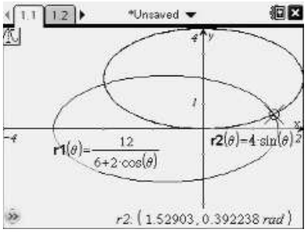
2. To find the value of θ when the curves intersect, on a Calculator page complete the entry line as:
   $\text{solve}(r2(\theta) - r1(\theta) = 0, \theta) \mid 0 \le \theta \le 2\pi$
   By limiting the values of θ, only the 2 desired solutions are found.



```
solve(r2(θ)-r1(θ)=0,θ)|0≤θ≤2·π
                    θ=0.392238 or θ=2.41332
More solutions may exist
```

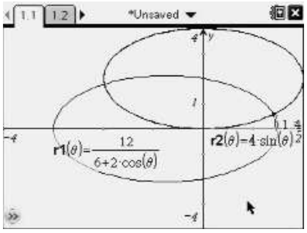
3. To find the point of intersection, return to the Graphs page (the intersection option doesn't work for polar graphs) press:
   - MENU menu
   - 5: Trace 5
   - 1: Graph trace. 1

   Immediately enter 0.392238 and press ENTER enter, the cursor will move to the point of intersection.

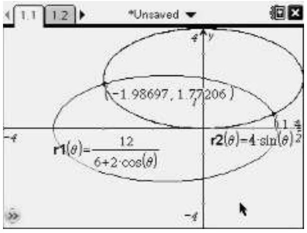


```
r2: (1.52903, 0.392238 rad)
```

4. To view the solution as rectangular coordinates, press the Click button twice.



5. Repeating the process will display the other coordinates.



```
(-1.98697, 1.77206)
```

6. Write the answers. | The solutions are (–1.99, 1.77) and (1.41, 0.58)

remember

1. The general form of the polar equation of a conic is:

   (a) $r = \dfrac{ek}{1 + e\cos\theta}$ when the directrix is perpendicular to the polar axis and k units to the right of the pole

   (b) $r = \dfrac{ek}{1 - e\cos\theta}$ when the directrix is perpendicular to the polar axis and k units to the left of the pole

   (c) $r = \dfrac{ek}{1 + e\sin\theta}$ when the directrix is parallel to the polar axis and k units above the pole

   (d) $r = \dfrac{ek}{1 - e\sin\theta}$ when the directrix is parallel to the polar axis and k units below the pole.

2. (a) If $0 < e < 1$, the locus is an ellipse.
   (b) If $e = 1$, the locus is a parabola.
   (c) If $e > 1$, the locus is a hyperbola.

EXERCISE 10H Conics in polar form

WORKED Example 16

1 Determine the nature of the following conics in polar form.

a $r = \dfrac{4}{1 + \cos\theta}$ **b** $r = \dfrac{5}{2 + \cos\theta}$ **c** $r = \dfrac{4}{2 + 3\cos\theta}$

d $r = \dfrac{3}{3 - 2\cos\theta}$ **e** $r = \dfrac{10}{3 - 3\cos\theta}$ **f** $r = \dfrac{6}{2 - \sin\theta}$

WORKED Example 17

2 State the polar equations of the following conics:

a $e = 1$, directrix 5 units to the right of the pole

b $e = \frac{1}{2}$, directrix a units to the right of the pole

c $e = 2$, directrix p units to the left of the pole.

WORKED Example 18

3 Express the following conics in Cartesian form.

a $r = \dfrac{3}{1 - \cos\theta}$ **b** $r = \dfrac{1}{1 + \cos\theta}$ **c** $r = \dfrac{2a}{1 + \cos\theta}$

d $r = \dfrac{5}{2 - 3\cos\theta}$ **e** $r = \dfrac{8}{3 - 5\cos\theta}$ **f** $r = \dfrac{9}{4 + 5\cos\theta}$

g $r = \dfrac{9}{5 - 4\cos\theta}$ **h** $r = \dfrac{1}{1 - 2\sin\theta}$ **i** $r = \dfrac{5}{2 + 3\sin\theta}$

WORKED Example 19

4 Using appropriate technology, graph and identify each of the following conics.

a $r = 2\sin\theta$ **b** $r = \dfrac{9}{5 + 4\cos\theta}$ **c** $r = \dfrac{5}{2 - 3\sin\theta}$

d $r = \dfrac{10}{3 + 3\cos\theta}$ **e** $r = \dfrac{9}{4 - 6\cos\theta}$ **f** $r = \dfrac{5}{2 + 2\sin\theta}$

g $r = \dfrac{8}{3 - 2\sin\theta}$ **h** $r = \dfrac{9}{5 + 2\sin\theta}$ **i** $r = \dfrac{5}{2 + 3\cos\theta}$

j $r = \dfrac{4}{6 + \sin\theta}$ **k** $r = \dfrac{2}{1 + \sin\theta}$ **l** $r = \dfrac{10}{2 + 3\sin\theta}$

WORKED Example 20

5 Using appropriate technology, determine the Cartesian points of intersection for each of the following curves.

a $r = 4$ and $r = \dfrac{4}{1 - \cos\theta}$

b $r = \dfrac{9}{5 + 4\cos\theta}$ and $r = 2\sin\theta$

c $r = \dfrac{4}{3 + 3\cos\theta}$ and $r = \dfrac{2}{1 - \sin\theta}$

d $r = \dfrac{2}{2 - 3\cos\theta}$ and $r = \dfrac{1}{2 + \sin\theta}$

e $r = \dfrac{3}{1 - 2\sin\theta}$ and $r = 4$

f $r = \dfrac{3}{1 - 2\sin\theta}$ and $r = \dfrac{3}{1 - \cos\theta}$

g $r = \dfrac{3}{2 + 3\cos\theta}$ and $r = 2(1 - \cos\theta)$

h $r = 2\sin\theta$ and $r = 2\cos(2\theta)$

6 The arc length, L, of a conic, r, may be calculated over the domain $\theta_1 \le \theta \le \theta_2$ by the integral $L = \displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$.

a If $r = 2\sin\theta + \cos\theta$, show that $L = \pi\sqrt{5}$, where $0 \le \theta \le \pi$.

b The cardiod family has polar equation $r = a(\cos\theta + 1)$ for some positive constant a.

  i Graph r for $0 \le \theta \le 2\pi$.

  ii Show that $L = 2a\sqrt{2}\displaystyle\int_0^{\pi} \sqrt{1 + \cos\theta}\ \mathrm{d}\theta$.

  iii Verify that $L = 8a$.

Some special curves and interesting problems

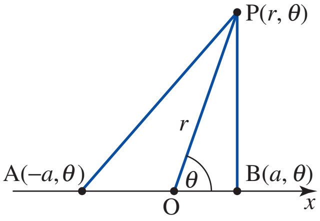
There are many other ‘special curves’ that mathematicians have studied over the centuries. The study of the general Cartesian, parametric and polar forms of these curves and how they may be constructed has added to the wealth of mathematical knowledge. *Encyclopaedia Britannica* has a good article on them under ‘curves, special’.

Some curves are very effectively expressed in polar form:

1. The Spiral of Archimedes $r = a\theta$
2. Fermat’s spiral $r^2 = a^2\theta$
3. The Hyperbolic spiral $r = \dfrac{a\pi}{\theta}$
4. The Ovals of Cassini $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$.

WORKED Example 21

Find the equation of the locus of a point that moves so that the product of its distances from the polar coordinates $(-a, 0)$ and $(a, 0)$ is a^2.



THINK	WRITE
① Use the information given in the question to write the relationship for the product of the distances from A to P and B to P. The product equals a^2.	$(AP)(BP) = a^2$ [1]
② Use the cosine rule with triangle AOP to write an equation involving AP.	$(AP)^2 = a^2 + r^2 - 2ar\cos(180° - \theta)$ $= a^2 + r^2 + 2ar\cos\theta$ [2]
③ Simplify by using the fact that $\cos(180° - \theta) = -\cos\theta$.	$(BP)^2 = a^2 + r^2 - 2ar\cos\theta$ [3]
④ Use the cosine rule with triangle BOP to write an equation involving BP.	Squaring [1] gives $(AP)^2(BP)^2 = a^4$
⑤ Square both sides of equation [1] and substitute for $(AP)^2$ and $(BP)^2$ using equations [2] and [3].	Substituting [2] and [3] gives $(a^2 + r^2 + 2ar\cos\theta)(a^2 + r^2 - 2ar\cos\theta) = a^4$
⑥ Simplify using the difference-of-2-squares identity.	$(a^2 + r^2)^2 - (2ar\cos\theta)^2 = a^4$
⑦ Expand and simplify.	$a^4 + 2a^2r^2 + r^4 - 4a^2r^2\cos^2\theta = a^4$ $2a^2r^2 + r^4 - 4a^2r^2\cos^2\theta = 0$
⑧ Divide both sides by r^2, assuming that $r \neq 0$.	$2a^2 + r^2 - 4a^2\cos^2\theta = 0$ $r^2 = 2a^2(2\cos^2\theta - 1)$
⑨ Use the identity $2\cos^2\theta - 1 = \cos 2\theta$ to simplify. This is the equation of a lemniscate.	$r^2 = 2a^2\cos 2\theta$

WORKED Example 22

Find the coordinates of the points of intersection of the following curves in polar form.

$r = 1 - \cos\theta,\ r = \sin\left(\frac{\theta}{2}\right)$

THINK	WRITE
① Write the two equations.	$r = 1 - \cos\theta$ [1] $r = \sin\left(\frac{\theta}{2}\right)$ [2]
② For the points of intersection, equate the two equations.	Equating [1] and [2] gives $1 - \cos\theta = \sin\left(\frac{\theta}{2}\right)$
③ Use the identity $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$ to produce an equation involving one trigonometric function only. (This is a variation of the identity $\cos 2\theta = 1 - 2\sin^2\theta$.)	$2\sin^2\left(\frac{\theta}{2}\right) = \sin\left(\frac{\theta}{2}\right)$
④ Rearrange and take out $\sin\left(\frac{\theta}{2}\right)$ as a common factor.	$2\sin^2\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) = 0$ $\sin\left(\frac{\theta}{2}\right)\left(2\sin\left(\frac{\theta}{2}\right) - 1\right) = 0$
⑤ Solve the equation for θ.	$\sin\left(\frac{\theta}{2}\right) = 0$ or $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}$ $\theta = 0$ or $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$
⑥ Substitute each value of θ into one of the original equations, say equation [2], to find the corresponding value of r.	When $\theta = 0$, $r = \sin 0$ $= 0$ When $\theta = \frac{\pi}{3}$, $r = \sin\frac{\pi}{6}$ $= \frac{1}{2}$ When $\theta = \frac{5\pi}{3}$, $r = \sin\frac{5\pi}{6}$ $= \frac{1}{2}$
⑦ Write the coordinates of the points of intersection.	The coordinates of the points of intersection are $(0, 0), \left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right)$

Areas bounded by polar functions

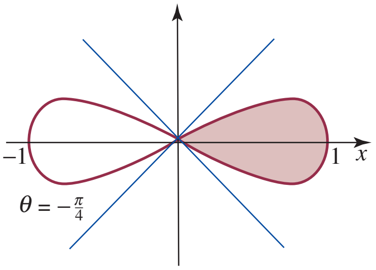
To calculate the area of a region determined by the graph of a polar equation $r = f(\theta)$ with the limits given as $[a, b]$ (between the rays $\theta = a$ and $\theta = b$), we can use the formula given by the definite integral ${}_aA_b = \frac{1}{2}\int_a^b [f(\theta)]^2 \, d\theta$.

The strategy we can use is as follows:

1. Sketch the curve using a graphing calculator (or any method available).
2. Look for symmetries to simplify the calculations.
3. Determine the limits of integration for the region whose area is to be calculated.
4. Multiply the integral by the constant that was determined by the use of symmetries in step 2.
5. Apply ${}_aA_b = \frac{1}{2}\int_a^b [f(\theta)]^2 \, d\theta$ and appropriate trigonometric identities.

WORKED Example 23

Find the area enclosed by the lemniscate $r^2 = \cos 2\theta$.



THINK

① Write the formula for the area of the right-hand lobe between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$. This is the area shaded in the diagram.

② The area required is twice the area shaded in the diagram, so write the formula needed for the total area.

③ Determine the definite integral and simplify.

WRITE

The area of the right-hand lobe

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$\text{The total area} = 2 \cdot \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$= \left[\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$= 1$$

remember

The area of a region determined by the graph of a polar equation $r = f(\theta)$ with the limits given as $[a, b]$ (between the rays $\theta = a$ and $\theta = b$) has the formula given by ${}_aA_b = \frac{1}{2}\int_a^b [f(\theta)]^2 \, d\theta$.

EXERCISE 10I Some special curves and interesting problems

WORKED Example 21

1 Find the equation of the locus of a point P(r, θ) such that its radius is proportional to its angle.

WORKED Example 22

2 Algebraically, find the coordinates (in Cartesian form) of the points of intersection of the following pairs of curves.

a $r = 4 + 4\cos\theta,\ r = \dfrac{3}{1 - \cos\theta}$

b $r = \sqrt{2}\cos\theta,\ r = \sin 2\theta$

WORKED Example 23

3 For each of the following, find the area of the region determined by the given polar equations and the limits specified.

a $r = 1 + \sin\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

b $r = 2\sin\theta,\ 0 \le \theta \le \pi$

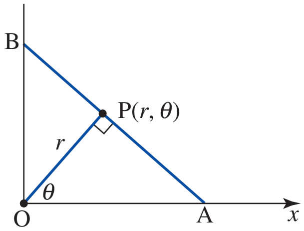
c $r = 2 + \sin\theta,\ 0 \le \theta \le \pi$

d $r = 4 + \sin\theta,\ \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

4 Find the area of the region enclosed by the graph of the given polar equations.

a $r = 2\cos\theta$ b $r = a\cos 2\theta$ c $r = a\cos 3\theta$

5 A line of length $2a$ has its extremities on two fixed perpendicular lines. Find the locus of the foot of the perpendicular from the intersection of the two fixed lines to the given line. (See diagram below.)



6 For each of the following general family of curves expressed in polar form, vary the constants a, b and n when they occur, and discuss the effect that they have on the shape of the family of curves. Use appropriate technology to assist in your analysis and to generate a hard copy of your work.

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Digital docs:
WorkSHEET 10.1
WorkSHEET 10.2

a The Rose family defined by $r = a\cos(n\theta)$.
b The Strophoid family defined by $r = a\cos(2\theta)\sec\theta$.
c The Nephroid family defined by $r = b + a\cos(\theta)$.
d The Lituus family defined by $r^2\theta = a^2$.
e The Cissoid of Diocles family defined by $r = a\sin(\theta)\tan(\theta)$.
f The Spiral family defined by $\log r = a\theta$.

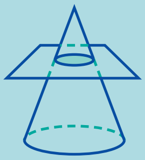
History of mathematics

THE HISTORY OF SOME MAJOR CURVES

The history of mathematics is full of themes that develop over a long time and the major curves are a good example of this. Many of these curves exist in nature and early mathematicians tried to find ways of defining and classifying them.

Omar Khayyam and Apollonius worked on the following conic sections, so called because you can generate them by cutting through a circular cone, using various approaches that led to the formulas below.

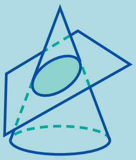
Circle



The circle occurs frequently in nature and is the shape of that most famous of all inventions — the wheel. One of the major areas of mathematical research has been finding a value for π, the value of the circumference of the circle divided by the diameter. Archimedes studied the properties of the circle and he knew that the value of π was not rational.

Euclid published a series of theorems relating to circles and the problem of relating them to polygons. One of the most famous ancient geometrical problems involving the circle was to develop a method of making a square of rational side-length the same area as a given circle. This was called 'squaring the circle', and has been proven to be impossible.

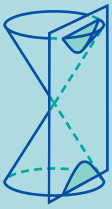
Ellipse



In his studies of geometry Euclid looked at the ellipse but it was Apollonius who gave it the name 'ellipse', which comes from the Greek word 'elleipsis' which means something with a defect.

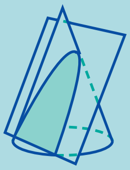
One of the first main uses of ellipses was in astronomy, with Kepler determining the path of the planets in the early 1600s. Halley determined that the comet, now known as 'Halley's comet', followed the path of an ellipse around the sun.

Hyperbola



This function was named by Apollonius and researched by Euclid. The word *hyperbola* comes from the Greek word *hyperbole* which means 'excess' or 'extension'. This was first described in words by Pappus and a formula was first proposed by Fermat.

Parabola



The first major study of the parabola was by Menaechmus who worked on it in his search to find a solution to the problem of doubling a cube; that is, creating a cube, the area of whose face was double that of another one. We can now do this using surds, but at the time, Menaechmus was unable to solve this problem.

Galileo worked on projectiles and the paths that they followed, and determined that this path was a parabola.

Questions

1. What are the main types of curves discussed here?
2. Who developed the name for the ellipse?
3. Where does the word *hyperbola* come from?
4. What is the connection between these curves and a circular cone?
5. What type of curve is the path of a projectile?

Research

Use the Internet or library to find the equations of several interesting mathematical curves and use a graphing program to explore the changes that occur to the curves when you alter the equation.

summary

Parametric equations

- To convert parametric equations into Cartesian equations, solve the set of equations to eliminate the parameter.
- If the parametric equations contain trigonometric (circular) functions, use trigonometric identities to assist in eliminating the parameter.

Tangents and normals

- Tangents and normals are perpendicular to each other. If the gradient of a line is m, then the gradient of a line perpendicular to this is $-\frac{1}{m}$.
- To find the equations of tangents and normals for implicit functions use:
  1. implicit differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$
  2. the gradient-point form of a straight line to write the equation; that is, $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a given point and m is the gradient of the line. Use the relationship for $\frac{\mathrm{d}y}{\mathrm{d}x}$ to find m.
- At the point $\mathrm{P}(x_1, y_1)$ on the parabola $y^2 = \pm 4ax$
  1. the equation of the tangent is $yy_1 = \pm 2a(x + x_1)$
  2. the equation of the normal is $xy_1 \pm 2ay = \pm x_1y_1 + 2ay$.
- At the point $\mathrm{P}(x_1, y_1)$ on the parabola $x^2 = \pm 4ay$
  1. the equation of the tangent is $xx_1 = \pm 2a(y + y_1)$
  2. the equation of the normal is $x_1y \pm 2ax = x_1y_1 \pm 2ax_1$.
- At the point $\mathrm{P}(x_1, y_1)$ on an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
  1. the equation of the tangent is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
  2. the equation of the normal is $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$.
- At the point $\mathrm{P}(x_1, y_1)$ on a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
  1. the equation of the tangent is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
  2. the equation of the normal is $a^2y_1x - b^2x_1y = (a^2 + b^2)x_1y_1$.
- At the point $\mathrm{P}(x_1, y_1)$ on a rectangular hyperbola of the form $(x - h)(y - k) = c^2$
  1. the equation of the tangent is $x(y_1 - k) + y(x_1 - h) = 2x_1y_1 - kx_1 - hy_1$
  2. the equation of the normal is $x(x_1 - h) - y(y_1 - k) = x_1{}^2 - y_1{}^2 - hx_1 + ky_1$.

Parametric equations

- The parametric equations that represent parabolas of the form $(y - k)^2 = \pm 4a(x - h)$ are $y = \pm 2at + k$ and $x = \pm at^2 + h$.

- The parametric equations that represent parabolas of the form $(x-h)^2 = \pm 4a(y-k)$ are $x = \pm 2at + h$, and $y = \pm at^2 + k$ where a (in this case) is the focal length.
- The parametric equations that represent an ellipse of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ are $x = a\cos\theta + h$, $y = b\sin\theta + k$, where a and b are the lengths of the semi-major and semi-minor axes respectively and the centre is located at (h, k).
- The parametric equations that represent a circle of the form $(x-h)^2 + (y-k)^2 = a^2$ are $x = a\cos\theta + h$, $y = a\sin\theta + k$ where a (in this case) is the radius of the circle and the centre is located at (h, k).
- The parametric equations that represent a hyperbola of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ are $x = a\sec\theta + h$ and $y = b\tan\theta + k$, where a and b are the lengths of the semi-transverse and semi-conjugate axes respectively and the centre is located at (h, k).
- The parametric equations for a hyperbola of the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ are $y = a\,\text{cosec}\,\theta + k$ and $x = b\cot\theta + h$.
- The parametric equations for a rectangular hyperbola of the form $(x-h)(y-k) = c^2$ are $x = ct + h$ and $y = \frac{c}{t} + k$ or $y = ct + k$ and $x = \frac{c}{t} + h$ where the centre is located at (h, k).

Polar equations

- Many plane curves can be more easily described in a coordinate system in which the coordinates of a point P(r, θ) measure its radial distance, r, along a ray from the origin (pole) that makes an angle of θ with some fixed reference ray (polar axis) — usually the x-axis.
- To convert between polar form and Cartesian form, the following relationships are useful:

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

- A circle with its centre at (r_1, θ_1) and radius a has the polar equation:

$$r^2 - 2rr_1\cos(\theta - \theta_1) + r_1^2 = a^2$$

- The general form of the polar equation of a conic, where e is the eccentricity, is:
  1. $r = \frac{ek}{1 + e\cos\theta}$ when the directrix is perpendicular to the polar axis and k units to the right of the pole
  2. $r = \frac{ek}{1 - e\cos\theta}$ when the directrix is perpendicular to the polar axis and k units to the left of the pole
  3. $r = \frac{ek}{1 + e\sin\theta}$ when the directrix is parallel to the polar axis and k units above the pole

4. $r = \dfrac{ek}{1 - e\sin\theta}$ when the directrix is parallel to the polar axis and k units below the pole.

Special curves in polar form

- The Spiral of Archimedes $r = a\theta$
- Fermat's spiral $r^2 = a^2\theta$
- The Hyperbolic spiral $r = \dfrac{a\pi}{\theta}$
- The Ovals of Cassini $r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4$

Areas bounded by polar functions

- To calculate the area of a region determined by the graph of a polar equation $r = f(\theta)$ with the limits given as $[a, b]$ (between the rays $\theta = a$ and $\theta = b$) we can use the formula given by the definite integral ${}_aA_b = \dfrac{1}{2}\displaystyle\int_a^b [f(\theta)]^2\, d\theta$.

CHAPTER review

Select the most appropriate answer (A, B, C, D or E) in each question.

1 **multiple choice** 10A

Find the Cartesian equation of the curve with parametric equations $x = -\frac{5}{t}$, $y = -5t$.

A $x^2 - y^2 = 25$
B $xy = 25$
C $y = -\frac{1}{x}$
D $xy = -25$
E $x^2 + y^2 = 25$

2 **multiple choice** 10A

The variable point P($3\cos\theta$, $2\sin\theta$) satisfies the equation:

A $\frac{x^2}{4} + \frac{y^2}{9} = 1$
B $\frac{y^2}{9} - \frac{x^2}{4} = 1$
C $\frac{x^2}{9} + \frac{y^2}{4} = 1$
D $\frac{x^2}{9} - \frac{y^2}{4} = 1$
E **A** and **C** only

3 **multiple choice** 10B

The equation of the tangent to the curve $y^2 = -8x$ at the point Q($-2q^2$, $-4q$) is given by:

A $y = \frac{x}{q} + 2q$
B $y = \frac{x}{q} - 2q$
C $y = -\frac{x}{q} - 2q$
D $y = -\frac{x}{q} + 2q$
E $y = \frac{2x}{q} - q$

4 **multiple choice** 10B

The tangent to the curve $x^2 = 12y$, at the variable point P($6t$, $3t^2$) meets the y-axis at T. The locus of the midpoint of PT is:

A x-axis
B y-axis
C $y = x(2x + 1)$
D $y = x(2x - 1)$
E $y = \frac{x}{6}(2x + 1)$

5 Show that the line $y = mx + c$ is tangent to the curve $Ax^2 + y^2 = 1$ if and only if the constants A, m and c satisfy the equation $A(c^2 - 1) = m^2$. 10B

6 Show that any tangent to the hyperbola $xy = c^2$ determines, with its asymptotes, a triangle of area $2c^2$. 10B

7 **multiple choice** 10C

The variable point P($2at$, at^2) lies on which of the following conics?

A $y^2 = 4ax$
B $(x - a)^2 = 4ay$
C $(x + a)^2 = 4ay$
D $x^2 = 4ay$
E None of these

8 Suppose that the line tangent to the parabola $y^2 = 4ax$ at a point Q crosses the parabola's axis at A and that the line through Q parallel to the parabola's axis crosses the directrix at D. Show that Q, A, D, and the parabola's focus, F, are four vertices of a rhombus. 10C

10C

9 A particle moves in the plane so that at time t it is at the point with coordinates $x(t) = t + 4$, $y(t) = 8 - t^2$. A second particle moves in the plane so that it is at a point with coordinates $x(t) = t + 4$, $y(t) = t + 6$ at time t.

- **a** Find the equation in the xy plane for each of the curves.
- **b** Do the paths cross? If so, where?
- **c** Do the particles collide? If so, where and at what time?

10D

10 **a** The points P$(5 \cos \theta, 5 \sin \theta)$ and Q$(5 \sin \theta, -5 \cos \theta)$ lie on the circle $x^2 + y^2 = 25$. Find the Cartesian equation of the locus of the point of intersection of the tangents at P and Q.

b **i** Show that the points P$(3 \cos \theta, 4 \sin \theta)$ and Q$(3 \sin \theta, -4 \cos \theta)$ lie on the same ellipse, and find the equation of the ellipse.

ii Find the Cartesian equation of the locus of the point of intersection of the tangents at P and Q.

10E

11 **multiple choice**

The hyperbola whose eccentricity is 2 and whose vertex is at (1, 0) has the polar equation:

A $r = \dfrac{3}{1 - 2 \sin \theta}$
B $r = \dfrac{3}{1 - 2 \cos \theta}$
C $r = \dfrac{3}{1 + 2 \cos \theta}$
D $r = \dfrac{3}{2 + 4 \cos \theta}$
E $r = \dfrac{3}{1 + 2 \sin \theta}$

10G

12 **multiple choice**

The polar equation $r = 3 \cos \theta$ in Cartesian form is:

A $(2x + 3)^2 + 4y^2 = 9$
B $(3x - 2)^2 + 4y^2 = 9$
C $(2x - 3)^2 - 4y^2 = 9$
D $(2x + 3)^2 - 4y^2 = 9$
E $(2x - 3)^2 + 4y^2 = 9$

10G

13 **multiple choice**

For the conic with polar equation $r = \dfrac{4}{2 + \sin \theta}$, the vertices are at:

A $V_1\left(\dfrac{4}{3}, \dfrac{\pi}{2}\right)$ $V_2\left(4, \dfrac{3\pi}{2}\right)$
B $V_1\left(2, \dfrac{\pi}{2}\right)$ $V_2\left(2, \dfrac{3\pi}{2}\right)$
C $V_1(2, 0)$ $V_2(2, \pi)$
D $V_1\left(0, -\dfrac{4}{3}\right)$ $V_2(0, -4)$
E None of these

10G

14 **multiple choice**

Which of the following is the polar form of $(x - 2)^2 + y^2 = 4$?

A $r^2 = 4$
B $r = 2 \cos \theta$
C $r = 2 \sin \theta$
D $r = 4 \cos \theta$
E $r = 4 \sin \theta$

10H

15 **multiple choice**

Determine the points of intersection between $r = \dfrac{3}{1 + 2 \cos \theta}$ and $r = \dfrac{3}{1 - \cos \theta}$

A (0.71, 1.42) (–3, 0)
B (0, 3) (0, –3)
C (3, 0) (–3, 0)
D (3, 0) (0, –3)
E (0, 3) (–3, 0)

Modelling and problem solving

1 The normal at P, a point on the parabola $y^2 = 4ax$, meets the curve again at Q. The axis divides the chord PQ in the ratio 1:3. Show that P is an extremity of the latus rectum. (The latus rectum is that focal chord which is parallel to the directrix.)

2 Find the equations of the three lines that pass through the point $(9a, 6a)$ and which are normal to the parabola $y^2 = 4ax$.

3 The normal at P, a point on the parabola, $y^2 = 4ax$, meets the curve again at Q. The x-axis divides the chord PQ in the ratio k:1. Show that the abscissa of P is $\frac{2ka}{1-k}$.

4 Two points, $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$, are on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the tangents to P and Q meet at the point A.

a Show that the line connecting A to the origin passes through the midpoint of PQ.

b If PQ is a focal chord, show that $\cos\left(\frac{\theta-\phi}{2}\right) = e\cos\left(\frac{\theta+\phi}{2}\right)$.

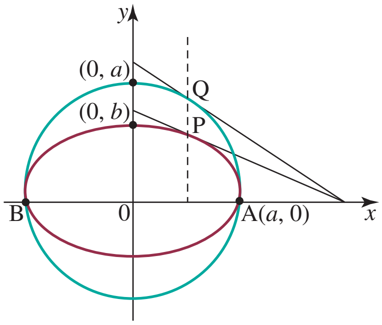
Hint: Use trigonometric identities from Chapter 7 summary.

5 The points A and B are two vertices on the major axis of an ellipse. CD is any line parallel to the minor axis. P is any point on the ellipse such that the lines PA and PB cut the line CD at H and K respectively. Show that the tangent at P passes through the midpoint of HK.

6 The normal at one end, C, of the latus rectum of the ellipse with the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ meets the minor axis at a point A. Point B is the foot of the perpendicular from C to the minor axis. Show that AB = a.

Note: The latus rectum is that focal chord which is parallel to the directrix.

7 Consider a variable point P on the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and a corresponding point Q (with the same x-coordinate as P) on the auxiliary circle (circle which circumscribes the ellipse; the radius of the circle is equal to the semi-major axis of the ellipse). Show that the tangents at P and Q intersect on the x-axis.



8 Show that the line $x + y = 1$ is a tangent to the hyperbola $2x^2 - 3y^2 = 6$, and find the point of contact.

9 $P(x_1, y_1)$ is a point on the hyperbola $\frac{x^2}{16}-\frac{y^2}{9} = 1$.

a Find the equation of the tangent at P.

b If Q is the point at which the tangent cuts the x-axis, show that $\frac{F_1P}{F_2P} = \frac{F_1Q}{F_2Q}$, where F_1 and F_2 are the foci of the hyperbola.

10 A wheel of radius 4 cm rolls along the x-axis with angular velocity of 2 radians/s. Find parametric equations for the curve described by a point on a spoke 2 cm from the centre of the wheel if it starts from the point (0, 2) at time $t = 0$.

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Test Yourself
Chapter 10

eBook*plus* ACTIVITIES

10F Applying technology to parametric equations

Interactivity

- int-0839: Investigate graphs of parametric equations. (*page 483*)

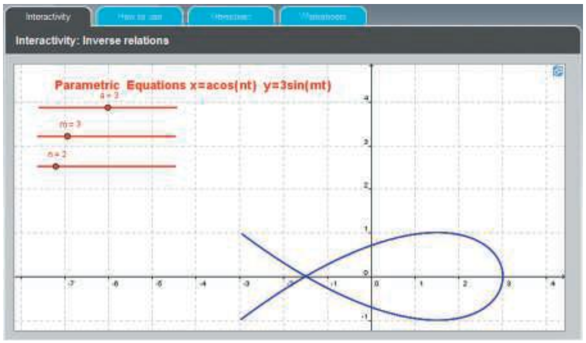
Digital doc

- SkillSHEET 10.1: Practise parametric forms of conics. (*page 484*)

10I Some special curves and interesting problems

Digital docs

- WorkSHEET 10.1: Express Cartesian equations in polar form and determine tangents and normals to points on given graphs. (*page 500*)
- WorkSHEET 10.2: Classify conics, write Cartesian equations of graphs given polar equations and sketch graphs. (*page 500*)



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 507*)

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Dynamics I

11

syllabus reference

Option topic:
Dynamics

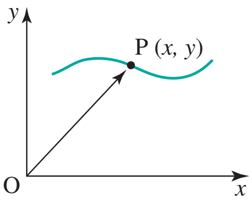
In this chapter

11A Displacement, velocity and acceleration
11B Projectile motion
11C Motion under constant acceleration

Displacement, velocity and acceleration

To explore the motion of a particle in a plane, we must consider both its magnitude and direction. As this motion may not be in a straight line, it is convenient to use a vector approach to calculate the displacement, velocity and acceleration at time *t*.

To illustrate this vector approach, consider the point P in the following diagram.



Relative to the origin, O, the position vector of the displacement of P from the origin, at time *t*, may be written as $\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$ where *x* and *y* denote magnitude of the displacement along the O*x* and O*y* axes respectively. The position of P varies with time so it is appropriate to think of $\overrightarrow{OP}$ as a vector which is time dependent. Displacement at time *t* is the change in position of the point P, and as the reference point is the origin, the displacement of P can be represented by the position vector.

If we write $\underset{\sim}{r}(t)$ to represent $\overrightarrow{OP}$ at time *t*, then we may say that $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ where $x(t)$ and $y(t)$ are the respective horizontal and vertical components of this displacement at time *t*. As a general rule, when we write the magnitude of the displacement or position vector $\underset{\sim}{r}(t)$, we will write *r* where it is understood that $r = |\underset{\sim}{r}(t)|$.

Vector expressions for velocity and acceleration

If the position vector $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ represents the displacement of P at time *t*, then $\frac{d\underset{\sim}{r}}{dt}$ represents the instantaneous velocity and $\frac{d^2\underset{\sim}{r}}{dt^2}$ represents the acceleration at time *t*. (It is often convenient to make use of the time-derivative notation of $\dot{\underset{\sim}{r}}(t)$ for velocity and $\ddot{\underset{\sim}{r}}(t)$ for the acceleration at time *t*.) We calculate the speed of a particle at time *t* as $\dot{r} = |\dot{\underset{\sim}{r}}(t)|$ and the magnitude of the acceleration at time *t* as $\ddot{r} = |\ddot{\underset{\sim}{r}}(t)|$. In our work, the distance or magnitude of the displacement will be measured in metres (m), the speed or magnitude of the velocity in metres per second (m/s) and the magnitude of the acceleration in metres per second per second (m/s^2).

WORKED Example 1

Tutorial:
Worked example 1
int-0411

Relative to the origin, O, the displacement (in metres) of a particle at time *t* seconds is given by $\underset{\sim}{r}(t) = 4t\underset{\sim}{i} + (10t - 5t^2)\underset{\sim}{j}$.

a Find expressions for the velocity and acceleration vectors at time *t* seconds.

b Determine the initial position and speed of the particle.

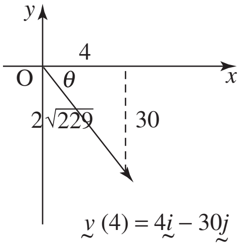
c When *t* = 4 seconds, calculate:

- **i the distance of the particle from the origin**
- **ii the velocity (both magnitude and direction) of the particle**
- **iii the measure of the angle between the velocity and displacement vectors.**

d Is the acceleration constant for this motion?

THINK	**WRITE**
a ① Write the position vector.	**a** $\underset{\sim}{r}(t) = 4t\underset{\sim}{i} + (10t - 5t^2)\underset{\sim}{j}$
② Find $\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt}$ for the velocity vector.	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt}$ $= 4\underset{\sim}{i} + (10 - 10t)\underset{\sim}{j}$ So $\underset{\sim}{v}(t) = 4\underset{\sim}{i} + (10 - 10t)\underset{\sim}{j}$ m/s
③ Find $\ddot{\underset{\sim}{r}}(t) = \frac{d\dot{\underset{\sim}{r}}}{dt}$ or $\ddot{\underset{\sim}{r}}(t) = \frac{d^2\underset{\sim}{r}}{dt^2}$ for the acceleration vector.	$\ddot{\underset{\sim}{r}}(t) = \frac{d\dot{\underset{\sim}{r}}}{dt}$ $= -10\underset{\sim}{j}$ So $\underset{\sim}{a}(t) = -10\underset{\sim}{j}$ m/s^2
b ① The initial position occurs when $t = 0$. Substitute $t = 0$ into the displacement vector expression.	**b** When $t = 0$, $\underset{\sim}{r}(0) = (4 \cdot 0)\underset{\sim}{i} + (10 \cdot 0 - 5 \cdot 0^2)\underset{\sim}{j}$ $= 0\underset{\sim}{i} + 0\underset{\sim}{j}$ This means that the initial position of the particle is at the origin.
② The initial speed occurs when $t = 0$. Substitute $t = 0$ into the velocity vector expression. Remember that speed is the magnitude of velocity.	When $t = 0$, $\underset{\sim}{v}(0) = 4\underset{\sim}{i} + (10 - 10 \cdot 0)\underset{\sim}{j}$ $= 4\underset{\sim}{i} + 10\underset{\sim}{j}$ The initial speed of the particle is the magnitude of the initial velocity. $\lvert\underset{\sim}{v}(0)\rvert = \sqrt{4^2 + 10^2}$ $= 2\sqrt{29}$ The initial speed is $2\sqrt{29}$ m/s.
c **i** ① Distance is the magnitude of displacement, so first calculate $\underset{\sim}{r}(4)$ to find the displacement at $t = 4$.	**c** **i** $\underset{\sim}{r}(t) = 4t\underset{\sim}{i} + (10t - 5t^2)\underset{\sim}{j}$ $\underset{\sim}{r}(4) = 16\underset{\sim}{i} - 40\underset{\sim}{j}$
② Find the magnitude of the resulting vector expression to calculate the distance.	The required distance is the magnitude of the displacement. $\lvert\underset{\sim}{r}(4)\rvert = \sqrt{16^2 + (-40)^2}$ $= 8\sqrt{29}$
③ State the answer.	The distance of the particle from the origin at $t = 4$ s is $8\sqrt{29}$ m.
ii ① Calculate $\underset{\sim}{v}(4)$ to find the velocity vector at $t = 4$.	**ii** $\underset{\sim}{v}(t) = 4\underset{\sim}{i} + (10 - 10t)\underset{\sim}{j}$ $\underset{\sim}{v}(4) = 4\underset{\sim}{i} - 30\underset{\sim}{j}$
② Find the magnitude of the velocity.	$\lvert\underset{\sim}{v}(4)\rvert = \sqrt{4^2 + (-30)^2}$ $= 2\sqrt{229}$ ≈ 30.27 m/s

Continued over page →

THINK	**WRITE**
③ To find the direction, draw a diagram to represent the velocity vector $\underset{\sim}{v}(4) = 4\underset{\sim}{i} - 30\underset{\sim}{j}$.	Let θ be the angle the velocity vector makes with the horizontal. 
④ Use trigonometry to obtain an equation to solve for θ.	$\tan\theta = \frac{30}{4}$ $\theta \approx 82°24'$
⑤ Write the answer.	When $t = 4$ s, the velocity of the particle is $2\sqrt{229}$ m/s (or approx. 30.27 m/s) downwards at an angle of 82°24′ to the horizontal.
iii ① Write the two vectors for velocity and displacement at $t = 4$ s.	**iii** When $t = 4$, $\underset{\sim}{r}(4) = 16\underset{\sim}{i} - 40\underset{\sim}{j}$ and $\underset{\sim}{v}(4) = 4\underset{\sim}{i} - 30\underset{\sim}{j}$
② To calculate the angle between the velocity and displacement vectors we can use the dot product of the two vectors. The dot product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given as $\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\| \cos\theta$ where θ is the angle included by $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	Let θ be the angle between the two vectors. Using the dot product: $\underset{\sim}{r}(4) \cdot \underset{\sim}{v}(4) = \|\underset{\sim}{r}(4)\|\,\|\underset{\sim}{v}(4)\| \cos\theta$
③ Calculate the product on the left of the equation by multiplying the corresponding components of $\underset{\sim}{i}$ and $\underset{\sim}{j}$, and substitute the magnitudes of $\underset{\sim}{r}(4)$ and $\underset{\sim}{v}(4)$ on the right of the equation.	$(16)(4) + (-40)(-30) = (8\sqrt{29})(2\sqrt{229}) \cos\theta$
④ Solve the equation to find θ.	$\cos\theta = \frac{1264}{16\sqrt{6641}}$ $\theta \approx 14°12'$
⑤ Write the answer.	The angle between the velocity and displacement vectors is 14°12′.
d ① Consider the vector expression for acceleration found in part **a**.	**d** $\underset{\sim}{a}(t) = -10\underset{\sim}{j}$
② Since $-10\underset{\sim}{j}$ does not contain t, it is a constant. State the answer.	Since $-10\underset{\sim}{j}$ is a constant, the acceleration is constant for this motion.

WORKED Example 2

At time t seconds, a particle has a position vector given by the expression $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (25 - t^2)\underset{\sim}{j}$ metres.

a **Use a graphics calculator to plot the trajectory of this particle across the interval $0 \le t \le 5$ seconds.**

b **Repeat part a above using an Excel spreadsheet.**

c **Determine the equation of this trajectory in the form $y = f(x)$.**

THINK	WRITE/DISPLAY
a ① Consider the components of the position vector. Assign x to the horizontal component and y to the vertical. This produces two parametric equations that we can use to graph the trajectory.	**a** $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (25 - t^2)\underset{\sim}{j}$ Let $x = 2t$ and $y = 25 - t^2$

② Use a graphics calculator to generate a graph of the particle's motion.

For the Casio *fx*-9860G AU

(a) Press:
- MENU
- 5 (GRAPH)
- F3 (TYPE)
- F3 (PARM)


```
Graph Func  :Param
Xt1:          [—]
Yt1:
Xt2:          [—]
Yt2:
Xt3:          [—]
Yt3:
```

(b) Complete the entry lines as:

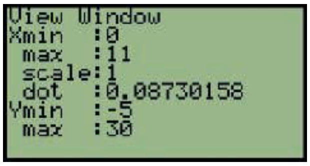
Xt1 = 2T

Yt1 = 25 – T^2

Press EXE after each line.

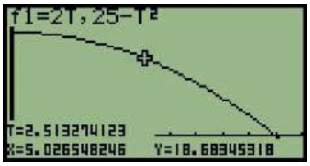

```
Graph Func  :Param
Xt1■2T        [—]
Yt1■25-T²
Xt2:          [—]
Yt2:
Xt3:          [—]
Yt3:
```

(c) Press F6 (DRAW) to display the graph.
To adjust the View Window Settings, press SHIFT F3 (V-WIN) and enter the settings as shown.


```
View Window
Xmin  :0
 max  :11
 scale:1
 dot  :0.08730158
Ymin  :-5
 max  :30
```

(d) Press EXE to return to the equations screen and then press F6 (DRAW) to display the graph.
To display the t, x and y values at different points on the graph, press:
- SHIFT
- F1 (TRACE)

Use the arrow keys to move along the curve.


```
f1=2T,25-T²
T=2.513274123
X=5.026548246     Y=18.68345318
```

Continued over page

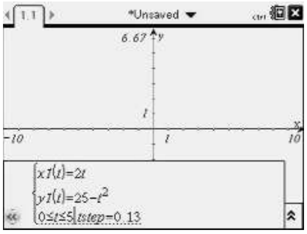
THINK

WRITE/DISPLAY

For the TI-Nspire CAS

(a) On a Graphs page, press:
- MENU menu
- 3: Graph Type 3
- 2: Parametric 2

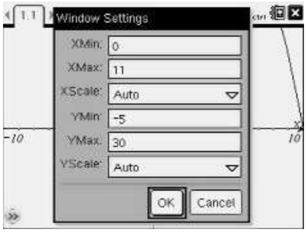
Complete the entry lines as:
$x1(t) = 2t$
$y1(t) = 25 - t^2$
$0 \le t \le 5$



$x1(t)=2t$
$y1(t)=25-t^2$
$0\le t\le 5$ tstep=0.13

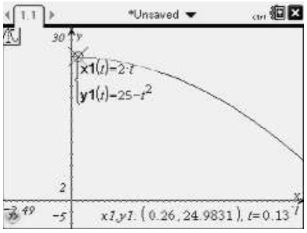
(b) To display the graph, press ENTER enter,
To adjust the Window Settings, press:
- MENU menu
- 4: Window/Zoom 4
- 1: Window Settings 1

Enter the settings as shown.



Window Settings
XMin: 0
XMax: 11
XScale: Auto
YMin: -5
YMax: 30
YScale: Auto

(c) Select OK and press ENTER enter.
To investigate coordinates along the graph, press:
- MENU menu
- 5: Trace 5
- 1: Graph Trace 1



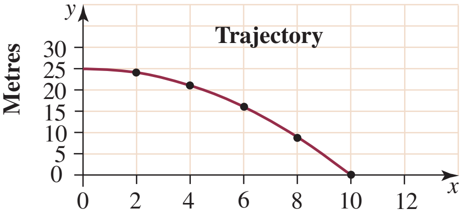
$x1(t)=2\cdot t$
$y1(t)=25-t^2$
x1,y1: (0.26, 24.9831), t=0.13

b ① Set up a spreadsheet with 3 columns titled t, x and y. In the first column, enter values from 0 to 5 for t. Enter the appropriate formula for both x and y and generate corresponding values in columns 2 and 3.

b

t	x	y
0	0	25
1	2	24
2	4	21
3	6	16
4	8	9
5	10	0

② Generate a graph of the particle’s trajectory.



c ① Write the parametric equations for the position vector.

c $x = 2t$ [1]
$y = 25 - t^2$ [2]

THINK	WRITE/DISPLAY
② Eliminate the shared parameter t between $x = 2t$ and $y = 25 - t^2$. Simplify to obtain an equation in the form $y = f(x)$.	Rearranging equation [1]: $t = \frac{x}{2}$ Substituting for t in equation [2]: $y = 25 - \left(\frac{x}{2}\right)^2$ $= 25 - \frac{x^2}{4}$ $= \frac{1}{4}(100 - x^2)$
③ Write the answer.	The equation of the trajectory is $y = \frac{1}{4}(100 - x^2)$.

WORKED Example 3

At time t seconds, the displacement (in metres) of a particle A is given by $\underset{\sim}{r}_A(t) = (8 - t)\underset{\sim}{i} + (8 - 4t + t^2)\underset{\sim}{j}$ and the displacement (in metres) of a particle B is given by $\underset{\sim}{r}_B(t) = (t + 2)\underset{\sim}{i} + (2 - 2t + t^2)\underset{\sim}{j}$.

a If these particles collide, determine when they collide.

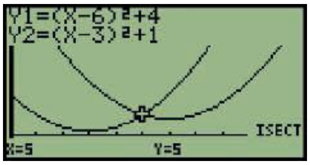
b What are the coordinates of the impact point?

c Find the Cartesian equations of the trajectories of the particles.

d Use a graphics calculator to verify the coordinates of the impact point.

THINK	WRITE/DISPLAY
a ① If the particles collide, they meet at a common point. So the horizontal components of the two vectors will be equal and the vertical components will be equal. Produce two equations to reflect this.	**a** At impact point, horizontal components are equal: $8 - t = t + 2$ [1] At impact point, vertical components are equal: $8 - 4t + t^2 = 2 - 2t + t^2$ [2]
② Solve equation [1] to find t.	Rearranging equation [1]: $2t = 6$ $t = 3$
③ Check the solution by substituting $t = 3$ into the left and right sides of equation [2].	Check by substituting $t = 3$ into equation [2]: LHS $= 8 - 4t + t^2 = 8 - 12 + 9 = 5$ RHS $= 2 - 2t + t^2 = 2 - 6 + 9 = 5$ LHS = RHS so solution is correct.
④ Write the answer.	Particles A and B collide after three seconds.

Continued over page →

THINK	**WRITE/DISPLAY**
b ① Substitute for t in the expression for the displacement of either particle A or B. (In this case, select particle A.)	**b** $\underset{\sim}{r}_A(t) = (8 - t)\underset{\sim}{i} + (8 - 4t + t^2)\underset{\sim}{j}$ At the point of impact, $t = 3$. So $\underset{\sim}{r}_A(3) = (8 - 3)\underset{\sim}{i} + (8 - 12 + 9)\underset{\sim}{j}$ $= 5\underset{\sim}{i} + 5\underset{\sim}{j}$
② Write the answer.	The coordinates of the impact point are (5, 5).
c In both expressions, assign the variables x and y respectively to the horizontal and vertical components. Eliminate the shared parameter t to obtain each Cartesian equation.	**c** Let x represent the horizontal component and y the vertical component of the displacement. For particle A: $x = 8 - t$ and $y = 8 - 4t + t^2$ Hence, $t = 8 - x$ and so $y = 8 - 4(8 - x) + (8 - x)^2$ $= x^2 - 12x + 40$ $y_A = (x - 6)^2 + 4$ For particle B: $x = t + 2$ and $y = 2 - 2t + t^2$ Hence, $t = x - 2$ and so $y = 2 - 2(x - 2) + (x - 2)^2$ $= x^2 - 6x + 10$ $y_B = (x - 3)^2 + 1$
d Verify that the coordinates of the impact point are (5, 5) by drawing the graphs of the displacement for each particle and locating the point of intersection.	**d**
For the Casio *fx*-9860G AU (a) Press: • MENU • 5 (GRAPH) • F3 (TYPE) • F1 (Y=) Complete the entry lines as: $Y1 = (X - 6)^2 + 4$ $Y2 = (X - 3)^2 + 1$ Press EXE after each line. Adjust the View Window settings if necessary.	 Graph Func :Y= Y1■(X−6)²+4 [—] Y2■(X−3)²+1 [—] Y3: [—] Y4: [—] Y5: [—] Y6: [—]
(b) Press: • F6 (DRAW) • SHIFT • F5 (G-Solv) • F5 (ISCT) The coordinates of the point of intersection will be displayed.	 Y1=(X−6)²+4 Y2=(X−3)²+1 ISECT X=5 Y=5 Point of intersection is (5, 5) as required.

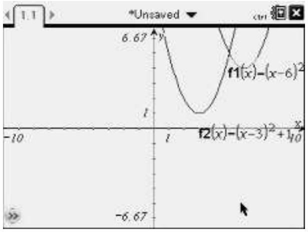
THINK | **WRITE/DISPLAY**

For the TI-Nspire CAS

(a) On a Graphs page, complete the entry lines as:
$f1(x) = (x - 6)^2 + 4$
$f2(x) = (x - 3)^2 + 1$

Press ENTER after each line.

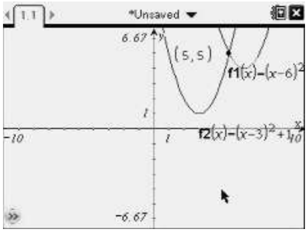
Adjust the Window Settings if necessary.



(b) To display the point of intersection, press:
- MENU
- 7: Points & Lines
- 3: Intersection Point(s)

Use the NavPad to move to each line and press ENTER.

The coordinates of the point of intersection will be displayed.



Point of intersection is (5, 5) as required.

remember

1. If the position vector $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ represents the displacement of P at time t, then $\frac{d\underset{\sim}{r}}{dt}$ represents the instantaneous velocity and $\frac{d^2\underset{\sim}{r}}{dt^2}$ represents the instantaneous acceleration at time t.
2. For the magnitude of a vector $x\underset{\sim}{i} + y\underset{\sim}{j}$: $|x\underset{\sim}{i} + y\underset{\sim}{j}| = \sqrt{x^2 + y^2}$
3. The dot product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given as $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$ where θ is the angle included by $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

EXERCISE 11A Displacement, velocity and acceleration

WORKED Example 1

1 Relative to the origin, O, the displacement (in metres) of a particle at time t seconds is given by $\underset{\sim}{r}(t) = 6t\underset{\sim}{i} + (3t - 7t^2)\underset{\sim}{j}$.

a Find expressions for the velocity and acceleration vectors at time t seconds.

b Determine the initial displacement and speed of the particle.

c When $t = 2$ seconds, calculate:

  i the distance of the particle from the origin

  ii the velocity (both magnitude and direction) of the particle

  iii the measure of the angle between the velocity and displacement vectors.

d Is the acceleration constant for this motion?

WORKED Example 2

2 At time t seconds, a particle has a position vector given by the expression $\underset{\sim}{r}(t) = 5t\underset{\sim}{i} + (49 - t^2)\underset{\sim}{j}$ metres.

- **a** Use a graphics calculator to plot the trajectory of this particle across the interval $0 \le t \le 7$ seconds.
- **b** Repeat part **a** above using an Excel spreadsheet.
- **c** Determine the equation of this trajectory in the form $y = f(x)$.

3 For each of the following position vectors (**a** to **h**) for a particle, where the displacement is measured in metres and time in seconds,

  - **i** calculate the corresponding vector expressions for the velocity and acceleration
  - **ii** find the initial position and speed
  - **iii** determine the Cartesian equation of the trajectory of the particle.

- **a** $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (5 - t)\underset{\sim}{j}$
- **b** $\underset{\sim}{r}(t) = t\underset{\sim}{i} + (6t - t^2)\underset{\sim}{j}$
- **c** $\underset{\sim}{r}(t) = -t\underset{\sim}{i} + (10 - 3t^2)\underset{\sim}{j}$
- **d** $\underset{\sim}{r}(t) = 3t\underset{\sim}{i} + (5t - 2t^2)\underset{\sim}{j}$
- **e** $\underset{\sim}{r}(t) = (4 + t)\underset{\sim}{i} + 3t^2\underset{\sim}{j}$
- **f** $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} - t\underset{\sim}{j}$
- **g** $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} - 4t\underset{\sim}{j}$
- **h** $\underset{\sim}{r}(t) = \sqrt{t}\,\underset{\sim}{i} - 3t^2\underset{\sim}{j}$

WORKED Example 3

4 At time t seconds, the displacement (in metres) of a particle A is given by $\underset{\sim}{r}_A(t) = (2 - t)\underset{\sim}{i} + (3 - 2t + t^2)\underset{\sim}{j}$ and the displacement (in metres) of a particle B is given by $\underset{\sim}{r}_B(t) = (t - 8)\underset{\sim}{i} + (t^2 + t - 12)\underset{\sim}{j}$.

- **a** If these particles collide, determine when they collide.
- **b** What are the coordinates of the impact point?
- **c** Find the Cartesian equations of the trajectories of the particles.
- **d** Use a graphics calculator to verify the coordinates of the impact point.

5 The displacement (in metres) of a particle at time t seconds is given by $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (4 + 2t - t^2)\underset{\sim}{j}$.

- **a** Find expressions for the velocity and acceleration vectors at time t seconds.
- **b** Determine the initial velocity of the particle.
- **c** When $t = 3$ seconds, calculate
  - **i** the distance of the particle from the origin
  - **ii** the velocity (both magnitude and direction) of the particle
  - **iii** the measure of the angle between the velocity and acceleration vectors.
- **d** Show that $\ddot{\underset{\sim}{r}}(t) + \dot{\underset{\sim}{r}}(t) = 2\underset{\sim}{i} - 2t\underset{\sim}{j}$.

6 At time t seconds, a particle has a position vector given by the expression $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (5 - t)^2\underset{\sim}{j}$ metres.

- **a** Use a graphics calculator to plot the trajectory of this particle across the interval $0 \le t \le 10$ seconds.
- **b** Repeat part **a** using an Excel spreadsheet.
- **c** For what values of t are the velocity vectors and acceleration vectors perpendicular?

7 P and Q are particles with displacements (in metres) at time t seconds given by

$\underset{\sim}{r}_{\mathrm{P}}(t) = (t+2)\underset{\sim}{i} + (5-t)^3\underset{\sim}{j}$ and $\underset{\sim}{r}_{\mathrm{Q}}(t) = \dfrac{12}{t-2}\underset{\sim}{i} + (t-3)\underset{\sim}{j}$.

a If these particles collide, determine when and where they collide.

b Verify your response to part **a** by using suitable technology.

c Express the displacement of Q as an equation written in Cartesian form.

8 The displacement (in metres) of a particle at time t seconds is given by

$\underset{\sim}{r}(t) = \dfrac{2}{t}\underset{\sim}{i} + 3t\underset{\sim}{j}$.

a Use a graphics calculator to plot the path of this particle across the interval $1 \le t \le 10$ seconds.

b Calculate the average velocity across the interval $1 \le t \le 10$ seconds.

c Calculate the instantaneous velocity and acceleration of the particle when $t = 5$ seconds.

d What can be said about the acceleration as t increases?

9 The displacement (in metres) of an object at time t seconds is given by $\underset{\sim}{r}(t) = (2+t)\underset{\sim}{i} + (3 + 12t - 5t^2)\underset{\sim}{j}$.

a From what position is this object projected?

b Calculate the speed of projection.

c At what angle to the horizontal is this object projected?

d When $t = 2$ seconds,

  i calculate the velocity (both magnitude and direction) of the object

  ii determine the position of the object relative to the origin.

10 Two particles are projected from the origin, O, such that at time t seconds the displacement (in metres) of the first particle is given by $\underset{\sim}{r}_1(t) = 20t\underset{\sim}{i} + (15t - 5t^2)\underset{\sim}{j}$ and the displacement (in metres) of the second particle is given by $\underset{\sim}{r}_2(t) = (80 - 12t)\underset{\sim}{i} + (a + 9t - 5t^2)\underset{\sim}{j}$ where a is a real number.

a What are the initial positions of the particles?

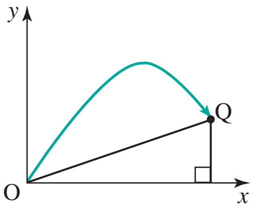
b If these two particles meet after $2\frac{1}{2}$ seconds of flight, what is the value of a in the expression for $\underset{\sim}{r}_2(t)$?

c Locate the point of collision.

d Calculate the angle between the velocity vectors at this time.

11 A particle is projected from the origin so that at time t seconds its displacement (in metres) is given by $\underset{\sim}{r}(t) = 20t\underset{\sim}{i} + (c + bt - 5t^2)\underset{\sim}{j}$ where both b and c are real numbers.

The point Q is a point on an inclined plane OQ where Q has coordinates (80, 15). When the particle reaches Q, its velocity vector is given by $\dot{\underset{\sim}{r}}(t) = 20\underset{\sim}{i} - 15\underset{\sim}{j}$ m/s.

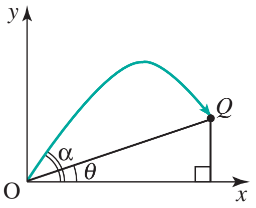


a What are the values of b and c?

b Find the equation of the trajectory of the particle.

c How far up the incline does the projectile land?

12 A particle is projected from the origin up a plane that is inclined at an angle of θ to the horizontal such that at time t seconds, its displacement (in metres) is given by $\underset{\sim}{r}(t) = 35\cos\alpha t\underset{\sim}{i} + (35\sin\alpha t - 5t^2)\underset{\sim}{j}$ where α is the angle of projection of the particle.



- **a** If the point Q is located a distance R metres along the incline, show that $R\cos\theta = 35\cos\alpha t$ and that $R\sin\theta = 35\sin\alpha t - 5t^2$.
- **b** Deduce that if $\tan\theta = \frac{3}{4}$, then $t = \dfrac{4R}{175\cos\alpha}$.
- **c** Now show that $(4\sin\alpha - 3\cos\alpha)\cos\alpha = \dfrac{16R}{1225}$.
- **d** By making suitable use of a graphics calculator, find the greatest value that R may have and the value of α when this occurs if $0 < \alpha < 90°$. Give your answer correct to the nearest minute.
- **e** Use an Excel spreadsheet to validate your solution obtained in part **d** above.

Projectile motion

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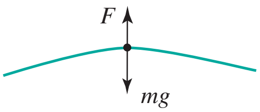
Projectile motion
int-0980

A *projectile* is a body that is projected at or near the Earth's surface, and which has no mechanical means of propulsion after leaving a firing mechanism. The only external force acting on such a body is its weight force. For speeds that are relatively low, we will assume that air-resistance is negligible and that the projectile behaves like a point mass. (In reality, shape, size, mass and speed are indeed significant, particularly so for high-speed motion.) It should be understood that the motion of a projectile occurs in the vertical plane containing an initial velocity vector, and that the motion is parabolic.

When the projectile is set in motion, the only external force is the particle's weight force (by Newton's Second Law of Motion, force = mass · acceleration). If we write g to represent gravitational acceleration, then weight = mass · g; that is, $W = mg$.

In the ideal situation where there are no resisting forces acting on the projectile, the net force of the system is zero. Hence, if F represents the upwards force of motion, then the equation $F + W = 0$ describes the sum of the forces acting vertically on the projectile.



Since we can write $F = ma$, then $F + W = 0$ becomes $ma + mg = 0$ so $a = -g$ m/s^2.
Note: Various approximations for g exist, and in our work we shall use $g = 9.8$ m/s^2 unless stated otherwise. The unit of force is the newton (N).

In the vertical plane, we may express the sum of the forces acting using vector notation.

$$\begin{aligned}\underset{\sim}{F} + mg\underset{\sim}{j} &= 0\\ m\underset{\sim}{a} + mg\underset{\sim}{j} &= 0\\ m\underset{\sim}{a} &= -mg\underset{\sim}{j}\\ \underset{\sim}{a} &= -g\underset{\sim}{j}\end{aligned}$$

Since $\underset{\sim}{a} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$, the horizontal component of acceleration (a_x) is zero and hence the horizontal component of the velocity is always a constant.

Now if $\underset{\sim}{r}(t)$ represents the displacement of a particle at time t, then $\dot{\underset{\sim}{r}}(t)$ represents the associated velocity vector and $\ddot{\underset{\sim}{r}}(t)$ represents the acceleration vector.

$$\begin{aligned}\text{Since } \underset{\sim}{a} = -g\underset{\sim}{j} &\text{ and } \underset{\sim}{a} = \ddot{\underset{\sim}{r}}(t)\\ \ddot{\underset{\sim}{r}}(t) &= -g\underset{\sim}{j}\\ \dot{\underset{\sim}{r}}(t) &= \int(-g\underset{\sim}{j})\,\mathrm{d}t = -gt\underset{\sim}{j} + \underset{\sim}{c}_1\\ \text{so } \underset{\sim}{v}(t) &= -gt\underset{\sim}{j} + \underset{\sim}{c}_1\end{aligned}$$

This is the velocity vector at time t where $\underset{\sim}{c}_1$ is a vector constant of integration.

If we integrate this expression for velocity, we obtain an expression for the displacement of the particle.

$$\underset{\sim}{r}(t) = -\tfrac{1}{2}gt^2\underset{\sim}{j} + \underset{\sim}{c}_1 t + \underset{\sim}{c}_2$$

The constant vectors $\underset{\sim}{c}_1$ and $\underset{\sim}{c}_2$ can be determined by any initial data.

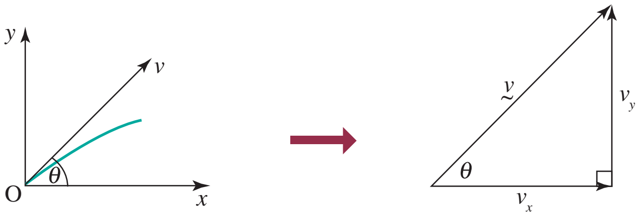
WORKED Example 4

A particle moves so that its acceleration at time *t* is given by $\underset{\sim}{a} = 2\underset{\sim}{i} + (3 + 4t)\underset{\sim}{j}$. Find vector expressions for the velocity and displacement of the particle given that the initial velocity is $3\underset{\sim}{i} + 4\underset{\sim}{j}$ and the initial position is $0\underset{\sim}{i} + 0\underset{\sim}{j}$.

THINK	WRITE
1 Integrate the expression for $\underset{\sim}{a}$ to find $\underset{\sim}{v}$.	$\underset{\sim}{v} = \int \underset{\sim}{a}\, dt$ $= \int (2\underset{\sim}{i} + (3 + 4t)\underset{\sim}{j})\, dt$ $= 2t\underset{\sim}{i} + (3t + 2t^2)\underset{\sim}{j} + \underset{\sim}{c}_1$
2 Apply the initial condition (initial velocity) to determine the constant vector of integration.	When $t = 0$, $\underset{\sim}{v} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ and so $3\underset{\sim}{i} + 4\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{c}_1$ $\underset{\sim}{c}_1 = 3\underset{\sim}{i} + 4\underset{\sim}{j}$
3 Substitute the expression for $\underset{\sim}{c}_1$ into $\underset{\sim}{v}$.	$\underset{\sim}{v} = 2t\underset{\sim}{i} + (3t + 2t^2)\underset{\sim}{j} + 3\underset{\sim}{i} + 4\underset{\sim}{j}$
4 Simplify to write the vector expression for the velocity of the particle.	Hence the velocity vector is $\underset{\sim}{v} = (2t + 3)\underset{\sim}{i} + (4 + 3t + 2t^2)\underset{\sim}{j}$
5 Integrate the expression for $\underset{\sim}{v}$ to find $\underset{\sim}{r}$.	$\underset{\sim}{r} = \int \underset{\sim}{v}\, dt$ $= \int [(2t + 3)\underset{\sim}{i} + (4 + 3t + 2t^2)\underset{\sim}{j}]\, dt$ $= (t^2 + 3t)\underset{\sim}{i} + (4t + \frac{3}{2}t^2 + \frac{2}{3}t^3)\underset{\sim}{j} + \underset{\sim}{c}_2$
6 Apply the initial condition (initial position) to determine the constant vector of integration.	When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$ and so $0\underset{\sim}{i} + 0\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{c}_2$ $\underset{\sim}{c}_2 = 0\underset{\sim}{i} + 0\underset{\sim}{j}$
7 Substitute the expression for $\underset{\sim}{c}_2$ into $\underset{\sim}{r}$ and write the vector expression for the displacement of the particle.	Hence the displacement vector is $\underset{\sim}{r} = (t^2 + 3t)\underset{\sim}{i} + \left(4t + \frac{3}{2}t^2 + \frac{2}{3}t^3\right)\underset{\sim}{j}$

Components of velocity and displacement

For projectile motion where resistance to motion is ignored, we have established that $\underset{\sim}{a} = -g\underset{\sim}{j}$ and hence $\underset{\sim}{v} = -gt\underset{\sim}{j} + \underset{\sim}{c}$. The initial velocity data will determine the vector constant, $\underset{\sim}{c}$.



For an object projected at an angle of θ to the horizontal, the velocity vector can be expressed as $\underset{\sim}{v} = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$.

Using trigonometric ratios, the initial velocity (at $t = 0$) can be written as

$\underset{\sim}{v} = v\cos\theta\,\underset{\sim}{i} + v\sin\theta\,\underset{\sim}{j}$.

To find $\underset{\sim}{c}$, we substitute $t = 0$ and the initial velocity into $\underset{\sim}{v} = -gt\underset{\sim}{j} + \underset{\sim}{c}$.

So $v\cos\theta\,\underset{\sim}{i} + v\sin\theta\,\underset{\sim}{j} = 0\,\underset{\sim}{j} + \underset{\sim}{c}$

$\underset{\sim}{c} = v\cos\theta\,\underset{\sim}{i} + v\sin\theta\,\underset{\sim}{j}$

Hence $\underset{\sim}{v} = -gt\,\underset{\sim}{j} + \underset{\sim}{c}$ becomes

$$\underset{\sim}{v} = -gt\,\underset{\sim}{j} + v\cos\theta\,\underset{\sim}{i} + v\sin\theta\,\underset{\sim}{j}$$

$$\underset{\sim}{v} = v\cos\theta\,\underset{\sim}{i} + (v\sin\theta - gt)\underset{\sim}{j}$$

The horizontal component of the velocity, v_x, is given by $v_x = v\cos\theta$ and the vertical component of the velocity, v_y, is given by $v_y = v\sin\theta - gt$.

In a similar way, the components of the displacement can be established by integrating $\underset{\sim}{v} = v\cos\theta\,\underset{\sim}{i} + (v\sin\theta - gt)\underset{\sim}{j}$ with respect to t.

That is,

$$\underset{\sim}{r} = \int[v\cos\theta\,\underset{\sim}{i} + (v\sin\theta - gt)\underset{\sim}{j}]\,dt$$

$$= vt\cos\theta\,\underset{\sim}{i} + (vt\sin\theta - \tfrac{1}{2}gt^2)\underset{\sim}{j} + \underset{\sim}{c}$$

If the origin is the launch point of the projectile, then

$\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$ and so $\underset{\sim}{c} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$.

Hence $\underset{\sim}{r} = vt\cos\theta\,\underset{\sim}{i} + (vt\sin\theta - \tfrac{1}{2}gt^2)\underset{\sim}{j}$.

The horizontal component of the displacement, r_x, is given by $r_x = vt\cos\theta$ and the vertical component of the displacement, r_y, is given by $r_y = vt\sin\theta - \tfrac{1}{2}gt^2$.

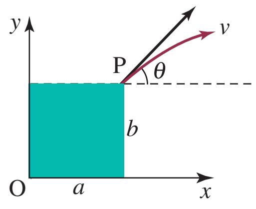
What do we do if the launch point is not at the origin?

Now if the launch point is at (a, b), then $\underset{\sim}{r}(0) = a\underset{\sim}{i} + b\underset{\sim}{j}$

and so $\underset{\sim}{c} = a\underset{\sim}{i} + b\underset{\sim}{j}$.

$\underset{\sim}{r} = vt\cos\theta\,\underset{\sim}{i} + (vt\sin\theta - \tfrac{1}{2}gt^2)\underset{\sim}{j} + \underset{\sim}{c}$ becomes

$\underset{\sim}{r} = (a + vt\cos\theta)\,\underset{\sim}{i} + (b + vt\sin\theta - \tfrac{1}{2}gt^2)\underset{\sim}{j}$



Characteristics of flight

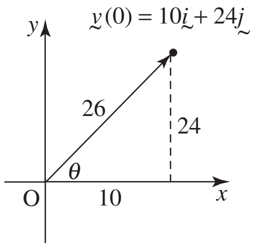
The launch velocity and angle of projection determine such things as the time of flight, greatest height reached above the point of projection and the range.

The period of time for which the projectile is moving under the influence of gravity alone is called the *time of flight* of the projectile. The *range* is the horizontal displacement of the projectile from its launch point. To determine the greatest height reached, we must find when the vertical component of velocity is zero; that is, solve $v_y = 0$ for t. The vertical component of the displacement using this value for t determines the greatest height reached.

WORKED Example 5

A projectile is fired from the foot of a cliff and its displacement (in metres) at time t seconds is given by $\underset{\sim}{r}(t) = 10t\underset{\sim}{i} + (24t - \frac{1}{2}gt^2)\underset{\sim}{j}$.
Use $g = 9.8$ m/s^2 in your calculations.

- **a** **Determine an expression for the velocity vector of the projectile at time t seconds.**
- **b** **Calculate the launch speed and angle of projection of the projectile.**
- **c** **Find the greatest height reached and the time of flight of the projectile.**
- **d** **Calculate the impact velocity of the projectile when it returns to the ground (as a horizontal plane through the origin).**

THINK	WRITE
a Find the derivative of the displacement vector to obtain an expression for the velocity vector.	**a** $\underset{\sim}{r}(t) = 10t\underset{\sim}{i} + (24t - \frac{1}{2}gt^2)\underset{\sim}{j}$ $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t)$ $\underset{\sim}{v}(t) = 10\underset{\sim}{i} + (24 - gt)\underset{\sim}{j}$
b ① To find the launch speed, substitute $t = 0$ into the velocity vector and then calculate the magnitude.	**b** At launch, $t = 0$. Therefore, $\underset{\sim}{v}(0) = 10\underset{\sim}{i} + 24\underset{\sim}{j}$ $\lvert\underset{\sim}{v}(0)\rvert = \sqrt{10^2 + 24^2}$ $= 26$
② Draw a diagram of the velocity vector at $t = 0$ to assist in calculating the angle of projection or launch angle.	Let θ be the angle of projection. 
③ Use trigonometry to obtain an equation to solve for θ.	$\tan\theta = \frac{24}{10}$ $\theta \approx 67°23'$
④ Write the answer.	The projectile is launched with a speed of 26 m/s upwards at an angle of 67°23′ to the horizontal.
c ① The greatest height occurs when the vertical component of the velocity is 0.	**c** If the vertical component of $\underset{\sim}{v}$ is v_y then $v_y = 24 - gt$ At the greatest height, $v_y = 0$. $0 = 24 - gt$ $t = \frac{24}{g}$ ≈ 2.45 s

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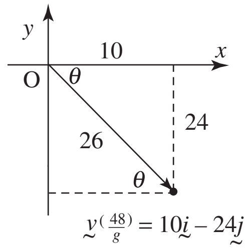
THINK	**WRITE**
2 To calculate the greatest height, substitute for t in the expression for the vertical component of the displacement vector.	When $t = \frac{24}{g}$, the vertical component of displacement, r_y, becomes: $r_y\left(\frac{24}{g}\right) = 24 \cdot \frac{24}{g} - \frac{1}{2} \cdot g \cdot \left(\frac{24}{g}\right)^2$ $= \frac{288}{g}$ $= \frac{288}{9.8}$ ≈ 29.4 m The greatest height reached is 29.4 m.
3 For the time of flight, calculate when the vertical component of the displacement is 0; that is, the time when the projectile returns to the ground. Note that there are two solutions for t when the displacement is 0.	Projectile returns to the ground when $r_y(t) = 0$. $24t - \frac{1}{2}gt^2 = 0$ $t(24 - \frac{1}{2}gt) = 0$ $t = 0$ or $t = \frac{48}{g}$ The solution $t = 0$ gives the time when the projectile is launched from the ground so the projectile returns to the ground at $t = \frac{48}{g}$ s (or ≈ 4.90 s). Hence, the time of flight of the projectile is 4.90 s.
d 1 Since we know that the projectile returns to the ground at $t = \frac{48}{g}$ s, substitute for t in the velocity vector.	**d** $\underset{\sim}{v}(t) = 10\underset{\sim}{i} + (24 - gt)\underset{\sim}{j}$ $\underset{\sim}{v}\left(\frac{48}{g}\right) = 10\underset{\sim}{i} + (24 - g \cdot \frac{48}{g})\underset{\sim}{j}$ $= 10\underset{\sim}{i} - 24\underset{\sim}{j}$
2 Calculate the magnitude of the velocity at this time. (This is the speed.)	The magnitude of the velocity $= \sqrt{10^2 + (-24)^2}$ $= 26$ m/s
3 Find the direction of the projectile at impact. Use a diagram showing the velocity vector at $t = \frac{48}{g}$ s and trigonometry to find the required angle.	Let θ be the angle the velocity vector at $t = \frac{48}{g}$ s makes with the horizontal.  $\tan \theta = \frac{24}{10}$ $\theta \approx 67°23'$
4 Write the answer.	At impact, the velocity of the projectile is 26 m/s downwards at an angle of 67°23′ to the horizontal.

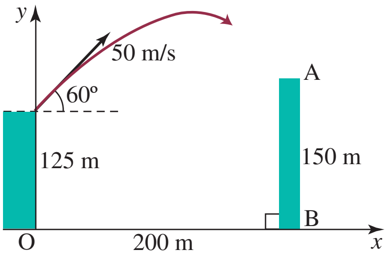
Consider the calculations in Worked example 5. What conclusions can you draw when you compare the launch velocity and the impact velocity of the projectile? What can you conclude about the time taken for the projectile to reach its greatest height compared with the total time of the flight? What condition must you have for these observations to be true in other situations?

WORKED Example 6

eBook*plus*

Tutorial:
Worked example 6
int-0415

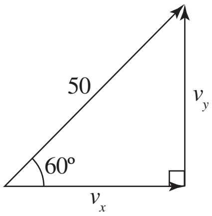
A particle is fired from the top of a 125 m high cliff with a velocity of 50 m/s inclined at an angle of 60° to the horizontal. Use $g = 9.8$ m/s^2 in your calculations.



a **Show that the velocity vector of the projectile at time *t* seconds is given by $\underset{\sim}{v}(t) = 25\underset{\sim}{i} + (25\sqrt{3} - gt)\underset{\sim}{j}$ m/s.**

b **Develop a vector expression for the displacement, $\underset{\sim}{r}(t)$, of the particle at time *t* seconds.**

c **Determine if this projectile will clear a 150 m high building situated 200 m from the base of the cliff and in the same horizontal plane.**

THINK	WRITE
a 1 Write an expression for the acceleration vector. In projectile motion, the horizontal component of $\underset{\sim}{a}$ is 0 and the vertical component is $-g$. (The negative signifies a downward direction.)	**a** For projectile motion, $\underset{\sim}{a} = -g\underset{\sim}{j}$
2 Integrate the expression for $\underset{\sim}{a}$ to find $\underset{\sim}{v}$.	$\underset{\sim}{v} = \int \underset{\sim}{a}\ dt$ $= \int (-g\underset{\sim}{j})\ dt$ $= -gt\underset{\sim}{j} + \underset{\sim}{c}_1$
3 Determine the initial velocity. Draw a diagram to represent the launch velocity and its horizontal and vertical components.	
4 Write expressions for the horizontal component, v_x, and the vertical component, v_y.	$v_x = 50 \cos 60°$ $v_y = 50 \sin 60°$

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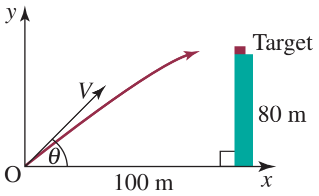
THINK	WRITE
⑤ Use the components to write the expression for the velocity vector at $t = 0$. This is the initial or launch velocity of the particle.	Since $\underset{\sim}{v} = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$ $\underset{\sim}{v}(0) = 50\cos 60°\,\underset{\sim}{i} + 50\sin 60°\,\underset{\sim}{j}$ $= 25\underset{\sim}{i} + 25\sqrt{3}\,\underset{\sim}{j}$
⑥ Apply the initial condition (launch velocity) to determine the constant vector of integration.	When $t = 0$, $\underset{\sim}{v} = 25\underset{\sim}{i} + 25\sqrt{3}\,\underset{\sim}{j}$ and so $25\underset{\sim}{i} + 25\sqrt{3}\,\underset{\sim}{j} = 0\underset{\sim}{j} + \underset{\sim}{c}_1$ $\underset{\sim}{c}_1 = 25\underset{\sim}{i} + 25\sqrt{3}\,\underset{\sim}{j}$
⑦ Substitute the expression for $\underset{\sim}{c}_1$ into $\underset{\sim}{v}$.	$\underset{\sim}{v} = -gt\underset{\sim}{j} + 25\underset{\sim}{i} + 25\sqrt{3}\,\underset{\sim}{j}$
⑧ Simplify to write the vector expression for the velocity of the particle.	Hence the velocity vector is $\underset{\sim}{v} = 25\underset{\sim}{i} + (25\sqrt{3} - gt)\underset{\sim}{j}$
b ① Integrate the expression for $\underset{\sim}{v}$ to find $\underset{\sim}{r}$.	**b** $\underset{\sim}{r} = \int \underset{\sim}{v}\ dt$ $= \int [25\underset{\sim}{i} + (25\sqrt{3} - gt)\underset{\sim}{j}]\ dt$ $= 25t\underset{\sim}{i} + (25\sqrt{3}\,t - \frac{1}{2}gt^2)\underset{\sim}{j} + \underset{\sim}{c}_2$
② Apply the initial condition to determine the constant vector of integration. The initial position of the particle is at the point (0, 125).	When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 125\underset{\sim}{j}$ and so $0\underset{\sim}{i} + 125\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{c}_2$ $\underset{\sim}{c}_2 = 0\underset{\sim}{i} + 125\underset{\sim}{j}$
③ Substitute the expression for $\underset{\sim}{c}_2$ into $\underset{\sim}{r}$ and write the vector expression for the displacement of the particle.	Hence the displacement vector is $\underset{\sim}{r} = 25t\underset{\sim}{i} + (125 + 25\sqrt{3}\,t - \frac{1}{2}gt^2)\underset{\sim}{j}$
c ① Write an expression for the range of the projectile. The horizontal component of $\underset{\sim}{r}$ gives the range.	**c** Range $= r_x$ $= 25t$
② Calculate when the range is 200 m.	When the range is 200 m, $25t = 200$ $t = 8$ s
③ Write an expression for the height of the projectile. The vertical component of $\underset{\sim}{r}$ gives the height.	Height of projectile $= r_x$ $= 125 + 25\sqrt{3}\,t - \frac{1}{2}gt^2$
④ Substitute $t = 8$ to calculate the height of the projectile at this time.	When $t = 8$, $r_y = 125 + 25\sqrt{3} \cdot 8 - \frac{1}{2} \cdot g \cdot 8^2$ ≈ 157.8 m
⑤ Write the answer.	Since the height of the projectile is about 157.8 m when it reaches a range of 200 m, the projectile easily clears the 150 m high building.

In the next worked example, we examine how to find the angle of projection of a particle, given prescribed data. Even though the principles and techniques are the same as those used in solving purely numeric problems, we need to employ some trigonometry in order to reach the solution.

Note: The angle of projection that gives a projected particle its greatest range is often intuitively accepted as an angle of elevation of 45° … but this is true only when the point of projection and point of impact are on the same horizontal plane (or more accurately, at the same elevation).

WORKED Example 7

A missile is fired from a point on level ground with a velocity V m/s and an angle of elevation of θ to the horizontal. The target is positioned on top of an 80 m high tower which is located 100 m away. The base of the tower is in the same horizontal plane as the point of projection of the missile.

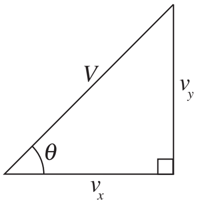


a If g is the gravitational acceleration of the missile, then show that the motion of this missile satisfies $V^2 \cos\theta\,(5\sin\theta - 4\cos\theta) = 250g$.

b Use a graphics calculator to show those values of θ in the domain $0 < \theta < 90°$ for which the expression $\cos\theta\,(5\sin\theta - 4\cos\theta)$ is positive. State the greatest possible value this expression has and the value of θ which produces it. Provide supporting argument for your solutions.

c Extend from your answer to part b above to find the least value of V for which the missile can reach the target. (Use $g = 9.8$ m/s^2.)

d Generate a spreadsheet to validate the solution obtained using a graphics calculator.

THINK	WRITE
a 1 Use a first principles approach to develop an expression for the velocity of the missile at time t seconds after launch.	**a** $\underset{\sim}{a} = -g\underset{\sim}{j}$ $\underset{\sim}{v} = \int \underset{\sim}{a}\ dt$ $= \int (-g\underset{\sim}{j})\ dt$ $= -gt\underset{\sim}{j} + \underset{\sim}{c}_1$
2 Determine the initial velocity. Draw a diagram to represent the launch velocity and write expressions for its horizontal and vertical components.	 $v_x = V\cos\theta$ and $v_y = V\sin\theta$ Since $\underset{\sim}{v} = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$ $\underset{\sim}{v}(0) = V\cos\theta\underset{\sim}{i} + V\sin\theta\underset{\sim}{j}$

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THINK	WRITE
3 Apply the initial condition (launch velocity) to determine the constant vector of integration in $\underset{\sim}{v} = -gt\underset{\sim}{j} + \underset{\sim}{c}_1$.	When $t = 0$, $\underset{\sim}{v} = V\cos\theta\underset{\sim}{i} + V\sin\theta\underset{\sim}{j}$ and $\underset{\sim}{v} = -(g\cdot\ 0)\underset{\sim}{j} + \underset{\sim}{c}_1$ so $V\cos\theta\underset{\sim}{i} + V\sin\theta\underset{\sim}{j} = 0\underset{\sim}{j} + \underset{\sim}{c}_1$ $\underset{\sim}{c}_1 = V\cos\theta\underset{\sim}{i} + V\sin\theta\underset{\sim}{j}$
4 Substitute the expression for $\underset{\sim}{c}_1$ into $\underset{\sim}{v} = -gt\underset{\sim}{j} + \underset{\sim}{c}_1$.	$\underset{\sim}{v} = -gt\underset{\sim}{j} + V\cos\theta\underset{\sim}{i} + V\sin\theta\underset{\sim}{j}$
5 Simplify to write the vector expression for the velocity of the missile at time t seconds.	Hence the velocity vector is $\underset{\sim}{v}(t) = V\cos\theta\underset{\sim}{i} + (V\sin\theta - gt)\underset{\sim}{j}$
6 Integrate the expression for $\underset{\sim}{v}$ to find $\underset{\sim}{r}$.	$\underset{\sim}{r} = \int\underset{\sim}{v}\ dt$ $= \int[V\cos\theta\underset{\sim}{i} + (V\sin\theta - gt)\underset{\sim}{j}]\ dt$ $= Vt\cos\theta\underset{\sim}{i} + (Vt\sin\theta - \frac{1}{2}gt^2)\underset{\sim}{j} + \underset{\sim}{c}_2$
7 Apply the initial condition to determine the constant vector of integration. The initial position of the missile is at the point (0, 0).	When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$ and so $0\underset{\sim}{i} + 0\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{c}_2$ $\underset{\sim}{c}_2 = 0\underset{\sim}{i} + 0\underset{\sim}{j}$
8 Substitute for $\underset{\sim}{c}_2$ and write the vector expression for the displacement of the missile at time t seconds.	Hence the displacement vector is $\underset{\sim}{r}(t) = Vt\cos\theta\underset{\sim}{i} + (Vt\sin\theta - \frac{1}{2}gt^2)\underset{\sim}{j}$
9 Write an expression for the range of the missile at time t. Use the given information that the range to the target is 100 m to obtain an expression for t.	Range of missile is given by $r_x = Vt\cos\theta$ When $r_x = 100$, $Vt\cos\theta = 100$ $t = \frac{100}{V\cos\theta}$
10 Write an expression for the height of the missile at time t. Use the information that the height of the target at the given range is 80 m and that this occurs at $t = \frac{100}{V\cos\theta}$ seconds.	Height of missile is given by $r_y = Vt\sin\theta - \frac{1}{2}gt^2$ When $r_y = 80$, $t = \frac{100}{V\cos\theta}$ and so $Vt\sin\theta - \frac{1}{2}gt^2 = 80$ becomes $V\cdot\frac{100}{V\cos\theta}\cdot\sin\theta - \frac{1}{2}g\cdot\left(\frac{100}{V\cos\theta}\right)^2 = 80$

THINK

⑪ Continue simplifying. Multiply both sides of the equation by $V^2 \cos^2 \theta$.

WRITE

$$\frac{100 \sin \theta}{\cos \theta} - \frac{5000g}{V^2 \cos^2 \theta} = 80$$

$$\frac{100 \sin \theta \cdot V^2 \cos^2 \theta}{\cos \theta} - \frac{5000g \cdot V^2 \cos^2 \theta}{V^2 \cos^2 \theta}$$

$$= 80\ V^2 \cos^2 \theta$$

$$100V^2 \sin\ \theta \cos\ \theta - 5000g = 80\ V^2 \cos^2 \theta$$

$$20V^2 \cos\ \theta\ (5 \sin\ \theta - 4 \cos\ \theta) = 5000g$$

$$V^2 \cos\ \theta\ (5 \sin\ \theta - 4 \cos\ \theta) = 250g$$

(as required)

b ① Use a graphics calculator to graph the function $f(\theta) = \cos\ \theta\ (5 \sin\ \theta - 4 \cos\ \theta)$. You will need to use x in place of the variable θ.

b

For the Casio *fx*-9860G AU

(a) Ensure that the calculator is set in degree mode, then press:

- MENU
- 5 (GRAPH)
- F3 (TYPE)
- F1 (Y=)

Complete the entry line as:

Y1 = (cos x)(5 sin x – 4 cos x)

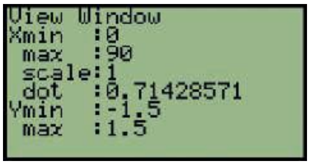
Then press EXE.

```
Graph Func  :Y=
Y1■(cos X)(5sin [—]
Y2:             [—]
Y3:             [—]
Y4:             [—]
Y5:             [—]
Y6:             [—]
```

(b) To adjust View Window settings, press:

- SHIFT
- F3 (V-Window)

Enter the settings as shown and then press EXE.



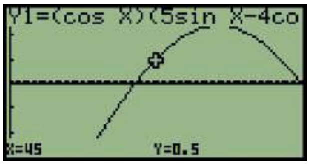
```
View Window
Xmin  :0
 max  :90
 scale:1
 dot  :0.71428571
Ymin  :-1.5
 max  :1.5
```

(c) To display the graph, press F6 (DRAW).

To investigate points along the curve, press:

- SHIFT
- F1 (TRACE)

Use the arrow keys to move along the curve.

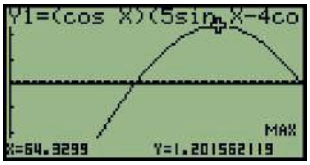


```
Y1=(cos X)(5sin X-4co
X=45          Y=0.5
```

(d) To find the maximum value, press:

- SHIFT
- F5 (G-Solv)
- F2 (MAX)

The coordinates of the required point are displayed.



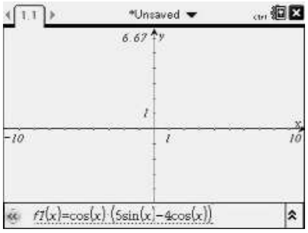
```
Y1=(cos X)(5sin X-4co
                  MAX
X=64.3299     Y=1.201562119
```

Continued over page →

THINK **WRITE**

For the TI-Nspire CAS

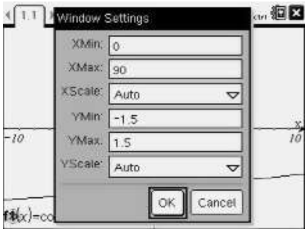
(a) Ensure that the calculator is set in degree mode.
On a Graphs page, complete the entry line as:
$f1(x) = \cos(x) \cdot\ (5\ \sin(x) - 4\ \cos(x))$
Then press ENTER ⌤.



f1(x)=cos(x)·(5sin(x)−4cos(x))

(b) To adjust the Window Settings, press:
- MENU (menu)
- 4: Window/Zoom ⟨4⟩
- 1: Window Settings ⟨1⟩

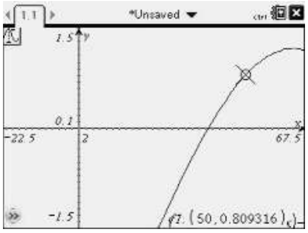
Enter the settings as shown.


Window Settings

XMin: 0
XMax: 90
XScale: Auto
YMin: -1.5
YMax: 1.5
YScale: Auto

(c) Select OK and press ENTER ⌤ to display the graph.
To investigate points along the curve, press:
- MENU (menu)
- 5: Trace ⟨5⟩
- 1: Graph Trace ⟨1⟩

Use the NavPad to move along the curve.



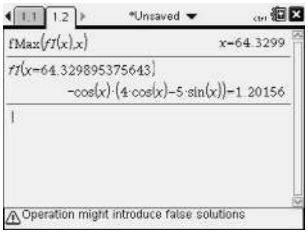
f1: (50, 0.809316)

(d) To find the value of x when the function is a maximum, on a Calculator page, complete the entry line as:
fMax$(f1(x), x)$
Then press ENTER ⌤.

fMax(f1(x),x) x=64.3299

1/99

(e) To find the maximum value of the function, complete the entry line as:
$f1(x = 64.329907948163)$
Then press ENTER ⌤.
Note: There is no need to reenter $x = 64.3299$, use the NavPad to highlight any previous value and then press ENTER ⌤ to place it into the current entry line.



fMax(f1(x),x) x=64.3299

f1(x=64.329895375643)
−cos(x)·(4·cos(x)−5·sin(x))=1.20156

⚠ Operation might introduce false solutions

② Write a conclusion from your observations and calculations.

The graph shows where the expression $\cos\ \theta\ (5\ \sin\ \theta - 4\ \cos\ \theta)$ is positive in the domain $0 < \theta < 90°$. There is one local maximum in this domain. The greatest possible value of this expression is approx. 1.202 when the angle is approx. 64.3°.

THINK **WRITE**

c ① Rearrange the equation from part **a** to make V^2 the subject of the equation.

c $V^2 \cos\theta\,(5\sin\theta - 4\cos\theta) = 250g$

$$V^2 = \frac{250g}{\cos\theta\,(5\sin\theta - 4\cos\theta)}$$

② The least value of V^2 occurs when the denominator is greatest. Use the maximum value for $\cos\theta\,(5\sin\theta - 4\cos\theta)$ found in part **b**.

The least value of V^2 occurs when $\cos\theta\,(5\sin\theta - 4\cos\theta)$ is greatest.

The maximum value of $\cos\theta\,(5\sin\theta - 4\cos\theta)$ is 1.202 (from part **b**).

③ Substitute the maximum value for the denominator and the value for g to calculate the least value for V for which the missile can reach the target.

$$V^2 = \frac{250 \cdot\ 9.8}{1.202}$$

Since $V > 0$, then $V \approx 45.1$.

④ Write the answer.

The required least velocity of the missile to reach the target is approx. 45.1 m/s at an angle of elevation of approx. 64.3°.

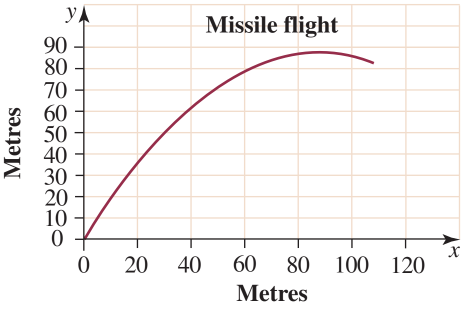
d ① Set up a spreadsheet with 3 columns titled t, x and y. In the first column, enter values for t. Enter the appropriate parametric equations of the displacement as $x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$ using the calculated value for V and θ and substituting 9.8 for g. Generate corresponding values in columns 2 and 3.

d

Theta	64.3299
Velocity	45.1000
Delta t	0.2500

t	x	y
0	0	0
0.25	4.884	9.856
0.50	9.768	19.099
0.75	14.653	27.730
⋮	⋮	⋮
4.75	92.800	82.525
5.00	97.684	80.744
5.25	102.568	78.350

② Generate the graph to provide a visual validation of the results.



remember

1. Finding the velocity given the acceleration: $\underset{\sim}{a} = \frac{d\underset{\sim}{v}}{dt} \rightarrow \underset{\sim}{v} = \int \underset{\sim}{a}\ dt$.
2. Finding the displacement given the velocity: $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} \rightarrow \underset{\sim}{r} = \int \underset{\sim}{v}\ dt$.
   Remember that the constant of integration is always a vector constant.
3. The period of time for which the projectile is moving under the influence of gravity alone is called the *time of flight* of the projectile.
4. The *range* is the horizontal displacement of the projectile from its launch point.

EXERCISE 11B Projectile motion

Unless otherwise stated, use $g = 9.8$ m/s^2.

1 Given the following information about the motion of a particle at time t, determine a vector expression for the particle's displacement.

a $\underset{\sim}{v}(t) = \underset{\sim}{i}$ and $\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

b $\underset{\sim}{v}(t) = 2t\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j}$

c $\frac{d\underset{\sim}{r}}{dt} = 4\underset{\sim}{i} + 7t\underset{\sim}{j}$ and $\underset{\sim}{r}(1) = 2\underset{\sim}{i} - \underset{\sim}{j}$

d $\frac{d\underset{\sim}{r}}{dt} = (3 + t)\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{r}(2) = \underset{\sim}{i} + 5\underset{\sim}{j}$

e $\dot{\underset{\sim}{r}}(t) = 3t\underset{\sim}{i} + (4 + t)\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = -\underset{\sim}{i} + \underset{\sim}{j}$

f $\dot{\underset{\sim}{r}}(t) = (2 - t)\underset{\sim}{i} + (6 + t^2)\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

g $\underset{\sim}{v}(t) = 15\underset{\sim}{i} + (10 - 3t)\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 20\underset{\sim}{j}$

h $\frac{d\underset{\sim}{r}}{dt} = -2t\underset{\sim}{i} + \left(5 + \frac{1}{t^2}\right)\underset{\sim}{j}$ and $\underset{\sim}{r}(1) = 2\underset{\sim}{i} - 7\underset{\sim}{j}$

WORKED Example 4

2 Determine vector expressions at time t for the velocity, $\underset{\sim}{v}$, and displacement, $\underset{\sim}{r}$, of a particle which moves according to the following relationships involving acceleration, $\underset{\sim}{a}$.

a $\underset{\sim}{a}(t) = 2\underset{\sim}{i}$ and when $t = 0$, $\underset{\sim}{v} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{r} = -\underset{\sim}{i} + 5\underset{\sim}{j}$

b $\frac{d\underset{\sim}{v}}{dt} = 3\underset{\sim}{i} - 5\underset{\sim}{j}$ and when $t = 0$, $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{r} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

c $\ddot{\underset{\sim}{r}} = -\underset{\sim}{i} + t\underset{\sim}{j}$, $\underset{\sim}{v}(0) = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

d $\ddot{\underset{\sim}{r}} = 2t\underset{\sim}{i} - 3\underset{\sim}{j}$, $\dot{\underset{\sim}{r}}(1) = -4\underset{\sim}{j}$ and $\underset{\sim}{r}(0) = \underset{\sim}{i} + \underset{\sim}{j}$

e $\frac{d\underset{\sim}{v}}{dt} = -5\underset{\sim}{i} + (t + 1)\underset{\sim}{j}$, $\underset{\sim}{r}(0) = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{r}(2) = 8\underset{\sim}{i} + 3\underset{\sim}{j}$

f $\underset{\sim}{a}(t) = \underset{\sim}{i} + (6 - t)\underset{\sim}{j}$, $\underset{\sim}{r}(0) = -5\underset{\sim}{i} + 12\underset{\sim}{j}$ and $\underset{\sim}{r}(1) = -\underset{\sim}{i} + 4\underset{\sim}{j}$

g $\frac{d\underset{\sim}{v}}{dt} = (2t + 1)\underset{\sim}{i} - (3 - 5t)\underset{\sim}{j}$, $\underset{\sim}{r}(0) = 8\underset{\sim}{j}$ and $\underset{\sim}{r}(1) = -3\underset{\sim}{i} + 9\underset{\sim}{j}$

h $\ddot{\underset{\sim}{r}} = 3\underset{\sim}{i} + (12 - 5t)\underset{\sim}{j}$, $\underset{\sim}{v}(3) = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{r}(5) = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

WORKED Example 5

3 A projectile is fired from the foot of a cliff and its displacement (in metres) at time t seconds is given by $\underset{\sim}{r}(t) = 12t\underset{\sim}{i} + (15t - \frac{1}{2}gt^2)\underset{\sim}{j}$.

a Determine an expression for the velocity vector of the projectile at time t seconds.

b Calculate the launch speed and angle of projection of the projectile.

c Find the greatest height reached and the time of flight of the projectile.

d Calculate the impact velocity of the projectile when it returns to the ground in a horizontal plane through the origin.

4 A projectile is fired from the origin, O, and its displacement (in metres) at time t seconds is given by $\underset{\sim}{r}(t) = 15t\underset{\sim}{i} + (30t - \frac{1}{2}gt^2)\underset{\sim}{j}$.

a Determine an expression for the velocity vector of the projectile at time t seconds.

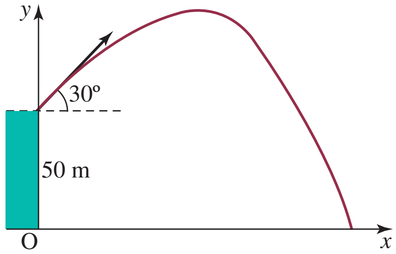
b Calculate the launch speed and angle of projection of the projectile.

c Determine the greatest height reached by the projectile.

d Will this projectile clear a 4 m high wall located 90 m from the launch point in the same horizontal plane?

WORKED Example 6

5 A particle is fired from the top of a 50 m high cliff with a velocity of 36 m/s inclined at an angle of 30° to the horizontal.



a Show that the velocity vector of the projectile at time t seconds is given by $\underset{\sim}{v}(t) = 18\sqrt{3}\,\underset{\sim}{i} + (18 - gt)\underset{\sim}{j}$ m/s.

b Develop a vector expression for the displacement, $\underset{\sim}{r}(t)$, of the particle at time t seconds.

c Calculate the greatest height reached by the projectile.

d What is the impact velocity of the projectile when it reaches a point in the horizontal plane through the origin, O?

e At this impact point, what is the range of the projectile?

6 At time t, the velocities of two particles A and B are given by

$$\underset{\sim}{v}_A(t) = 4\underset{\sim}{i} + (5 - 2t)\underset{\sim}{j} \text{ and } \frac{d\underset{\sim}{r}_B}{dt} = -3\underset{\sim}{i} + (4t - 7)\underset{\sim}{j}.$$

When $t = 0$, $\underset{\sim}{r}_A = -4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{r}_B = 24\underset{\sim}{i} + 6\underset{\sim}{j}$.

a Determine vector expressions for the displacement of both particles at time t.

b Do these particles ever collide? If so, calculate when and where. If not, explain what happens.

c Find the Cartesian equations of the trajectories of the particles.

d Use a graphics calculator to verify your response to part **b** above.

7 The acceleration of a particle λ, at time t, is given by $\underset{\sim}{a}_\lambda(t) = -8\underset{\sim}{j}$. For this particle, $\underset{\sim}{r}_\lambda(2) = -4\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{r}_\lambda(5) = 11\underset{\sim}{i} + 15\underset{\sim}{j}$. The displacement of a second particle, ξ, at time t, is given by $\underset{\sim}{r}_\xi(t) = (6 + t)\underset{\sim}{i} + (15t - 10 - 2t^2)\underset{\sim}{j}$.

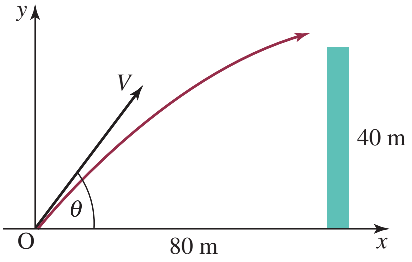
a Determine the vector expression for the displacement of the particle, λ, at time, t.

b Do these particles ever collide? If so, calculate when and where.

c Validate your response to part **b** above by using suitable technology.

WORKED Example 7

8 A missile is fired from a point on level ground with a velocity V m/s and an angle of elevation of θ to the horizontal. The target is positioned on top of a 40 m high tower which is located 80 m away. The base of the tower is in the same horizontal plane as the point of projection of the missile.



a If g is the gravitational acceleration of the missile, then show that the motion of this missile satisfies $V^2 \cos \theta\,(2 \sin \theta - \cos \theta) = 80g$.

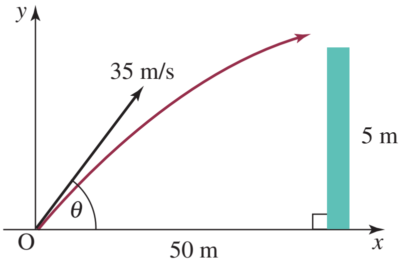
b Use a graphics calculator to show those values of θ in the domain $0 < \theta < 90°$ for which the expression $\cos \theta\,(2 \sin \theta - \cos \theta)$ is positive. State the greatest possible value this expression has, and the value of θ which produces it. Provide supporting argument for your solutions.

c Extend from your answer to part **b** above to find the least value of V for which the missile can reach the target. (Use $g = 9.8$ m/s^2.)

d Generate a spreadsheet to validate the solutions obtained using a graphics calculator.

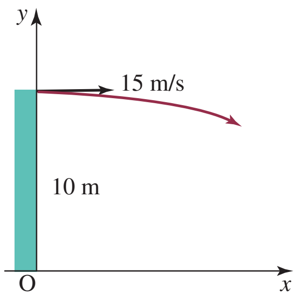
9 A particle is projected with an initial speed of 35 m/s in such a direction that it just passes over the top of a 5 m high wall. The foot of the wall is situated 50 m from the point of projection of the particle and, as shown at right, is in the same horizontal plane through the point of projection.

Determine all possible angles of projection, θ.



10 A particle is projected horizontally from the top of a tower 10 m high with an initial speed of 15 m/s.

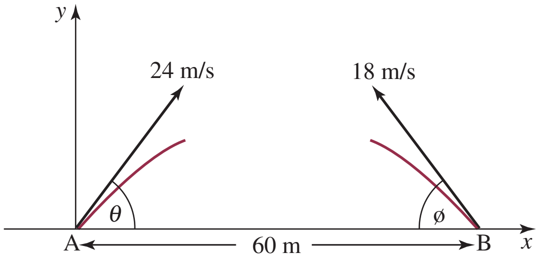
If the surrounding ground is horizontal, how far from the base of the tower does the particle land?



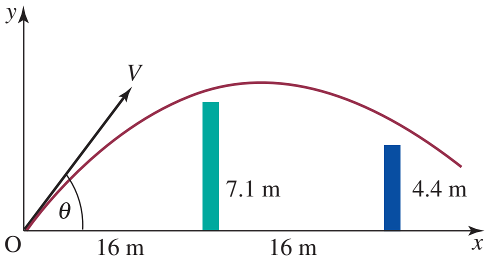
11 A projectile is fired at an angle θ below the horizontal, where $\tan \theta = \frac{7}{24}$, from the top of a cliff, 75.6 m high, which overlooks the sea. If the projectile reaches the sea at the instant 2 s after projection, find its initial speed and the distance from the base of the cliff to the point of impact of the projectile with the sea.

12 Two particles, P and Q, are fired simultaneously from points A and B respectively which are located 60 m apart on horizontal ground, as shown below. The particle P has an initial speed of 24 m/s and an angle of projection θ, where $\theta = \tan^{-1}\left(\frac{3}{4}\right)$. The particle Q has an initial speed of 18 m/s with an angle of projection ϕ where $\phi = \tan^{-1}\left(\frac{4}{3}\right)$.

Do these particles collide? If so, when and where do they collide?



13 Two walls are situated 16 m apart on level ground. One wall is 7.1 m high and the other is 4.4 m high. A projectile is to be fired from the point on the ground which is 16 m from the 7.1 m high wall and on the side remote from the second wall as seen in the figure at right.



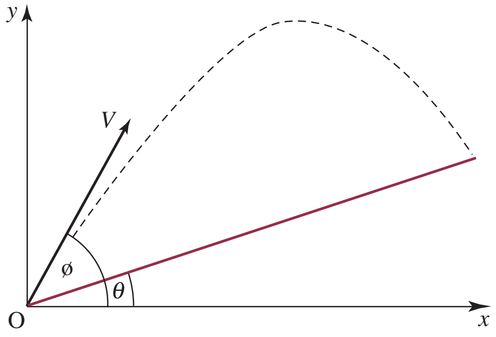
Determine the initial speed and angle of projection required if this projectile will just clear both walls.

14 A particle is projected with an initial speed of 35 m/s from the base of a plane inclined at an angle ϕ to the horizontal where $\phi = \tan^{-1}\left(\frac{3}{4}\right)$. If the projectile is initially angled at θ to the horizontal, where $\tan\theta = \frac{5}{2}$, find how far along the plane the projectile lands, and the time of flight.

15 A particle is projected up a plane inclined at an angle θ to the horizontal, the angle of projection being ϕ above the horizontal.
If the initial speed of projection is V,

a show that the range, R, on this inclined plane is given by

$$R = \frac{2V^2 \sin(\phi - \theta)\cos\phi}{g\cos^2\theta}$$



b determine the maximum range ($R_{\max}$) on the plane and the value of ϕ for which this occurs.

16 A sprinkler sprays water symmetrically about its vertical axis at a constant speed of V. The initial direction of the spray varies continuously between angles of 15° and 60° to the horizontal.

a Prove that from a fixed position, O, on level ground, the sprinkler will wet the surface of an annular region with respective internal and external radii $\frac{V^2}{2g}$ and $\frac{V^2}{g}$.

b Now show that if the sprinkler is located appropriately to a rectangular garden bed of size 6 m by 3 m, the entire garden will be watered provided $\frac{V^2}{2g} \geq 1 + \sqrt{7}$.

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WorkSHEET 11.1

Motion under constant acceleration

Even though our study in this course is restricted to motion involving constant acceleration, this study does include many situations of practical significance.

The common example of bodies in free fall towards the Earth's surface, regardless of their size, mass or composition, involves motion under constant acceleration — that of acceleration due to gravity.

Galileo illustrated this property by dropping various sized cannon balls from the top of the Leaning Tower of Pisa. Provided the effects of air resistance are minimised (that is, the object should not be too large in area or fall too far) all bodies will fall with constant acceleration. Other variations will be considered later in your studies.

Since the acceleration due to gravity depends on the distance from the centre of the Earth, it varies slightly at different places on the Earth's surface — being greater at the poles than at the equator and less at high altitudes. It is usual to state the constant of acceleration due to gravity as 9.8 m/s^2 unless you are told otherwise.

Since acceleration is constant, the displacement of a falling object can be modelled by the general quadratic expression

$$lt^2 + mt + n$$

This can be used to model the motion of a body in free fall or one tossed upwards. The important difference is that the velocity and acceleration are in different directions. When vector notation is used the vertical displacement vector is written as

$$\underset{\sim}{r} = (lt^2 + mt + n)\underset{\sim}{j}$$

Because $\underset{\sim}{j}$ does not depend on t the derivative of this function is as follows:

$$\underset{\sim}{v} = (2lt + m)\underset{\sim}{j}$$

and $$\underset{\sim}{a} = 2l\underset{\sim}{j}$$

The following analysis examines the motion of a ball as it rests at the top of a cliff. As soil falls away the ball begins to move.

(a) At $t = 0$, find v.
At $t = 0$ the ball is at rest, therefore $v = 0$.

(b) If the bottom of the cliff is the origin and the cliff is h metres high, find r when $t = 0$.
At $t = 0$, $\underset{\sim}{r} = h\underset{\sim}{j}$

(c) Given the constant acceleration towards the origin equals -9.8 m/s^2 (a negative value is used to indicate that the direction of the acceleration is downwards), find m and l.

Use the original equation for displacement and differentiate:

$\underset{\sim}{r} = (lt^2 + mt + n)\underset{\sim}{j}$ [1]

$\underset{\sim}{v} = (2lt + m)\underset{\sim}{j}$ [2]

$\underset{\sim}{a} = 2l\underset{\sim}{j}$ [3]

From part (a) above when $t = 0$

$\underset{\sim}{v} = [2l(0) + m]\underset{\sim}{j}$ in [2]

but $\underset{\sim}{v} = 0$ (initially at rest)

Therefore, $m = 0$

From part (b) above, if at $t = 0$, $\underset{\sim}{r} = h\underset{\sim}{j}$

$\underset{\sim}{r} = (lt^2 + mt + n)\underset{\sim}{j}$ [1]

$\underset{\sim}{r} = [l(0)^2 + m(0) + n]\underset{\sim}{j}$

$= h\underset{\sim}{j}$ (as given at $t = 0$)

Therefore, $n = h$

From (c) $g = -9.8$ m/s^2 so $\underset{\sim}{a} = -9.8\underset{\sim}{j}$ (as $\underset{\sim}{a} = g\underset{\sim}{j}$)

Also $\underset{\sim}{a} = 2l\underset{\sim}{j}$

So $2l\underset{\sim}{j} = -9.8\underset{\sim}{j}$

Therefore, $l = -4.9$

Therefore the general equation for this motion can be written:

$\underset{\sim}{r} = (-4.9t^2 + h)\underset{\sim}{j}$

Do not attempt to memorise these equations. Equations [1], [2] and [3] are the basic equations and all others are specific rearrangements of these.

In general,

1. l can be readily found if the acceleration is given: $a = 2l$.
2. m can be readily found if the initial velocity is given: $v = 2l(0) + m$.
3. n can be readily found if the initial displacement is given: $r = l(0)^2 + m(0) + n$

WORKED Example 8

Examine the motion of a stone thrown upwards with a velocity of 6 m/s from the top of a cliff 60 m high. Find the time taken for it to reach the ground and its velocity on impact.

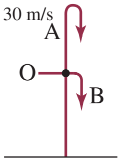
THINK	WRITE
1 As the acceleration is constant, the vertical displacement is quadratic. Write the three equations of motion.	$\underset{\sim}{r} = (lt^2 + mt + n)\underset{\sim}{j}$ $\underset{\sim}{v} = (2lt + m)\underset{\sim}{j}$ $\underset{\sim}{a} = 2l\underset{\sim}{j}$
2 The position of the origin needs to be stated. Generally where the motion begins or finishes can be thought of as the origin.	Let the origin be the top of the cliff.
3 Use the given information to find each value of n, m and l. First use the initial position to find n.	At $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{j}$ and $\underset{\sim}{r} = [l(0)^2 + m(0) + n]\underset{\sim}{j}$ So $0 = 0 + 0 + n$ therefore $n = 0$
4 Rewrite the equation for $\underset{\sim}{r}$ with the new information.	$\underset{\sim}{r} = (lt^2 + mt)\underset{\sim}{j}$

Continued over page →

THINK	WRITE
5 Use the equations for $\underset{\sim}{a}$ to find l. Constant downwards acceleration away from the origin so the acceleration is negative.	$\underset{\sim}{a} = 2l\underset{\sim}{j}$ and $\underset{\sim}{a} = -9.8\underset{\sim}{j}$ So $2l = -9.8$ $l = -4.9$ $\underset{\sim}{r} = (-4.9t^2 + mt)\underset{\sim}{j}$
6 Use the initial velocity to find m. Initial velocity at $t = 0$ is 6 m/s up (positive sign).	At $t = 0$, $v = 6$ or $\underset{\sim}{v} = 6\underset{\sim}{j}$ $6\underset{\sim}{j} = [2l(0) + m]\underset{\sim}{j}$ $m = 6$
7 Rewrite $\underset{\sim}{r}$ with all current information.	$\underset{\sim}{r} = (-4.9t^2 + 6t)\underset{\sim}{j}$
8 Find the time when the stone reaches the ground at $\underset{\sim}{r} = -60\underset{\sim}{j}$. Drop the vector notation. Solve for t using the quadratic formula where $a = -4.9$, $b = 6$ and $c = 60$.	At $\underset{\sim}{r} = -60\underset{\sim}{j}$ $-60\underset{\sim}{j} = (-4.9t^2 + 6t)\underset{\sim}{j}$ $0 = -4.9t^2 + 6t + 60$ $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-6 \pm \sqrt{36 - 4(-4.9)(60)}}{2(-4.9)}$ $t = 4.16$ s or -2.94 s Reject the negative solution. Therefore the stone reaches the ground 4.16 s after it was thrown upwards.
9 Find the velocity on impact by finding v when $t = 4.16$. Note that the negative sign indicates a vector moving downwards, in the same direction as the negative acceleration.	$\underset{\sim}{v} = (-9.8t + 6)\underset{\sim}{j}$ At $t = 4.16$ s $\underset{\sim}{v} = [-9.8(4.16) + 6]\underset{\sim}{j}$ $= -34.81\underset{\sim}{j}$
10 State the solution.	Therefore, the stone reaches the ground with a velocity of 34.81 m/s downwards 4.16 s after launching.

WORKED Example 9

A stone (A) is thrown upwards from a cliff with a velocity of 30 m/s. After stone A has been in motion for 4 s another stone (B) is dropped from the same point. Find when and where the two stones will meet.

THINK	WRITE/DRAW
1 Draw a diagram and state the origin position and direction of positive motion. Write the 3 equations of motion.	 $\underset{\sim}{r} = (lt^2 + mt + n)\underset{\sim}{j}$ $\underset{\sim}{v} = (2lt + m)\underset{\sim}{j}$ $\underset{\sim}{a} = 2l\underset{\sim}{j}$ The origin is at the top of the cliff and there is positive motion up.

THINK	WRITE/DRAW
2 Stone A: Substitute the given information to find $\underset{\sim}{r}$.	Stone A: At $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{j}$ and $\underset{\sim}{r} = [l(0)^2 + m(0) + n]\underset{\sim}{j}$ So $0 = 0 + 0 + n$ therefore $n = 0$ $\underset{\sim}{r} = (lt^2 + mt)\underset{\sim}{j}$
3 Use information about $\underset{\sim}{v}$ to find m.	At $t = 0$, $\underset{\sim}{v} = 30\underset{\sim}{j}$ and $\underset{\sim}{v} = [2l(0) + m]\underset{\sim}{j}$ So $30 = 0 + m$ $m = 30$
4 Rewrite $\underset{\sim}{r}$.	$\underset{\sim}{r} = (lt^2 + 30t)\underset{\sim}{j}$
5 Use information about a to find l.	At $t = 0$, $\underset{\sim}{a} = -9.8\underset{\sim}{j}$ and $\underset{\sim}{a} = 2l\underset{\sim}{j}$ So $-9.8 = 2l$ $l = -4.9$ $\underset{\sim}{r} = (-4.9t^2 + 30t)\underset{\sim}{j}$
6 For stone B: Repeat the method used for stone A. For many of your problems $l = -4.9$.	Stone B: At $t = 0$, $a = -9.8$ and $l = -4.9$ At $t = 0$, $\underset{\sim}{v} = 0\underset{\sim}{j}$ and $\underset{\sim}{v} = [2l(0) + m]\underset{\sim}{j}$ therefore $m = 0$ At $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{j}$ and $\underset{\sim}{r} = [l(0)^2 + m(0) + n]\underset{\sim}{j}$ therefore $n = 0$
7 Rewrite the equation for $\underset{\sim}{r}$ using all current information.	$\underset{\sim}{r} = -4.9t^2\underset{\sim}{j}$
8 Find where the two stones meet: when $\underset{\sim}{r}_A = \underset{\sim}{r}_B$; but $t_B = t_A - 4$. Express all information in terms of A.	The two stones meet where $\underset{\sim}{r}_A = \underset{\sim}{r}_B$ $(-4.9t_A^2 + 30t_A)\underset{\sim}{j} = -4.9t_B^2\underset{\sim}{j}$ $-4.9t_A^2 + 30t_A = -4.9(t_A - 4)^2$ $= -4.9(t_A^2 - 8t_A + 16)$ $0 = 39.2t_A - 30t_A - 78.4$ $= 9.2t_A - 78.4$ $t_A = 8.52$ s
9 Find the displacement of B at $t_A = 8.52$ s and $t_B = t_A - 4 = 4.52$ s. The information for stone B is easier to use to find the displacement because A has had upwards and downwards motion (both unknown) whereas B has had downwards motion only.	$r_B = -4.9\ (4.52)^2$ $= -100.1$ m The stones meet about 100 m below the top of the cliff 8.52 seconds after stone A is thrown.

In general this procedure can be shortened. If an object starts at the origin, $n = 0$ and m has the value of the initial velocity (upwards is positive, downwards is negative). If the body is influenced by gravity then $l = -4.9$.

If an object is initially at rest then $m = 0$, but if an object is part of a system that is itself moving, and the object is released from that moving system, then the initial velocity (m) of the object will be the velocity of the whole system at the instant prior to release.

remember

1. If the origin is the starting point, then $n = 0$.
2. If the body is initially at rest, then $m = 0$.
3. In vertical motion, displacement up is positive; displacement down is negative.
4. Velocity up is positive; velocity down is negative.
5. Acceleration up is positive; acceleration down is negative.

EXERCISE 11C Motion under constant acceleration

1 A particle moving from rest with constant acceleration reaches a speed of 16 m/s in 4 seconds. Find:

a the acceleration

b the distance travelled.

2 An object travelling at 8 m/s accelerates uniformly over a distance of 20 metres until it reaches a speed of 18 m/s. Find:

a the acceleration

b the time taken.

3 **a** A racing car accelerates constantly from rest and covers a distance of 400 metres in 10 seconds. Find its velocity at the end of the 400 metres.

b Another car travels the 400 metres with a constant acceleration of 10 m/s^2. Find its time for the 400 metres.

4 A train travelling at a constant speed decelerates uniformly for 30 seconds over a distance of 270 metres, coming to a stop. Find:

a the initial speed

b the acceleration.

5 A parachutist free-falls from an aircraft for 6 seconds. If the acceleration due to gravity is 9.8 m/s^2, find:

a the speed of the parachutist after 6 seconds

b the distance travelled after 6 seconds.

6 A ball is thrown up from the ground with an initial velocity of 19.6 m/s. The acceleration due to gravity is 9.8 m/s^2 downwards, that is, –9.8 m/s^2. Find:

a the maximum height attained by the ball

b the total time taken for the ball to return to the ground.

WORKED Example 8

7 A stone is dropped from a bridge which is 39.2 metres above a river.

a How long does it take the stone to reach the water?

b What is its speed on impact?

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8 A ball is dropped from a tower and reaches the ground in 4 seconds. Find:

a the height of the tower

b the velocity of the ball when it hits the ground.

9 A particle is projected vertically up from the top of a building that is 50 metres above the ground. If the initial speed of the particle is 28 m/s, find:

a the maximum height, above the ground, that it reaches

b total time taken to reach the ground

c the speed of the particle when it reaches the ground.

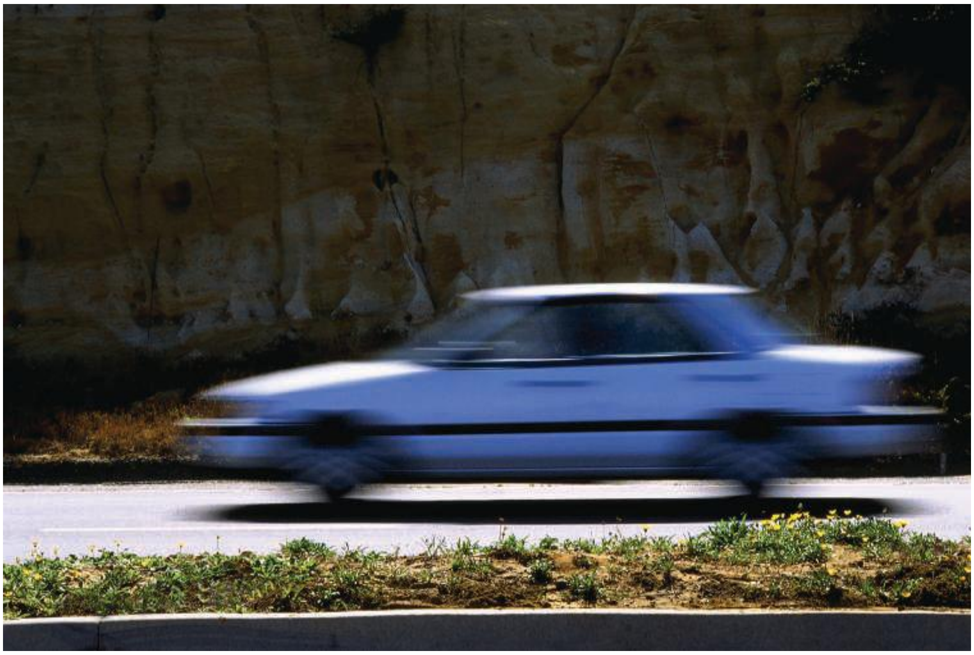
10 A train travels a distance of 1800 metres in 90 seconds while accelerating uniformly from rest. What is its velocity at the end of 500 metres?

11 A car accelerates uniformly from rest, increasing its speed from 5 m/s to 25 m/s in 10 seconds. Find:

a the acceleration

b the distance travelled, from rest, in 12 seconds

c the time taken to increase its speed from 15 m/s to 30 m/s.



12 A sprinter accelerates uniformly to his top speed after running 30 metres of a 100-metre race. He maintains this speed for the remainder of the race and takes 10.4 seconds to complete it. Find:

a the top speed of the athlete

b the time taken to reach the top speed.

13 A bus is travelling at 16 m/s when the brakes are applied, reducing the speed to 6 m/s in 2 seconds. Assuming the retardation is constant, find:

a the acceleration

b the distance travelled 2 seconds after the brakes are applied

c how long after applying the brakes the tram comes to a stop

d the braking distance of the bus.

14 A car moving from rest with uniform acceleration takes 12 seconds to travel 144 metres. What is its speed after 6 seconds?

15 A bus travels 60 metres in 10 seconds and the next 60 metres in 15 seconds. If the acceleration is constant, find:

a how much further it will travel before coming to rest

b how many more seconds it takes before coming to rest.

16 A juggler throws balls vertically into the air so that they rise to a height of 4.4 metres above the ground. He fails to catch one and it hits the ground with a speed of 1.155 times that of its initial speed. Find:

a the speed of projection of the ball

b the height from which the ball is thrown

c the total time the ball is in the air.

17 An object is projected vertically up from a 14 metre tower and reaches the ground 4 seconds later.

a What is the projection speed of the object?

b What is the maximum height above the ground that is attained by the object?

18 a An object is dropped from the top of a building 39.2 m high. Calculate:

- **i** its velocity when it is halfway to the ground
- **ii** its velocity on striking the ground
- **iii** the time taken to each the ground.

b Repeat the above questions for the case when the object is thrown to the ground with a velocity of 4.9 m/s.

19 **a** A stone rolls off the top of a cliff and is found to take 4 seconds to reach the sea below. What is the height of the cliff?

b What is the difference in time to reach the bottom between part **a** and if the stone were launched vertically upwards from this cliff with a velocity of 20 m/s?

20 A vertical slit 1.5 m long is positioned in a stone wall 9.8 m below the top of the wall. A small object is dropped from the top of the wall so that it falls in line with the slit. For what length of time is the falling object visible through the slit?

WORKED Example 9

21 A body is dropped from the top of a 100 m high tower at the same time as a body is launched vertically upwards from the bottom of the tower with a velocity of 25 m/s. Find when and where the two bodies are at the same height above the ground.

22 A body is rising with a velocity of 20 m/s and it releases a small particle when it is 50 m above the ground. How long will it take for the small particle to reach the ground?

23 A cage is descending into a well at a constant velocity of 2 m/s when an object falls through the wire of the cage. If the object reaches the water at the bottom of the well 10 seconds before the cage find the height above the water level at which the object fell out of the cage.

24 A fireworks rocket is fired vertically upwards with uniform acceleration of 19.6 m/s^2. After 2 seconds a small particle is released from the rocket. How long after release will the particle fall to the ground?

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summary

Displacement, velocity and acceleration

- Displacement gives the position of a particle relative to some reference point (usually the origin).
- Instantaneous velocity is the rate of change of displacement with respect to time and is represented by the derivative of the displacement variable.
- Average velocity during a time interval is the change in velocity across that interval.
- Speed is the magnitude of velocity just as distance is the magnitude of displacement.
- Instantaneous acceleration is the rate of change of velocity with respect to time and is represented by the derivative of the velocity variable.

Vector relationships

- If the position vector $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ represents the displacement of P at time t, then $\frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t}$ (or $\dot{\underset{\sim}{r}}(t)$) represents the instantaneous velocity and $\frac{\mathrm{d}^2\underset{\sim}{r}}{\mathrm{d}t^2}$ $\left(\text{or } \ddot{\underset{\sim}{r}}(t) \text{ or } \frac{\mathrm{d}\underset{\sim}{v}}{\mathrm{d}t}\right)$ represents the instantaneous acceleration at time t.
- Finding the velocity given the acceleration: $\underset{\sim}{a} = \frac{\mathrm{d}\underset{\sim}{v}}{\mathrm{d}t} \rightarrow \underset{\sim}{v} = \int \underset{\sim}{a}\ \mathrm{d}t$.
- Finding the displacement given the velocity: $\underset{\sim}{v} = \frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t} \rightarrow \underset{\sim}{r} = \int \underset{\sim}{v}\ \mathrm{d}t$.
  Remember that the constant of integration is always a vector constant.
- For the magnitude of a vector $x\underset{\sim}{i} + y\underset{\sim}{j}$: $\left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$.
- The dot product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given as $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\ |\underset{\sim}{b}|\ \cos\theta$ where θ is the angle included by $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Projectile motion

- The period of time for which the projectile is moving under the influence of gravity alone is called the *time of flight* of the projectile.
- The *range* is the horizontal displacement of the projectile from its launch point.

Motion under constant acceleration

- $\underset{\sim}{r} = (lt^2 + mt + n)\underset{\sim}{j}$, $\underset{\sim}{v} = (2lt + m)\underset{\sim}{j}$, $\underset{\sim}{a} = 2l\underset{\sim}{j}$
- If the origin is the starting point, then $n = 0$.
- If the body is initially at rest, then $m = 0$.
- In vertical motion, displacement up is positive; displacement down is negative.
- Velocity up is positive; velocity down is negative.
- Acceleration up is positive; acceleration down is negative.

CHAPTER review

Questions **1** to **5** refer to a particle that has a displacement $\underset{\sim}{r}(t)$ at time t seconds, where $\underset{\sim}{r}(t) = 4t\underset{\sim}{i} + (8t - t^2)\underset{\sim}{j}$ metres.

1 **multiple choice** 11A

The initial speed of the particle in m/s is:

A 4 **B** $4\sqrt{5}$ **C** 0 **D** $2\sqrt{13}$ **E** $2\sqrt{10}$

2 **multiple choice** 11A

The vector which represents the average displacement (in m) in the first two seconds is:

A $4\underset{\sim}{i} + 6\underset{\sim}{j}$ **B** $2\underset{\sim}{i} + 3.5\underset{\sim}{j}$ **C** $4\underset{\sim}{i} + 7\underset{\sim}{j}$ **D** $8\underset{\sim}{i} + 12\underset{\sim}{j}$ **E** $4\underset{\sim}{i} + 4\underset{\sim}{j}$

3 **multiple choice** 11A

The particle reaches its greatest height after how many seconds?

A 0 **B** 2 **C** 3 **D** 4 **E** 5

4 **multiple choice** 11A

The acceleration of the particle in m/s^2 is given by:

A $2\underset{\sim}{i}$ **B** $-2\underset{\sim}{j}$ **C** $2\underset{\sim}{i} - 2\underset{\sim}{j}$ **D** $0\underset{\sim}{i} + 0\underset{\sim}{j}$ **E** $2\underset{\sim}{j}$

5 **multiple choice** 11A

The Cartesian equation of the trajectory of the particle is:

A $y = 4x$ **B** $y = 8x - x^2$ **C** $y = \frac{x}{16}(32 - x)$ **D** $y = \frac{x}{8}(2 - x)$ **E** $y = x^2 - 16x$

6 Relative to the origin, O, the displacement of a particle at time t seconds is given by $\underset{\sim}{r}(t) = 6t\underset{\sim}{i} + 3t(4 - t)\underset{\sim}{j}$ metres. 11A

a Find expressions for the velocity and acceleration vectors at time t seconds.

b Determine the initial displacement and speed of the particle.

c When $t = 2$ seconds, calculate:

- **i** the distance of the particle from the origin
- **ii** the velocity (both magnitude and direction) of the particle
- **iii** the measure of the angle between the velocity and displacement vectors.

7 At time t, a particle A has a displacement given by $\underset{\sim}{r}_A(t) = t\underset{\sim}{i} + (4t - t^2)\underset{\sim}{j}$, while a second particle B has a displacement given by $\underset{\sim}{r}_B(t) = t\underset{\sim}{i} + (t - 10)\underset{\sim}{j}$. 11A

a If these particles collide, determine when they collide.

b What are the coordinates of the impact point?

c Find the Cartesian equations of the trajectories of the particles.

d Use a graphics calculator to verify the coordinates of the impact point.

8 At time t seconds, a particle has a position vector given by the expression $\underset{\sim}{r}(t) = (t + 1)\underset{\sim}{i} + (25 - t^2)\underset{\sim}{j}$ metres. 11A

a Use a graphics calculator to plot the trajectory of this particle across the interval $0 \le t \le 5$ seconds.

b Repeat part **a** using an Excel spreadsheet.

c Determine the equation of this trajectory in the form $y = f(x)$.

Questions **9** to **11** refer to a projectile that has velocity $\underset{\sim}{v}(t)$ at time t seconds, where $\underset{\sim}{v}(t) = 6\underset{\sim}{i} + (10 - 2t)\underset{\sim}{j}$ m/s. The initial displacement of this projectile is given by $\underset{\sim}{r}(0) = 4\underset{\sim}{j}$ m.

11B

9 **multiple choice**

The angle of projection of this particle, correct to the nearest degree, is:

A 39° **B** 49° **C** 51° **D** 59° **E** 61°

11B

10 **multiple choice**

The range (in m) of the projectile through the foot of the launch platform is nearest in value to:

A 45 **B** 50 **C** 60 **D** 70 **E** 75

11B

11 **multiple choice**

The greatest height (in m) reached by the particle is closest in value to:

A 29 **B** 32 **C** 35 **D** 39 **E** 95

11B

12 A particle moves so that its acceleration at time t is given by $\underset{\sim}{a} = \underset{\sim}{i} + (3 - 2t)\underset{\sim}{j}$. Find vector expressions for the velocity and displacement of the particle given that $\underset{\sim}{v} = 0$ and $\underset{\sim}{r} = 0$ when $t = 0$.

11B

13 A projectile is fired from the foot of a cliff and its displacement (in metres) at time t seconds is given by $\underset{\sim}{r}(t) = 6t\underset{\sim}{i} + (45t - \frac{1}{2}gt^2)\underset{\sim}{j}$.
Use $g = 10$ m/s^2 in your calculations.

- **a** Determine an expression for the velocity vector of the projectile at time t seconds.
- **b** Calculate the launch speed and angle of projection of the projectile.
- **c** Find the greatest height reached and the time of flight of the projectile.
- **d** Calculate the impact velocity of the projectile when it returns to the ground in a horizontal plane through the origin.

11B

14 A particle is fired from the top of a 60 m high building with a velocity of 24 m/s inclined at an angle of 30° to the horizontal.

- **a** Show that the velocity vector of the projectile at time t seconds is given by $\underset{\sim}{v}(t) = 12\sqrt{3}\,\underset{\sim}{i} + (12 - gt)\underset{\sim}{j}$ m/s.
- **b** Develop a vector expression for the displacement, $\underset{\sim}{r}(t)$ metres, of the particle at time t seconds.
- **c** Determine if this projectile will clear a 14 m high tower located 94 m from the launch point and in the same horizontal plane.

11B

15 A missile is fired from a point on level ground with a velocity V m/s and with an angle of elevation of θ to the horizontal. The target is positioned on top of a cliff 30 m above the ground in the same horizontal plane as the point of projection of the missile, and with a range of 60 m from it.

- **a** If g is the gravitational acceleration of the missile, show that the motion of this missile satisfies $V^2 \cos\theta\,(2\sin\theta - \cos\theta) = 60g$.
- **b** Use a graphics calculator to find those values of θ in the domain $0 < \theta < 90°$ for which the expression $\cos\theta\,(2\sin\theta - \cos\theta)$ is positive. State the greatest possible value this expression has, and the value of θ that produces it. Provide supporting argument for your solutions.
- **c** Extend from your answer to part **b** above to find the least value of V for which the missile can reach the target. (Use $g = 9.8$ m/s^2.)
- **d** Generate a spreadsheet to validate the solution obtained using a graphics calculator.

16 A stone is projected vertically up from the ground with an initial velocity of 24.5 m/s. Taking the acceleration due to gravity to be –9.8 m/s^2, find:

a the maximum height reached by the stone

b the times at which its height is 20 metres above the ground.

11C

17 **multiple choice**

An object is dropped from the top of a 200 metre high building. If the acceleration due to gravity is 9.8 m/s^2, what will be the height of the object after 5 seconds?

A 151 m **B** 49 m **C** 122.5 m **D** 20 m **E** 77.5 m

11C

18 **multiple choice**

A particle initially moving at 6 m/s is subject to a constant retardation of 2 m/s^2. The distance, in metres, travelled before coming to rest is:

A 27 **B** 8 **C** 9 **D** 10 **E** 12

11C

19 **multiple choice**

A bus travels 500 metres in 25 seconds when accelerating uniformly from rest. The acceleration, in m/s^2, is:

A 0.4 **B** 1.6 **C** 1.25 **D** –1.2 **E** 0.625

11C

20 A parachutist drops from an aeroplane so that the constant acceleration during free fall due to gravity and air resistance is 8 m/s^2. The parachute is released after 6 seconds, uniformly retarding the parachutist in 28 seconds to a constant speed of 2.5 m/s. This speed is maintained until the parachutist reaches the ground which is 1101 metres below the point of release.

a How long is the parachutist in the air?

b After how long has the parachutist fallen half the distance (answer to the nearest tenth of a second)?

11C

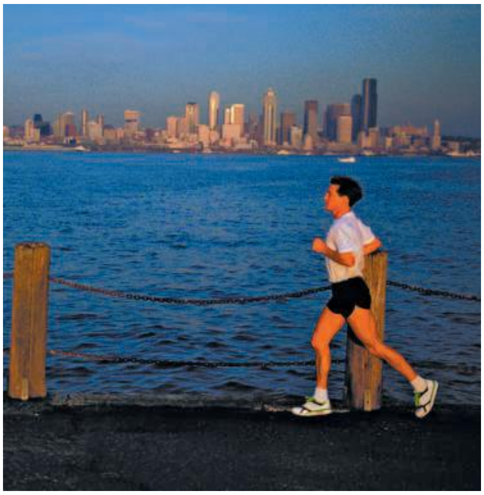
21 Jogger A is running in a straight line at a constant speed of 4 m/s when passing jogger B who has stopped to tie a lace. Jogger B heads off in the same direction as jogger A 6 seconds later, accelerating uniformly at 2 m/s^2 until reaching a speed of 5 m/s.

a Sketch a velocity–time graph showing the motion of both joggers.

b How long is it after jogger A first passes jogger B until B catches up to A?

c How far has jogger B travelled to catch jogger A?

d How far ahead will jogger B be after jogger B has travelled 225 metres?



11C

Modelling and problem solving

1 Car A is 600 metres from the centre of the intersection when it starts from rest and accelerates uniformly at 4 m/s^2, reaching a speed of 24 m/s which it maintains. At the instant car A takes off, car B is 780 metres from the centre of the intersection and travelling at a constant speed of 28 m/s. When car B is 52 metres from the centre of the intersection it decelerates uniformly at 5 m/s^2.

a Which car gets to the centre of the intersection first?

b How far past the centre of the intersection is the first car when the second car reaches it?

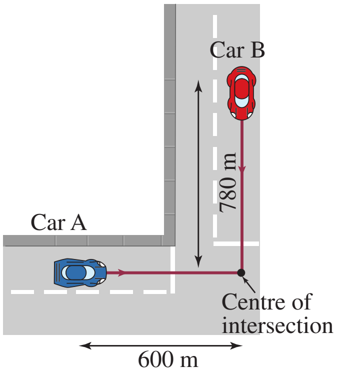
c If all other conditions remain the same, what constant acceleration would:

- **i** the second car need to have for a collision to occur?
- **ii** the first car need to have for a collision to occur?

d If all other conditions remain the same, at what constant speed would:

- **i** the first car need to travel for a collision to occur? (Use a graphics calculator or a numerical method to assist.)
- **ii** the second car need to travel for a collision to occur?

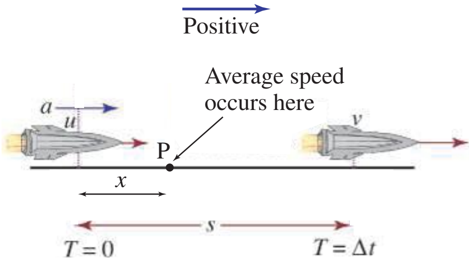
(Give answers correct to 2 decimal places where appropriate.)



2 Martin is trying out his new spaceship by challenging a local alien to a race. He places his hand on the throttle and his spaceship starts to accelerate at a constant rate a for a time Δt.

During that time Δt, he travels a distance s through space. Martin's spaceship has an initial speed u at the beginning of the time interval and a final speed v after a time Δt.

The local space police are on to Martin and want to photograph him just as he breaks the speed limit. However, they need to set their camera up at the precise point, P, where Martin is travelling at his average speed. The police know *when* the average speed will occur; it will occur at a time $T = \frac{\Delta t}{2}$. They want to find out *where* the average speed will occur. Let x be the distance travelled when the spaceship reaches its average velocity for the time interval Δt. All distances and times are in metres (m) and seconds (s). It is clear that x is a fraction of s, the total distance covered during the time interval Δt.



a The initial velocity (u) is 0, the acceleration (a) is 1.0 and the time interval (Δt) is 10. Find the value of x and hence state the value of the fraction $\frac{x}{s}$.

b Find equations for x and s in terms of the acceleration, a; the initial speed, u; and the time interval Δt.

c Under what conditions will $x = \frac{s}{2}$?

3 When a projectile is fired from a body out into space, it decelerates due to the gravitational pull of the body. The general equation for this deceleration is $a = -\frac{k}{r^2}$ where r is the distance from the centre of the body and k is a positive constant.

a On the surface of the moon ($r = 1760$ km), the deceleration is $a = 1.6$ m/s^2. Find the value of k.

b If a body is launched from the surface of the moon with a velocity of 500 m/s, find the velocity of the body at a distance r from the centre of the moon; that is, find $v(r)$.

c Find the distance from the centre of the moon when the body is momentarily stationary.

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Test Yourself
Chapter 11

eBook*plus* ACTIVITIES

11A Displacement, velocity and acceleration

Tutorial

- int-0411: Watch a tutorial on how to find the velocity vector, the speed at a given time, and the average speed over a period of time, given the position vector. (*page 510*)

11B Projectile motion

Interactivity

- int-0980: Use the interactivity to explore projectile motion. (*page 520*)

Tutorials

- int-0415: Watch a tutorial on make calculations about the movement of a golf ball. (*page 525*)

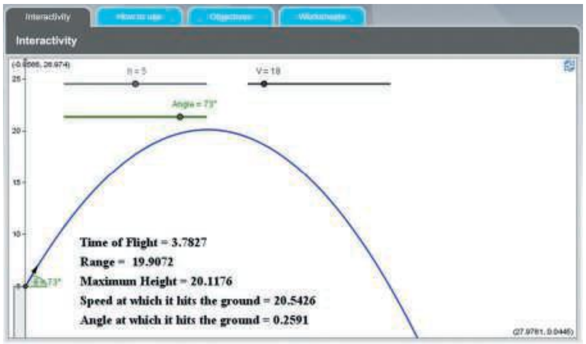
Digital doc

- WorkSHEET 11.1: Determine displacement, velocity and acceleration of bodies in motion. (*page 535*)

11C Motion under constant acceleration

Digital doc

- SkillSHEET 11.1: Practise calculations with vectors. (*page 541*)
- WorkSHEET 11.2: Calculate time, distance and velocity for the motion of objects due to gravity. (*page 543*)



Time of Flight = 3.7827
Range = 19.9072
Maximum Height = 20.1176
Speed at which it hits the ground = 20.5426
Angle at which it hits the ground = 0.2591

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 549*)

To access eBookPLUS activities, log on to

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12

Dynamics II

syllabus reference

Option topic:
Dynamics

In this chapter

12A Position, velocity and acceleration
12B Motion under constant acceleration
12C Applying differential equations to rectilinear motion
12D Vector calculus
12E Newton's Second Law of Motion
12F Applications of Newton's Laws of Motion
12G Variable forces

Position, velocity and acceleration

eBookplus

Interactivity:
Motion graphs (Kinematics)
int-0267

Differentiation with position, velocity and acceleration

Two of the bodies in our solar system, Neptune and Pluto, were discovered more by mathematicians using pencil and paper than by astronomers using a telescope. Earlier, when Uranus was discovered, mathematicians began to calculate its trajectory and predict its future position. It turned out that the predictions failed to agree with observation. They concluded that another planet was exerting a gravitational pull on Uranus and that this caused the discrepancy between theory and observation. Independently, German and English mathematicians predicted the position and size of the new planet and when astronomers looked at the predicted position, they discovered the new planet, Neptune. Pluto was found in a similar way.

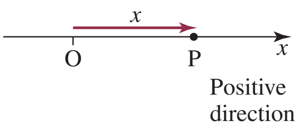
As we continue our study of Dynamics, the area of mathematics that deals with forces and motion, you would appreciate that the arrangement of planets exerting forces on one another can become very complex. In our course of study we'll be investigating some rules that determine how bodies move.

Kinematics

Kinematics is the name given to the study of the motion of bodies, objects or particles. This chapter will deal with motion in a straight line only — rectilinear motion — and all objects considered will be treated as points for mathematical convenience; that is, the objects do not rotate or change shape. First, some basic concepts are explained.

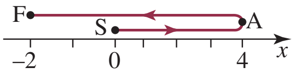
Position

The position of a particle moving in a straight line measures its location from a fixed point of reference, usually the origin O on the line. Positions to the right of O are usually taken as positive. The point P in this diagram has a position coordinate x.



Displacement

The displacement of a moving particle is its change in position relative to a fixed point. Displacement gives both the distance and direction that a particle is from a fixed point.



For example, a particle which moves from S to F via A is shown here on a displacement–time line (or position–time line).

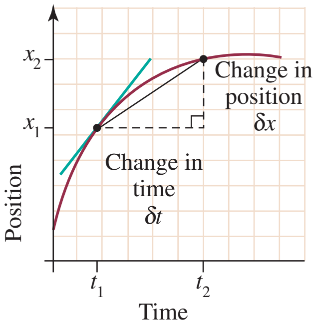
The distance travelled from S to F is 10 units. The displacement of F from S is −2.

Displacement and position have the same value if a body is initially at the origin, so the two terms are often used to mean the same thing. Hence, the same variable is used, namely x.

Note that the displacement of an object is not necessarily the same as the distance that it has travelled.

Velocity

The average velocity of a particle is the rate of change of its position with respect to time. This can be illustrated on a position–time graph. For this curve the position x at time t is shown. The curve $x(t)$ is shown at right.



$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}}$$

$$= \frac{x_2 - x_1}{t_2 - t_1}$$

$$= \frac{\delta x}{\delta t}$$

The instantaneous velocity, $v(t)$, at time t is defined as the limiting value of the average velocity as δt approaches zero.

That is, it is the gradient of a position–time graph

or $$v(t) = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} \quad \text{or} \quad v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$$

Units of velocity

The units of velocity are most commonly cm/s, m/s, or km/h.

Note: 1 m/s = 3.6 km/h (verify this).

Speed

Instantaneous speed is the magnitude of instantaneous velocity and is always positive.

$$\text{Average speed} = \frac{\text{Distance travelled}}{\text{Time taken}}$$

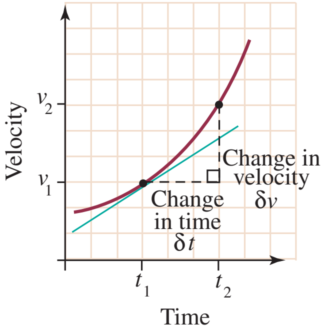
Remember that distance travelled is not necessarily the same as displacement.

Acceleration

The average acceleration of a particle a during a time interval $t_2 - t_1$ is the rate of change of its velocity with respect to time.

That is, average acceleration $$= \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{\delta v}{\delta t}$$



The instantaneous acceleration, a, at time t is the limiting value of the average acceleration as t approaches 0.

That is, it is the gradient of a velocity–time graph; or

$$a = \lim_{\delta t \to 0} \frac{\delta v}{\delta t}$$

$$= \frac{\mathrm{d}v}{\mathrm{d}t}$$

Units of acceleration

The units of acceleration match their corresponding velocities and are usually expressed as cm/s^2 or m/s^2.

Note: The acceleration due to gravity is $g = 9.8$ m/s^2 in a downward direction. For objects travelling through the air, air resistance is ignored unless it is specified in a given problem.

Symbols

Throughout this chapter the following symbols are used for kinematic variables.

Displacement: x

Velocity: v or $\dfrac{dx}{dt}$ or $\dot{x}$

Acceleration: a or $\dfrac{dv}{dt}$ or $\dfrac{d^2x}{dt^2}$ or $\ddot{x}$

WORKED Example 1

The position of a particle moving in a straight line is given by:

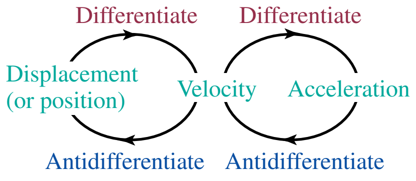
$$x(t) = 2t^3 + t\log_e t - 4,\ t > 0$$

Find:

a **the velocity at any time t** **b** **the acceleration at any time t.**

THINK	**WRITE**
a ① Differentiate x with respect to t, using the product rule on $t\log_e t$.	**a** $x = 2t^3 + t\log_e t - 4$ $v = \dfrac{dx}{dt}$ $= 6t^2 + \log_e t + t\left(\dfrac{1}{t}\right)$
② Simplify.	$= 6t^2 + \log_e t + 1$
b Differentiate v with respect to t.	**b** $a = \dfrac{dv}{dt}$ $= 12t + \dfrac{1}{t}$

Using antidifferentiation



If the acceleration, a, of a particle is known in terms of time, t, then the velocity, $v(t)$, can be found by antidifferentiation. Thus,

$$v(t) = \int a\ dt + c_1$$

where c_1 is a constant due to integration; generally it is the initial velocity, $v(0)$.

Similarly, the position, $x(t)$, can be found by antidifferentiating velocity with respect to time:

$$x(t) = \int v\ dt + c_2$$

where c_2 is a constant due to integration; generally it is the initial position, $x(0)$.

The constant of antidifferentiation can be determined if initial conditions are given.

Velocity: $v(t) = \int a\ dt + c_1$
Displacement: $x(t) = \int v\ dt + c_2$

WORKED Example 2

The velocity of an object which is initially 3 metres left of O is given by:

$$v(t) = 3t^2 - 2t - 5 \text{ m/s}$$

Find:

- **a** **the displacement from O at any time *t***
- **b** **the acceleration at any time *t***
- **c** **when the object is at rest**
- **d** **the distance travelled in the first second**
- **e** **the acceleration when the velocity is 0.**

THINK	WRITE
a ① Antidifferentiate with respect to t, to find the displacement, x.	**a** $v(t) = 3t^2 - 2t - 5$ $x(t) = \int (3t^2 - 2t - 5)\ \mathrm{d}t$ $= t^3 - t^2 - 5t + c$
② To find c, substitute the initial condition $t = 0$ and $x = -3$ into the equation.	When $t = 0$, $x = -3$. $-3 = 0^3 - 0^2 - 5(0) + c$
③ Solve the equation for c.	$-3 = c$
④ State the rule for x.	$x(t) = t^3 - t^2 - 5t - 3$
b Find the acceleration, $\dot{v}(t)$, by differentiation. That is, $\dot{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}[v(t)]$.	**b** $\dot{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(3t^2 - 2t - 5)$ $\dot{v}(t) = 6t - 2$
c ① Object is at rest when $v(t) = 0$.	**c** $v(t) = 3t^2 - 2t - 5 = 0$
② Factorise the LHS.	$(3t - 5)(t + 1) = 0$
③ Solve for t.	$t = \frac{5}{3}$ or $t = -1$, but the domain is $t \geq 0$, so only the first solution is valid.
④ State the answer (t cannot be negative).	The object is at rest after $1\frac{2}{3}$ seconds.
⑤ Verify using the graph of the velocity function on a graphics calculator.	
d ① The velocity is always negative during the first second and so the distance travelled will equal the magnitude of the displacement during the first second. The object is always moving to the left during the first second.	**d** At $t = 0$, $v = -5$ m/s and at $t = 1$, $v = -4$ m/s. Also, $v \neq 0$ during the first second. Thus, the distance travelled will equal the magnitude of the displacement during the first second.
② The displacement is given by $x(t) = t^3 - t^2 - 5t - 3$. The distance is thus $x(1) - x(0)$.	$x(t) = t^3 - t^2 - 5t - 3$ Distance travelled $= \lvert x(1) - x(0) \rvert$ $= \lvert (1^3 - 1^2 - 5 - 3) - (-3) \rvert$ $= \lvert -5 \rvert$ $= 5$ The object moves 5 m to the left during the first second.

Continued over page →

THINK	WRITE
e ① Find when the velocity is 0 by solving $v(t) = 0$ as in part **c**.	**e** $v(t) = 3t^2 - 2t - 5 = 0$ $(3t - 5)(t + 1) = 0$
② Accept only positive solutions.	$t = \frac{5}{3}$ or $t = -1$, but the domain is $t \geq 0$, so only $t = \frac{5}{3}$ is valid.
③ Substitute the solution into the equation for acceleration.	$a = \frac{dv}{dt}$ $v(t) = 3t^2 - 2t - 5$ $a(t) = 6t - 2$ When $t = \frac{5}{3}$, $a = 6 \cdot \frac{5}{3} - 2$.
④ Evaluate for a.	$a = 10 - 2$ $= 8$
⑤ State the answer.	When the velocity is 0, the acceleration is 8 m/s^2.

WORKED Example 3

The acceleration of a particle moving in a straight line is given by:

$$\frac{dv}{dt} = 5e^t - 6t + 4 \text{ cm/s}^2,$$ **where v is the velocity at any time.**

If the particle starts at the origin with a velocity of –1 cm/s, find:

a **the velocity at any time t**

b **the displacement $x(t)$ from the origin at any time t**

c **the displacement from the origin after 1 second.**

THINK	WRITE
a ① Antidifferentiate the acceleration, $\frac{dv}{dt}$, to find v.	**a** $\frac{dv}{dt} = 5e^t - 6t + 4$ $v(t) = \int (5e^t - 6t + 4)\, dt + c$ $= 5e^t - 3t^2 + 4t + c$
② Substitute the initial condition $v = -1$ when $t = 0$.	When $t = 0$, $v = -1$. $-1 = 5e^0 - 3(0)^2 + 4(0) + c$
③ Solve for the constant of antidifferentiation.	$-1 = 5 + c$ $c = -6$
④ State the velocity.	Therefore, $v(t) = 5e^t - 3t^2 + 4t - 6$.
b ① Antidifferentiate v with respect to t.	**b** $x(t) = \int (5e^t - 3t^2 + 4t - 6)\, dt$ $= 5e^t - t^3 + 2t^2 - 6t + c$
② Substitute the initial condition $x = 0$ when $t = 0$.	When $t = 0$, $x = 0$. $0 = 5e^0 - (0)^3 + 2(0)^2 - 6(0) + c$
③ Solve for the constant of antidifferentiation.	$0 = 5 + c$ $c = -5$
④ State the displacement.	Therefore, $x(t) = 5e^t - t^3 + 2t^2 - 6t - 5$.

THINK	WRITE
c ① Substitute $t = 1$ into the rule for x.	**c** $x(1) = 5e^1 - (1)^3 + 2(1)^2 - 6(1) - 5$
② Evaluate.	$= 5e - 1 + 2 - 6 - 5$ $= 5e - 10$
③ Give an approximate answer correct to a useful number of decimal places.	Therefore, the displacement from O after 1 second is approximately 3.59 cm.
④ Verify, using a graph of $x(t)$ on a graphics calculator.	

Graphics Calculator tip!

Graphing a *v–t* graph and the corresponding *x–t* graph

Plotting the *x–t* graph

To plot the x–t graph given $v(t) = 2t - t^2$, $0 \le t \le 4$, and the corresponding x–t graph for which $x = 2$ at $t = 0$

For the CASIO *fx*-9860G AU

The Casio graphics calculator does not have the capability to graph an integral function. To do this example we need to integrate the function manually, and evaluate the constant, given that $x = 2$ when $t = 0$.

1. Press:
   - MENU
   - 5 (GRAPH)

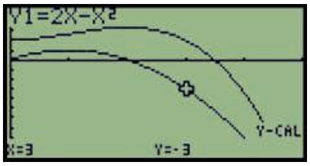
   Complete the entry lines as:
   $Y1 = 2x - x^2$
   $Y2 = x^2 - x^3 \mid 3 + 2$
   Press EXE after each line.
   To set V-Window, press:
   - SHIFT
   - F3 (V-Window)

   Enter the settings as shown and press EXE.
   To draw the graph press F6 (DRAW).

```
View Window
Xmin  :0
 max  :5
 scale:1
 dot  :0.03968253
Ymin  :-10
 max  :5
INIT TRIG STD STO RCL
```

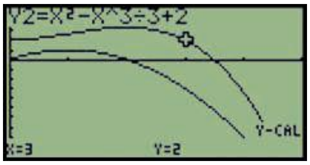
2. To determine the velocity at $x = 3$, press:
   - SHIFT
   - F5 (G-Solv)
   - F6
   - F1 (Y-CAL)

   Using the arrow keys, choose Y1, and then press EXE.
   Enter the x-value as 3 and the corresponding y-value will be displayed.
   When $t = 3$, the velocity is −3.



3. To determine the displacement at $x = 3$, press:
   - SHIFT
   - F5 (G-Solv)
   - F6
   - F1 (Y-CAL)

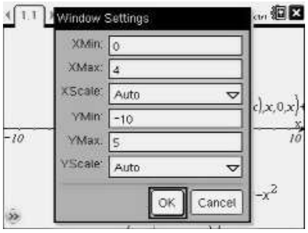
   Using the arrow keys, choose Y2, and then press EXE.
   Enter the x-value as 3 and the corresponding y-value will be displayed.
   When $t = 3$ the position will be 2.


Y2=X²-X^3÷3+2
X=3 Y=2

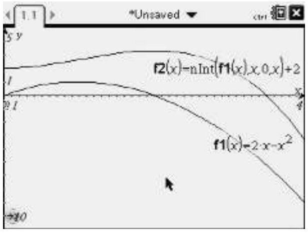
For the TI-Nspire CAS

1. On a Graphs page complete the entry line as:
   $f1(x) = 2 \cdot x - x^2$
   then press ENTER enter.
2. To graph the integral of $f1(x)$, complete the entry line as:
   nInt(f1(x),x,0,x)+2
   Then press ENTER enter.
   Note: The position is 2 when $t = 0$, so 2 is the constant of integration.
3. To set the specified domain, press:
   - MENU menu
   - 4: Window/Zoom 4
   - 1: Window Settings 1

   Enter the settings as shown and select OK.

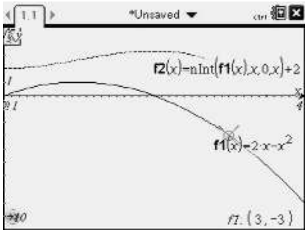

Window Settings
XMin: 0
XMax: 4
XScale: Auto
YMin: -10
YMax: 5
YScale: Auto

4. The graphs shown are $v(t)$ and $x(t)$.



5. To determine the velocity at $t = 3$, press:
   - MENU menu
   - 5: Trace 5
   - 1: Graph Trace 1

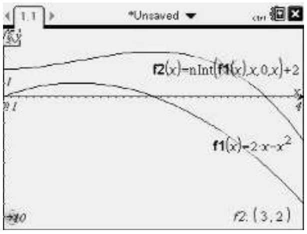
   Move the cursor to $f1(x)$, press 3 as the desired x-value and then press ENTER enter. The point (3, –3) is displayed.
   The velocity is –3 when $t = 3$.



6. To determine the position at $t = 3$, press:
   - MENU (menu)
   - 5: Trace (5)
   - 1: Graph Trace (1)

   The flashing cursor will appear on $f1(x)$. Press the Up ▲ arrow and the cursor will move to $f2(x)$ at the point (3, 2).

   The position is 2 when $t = 3$.



remember

1. Position gives the location of a particle relative to a reference point (usually the origin). The variable used is x.
2. Displacement is change in position δx or s. Displacement and position have the same value if a body is initially at the origin and so the two terms are often used to mean the same thing. Hence, the same variable is used, namely x.
3. Instantaneous velocity, v or $\frac{\mathrm{d}x}{\mathrm{d}t}$ or $\dot{x}$, is the time rate of change of position or the rate of displacement.
4. Average velocity during time interval $\delta t = \frac{\delta x}{\delta t} = \frac{s}{\delta t} = \frac{\text{displacement}}{\text{time}}$.
5. Speed $= \frac{\text{Distance travelled}}{\text{Time taken}}$
6. Instantaneous acceleration, a or $\ddot{x}$ or $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ or $\frac{\mathrm{d}v}{\mathrm{d}t}$ is the time rate of change of velocity.
7. $v(t) = \int a\,\mathrm{d}t + c$. The instantaneous velocity can be found by antidifferentiating the instantaneous acceleration. The value of c is found by knowing the velocity at a specific time, usually at time $t = 0$.
8. $x(t) = \int v\,\mathrm{d}t + c$. The displacement and hence the position can be found by antidifferentiating the instantaneous velocity. The value for c is found by knowing the position at a specific time, usually at time $t = 0$.

EXERCISE 12A Position, velocity and acceleration

A graphics calculator may be used to assist in calculations, or verify answers in the following exercise where appropriate.

WORKED Example 1

1 For each of the following, find **i** the velocity and **ii** the acceleration at any time t.

a $x = t^3 + 2t^2 - 5t$ **b** $x = 4t - \sqrt{t+2}$ **c** $x = e^t + 2e^{-t}$

d $x = t^2 - \log_e(t + 3)$ **e** $x = \text{Sin}^{-1}\frac{t}{3}$ **f** $x = t(5 - t)(t + 2)$

2 The displacement from O of a particle travelling in a straight line is:

$x = t^3 - 6t^2 + 11t - 6$, where x is in cm and t is in seconds.

a At what time does it pass through the origin?
b Find the velocity at any time.
c When is the velocity 0?
d Find the minimum velocity.
e Find the distance travelled in the first second.

3 An object travelling in a straight line has its displacement given by:

$$x = 2\cos(3t - 1) + 3$$

a Find the minimum and maximum displacement.
b Find when the velocity is first equal to 0.
c How long after it is first at rest is it next at rest?
d Find the acceleration in terms of the displacement.

WORKED Example 2

4 The velocity of an object which is initially 2 m right of O is given by:

$$v(t) = 6t^2 - 2t - 8 \text{ m/s}$$

Find:
a the displacement $x(t)$ from the origin at any time t
b the acceleration $\ddot{x}$ at any time t
c when the object is at rest
d the distance travelled in the first second
e the acceleration when the velocity is 0.

5 The velocity of an object moving in a straight line, which is initially positioned at $x = -1$, is given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t^2 - 9t + 4 \text{ cm/s}$$

Find:
a the displacement at time t
b the acceleration at time t
c when the object is at rest
d the distance travelled during $t = 3$ to $t = 5$
e the average speed during the time interval $t = 3$ to $t = 5$
f the average velocity during the time interval $t = 3$ to $t = 5$.

6 The acceleration of a body moving in a straight line is given by:

$$a(t) = 6 - 12t \text{ m/s}^2$$

Initially, the body is at rest at $x = 4$. Find:
a the velocity at any time
b the position at any time
c the time and position when the velocity is 0
d the distance travelled in the first 3 seconds.

WORKED Example 3

7 The acceleration of a particle travelling in a straight line is given by:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 4e^{2t} - 2t + 3 \text{ cm/s}^2$$

If the particle starts from the origin with a velocity of 2 cm/s, then find:
a the velocity at any time t
b the displacement from the origin at any time t
c the displacement from the origin after 3 seconds (correct to 2 decimal places).

8 **multiple choice**

A particle travelling in a straight line has an initial velocity of −1 m/s and initial displacement of −2 m. The acceleration of the particle is given by:

$$\ddot{x} = 6t - 2$$

a The particle is stationary when t is equal to:

A $\frac{1}{3}$ s **B** 2 s **C** $-\frac{1}{3}$ s **D** 3 s **E** 1 s

b The particle is at the origin when t is equal to:

A 0 s **B** 2 s **C** −1 s **D** 3 s **E** 1 s

9 **multiple choice**

If the velocity of a body moving in a straight line is:

$$\dot{x} = t^2 + 2t - 3 \text{ cm/s}$$

then the distance, in cm, travelled in the first 2 seconds is equal to:

A $1\frac{2}{3}$ **B** $2\frac{1}{3}$ **C** 4 **D** 1 **E** 3

10 A body is travelling in a straight line with a velocity given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t}{t^2 + 1} \text{ m/s}$$

If the body starts at the origin, find:

a the initial velocity
b the displacement at any time
c the acceleration at any time
d the maximum velocity
e the minimum acceleration.

11 The acceleration of a stone falling from rest from a height of 100 m is given by:

$$\ddot{x} = -10e^{-t} \text{ m/s}^2$$

a Find the velocity at any time t.
b Sketch a velocity versus time graph.
c Estimate the time taken, to the nearest second, to reach the ground.

12 The velocity of a particle is given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{4}{1 + t^2} \text{ m/s}$$

where the initial displacement is 1. Find:

a the displacement at any time t
b the maximum velocity and when it occurs
c the acceleration at any time t
d the acceleration when $t = 1$.

13 The acceleration of a particle is:

$$a(t) = 4 \sin 2t \text{ where } t = 0,\ v = -2 \text{ and } x = 0$$

Find:

a the velocity at any time t
b the displacement at any time t
c the acceleration in terms of the displacement x
d the acceleration in terms of the velocity v.

Motion under constant acceleration

Having investigated the link between position, velocity and acceleration, we shall now review the situation where acceleration is constant. These quantities are vectors, but as motion is in a straight line, we are going to use the sign of the quantity to indicate direction.

Suppose a particle is travelling in a straight line with constant acceleration a. The equation of motion can be found using the quadratic expression $x = lt^2 + mt + n$

The derivative of this function results in $v = 2lt + m$

and $a = 2l$ (a constant value)

Unless otherwise stated, assume that the object starts at 0.

WORKED Example 4

A particle moves from rest with a constant acceleration, reaching a speed of 20 m/s in 8 seconds. Find

a **the acceleration** **b** **the distance travelled by the particle.**

THINK	WRITE
a ① List the given information and what has to be found.	**a** $v_0 = 0$ m/s $v_t = 20$ m/s $t = 8$ s $x = 0$ m Require a
② Acceleration is constant; position will be a quadratic.	$x = lt^2 + mt + n$
③ Derive the equation for velocity and substitute known values to solve for l and m.	$v = 2lt + m$ Initially, $t = 0$ and $v = 0$; therefore $m = 0$ When $t = 8$, $20 = 2l \cdot 8$ $l = 1.25$
State the equation for v.	$v = 2.5\ t$
④ Differentiate v to find a.	$a = 2.5$ The acceleration is 2.5 m/s^2.
b ① Substitute the values for l, m, n into the equation for x. Assume that the initial position is the origin.	**b** Assume that the particle starts at the origin. Therefore $n = 0$ $x = 1.25\ t^2$
② As we are finding distance, check if the object changes direction in the time period.	The equation $x = 1.25\ t^2$ is increasing and therefore does not change direction. The distance covered will be the same as position.
③ Solve for $t = 8$ s. State the solution.	$t = 8$ s, $x = 1.25 \cdot 8^2 = 80$ The distance covered is 80 m.

WORKED Example 5

A ball is thrown directly upward from a platform 16 metres above the ground, and has an initial velocity of 24.5 m/s. Find:

a **the time taken to reach the maximum height**

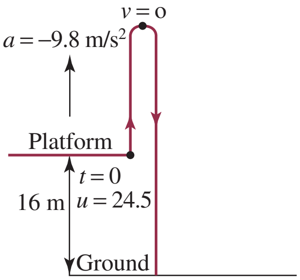
b **the maximum height above the ground that the ball reaches**

c **the total time the ball is in the air**

d **the time taken for the ball to fall to the ground from the maximum height**

e **the speed of the ball when it hits the ground**

f **the speed of the ball when it returns to the level of the platform.**

THINK	WRITE
a ① Sketch a diagram of the motion of the ball. Consider *up* to be positive and *down* to be negative.	**a** $v_0 = 24.5$ and $v_t = 0$ 
② At its maximum height, the velocity of the ball is 0 m/s.	$v_t = 0$ at the ball's maximum height.
③ List what is given and what has to be found (acceleration is due to gravity).	$v_0 = 24.5$ m/s $a = -9.8$ m/s^2 $v_t = 0$ Require x_t,
④ Acceleration is constant; position will be a quadratic.	$x = lt^2 + mt + n$
⑤ Derive the equations for velocity and acceleration. Substitute known values to solve for l and m.	$v = 2lt + m$ Initially, $t = 0$ and $v = 24.5$; therefore $m = 24.5$ $a = 2l$ $-9.8 = 2l$ $l = -4.9$
State the equation for v.	$v = -9.8\ t + 24.5$
⑥ Find when $v = 0$. Solve for t. State the solution.	$0 = -9.8\ t + 24.5$ $t = 2.5$ It takes 2.5 s for the ball to reach its maximum height.
b ① Make the ground position 0. State the equation for x. Find the height when $t = 2.5$ s.	**b** Initially, $x = 16$, therefore $n = 16$ $x = -4.9\ t^2 + 24.5\ t + 16$ $t = 2.5,\ x = -4.9 \cdot 2.5^2 + 24.5 \cdot 2.5 + 16$ $= 46.625$
State the solution.	The maximum height above the ground is 46.625 m.
c When the ball lands, $x = 0$. Solve for t.	**c** When the ball lands, $x = 0$. $0 = -4.9\ t^2 + 24.5t + 16$
Use the quadratic formula or graphics calculator to solve. State the solution.	$t = 5.585$ or $t = -0.584$ Ignore the negative solution. The ball takes 5.585 s to reach the ground.
d The ball took 2.5 s to reach its maximum height, and 5.585 s to land. State the solution.	**d** Time to fall from the maximum height $= 5.585 - 2.5$ $= 3.025$ It takes 3.025 s to fall from the maximum height.

Continued over page →

THINK	WRITE
e It takes 5.585 s for the ball to reach the ground. Find the velocity. State the solution (speed doesn't have direction).	**e** When $t = 5.585$ s, $v = -9.8 \cdot 5.585 + 24.5$ $= -30.23$ The speed when it hits the ground is 30.23 m/s.
f ① Find when the position is 16 m.	**f** $x = 16,\ 16 = -4.9\ t^2 + 24.5\ t + 16$ $0 = -4.9\ t^2 + 24.5\ t$ $= -4.9\ t(t - 5)$ $t = 0$ or $t = 5$ The ball will return to the platform level after 5 s.
② Find the velocity after 5 s. State the solution (speed doesn't have direction).	$t = 5,\ v = -9.8 \cdot 5 + 24.5$ $= -24.5$ The speed when it passes the platform will be 24.5 m/s.

remember

When the acceleration is a constant, the equation of position will be a quadratic.

1. $x = lt^2 + mt + n$
2. $v = 2lt + m$
3. $a = 2l$

EXERCISE 12B Motion under constant acceleration

1 Copy the following table, which displays data for objects travelling with constant acceleration. Complete your copy by using the kinematic equations.

	v_0	v_t	a	x_t	t
a	0	10			5
b	30	45	2		
c	−6	8		20	
d		18	−1	12	
e	4		1.5		4
f	50		−3	24	

2 **multiple choice**

A particle moves from rest with constant acceleration of 2 m/s^2. Answer the following questions on the particle, which has travelled 25 metres.

a The velocity at this time is equal to:
A 12.5 m/s **B** 10 m/s **C** 25 m/s **D** 5 m/s **E** 50 m/s

b The time taken for the particle to reach a speed of 20 m/s is:
A 5 s **B** 40 s **C** 15 s **D** 10 s **E** 2 s

c The distance travelled when it reaches 20 m/s is:
A 100 m **B** 20 m **C** 1600 m **D** 5 m **E** 200 m

WORKED Example 4

3 A particle moving from rest with constant acceleration reaches a speed of 20 m/s in 5 seconds. Find:
a the acceleration **b** the distance travelled.

4 A train decelerates uniformly for 20 seconds over a distance of 320 metres before coming to a stop. Find:
a acceleration **b** the initial speed.

5 A car accelerates from rest at 12 m/s^2 over a distance of 400 metres. Calculate:
a the time taken to travel the 400 metres **b** the final velocity of the car.

6 An object accelerates at 9 m/s^2 over a distance of 40 metres until it reaches a speed of 60 m/s. Find:
a the initial velocity **b** the time taken.

WORKED Example 5

7 An object is thrown from the ground at 39.2 m/s. Calculate:
a the maximum height reached by the object
b the total time that the object is in flight.

8 A stone is dropped from a tower that is 50 metres above ground level.
a How long does it take the stone to reach the ground?
b What is the speed of the stone on impact?

9 A stone is dropped down a deep well and it takes 4 seconds for it to hit the water at the bottom.
a How deep is the well?
b What is the speed of the stone on impact with the water?

Applying differential equations to rectilinear motion

In the previous sections of this chapter the position, velocity and acceleration of bodies moving in straight lines were all represented as functions of time. We will now consider situations where the velocity or acceleration of the body is, for example, a function of its position. This requires the use of various differential equations, depending on the information given and the information required.

Velocity given as a rate of change of position

If the velocity is given as a function of position and the position is required as a function of time, we proceed as follows:

$$v = f(x)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \text{ since } v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

or $$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{f(x)}, \text{ inverting}$$

$$\Rightarrow t = \int \frac{1}{f(x)}\,\mathrm{d}x$$

Remember that:

1. $t = \int \frac{1}{f(x)}\,\mathrm{d}x$
2. **After the antiderivative is found, the equation may then be rearranged to make x the subject.**

WORKED Example 6

A particle moves in a straight line such that its velocity, v cm/s, is

$$v(x) = \frac{1}{x+2} \text{ cm/s}, x > -2. \text{ When } t = 0, x = 0.$$

Find:

a **the position, x, as a function of time, t**

b **the position after 2.5 seconds**

c **the distance travelled in the first 2.5 seconds.**

THINK	**WRITE**
a ① Express v as $\frac{\mathrm{d}x}{\mathrm{d}t}$.	**a** $v = \frac{1}{x+2}, x > -2$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x+2}$
② Invert both sides.	$\frac{\mathrm{d}t}{\mathrm{d}x} = x + 2$
③ Express t in integral notation.	$t = \int (x+2)\,\mathrm{d}x$
④ Antidifferentiate the integrand with respect to x.	$t = \frac{1}{2}x^2 + 2x + c$
⑤ Substitute $t = 0$ and $x = 0$.	When $t = 0, x = 0$ $0 = 0 + 0 + c$
⑥ Solve the equation for the integration constant c.	$c = 0$
⑦ State the equation for t as a function of x.	So $t = \frac{1}{2}x^2 + 2x$
⑧ Multiply both sides by 2.	or $2t = x^2 + 4x$
⑨ Express the RHS as a perfect square to get x by itself, so that x can be expressed as a function of t.	$= (x+2)^2 - 4$

THINK	**WRITE**
10 Add 4 to both sides.	or $(x + 2)^2 = 2t + 4$
11 Take the square root of both sides.	$x + 2 = \pm\sqrt{2t + 4}$
12 Subtract 2 from both sides and keep the positive square root only as $x > -2$ (given.)	$x = -2 + \sqrt{2t + 4}$, as $x > -2$
b 1 Substitute $t = 2.5$.	**b** When $t = 2.5$, $x = -2 + \sqrt{2(2.5) + 4}$
2 Evaluate x.	$= -2 + \sqrt{9}$ $= -2 + 3$ $x = 1$
3 State the answer.	Therefore, the position after 2.5 seconds is 1, or 1 cm right of O.
c 1 Since $v = \dfrac{1}{x + 2}$ and $x > -2$, the velocity is always positive.	**c** Since $v = \dfrac{1}{x + 2}$ and $x > -2$, $v > 0$ at all times.
2 Since the velocity is positive the object is moving to the right only and does not turn.	Therefore, the object is travelling to the right (or in one direction) at all times.
3 The distance travelled is the position at $t = 2.5$ minus the position at $t = 0$.	Distance $= x(2.5) - x(0)$
4 Calculate the distance.	$= 1 - (-2 + \sqrt{4})$ $= 1 - (-2 + 2)$ $= 1 - 0$ $= 1$
5 State the answer.	Therefore, the object travels 1 cm in the first 2.5 seconds.

If the velocity is given as a function of position x, and the acceleration is required as a function of x, we proceed as follows.

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{dx} \cdot \frac{dx}{dt}, \text{ using the chain rule.} \end{aligned}$$

But $\dfrac{dx}{dt}$ is the velocity, v.

Therefore $$a = v\frac{dv}{dx}$$

WORKED Example 7

The velocity of a particle moving in a straight line is given by $v(x) = 2 + \log_e x$, $x > 0$, where x is the position at any time, t.

What is the acceleration in terms of x?

THINK	WRITE
① Write down the velocity.	$v = 2 + \log_e x$
② Find $\frac{dv}{dx}$.	$\frac{dv}{dx} = \frac{1}{x}$
③ Multiply v by $\frac{dv}{dx}$ to find the acceleration.	So $a = v\frac{dv}{dx}$
④ Simplify the RHS.	$a = (2 + \log_e x)\ \frac{1}{x}$ $a = \frac{2 + \log_e x}{x}$

Acceleration given as a rate of change of velocity

If the acceleration is given as a function of velocity then the position, velocity and acceleration can be found in terms of time, t.

That is, if $a = g(v)$, since

$$a = \frac{dv}{dt}$$

then

$$\frac{dt}{dv} = \frac{1}{g(v)}$$

$$t = \int \frac{1}{g(v)}\ dv$$

Remember that:

1. $t = \int \frac{1}{g(v)}\ dv$
2. **The antiderivative can be rearranged to make v the subject, giving v as a function of time. The position and acceleration can be found by antidifferentiating and differentiating the velocity respectively, as seen earlier.**

WORKED Example 8

The acceleration of a particle travelling in a straight line is given by $\frac{dv}{dt} = v + 1$, where v is the velocity at any time t. At time $t = 0$, the body is at rest at the origin.

Find, as a function of time:

a the velocity **b** the acceleration **c** the position.

THINK	WRITE
a ① Write the acceleration as given.	**a** $\frac{dv}{dt} = v + 1$
② Invert both sides of the differential equation.	or $\frac{dt}{dv} = \frac{1}{v + 1}$
③ Express t in integral.	$t = \int \frac{1}{v + 1}\, dv$
④ Antidifferentiate the integrand.	$t = \log_e(v + 1) + c$
⑤ Substitute $t = 0$ and $v = 0$ (the initial conditions given) into the equation.	When $t = 0, v = 0$ $0 = \log_e 1 + c$
⑥ Find the constant of antidifferentiation.	$0 = 0 + c$ $c = 0$
⑦ Rewrite the rule for t.	$t = \log_e(v + 1)$
⑧ Raise both sides to the power of e.	$e^t = v + 1$
⑨ Subtract 1 from both sides to obtain the velocity.	$v = e^t - 1$
b ① Differentiate v with respect to t to find the acceleration (or we could substitute v into the original equation for acceleration).	**b** $a = \frac{dv}{dt}$ $= \frac{d}{dt}(e^t - 1)$
② State the answer.	$a = e^t$
c ① Express x as the antiderivative of v.	**c** $x = \int (e^t - 1)\, dt$
② Antidifferentiate the integrand.	$x = e^t - t + c$
③ Substitute $t = 0$ and $x = 0$, the initial condition given.	When $t = 0, x = 0$ $0 = e^0 - 0 + c$
④ Solve for c.	$0 = 1 + c$ $c = -1$
⑤ State the answer.	$x = e^t - t - 1$

Acceleration given as a function of position

The following relation is used if the acceleration is given as a function of position, x, and the velocity as a function of x is required:

$$v = \frac{d}{dv}\left(\tfrac{1}{2}v^2\right)$$

Now $$a = v\frac{dv}{dx} \text{ (seen earlier).}$$

Substituting $\frac{d}{dv}\left(\tfrac{1}{2}v^2\right)$ for v, gives:

$$a = \frac{d}{dv}\left(\tfrac{1}{2}v^2\right) \cdot \frac{dv}{dx}$$

$$a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

WORKED Example 9

The acceleration of an object travelling in a straight line is given by

$a(x) = e^x + 1$ cm/s^2, $x \geq 0$, where x is the position at any time, t.

The velocity is 1 when $x = 0$.

a **Explain why the velocity is always positive.**

b **Find the velocity $v(x)$ in terms of the position x.**

THINK	WRITE
a ① The acceleration, $a = e^x + 1$, is > 0 for all values of x since $e^x > 0$ for all x.	**a** $a = e^x + 1 > 0$ for all x (since $e^x > 0$ for all x)
② The velocity is always increasing as x increases since $a > 0$.	The velocity, v, is always increasing as x increases.
③ The velocity is initially equal to 1 where $x = 0$.	At $x = 0$, $v = 1$.
④ The velocity must be greater than or equal to 1.	As x increases, $v > 1$.
⑤ The velocity is always positive.	Therefore, the velocity is always positive.
b ① Use the rule $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to set up a differential equation for acceleration.	**b** $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = e^x + 1$
② Express $\frac{1}{2}v^2$ in integral notation.	$\frac{1}{2}v^2 = \int (e^x + 1)\, dx$
③ Antidifferentiate the integrand.	$= e^x + x + c$
④ Substitute $x = 0$ and $v = 1$ into the equation.	When $x = 0$, $v = 1$. $\frac{1}{2}(1)^2 = e^0 + 0 + c$
⑤ Solve the equation for c.	$\frac{1}{2} = 1 + c$ $c = -\frac{1}{2}$
⑥ Rewrite the rule for $\frac{1}{2}v^2$.	So $\frac{1}{2}v^2 = e^x + x - \frac{1}{2}$
⑦ Multiply both sides by 2.	or $v^2 = 2e^x + 2x - 1$
⑧ Take the square root of both sides.	$v = \pm\sqrt{2e^x + 2x - 1}$
⑨ Dismiss the negative square root, as $v > 0$, and state the solution.	but $v > 1$ $v = \sqrt{2e^x + 2x - 1}$

remember

1. If $v = f(x)$, then $t = \int \frac{1}{f(x)}\, dx$.
2. $a = v\frac{dv}{dx}$ and thus: $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.
3. If $a = g(v)$, then $t = \int \frac{1}{g(v)}\, dv$.

EXERCISE 12C

Applying differential equations to rectilinear motion

A graphics calculator may be used to assist in calculations, or verify answers in the following exercise where appropriate.

WORKED Example 6

1 A particle moves in a straight line such that its velocity, v cm/s, is:

$$v(x) = \frac{1}{2x-6}; x > 3; x = 4 \text{ when } t = 0.$$

Find:

a the position, x, at any time, t

b the position after 15 seconds

c the distance travelled in the first 15 seconds.

2 An object initially at the origin is moving in a straight line and has a velocity given by:

$$v(x) = 2x + 1 \text{ cm/s}$$

Find:

a the position at any time, t

b the velocity at any time, t

c the initial acceleration.

3 The velocity of a body moving in a straight line is given by:

$$v(x) = \frac{x}{2\log_e x}, x \geq e; x = e \text{ when } t = 0.$$

Find:

a x in terms of t

b v in terms of t

c a in terms of t.

WORKED Example 7

4 The velocity, v m/s, of a particle moving in a straight line is given by:

$$v(x) = (x - 1)^2, x > 1$$

Find the acceleration when the velocity is 1 m/s.

5 **multiple choice**

A particle moves in a straight line so that the velocity is given by $v = 1 + e^{-x}$ m/s, and $x = 0$ when $t = 0$.

a The acceleration in terms of x is:

A $\frac{1}{2}x - e^{-x} - \frac{1}{4}e^{-2x}$ **B** $x - e^{-x}$ **C** $e^{-2x} - e^{-x}$

D $-e^{-x} - e^{-2x}$ **E** $x + e^{-x}$

b The position, x m, in terms of t is:

A $2e^t - 1$ **B** $\log_e(e^t + 1)$ **C** $\log_e(2e^t - 1)$ **D** $2t - 1$ **E** $e^{-x} + t$

c The initial acceleration in m/s^2 is:

A -1 **B** $-\frac{5}{4}$ **C** 0 **D** 1 **E** -2

6 An object moving in a straight line has a velocity given by:

$$v(x) = 4\sqrt{1 - x^2}$$

Find:

a the acceleration in terms of x

b the position in terms of t if $x = 1$ when $t = 0$.

7 The velocity of a body moving in a straight line is given by $v(x) = 4 + x^2$ cm/s, where at time $t = 0$, $x = 2$.
Find:
- **a** the acceleration in terms of position x
- **b** the position in terms of t
- **c** the velocity when $t = \frac{\pi}{24}$.

WORKED Example 8

8 The acceleration of a particle travelling in a straight line is given by:

$$\frac{dv}{dt} = 2 + v.$$

Initially, the object is at rest at the origin. Find:
- **a** the velocity in terms of t
- **b** the acceleration in terms of t
- **c** the position at any time t.

9 The acceleration of an object moving in a straight line is given by:

$$\frac{dv}{dt} = 4 - v \text{ and } v = 3 \text{ when } t = 2.$$

Find:
- **a** v at any time t
- **b** the acceleration when $t = 3$.

10 A body moving in a straight line has an acceleration given by:

$$a = 15 - \frac{v^2}{60}$$

Initially the body is at rest at the origin.
- **a** Find the velocity in terms of t.
- **b** Find the acceleration in terms of t.
- **c** Find the displacement in terms of t.

WORKED Example 9

11 The acceleration of an object travelling in a straight line is given by:

$$a = \frac{1}{(x-2)^2} \text{ cm/s}^2,\ 0 \le x < 2, \text{ where } x \text{ is the position at any time.}$$

The velocity is 0 when $x = 0$.

a Explain why the velocity is always positive.
b Find the velocity $v(x)$ in terms of the position x.
c Find where the particle has a velocity of 3 cm/s.

12 A body moves in a line so that its acceleration is $(x + 1)$ m/s^2 and when $x = 2$, $v = 3$.

a Find where the body is when the velocity is 0.
b Explain why v is increasing if $x > -1$.
c Find v in terms of x.
d Find the body's velocity when it is at the origin.

13 The acceleration of an object moving in a straight line is $4x$ cm/s^2 and when $x = 0$ the velocity is 1. What is the velocity when $x = \sqrt{2}$?

14 An object is dropped from rest so that the acceleration due to air resistance is $0.2v$ where v is the speed of the object. The acceleration due to gravity is g m/s^2. Take 'downwards' as positive.

a Show that $\frac{dv}{dt} = g - 0.2v$.
b Find v in terms of t.
c Find $\lim_{t \to \infty} v(t)$, that is, the maximum velocity.
d Find the distance fallen after 5 seconds (to the nearest metre).

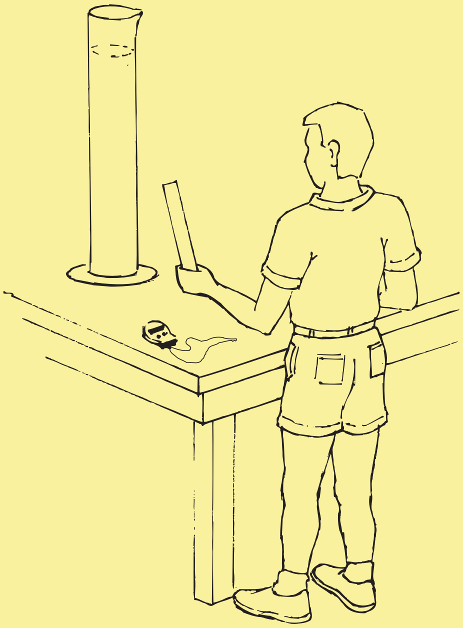
15 A cork is projected vertically upwards (take 'upwards' to be positive) with an initial speed u m/s against an air resistance kv^2, where v is the velocity in m/s at any instant and k is a constant. If x is the height (in metres) of the cork at any time show that:

a $a = -(g + kv^2)$

and hence find:

b the velocity in terms of x
c the maximum height reached
d the maximum height if $k = 0.002$ and $u = 30$.

The falling ball bearing



As part of a mathematics project, two students are investigating the motion of ball bearings as they fall from rest in a cylinder of motor oil. They aim to model the motion using an expression for the acceleration of the type $a = g - kv$ where g is the acceleration in the absence of drag due to gravity, v is the velocity of the ball bearing and k is a constant related to the viscosity of the oil. A schematic is shown at right.

One student suggests that it would be best to try to measure the speed of the falling ball bearing at a variety of different depths. They decide to make the depth at which they take measurements be the variable x. The velocity at a depth x is then the function $v(x)$. The other student suggests that it would be best to try to measure the speed of the falling ball bearing at a variety of different times. They decide to make the time after which they release the ball from rest at the top of the oil the variable t. The velocity at a time t is then the function $v(t)$.

a Using the model $a = g - kv$ with $v = 0$, $x = 0$ initially (that is, when $t = 0$), find an expression which contains the variables x and v and constants g and k. Note that finding the function $v(x)$ is beyond this course.

b Using the model $a = g - kv$ with $v = 0$, $x = 0$ initially (that is, when $t = 0$) find an expression for $v(t)$ in terms of g, k and the variable t.

c Using the model $a = g - kv$ with $v = 0$, $x = 0$ initially (that is, when $t = 0$) find an expression for $x(t)$ in terms of g, k and the variable t.

d Use a graphics calculator or a computer-based graphics package to plot $x(t)$ on one axis for a selection of values of k.

e Plot $v(t)$ on one axis for a selection of values of k.

f The terminal velocity of a falling ball bearing occurs when the velocity is a constant. Find an expression for the terminal velocity of a falling ball bearing in terms of the constants g and k. Compare and comment on this result and the findings illustrated in parts **e** and **d**.

g In an experiment the two students release a ball from rest at the top of the oil. The ball is found to have a terminal velocity of 0.1 m/s. Find the value of k and hence plot a graph of $v(t)$. Find the time, t, and location, x, when the ball is travelling at half the terminal velocity. Comment on how rapidly terminal velocity is established.

h In a second experiment the two students again release a ball bearing, this time in water which has $k = 196$. Using estimation techniques, find the speed of the ball bearing when it is at a depth of 0.15 m.

Vector calculus

When any object — from a golf ball to an athlete — travels in two or more dimensions, vectors must be used if we are to fully record and study its motion. The general principles of kinematics still apply; but the position, velocity and acceleration of the object are vector quantities and these involve both magnitude and direction.

In addition, it is worth recalling the following.

1. **The dot product or scalar product of two vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ where**

$$\underset{\sim}{b} = b_x\underset{\sim}{i} + b_y\underset{\sim}{j} \text{ and } \underset{\sim}{c} = c_x\underset{\sim}{i} + c_y\underset{\sim}{j} \text{ is given by } \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{b} = b_xc_x + b_yc_y.$$

2. **The magnitude or length of a vector, $\underset{\sim}{b}$ is given by**

$$b = |\underset{\sim}{b}| = \sqrt{\underset{\sim}{b} \cdot \underset{\sim}{b}} = \sqrt{b_x^2 + b_y^2}.$$

3. **The dot or scalar product of two vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ is related to their respective magnitudes and the angle θ between them:**

$$\underset{\sim}{b} \cdot \underset{\sim}{c} = |\underset{\sim}{b}|\ |\underset{\sim}{c}|\ \cos\theta$$

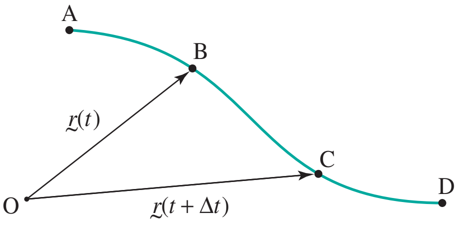
4. **Two non-zero vectors are perpendicular when $\underset{\sim}{b} \cdot \underset{\sim}{c} = 0$. This is because**

$$\begin{aligned}\underset{\sim}{b} \cdot \underset{\sim}{c} &= |\underset{\sim}{b}||\underset{\sim}{c}|\ \cos\theta \\ &= 0 \text{ if and only if } \cos\theta = 0\end{aligned}$$

that is, if $\theta = 90°$ and hence $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are perpendicular.

Position, velocity and acceleration

Position



Let us first consider a particle which moves along the curved path ABCD as shown at right.

The point O is the origin. The position of the particle from the origin at any time, t, is given by the vector $\underset{\sim}{r}(t)$. At a time t, the particle is at the point B on the curve and at a later time, $t + \Delta t$, the particle has moved to the point C. The *position vector* at time t, when the particle is at B, is $\underset{\sim}{r}(t)$; likewise, at C it is $\underset{\sim}{r}(t+\Delta t)$.

Displacement

The *displacement* of a particle is the change of its position over a given time interval. The displacement of the particle in the diagram above during the time interval Δt is given by:

$$\textbf{displacement} = \underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)$$

which is the difference between the final position at the time $t + \Delta t$ and the initial position at time t.

Distance

The *distance* from the origin of a particle at any time, t, is given by the magnitude of $\underset{\sim}{r}(t)$:

$$r(t) = |\underset{\sim}{r}(t)| = \sqrt{\underset{\sim}{r}(t) \cdot \underset{\sim}{r}(t)}$$

WORKED Example 10

Find the distance from the origin of the body described by the position vector $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + (t^2 - 1)\underset{\sim}{j} - 5\underset{\sim}{k}$ at $t = 2$.

THINK	WRITE
1 Substitute $t = 2$ into $\underset{\sim}{r}(t)$ to find $\underset{\sim}{r}(2)$.	$\underset{\sim}{r}(2) = 4\underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$
2 Evaluate $\lvert\underset{\sim}{r}(2)\rvert = \sqrt{\underset{\sim}{r}(2) \cdot \underset{\sim}{r}(2)}$ to find the distance from the origin.	$\lvert\underset{\sim}{r}(2)\rvert = \sqrt{4^2 + 3^2 + (-5)^2}$ $= \sqrt{50}$
3 Simplify the surd.	$= 5\sqrt{2}$
4 State the distance from the origin when $t = 2$.	The body is $5\sqrt{2}$ units from the origin when $t = 2$.

Average velocity

The *average velocity* $\underset{\sim}{v}_{\text{av}}$ during the time interval Δt is given by the displacement vector divided by the magnitude of the time interval. Thus:

$$\underset{\sim}{v}_{\text{av}} = \frac{\underset{\sim}{r}(t + \Delta t) - \underset{\sim}{r}(t)}{\Delta t}$$

WORKED Example 11

Find the average velocity of the body with position vector

$$\underset{\sim}{r}(t) = \sqrt{t+1}\,\underset{\sim}{i} - \left(\frac{t^2}{3}\right)\underset{\sim}{j} + 5\underset{\sim}{k} \text{ between } t = 0 \text{ and } t = 3.$$

THINK	WRITE
1 Find $\underset{\sim}{r}$ when $t = 0$ and when $t = 3$.	$\underset{\sim}{r}(0) = \underset{\sim}{i} + 5\underset{\sim}{k}$ $\underset{\sim}{r}(3) = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$
2 Simplify $\underset{\sim}{r}(3) - \underset{\sim}{r}(0)$.	$\underset{\sim}{r}(3) - \underset{\sim}{r}(0) = (2 - 1)\underset{\sim}{i} + (-3 - 0)\underset{\sim}{j} + (5 - 5)\underset{\sim}{k}$ $= \underset{\sim}{i} - 3\underset{\sim}{j}$
3 Calculate the average velocity by simplifying $\dfrac{\underset{\sim}{r}(3) - \underset{\sim}{r}(0)}{3 - 0}$.	$\underset{\sim}{v}_{\text{av}} = \dfrac{\underset{\sim}{i} - 3\underset{\sim}{j}}{3 - 0}$ $= \dfrac{\underset{\sim}{i} - 3\underset{\sim}{j}}{3}$ $= \frac{1}{3}\underset{\sim}{i} - \underset{\sim}{j}$
4 State the average velocity.	The average velocity is $\frac{1}{3}\underset{\sim}{i} - \underset{\sim}{j}$.

Instantaneous velocity

To obtain the instantaneous velocity, $v(t)$, of the particle at the point B, we apply a limit and make $\Delta t \to 0$. Thus the *instantaneous velocity* of the particle at time t is:

$$\begin{aligned} \underset{\sim}{v}(t) &= \lim_{\Delta t \to 0} \frac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t} \\ &= \frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t} \end{aligned}$$

This means that the instantaneous velocity vector $\underset{\sim}{v}(t)$ is given by the time derivative of the position vector $\underset{\sim}{r}(t)$. As $\Delta t \to 0$ the point C approaches B and so the velocity vector has a direction parallel to the tangent of the curved path of the particle.

The direction of motion of a particle is in the direction parallel to the velocity vector.

Rules for differentiating vectors

There are four intuitively obvious results which are used in differentiating vector quantities.

Rule 1. The derivative of a unit vector equals zero:

$$\frac{\mathrm{d}\underset{\sim}{i}}{\mathrm{d}t} = 0\,;\ \frac{\mathrm{d}\underset{\sim}{j}}{\mathrm{d}t} = 0\,.$$

Since both the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ have constant magnitude and direction they are constants and hence their derivative equals zero.

Rule 2. The derivative of a constant vector $a\underset{\sim}{i}$ equals zero. Using the product rule:

$$\begin{aligned} \frac{\mathrm{d}(a\underset{\sim}{i})}{\mathrm{d}t} &= a\frac{\mathrm{d}\underset{\sim}{i}}{\mathrm{d}t} + \left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)\underset{\sim}{i} \\ &= a \cdot\ 0 + \underset{\sim}{i} \cdot\ 0 \\ &= 0 \end{aligned}$$

Rule 3. The derivative of a vector $x\underset{\sim}{i}$ whose scalar coefficient $x(t)$ is a function of time is given as:

$$\begin{aligned} \frac{\mathrm{d}(x\underset{\sim}{i})}{\mathrm{d}t} &= x\left(\frac{\mathrm{d}\underset{\sim}{i}}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\underset{\sim}{i} \\ &= 0 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\underset{\sim}{i} \\ &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\underset{\sim}{i} \end{aligned}$$

Rule 4. The derivative of the sum of two vectors is the sum of the derivative of each vector. In the case of the position vector $\underset{\sim}{r} = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{i}$:

$$\begin{aligned} \frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t} &= \frac{\mathrm{d}(x\underset{\sim}{i} + y\underset{\sim}{j})}{\mathrm{d}t} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\underset{\sim}{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\underset{\sim}{j} \\ &= \underset{\sim}{v}(t) \end{aligned}$$

Speed

The speed of a particle is a scalar quantity; that is, it has magnitude but not direction. Therefore, the speed of a particle is the magnitude of the velocity vector.

$$\textbf{Speed} = |\underset{\sim}{v}(t)| = \sqrt{\underset{\sim}{v}(t) \cdot \underset{\sim}{v}(t)} = v(t)$$

WORKED Example 12

An object has a position vector in metres $\underset{\sim}{r}(t) = (3t^2 - \sin \pi t)\underset{\sim}{i} + (e^{-t} + 2t)\underset{\sim}{j}$; $t \geq 0$ seconds.

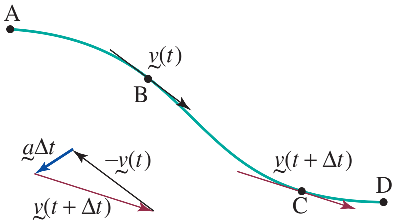
a **Find the velocity vector $\underset{\sim}{v}(t)$.**
b **Find the velocity of the object at $t = 2$ s.**
c **Find the speed of the object at $t = 2$ s.**
d **Find the average velocity in the first 2 seconds of the body's motion.**

THINK	WRITE
a ① The velocity vector $\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt}$ is found by differentiating each vector component of $\underset{\sim}{r}(t)$ with respect to time.	**a** $\underset{\sim}{v}(t) = \dfrac{d\underset{\sim}{r}}{dt}$ $= \dfrac{d}{dt}[(3t^2 - \sin \pi t)\underset{\sim}{i} + (e^{-t} + 2t)\underset{\sim}{j}]$ $= (6t - \pi \cos \pi t)\underset{\sim}{i} + (-e^{-t} + 2)\underset{\sim}{j}$
② State the velocity.	The velocity of the body is given by $\underset{\sim}{v}(t) = (6t - \pi \cos \pi t)\underset{\sim}{i} + (-e^{-t} + 2)\underset{\sim}{j}$.
b ① The velocity at time $t = 2$ is found by substitution into the expression for $\underset{\sim}{v}(t)$.	**b** $\underset{\sim}{v}(2) = (12 - \pi \cos 2\pi)\underset{\sim}{i} + (-e^{-2} + 2)\underset{\sim}{j}$
② Simplify the value of $\underset{\sim}{v}(2)$.	$= (12 - \pi)\underset{\sim}{i} + \left(2 - \dfrac{1}{e^2}\right)\underset{\sim}{j}$
③ State the velocity.	The velocity of the object at a time $t = 2$ s is $\underset{\sim}{v}(2) = (12 - \pi)\underset{\sim}{i} + \left(2 - \dfrac{1}{e^2}\right)\underset{\sim}{j}$.
c ① The speed of the object at time $t = 2$ is given by the magnitude of the vector $\underset{\sim}{v}(2)$ — that is, the speed is equal to $\sqrt{\underset{\sim}{v}(2) \cdot \underset{\sim}{v}(2)}$.	**c** speed, $|\underset{\sim}{v}(2)| = \sqrt{\underset{\sim}{v}(2) \cdot \underset{\sim}{v}(2)}$
② Calculate the speed using the velocity vector, $\underset{\sim}{v}(2)$, found in part **b**.	$= \sqrt{(12 - \pi)^2 + \left(2 - \dfrac{1}{e^2}\right)^2}$
③ Evaluate, correct to 2 decimal places.	≈ 9.05
④ State the approximate speed.	The speed of the object is 9.05 m/s.

THINK	**WRITE**
d ① The average velocity in the first 2 seconds is the displacement of the object divided by the time interval (that is, 2 seconds).	**d** $\underset{\sim}{v}_{\text{av}} = [\underset{\sim}{r}(2) - \underset{\sim}{r}(0)] \mid 2$
② The displacement is given by $\underset{\sim}{r}(2) - \underset{\sim}{r}(0)$, this being the difference in the position of the object from $t = 0$ until $t = 2$.	$= \left[(12 - \pi)\underset{\sim}{i} + \left(2 - \frac{1}{e^2}\right)\underset{\sim}{j} - (0\underset{\sim}{i} + 1\underset{\sim}{j})\right] \mid 2$
③ Simplify.	$= \left[(12 - \pi)\underset{\sim}{i} + \left(1 - \frac{1}{e^2}\right)\underset{\sim}{j}\right] \mid 2$
④ Calculate each component correct to 3 decimal places.	$\approx 4.429\underset{\sim}{i} + 0.432\underset{\sim}{j}$
⑤ State the approximate average velocity.	The average velocity in the first two seconds is $\underset{\sim}{v}_{\text{av}} = 4.429\underset{\sim}{i} + 0.432\underset{\sim}{j}$ m/s.

Acceleration

We can, in a similar way to that shown in the previous section, obtain a vector which describes the acceleration of an object from the instantaneous velocity vector. Consider a particle travelling on the path ABCD, as before. At the point B the object has a velocity given by $\underset{\sim}{v}(t)$. At a later time, $t + \Delta t$, the body is now at the point C with a velocity $\underset{\sim}{v}(t + \Delta t)$. This situation is shown below.



Average acceleration

The average acceleration of the object, $\underset{\sim}{a}_{\text{av}}$, is given as the change in velocity, $\Delta\underset{\sim}{v}$, divided by the time interval, Δt. Thus:

$$\underset{\sim}{\boldsymbol{a}}_{\textbf{av}} = \frac{\underset{\sim}{\boldsymbol{v}}(\boldsymbol{t} + \boldsymbol{\Delta t}) - \underset{\sim}{\boldsymbol{v}}(\boldsymbol{t})}{\boldsymbol{\Delta t}}.$$

Instantaneous acceleration

To obtain the instantaneous acceleration, $\underset{\sim}{a}(t)$, of the particle at the point B, we apply a limit and make $\Delta t \to 0$. Thus, the instantaneous acceleration of the particle at time t is:

$$\begin{aligned} \underset{\sim}{\boldsymbol{a}}(\boldsymbol{t}) &= \lim_{\Delta t \to 0} \frac{\underset{\sim}{\boldsymbol{v}}(\boldsymbol{t} + \boldsymbol{\Delta t}) - \underset{\sim}{\boldsymbol{v}}(\boldsymbol{t})}{\boldsymbol{\Delta t}} \\ &= \frac{\mathbf{d}\underset{\sim}{\boldsymbol{v}}}{\mathbf{d}\boldsymbol{t}} \end{aligned}$$

This means that the instantaneous acceleration vector $\underset{\sim}{a}(t)$ is given by the time derivative of the velocity vector $\underset{\sim}{v}(t)$.

WORKED Example 13

Find the acceleration vector $\underset{\sim}{a}(t)$ for the object whose position vector is

$$\underset{\sim}{r}(t) = e^{2t}\underset{\sim}{i} - (t^3 + t)\underset{\sim}{j} + (t \log_e t)\underset{\sim}{k}.$$

THINK	WRITE
① Find $\underset{\sim}{v}(t)$ by differentiating each vector component of $\underset{\sim}{r}(t)$ with respect to time.	$\underset{\sim}{v}(t) = 2e^{2t}\underset{\sim}{i} - (3t^2 + 1)\underset{\sim}{j} + (1 + \log_e t)\underset{\sim}{k}$
② Find $\underset{\sim}{a}(t)$ by differentiating each vector component of $\underset{\sim}{v}(t)$ with respect to time.	$\underset{\sim}{a}(t) = 4e^{2t}\underset{\sim}{i} - 6t\underset{\sim}{j} + \left(\frac{1}{t}\right)\underset{\sim}{k}$

WORKED Example 14

A body moves in such a way that its position in metres at an instant t seconds is given by:

$$\underset{\sim}{r}(t) = (15t - \tfrac{1}{2}t^2 + 1)\underset{\sim}{i} + (12t - \tfrac{1}{3}t^3)\underset{\sim}{j},\ t \geq 0.$$

Find:

a **the velocity vector $\dot{\underset{\sim}{r}}(t)$**

b **the acceleration vector $\ddot{\underset{\sim}{r}}$**

c **the angle between the velocity vector and the acceleration vector of the body at a time $t = 1$ s, to the nearest degree**

d **the time when the body has an acceleration of magnitude 9.8 m/s^2.**

THINK	WRITE
a ① The velocity vector $\dot{\underset{\sim}{r}}$ is the time derivative of the displacement $\underset{\sim}{r}(t)$.	**a** $\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt}$ $= \frac{d}{dt}\left[(15t - \tfrac{1}{2}t^2 + 1)\underset{\sim}{i} + (12t - \tfrac{1}{3}t^3)\underset{\sim}{j}\right]$
② Simplify.	$= (15 - t)\underset{\sim}{i} + (12 - t^2)\underset{\sim}{j}$
b ① The acceleration vector $\ddot{\underset{\sim}{r}}$ is the time derivative of the velocity $\dot{\underset{\sim}{r}}(t)$.	**b** $\ddot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{v}}{dt}$ $= \frac{d}{dt}\left[(15 - t)\underset{\sim}{i} + (12 - t^2)\underset{\sim}{j}\right]$
② Simplify.	$= -\underset{\sim}{i} - 2t\underset{\sim}{j}$
c ① Calculate the velocity vector when $t = 1$, $\dot{\underset{\sim}{r}}(1)$.	**c** $\dot{\underset{\sim}{r}}(1) = 14\underset{\sim}{i} + 11\underset{\sim}{j}$
② Calculate the acceleration vector when $t = 1$, $\ddot{\underset{\sim}{r}}(1)$.	$\ddot{\underset{\sim}{r}}(1) = -\underset{\sim}{i} - 2\underset{\sim}{j}$
③ The angle, θ, between the two vectors $\ddot{\underset{\sim}{r}}(1)$ and $\dot{\underset{\sim}{r}}(1)$ is found using the equation: $\ddot{\underset{\sim}{r}}(1) \cdot \dot{\underset{\sim}{r}}(1) = \lvert\ddot{\underset{\sim}{r}}(1)\rvert \cdot \lvert\dot{\underset{\sim}{r}}(1)\rvert \cdot \cos\theta$.	$\ddot{\underset{\sim}{r}}(1) \cdot \dot{\underset{\sim}{r}}(1) = \lvert\ddot{\underset{\sim}{r}}(1)\rvert \cdot \lvert\dot{\underset{\sim}{r}}(1)\rvert \cdot \cos\theta$

THINK	WRITE
④ Substitute $\ddot{\underset{\sim}{r}}(1)$ and $\dot{\underset{\sim}{r}}(1)$ into the equation.	$(-\underset{\sim}{i}-2\underset{\sim}{j})\cdot(14\underset{\sim}{i}+11\underset{\sim}{j})$ $=\sqrt{(-1)^2+(-2)^2}\sqrt{14^2+11^2}\cos\theta$
⑤ Simplify the equation.	$-14-22=\sqrt{5}\sqrt{317}\cos\theta$ $-36=\sqrt{1585}\cos\theta$
⑥ Express cos θ as the subject of the equation.	$\cos\theta=\frac{-36}{\sqrt{1585}}$ $=-0.90425$
⑦ Express θ as the subject of the equation.	$\theta=\cos^{-1}(-0.90425)$
⑧ Give the angle θ to the nearest degree.	$\theta\approx155°$.
⑨ State the solution.	The angle between the two vectors is approximately 155°.
d ① State the acceleration, $\ddot{\underset{\sim}{r}}(t)$ at any time, t.	**d** $\ddot{\underset{\sim}{r}}(t)=-\underset{\sim}{i}-2t\underset{\sim}{j}$
② Obtain an expression for the magnitude of the vector $\ddot{\underset{\sim}{r}}(t)$ and equate to 9.8.	$\lvert\ddot{\underset{\sim}{r}}(t)\rvert=\sqrt{1+4t^2}=9.8$
③ Square both sides of the equation.	$1+4t^2=9.8^2$ $=96.04$
④ Make t^2 the subject of the equation.	$4t^2=95.04$ $t^2=23.76$
⑤ Solve the equation for t by taking the square root of both sides. Note that only positive solutions for t are allowed.	$t\approx4.87,\ (t\geq0)$
⑥ State the solution.	When $t=4.87$ s the magnitude of the acceleration is 9.8 m/s^2.

Graphics Calculator tip!

Vector functions of time — derivatives

For the CASIO *fx*-9860G AU

1. Check the SET UP and ensure the calculator is in Radian Mode and that the derivative is ON, then press:
   - MENU
   - 5 (GRAPH)
   - F3 (TYPE)
   - F3 (PARM)

   Complete the entry lines as:
   $Xt1=2T$
   $Yt1=10-T^2$
   Press EXE after each line.

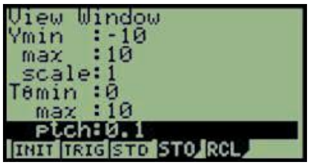
View Window
Xmin :0
max :10
scale:1
dot :0.07936507
Ymin :-10
max :10

Then press:

- SHIFT
- F3 (V-Window)

Enter the settings as shown.

2. Press the down arrow and enter the settings for t, and then press EXIT.

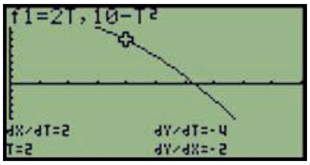


```
View Window
Ymin  :-10
 max  :10
 scale:1
Tθmin :0
 max  :10
 ptch:0.1
```

3. To find the value of the derivative, press:
   - F6 (DRAW)
   - F1 (Trace)

   Move the cursor to the point where $t = 2$.
   All derivative values are displayed; however, $\dot{\underset{\sim}{r}}$ is found by combining $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 4$.

   Therefore, at $t = 2$, $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$.

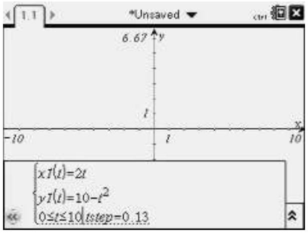


```
dX/dT=2     dY/dT=-4
T=2         dY/dX=-2
```

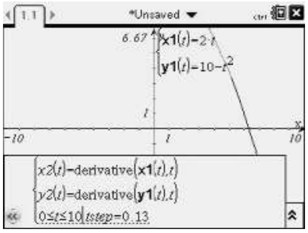
For the TI-Nspire CAS

1. To draw the path of the object, on a Graphs page, press:
   - MENU menu
   - 3: Graph Type 3
   - 2: Parametric 2

   Complete the entry lines as:
   $x1(t) = 2t$
   $y1(t) = 10 - t^2$
   $0 \le t \le 10$ *tstep* $= 0.13$
   Use Tab tab to move between lines.
   Then press ENTER enter.



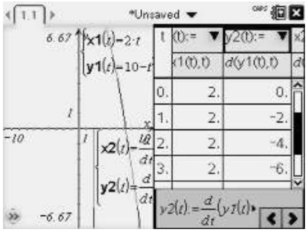
2. Complete the entry lines as:
   $x2(t) = $ derivative$(x1(t), t)$
   $y2(t) = $ derivative$(y1(t), t)$
   $0 \le t \le 10$ *tstep* $= 0.13$
   Then press ENTER enter.
   The function has been set to stay hidden (icon at bottom left).



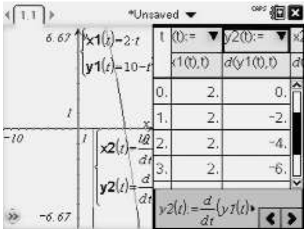
3. To create a function table, press:
   - MENU menu
   - 2: View 2
   - 9: Show Table 9

   To display the values of a particular function, place the cursor in the top cell of column 2, press:
   - ENTER enter
   - 2: Choose . . . 2

   And select $f(x)x2$. In a similar fashion set column 3 to display $f(x)y2$.



t	x2(t):=	y2(t):=
	d(x1(t),t)	d(y1(t),t)
0.	2.	0.
1.	2.	-2.
2.	2.	-4.
3.	2.	-6.



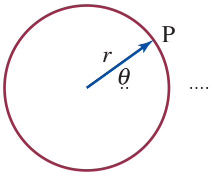
t	(t):=	y2(t):=
	x1(t),t)	d(y1(t),t)
0.	2.	0.
1.	2.	-2.
2.	2.	-4.
3.	2.	-6.

$y2(t):=\frac{d}{dt}(y1(t)$

4. To make the table easier to read, press:
    - Ctrl ctrl
    - HOME ⌂
    - 5: Page Layout 5
    - 1: Custom Split 1

    Use the left arrow ◀ to move the table to the left.
    It is now possible to read the derivatives when $t = 2$.
    Therefore at $t = 2$, $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$.

Circular motion



If an object at P moves in a circle of radius r at a constant speed, its position at any time is given by the equation:

$$\underset{\sim}{r} = r\cos\theta\underset{\sim}{i} + r\sin\theta\underset{\sim}{j}.$$

Suppose that the object is rotating anticlockwise at a constant angular velocity of ω radians per second.

This means that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$. The time it takes to make one complete revolution is called the *period of rotation* and is denoted by T. The period is related to the angular velocity by the formula:

$$\omega = \frac{2\pi}{T}$$

Let us calculate the velocity, $\underset{\sim}{v}$, of P.

$$\begin{aligned}\underset{\sim}{v} &= \frac{\mathrm{d}\underset{\sim}{r}}{\mathrm{d}t}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(r\cos\theta\underset{\sim}{i}) + \frac{\mathrm{d}}{\mathrm{d}t}(r\sin\theta\underset{\sim}{j})\end{aligned}$$

Using the chain rule we see that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\\ &= -\sin\theta\cdot\omega.\end{aligned}$$

As the radius of the circle, r, is constant, $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$.

Therefore $\underset{\sim}{v} = -r\omega\sin\theta\underset{\sim}{i} + r\omega\cos\theta\underset{\sim}{j}$.
The velocity is perpendicular to the position vector, $\underset{\sim}{r}$:

$$\begin{aligned}\underset{\sim}{v}\bullet\underset{\sim}{r} &= (-r\omega\sin\theta\underset{\sim}{i} + r\omega\cos\theta\underset{\sim}{j})\bullet(r\cos\theta\underset{\sim}{i} + r\sin\theta)\\ &= -r^2\omega\sin\theta\cos\theta + r\omega^2\sin\theta\cos\theta\\ &= 0\end{aligned}$$

The acceleration, $\underset{\sim}{a}$, of the object at P is calculated from $\frac{\mathrm{d}\underset{\sim}{v}}{\mathrm{d}t}$.

$$\begin{aligned}\underset{\sim}{a} &= \frac{\mathrm{d}}{\mathrm{d}t}(-r\omega\sin\theta\underset{\sim}{i} + r\omega\cos\theta\underset{\sim}{j})\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(-r\omega\sin\theta\underset{\sim}{i}) + \frac{\mathrm{d}}{\mathrm{d}t}(r\omega\cos\theta\underset{\sim}{j})\\ &= -r\omega^2\cos\theta\underset{\sim}{i} + -r\omega^2\sin\theta\underset{\sim}{j}\end{aligned}$$

$$= -\omega^2(r\cos\theta\underset{\sim}{i} + r\sin\theta\underset{\sim}{j})$$
$$= -\omega^2\underset{\sim}{r}.$$

If we write the magnitude of the velocity of P as v then

$$\begin{aligned} v &= |\underset{\sim}{v}| \\ &= \sqrt{(-r\omega\sin\theta)^2 + (r\omega\cos\theta)^2} \\ &= \sqrt{r^2\omega^2(\sin^2\theta + \cos^2\theta)} \\ &= r\omega \end{aligned}$$

In summary, the magnitude of displacement, velocity and acceleration are related as follows:

$$v = r\omega$$
$$a = r\omega^2$$

By combining these two equations we can formulate another expression for a:

$$a = \frac{v^2}{r}$$

WORKED Example 15

A rock is attached to a string and is swung around in a circle whose radius, r, is 0.5 m. The rock completes 2 revolutions every second.

For this motion calculate:

a **the speed of the rock**

b **the acceleration of the rock.**

THINK	WRITE
a 1 Calculate the period of rotation; that is, the time taken for the rock to complete one revolution.	**a** The time taken to complete one revolution is $T = 0.5$ s.
2 Calculate the angular velocity, ω, using the value for T.	Now $\omega = \frac{2\pi}{T}$ $= \frac{2\pi}{0.5}$ $= 4\pi$
3 Use the radius of the circle, r, and the angular velocity, ω, to find the speed, v, using the formula $v = r\omega$.	$v = r\omega$ $= 0.5 \cdot 4\pi$ $= 2\pi$ m/s The speed of the rock is 2π m/s.
b 1 Calculate the acceleration, a, using either $a = \frac{v^2}{r}$ or $a = r\omega^2$.	**b** $a = r\omega^2$
2 For the formula $a = r\omega^2$, substitute the values for r and ω and simplify.	$= 0.5 \cdot 16\pi^2$ $= 8\pi^2$ m/s^2. The acceleration of the rock is $8\pi^2$ m/s^2.

remember

1. Two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular when $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.
2. The angle between two vectors θ is given by $\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$ where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors.
3. Position $\underset{\sim}{r}(t)$, velocity $\underset{\sim}{v}(t)$ and acceleration $\underset{\sim}{a}(t)$ are vector quantities. In two dimensions they can be written as:

   $\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$

   $\underset{\sim}{v}(t) = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$ [*Note*: $v(t) = \dot{\underset{\sim}{r}}(t) = \dfrac{d\underset{\sim}{r}}{dt} = \dfrac{dx}{dt}\underset{\sim}{i} + \dfrac{dy}{dt}\underset{\sim}{j}$]

   $\underset{\sim}{a}(t) = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$, respectively. [*Note*: $\underset{\sim}{a}(t) = \ddot{\underset{\sim}{r}}(t) = \dfrac{d^2\underset{\sim}{r}}{dt^2} = \dfrac{d^2x}{dt^2}\underset{\sim}{i} + \dfrac{d^2y}{dt}\underset{\sim}{j}$]
4. The average velocity during the time interval Δt is $\underset{\sim}{v}_{\text{av}} = \dfrac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t}$.
5. The instantaneous velocity $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \dfrac{d\underset{\sim}{r}}{dt} = \dfrac{dx}{dt}\underset{\sim}{i} + \dfrac{dy}{dt}\underset{\sim}{j}$.
6. The instantaneous speed is given by $\sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}} = \sqrt{v_x^2 + v_y^2}$ which is the magnitude of the velocity vector $\underset{\sim}{v}$.
7. The direction of motion is in the direction parallel to the velocity vector.
8. The instantaneous acceleration $\underset{\sim}{a}(t) = \dfrac{d\underset{\sim}{v}}{dt} = \dfrac{dv_x}{dt}\underset{\sim}{i} + \dfrac{dv_y}{dt}\underset{\sim}{j} = \dfrac{d^2x}{dt^2}\underset{\sim}{i} + \dfrac{d^2y}{dt^2}\underset{\sim}{j}$.
9. For motion in a circle of radius r, with constant angular velocity ω:

$$v = r\omega$$
$$a = r\omega^2$$
$$a = \frac{v^2}{r}$$

EXERCISE 12D Vector calculus

WORKED Example 10

1 For the body described by the following position vectors, find the distance from the origin at the times listed using distance $= \sqrt{\underset{\sim}{r} \cdot \underset{\sim}{r}} = |\underset{\sim}{r}|$.

a $\underset{\sim}{r}(t) = 3\underset{\sim}{i} + 2t^2\underset{\sim}{j}$ at $t = \sqrt{2}$

b $\underset{\sim}{r}(t) = t\underset{\sim}{i} - (2t+1)^2\underset{\sim}{j} + 4\underset{\sim}{k}$ at $t = 3$

c $\underset{\sim}{r}(t) = \cos 4t\underset{\sim}{i} - \sin 4t\underset{\sim}{j}$ at $t = 3\pi$

d $\underset{\sim}{r}(t) = 2t^2\underset{\sim}{i} + \log_e(10t+1)\underset{\sim}{j} - (2t-1)\underset{\sim}{k}$ at $t = 1$

WORKED Example 11

2 Find the average velocity of the bodies described in question **1** from $t = 0$ to the given time.

WORKED Example 12a

3 Find the velocity vector for each of the following position vectors.

a $\underset{\sim}{r}(t) = 3\underset{\sim}{i} - 2t\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + -(2t+1)\underset{\sim}{j} + t^3\underset{\sim}{k}$

c $\underset{\sim}{r}(t) = \sin 3t\underset{\sim}{i} + 4\cos(-2t)\underset{\sim}{j}$

d $\underset{\sim}{r}(t) = 4e^{-2t}\underset{\sim}{i} - 2t^{-1}\underset{\sim}{j}$

e $\underset{\sim}{r}(t) = \log_e(t-1)\underset{\sim}{i} + (3t^3+2t)\underset{\sim}{j} - (4-t^2)\underset{\sim}{k}$

f $\underset{\sim}{r}(t) = t^2 \sin t\underset{\sim}{i} + te^t\underset{\sim}{j}$

g $\underset{\sim}{r}(t) = 3\cos 2t\underset{\sim}{i} + 3\sin 2t\underset{\sim}{j}$

WORKED Example 12b, c, d

4 For the following position vectors, find

i the speed, by calculating $\sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}$

ii the speed at $t = 2$

iii the average velocity in the first 2 seconds of the object's motion.

a $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} - (2t+1)\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = (t^3 - t)\underset{\sim}{i} - 2t\underset{\sim}{j} + t^2\underset{\sim}{k}$

c $\underset{\sim}{r}(t) = \log_e(3t+1)\underset{\sim}{i} - 4t\underset{\sim}{j}$

d $\underset{\sim}{r}(t) = \cos 4t\underset{\sim}{i} - \sin 4t\underset{\sim}{j}$

WORKED Example 13

5 Find the acceleration vector $\underset{\sim}{a}(t)$ for the following objects whose position vectors are:

a $\underset{\sim}{r}(t) = 3\underset{\sim}{i} - 2t\underset{\sim}{j}$

b $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} - (2t+1)\underset{\sim}{j} + 12t\underset{\sim}{k}$

c $\underset{\sim}{r}(t) = \sin 3t\underset{\sim}{i} + 4\cos(-2t)\underset{\sim}{j} - (t^3+t)\underset{\sim}{k}$

d $\underset{\sim}{r}(t) = 4e^{-2t}\underset{\sim}{i} - 2t^{-1}\underset{\sim}{j}$

e $\underset{\sim}{r}(t) = \log_e(t-1)\underset{\sim}{i} + (3t^3+2t)\underset{\sim}{j} - \left(\frac{1}{t}\right)\underset{\sim}{k}$

f $\underset{\sim}{r}(t) = \sin t\underset{\sim}{i} + te^t\underset{\sim}{j}$

6 **multiple choice**

The position of an object is given by $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} - 6t\underset{\sim}{j}$.

a The velocity at $t = 2$ is equal to:

A $\sqrt{52}$ B $4\underset{\sim}{i} - 12\underset{\sim}{j}$ C $4\underset{\sim}{i} - 6\underset{\sim}{j}$ D $2\underset{\sim}{i} - 3\underset{\sim}{j}$ E $4\sqrt{10}$

b The average velocity for the interval [0, 2] is:

A $2\underset{\sim}{i} - 3\underset{\sim}{j}$ B $2\underset{\sim}{i} - 6\underset{\sim}{j}$ C $\sqrt{13}$ D $4\underset{\sim}{i} - 12\underset{\sim}{j}$ E $4\underset{\sim}{i} - 6\underset{\sim}{j}$

c The acceleration at any time, t, is equal to:

A $2t\underset{\sim}{i} - 6\underset{\sim}{j}$ B $2\underset{\sim}{i} - 6\underset{\sim}{j}$ C $2\underset{\sim}{i}$ D 2 E $\underset{\sim}{i} - 3\underset{\sim}{j}$

WORKED Example 14

7 The position of an object at any time, t, is given as follows: $\underset{\sim}{r}(t) = 3t^2\underset{\sim}{i} + t^3\underset{\sim}{j} - 6t\underset{\sim}{k}$.

Find:

a the velocity vector $\underset{\sim}{v}(t)$

b the acceleration vector $\underset{\sim}{a}(t)$

c the angle between the velocity and acceleration vectors when $t = 1$

d when the object is moving in a direction parallel to the unit vector $\underset{\sim}{k}$.

8 **multiple choice**

The velocity of a particle at any time, t, is given by $\dot{\underset{\sim}{r}}(t) = 2\sin \pi t\underset{\sim}{i} + \frac{9}{(t+1)^2}\underset{\sim}{j}$.

a The acceleration at $t = 2$ is equal to:

A $-3\underset{\sim}{j}$ B $2\underset{\sim}{i} - 3\underset{\sim}{j}$ C $2\pi\underset{\sim}{i} + 3\underset{\sim}{j}$ D $2\underset{\sim}{i} + 3\underset{\sim}{j}$ E $2\pi\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j}$

b The angle between the velocity vector and acceleration vector when $t = 2$, to the nearest degree, is:

A 96° B 90° C 64° D 146° E 34°

9 An object moves with a position vector $\underset{\sim}{r}(t) = (t^3 - t)\underset{\sim}{i} + 2t^{-1}\underset{\sim}{j}$ for $t > 0$.

a At $t = 3$, determine the position vector and hence the distance of the object from the origin.

b Find the velocity vector $\underset{\sim}{v}(t)$.

c Find the speed of the object at $t = 2$.

d At what time is the object travelling parallel to the unit vector $\underset{\sim}{j}$?

10 An object moves with a position vector $\underset{\sim}{r}(t) = \sin 3\pi t\underset{\sim}{i} + t\underset{\sim}{j}$ for $t \geq 0$.

a At $t = 3$, determine the position vector and hence the distance of the object from the origin.

b Find the velocity vector $\underset{\sim}{v}(t)$.

c Find the speed of the object at $t = 2$.

d What is the maximum speed of the object?

11 An object moves with a position vector $\underset{\sim}{r}(t)$, expressed in metres, at a time, t seconds:

$$\underset{\sim}{r}(t) = \frac{1}{t+1}\underset{\sim}{i} + t\underset{\sim}{j} \text{ for } t \geq 0$$

a At $t = 2$, determine the position vector and hence the distance of the object from the origin.

b Find the velocity vector $\underset{\sim}{v}(t)$.

c Find the speed of the object at $t = 2$.

d Show that after a long time the body travels at a speed of 1 m/s parallel to the unit vector $\underset{\sim}{j}$.

12 A body moves with a velocity $\dot{\underset{\sim}{r}}(t) = 4 \cos 3t\underset{\sim}{i} + 4 \sin 3t\underset{\sim}{j}$.

a Show that the speed is constant.

b Find the magnitude of the acceleration at any time, t.

c When is the velocity first equal to $-2\sqrt{2}\underset{\sim}{i} + 2\sqrt{2}\underset{\sim}{j}$?

d Find the angle between $\dot{\underset{\sim}{r}}(t)$ and $\ddot{\underset{\sim}{r}}(t)$ when $t = \frac{\pi}{6}$.

WORKED Example 15

13 An object travels in a circle of radius 2 metres at a constant speed. The object completes 3 revolutions each second. Calculate:

a the speed of the object

b the acceleration of the object.

14 A car travels around a circular track at a constant speed of 25 m/s. The radius of the circle is 50 m. Calculate:

a the angular velocity, ω, of this motion

b the acceleration of the car.

Newton's Second Law of Motion

In the study of motion so far, we have looked at the relationship between displacement, velocity and acceleration. Calculus and vectors have been used to describe the motion. In this section we see that motion is caused by force, and we look at the relationship between force and acceleration. The interaction between force and motion is governed by Newton's Laws.

Historically, Newton's First Law of Motion had its origins in some of the work carried out by Galileo. This law concerns itself with the motion of a particle when the resultant force on it is zero. It was once a commonly held belief that in order to have motion, an unbalanced force was required. This belief seemed quite natural: clearly, carts need to be pulled by horses to have movement; balls rolling across a lawn eventually come to rest.

However, experiments showed that the force of friction could be used in both cases to explain the apparent necessity of an applied force to maintain constant motion. It was Newton who stated that in the absence of an unbalanced force acting on the body, the body would continue in a uniform state of motion. The phrase 'uniform motion' is used to describe the motion of a body with zero acceleration.

Newton's First Law states that if the resultant force acting on a body is zero then its acceleration is zero. This law was examined in detail in Year 11.

Newton's Second Law describes the relationship between the resultant force, $\underset{\sim}{R}$, on a body of mass m, and the acceleration, $\underset{\sim}{a}$, of that body.

The sum of all physical forces acting on the object.
Unit: N

The acceleration of the object.
Unit: m/s^2

$$\underset{\sim}{R} = m\underset{\sim}{a}$$

The mass of the object.
Unit: kg

It is this form of Newton's Second Law that we will use. In essence the law tells us that the vector sum of all forces acting on an object, $\underset{\sim}{R}$, has the effect of causing an acceleration $\underset{\sim}{a}$. Note that the vector, $\underset{\sim}{R}$, is parallel to $\underset{\sim}{a}$ since they are proportional to each other; accelerations occur in the direction in which the resultant force acts. The constant of proportionality is the mass, m, of the body. Two bodies which experience resultant forces of the same size will have accelerations which are inversely proportional to their masses: large masses have small accelerations, while small masses accelerate rapidly under the action of an identical resultant force. Both the resultant force, $\underset{\sim}{R}$, and the quantity $m\underset{\sim}{a}$ are *not* physical forces. The resultant force is the *vector sum of all physical forces* which act on a body. This can best be determined from a vector diagram showing all forces which act on the object. The quantity $m\underset{\sim}{a}$ is the mass of the body multiplied by its acceleration. Newton's second law asserts that $\underset{\sim}{R}$ and $m\underset{\sim}{a}$ are equal vectors.

We can readily see that when $\underset{\sim}{R} = 0 \Leftrightarrow \underset{\sim}{a} = 0$; consequently, Newton's first law is consistent with, and a special case of, his second law.

To solve dynamics and related kinematics problems there is a clear strategy when using Newton's Second Law of Motion.

1. **Read the question carefully.**
2. **Draw a clear diagram which has all the information.**
3. **Superimpose arrows depicting force vectors that act on the body in question.**
4. **Find a value or expression for the resultant force vector which is the sum of all forces acting on the body.**
5. **Equate this to the product of inertial mass and acceleration.**

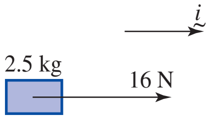
WORKED Example 16

A resultant force of 16 N acts for 5 seconds on an object of mass 2.5 kg which is initially at rest.

a **What acceleration will this produce?**

b **What will be its speed after 5 seconds?**

c **How far will it travel in 15 seconds?**

THINK	**WRITE**
Draw a force vector diagram.	
a ① Use Newton's second law: $\underset{\sim}{R} = m\underset{\sim}{a}$.	**a** $\underset{\sim}{R} = m\underset{\sim}{a}$
② Substitute $\underset{\sim}{R} = 16\underset{\sim}{i}$ and $m = 2.5$ kg.	$16\underset{\sim}{i} = 2.5\underset{\sim}{a}$
③ Solve for a.	$\underset{\sim}{a} = 6.4\underset{\sim}{i}$
④ State the solution.	The acceleration is 6.4 m/s^2 in the direction of the resultant force $\underset{\sim}{R}$, parallel to the unit vector $\underset{\sim}{i}$.
b ① The object accelerates uniformly for 5 s. The equation of position will be a quadratic. Derive the equations for v and a.	**b** $x = lt^2 + mt + n$ $v = 2lt + m$ $a = 2l$
② $a = 6.4$ m/s^2, $v_0 = 0$ m/s Evaluate l, m and n.	As $a = 6.4$, $l = 3.2$ As $v_0 = 0$, $m = 0$ Assume the object starts at the origin, $n = 0$.
③ Calculate the velocity at $t = 5$ s.	$v = 6.4t$ $t = 5$, $v = 6.4 \cdot 5 = 32$
State the solution.	The speed after 5 s will be 32 m/s.
c ① To find the distance covered, we need to know how far the object travelled in the first 5 s.	**c** $x = 3.2t^2$ $t = 5$, $x = 3.2 \cdot 5^2 = 80$
② In the second stage ($t > 5$), $a = 0$. This will mean that $l = 0$. Find the new equations.	Stage 2 occurs for $t > 5$. $a = 0$. The equations become: $x = mt + n$ $v = m$ $v = 32$ (from part **b**) therefore $m = 32$. When $t = 5$, $x = 80$ $80 = 32 \cdot 5 + n$ $n = -80$
③ State the new equation of motion. Find the position when $t = 15$.	$x = 32t - 80$ $t = 15$, $x = 32 \cdot 15 - 80$ $= 400$
State the solution.	The total distance travelled is 400 m.

WORKED Example 17

A 75-kg athlete running on a horizontal track increases his speed from 7.0 m/s to 9.5 m/s in 2.0 seconds.

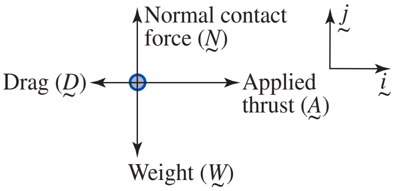
a **Draw a force vector diagram for the athlete. Use $\underset{\sim}{W}$ to denote the weight, $\underset{\sim}{N}$ to denote the normal contact force, $\underset{\sim}{A}$ to denote the applied thrust and $\underset{\sim}{D}$ to denote air drag.**

b **Write down the equation of motion using Newton's second law from the vector diagram.**

c **Calculate the acceleration of the athlete.**

d **Calculate the magnitude of his weight and consequently the normal contact force.**

e **Calculate the magnitude of the resultant force acting on him.**

THINK	WRITE
a The weight force acts vertically downwards and is opposed by a normal contact force. In the horizontal, the applied thrust is a force which pushes the athlete forward while air drag retards his motion.	**a** 
b ① Use $\underset{\sim}{R} = m\underset{\sim}{a}$ and recognise that applied thrust is in the direction of $\underset{\sim}{i}$ while drag opposes this in the direction of $-\underset{\sim}{i}$. Similarly, the two vertical forces, $\underset{\sim}{N}$ and $\underset{\sim}{W}$, act in the $\underset{\sim}{j}$ direction.	**b** $\underset{\sim}{R} = m\underset{\sim}{a}$ From diagram, $\underset{\sim}{R} = (A - D)\underset{\sim}{i} + (N - W)\underset{\sim}{j}$
② Let $\underset{\sim}{a} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$ where a_x is the component of the acceleration in the $\underset{\sim}{i}$ direction and a_y is the component of the acceleration in the $\underset{\sim}{j}$ direction.	Let $\underset{\sim}{a} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$
③ Write the equation of motion.	So equation of motion is: $\underset{\sim}{R} = (A - D)\underset{\sim}{i} + (N - W)\underset{\sim}{j} = ma_x\underset{\sim}{i} + ma_y\underset{\sim}{j}$
c ① The object accelerates uniformly for 5 s. The equation of position in the horizontal direction will be a quadratic. Derive the equations for the horizontal component of v and a.	**c** $x = lt^2 + mt + n$ $v = 2lt + m$ $a = 2l$
② $v_0 = 7$ m/s, $v_t = 9.5$ m/s $t = 2.0$ s Solving for $2l$ will give a_x.	$v_0 = 7,\ m = 7$ Substitute $v_t = 9.5,\ t = 2.0$ into the velocity equation. $9.5 = 2l \cdot\ 2.0 + 7$ $2l = 1.25$ $a_x = 2l = 1.25$
③ The acceleration in the $\underset{\sim}{j}$ direction, a_y, is zero since the athlete is not moving vertically.	$a_y = 0$
④ State the acceleration vector.	The acceleration vector can be written as $\underset{\sim}{a} = 1.25\underset{\sim}{i} + 0\underset{\sim}{j}$.

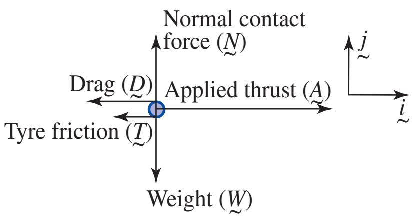
THINK	**WRITE**
d ① The weight of the athlete is easily found using $W = mg$ with $m = 75$ kg and $g = 9.8$.	**d** $W = 75 \cdot\ 9.8 = 735$ N
② Now, to find the magnitude of the normal contact force we use the $\underset{\sim}{j}$ component of the equation of motion.	$(N - W)\underset{\sim}{j} = ma_y\underset{\sim}{j}$ with $W = 735$ and $a_y = 0$.
③ Solve for N.	$N - 735 = 0$ $N = 735$ N
e ① Find the resultant force using $\underset{\sim}{R} = m\underset{\sim}{a}$ directly since $m = 75$ kg and $a = 1.25\underset{\sim}{i}$ m/s^2.	**e** $\underset{\sim}{R} = 75 \cdot\ 1.25\underset{\sim}{i}$ $= 93.75\underset{\sim}{i}$ N
② State the magnitude of $\underset{\sim}{R}$, R.	$R \approx 94$ N (to 2 significant figures)

It is worth noting here that if you know the mass and acceleration of an object you can calculate the resultant force. This, however, does not necessarily give you information about the sizes of individual forces on the object. For this you need a force–vector diagram which allows you to write the equation for the resultant force since it is the sum of the physical forces that are acting.

WORKED Example 18

A car of mass 1200 kg accelerates uniformly from rest to 28 m/s in a time of 7.0 s. The applied force of the engine through the *rear tyres* onto the road is 5400 N. The friction force retarding the motion of the car acting on the front tyres is 2.0% of the car's weight.

a Draw a force vector diagram depicting the five forces acting on the car.
b Determine the vector for the net force, $\underset{\sim}{R}$, acting on the car.
c Calculate the size of the average force of air drag acting on the car.

THINK	**WRITE**
a Five forces act on the car. The weight and normal contact forces act in the vertical direction. In the horizontal direction the applied force is opposed by two other forces: air drag and the force of friction of the non-driving tyres on the road.	**a** 

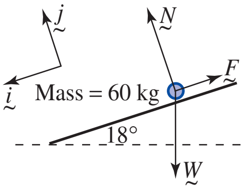
Continued over page

THINK	**WRITE**
b ① Take the $\underset{\sim}{i}$ direction to be the direction of acceleration of the car.	**b**
② Use Newton's second law with $m = 1200$ and $\underset{\sim}{a} = a_x \underset{\sim}{i}$, since the acceleration in the $\underset{\sim}{j}$ direction is zero.	$\underset{\sim}{R} = ma_x\underset{\sim}{i}$
③ Evaluate a_x using $a = \dfrac{v-u}{t}$ with $v = 28$, $u = 0$ and $t = 7.0$.	$a_x = \dfrac{28-0}{7.0}$ $= 4.0$
④ Substitute $a_x = 4.0$ into the equation of motion and calculate $\underset{\sim}{R}$.	$\underset{\sim}{R} = 1200 \cdot\ 4.0\underset{\sim}{i}$ $\underset{\sim}{R} = 4800\underset{\sim}{i}$ N
c ① Write the equation of motion.	**c** $\underset{\sim}{R} = (A - D - T)\underset{\sim}{i} + (N - W)\underset{\sim}{j} = 4800\underset{\sim}{i}$
② Equate the $\underset{\sim}{i}$ components of the equation of motion.	$A - D - T = 4800$
③ Calculate the tyre friction, T, which is 2% of the weight ($W = 1200$ g).	$T = 0.02 \cdot\ 1200 \cdot\ 9.8$ $= 235$ N
④ The applied force, A, is 5400 N.	$A = 5400$ N
⑤ Substitute A and T into the $\underset{\sim}{i}$ component of the equation of motion.	$5400 - D - 235 = 4800$
⑥ Solve the equation for the air drag, D.	$D = 365$ N

WORKED Example 19

A skier of mass 60 kg begins a ski run from the top of a mountain down an 18° slope. A constant friction force of 37 N acts on the skier as he descends.

- **a** **Draw a vector force diagram depicting the three forces which act on the skier.**
- **b** **Write the equation of motion using Newton's Second Law of Motion.**
- **c** **Determine the magnitude of the normal contact force.**
- **d** **Determine the magnitude of the component of the skier's weight parallel to the slope.**
- **e** **Calculate the resultant force acting on the skier.**
- **f** **Calculate the acceleration of the skier.**
- **g** **If there were no frictional force present, find the acceleration of the skier.**

THINK	**WRITE**
a Three significant forces act on the skier: the weight force $\underset{\sim}{W}$, the normal contact force $\underset{\sim}{N}$ and the force of friction $\underset{\sim}{F}$.	**a** 
b ① Use Newton's second law.	**b** $\underset{\sim}{R} = m\underset{\sim}{a}$
② The vector $\underset{\sim}{W}$ is to be resolved into components parallel to $\underset{\sim}{i}$ and $\underset{\sim}{j}$.	$\underset{\sim}{R} = (W \sin q - F)\underset{\sim}{i} + (N - W \cos q)\underset{\sim}{j}$ $\underset{\sim}{R} = ma_x\underset{\sim}{i} + ma_y\underset{\sim}{j}$
③ We know that $m = 60$ and $a_y = 0$.	$\underset{\sim}{R} = 60a_x\underset{\sim}{i} + 0\underset{\sim}{j}$

THINK	**WRITE**
c ① Equate the $\underset{\sim}{j}$ components of the equation of motion.	**c** $N - W\cos\theta = 0$
② Substitute $W = 60g$ and $\theta = 18°$.	$N - 60g\cos 18° = 0$
③ Solve for N.	$N \approx 559$ N (to the nearest whole number).
d The component of the weight vector parallel to the slope is equal to $W\sin\theta$ where $W = 60g$ and $\theta = 18°$.	**d** Component of $\underset{\sim}{W}$ parallel to $\underset{\sim}{i}$ $= W\sin\theta$ $= 60g\sin 18°$ ≈ 182 N (to the nearest whole number).
e ① Substitute $W\sin\theta = 182$, $F = 37$ and $N - W\cos\theta = 0$ into the equation for the resultant force $\underset{\sim}{R} = (W\sin\theta - F)\underset{\sim}{i} + (N - W\cos\theta)\underset{\sim}{j}$.	**e** $\underset{\sim}{R} = (W\sin\theta - F)\underset{\sim}{i} + (N - W\cos\theta)\underset{\sim}{j}$ $\underset{\sim}{R} = (182 - 37)\underset{\sim}{i} + 0\underset{\sim}{j}$
② Simplify.	$\underset{\sim}{R} = 145\underset{\sim}{i}$ N
f ① Again use Newton's second law $\underset{\sim}{R} = m\underset{\sim}{a}$ where $\underset{\sim}{R} = 145\underset{\sim}{i}$ and $m = 60$.	**f** $\underset{\sim}{R} = 145\underset{\sim}{i} = 60a_x\underset{\sim}{i}$
② Solve for a_x.	$a_x = 2.42$ m/s^2
g ① Again use Newton's second law $\underset{\sim}{R} = m\underset{\sim}{a}$ but with $\underset{\sim}{F} = 0$ to comply with no friction. In this case $\underset{\sim}{R} = 182\underset{\sim}{i}$.	**g** $\underset{\sim}{R} = 182\underset{\sim}{i}$ $= 60a_x\underset{\sim}{i}$
② Solve for a_x.	$a_x = 3.03$ m/s^2

Note: By reducing friction the skier can accelerate more rapidly down the slope.

remember

1. Force is a vector quantity. Its derived unit is the newton.
$$1\text{ N} = 1\text{ kg m/s}^2$$
2. Newton's Second Law of Motion for objects is:
$$\underset{\sim}{R} = m\underset{\sim}{a}$$
(the resultant force is equal to the product of mass and acceleration).
3. The weight force parallel to an incline of elevation angle θ is $mg\sin\theta$.
4. The normal contact force for a mass, m, on an incline of elevation angle θ is:
$$N = mg\cos\theta.$$
5. To solve dynamics and related kinematics problems:
   (a) read the question carefully
   (b) draw a clear diagram which has all the information
   (c) superimpose arrows depicting vectors which act on the body in question
   (d) find a value or expression for the resultant force vector which is the sum of all forces acting on the body
   (e) equate this to the product of inertial mass and acceleration.

EXERCISE 12E Newton's Second Law of Motion

1 A car travelling in a straight line brakes uniformly from 54 km/h to 36 km/h in a time of 2.0 seconds.

- **a** Calculate the acceleration, a, of the car in m/s^2.
- **b** Calculate the distance, s, travelled by the car in that time.
- **c** Calculate the speed, v, of the car, 1.5 seconds after it began to brake.

2 a A car travelling along a residential street comes to a stop in a time t. What can be said about the magnitude and the direction of the resultant force acting on the car during that time?

b A skier races down a slope at an increasing velocity. What can be said about the magnitude and the direction of the resultant force acting on the skier during that time?

3 Copy and complete the following table:

Force (N)	Mass (kg)	Acceleration (m/s^2)
10	2	
5		10
	2	1

WORKED Example 16

4 For each of the following forces acting on a mass of 2.5 kg which is initially at rest, determine:

- **i** the magnitude of the acceleration
- **ii** the speed of the mass 2 seconds after accelerating from rest
- **iii** the distance travelled by the mass during that time.

a 4.0 N **b** 8.0 N **c** 12 N **d** 16 N

5 In a maths classroom two students are conducting an experiment. They pull a block of wood across a rough horizontal table top. The block experiences a constant force of friction, $\underset{\sim}{F}$. For each applied force, $\underset{\sim}{T}$, the acceleration, a, of the block is measured. The results of the experiment are listed below in the table.

Applied force, *T* (N)	1.5	2.0	2.5	3.0	3.5
Acceleration, *a* (m/s^2)	0.20	0.53	0.67	1.00	1.33

- **a** Draw a vector force diagram for forces acting on the block of wood as it is pulled across the table.
- **b** Explain why Newton's second law, when applied to this situation, gives the equation $T - F = ma$.
- **c** Using a spreadsheet, or other method, plot a graph of the applied force, T, against acceleration, a.
- **d**
  - **i** Which aspect of the graph allows you to calculate the magnitude of the frictional force?
  - **ii** What is the estimated value of the frictional force acting on the block as it slides across the table?
- **e**
  - **i** Which aspect of the graph allows you to calculate the mass of the block?
  - **ii** What is the estimated value for the mass of the block?

6 **multiple choice**

A car of mass 800 kg accelerates from rest to a speed of 20 m/s in 4 seconds.

a The average acceleration of the car is:

A 80 m/s^2 **B** 5 m/s^2 **C** 0.25 m/s^2 **D** –5 m/s^2 **E** 0.2 m/s^2

b The magnitude of the average resultant force acting on the car during this time is:

A 160 N **B** 800g N **C** 200 N **D** 400 N **E** 4000 N

After 4 seconds the car's engine is shut off and it rolls along subject to a resistive force of 2000 N until it comes to rest.

c The deceleration of the car is of magnitude:

A 0.2 m/s^2 **B** –2.5 m/s^2 **C** –0.2 m/s^2 **D** 2.5 m/s^2 **E** 7.5 m/s^2

d The total distance travelled by the car is:

A 88 m **B** 48 m **C** 120 m **D** 200 m **E** 40 m

WORKED Example 17

7 A 60-kg athlete accelerates from rest to a speed of 11 m/s in a time of 1.5 seconds.

a Draw a force vector diagram for the athlete. Use $\underset{\sim}{W}$ to denote the weight, $\underset{\sim}{N}$ to denote the normal contact force, $\underset{\sim}{A}$ to denote the applied thrust and $\underset{\sim}{D}$ to denote air drag.

b Write down the equation of motion using Newton's second law from the vector diagram.

c Calculate the acceleration of the athlete.

d Calculate the magnitude of her weight and consequently the normal contact force.

e Calculate the magnitude of the resultant force acting on the athlete during that time interval.

8 A Volvo with mass, *m*, 1200 kg, brakes suddenly to avoid an accident, slowing from 100 km/h to a safe 20 km/h over a distance of 40 m.

a Find the time taken to stop.

b Calculate the acceleration of the car. (Be careful here with the sign.)

c Calculate the magnitude of the resultant force acting on the Volvo during braking.

9 A 30-g bullet travelling at a speed of 1200 m/s strikes a practice target and stops when embedded 3.0 cm into the target.

a Calculate the time taken for the bullet to stop.

b Calculate the acceleration of the bullet. (Be careful here with the sign.)

c Calculate the magnitude of the resultant force acting on the bullet.

10 A constant 20-N force acts on a cart which starts from rest at $t = 0$ such that during the third second it travels a distance of 4.7 m. Calculate the mass of the cart.

11 In each case below, find:

i the resultant force, $\underset{\sim}{R}$, acting on the body

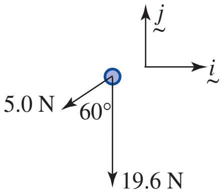
ii the magnitude of the resultant force, $|\underset{\sim}{R}|$

iii the acceleration of the body.

a $\underset{\sim}{F}_1 = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ N, $\underset{\sim}{F}_2 = 3\underset{\sim}{i} + 5\underset{\sim}{j}$ N, $m = 2.0$ kg

b $\underset{\sim}{F}_1 = 200$ N East, $\underset{\sim}{F}_2 = 450$ N North-west, $m = 75$ kg

c See the diagram at right. The mass is 2.4 kg.



WORKED Example 18

12 A sports car of mass 700 kg increases its speed from 90 km/h to 100 km/h during a 5-second interval. The average combined forces of air resistance and road friction, $\underset{\sim}{D}$, at that speed is 200 N.

- **a** Draw a force vector diagram depicting the forces acting on the car.
- **b** Determine the vector for the net force, $\underset{\sim}{R}$, acting on the car.
- **c** Calculate the magnitude of the thrust due to the car's engine.
  (*Hint*: To convert from km/h to m/s, divide by 3.6.)

13 When a ball bearing drops vertically through a liquid such as oil, three forces act on it. These are the weight force $\underset{\sim}{W}$, and two upward forces — drag, $\underset{\sim}{D}$, and buoyancy, $\underset{\sim}{B}$. A student drops a 12-gram ball bearing into a cylinder of oil and measures the velocity at two points separated by 15 cm. When the ball is in the oil it experiences a constant buoyancy force of 0.04 N.

- **a** Draw a force vector diagram to illustrate the forces acting on the ball.
- **b** Calculate the size of the resultant force and hence the drag force under the following conditions:
  - **i** the velocity of the ball is the same at both measurement points
  - **ii** the velocity increases from 12 cm/s to 16 cm/s over the 15 cm interval.

14 A mass of 1.5 kg is suspended from a light spring and is vibrating up and down in a vertical line. At a particular instant the spring exerts an upward force of 44.1 N on the mass.

- **a** Draw a vector force diagram to represent the situation.
- **b** Determine the net force acting on the mass.
- **c** Calculate the upward acceleration of the mass.

15 Anita, who has a mass of 60 kg, is a skydiver. She is accelerating toward the Earth at 6.8 m/s^2 with her parachute still closed.

- **a** Draw a vector force diagram to represent the two significant forces which act on Anita.
- **b** What is the magnitude of the resultant force acting on Anita?
- **c** Calculate Anita's weight.
- **d** Determine the magnitude of the upward force of air resistance acting on Anita.

At a later time Anita has opened her parachute and is drifting down at a constant velocity. Ignoring the force of air resistance on her:

- **e** calculate the magnitude and direction of the force that the parachute, via her harness, exerts on Anita.

WORKED Example 19

16 A cart of mass 1.2 kg is launched up an inclined plane with an initial speed of 3.0 m/s. The inclined plane makes an angle of 15° to the horizontal. In this question ignore the effects due to friction.

- **a** Draw a vector force diagram representing forces acting on the cart.
- **b** Calculate the component of the weight force acting on the cart parallel to the plane.
- **c** Calculate the component of the weight force acting on the cart perpendicular to the plane.
- **d** Calculate the size and direction of the resultant force acting on the cart.
- **e** Explain why your answers to **b** and **d** are the same.
- **f** Calculate the acceleration of the cart. (Assume that moving *up* the inclined plane is the positive direction.)
- **g** Calculate the distance that the cart travelled up the inclined plane before coming to rest.

17 The cart from question **16** is again launched up the inclined plane at an initial speed of 3.0 m/s. This time the force due to frictional effects is a constant 0.64 N.

- **a** Draw a vector force diagram representing forces acting on the cart.
- **b** Calculate the size and direction of the acceleration of the cart.
- **c** Find the distance that the cart travelled up the inclined plane before coming to rest.

18 An 80-kg man is standing in a lift. The lift can move upwards or downwards. (In this question take the upward direction to be the direction of the unit vector $\underset{\sim}{j}$.)

- **a** Draw a force vector diagram to illustrate the two forces acting on the man.
- **b** Write an equation for the normal contact force using Newton's Second Law of Motion.
- **c** Show that when the lift is stationary or moving at a constant velocity, the normal contact force has the same magnitude as the weight force acting on him.

The lift now accelerates upwards at $2.0\underset{\sim}{j}$ m/s^2.

- **d** Calculate the magnitude of the resultant force acting on the man.
- **e** Calculate the magnitude of the weight and normal contact force acting on him.
- **f** Comment on the result.

The lift now accelerates downwards at $-2.0\underset{\sim}{j}$ m/s^2.

- **g** Calculate the magnitude of the resultant force acting on the man.
- **h** Calculate the magnitude of the weight and normal contact force acting on him.
- **i** Comment on the result.
- **j** Under what circumstances will he have a normal contact force equal in magnitude to twice his weight?
- **k** Under what circumstances will he have a zero normal contact force?

Applications of Newton's Laws of Motion

In this next section we shall extend the application of Newton's first two laws to include more detailed frictional effects.

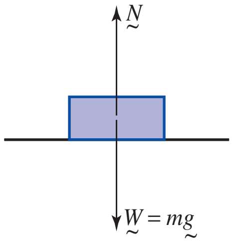
Static and dynamic friction

Frictional forces act when:

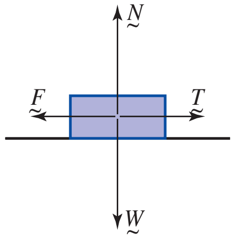
1. two surfaces move in relation to each other, or
2. when an applied force attempts to move an object across a surface.

The former is called *dynamic friction* and the latter is called *static friction*. In general the amount of dynamic friction that acts on a sliding or skidding object is *less than* the amount of limiting static friction which can, for example, support an object on a slope against its weight. One piece of evidence for this fact is that cars which come to a stop without skidding do so in a distance which is less than the stopping distance when skidding. In recent years cars have become equipped with braking systems that inhibit skidding. In this exercise we shall make the simplifying approximation, however, that dynamic friction is equal in magnitude to the limiting static friction.

The diagram at right shows the forces which act on an object resting on a horizontal plane.



Consider now an applied force $\underset{\sim}{T}$ pulling to the right. To oppose this force a frictional force $\underset{\sim}{F}$ is generated which balances the applied force. This is shown at right.



As $\underset{\sim}{T}$ increases, so does $\underset{\sim}{F}$ (to maintain equilibrium) until it reaches a maximum value at which point the block moves in the direction of the applied force $\underset{\sim}{T}$. The block then slides across the plane. The magnitude of the limiting static friction that is produced just prior to slipping is well modelled by the equation:

$$|\underset{\sim}{F}| = \propto|\underset{\sim}{N}|$$

or

$$\boldsymbol{F = \propto N}$$

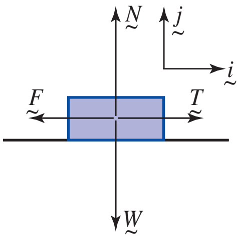
The vector $\underset{\sim}{N}$ is the normal contact force, the vector $\underset{\sim}{F}$ is the limiting friction and the quantity $\propto$ is called the *coefficient of static friction*. The magnitude of the limiting static friction is proportional to the magnitude of the normal contact force. This was first reported by Leonardo da Vinci. If we write the coefficient of friction $\propto$ as the subject of the above equation we get $\propto = \dfrac{F}{N}$.

Smooth surfaces are those where static friction can be ignored ($\underset{\sim}{F} = 0$) in which case the value of $\propto$ is 0. Typical surfaces such as floors and table tops have values of $\propto$ ranging 0.1 to 0.5. Very rough surfaces will have values of $\propto$ in excess of 1.0.

WORKED Example 20

A 0.5-kg weight rests on a table. A student applies an increasing horizontal force on the block. The block first slides across the table when the applied force, $\underset{\sim}{T}$, reaches 2.94 N. Find the magnitudes of:

a the normal contact force **b the force of friction** **c the coefficient of friction.**

THINK	**WRITE**
① At the point of slipping, the body is in equilibrium and so the resultant force $\underset{\sim}{R} = 0$.	$\underset{\sim}{R} = 0$
② Draw a vector force diagram for the four forces which act on the block.	

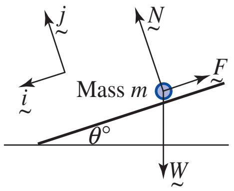
THINK	**WRITE**
③ Use Newton's second law to write the equation of motion.	$\underset{\sim}{R} = (T-F)\underset{\sim}{i} + (N-mg)\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$
④ Substitute $T = 2.94$, $m = 0.5$ and $g = 9.8$ into the equation.	$(2.94-F)\underset{\sim}{i} + (N-0.5 \cdot 9.8)\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$
⑤ Simplify.	$(2.94-F)\underset{\sim}{i} + (N-4.9)\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$
a ① Equate the $\underset{\sim}{j}$ components of the equation of motion.	**a** $\underset{\sim}{j}$ components ⇒ $N - 4.9 = 0$
② Solve for N.	$N = 4.9$ N
b ① Equate the $\underset{\sim}{i}$ components of the equation of motion.	**b** $\underset{\sim}{i}$ components ⇒ $2.94 - F = 0$
② Solve the equation for F.	$F = 2.94$ N
c ① Since the body is on the point of slipping, $\propto\ = \dfrac{F}{N}$.	**c** $\propto\ = \dfrac{F}{N}$
② Evaluate ∝.	$\propto\ = \dfrac{2.94}{4.9}$ $= 0.6$

This value is a typical value for a classroom desk. A rough surface may have a value of ∝ that exceeds 1.

Limiting static friction and dynamic friction on inclined planes

Let us now consider the forces acting on a body on an *inclined* plane. If the body is resting, static friction opposes the component of the body's weight. If the body is sliding down the inclined plane, dynamic friction opposes the component of the body's weight. If the body is sliding *up* the inclined plane, the friction force will act down the plane in the direction opposite to that of the motion of the body.

Friction always acts in a direction opposite to that of the direction of the motion or intended motion.



The diagram at right illustrates forces that act on an object of mass m on an inclined plane which is not smooth.

The resultant force acting on the object is $\underset{\sim}{R}$ where (using Newton's second law):

$$\begin{aligned}\underset{\sim}{R} &= (W\sin\theta - F)\underset{\sim}{i} + (N - W\cos\theta)\underset{\sim}{j}\\ &= m\underset{\sim}{a}\end{aligned}$$

If the object is in equilibrium then $\underset{\sim}{a} = 0$ and thus

$$\underset{\sim}{R} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$$

Equating the coefficients leads to

$$\underset{\sim}{i}:\ W\sin\theta = F$$

$$\underset{\sim}{j}:\ N = W\cos\theta$$

For an object in equilibrium on an inclined plane then, as the angle of the plane, θ, is increased the normal contact force, N, decreases in magnitude while the static friction force, $\underset{\sim}{F}$, increases in magnitude up to its limiting value. When this limiting value is exceeded, the object will begin to move down the plane under the action of unbalanced forces.

There is a critical angle, ϕ, sometimes referred to as the *angle of friction*, which is related to the coefficient of static friction for the pair of surfaces, namely that of the plane and block. At that angle, the block is *about to slip*. We refer to the previous diagram, where the block is in equilibrium but just about to slip. The sum of the three forces is equal to zero, that is:

$$\underset{\sim}{F} + \underset{\sim}{N} + \underset{\sim}{W} = 0,$$

or alternatively:

$$F = W \sin \phi$$

and

$$N = W \cos \phi.$$

Using the equation which defines the coefficient of static friction:

$$F = \propto N$$
$$W \ \sin \phi = \propto W \cos \phi$$
$$\propto = \tan \phi$$

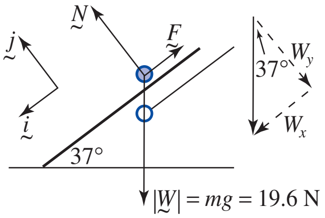
As a consequence of the above, the coefficient of static friction can be determined by placing an object on a ramp and increasing the angle of elevation up to the point of slipping and then measuring the angle of elevation at this point. Alternatively, if $\propto$ is known, then the angle of friction can be established:

$$\phi = \tan^{-1} \propto$$

WORKED Example 21

A 2.0-kg mass rests on an inclined plane such that it is just about to slip. The plane is at an angle of 37° to the horizontal.

- **a** **Draw a force vector diagram depicting the situation.**
- **b** **Calculate the component of the weight of the mass parallel to the plane.**
- **c** **Calculate the magnitude of the normal contact force acting on the mass.**
- **d** **Calculate the magnitude of the force of static friction.**
- **e** **Calculate the coefficient of friction $\propto$.**

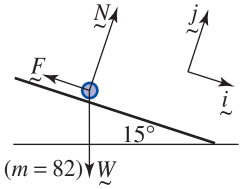
THINK	WRITE
a Draw a force vector diagram for the situation and resolve the weight vector $\underset{\sim}{W}$ into components (a) parallel to $\underset{\sim}{i}$, in the direction of intended motion down the plane and (b) parallel to $\underset{\sim}{j}$, perpendicular to the plane.	**a** 

THINK	WRITE
b ① The component of the weight force parallel to the plane is $W_x = mg\sin\theta$.	**b** $W_x = mg\sin\theta$
② Substitute $m = 2$, $g = 9.8$ and $\theta = 37°$.	$= 2(9.8)\sin 37°$
③ Simplify.	$= 11.8$ N
c ① The resultant force is zero.	**c** $\underset{\sim}{R} = (mg\sin\theta - F)\underset{\sim}{i} + (N - mg\cos\theta)\underset{\sim}{j}$
② Use Newton's Second Law to write the equation of motion for the situation.	$= 0\underset{\sim}{i} + 0\underset{\sim}{j}$
③ Substitute $m = 2$, $g = 9.8$ and $\theta = 37°$ into the equation.	$[2(9.8)\sin 37° - F]\underset{\sim}{i} + [N - 2(9.8)\cos 37°]\underset{\sim}{j}$ $= 0\underset{\sim}{i} + 0\underset{\sim}{j}$
④ Simplify.	$(11.8 - F)\underset{\sim}{i} + (N - 15.65)\underset{\sim}{j} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$
⑤ Equate the $\underset{\sim}{j}$ components of the equation of motion.	The $\underset{\sim}{j}$ component of $\underset{\sim}{R}$: $N - 15.65 = 0$
⑥ Solve for N.	$N = 15.65$ N
d ① Equate the $\underset{\sim}{i}$ components of the equation of motion.	**d** $\underset{\sim}{i}$ component of $\underset{\sim}{R}$: $11.8 - F = 0$
② Solve for F.	$F = 11.8$ N
e ① Since the mass is on the point of slipping, $\propto = \dfrac{F}{N}$.	**e** $\propto = \dfrac{F}{N}$
② Evaluate $\propto$.	$\propto = \dfrac{11.8}{15.65}$ $\propto = 0.75$

Note: The coefficient in this example could also be found using the angle-of-friction principle; that is, $\propto = \tan\theta = \tan 37° = 0.75$.

WORKED Example 22

David is about to slide down a 15° slope on a toboggan. The combined mass of David and the toboggan is 82 kg. He starts from rest and travels 36 m in the first 10 s during which he accelerates uniformly. Calculate the magnitude of the combined forces which retard the motion.

THINK	WRITE
① Draw a force vector diagram showing all the forces acting on David and the toboggan.	

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THINK	WRITE
2 Acceleration is uniform down the slope; position will be a quadratic. Initial velocity and position = 0.	$x = lt^2 + mt + n$ $v = 2lt + m$ $a = 2l$ $x_0 = 0,\ n = 0$ $v_0 = 0,\ m = 0$
State the equation of position.	$x = lt^2$
3 Use $x = 36$ m when $t = 10$ s to find a.	$x = 36,\ t = 10$ $36 = l \cdot 10^2$ $l = 0.36$
4 Find the acceleration down the slope.	$a = 2l = 0.72$ m/s^2 down the slope
5 Use Newton’s second law to write the equation of motion in the $\underset{\sim}{i}$ direction only.	$\underset{\sim}{R} = (mg \sin \theta - F)\underset{\sim}{i}$ $= ma\underset{\sim}{i}$
6 Substitute $m = 82$, $g = 9.8$ and $a = 0.72$ into the equation.	$82(9.8) \sin 15° - F = 82(0.72)$
7 Simplify.	$208 - F = 59$
8 Solve for the retardation forces, F.	$F = 149$
9 State the solution.	Thus the combined force of air resistance and dynamic friction is 149 N acting on David and his toboggan.

remember

1. When solving problems, draw a force vector diagram of the situation and apply Newton’s First or Second Law as appropriate.
2. The weight force parallel to an incline of elevation angle θ is $mg \sin \theta$.
3. The normal contact force for a mass m on an incline of elevation angle θ is $mg \cos \theta$.
4. Friction forces oppose motion.
5. Limiting static friction is given by $F = \propto N$ where the coefficient of friction is $\propto$.
6. Dynamic friction is approximated to be equal to limiting static friction.

EXERCISE 12F Applications of Newton’s Laws of Motion

For the questions in this exercise use $\underset{\sim}{g} = 9.8$ m/s^2 downward. For simplicity, assume the magnitude of dynamic friction to be equal to the magnitude of limiting static friction.

WORKED Example 20

1 A horizontal force of 10 N acts on a 2.0-kg block, resulting in uniform motion across a flat table top. Find the magnitudes of:

a the normal contact force

b the force of friction

c the coefficient of friction.

2 A 2.0-kg block is dragged across a table top at a constant velocity by a horizontal applied force. The coefficient of friction is 0.40. Calculate the magnitude of the applied force.

3 An applied force of 12 N acting at an angle of 30° to the horizontal pulls a 2.0-kg mass across a table top at a constant velocity.

a Draw a force vector diagram depicting the situation.
b Calculate the magnitude of the normal contact force.
c Calculate the magnitude of the force of friction.
d Determine the coefficient of friction for the table top and mass.

4 **a** A 2.5-kg block is slid across the floor at an initial speed of 4.0 m/s in a straight line. It comes to rest over a distance of 10 m. Calculate the coefficient of friction for the floor.

b A second mass of 5.0 kg is slid across the same floor, again with an initial speed of 4.0 m/s. Calculate the distance that the 5.0-kg mass will travel before it stops.

5 **multiple choice**

A 20-kg crate is dragged across a horizontal floor at a constant speed of 2 m/s by a horizontal force of 98 N.

a The coefficient of friction is:
A 0.5 **B** 4.9 **C** 2 **D** 1 **E** 0

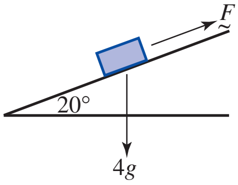
b If the pulling force is stopped, the crate will travel a further:
A 2.45 m **B** 0.82 m **C** 4.9 m **D** 0.41 m **E** 11 m

6 **multiple choice**

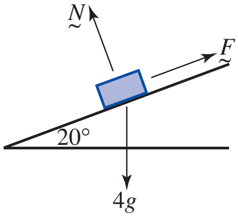
A 4-kg box sitting on a plane inclined at 20° from the horizontal is on the point of sliding down the plane.

a The force diagram depicting the situation is:

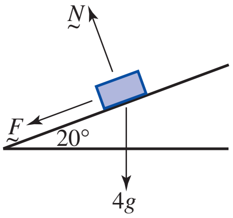
A



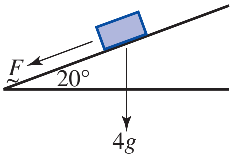
B



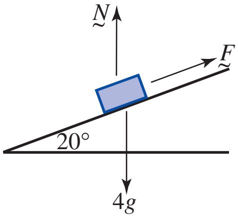
C



D



E



b The coefficient of friction is nearest to:
A 2.75 **B** 1.37 **C** 3.76 **D** 13.41 **E** 0.36

7 In the following situations, objects of mass m are placed on planes with angles of inclination θ and coefficients of friction $\propto$. In each case calculate:

- **i** normal contact force to the nearest unit
- **ii** the frictional force to the nearest tenth
- **iii** the acceleration to the nearest tenth.

a $\propto = 0.10$ $\theta = 20°$ $m = 3.5$ kg **b** $\propto = 0.10$ $\theta = 40°$ $m = 3.5$ kg

c $\propto = 0.20$ $\theta = 20°$ $m = 3.5$ kg **d** $\propto = 0.10$ $\theta = 20°$ $m = 7.0$ kg

WORKED Example 21

8 A 0.75-kg mass is resting on a plane inclined at 18° such that it is just about to slide.

- **a** Draw a force vector diagram depicting the situation.
- **b** Calculate the component of the weight force of the mass parallel to the plane.
- **c** Calculate the magnitude of the normal contact force.
- **d** Calculate the magnitude of the force of static friction.
- **e** Determine the coefficient of friction for the plane and mass.

The angle of the inclined plane is now increased to 30° and the mass accelerates down the plane.

- **f** Calculate the magnitude of the normal contact force.
- **g** Calculate the magnitude of the component of weight acting down the plane.
- **h** Calculate the magnitude of the force of static friction.
- **i** Determine the acceleration of the mass down the plane.

9 A 1.0-kg mass is just in equilibrium on a 12° inclined plane. The angle of inclination is increased to 24°. What is the size of an applied force on the mass parallel to the plane so that it accelerates upwards at 1.0 m/s^2.

10 Sally is pushing a crate up a ramp into a truck at a constant speed. The crate has a mass of 30 kg and the ramp has an incline angle of 8.0° and a coefficient of friction of $\propto = 0.10$.

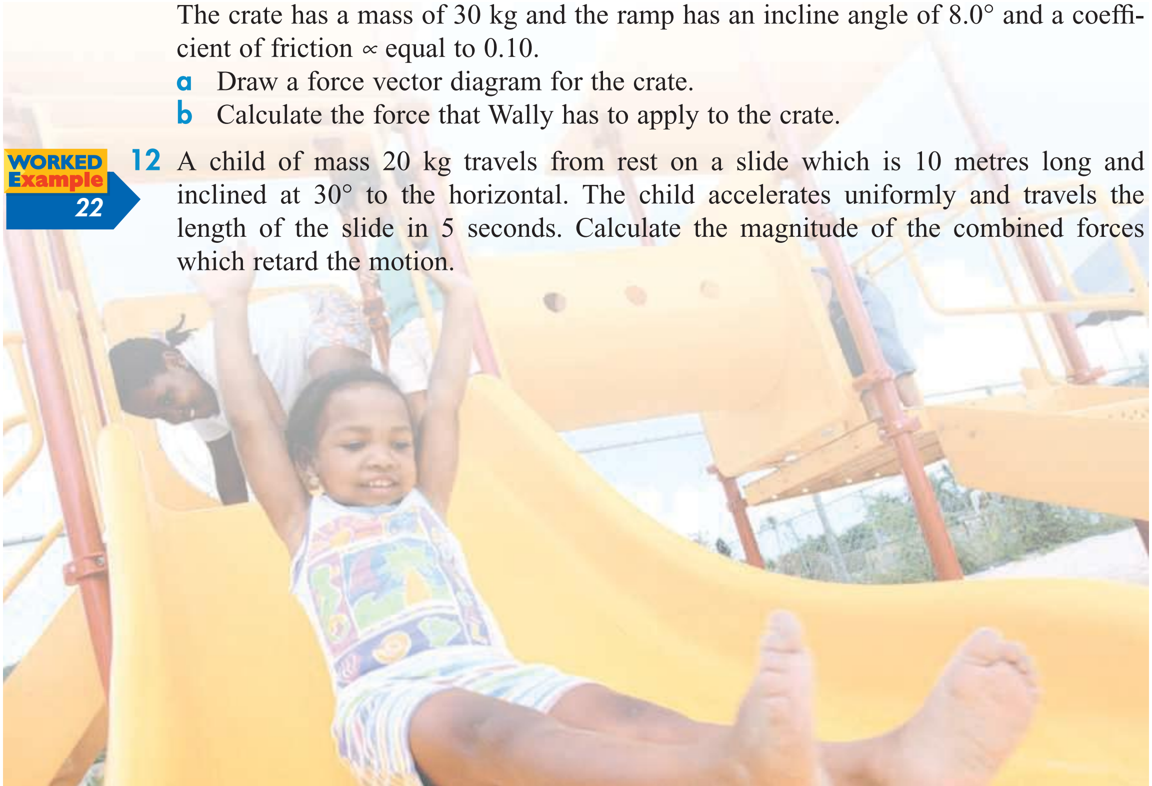
- **a** Draw a force vector diagram for the crate.
- **b** Calculate the force that Sally has to apply to the crate.

11 Wally is pushing a crate up a ramp into a truck at a constant acceleration of 0.50 m/s^2. The crate has a mass of 30 kg and the ramp has an incline angle of 8.0° and a coefficient of friction $\propto$ equal to 0.10.

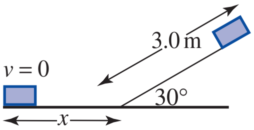
- **a** Draw a force vector diagram for the crate.
- **b** Calculate the force that Wally has to apply to the crate.

WORKED Example 22

12 A child of mass 20 kg travels from rest on a slide which is 10 metres long and inclined at 30° to the horizontal. The child accelerates uniformly and travels the length of the slide in 5 seconds. Calculate the magnitude of the combined forces which retard the motion.



13 A 2.0-kg body slides down a ramp at 30°. The coefficient of friction is 0.10. After sliding 3.0 m down the plane, the body moves onto a horizontal plane whose coefficient of friction is 0.20, and comes to rest in a distance *x*. This is illustrated at right.
Calculate the distance travelled, *x*, before coming to rest.



14 A 1.0-kg mass is projected from the base of a plane inclined at 15° with an initial speed of 2.0 m/s. The plane has a coefficient of friction equal to 0.20. The mass comes to rest and then slides back down the incline to where it was released.

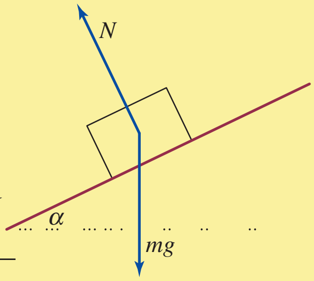
- **a** Draw two force vector diagrams illustrating the forces acting on the mass when travelling up the incline and down the incline.
- **b** Find the distance travelled by the mass up the inclined plane.
- **c** Find the time taken for the mass to return to the base of the incline where it was launched.
- **d** For the mass to stay at rest on the incline, what minimum value of ∝ is required?

Banking of tracks

As a car drives around a circular track, a certain force is needed to keep the car travelling in the circle. The force is usually provided by the friction between the tyres and the road surface. However, if the track is banked, the force needed to maintain the circular motion can be provided in full or in part by the normal reaction of the road on the car.

You will recall that the acceleration of an object moving in a circle can be calculated by:

$$a = \frac{v^2}{r}.$$



Consider the forces on a car of mass *m* travelling around a circle of radius *r* at a speed *v*, where the angle at which the track is banked is α. The direction of motion of the car is into the page.

For certain values of α, no frictional force is needed to maintain the circular motion of the car. The two forces — the normal reaction, *N*, and the weight, *mg* — combine to provide sufficient force.

By considering the two forces, *N* and *mg*, and using $R = ma$ for the car moving in the circle, show that the angle α is given by:

$$\tan\alpha = \frac{v^2}{rg}.$$

Variable forces

Up to now we have considered situations involving constant forces. In this section we deal with problems involving variable forces; that is, forces that change with respect to time, displacement or velocity.

Firstly, recall from kinematics that acceleration can be expressed in each of the following ways:

$$a,\ \frac{dv}{dt},\ \frac{d^2x}{dt^2},\ v\frac{dv}{dx},\ \text{or}\ \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

where **x is the displacement,**
v is the velocity, and
a is the acceleration at any time t.

Newton's first and second laws can still be applied to problems involving variable forces.

Note: In this section we consider only those objects that move in straight lines. Therefore all vector quantities will have associated direction as either positive or negative.

WORKED Example 23

An object of mass 10 kg, initially at rest, is acted on by a force $F = t(t + 1)$ N where $0 \le t \le 5$ seconds.
Find the speed of the object after 4 seconds and the distance travelled.

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Tutorial:
Worked example 23
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THINK	WRITE
1 Use Newton's Second Law to write the equation of motion.	$F = ma$ $t(t + 1) = 10a$
2 Express the acceleration as the subject of the equation.	$a = \dfrac{t(t+1)}{10}$
3 Replace a by its equivalent $\dfrac{dv}{dt}$ since the equation is dependent on t.	$\dfrac{dv}{dt} = \dfrac{t^2}{10} + \dfrac{t}{10}$
4 Antidifferentiate with respect to t to obtain v as a function of t.	$v(t) = \dfrac{t^3}{30} + \dfrac{t^2}{20} + c$
5 Substitute $t = 0$ and $v = 0$ into the equation to find the constant c, since the object is initially at rest.	When $t = 0, v = 0$: $0 = c$
6 Replace c with 0 and rewrite the rule for $v(t)$.	$v(t) = \dfrac{t^3}{30} + \dfrac{t^2}{20}$
7 Substitute $t = 4$ into the equation and evaluate v.	When $t = 4$, $v = \dfrac{64}{30} + \dfrac{16}{20}$ $= 2\frac{14}{15}$
8 Replace $v(t)$ by $\dfrac{dx}{dt}$.	$\dfrac{dx}{dt} = \dfrac{t^3}{30} + \dfrac{t^2}{20}$

THINK	WRITE
⑨ Antidifferentiate with respect to t to obtain $x(t)$.	$x(t) = \frac{t^4}{120} + \frac{t^3}{60} + d$
⑩ Substitute $t = 0$ and $x = 0$ to find d.	When $t = 0, x = 0$ $0 = d$
⑪ Replace $d = 0$ and rewrite the rule for $x(t)$.	$x(t) = \frac{t^4}{120} + \frac{t^3}{60}$
⑫ Substitute $t = 4$ into the equation and evaluate x.	When $t = 4$ $x = \frac{256}{120} + \frac{64}{60}$ $= 3\frac{1}{5}$
⑬ State the solution.	After 4 seconds the object has a speed of $2\frac{14}{15}$ m/s and has travelled $3\frac{1}{5}$ m.

WORKED Example 24

A resultant force of $4x + 5$ acts on a particle of mass 2 kg moving in a straight line where x is the displacement from a fixed origin.
If the velocity $v = 3$ when $x = 0$, find v when $x = 1$.

THINK	WRITE
① Write the equation of motion using Newton's Second Law.	$R = 4x + 5 = 2a$
② Make a the subject of the equation.	$a = 2x + 2.5$
③ Notice a is positive for $x \geq 0$, meaning that velocity is increasing.	As x increases from 0, a is positive. v will increase as x increases.
④ Replace a by its equivalent $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$ since the acceleration depends on x.	or $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = 2x + 2.5$
⑤ Antidifferentiate with respect to x to obtain $\frac{1}{2}v^2$.	$\frac{1}{2}v^2 = x^2 + 2.5x + c$
⑥ Substitute the initial conditions $x = 0$ and $v = 3$ and evaluate c.	When $x = 0, v = 3$: $4.5 = c$
⑦ Rewrite the expression for $\frac{1}{2}v^2$ by replacing $c = 4.5$.	$\frac{1}{2}v^2 = x^2 + 2.5x + 4.5$
⑧ Multiply the equation by 2 to simplify.	$v^2 = 2x^2 + 5x + 9$
⑨ Substitute $x = 1$ and evaluate v.	When $x = 1$ $v^2 = 16$ $v = \pm 4$
⑩ Remember that v is increasing.	As the velocity is increasing, $v = 4$.

WORKED Example 25

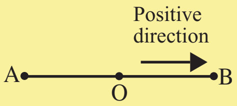
A van of mass 1500 kg is subject to a resistance of 100*v*, where *v* m/s is the velocity at any time *t* seconds.
If the velocity is $30(1 - e^{-0.1t})$ find the driving force, *A*, of the van at any time.

THINK	WRITE
1 Write down the velocity function.	$v = 30(1 - e^{-0.1t})$
2 Differentiate to obtain the acceleration.	$a = 3e^{-0.1t}$
3 Use Newton's Second Law to write down the equation of motion for the van.	$R = A - 100v = 1500a$
4 Substitute the functions for the velocity and acceleration into the equation.	$A - 3000(1 - e^{-0.1t}) = 4500e^{-0.1t}$
5 Transpose the equation to make *A* the subject.	$A = 4500e^{-0.1t} + 3000(1 - e^{-0.1t})$
6 Simplify the equation.	$A = 1500e^{-0.1t} + 3000$ newtons or $1500\,(2 + e^{-0.1t})$

Simple harmonic motion

In many physical situations, objects oscillate in a straight line between two extreme points about a mean position. The motion repeats itself in cycles of equal time intervals or periods. This type of motion is known as *simple harmonic motion*. Examples that can be modelled as simple harmonic motion are the swing of a pendulum, a bungee jumper (although only for a short time) and a vibrating guitar string.

Suppose a particle moves periodically in a straight line between the two points A and B about a centre (or mean position O) as shown below.



The displacement of the particle can be defined in terms of *t* as:

$$x = a \sin nt$$

1 State the period and amplitude of the motion.

2 Find the acceleration in terms of *t*.

3 When and where is the acceleration:
  i equal to zero?
  ii of maximum magnitude?

4 Show that $\ddot{x} = -n^2x$, which is a necessary condition for simple harmonic motion.

5 i Find the velocity, *v*, in terms of *t*.
  ii What is the maximum speed and when and where does it occur?

6 Find an expression for v^2 in terms of *x*.

remember

1. Newton's first and second laws can be applied to problems involving variable forces.
2. If the forces and (consequently) the acceleration are variable then the equation of motion can have acceleration replaced, as appropriate, by either:

$$\frac{dv}{dt}, \quad \frac{d^2x}{dt^2}, \quad v\frac{dv}{dx} \quad \text{or} \quad \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$

EXERCISE 12G Variable forces

1 A particle of mass 1 kg is acted on by a force of magnitude $(4t - 3t^2)$ where $0 \le t \le 5$. If the particle is initially at rest at the origin, find:

a the velocity at any time t

b the displacement at any time t.

2 An object of mass 3 kg is acted on by a force of magnitude $t - \cos\frac{t}{2}$ where $0 \le t \le \pi$. If the object is initially at rest at the origin, find:

a the velocity at any time t

b the displacement at any time t.

WORKED Example 23

3 An object of mass 4 kg, initially at rest, is acted on by a force $F = (2 - t)^2$ N where $0 \le t \le 7$ seconds. Find:

a the speed of the object after 5 seconds

b the distance travelled.

4 A body of mass 6 kg, initially at rest, is acted on by a force $F = 1 + \sin 2t$ N where $0 \le t \le 4$ seconds. Find:

a the speed of the body after $\frac{\pi}{2}$ seconds

b the distance travelled.

5 A 2-kg mass initially travelling at 2 m/s is brought to rest by a force $F = \dfrac{40}{(t+5)^2}$ N where $t \ge 0$.

a Calculate the time at which the body comes to rest.

b Calculate the distance travelled in this time.

6 A car of mass 1 tonne starts from rest and moves against a constant resistance force of 400 N. The driving force decreases uniformly from 2400 N so that after 10 seconds it has a value of 2000 N. Find:

a the resultant force at any time t

b **i** the velocity and
  ii distance travelled after 5 seconds.

WORKED Example 24

7 A resultant force of $2x + 3$ acts on a particle of mass 4 kg which is moving in a straight line, where x is the displacement from a fixed origin. If the velocity $v = 2$ when $x = 0$, find v when $x = 2$.

8 A resultant force of $x^2 + 1$ acts on a body of mass 2 kg which is moving in a straight line, where x is the displacement from a fixed origin. If the velocity $v = 4$ when $x = 3$, find v when $x = 1$.

9 A crane is used to lift a crate of mass 400 kg vertically upwards from the ground. The tension force in the cable has a decreasing magnitude of $(6000 - 4x)$ N where x metres is the height of the crate above the ground and $0 \le x \le 50$. Find:

a the acceleration of the crate when at a height of 20 metres

b the speed of the crate when it is 40 metres above the ground.

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WORKED Example 25

10 A car of mass 800 kg is subject to a resistance of $60v$ where v m/s is the velocity at any time t seconds. If the velocity is $v = 40(1 - e^{-0.2t})$ m/s, find the driving force at any time.

11 A train of mass 100 tonnes is subject to a resistance of $500v$ where v m/s is the velocity at any time t seconds. If the velocity is $v = 25(1 - e^{-0.05t})$ m/s, find:

a the driving force at any time

b the time, to the nearest second, when the driving force is 25 000 N.

12 A body of mass 1.5 kg, initially at rest, is acted on by a force $\left(30 - \frac{v}{2}\right)$ N. Find:

a the velocity at any time t

b the terminal velocity of the mass

c the distance, to the nearest metre, travelled in the first 6 seconds.

summary

Differentiation and displacement, velocity and acceleration

- Position gives the location of a particle relative to a reference point (usually the origin). The variable used is x.
- Displacement is change in position δx or s. Displacement and position have the same value if a body is initially at the origin and so the two terms are often used to mean the same thing. Hence, the same variable is used, namely x.
- Instantaneous velocity, v or $\frac{dx}{dt}$ or $\dot{x}$, is the time rate of change of position or the rate of displacement.
- Average velocity during time interval $\delta t = \frac{\delta x}{\delta t} = \frac{s}{\delta t} = \frac{\text{displacement}}{\text{time}}$.
- Speed $= \frac{\text{Distance travelled}}{\text{Time taken}}$.
- Instantaneous acceleration, a or $\ddot{x}$ or $\frac{d^2x}{dt^2}$ or $\frac{dv}{dt}$ is the time rate of change of velocity.

Using antidifferentiation

- $v(t) = \int a \, dt + c$. The instantaneous velocity can be found by antidifferentiating the instantaneous acceleration. The value for c is found by knowing the velocity at a specific time, usually at time $t = 0$.
- $x(t) = \int v \, dt + c$. The displacement and hence the position can be found by antidifferentiating the instantaneous velocity. The value for c is found by knowing the position at a specific time, usually at time $t = 0$.

Motion under constant acceleration

- When the acceleration is a constant, the equation of position will be a quadratic.
  1. $x = lt^2 + mt + n$
  2. $v = 2lt + m$
  3. $a = 2l$

Applying differential equations to rectilinear motion

- If $v = f(x)$, then $t = \int \frac{1}{f(x)} \, dx$.
- $a = v\frac{dv}{dx}$ and thus:

  $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.
- If $a = g(v)$, then $t = \int \frac{1}{g(v)} \, dv$.

Position, velocity and acceleration

- Position, $\underset{\sim}{r}(t)$, velocity, $\underset{\sim}{v}(t)$ and acceleration, $\underset{\sim}{a}(t)$, are vector quantities. In two dimensions they can be written as:

  $$\underset{\sim}{r}(t) = x\underset{\sim}{i} + y\underset{\sim}{j}$$

$\underset{\sim}{v}(t) = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$ [*Note*: $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j}$]

$\underset{\sim}{a}(t) = a_x\underset{\sim}{i} + a_y\underset{\sim}{j}$, respectively. [*Note*: $\underset{\sim}{a}(t) = \ddot{\underset{\sim}{r}}(t) = \frac{d^2\underset{\sim}{r}}{dt^2} = \frac{d^2x}{dt^2}\underset{\sim}{i} + \frac{d^2y}{dt}\underset{\sim}{j}$]

- The average velocity during the time interval Δt is $\underset{\sim}{v}_{av} = \frac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t}$.
- The instantaneous velocity $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j}$.
- The instantaneous speed is given by $\underset{\sim}{v} \cdot \underset{\sim}{v} = \sqrt{v_x^2 + v_y^2}$ which is the magnitude of the velocity vector $\underset{\sim}{v}$.
- The direction of motion is in the direction parallel to the velocity vector.
- The instantaneous acceleration $\underset{\sim}{a}(t) = \frac{d\underset{\sim}{v}}{dt} = \frac{dv_x}{dt}\underset{\sim}{i} + \frac{dv_y}{dt}\underset{\sim}{j} = \frac{d^2x}{dt^2}\underset{\sim}{i} + \frac{d^2y}{dt^2}\underset{\sim}{j}$.
- Two non-zero vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, are perpendicular when $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.
- The angle, θ, between two vectors is given by $\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$ where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are vectors.
- For circular motion with constant speed:

$$v = r\omega$$

$$a = r\omega^2 = \frac{v^2}{r}$$

Newton's First Law of Motion

- Newton's First Law of Motion is used for equilibrium or statics problems:

$\underset{\sim}{R} = 0$ (the resultant force is zero)
$a = 0$ (the magnitude of acceleration is zero)
$v = c$ (the magnitude of velocity is a constant).

Newton's Second Law of Motion

- $\underset{\sim}{R} = m\underset{\sim}{a}$ (the resultant force is equal to the product of mass and acceleration).
- The weight force parallel to an incline of elevation angle θ is $mg\sin\theta$.
- The normal contact force for a mass m on an incline of elevation angle θ is $mg\cos\theta$.
- Friction forces oppose motion. Limiting static friction is given by $|\underset{\sim}{F}| = \propto|\underset{\sim}{N}|$. The coefficient of friction is $\propto$. Dynamic friction is approximated to be equal to limiting static friction.
- To solve dynamics and related kinematics problems:
  1. read the question carefully
  2. draw a clear diagram which has all the information
  3. superimpose arrows depicting vectors which act on the body in question
  4. find a value or expression for the resultant force vector which is the sum of all forces acting on the body
  5. equate this to the product of inertial mass and acceleration.

Variable forces

- Acceleration:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = v\frac{dv}{dx}$$

CHAPTER review

Questions **1** to **3** refer to a particle moving in a straight line, which has a displacement, x cm, from the origin at any time, t seconds, given by:

$$x(t) = t^2 - 6t + 5$$

1 **multiple choice** 12A

The initial velocity in cm/s is:

A 4
B 0
C −2
D 5
E −6

2 **multiple choice** 12A

The distance travelled in the first 4 seconds, in cm, is:

A 8
B 9
C 10
D 3
E 7

3 **multiple choice** 12A

The number of times the particle passes through the origin is:

A 1
B 2
C 0
D 4
E 3

4 **multiple choice** 12A

A particle moves in a straight line so that its position at any time is given by:

$$x = t^3 - 2t$$

The acceleration of the particle when $t = 1$ is:

A 1
B 4
C 0
D 6
E 8

5 **multiple choice** 12A

The velocity of a body moving in a straight line is $v = 3t^2 + 5$. Initially the body is at −2. The position at time $t = 2$ is:

A 16
B 17
C −4
D 15
E 10

6 **multiple choice** 12A

A body starts from the origin, travelling in a straight line with a velocity given by:

$$\dot{x} = \frac{t+1}{t^2 + 2t + 3}$$

The position of the body at time $t = 3$ is closest to:

A 0.90
B 3.21
C 9
D −2.5
E 0

12A **7** The acceleration of an object moving in a straight line is given by:

$$a = 3 - 4t \text{ m/s}^2$$

Initially, its position is $x = 3$ and the velocity is 2 m/s. What is its position when $t = 6$?

12A **8** The velocity of a particle travelling in a straight line is given by:

$$v = 2t^2 - 5t + 2 \text{ cm/s}$$

Find the distance travelled by the particle between the times when it is momentarily at rest.

12A **9** For each acceleration given below, find:

- **i** the velocity
- **ii** the displacement at any time t.

a $\ddot{x} = 6t - 4$; when $t = 0$, $\dot{x} = 1$ and $x = 0$

b $\ddot{x} = 3t + 4$; when $t = 0$, $\dot{x} = 6$ and $x = 0.8$

c $\dfrac{dv}{dt} = 2e^{\frac{t}{2}}$; when $t = 0$, $v = -1$ and $x = 0$

d $\dfrac{dv}{dt} = \dfrac{t}{(4 - t^2)^{\frac{3}{2}}}$, $0 \leq t < 2$; when $t = 0$, $v = 1$ and $x = -2$

e $a = \dfrac{-3}{(t + 1)^2}$; when $t = 0$, $v = 0$ and $x = 5$

f $a = \dfrac{e^t}{e^t + 1}$; when $t = 0$, $v = 0$.

Find the velocity only.

12A **10** A missile is fired vertically up from the ground with an initial velocity of 20 m/s. The acceleration at any time is given by:

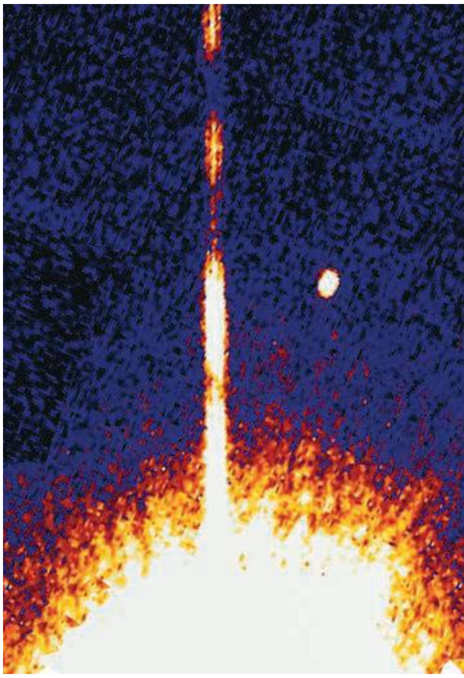
$$\ddot{x} = -10 + \frac{t}{2} \text{ m/s}^2$$

Find:

a the height of the missile after 2 seconds

b the height of the missile after 4 seconds

c the direction the missile is travelling when $t = 4$.



12A **11** The acceleration of a body travelling in a straight line is given by:

$$\ddot{x} = \frac{12}{(2t + 1)^3} \text{ cm/s}^3$$

and $x = 0$ cm and $\dot{x} = 0$ cm/s when $t = 0$.

a Find the displacement at any time t.

b Find the distance travelled in the first 2 seconds.

12 A cyclist accelerates uniformly from 4 m/s to 12 m/s in 10 seconds. Find: 12B

a the acceleration

b the distance travelled in the first 5 seconds

c the time taken to reach a speed of 18 m/s.

13 A ball is thrown vertically upwards, at a speed of 30 m/s, from a tower 22 metres above ground level. 12B

a When does the ball reach ground level?

b What total distance does the ball travel?

14 A car travels 80 metres in the first 10 seconds and then 80 metres in the next 20 seconds. If the acceleration is constant, find: 12B

a how much further it will travel before coming to rest

b how many more seconds it will travel before coming to rest.

15 An object is projected vertically upwards from a tower that is 20 metres above ground level. If the object takes 4.5 seconds to reach the ground, calculate the initial velocity of the object. 12B

16 **multiple choice** 12C

Given that $\frac{dv}{dt} = \frac{1}{2v-4}$, $v > 2$; and that $v = 3$ when $t = 1$ for an object moving in a straight line, the acceleration when $t = 9$ is:

A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** $\frac{1}{6}$ **D** $1\frac{1}{2}$ **E** $\frac{2}{3}$

17 **multiple choice** 12C

A body moving in a straight line is initially at rest at the origin and has an acceleration given by:

$$a = x + 4 \text{ cm/s}^2, x \geq 0$$

The velocity of the body when $x = 1$, in cm/s, is:

A 2 **B** 1 **C** 4 **D** 5 **E** 3

18 A particle moves in a straight line so that its displacement from a fixed given origin is given by: 12C

$$x = t^3 - 5t^2 - 8t + 33$$

a Find when and where the particle first comes to rest.

b When is the velocity negative?

c Illustrate the motion using a displacement–time line.

19 An object moving in a straight line has an acceleration of $(x - 4)$. When $x = 4$, the velocity $v = 0$. 12C

a Explain why the velocity is decreasing from $x = 0$ to $x = 4$.

b Hence, find v at $x = 0$.

12C

20 The velocity of a particle moving in a straight line is given by:

$$v = \sqrt{4 - x^2},\ x > 0.$$

Find the acceleration when $v = \sqrt{3}$.

12C

21 The velocity of a body moving in a straight line which is initially at the origin is given by:

$$v = 10\cos^2\frac{\pi t}{8}.$$

When $t = 2$, find:

a the acceleration

b the displacement.

12C

22 **multiple choice**

An object which is initially at the origin moves in a straight line so that its velocity is $v(x) = x + 1$. The position, x, at any time, t, is given by:

A $x = \log_e t$ **B** $x = e^t + 1$ **C** $x = \sqrt{2t} - 1$ **D** $x = e^t - 1$ **E** $x = \sin t$

12C

23 The acceleration of an object starting at rest from the origin is $\frac{dv}{dt} = 3 + v$, where v is the velocity.

a Find the velocity as a function of time.

b Find the displacement as a function of time.

12C

24 The velocity of a particle starting at the origin is given by $\frac{dx}{dt} = \frac{\sqrt{4 - x^2}}{3}$, where x is the displacement.

a Find x as a function of t.

b Find the maximum displacement from O.

12D

25 **multiple choice**

The position vector of an object is given by $\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + t^2\underset{\sim}{j}$. The average velocity in the time interval [1, 2] is:

A $3\underset{\sim}{i} + 3\underset{\sim}{j}$ **B** $3\underset{\sim}{i} + 2\underset{\sim}{j}$ **C** $3\underset{\sim}{i} - 3\underset{\sim}{j}$ **D** $2\underset{\sim}{i} + 3\underset{\sim}{j}$ **E** $3\underset{\sim}{i} - 2\underset{\sim}{j}$

12D

26 **multiple choice**

The position vector of an object is given by $\underset{\sim}{r}(t) = (t^3 - t)\underset{\sim}{i} + (t - t^2)\underset{\sim}{j}$. The magnitude of the velocity at $t = 2$ is:

A $\sqrt{112}$ **B** $\sqrt{40}$ **C** $\sqrt{130}$ **D** 32 **E** $\sqrt{153}$

12D

27 **multiple choice**

If $\underset{\sim}{r}(t) = t\underset{\sim}{i} - (3t^2 + \cos\pi t)\underset{\sim}{j}$ then $\ddot{\underset{\sim}{r}}(2)$ is equal to:

A $(\pi^2 - 6)\underset{\sim}{j}$ **B** $(6 + \pi^2)\underset{\sim}{j}$ **C** $(6 - \pi^2)\underset{\sim}{j}$

D $(-6 - \pi^2)\underset{\sim}{j}$ **E** $-(6 + \pi^2)\underset{\sim}{j}$

28 **multiple choice** 12D

Two objects, M and N, have position vectors given by:

$\underset{\sim}{r}_{\text{M}} = (4t+6)\underset{\sim}{i} - \frac{1}{2}t^2\underset{\sim}{j}$ $\underset{\sim}{r}_{\text{N}} = (2t-1)\underset{\sim}{i} + 3t\underset{\sim}{j}$

The nearest whole angle between their respective directions at $t = 2$ is:

A 37° **B** 53° **C** 127° **D** 143° **E** 83°

29 The velocity of a projectile in m/s is given by $\underset{\sim}{v}(t) = (4t^2 - 6t)\underset{\sim}{i} - 8\cos 7\pi t\underset{\sim}{j}$. Find the magnitude of the acceleration of the projectile at time $t = 3$ s. 12D

30 **multiple choice** 12E

An 800-kg car has two horizontal opposing forces acting on it as it travels along a horizontal road. The first is a retarding force of 400 N; the second is the applied force of the engine which is 800 N in magnitude. The acceleration of the car, in m/s^2, is:

A 1.5 **B** 1.0 **C** 0.50 **D** 0 **E** –0.50

31 A bus of mass 8 tonnes accelerates from rest to a speed of 18 m/s in 12 seconds. Find: 12E

a the average acceleration of the bus

b the magnitude of the average resultant force acting on the bus during this time

c the distance travelled by the bus during this time.

32 **multiple choice** 12F

Two opposing forces act on a 2.0-kg body to produce an acceleration of 3.0 m/s^2. If the magnitude of one force is 8.0 N, the magnitude of the second force could be:

A 12.0 N **B** 6.0 N **C** 10 N **D** 14 N **E** 16 N

33 **multiple choice** 12F

A particle of mass 2 kg rests on a rough plane inclined at 60° to the horizontal. The friction force which prevents the mass slipping down the plane is:

A $4g$ N **B** g N **C** $2g$ N **D** $\sqrt{3}\,g$ N **E** $2\sqrt{3}\,g$ N

34 **multiple choice** 12F

A 2.0-kg mass whose initial speed is 5.0 m/s slides to rest along a rough floor over 5.0 m. The coefficient of friction $\propto$ is (to 2 decimal places):

A 0.24 **B** 0.48 **C** 0.26 **D** 0.52 **E** 0.60

35 **multiple choice** 12F

The acceleration of a 5.0-kg mass down an inclined plane elevated at 10° to the horizontal is 1.50 m/s^2. The magnitude of the force of friction is:

A 1.0 N **B** 1.2 N **C** 1.4 N **D** 1.6 N **E** 1.8 N

12F

36 **a** A body of mass 5.0 kg rests on a plane of inclination 20° in limiting equilibrium. Find the coefficient of friction.

b If the inclination is increased to 30°, find the acceleration down the plane.

Use the following information for questions **37** and **38**.
A resultant force of $6t + 3$ newtons acts on a mass of 3 kg which is initially at rest.

12G

37 **multiple choice**

The velocity in m/s of the mass at any time t is:

A 6 **B** $t^2 + t$ **C** $\frac{1}{2}t^2 + 2t$ **D** $t^2 + 3t$ **E** $t^2 + t + 1$

12G

38 **multiple choice**

The distance travelled by the mass in the first 6 seconds is:

A 45 m **B** 72 m **C** 90 m **D** 54 m **E** 40 m

12G

39 A particle of mass 2 kg moves in a straight line and, at time t, its displacement from a fixed origin is x and its speed is v. If the resultant force is $4x - 2$, and $v = 1$ when $x = 0$, find v when $x = 2$.

Modelling and problem solving

1 The velocity of a bus travelling on a straight line between two stops is given by:

$$v = 16 \sin\frac{\pi t}{30} \text{ m/min}$$

Find:

a the time lapse between stops

b the distance travelled between stops

c the maximum velocity of the bus and when it occurs.

2 A high diver jumps off a tower which is 44.1 metres above a deep tank of water. When the diver hits the water he is subject to an acceleration given by $a = -0.4(v + 0.6)^2$ m/s^2, $v \geq 0$ where v is the velocity in the water at any time t seconds after impact with the water.

a Find the velocity of the diver on impact with the water, assuming a constant acceleration of 9.8 m/s^2.

b Find the velocity of the diver $v(t)$ in the water at time t.

c How long after the diver leaves the tower does he come to rest in the tank?

d How far below the surface of the water does he descend?

3 An empty lift which has a mass of 800 kg is entered by a person of mass 70 kg. It then descends at its maximum acceleration of 1.2 m/s^2.

a Calculate:

  i the tension in the lift cable

  ii the force exerted on the person by the lift floor.

b The person's glasses fall off from a height of 1.7 metres. How long does it take for the glasses to hit the floor of the lift?

c The maximum allowable tension in the cable is 19 360 N. Determine the maximum number of people of average mass 80 kg that the lift can carry.
(*Hint*: Be careful as to which direction gives the maximum tension.)

4 A body of mass m is about to slide up a plane inclined at an angle θ to the horizontal due to the action of an applied force acting at an angle β to the incline. The plane has a coefficient of friction $\propto$.

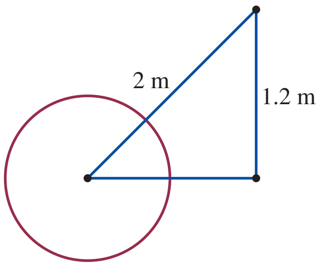
a Show that the magnitude of the normal contact force acting on the body is:

$$N = \frac{mg\cos(\theta + \beta)}{\cos\beta + \propto\sin\beta}$$

b What is the value of β when the applied force has the least magnitude?

5 A curve on a railway track has a radius of 352 m and is banked to suit a speed of 22 m/s (at this speed the train won't want to slip on the curve). Find the angle of the banking.

6 A man pushes a 100-kg garden roller over a level lawn. He holds the 2-m-long handle 1.2 m above the horizontal axis (as shown). He is able to push down on the handle with a force of 210 N.



a What is the size of the effective force pushing the roller along?

b What force is the roller exerting on the lawn?

c If the force applied keeps the roller moving at a constant speed, calculate the coefficient of friction between the roller and the lawn.

d If the man pulled the roller (holding the handle in the same position) with the same force, what force would the roller exert on the lawn?

e If the roller was caught in a small rut, would it be easier to push or pull it out? Justify your answer.

7 Olivia is spinning her 1-m-diameter umbrella at a rate of 2 revolutions in 5 seconds. If she holds the rim 1.225 m above the ground, find the diameter of the circle where the droplets meet the ground. (The drops leave the horizontal rim of the umbrella tangentially).

8 A pilot is travelling in a straight line at an angle of 45° to the ground and is aiming at a point X on the ground. When he is 700 m above the ground and travelling at 480 km/h he releases a package. Find how far away from X the package lands. (The package will have the same velocity as the plane when it leaves. Ignore air resistance.)

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Test Yourself
Chapter 12

eBook*plus* ACTIVITIES

12A Position, velocity and acceleration

Interactivity

- int-0267: Use the interactivity to explore kinematics. (*page 552*)

12G Variable forces

Tutorial

- int-0422: Watch a tutorial on how to use calculus to find the velocity of a particle given its variable force. (*page 606*)

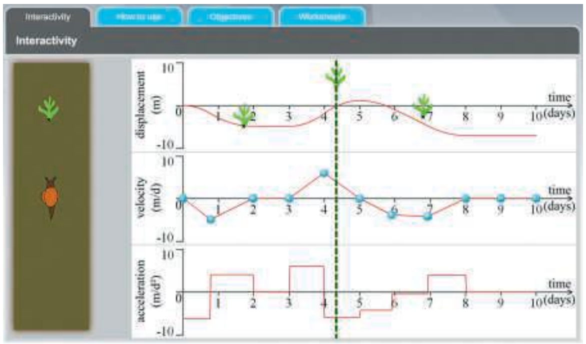
Digital doc

- WorkSHEET 12.1: Calculate magnitude and vector forces of bodies in motion. (*page 610*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. (*page 619*)



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Chapter 1 page 5

Graphics Calculator tip!

Complex Numbers

The graphics calculator can be used to good effect in performing calculations on complex numbers.

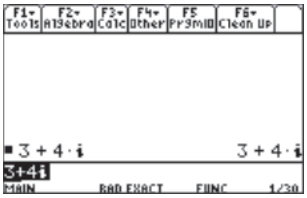
For the TI-89 Titanium

To enter the complex number 3 + 4i, on a HOME page, complete the entry line as:

3 + 4i

Then press **ENTER**.

Note: The complex number 'i' is found as the second function on the **CATALOG** key.



```
■3 + 4·i                3 + 4·i
3+4i
```

Chapter 1 page 6

WORKED Example 4

THINK

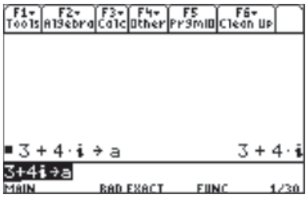
WRITE/DISPLAY

For the TI-89 Titanium

① As the complex number 3 + 4*i* is used frequently, it is worthwhile storing it as a variable instead of frequently re-entering it, to do this complete the entry line as:

3 + 4*i* (STO ▶) a

Then press (ENTER).



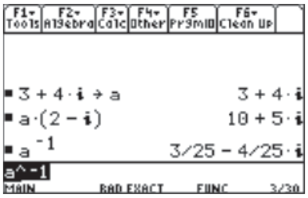
```
■3+4·i→a                    3+4·i
3+4i→a
MAIN      RAD EXACT     FUNC     1/30
```

② To answer parts **a** and **b**, complete the entry lines as:

$a \times (2 - i)$

a^−1

Press (ENTER) after each line.

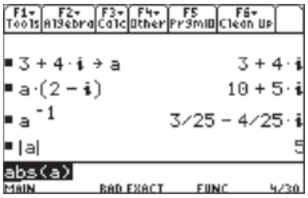


```
■3+4·i→a                    3+4·i
■a·(2−i)                   10+5·i
■a^-1                 3/25−4/25·i
a^-1
MAIN      RAD EXACT     FUNC     3/30
```

③ To calculate |3 + 4*i*| , press (CATALOG) and select the abs(function. Complete the entry line as:

abs(a)

Then press (ENTER).



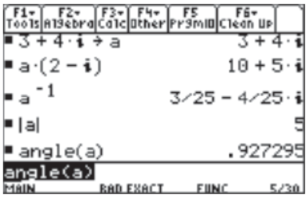
```
■3+4·i→a                    3+4·i
■a·(2−i)                   10+5·i
■a^-1                 3/25−4/25·i
■|a|                            5
abs(a)
MAIN      RAD EXACT     FUNC     4/30
```

④ To calculate Arg(3 + 4*i*), press (CATALOG) and select the angle(function.

Complete the entry line as:

angle(a)

Then press (ENTER).



```
■3+4·i→a                    3+4·i
■a·(2−i)                   10+5·i
■a^-1                 3/25−4/25·i
■|a|                            5
■angle(a)                 .927295
angle(a)
MAIN      RAD EXACT     FUNC     5/30
```

⑤ Write the answers.

For **a**, $(3 + 4i) \times (2 - i) = 10 + 5i$

For **b**, $(3 + 4i)^{-1} = \frac{3}{25} - \frac{4}{25}i$

For **c**, $|3 + 4i| = 5$

For **d**, $\text{Arg}(3 + 4i) = 0.9273$

Chapter 1 page 16

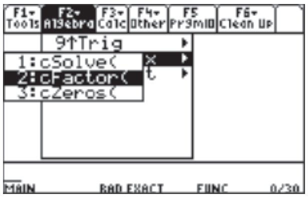
WORKED Example 9

THINK

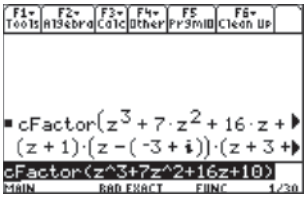
WRITE/DISPLAY

Using the TI-89 Titanium

① To find factors of $f(z)$ over C, on a HOME page press:
- F2
- A: Complex
- 2: cFactor



② Complete the entry line as:
cFactor $(z^3 + 7z^2 + 16z + 10)$
Then press ENTER.



```
■ cFactor(z³ + 7·z² + 16·z +▸
  (z + 1)·(z −(⁻3 + i))·(z + 3 +▸
cFactor(z^3+7z^2+16z+10)
```

③ Write the answer.

$f(z) = z^3 + 7z^2 + 16z + 10$
$f(z) = (z + 1)(z + 3 - i)(z + 3 + i)$

Chapter 1 page 18

WORKED Example 11

THINK

WRITE/DISPLAY

Using the TI-89 Titanium

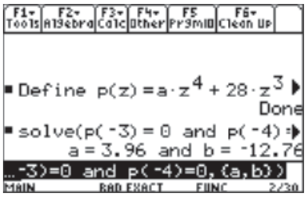
① On a HOME page, complete the entry line as:

Define $p(z) = a \times z^4 + 28z^3 + 49z^2 + 7 + b$

Press ENTER.
To solve for a and b, complete the entry line as:

solve($p(-3) = 0$ and $p(-4) = 0$, $\{a, b\}$)

Press ENTER.



```
■ Define p(z) = a·z⁴ + 28·z³ ▸
                         Done
■ solve(p(⁻3) = 0 and p(⁻4) =▸
      a = 3.96 and b = ⁻12.76
…⁻3)=0 and p(⁻4)=0,{a,b})
```

② Write the answer.

$P(z) = az^4 + 28z^3 + 49z^2 + 7 + b$
If $P(3) = 0$ and $P(-4) = 0$, solving for a and b gives $a = 3.96$ and $b = -12.76$.

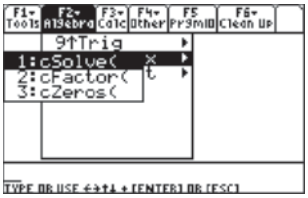
Chapter 1 page 21

WORKED Example 14

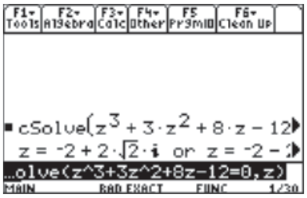
THINK

Using the TI-89 Titanium

① On a HOME page, press:
- (F2)
- A: Complex
- 1: cSolve(



② Complete the entry line as:
cSolve($z^3 + 3z^2 + 8z - 12 = 0, z$)
Then press (ENTER).



```
■ cSolve(z³ + 3·z² + 8·z − 12▸
  z = ⁻2 + 2·√2·i  or  z = ⁻2 − 2▸
...olve(z^3+3z^2+8z-12=0,z)
MAIN        RAD EXACT     FUNC       1/30
```

③ Write the answer.

WRITE/DISPLAY

$z^3 + 3z^2 + 8z - 12 = 0$
Solving for z over C gives
$z = 1$ or $z = -2 + 2\sqrt{2}i$ or
$z = -2 - 2\sqrt{2}i$

Chapter 2 page 48

WORKED Example 4

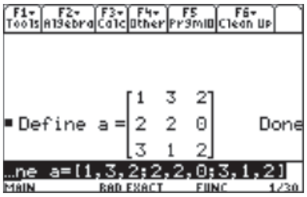
THINK

WRITE/DISPLAY

Using the TI-89 Titanium

① To define Matrix A, on a HOME page complete the entry line as:
Define a = [1,3,2;2,2,0;3,1,2]
Then press (ENTER).

Note: Elements in a matrix row need to be separated by a comma. The rows need to be separated by a semicolon.



```
            [1  3  2]
■Define a = [2  2  0]      Done
            [3  1  2]
...ne a=[1,3,2;2,2,0;3,1,2]
```

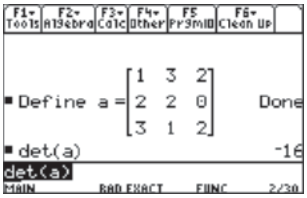
② To find the determinant of Matrix A, press:

- 2ND
- 5: MATH
- 4: Matrix
- 2:det(

Complete the entry line as:

det(a)

Then press (ENTER).



```
            [1  3  2]
■Define a = [2  2  0]      Done
            [3  1  2]
■det(a)                     -16
det(a)
```

③ Write the answer.

The determinant of Matrix A is −16.

Chapter 2 page 50

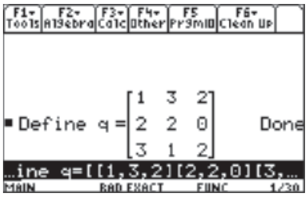
WORKED Example 6

THINK

WRITE/DISPLAY

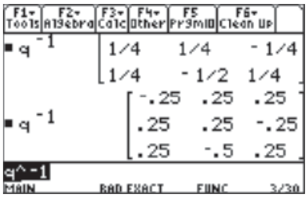
For the TI-89 Titanium

① To define Matrix Q, on a HOME page complete the entry line as:
Define q = [1,3,2;2,2,0;3,1,2]
Then press ENTER.



```
                 ┌1  3  2┐
■Define q =      │2  2  0│        Done
                 └3  1  2┘
...ine q=[[1,3,2][2,2,0][3,...
MAIN          RAD EXACT     FUNC     1/30
```

② To determine the inverse of Matrix Q, complete the entry line as:
q^{-1}
Then press ENTER.



```
■q⁻¹         │1/4    1/4    -1/4
             └1/4   -1/2    1/4 ┘
                ┌-.25   .25   .25 ┐
■q⁻¹            │.25    .25   -.25│
                └.25    -.5   .25 ┘
q^-1
MAIN          RAD EXACT     FUNC     3/30
```

③ Write the answer.

$$Q^{-1} = \begin{bmatrix} -0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & -0.25 \\ 0.25 & -0.50 & 0.25 \end{bmatrix}$$

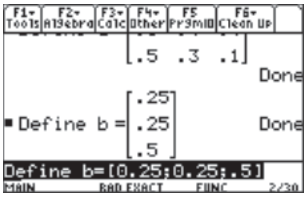
Chapter 2 page 59

WORKED Example 8

THINK **WRITE/DISPLAY**

For the TI-89 Titanium

② On a HOME page, complete the entry lines as:
Define a = [0.2,0.3,0.5;0.3,0.4,0.4;0.5,0.3,0.1]
Define b = [0.25;0.25;0.50]
Press (ENTER) after each line.

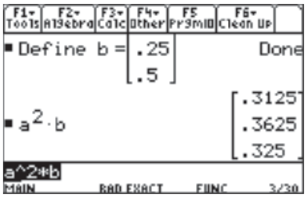


```
             [.5  .3  .1]
                           Done
             [.25]
■ Define b = [.25]         Done
             [.5 ]
Define b=[0.25;0.25;.5]
```

③ To multiply the matrices, complete the entry line as:

$a^2 \times b$

Then press (ENTER).



```
■ Define b = |.25          Done
             [.5 ]
                        [.3125]
■ a²·b                  [.3625]
                        [.325 ]
a^2*b
```

④ Write the answer.

The steady state will be

$$p = \begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.3250 \end{bmatrix}.$$

Chapter 2 page 69

WORKED Example 12

THINK

WRITE/DISPLAY

For the TI-89 Titanium

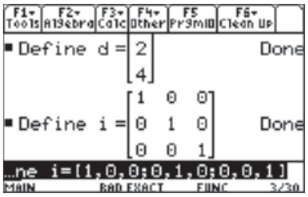
a ③ On a HOME page, complete the entry lines as:

Define $c = [0.2,0.3,0.3;0.4,0.1,0.3;$
$0.3,0.4,0.1]$

Define $d = [5;2;4]$

Define $i = [1,0,0;0,1,0;0,0,1]$

Press (ENTER) after each line.

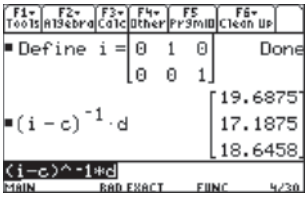


```
■Define d =│2 │           Done
           │4┘
           ┌1  0  0┐
■Define i =│0  1  0│       Done
           └0  0  1┘
...ne i=[1,0,0;0,1,0;0,0,1]
MAIN       RAD EXACT     FUNC      3/30
```

④ To perform the required calculation, complete the entry line as:

$(i - c)^{-1} \times d$

Then press (ENTER).



```
■Define i =│0  1  0│       Done
           └0  0  1┘
                          ┌19.6875┐
■(i - c)^-1·d             │17.1875│
                          └18.6458┘
(i-c)^-1*d
MAIN       RAD EXACT     FUNC      4/30
```

⑤ Write the answer.

The production that exactly matches the demand $E = 5$, $R = 2$ and $M = 4$ is $E = 19.69$, $R = 17.19$ and $M = 18.65$.

Chapter 3 page 112

WORKED Example 12

THINK

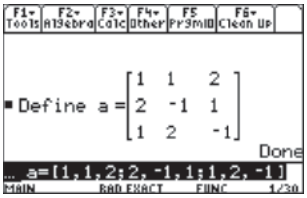
WRITE/DISPLAY

For the TI-89 Titanium

① To calculate the scalar triple product using matrices on a HOME page, complete the entry line as:

Define $a = [1, 1, 2; 2, -1, 1; 1, 2, -1]$

Then press ENTER.



■ Define a = $\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 1\\ 1 & 2 & -1\end{bmatrix}$ Done

... a=[1,1,2;2,-1,1;1,2,-1]

MAIN RAD EXACT FUNC 1/30

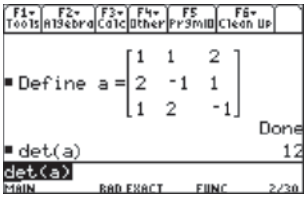
② To find the determinant of a, press:

- 2ND
- 5: MATH
- 4: Matrix
- 2: det(

Complete the entry line as:

det(a)

Then press ENTER.



■ Define a = $\begin{bmatrix}1 & 1 & 2\\ 2 & -1 & 1\\ 1 & 2 & -1\end{bmatrix}$ Done

■ det(a) 12

det(a)

MAIN RAD EXACT FUNC 2/30

Chapter 3 page 113

WORKED Example 13

THINK

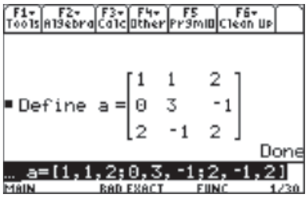
WRITE/DISPLAY

For the TI-89 Titanium

④ To calculate the scalar triple product using matrices on a HOME page, complete the entry line as:

Define a = [1 ,1, 2; 0, 3, −1; 2, −1, 2]

Then press (ENTER).



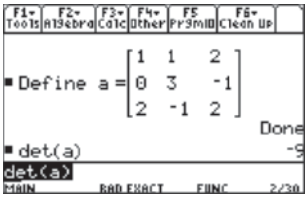
```
■ Define a = [1   1   2 ]
             [0   3  -1 ]
             [2  -1   2 ]
                            Done
... a=[1,1,2;0,3,-1;2,-1,2]
```

⑤ To find the determinant of *a*, press:
- 2ND
- 5: MATH
- 4: Matrix
- 2: det(

Complete the entry line as:

det(*a*)

Then press (ENTER).



```
■ Define a = [1   1   2 ]
             [0   3  -1 ]
             [2  -1   2 ]
                            Done
■ det(a)                      -9
det(a)
```

State the value obtained for the scalar triple product.

$\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c} = -9$

Use this value to calculate the required volume

$$\begin{aligned}\text{Volume} &= |\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c}| \\ &= |-9| \\ &= 9 \text{ cubic units}\end{aligned}$$

Chapter 4 page 133

WORKED Example 3

THINK

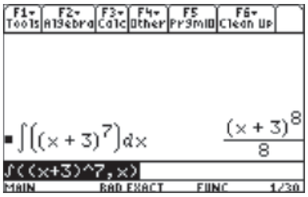
WRITE/DISPLAY

For the TI-89 Titanium

a, **b**, **c** (using calculator)

① Press

- (F3) – Calc
- 2: ∫ (integrate



$$\int\left((x+3)^7\right)dx \qquad \frac{(x+3)^8}{8}$$

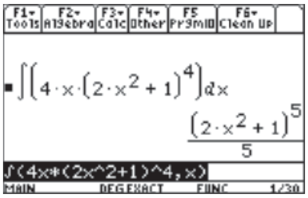
```
∫((x+3)^7,x)
MAIN          RAD EXACT     FUNC     1/30
```

Complete the entry line as:

$\int((x+3)^7, x)$

Press (ENTER).

Repeat these steps for parts **b** and **c**.

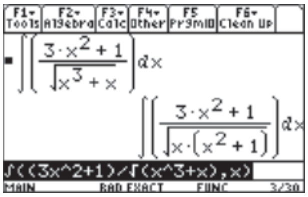


$$\int\left(4\cdot x\cdot\left(2\cdot x^2+1\right)^4\right)dx \qquad \frac{\left(2\cdot x^2+1\right)^5}{5}$$

```
∫(4x*(2x^2+1)^4,x)
MAIN          DEG EXACT     FUNC     1/30
```

Notes

1. The calculator cannot determine the solution for part **c**. It simply returns the input you entered. You will have to do this problem by hand.
2. Also, it does not include the constant for any of the anti-derivatives. You will have to do that yourself.



$$\int\left(\frac{3\cdot x^2+1}{\sqrt{x^3+x}}\right)dx \qquad \int\left(\frac{3\cdot x^2+1}{\sqrt{x\cdot\left(x^2+1\right)}}\right)dx$$

```
∫((3x^2+1)/√(x^3+x),x)
MAIN          RAD EXACT     FUNC     3/30
```

② Write your solutions, remembering to include the constant of integration.

$$\int (x+3)^7\,dx = \frac{(x+3)^8}{8} + c$$

$$\int 4x(2x^2+1)^4\,dx = \frac{(2x^2+1)^5}{5} + c$$

Chapter 4 page 135

WORKED Example 4

THINK

WRITE/DISPLAY

For the TI-89 Titanium

a ① Press:

- (F3) – Calc
- 2: $\int$ (integrate

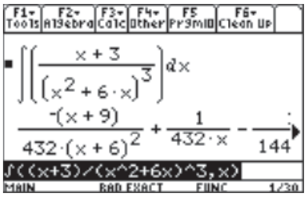
Complete the entry line as:

$\int((x + 3) \div (x^2 + 6x)^3, x)$

Press (ENTER)

The answer is in a form that is equivalent to, but more complex than, that of the solution found by hand.

You can collect the terms in factorised form over a common denominator as follows:



$$\blacksquare \int\left(\frac{x+3}{(x^2+6\cdot x)^3}\right)dx$$

$$\frac{^-(x+9)}{432\cdot(x+6)^2}+\frac{1}{432\cdot x}-\frac{\vdots}{144}$$

```
∫((x+3)/(x^2+6x)^3,x)
```

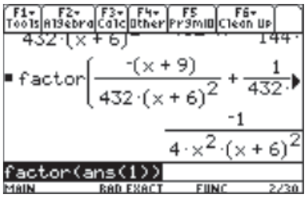
② Press:

- (F2) – Algebra
- 2: factor(
- 2nd ANS

This will select the most recent answer and complete the entry line as:

Factor(ans(1))

Press (ENTER) again to get the answer.



$$\blacksquare\ \text{factor}\left(\frac{^-(x+9)}{432\cdot(x+6)^2}+\frac{1}{432\cdot}\right.$$

$$\frac{-1}{4\cdot x^2\cdot(x+6)^2}$$

```
factor(ans(1))
```

③ Write your solution, remembering to include the constant of integration

$$\int\frac{x+3}{(x^2+6x)^3}dx = -\frac{1}{4x^2(x+6)^2}+c$$

Chapter 4 page 148

WORKED Example 9

THINK

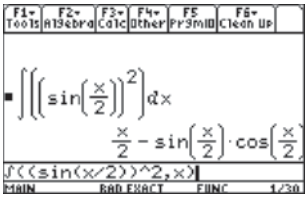
WRITE/DISPLAY

For the TI-89 Titanium

c (repeat part **a** using a calculator)

Note: If you use a calculator for problems such as these, you may find the answer expressed in a form that is different from that found by hand.

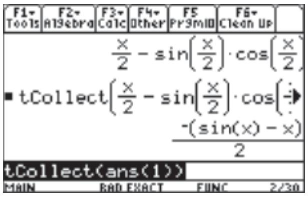
The screen dump shows the result, in the first line, of part **a** done by calculator.



$$\blacksquare \int \left(\left(\sin\left(\frac{x}{2}\right)\right)^2\right) dx \qquad \frac{x}{2} - \sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)$$

```
∫((sin(x/2))^2,x)
```

The compact form shown in the second line can be obtained as follows:



$$\frac{x}{2} - \sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)$$

$$\blacksquare\ \mathrm{tCollect}\left(\frac{x}{2} - \sin\left(\frac{x}{2}\right)\cdot\cos\left(\vdots\right.\right. \qquad \frac{-(\sin(x) - x)}{2}$$

```
tCollect(ans(1))
```

① Press:
- (F2) – Algebra
- 9: Trig
- 2: tCollect
- 2nd ANS.

Press (ENTER).

② Write your solution, remembering to include the constant of integration

$$\int \sin^2\left(\frac{x}{2}\right) dx = \frac{x - \sin x}{2} + c$$

Chapter 4 page 156

Investigation: The graph of a function and the graphs of its antiderivatives

For the TI-89 Titanium

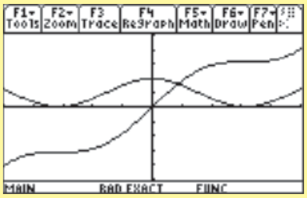
Given $f(x) = 2\cos^2\left(\frac{x}{2}\right)$, what do the graphs of its antiderivatives look like?

From the **APPS** menu press Y=, complete the entry lines as:

$$y1 = 2 \times \left(\cos\left(\frac{x}{2}\right)\right)^2$$

$$y2 = \int(y1(x), x)$$

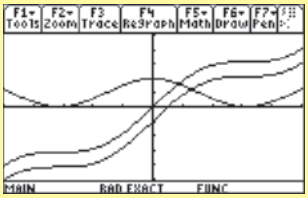
Press **ENTER** after each line.



Note: The antiderivative graph cuts 0 at $x = 0$, as the calculator defaults to the integral that has a constant of 0.

To see another antiderivative graph, complete the entry line by adding or subtracting a different constant.

$$y3 = \int(y1(x), x) - 1$$



1 Generate another two antiderivative graphs on your calculator. Sketch the function and the four antiderivative graphs. Describe any relationships you can find.

2 Choose another function and investigate the realtionship between the graph of the function and the graphs of its antiderivatives.

Chapter 4 page 158

WORKED Example 14

THINK

For the TI-89 Titanium

a ⑨ A CAS calculator can express in partial fraction form as follows:

Press:

- (F2) – Algebra
- 3: expand(

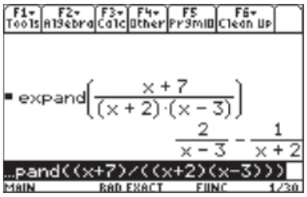
Complete the entry line as:

$$\text{expand}\left(\frac{x+7}{(x+2)\times(x-3)}\right)$$

Press (ENTER).

⑩ Write the answer.

WRITE/DISPLAY



$$\blacksquare\ \text{expand}\left(\frac{x+7}{(x+2)\cdot(x-3)}\right) \qquad \frac{2}{x-3}-\frac{1}{x+2}$$

```
...pand((x+7)/((x+2)(x-3)))
```

$$\frac{x+7}{(x+2)(x-3)} = \frac{2}{x-3} - \frac{3}{x+2}$$

Chapter 4 page 169

WORKED Example 22

THINK

For the TI-89 Titanium

a ⑫ Press:

- (F3) – Calc
- 2: $\int$(integrate

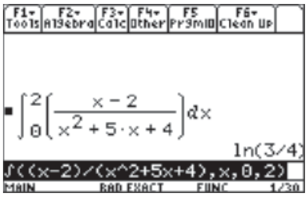
Complete the entry line as:

$$\int((x-2)\div(x^2+5x+4),\ x,\ 0,\ 2)$$

Press (ENTER).

Write your solution.

WRITE/DISPLAY



$$\blacksquare\ \int_0^2\left(\frac{x-2}{x^2+5\cdot x+4}\right)dx \qquad \ln(3/4)$$

```
∫((x-2)/(x^2+5x+4),x,0,2)
```

$$\int_0^2 \frac{x-2}{(x^2+5x+4}\,dx = \ln\left(\frac{3}{4}\right)$$

Chapter 4 page 177

Graphics Calculator tip!

Finding the volume of a solid of revolution

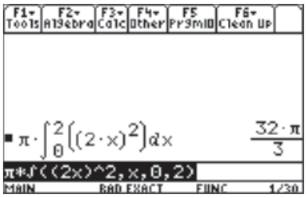
To find the volume of the solid of revolution formed in Worked example 24.

For the TI-89 Titanium

1. On a Calculator page, complete the entry line as:

   $\pi \times \int((2x)^2, x, 0, 2)$

   Press **ENTER**.



■ $\pi \cdot \int_0^2 \left((2\cdot x)^2\right) dx$ $\frac{32\cdot\pi}{3}$

```
π*∫((2x)^2,x,0,2)
```

2. Write the answer.

The volume of the solid of revolution formed when the region bounded by the graph of $y = 2x$, the x-axis and the lines $x = 0$ and $x = 2$ is rotated about the x-axis is $\frac{32\pi}{3}$ square units.

Chapter 5 page 209

Graphics Calculator tip!

Solving a first order differential equation

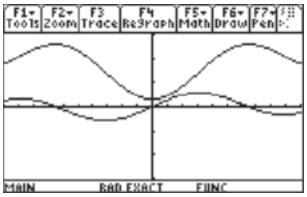
The TI-89 Titanium is able to use the nInt function. This function can be found by typing nInt into the calculator. The command nInt(*f*,*x*,*a*,*b*) finds the integral of function *f* with respect to *x* between the values *a* and *b*. If we integrate between 0 and *x*, the resulting function will have a *y*-intercept of 0. As we know the *y*-intercept, adding the intercept will result in the desired function.

To provide a general example, consider the equation in Worked example 7, using the variables *x* and *y* to stand for *t* and *A*. This means that we are solving

$$\frac{dy}{dx} = 0.02x + 0.5\sin\frac{\pi x}{6} \text{ where } y = 0.3 \text{ when } x = 0.$$

We want to find the value of *y* when $x = 9$. The solution can be found by using a graph or table.

Using a graph



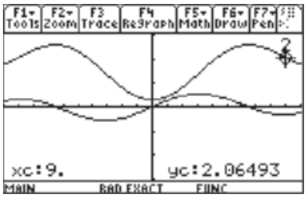
1. On a Graph page, press Y= and complete the entry lines as:

   $y1 = 0.02x + 0.5\sin\frac{\pi x}{6}$

   $y2 = \text{nInt}(y1(x), x, 0, x) + 0.3$

   Press ENTER after each line.
2. To find the value of *y* when $x = 9$, press:
   - F5 – Maths
   - 1: Value.

   Enter $x = 9$; after the value for *y*1 is shown, use the up arrow to select function *y*2. This gives a *y*-value of 2.06.



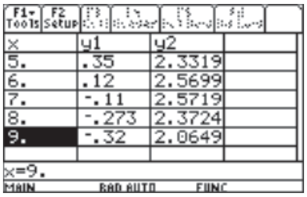
```
xc:9.                yc:2.06493
```

Using a table

1. To display the data in a table, press:

   ♦ F5 (TABLE)

   Set the second column to be F1 and the third column to be F2.



x	y1	y2
5.	.35	2.3319
6.	.12	2.5699
7.	-.11	2.5719
8.	-.273	2.3724
9.	-.32	2.0649

x=9.

2. Scroll down to find the solution when $x = 9$. (That is, $y = 2.06493$.)

Chapter 6 page 242

WORKED Example 6

THINK

WRITE/DISPLAY

For the TI-89 Titanium

a **i** ① First, identify whether the sequence is arithmetic or geometric, and list a and d or r as appropriate.

a **i** The sequence is arithmetic with $a = 3$ and $d = 2.5$.

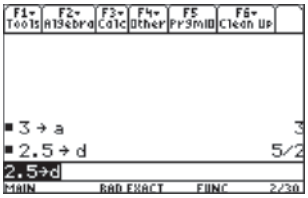
② Write the formula for the nth term.

Therefore, $T_n = 3 + (n - 1) \times 2.5$.

③ To find the 100th term:

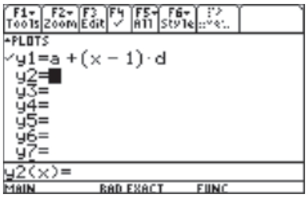
(a) set $a = 3$ and $d = 2.5$, open a HOME screen and press:

- 3 STO▶ ALPHA A ENTER
- 2.5 STO▶ ALPHA D ENTER



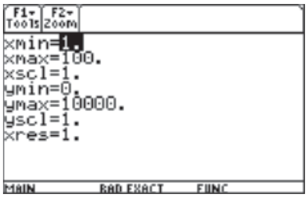
```
■3 → a                3
■2.5 → d            5/2
2.5→d
```

(b) Press ♦Y = and enter the equation $y1 = a + (x - 1)d$.



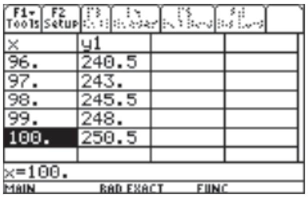
```
▲PLOTS
✓y1=a +(x − 1)·d
 y2=█
 y3=
 y4=
 y5=
 y6=
 y7=
y2(x)=
```

(c) Press ♦ WINDOW and enter the settings shown.



```
xmin=1.
xmax=100.
xscl=1.
ymin=0.
ymax=10000.
yscl=1.
xres=1.
```

(d) To find the 100th term, press ♦ TABLE and scroll down to where $x = 100$.



x	y1		
96.	240.5		
97.	243.		
98.	245.5		
99.	248.		
100.	250.5		

```
x=100.
```

④ Write the answer.

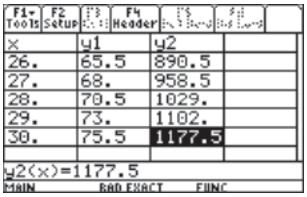
$T_{100} = 250.5$

ii ① To calculate the sum of the first 30 terms, press ♦ Y= and enter

$$y2 = \frac{x}{2}\,(2a + (n - 1)d)$$

Press ♦ TABLE and scroll down to $x = 30$.

ii



x	y1	y2	
26.	65.5	890.5	
27.	68.	958.5	
28.	70.5	1029.	
29.	73.	1102.	
30.	75.5	1177.5	

```
y2(x)=1177.5
```

② Write the answer.

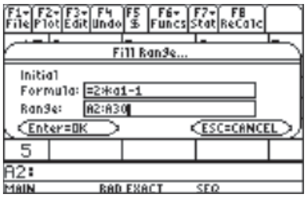
The sum of the first 30 terms is 1177.5.

THINK | **WRITE/DISPLAY**

b i ① For a recursion sequence such as this:

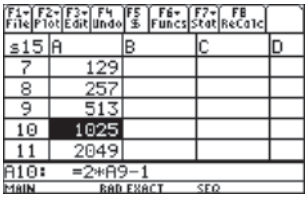
(a) Press **APPS** and open a new CellSheet.
(b) In cell A1 enter the initial value of 3.
(c) Press **F3** and select option 3: Fill Range.
(d) Enter the formula =2*a1 − 1 for the range A2:A30

b i



Fill Range...
Initial
Formula: =2*a1-1
Range: A2:A30
5
A2:

② Scroll down to the 10th term.



s15	A	B	C	D
7	129			
8	257			
9	513			
10	1025			
11	2049			

A10: =2*A9−1

③ Write the answer.

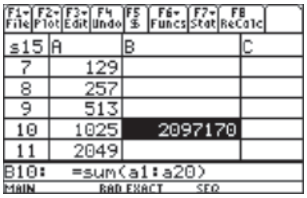
$T_{10} = 1025$

ii ① To calculate the sum of the first 20 terms, place the cursor in an empty cell and complete the entry line as:

= sum(a1:a20)

Note: You may need to widen the column to display the full answer. This option is found by pressing **F3** and selecting Option 8: Column format.

ii



s15	A	B	C
7	129		
8	257		
9	513		
10	1025	2097170	
11	2049		

B10: =sum(a1:a20)

② Write the answer.

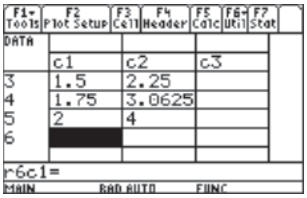
The sum of the first 20 terms is 2 097 170.

Chapter 6 page 257

Finite Differences

For the TI-89 Titanium

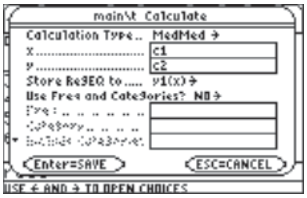
1. To generate a linear model for the data, on a CellSheet page, enter the *x*-values in Column A and the *y*-values in Column B.



	c1	c2	c3
3	1.5	2.25	
4	1.75	3.0625	
5	2	4	
6			

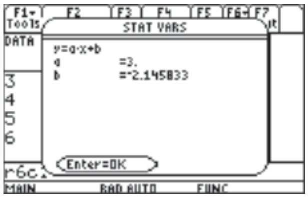
r6c1=

2. Press (F5) and enter the settings shown at right.



Calculation Type.. MedMed →
x........ c1
y........ c2
Store RegEQ to..... y1(x) →
Use Freq and Categories? NO →

3. Press (ENTER).
The calculator models the data using a linear model.



```
STAT VARS
y=a·x+b
a         =3.
b         =⁻2.145833
```

Chapter 6 page 260

WORKED Example 8

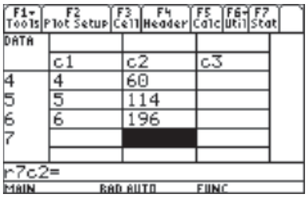
THINK

WRITE/DISPLAY

For the TI-89 Titanium

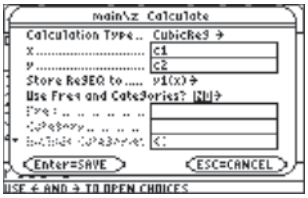
b ① (a) On a Data Matrix editor, enter the values of n in Column 1 and the values of t_n in Column 2.

b



```
DATA
     c1    c2     c3
4    4     60
5    5     114
6    6     196
7
r7c2=
```

(b) Press (F5) and enter the settings shown at right.



```
Calculation Type.. CubicReg →
x....................... c1
y....................... c2
Store RegEQ to..... y1(x)→
Use Freq and Categories? NO→
```

(c) Press (ENTER) to display the equation of the regression.

```
y=a·x³+b·x²+c·x+d
a          =1.
b          =⁻1.
c          =2.
d          =4.
R²         =1.
```

② Write the answer in terms of t and n.

The formula is $t_n = n^3 - n^2 + 2n + 4$.

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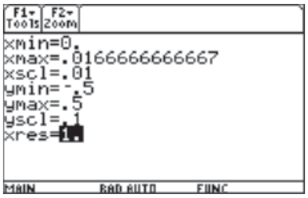
WORKED Example 8

THINK

WRITE/DISPLAY

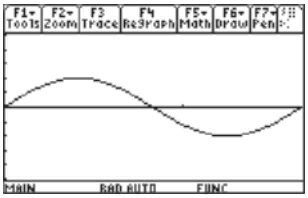
For the TI-89 Titanium

1. On a Graphs page, press ♦ WINDOW.
   Enter the values in the fields as shown.



```
xmin=0.
xmax=.0166666666667
xscl=.01
ymin=⁻.5
ymax=.5
yscl=.1
xres=1.
```

2. Press ♦ Y= and complete the entry line as:
   $y1 = 0.2\sin(120\pi x)$
   Then press ENTER.
   Press ♦ GRAPH to draw the function.
   Note: The calculator must be in radian mode.



Chapter 7 page 283

WORKED Example 9

THINK

WRITE/DISPLAY

For the TI-89 Titanium

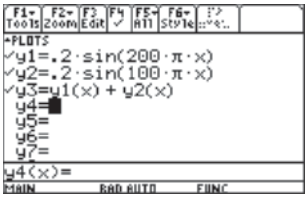
c Use a graphics calculator to draw the graphs of the waveforms from $t = 0$ to $t = 0.02$ s.

① Press ◆ Y= and complete the entry lines as:
$y1 = 0.2 \sin(200\pi x)$
$y2 = 0.2 \sin(100\pi x)$
$y3 = y1(x) + y2(x)$
Press ENTER after each line.

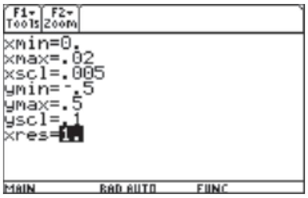
② To set the Window Settings, press ◆ WINDOW.
Enter the values in the fields as shown.

③ Press ◆ GRAPH to draw the graphs.

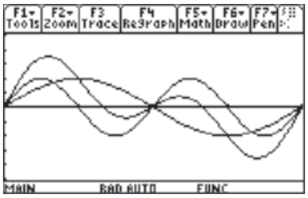
c



```
+PLOTS
✓y1=.2·sin(200·π·x)
✓y2=.2·sin(100·π·x)
✓y3=y1(x) + y2(x)
 y4=
 y5=
 y6=
 y7=
y4(x)=
MAIN      RAD AUTO      FUNC
```



```
xmin=0.
xmax=.02
xscl=.005
ymin=-.5
ymax=.5
yscl=.1
xres=1.
MAIN      RAD AUTO      FUNC
```



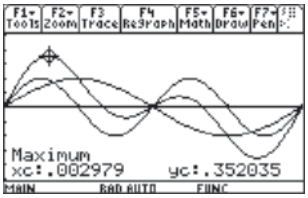
d Use a graphics calculator to check the amplitude of the composite wave.

① To find the maximum of the composite wave, press:
- F5
- 3: Maximum

Use the arrow key to select Function 3 and then set a lower and upper bound.

② Write the answer.

d



```
Maximum
xc:.002979        yc:.352035
MAIN      RAD AUTO      FUNC
```

The amplitude of the composite function is 0.35. The loudness is almost doubled.

Chapter 7 page 286

WORKED Example 10

THINK

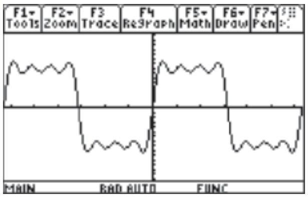
WRITE/DISPLAY

For the TI-89 Titanium

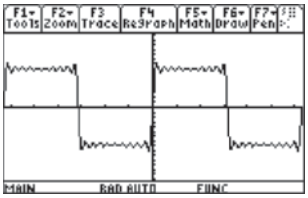
1. Press ◆ Y= and complete the entry lines as:

   $$y1 = \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7}$$

   Set the domain in the Window Settings as $-2\pi \leq x \leq 2\pi$.



2. Edit the graph to improve its 'squareness' by adding more terms



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WORKED Example 13

THINK

WRITE/DISPLAY

For the TI-89 Titanium

a ① Press ♦ Y= and complete the entry lines as:

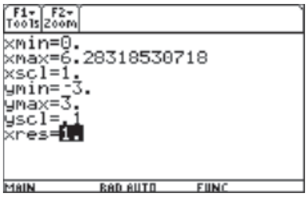
$y1 = -\cos(\pi x) + 1$

$y2 = \dfrac{-1}{\cos(\pi x)} + 1$

Press ENTER after each line.

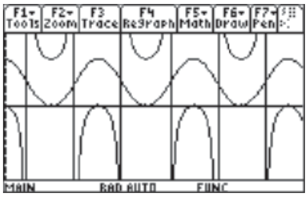
Enter Window Settings as shown.

a



```
xmin=0.
xmax=6.28318530718
xscl=1.
ymin=-3.
ymax=3.
yscl=.1
xres=1.
MAIN          RAD AUTO       FUNC
```

② Press ♦ GRAPH and the graph will be displayed.



b ① Press ♦ Y= and complete the entry lines as:

$y1 = \dfrac{-1}{\cos(\pi x)} + 1$

$y2 = 2.5$

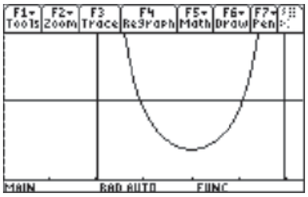
Press ENTER after each line.

To improve the clarity of the graph, press:

- F2
- 2: Zoom In

Move the centre close to the first point of intersection.

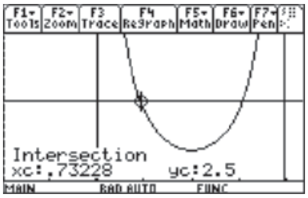
b



② To find the smallest value of x for which $-\sec(\pi x) + 1 = 2.5$, press:

- F5 – Math
- 5: Intersection

Move the pointer to select the two functions and mark an upper and a lower bound.

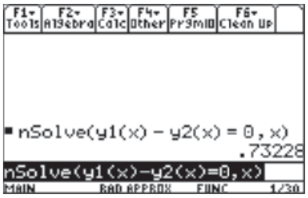


```
Intersection
xc:.73228          yc:2.5
```

③ Alternatively, on a HOME screen, to obtain a numerical solution press CATALOG and select the nSolve function.

Complete the entry line as:

nSolve($y1(x) - y2(x) = 0,x$)



```
■ nSolve(y1(x) - y2(x) = 0, x)
                          .73228
nSolve(y1(x)-y2(x)=0,x)
MAIN          RAD APPROX     FUNC     1/30
```

Chapter 7 page 311

WORKED Example 31

THINK

For the TI-89 Titanium

① To solve $\sqrt{3}\cos x + \sin x = 1.5$ for $0 < x < 2\pi$, press ◆ Y= and complete the entry lines as:

$$y1 = 2\sin\left(x + \frac{\pi}{3}\right)$$

$$y2 = 1.5$$

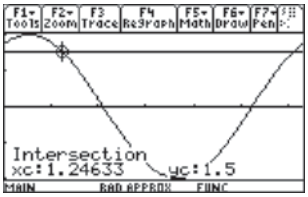
Press ENTER after each line.
Set Window Settings appropriately.

② To find the points of intersection, press:
- F5 – Math
- 5: Intersection

Follow the instructions to select both curves, and an upper and a lower bound. The intersection will then be displayed.

③ Write the answer.

WRITE/DISPLAY



Intersection
xc:1.24633 yc:1.5

Solving $\sqrt{3}\cos x + \sin x = 1.5$ for $0 < x < 2\pi$ gives $x = 1.246$ and $x = 6.084$.

Chapter 7 page 314

WORKED Example 33

THINK

For the TI-89 Titanium

① To find the area as described, first sketch the graph between $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

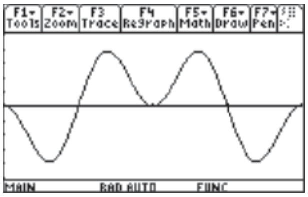
② To find the area of the four identical regions, return to the HOME screen and complete the entry line as:

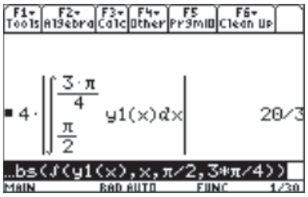
$4 \times \text{abs}(\int(y1(x), x, \pi/2, 3\pi/4))$

Then press ENTER.

③ Write the answer.

WRITE/DISPLAY





$$\blacksquare\ 4\cdot\left|\int_{\frac{\pi}{2}}^{\frac{3\cdot\pi}{4}} y1(x)\,dx\right| \qquad 20/3$$

```
...bs(∫(y1(x),x,π/2,3*π/4))
```

$$4 \times \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 5\sin(4x) \times \sin(2x)\,dx\right| = \frac{20}{3}$$

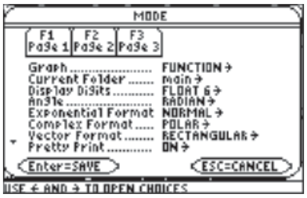
Chapter 8 page 336

Graphics Calculator tip! Using Euler's formula

The calculations in Worked example 5 can be performed on a graphics calculator.

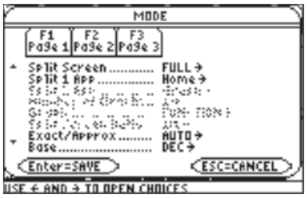
For the TI-89 Titanium

1. To ensure that the calculator is set in Complex Mode, open a HOME page and then press (MODE). Ensure that the Angle is in Radians and the complex format is set to polar.



```
Graph.................. FUNCTION →
Current Folder......... main →
Display Digits......... FLOAT 6 →
Angle.................. RADIAN →
Exponential Format      NORMAL →
Complex Format......... POLAR →
Vector Format.......... RECTANGULAR →
Pretty Print........... ON →
```

   Press (F2) to set Exact/Approx to Auto.



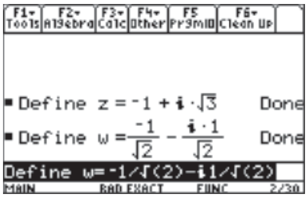
```
Split Screen........... FULL →
Split 1 App............ Home →
Exact/Approx........... AUTO →
Base................... DEC →
```

2. To define z and w, press (HOME) and complete the entry lines as:

   Define $z = -1 + i\sqrt{3}$

   Define $w = \dfrac{-1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}$

   Press (ENTER) after each line.



■ Define $z = {}^-1 + i \cdot \sqrt{3}$ Done

■ Define $w = \dfrac{{}^-1}{\sqrt{2}} - \dfrac{i \cdot 1}{\sqrt{2}}$ Done

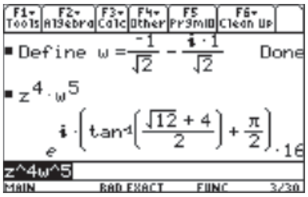
```
Define w=-1/√(2)-i1/√(2)
```

3. To calculate z^4w^5, complete the entry line as:

   z^4w^5

   Then press (ENTER).

   The calculator defaults to an exact value of θ.



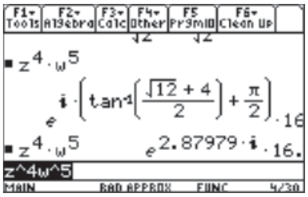
■ Define $w = \dfrac{{}^-1}{\sqrt{2}} - \dfrac{i \cdot 1}{\sqrt{2}}$ Done

■ $z^4 \cdot w^5$

$e^{i \cdot \left(\tan^{-1}\left(\frac{\sqrt{12}+4}{2}\right)+\frac{\pi}{2}\right)} \cdot 16$

```
z^4w^5
```

4. To find an approximate angle, press (MODE) then (F2) to change to Approx mode. Return to the HOME screen and press (ENTER) to perform the calculation again.



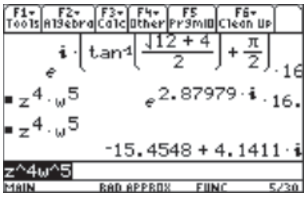
■ $z^4 \cdot w^5$

$e^{i \cdot \left(\tan^{-1}\left(\frac{\sqrt{12}+4}{2}\right)+\frac{\pi}{2}\right)} \cdot 16$

■ $z^4 \cdot w^5$ $e^{2.87979 \cdot i} \cdot 16.$

```
z^4w^5
```

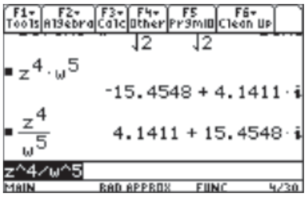
To express the result as rectangular coordinates, press **MODE** and change the complex format to rectangular. Return to the HOME screen and press **ENTER** to perform the calculation again. The answer will now be in rectangular form.



```
        i·(tan⁻¹(√12 + 4)/2 + π/2)
       e                          ·16
■ z⁴·w⁵              e^(2.87979·i)·16.
■ z⁴·w⁵
                  -15.4548 + 4.1411·i
z^4w^5
MAIN        RAD APPROX    FUNC     5/30
```

It can be seen that $z^4w^5 = 16e^{2.88i}$ $= -15.45 + 4.14i$.

5. Follow the steps above to find, $\dfrac{z^4}{w^5}$.



```
                √2       √2
■ z⁴·w⁵
                  -15.4548 + 4.1411·i
  z⁴
■ ──               4.1411 + 15.4548·i
  w⁵
z^4/w^5
MAIN        RAD APPROX    FUNC     4/30
```

Chapter 8 page 365

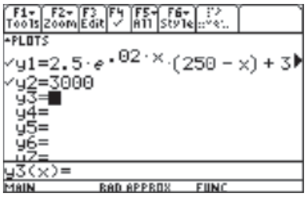
WORKED Example 19

THINK

WRITE/DISPLAY

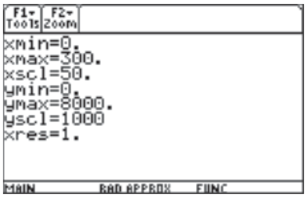
For the TI-89 Titanium

1. Press ♦ Y= and complete the entry lines as:
   $y1 = 2.5\ e^{0.02x}(250 - x) + 375$
   $y2 = 3000$
   Press ENTER after each line.



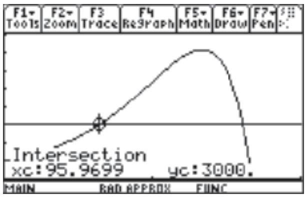
```
▲PLOTS
✓y1=2.5·e^.02·x·(250 − x) + 3▶
✓y2=3000
 y3=■
 y4=
 y5=
 y6=
 y7=
y3(x)=
MAIN          RAD APPROX     FUNC
```

2. Enter the Window Settings as shown.



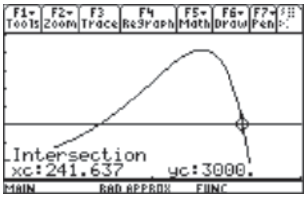
```
xmin=0.
xmax=300.
xscl=50.
ymin=0.
ymax=8000.
yscl=1000
xres=1.
MAIN          RAD APPROX     FUNC
```

3. Press ♦ GRAPH to draw the graph; to find the points of intersection, press:
   - F5 – Math
   - 5: Intersection



```
Intersection
xc:95.9699      yc:3000.
MAIN          RAD APPROX     FUNC
```

Select each curve using the arrow keys, then select an upper and lower bound around the first point of intersection. Press ENTER. The point of intersection will be marked.

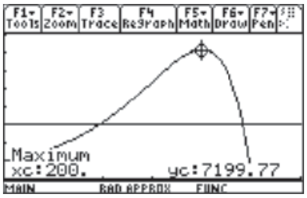


```
Intersection
xc:241.637      yc:3000.
MAIN          RAD APPROX     FUNC
```

Repeat the process, selecting the lower and upper bounds around the second point of intersection.

4. To find the maximum value, press:
   - F5 – Math
   - 4: Maximum

   Use the arrow keys to select a lower and an upper bound about which to find the maximum.



```
Maximum
xc:200.         yc:7199.77
MAIN          RAD APPROX     FUNC
```

This gives the time taken for the number of termites to reach its maximum (required for part **iii**).

Chapter 8 page 373

WORKED Example 20

THINK

WRITE/DISPLAY

Using the TI-89 Titanium

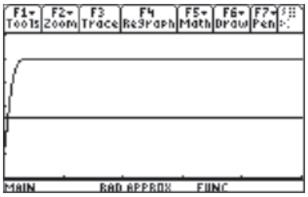
e ① To draw the graphs press (♦) Y= and complete the entry lines as:

$$y1 = \frac{500}{1 + 4e^{-x}}$$

$$y2 = 250$$

Press (ENTER) after each line. Adjust the Window settings to find an appropriate scale and press (♦) GRAPH to draw the graph.

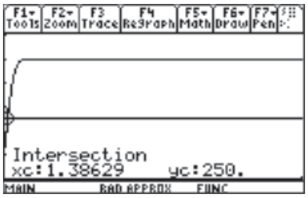
e



② To find the point of intersection, press:

- (F5) – Math
- 5: Intersection

Select each curve using the arrow keys, then select an upper and a lower bound around the first point of intersection. Press (ENTER). The point of intersection will be displayed.



```
Intersection
xc:1.38629     yc:250.
```

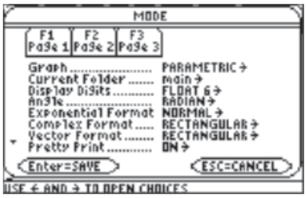
Chapter 10 page 479

Applying technology to parametric equations

By using appropriate technology, we can quickly generate graphs of conic sections in parametric form. The following examples illustrate the procedure using a graphics or CAS calculator.

For the TI-89 Titanium

1. Ensure that the calculator is in radian mode and that the graph mode is set to Parametric.



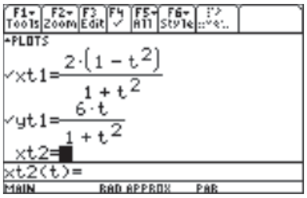
Graph.................... PARAMETRIC→
Current Folder....... main→
Display Digits.......... FLOAT 6→
Angle........................ RADIAN→
Exponential Format NORMAL→
Complex Format...... RECTANGULAR→
Vector Format........ RECTANGULAR→
Pretty Print............ ON→

2. To graph the conic section using parametric equations, press ♦ Y=, and complete the entry lines as:

$$xt1 = 2\left(\frac{1-t^2}{1+t^2}\right)$$

$$yt1 = \frac{6t}{1+t^2}$$

Press ENTER after each line.



+PLOTS

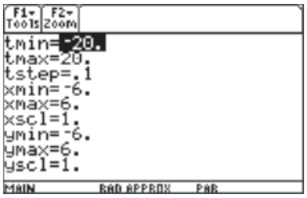
$\checkmark xt1 = \frac{2\cdot(1-t^2)}{1+t^2}$

$\checkmark yt1 = \frac{6\cdot t}{1+t^2}$

xt2=

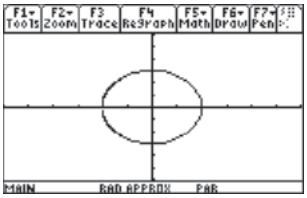
xt2(t)=

3. To change the Window Settings, press ♦ WINDOW. Enter the window settings as shown.



tmin=-20.
tmax=20.
tstep=.1
xmin=-6.
xmax=6.
xscl=1.
ymin=-6.
ymax=6.
yscl=1.

4. Press ♦ GRAPH to draw the graph. The graph produced is an ellipse.



Chapter 10 page 481

WORKED Example 12

THINK

WRITE/DISPLAY

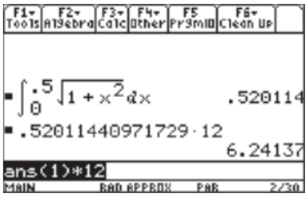
For the TI-89 Titanium

1. On a HOME page press:
   - (F3) – Calc
   - 2: integrate

   Then complete the entry line as:

   $\int \sqrt{(1+x^2)}, x, 0, 0.5)$

   Press (ENTER).
   Multiply by 12 to obtain the result for *L*.



```
■ ∫(0 to .5) √(1 + x²) dx          .520114
■ .52011440971729·12
                                 6.24137
ans(1)*12
```

2. Write the answer

The length, *L*, of the curve for $-3 \le x \le 3$ is 6.241 units.

Chapter 10 page 492

WORKED Example 19

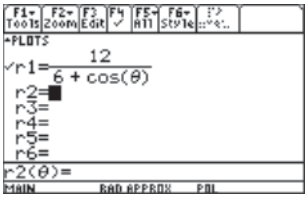
Graph and identify the curve that has $r = \dfrac{12}{6 + \cos\theta}$ as its polar equation.

THINK

WRITE/DISPLAY

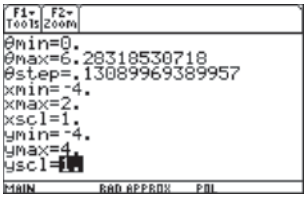
For the TI-89 Titanium

① Press MODE and set the graph type to Polar. On a Graph page press ◆ Y=, and complete the entry line as shown.



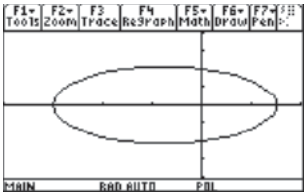
```
✓r1= 12/(6 + cos(θ))
 r2=
 r3=
 r4=
 r5=
 r6=
r2(θ)=
MAIN        RAD APPROX      POL
```

② To change the Window Settings, press ◆ WINDOW.
Enter the window settings as shown.



```
θmin=0.
θmax=6.28318530718
θstep=.13089969389957
xmin=⁻4.
xmax=2.
xscl=1.
ymin=⁻4.
ymax=4.
yscl=1.
MAIN        RAD APPROX      POL
```

③ Press ◆ GRAPH. The graph shown at right is the result.



④ Write the answer. The curve is an ellipse.

Chapter 11 page 513

WORKED Example 2

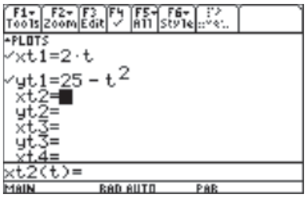
THINK

WRITE/DISPLAY

For the TI-89 Titanium

a Set the Graph mode on the calculator to Parametric.
Press ◆ Y=
Complete the entry lines as:
$xt1 = 2t$
$yt1 = 25 - t^2$

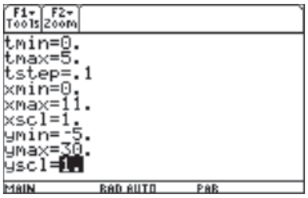
a



```
▸PLOTS
✓xt1=2·t
✓yt1=25 - t²
 xt2=
 yt2=
 xt3=
 yt3=
 xt4=
xt2(t)=
```

b To adjust the Window Settings, press ◆ WINDOW and enter the settings as shown.

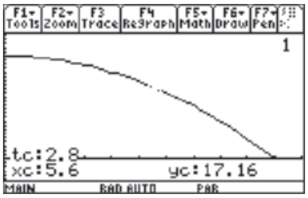
b



```
tmin=0.
tmax=5.
tstep=.1
xmin=0.
xmax=11.
xscl=1.
ymin=-5.
ymax=30.
yscl=1.
```

c Press ◆ GRAPH to draw the graphs. To investigate coordinates along the graph, press F3.

c



```
tc:2.8
xc:5.6          yc:17.16
```

Chapter 11 page 516

WORKED Example 3

THINK

d Verify that the coordinates of the impact point are (5, 5) by drawing the graphs of the displacement for each particle and locating the point of intersection.

For the TI-89 Titanium

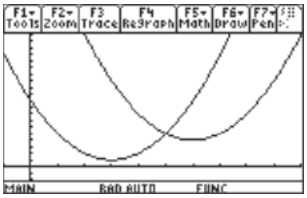
(a) Press ◆ Y= and complete the entry lines as:
$y1 = (x - 6)^2 + 4$
$y2 = (x - 3)^2 + 1$
Press ENTER after each line.
Adjust the Window Settings if necessary.

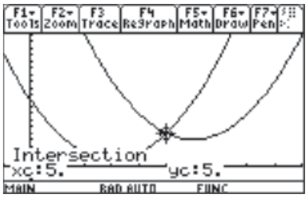
(b) To display the point of intersection, press F5 and select Option 5: Intersection. Use the arrow keys to select each curve as well as setting a lower and an upper bound.

The coordinates of the point of intersection will be displayed.

WRITE/DISPLAY

d




Intersection
xc:5.
yc:5.

Point of intersection is (5, 5) as required.

Chapter 11 page 529

WORKED Example 7

THINK

WRITE/DISPLAY

For the TI-89 Titanium

a Ensure that the calculator is set in degree mode.

Press ♦ Y= and complete the entry line as:

$y1 = \cos(x) \times (5\sin(x) - 4\cos(x))$

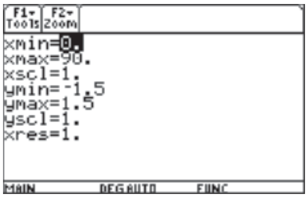
Then press ENTER.

a

```
✓y1=cos(x)·(5·sin(x) − 4·cos▸
 y2=■
 y3=
 y4=
 y5=
 y6=
 y7=
y2(x)=
```

b To adjust the Window Settings, press ♦ WINDOW.
Enter the settings as shown.

b



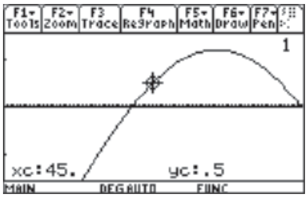
```
xmin=0.
xmax=90.
xscl=1.
ymin=-1.5
ymax=1.5
yscl=1.
xres=1.
```

c Press ♦ GRAPH to display the graph.

To investigate points along the curve, press F3.

Use the Arrow keys to move along the curve.

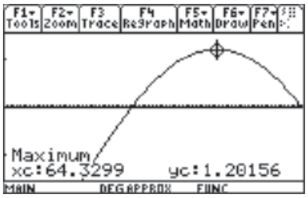
c



```
xc:45.          yc:.5
```

d To find the value of x when the function is a maximum, press F5-Math and set a lower and an upper bound. The x-value tells you *when* the function is a maximum, while the y-value gives the maximum value.

d



```
Maximum
xc:64.3299      yc:1.20156
```

Chapter 12 page 557

Graphics Calculator tip!

Graphing a *v–t* graph and the corresponding *x–t* graph

Plotting the *x–t* graph

To plot the *x–t* graph, given that $v(t) = 2t - t^2$, $0 \leq t \leq 4$, and the corresponding *x–t* graph for which $x = 2$ at $t = 0$

For the TI-89 Titanium

1. Press ◆ Y= and complete the entry line as:

   $y1 = 2{\cdot}x - x^2$

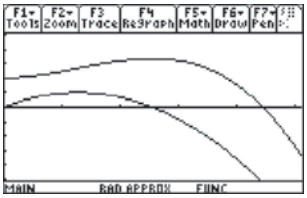
   nInt($y1(x)$, x, 0, x) + 2

   Press ENTER after each line.

```
PLOTS
✓y1=2·x - x²
✓y2=nInt(y1(x),x,0,x)+2
 y3=
 y4=
 y5=
 y6=
 y7=
y3(x)=
```

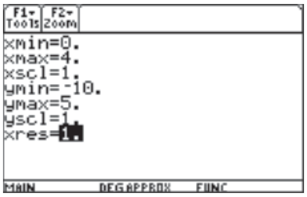
2. Press ◆ GRAPH to display the graph, then press ENTER.

   Note: The position is 2 when $t = 0$, so 2 is the constant of integration.



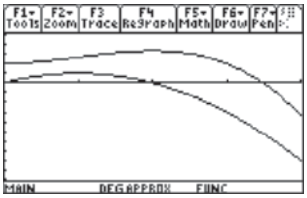
3. To set the specified domain, press ◆ WINDOW.

   Enter the settings as shown.



```
xmin=0.
xmax=4.
xscl=1.
ymin=-10.
ymax=5.
yscl=1.
xres=1.
```

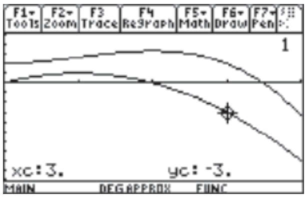
4. The graphs shown are $v(t)$ and $x(t)$.



5. To determine the velocity at $t = 3$, press:
   - F5 –Math
   - 1: Value.

   Enter $x = 3$ and the point (3, –3) is displayed.

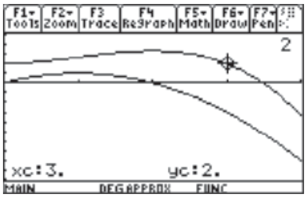
   The velocity is –3 when $t = 3$.



```
xc:3.          yc:-3.
```

6. To determine the position at $t = 3$, press the up arrow to switch to curve 2. On curve $y2(x)$, the point (3, 2) is displayed.

   The position is 2 when $t = 3$.



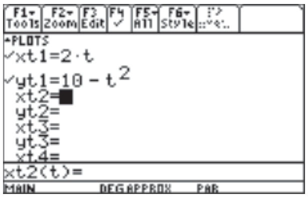
```
xc:3.          yc:2.
```

Chapter 12 page 581

Graphics Calculator tip! Vector functions of time — derivatives

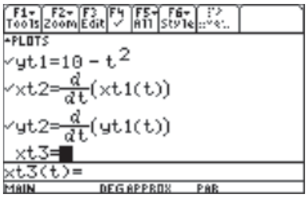
For the TI-89 Titanium

1. To draw the path of the object, on a Graph page, press MODE and set the graph mode to Parametric.
   Press ◆ Y=, and complete the entry lines as:
   $xt1 = 2t$
   $yt1 = 10 - t^2$



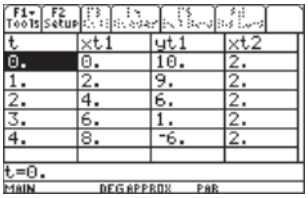
```
PLOTS
✓xt1=2·t
✓yt1=10 - t²
 xt2=
 yt2=
 xt3=
 yt3=
 xt4=
xt2(t)=
```

2. Complete the entry line as:
   $xt2(t) = d(x1(t), t)$
   $yt2(t) = d(y1(t), t)$



```
PLOTS
✓yt1=10 - t²
✓xt2=d/dt(xt1(t))
✓yt2=d/dt(yt1(t))
 xt3=
xt3(t)=
```

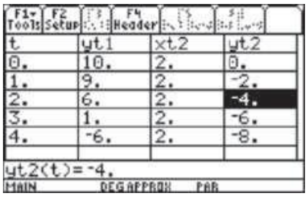
3. To view a function table, press ◆ TABLE.



```
t     xt1   yt1   xt2
0.    0.    10.   2.
1.    2.    9.    2.
2.    4.    6.    2.
3.    6.    1.    2.
4.    8.    -6.   2.

t=0.
```

4. Scroll to read the derivatives when $t = 2$.
   Therefore, at $t = 2$, $\dot{\underset{\sim}{r}} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$.



```
t     yt1   xt2   yt2
0.    10.   2.    0.
1.    9.    2.    -2.
2.    6.    2.    -4.
3.    1.    2.    -6.
4.    -6.   2.    -8.

yt2(t)=-4.
```

Answers

CHAPTER 1 Complex numbers

Exercise 1A — Operations on and representations of complex numbers

1 **a** $12 - 8i$ **b** $5 + 2i$ **c** $12 + 8i$
d $1 - 6i$ **e** $14 + 8i$ **f** $-0.1 - 0.8i$
g $5 - 12i$

2 **a** 2.24 **b** 9 **c** $-2 - i$
d $-3 - 6i$ **e** 4.12 **f** 5
g 68 **h** $-0.4 - 0.2i$ **i** $0.06 - 0.24i$

3 Check with your teacher.

4 **a** $4.24 \text{ cis } \frac{\pi}{4}$ **b** $4.24 \text{ cis } -\frac{\pi}{4}$
c $4.24 \text{ cis } \frac{3\pi}{4}$ **d** $5.29 \text{ cis } (0.86)$
e $5.29 \text{ cis } (-0.86)$ **f** $5 \text{ cis } (0.64)$
g $5 \text{ cis } (-2.50)$ **h** $5 \text{ cis } (2.50)$
i $5 \text{ cis } (0.93)$

5 **a** $2 + 3.46i$ **b** $2.6 + 1.5i$
c $-10 + 17.32i$ **d** $5.66 - 5.66i$
e $-i$ **f** $2.16 + 3.37i$
g $-1.25 + 2.73i$ **h** $1.08 - 1.68i$

6 Check answers for question **1**.

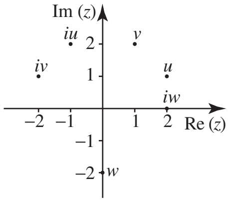
7 Check answers for question **2**.

8 **a** $32 \text{ cis } \frac{11\pi}{12}$ **b** $0.5 \text{ cis } \left(-\frac{5\pi}{12}\right)$
c $2 \text{ cis } \frac{5\pi}{12}$ **d** $16 \text{ cis } \frac{\pi}{2}$
e 512

9 **a** $2.24 \text{ cis } (0.46)$ **b** $1.5 \text{ cis } (0.23)$
c $1.46 + 0.34i$

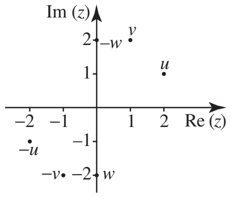
10 **a** $0.56 - 0.35i$ **b** $0.71 + 0.71i$
c $-0.5 + 0.866i$

11



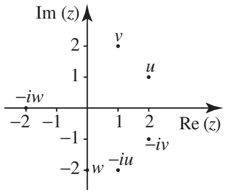
Each point has been rotated anticlockwise through 90°.

12



Each point has been rotated through 180°.

13



Each point has been rotated clockwise through 90°.

14

Operation on z	Geometric relationship to z
a $\bar{z}$	Reflection in x-axis
b iz	Anticlockwise rotation of 90°
c $-z$	Rotation of 180°
d $-iz$	Clockwise rotation of 90°

Investigation — Complex numbers and matrices

1 **a** $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}, z^2$

c $\frac{1}{25}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}, z^{-1}$

2 **a** Transpose — Conjugate
b Inverse — Inverse
c Determinant — Modulus squared

Investigation — $e^{i\theta}$ and de Moivre's theorem

3 **a** $i^i = e^{-\frac{\pi}{2}}$
b $e^{1 - \pi i} = -e$
c $\log_e 4 - \frac{i\pi}{6}$

Exercise 1B — Factorisation of polynomials in *C*

1 **a** $(z + 2i)(z - 2i)$
b $(z + \sqrt{7}\, i)(z - \sqrt{7}\, i)$
c $(z + 4 - 3i)(z + 4 + 3i)$
d $\left(z - \frac{3}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{7}}{2}i\right)$
e $(2z - 1 + 4i)(2z - 1 - 4i)$
f $-(3z - 4 + 4i)(3z - 4 - 4i)$

2 **a** $(z - 3)(z + 3)(z - 3i)(z + 3i)$
b $\left(z + \sqrt{3}\right)\left(z - \sqrt{3}\right)\left(z + i\right)\left(z - i\right)$
c $(z + 4i)(z - 4i)(z + 2i)(z - 2i)$
d $\left(z + \sqrt{5}i\right)\left(z - \sqrt{5}i\right)\left(z + \sqrt{2}\right)\left(z - \sqrt{2}\right)$

3 **a** $z+2, z-3\pm\sqrt{5}i$

b $z-1, z+1\pm\sqrt{3}i$

c $z+2, z-\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$

d $z+3, 2z-5, z+1$

e $z-3, z+1, z+1\pm 2i$

f $z\pm 1, z-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i, z+\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

4 **a** $1-i, -6$ **b** $-2-i, -\frac{1}{2}$ **c** $4+i, 2$

5 $z+3i, z-1\pm\frac{1}{\sqrt{2}}i$

6 E **7** D **8** E **9** C

10 **a** 6 **b** −4 **c** −5 **d** 16

11 **a** $a=15, b=-84$ **b** $a=-3, b=-3$

c $a=-3, b=4$

12 **a** $a=-2, b=1$ **b** $a=12, b=3$

c $a=1, b=-4$

13 Since complex roots occur in conjugate pairs, an odd number of roots must have at least one real root.

14 $z^3-2z^2+16z-32$

15 **a** ±6 **b** −1, −2, −3

16 $(z-i)(z^2+iz-1)$

17 **a** Solution will vary.

b $Q(z)=z^3-3z^2i-3z+4+i$

c $a=i, b=4$

18 $(z+\sqrt{5}i)(z-\sqrt{5}i)(z+1+\sqrt{2}i)(z+1-\sqrt{2}i)$

19 $(3z-2-i)(3z-2+i)(z+i)$ **20** $a=5$

Investigation — The exact values of cos $\frac{2\pi}{5}$ and sin $\frac{2\pi}{5}$

1 $x^5+5x^4yi-10x^3y^2-10x^2y^3i+5xy^4+y^5i$

2 $x^5-10x^3y^2+5xy^4=1$

3 $y^2=1-x^2$

4 $16x^5-20x^3+5x-1=0$

5 1 and $\frac{\sqrt{5}-1}{4}$

6 and **7** Solution will vary.

Exercise 1C — Solving equations in *C*

1 **a** $-1\pm 2i$ **b** $4\pm 3i$ **c** $7\pm 10i$

d $\frac{3}{2}\pm i$ **e** $\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i$

2 **a** $-2, \frac{3}{2}\pm\frac{\sqrt{11}}{2}i$ **b** $1, \frac{1}{2}\pm\frac{\sqrt{7}}{2}i$

c $2, \frac{3}{4}\pm\frac{\sqrt{23}}{4}i$ **d** $4, \frac{1}{6}\pm\frac{\sqrt{11}}{6}i$

e $2, \frac{3}{2}\pm\frac{1}{2}i$

3 $4, \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

4 **a** $\pm 4i, \pm 3i$ **b** $\pm 2, \pm i$

c $\pm\frac{1}{3}, \pm 2i$ **d** $x=\pm\frac{\sqrt{6}}{2}i$

5 B

6 **a** $\pm\frac{1}{2}(\sqrt{6}+\sqrt{2}\,i)$ **b** $\pm(6+5i)$

c $\pm\frac{1}{\sqrt{2}}(9+7i)$

7 $\pm\frac{1}{\sqrt{2}}(1+i)$ **8** C

9 **a** $\sqrt{2}\text{ cis}\left(-\frac{\pi}{12}\right), \sqrt{2}\text{ cis }\frac{11\pi}{12}$

b $2^{\frac{5}{4}}\text{ cis }\frac{\pi}{8}, 2^{\frac{5}{4}}\text{ cis}\left(-\frac{7\pi}{8}\right)$

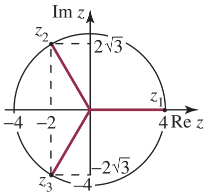
c $2\text{ cis }\frac{2\pi}{9}, 2\text{ cis }\frac{8\pi}{9}, 2\text{ cis}\left(-\frac{4\pi}{9}\right)$

d $\text{cis }\frac{\pi}{6}, \text{cis }\frac{5\pi}{6}, \text{cis}\left(-\frac{\pi}{2}\right)$

e $\sqrt[6]{2}\text{ cis}\left(-\frac{\pi}{4}\right), \sqrt[6]{2}\text{ cis }\frac{5\pi}{12}, \sqrt[6]{2}\text{ cis}\left(-\frac{11\pi}{12}\right)$

10 $5i, \pm 5\frac{\sqrt{3}}{2}-\frac{5}{2}i$, sum = 0

11 **a** $4, -2\pm 2\sqrt{3}i$ **b**



12 **a** $\pm 2, \pm 2i$ **b** $\pm\sqrt{5}, \pm\sqrt{5}i$

c $\pm 2, -1\pm\sqrt{3}i, 1\pm\sqrt{3}i$

d $\pm\sqrt{3}, \pm\frac{\sqrt{3}}{2}+\frac{3}{2}i, \pm\frac{\sqrt{3}}{2}-\frac{3}{2}i$

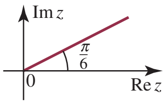
13 $\text{cis }0, \text{cis }\frac{2\pi}{5}, \text{cis }\frac{4\pi}{5}, \text{cis}\left(-\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right)$

Investigation — Current and voltage

1 $V=400\cos 130t+12\,720\sin 130t$

Exercise 1D — Graphs of complex relations: rays, lines and circles

1 **a**



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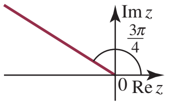

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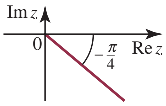

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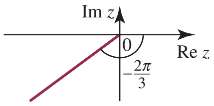

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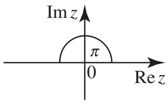

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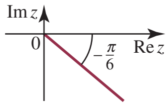
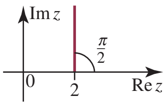


2 **a**



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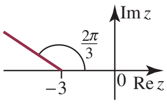

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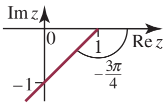

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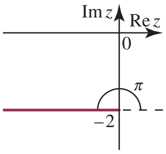

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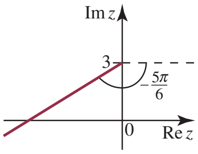

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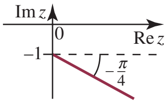

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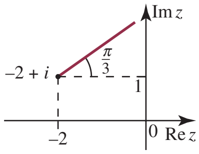

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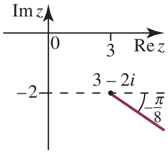

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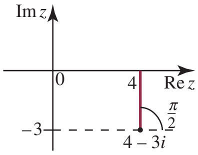

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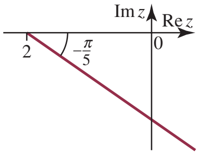
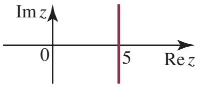
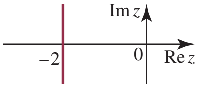


3 **a** C **b** E **4** B **5** D

6 **a**



b



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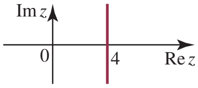

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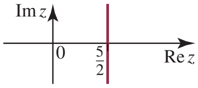

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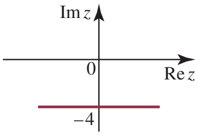

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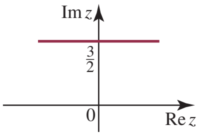

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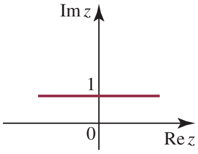

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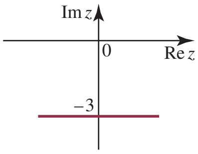
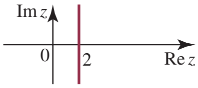


7 **a**

b



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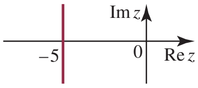

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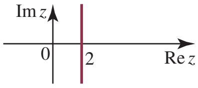

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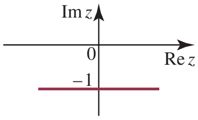

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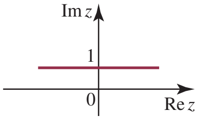

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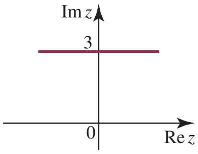

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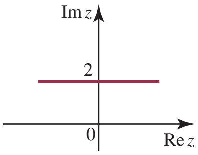
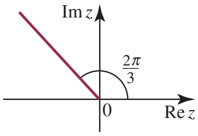


8 **a**



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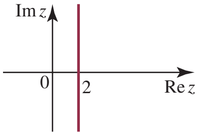

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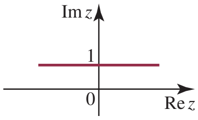

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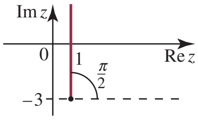

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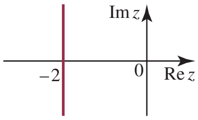

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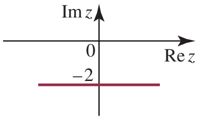

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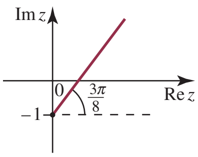

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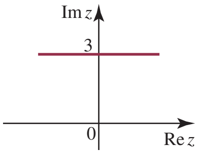
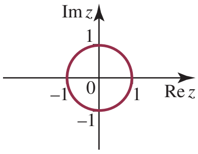


9 **a**



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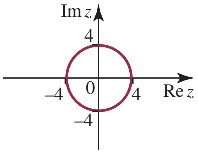

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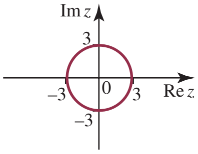

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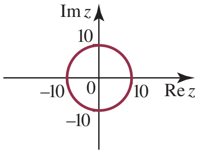

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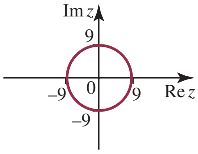

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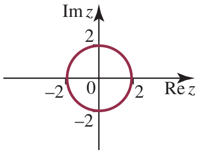
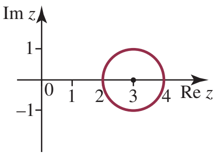


10 B **11** D

12 **a**



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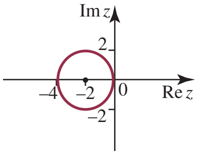

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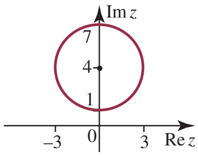

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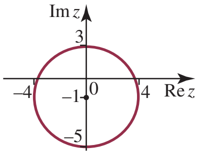

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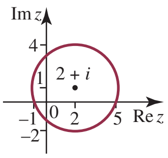

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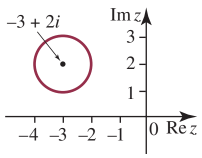

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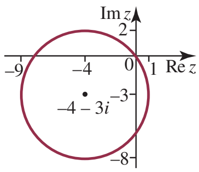

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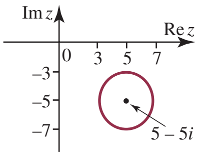


Chapter review

1 **a** $8-6i$ **b** $6+2i$ **c** $23+14i$

d $-\frac{7}{29}-\frac{26}{29}i$ **e** $16+i$ **f** $\sqrt{257}\approx 16.03$

2 **a** 5 **b** $\sqrt{5}$ **c** $3-4i$

d 6.32 **e** $5-10i$ **f** $-\frac{1}{5}-\frac{2}{5}i$

3 $2\sqrt{2}\ \text{cis}\ \frac{\pi}{4}$

4 $\frac{3\sqrt{3}}{2}+\frac{3}{2}i$

5 **a** $6\ \text{cis}\left(-\frac{11\pi}{12}\right)$ **b** $\frac{3}{2}\ \text{cis}\ \frac{5\pi}{12}$

c $8\ \text{cis}\ \pi$ **d** $\sqrt{3}\ \text{cis}\ \frac{3\pi}{8}$

6 **a** 1 **b** $0.45+0.19i$

7 A **8** A

9 **a** $(z-8+5i)(z-8-5i)$

b $2(z+4)(z-\frac{1}{2}+\frac{1}{2}i)(z-\frac{1}{2}-\frac{1}{2}i)$

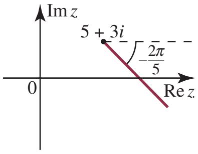
10 $a=8$

11 D **12** C **13** E

14 **a** $\sqrt{5}\pm 2\sqrt{2}i$ **b** $\frac{1}{\sqrt{2}}(5\pm 3i)$ **c** $-2,\ 1\pm\sqrt{3}i$

15 E **16** B **17** C **18** A **19** D

20



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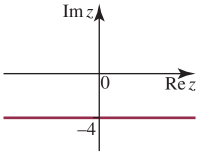

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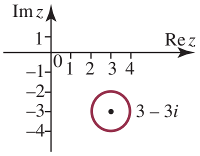


Modelling and problem solving

1 **a** $\cos\frac{\pi}{12}+\sin\frac{\pi}{12}i$

b $z=1+\sqrt{3}i,\ w=\sqrt{2}+\sqrt{2}i$

c $\frac{\sqrt{2}+\sqrt{6}+\left(\sqrt{6}-\sqrt{2}\right)i}{4}$

d **i** $\frac{\left(\sqrt{6}+\sqrt{2}\right)}{4}$ **ii** $\frac{\left(\sqrt{6}-\sqrt{2}\right)}{4}$ **iii** $2-\sqrt{3}$

e Solution will vary.

2 **a** $(z+8i)(z-8i)$

b $(z^2+8i)(z^2-8i)$

c Solution will vary.

d $(z+2-2i)(z-2+2i)(z+2+2i)(z-2-2i)$

e $(z^2+4z+8)(z^2-4z+8)$

CHAPTER 2 Matrices and applications

Exercise 2A — Review of matrix operations

1 **a** $\begin{bmatrix}0 & 3\\ -9 & -2\end{bmatrix}$ **b** Not possible

c $\begin{bmatrix}13 & -1 & 12\\ 4 & -3 & 8\end{bmatrix}$ **d** $\begin{bmatrix}2 & 3\\ 6 & 16\end{bmatrix}$

e Not possible **f** $\begin{bmatrix}5 & 18\\ -6 & 5\end{bmatrix}$

2 **a** $\begin{bmatrix}3 & 9 & 6\\ 6 & 0 & 3\\ 3 & -9 & 9\end{bmatrix}$ **b** $\begin{bmatrix}9 & -3 & 11\\ 3 & 3 & 7\\ -2 & -6 & 8\end{bmatrix}$

c Not possible **d** $\begin{bmatrix}-1 & 3 & 5\end{bmatrix}$

e $\begin{bmatrix}11\\ 4\\ -15\end{bmatrix}$ **f** Not possible

g $\begin{bmatrix}-4\end{bmatrix}$

3 **a** 2 **b** −2 **c** 0 **d** 2

e 0 **f** 2 **g** −4 **h** $2a-3$

i 1 **j** a

4 $x=-3$ or $x=2$

5 See answers for question **3**.

6 **a** $\frac{1}{2}\begin{bmatrix}3 & -4\\ -1 & 2\end{bmatrix}$ **b** $\frac{1}{2}\begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}$

c Not possible **d** $\frac{1}{2}\begin{bmatrix}2 & -2\\ 0 & 1\end{bmatrix}$

e Not possible **f** $\frac{1}{2}\begin{bmatrix}2 & 4\\ -2 & -3\end{bmatrix}$

g $\frac{-1}{4}\begin{bmatrix}2 & -2\\ -5 & 3\end{bmatrix}$ **h** $\frac{1}{2a-3}\begin{bmatrix}2 & -3\\ -1 & a\end{bmatrix}$

i $\begin{bmatrix}\cos A & \sin A\\ -\sin A & \cos A\end{bmatrix}$ **j** $\begin{bmatrix}\frac{1}{a}-1 & 1\\ -1 & 1\end{bmatrix}$

7 **a** 2 **b** 8

8 a $\begin{bmatrix} 1.5 & -1.5 & -0.5 \\ 0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 0.5 \end{bmatrix}$

b $\begin{bmatrix} 0.875 & -2 & -0.125 & 0.25 \\ -0.625 & 1 & 0.375 & 0.25 \\ 0.375 & 0 & -0.625 & 0.25 \\ 0.125 & 0 & 0.125 & -0.25 \end{bmatrix}$

9 a $\begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix}$ **b** $\frac{1}{3}\begin{bmatrix} -2 & 3 \\ -4 & 3 \end{bmatrix}$ **c** $\frac{1}{20}\begin{bmatrix} 5 & 5 \\ -3 & 1 \end{bmatrix}$

10 a

×	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	C	D	E	I
B	B	C	D	E	I	A
C	C	D	E	I	A	B
D	D	E	I	A	B	C
E	E	I	A	B	C	D

b i B **ii** C **iii** D **iv** E **v** I **vi** D

11 Check with your teacher.

12 a $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

d $\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$

Investigation — Determinants

1 1

2 a 3 **b** It is three times det *P*.

3 6

4 a –1 **b** It is the negative of det *P*.

5 a 1 **b** It does not change the determinant.

6 Multiplying a row by a constant, *k*, the determinant is *k* times the original; adding rows leaves the determinant unchanged.

Investigation — Steady state

You should have obtained $\begin{bmatrix} 0.3095 \\ 0.4286 \\ 0.2619 \end{bmatrix}$.

Exercise 2B — Markov chains and eigenvalues

1 a 0.6

b i $\begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$ **ii** $\begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$ **iii** $\begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$ **iv** $\begin{bmatrix} 0.353 \\ 0.255 \\ 0.392 \end{bmatrix}$

c i $\begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$ **ii** $\begin{bmatrix} 0.35 \\ 0.25 \\ 0.40 \end{bmatrix}$ **iii** $\begin{bmatrix} 0.355 \\ 0.255 \\ 0.390 \end{bmatrix}$ **iv** $\begin{bmatrix} 0.353 \\ 0.255 \\ 0.393 \end{bmatrix}$

d i $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$ **ii** $\begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$ **iii** $\begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$ **iv** $\begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$

2 a $a = 0.5$, $b = 0.7$, $c = 0.55$

b i $\begin{bmatrix} 0.445 \\ 0.345 \\ 0.210 \end{bmatrix}$ **ii** $\begin{bmatrix} 0.274 \\ 0.428 \\ 0.299 \end{bmatrix}$ **iii** $\begin{bmatrix} 0.305 \\ 0.457 \\ 0.239 \end{bmatrix}$ **iv** $\begin{bmatrix} 0.284 \\ 0.470 \\ 0.246 \end{bmatrix}$

c i $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$ **ii** $\begin{bmatrix} 0.375 \\ 0.395 \\ 0.230 \end{bmatrix}$ **iii** $\begin{bmatrix} 0.280 \\ 0.447 \\ 0.273 \end{bmatrix}$ **iv** $\begin{bmatrix} 0.296 \\ 0.465 \\ 0.240 \end{bmatrix}$

3 a –2, 7 **b** $\begin{bmatrix} 5a \\ -4a \end{bmatrix}$ or $\begin{bmatrix} a \\ a \end{bmatrix}$

4 a 1, –0.2 **b** $\begin{bmatrix} 5a \\ 7a \end{bmatrix}$ or $\begin{bmatrix} a \\ -a \end{bmatrix}$

5 Let $M = \begin{bmatrix} t & s \\ 1-t & 1-s \end{bmatrix}$ then show $\lambda = 1$ is a solution of $\det(M - \lambda I) = 0$.

6 Check with your teacher.

Exercise 2C — Leontief matrices and Leslie matrices

1 a 0.5 units of *G*, 0.4 units of *H*, 0.1 units of *T*
b 17.1 units of *G*, 17.4 units of *H*, 11.7 units of *T*

2 a 0.6 units of *G*, 0.6 units of *H*, 1.2 units of *T*
b 7.6 units of *G*, 14.6 units of *H*, 6.7 units of *T*

3 a 109 units of *G*, 171 units of *H*, 119 units of *T*
b $|I - C|$ is not zero.

4 a 83 units of *G*, 143 units of *H*, 81 units of *T*
b $|I - C|$ is not zero.

5 a To produce 1 unit of *A* requires 0.5 units of *C*.
b The required output is 37 units of *A*, 50 units of *B*, 59 units of *C*

6 a $\begin{bmatrix} 0.3 & 0.45 \\ 0.4 & 0.20 \end{bmatrix}$

b 124 tonnes of coal, 93 tonnes of steel

7 a $\begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}$

b 16 units of *A*, 13 units of *B*, 23 units of *C*
c 90 units of *A*, 83 units of *B*, 94 units of *C*

8 a 6 **b** 3.6

9 a A two-year-old female rat produces, on average, 1.4 females per year.
b 50% of two-year-olds live to age 3.
c 673 rats in total

d $\begin{bmatrix}288\\93\\37\end{bmatrix}$ 418 female rats

836 rats in total

10 a 0 **b** 3.1 **c** 0.2

d $\begin{bmatrix}241\\77\\20\end{bmatrix}$ 338 female birds

Investigation — Mono-alphabetic substitution

In the same hour came forth …

Exercise 2D — Cryptology

1 a 14 **b** 15 **c** 12
d 18 **e** Not possible **f** 21
g 15 **h** 7 **i** 11

2 MKPGKXMZVYUNCSUG

3 BHCXGCIPYSNMTQ

4 a $\begin{bmatrix}8 & 11\\11 & 4\end{bmatrix}$ **b** RELAX BE HAPPY

5 a $\begin{bmatrix}17 & 19\\7 & 12\end{bmatrix}$ **b** YOU SHOULD SEE ENIGMA

6 CZOAWJRXKKRDODONEZBPP

Chapter review

1 a $\begin{bmatrix}11 & 18\\2 & 15\end{bmatrix}$ **b** $\begin{bmatrix}42 & 27\\42 & -24\end{bmatrix}$ **c** $\begin{bmatrix}-4 & 79\\26 & 22\end{bmatrix}$

d $\frac{1}{42}\begin{bmatrix}8 & -5\\2 & 4\end{bmatrix}$ **e** $\frac{1}{51}\begin{bmatrix}1 & 8\\6 & -3\end{bmatrix}$

2 a 61 **b** $\frac{1}{61}\begin{bmatrix}-7 & 12 & 8\\-2 & -14 & 11\\17 & -3 & -2\end{bmatrix}$

3 a 0.45 **b** $\begin{bmatrix}0.386\\0.269\\0.345\end{bmatrix}$

4 a 3430 **b** 2244
c Car 2333.3; bus 666.7; train 4000

5 a $\lambda = -0.1, 1$ **b** $\begin{bmatrix}a\\-a\end{bmatrix}, \begin{bmatrix}a\\\frac{7}{4}a\end{bmatrix}$

6 a 0.5G, 0.4H, 0.1I **b** 17G, 17H, 11I
c 445G, 427H, 290I

7 a 0.6C is needed to produce 1C
b 224A, 207B, 358C

Modelling and problem solving

1 a $\begin{bmatrix}0.7 & 0.6\\0.3 & 0.4\end{bmatrix}$ **b i** 0.7 **ii** 0.67 **iii** 0.67 **c** 0.67

2 a $\begin{bmatrix}0.4 & 0.8\\0.6 & 0.2\end{bmatrix}$ **b i** 0.4 **ii** 0.573

3 a 2600 **b** 3300 (Bus), 2200 (Train)

4 Croke (20%), Pepola (26%), Sprote (13%), Funta (42%). Croke and Sprote will not survive.

5 Tellya (18.5%), Oops (37%), Yodacall (44.5%). Tellya will not survive.

6 a $\begin{bmatrix}0 & 0.50 & 0.45 & 0.4\\0.9 & 0 & 0 & 0\\0 & 0.85 & 0 & 0\\0 & 0 & 0.50 & 0\end{bmatrix}$ **b** $\begin{bmatrix}1136\\1087\\930\\577\end{bmatrix}$ 7460 sheep

7 a i 0.85 **ii** 0.99 **iii** 0
b i 0 **ii** 0.4 **iii** 0.08

c $\begin{bmatrix}0 & 0.4 & 0.08\\0.85 & 0 & 0\\0 & 0.99 & 0\end{bmatrix}$ **d** $\begin{bmatrix}182\\309\\278\end{bmatrix}$ 769 female sheep

8 a 0 **b** 5.3 **c** 0.2 **d** $\begin{bmatrix}77.4\\64.2\\1.8\end{bmatrix}$

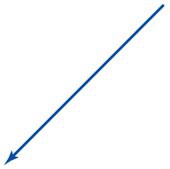
9 FANXOJKRCZYLMEYL

10 a $\begin{bmatrix}1 & 25\\5 & 16\end{bmatrix}$ **b** THESKYISBLUE

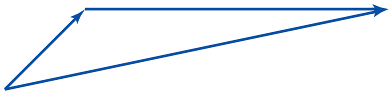
CHAPTER 3 Vectors

Exercise 3A — Review of vectors

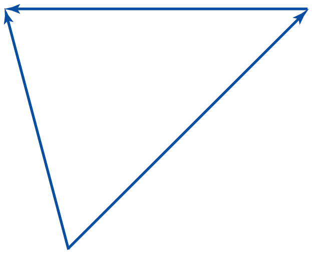
1 a



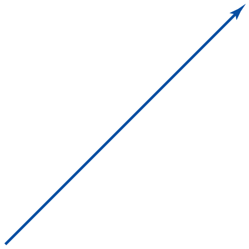
b



c

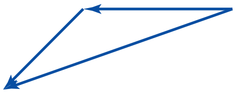


d



e

f



2 **a** $-6\underset{\sim}{j}$ **b** $5\underset{\sim}{i}$ **c** $7.7\underset{\sim}{i} + 6.4\underset{\sim}{j}$

d $-4.5\underset{\sim}{i} + 7.8\underset{\sim}{j}$ **e** $-12.4\underset{\sim}{i} - 10.1\underset{\sim}{j}$ **f** $13.6\underset{\sim}{i} - 6.3\underset{\sim}{j}$

g $13.5\underset{\sim}{i} + 6.6\underset{\sim}{j}$ **h** $22.6\underset{\sim}{i} - 22.6\underset{\sim}{j}$

3 **a** 38.3 in a direction of 33° north of east

b 34.7 in a direction of 36.4° north of east

c 32.5 in a direction of 17.2° south of east

d 14.4 in a direction 23.7° north of east

e 10.4 in a direction of 46.5° north of east

f 24.8 in a direction of 12.5° south of east

g 68.7 in a direction of 16.9° south of east

4 4316 N

5 **a** 6.32 **b** 7.2 **c** 5.4

d 6 **e** 6.5

6 543 N

7 **a** 69.3 **b** −138.6 **c** 0

8 **a** −4 **b** 13 **c** 0

9 **c**

10 **a** 0 **b** 6 **c** −2

d 6 **e** −11

11 **a**

12 **a** −1.5 **b** 4 or −4 **c** −3

d −6

13 **a** 41.3 N in the direction of 0.5° south of east

b 38.5 N in the direction of 29.8° south of east

Exercise 3B — Vector product

1 **a** **i** 40.5 in a downward direction (into the page)

ii 40.5 in an upward direction (or out of the page)

b **i** 32.1 in an upward direction (out of the page)

ii 32.1 in a downward direction (or into the page)

c **i** 0 **ii** 0

2 **a** $-3\underset{\sim}{j} + 2\underset{\sim}{k}$ **b** $2\underset{\sim}{i} + 2\underset{\sim}{k}$ **c** $-6\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

3 **a** $2\underset{\sim}{i} - 2\underset{\sim}{j} - 5\underset{\sim}{k}$ **b** $-\underset{\sim}{i} - 5\underset{\sim}{j} + 13\underset{\sim}{k}$

c $-10\underset{\sim}{i} + 15\underset{\sim}{j} + 15\underset{\sim}{k}$

4 **a** 3.45 square units **b** 4.36 square units

5 11 Nm

6 **a** 37.6 Nm **b** 34.6 Nm

7 **a** $-\underset{\sim}{j}$ **b** $\underset{\sim}{i}$

c 0 **d** $\underset{\sim}{j}$

8 **a** $0.577(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k})$ **b** $0.707(-\underset{\sim}{j} + \underset{\sim}{k})$

9 **a** $16\underset{\sim}{k}$ **b** $4\underset{\sim}{b} - 8\underset{\sim}{c}$

10 230.6 Nm

11 72 Nm

Exercise 3C — Scalar triple product

1 **a** 27 **b** 1 **c** 2

d 0 **e** 0 **f** 3

g −6

2 **a** 12 **b** 19 **c** 0 **d** 4

3 **a** 24 **b** −1.5 **c** $\frac{24}{11}$

4 **a** 735.4 **b** 1022

5 **a** 1 **b** 0 **c** 1 **d** 0

e −1 **f** 1

6 If $\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c}$ is a clockwise ordering of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, the result is 1. If it is anticlockwise, it is −1.

7 **a** If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are coplanar, the volume of the parallelepiped formed is 0.

b If $\underset{\sim}{a}$ is in the same plane as $\underset{\sim}{b}$ and $\underset{\sim}{c}$, then $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b} \times \underset{\sim}{c}$. Therefore, the scalar product will be 0.

Exercise 3D — Applications of the vector product

1 **a** $x - y - 6z = -5$ **b** $x + y + z = 1$

c $z = 1$ **d** $-y + z = 2$

e $-3x + y + z = -5$ **f** $x + 4y + z = 12$

2 **a** $x + 2z = 4$ **b** $2x + y + 2z = 5$

c $-x + 2y - z = 4$ **d** $3x - y + 2z = 2$

3 $2y - z = 0$

4 **a** $5\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$ **b** $5\underset{\sim}{i}$

c $2\underset{\sim}{j} + 3\underset{\sim}{k}$ **d** $2y + 3z = 0$

5 $y = 1$

6 **a** P = (3, 0), Q = (0, 2)

b $2x + 3y + 6z = 6$

7 $\frac{1}{13}(3\underset{\sim}{i} + 4\underset{\sim}{j} + 12\underset{\sim}{k})$

8 $\frac{1}{\sqrt{14}}(3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$

9 $12\underset{\sim}{k}$

10 **a** $-3\underset{\sim}{i} + 9\underset{\sim}{j}$ **b** $36\underset{\sim}{k}$

11 **a** $\underset{\sim}{r} \cdot \underset{\sim}{v} = 0$ **c** v is halved **d** v is doubled

12 **b** 3.2

13 **b** $4\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}$

Chapter review

1 **a** $-15.3\underset{\sim}{i} + 12.9\underset{\sim}{j}$ **b** $5.8\underset{\sim}{i} - 6.9\underset{\sim}{j}$

2 23.3 in a direction of 12.1° south of east

3 −9

4 0

5 17 N in a direction of 3.9° south of west

6 82.7 out of page

7 **a** $-11\underset{\sim}{i} + 9\underset{\sim}{j} + \underset{\sim}{k}$ **b** $8\underset{\sim}{i} - 2\underset{\sim}{j} - 5\underset{\sim}{k}$

8 3.5 square units

9 8.72 Nm

10 31.2 Nm

11 $0.42\underset{\sim}{i} - 0.57\underset{\sim}{j} - 0.71\underset{\sim}{k}$

13 **a** 19 **b** 13

14 $\frac{4}{3}$

15 5192 cubic units

16 The vectors are coplanar.

17 $x + 5y + 10z = 22$

18 $2x + y + z = 5$

Modelling and problem solving

1 **a** 1.22 **b** 490 N

2 407 J

3 **a** 960 Nm **b** 1.48 m from centre

4 3.5 m^2

5 43.5 m^2

6 **a** $-\underset{\sim}{p}$ **b** −4

c The magnitude remains constant. A clockwise ordering of a, b and c gives the same answer. An anticlockwise ordering of a, b and c gives the negative of $\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c}$.

7 $21\underset{\sim}{k}$

8 Answers will vary.

CHAPTER 4 Integral calculus and numerical methods

Exercise 4A — Approximation using the derivative

1 b $\frac{\pi}{2}$ **c** 4.7%

2 a x **b** $\frac{\pi}{8}$

3 a $3x + 1$ **b** 1.27

4 $9\frac{1}{6}$

5 a 9.8 **b** 1.02 **c** 5.1 **d** $3\frac{1}{27}$

6 b 160.8 cm^2

7 1%

8 1.5%

Investigation — Polynomial expansions

2 a $f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

b $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

c $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

3 a 0.91 **b** −0.211 **c** 20.0855 **d** 0.405

Exercise 4B — Substitution where the derivative is present in the integrand

1 a $\frac{(x^2+3)^5}{5} + c$ **b** $\frac{1}{2(6-x^2)^2} + c$

c $\frac{(x^3-2)^6}{6} + c$ **d** $-\frac{1}{2(x^2+4x)^2} + c$

e $\frac{2(x^2+5x)^{\frac{3}{2}}}{3} + c$ **f** $-\frac{1}{3(x^2-3x)^3} + c$

g $\frac{(x^3-5)^3}{3} + c$ **h** $2\sqrt{x^3+2x^2} + c$

i $e^{x^4} + c$ **j** $-\cos(x^2 + 3x - 2) + c$

k $\sin(x^3 + 5x) + c$ **l** $\frac{\sin^4 x}{4} + c$

m $\frac{-\sin^5 x}{5} + c$ **n** $\frac{(\log_e x)^2}{2} + c$

o $\frac{\tan^4 x}{4} + c$ **p** $\frac{(\mathrm{Sin}^{-1}x)^3}{3} + c$

2 A

3 a D **b** B **c** E

4 a $\frac{(x^3-5)^6}{3} + c$ **b** $-\frac{(4-x^2)^4}{8} + c$

c $\frac{(x^3-1)^8}{24} + c$ **d** $\frac{(x^2+6x-2)^5}{10} + c$

e $-\frac{(x^2+2x+3)^{-3}}{6} + c$ **f** $4\sqrt{x^2+3x} + c$

g $-\frac{(x^2-5x+2)^{-5}}{5} + c$ **h** $\frac{2(4-3x+x^3)^{\frac{3}{2}}}{9} + c$

i $3e^{(x^2-x+3)} + c$ **j** $\frac{e^{x^3+2}}{3} + c$

k $-\frac{\cos(x^2+2x-3)}{2} + c$ **l** $-\frac{\sin(6x-x^3)}{3} + c$

m $-\frac{\cos^5 2x}{10} + c$ **n** $\frac{\sin^3 3x}{9} + c$

o $\frac{(\log_e 3x)^2}{4} + c$ **p** $[\log_e(x^2 - x)]^2 + c$

5 a $\frac{(x^2+1)^{\frac{7}{2}}}{7} + c$ **b** $-\frac{(1-x^2)^{\frac{3}{2}}}{3} + c$

c $\frac{(3+2e^x)^5}{10} + c$ **d** $\frac{1}{2\cos^2 x} + c$

e $\frac{-\cos x^3}{3} + c$ **f** $-e^{\cos x} + c$

g $\frac{[\log_e(\sin x)]^2}{2} + c$ **h** $-\frac{(1-e^{3x})^3}{9} + c$

i $\left(\mathrm{Cos}^{-1}\frac{x}{3}\right)^2 + c$ **j** $\frac{2(x+x^2-3)^{\frac{3}{2}}}{3} + c$

k $\frac{\sin(x^2+4x)}{2} + c$ **l** $2e^{\sqrt{x+1}} + c$

m $\frac{(\mathrm{Sin}^{-1}4x)^2}{8} + c$ **n** $\frac{(\mathrm{Tan}^{-1}x)^2}{2} + c$

o $-\log_e\frac{(1-4x^2)}{8} + c$

6 a $\frac{2\sqrt{1+3\sin x}}{3} + c$ **b** $\frac{2(2+\tan x)^{\frac{3}{2}}}{3} + c$

c $\frac{1}{2}\sec^2 x + c$ **d** $-\frac{1}{2(e^{2x}-3)} + c$

e $\frac{1}{2(5-\tan x)^2} + c$ **f** $4\log_e(\log_e x) + c$

g $\frac{(\log_e x)^4}{4} + c$ **h** $e^{\tan x} + c$

i $2\sqrt{e^x + e^{-x}} + c$ **j** $-\log_e(\sin x + \cos x) + c$

k $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$ **l** $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c$

m $\frac{[\log_e(\tan x)]^2}{2} + c$

7 $f(x) = \sqrt{x^2+5} - 2$ **8** $f(x) = 2e^{\sqrt{x}} + 1$

9 $g(x) = (\log_e x^2)^2 - 2$ **10** $g(x) = 1 - 4\cos^4 x$

Exercise 4C — Linear substitution

1 a i $\int \frac{4}{u}\,du$ **ii** $4\log_e(x - 3) + c$

b **i** $\int \frac{2}{3u}\,du$ **ii** $\frac{2\log_e(3x+5)}{3}+c$

c **i** $\int \frac{\sqrt{u}}{4}\,du$ **ii** $\frac{(4x+1)^{\frac{3}{2}}}{6}+c$

d **i** $\int -\frac{\sqrt{u}}{2}\,du$ **ii** $-\frac{(3-2x)^{\frac{3}{2}}}{3}+c$

e **i** $\int (u^4-u^3)\,du$ **ii** $\frac{(4x-1)(x+1)^4}{20}+c$

f **i** $\int (4u^5+12u^4)\,du$ **ii** $\frac{2(5x+3)(x-3)^5}{15}+c$

g **i** $\int \frac{u^5-u^4}{2}\,du$ **ii** $\frac{(10x-1)(2x+1)^5}{60}+c$

h **i** $\int \frac{u^6-u^5}{3}\,du$ **ii** $-\frac{(18x+1)(1-3x)^6}{126}+c$

i **i** $\int \frac{2}{3}\left(u^{\frac{7}{4}}+2u^{\frac{3}{4}}\right)du$ **ii** $\frac{8}{231}(3x-2)^{\frac{7}{4}}(21x+8)+c$

j **i** $\int \frac{\left(u^{\frac{4}{3}}-7u^{\frac{1}{3}}\right)}{4}\,du$ **ii** $\frac{3(8x-21)(2x+7)^{\frac{4}{3}}}{112}+c$

k **i** $\int \left(u^{\frac{3}{2}}-3u^{\frac{1}{2}}\right)du$ **ii** $\frac{2(x-2)(x+3)^{\frac{3}{2}}}{5}$

l **i** $\int \frac{\left(u^{\frac{3}{2}}+4u^{\frac{1}{2}}\right)}{9}\,du$ **ii** $\frac{2}{135}(3x-4)^{\frac{3}{2}}(9x+8)+c$

m **i** $\int \left(u^{\frac{5}{2}}+6u^{\frac{3}{2}}\right)du$ **ii** $\frac{2(5x+22)(x-4)^{\frac{5}{2}}}{35}+c$

n **i** $\int \frac{\left(u^{\frac{7}{2}}-7u^{\frac{5}{2}}\right)}{4}\,du$ **ii** $\frac{(x-4)(2x+1)^{\frac{7}{2}}}{9}+c$

o **i** $\int \left(2u^{\frac{1}{2}}+12u^{-\frac{1}{2}}\right)du$ **ii** $\frac{4(x+12)(x-6)^{\frac{1}{2}}}{3}+c$

p **i** $\int \left(3u^{\frac{1}{2}}-24u^{-\frac{1}{2}}\right)du$ **ii** $-2(x+16)(8-x)^{\frac{1}{2}}+c$

2 **a** C **b** E

3 **a** B **b** D

4 **a** $\frac{1}{7}(x-4)^7+\frac{4}{3}(x-4)^6+\frac{16}{5}(x-4)^5+c$

b $2(5-x)^5-\frac{25}{4}(5-x)^4-\frac{1}{6}(5-x)^6+c$

c $\frac{2}{7}(x-1)^{\frac{7}{2}}+\frac{4}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$

d $-6(3-x)^{\frac{3}{2}}+\frac{12}{5}(3-x)^{\frac{5}{2}}-\frac{2}{7}(3-x)^{\frac{7}{2}}+c$

e $\frac{3}{13}(x+2)^{\frac{13}{3}}-\frac{6}{5}(x+2)^{\frac{10}{3}}+\frac{12}{7}(x+2)^{\frac{7}{3}}+c$

f $-\frac{4}{7}(1-x)^{\frac{7}{4}}+\frac{8}{11}(1-x)^{\frac{11}{4}}-\frac{4}{15}(1-x)^{\frac{15}{4}}+c$

g $\frac{2}{7}(x-2)^{\frac{7}{2}}+\frac{12}{5}(x-2)^{\frac{5}{2}}+6(x-2)^{\frac{3}{2}}+c$

h $\frac{2}{7}(x+1)^{\frac{7}{2}}-\frac{16}{5}(x+1)^{\frac{5}{2}}+\frac{32}{3}(x+1)^{\frac{3}{2}}+c$

i $\log_e(e^x+1)+c$

j $\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{4}{3}(x+1)^{\frac{3}{2}}+2(x+1)^{\frac{1}{2}}+c$

k $-36(3-x)^{\frac{1}{2}}+8(3-x)^{\frac{3}{2}}-\frac{4}{5}(3-x)^{\frac{5}{2}}+c$

l $\frac{2}{9}(x-1)^{\frac{9}{2}}+\frac{6}{7}(x-1)^{\frac{7}{2}}+\frac{6}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$

m $\frac{2}{7}(x+4)^{\frac{7}{2}}-\frac{24}{5}(x+4)^{\frac{5}{2}}+32(x+4)^{\frac{3}{2}}-128(x+4)^{\frac{1}{2}}+c$

n $-4(1-x)^{\frac{1}{2}}+4(1-x)^{\frac{3}{2}}-\frac{12}{5}(1-x)^{\frac{5}{2}}+\frac{4}{7}(1-x)^{\frac{7}{2}}+c$

o $\log_e(x-2)-\frac{5}{x-2}+c$ **p** $\frac{3}{2(x+1)^2}-\frac{2}{x+1}+c$

q $4\log_e(x+2)+\frac{8}{x+2}+c$

r $(x-1)+2\log_e(x-1)-\frac{1}{x-1}+c$

s $\frac{2}{5}(x+2)^{\frac{5}{2}}+\frac{4}{3}(x+2)^{\frac{3}{2}}+2(x+2)^{\frac{1}{2}}+c$

t $-\frac{2}{5}(2-x)^{\frac{5}{2}}+c$ **u** $e^x+2-2\log_e(e^x+2)+c$

v $\frac{1}{2}(e^x-1)^2+2(e^x-1)+\log_e(e^x-1)+c$

5 **a** $f(x)\ =\ \frac{2}{3}(5-x)^{\frac{3}{2}}-20(5-x)^{\frac{1}{2}}+32\frac{2}{3}$ **b** $x\le 5$

6 **a** $f(x)\ =\ (x+1)^{\frac{5}{2}}-2(x+1)^{\frac{3}{2}}+(x+1)^{\frac{1}{2}}+1$
b $x\ge -1$

7 **a** $g(x)\ =\ 2\log_e(x-1)-\frac{3}{x-1}+3$ **b** $x>1$

8 **a** $g(x)\ =\ e^x+1-\log_e(e^x+1)$ **b** R

Exercise 4D — Antiderivatives involving trigonometric identities

1 **a** $\frac{1}{2}x+\frac{1}{4}\sin 2x+c$

b $\frac{1}{2}x-\frac{1}{8}\sin 4x+c$

c $x+\frac{1}{8}\ \sin 8x+c$

d $2x-\frac{1}{3}\sin 6x+c$

e $\frac{1}{2}x+\frac{1}{20}\sin 10x+c$

f $\frac{1}{2}x-\frac{1}{24}\sin 12x+c$

g $\frac{1}{2}(x+\sin x)+c$

h $\frac{1}{2}x-\frac{3}{4}\sin\frac{2x}{3}\ +c$

i $\frac{3}{2}x + \frac{9}{2}\sin\frac{x}{3} + c$

j $x - 2\sin\frac{x}{2} + c$

k $\frac{1}{2}x + \frac{3}{8}\sin\frac{4x}{3} + c$

l $\frac{1}{2}x - \frac{1}{6}\sin 3x + c$

2 a $\sin^2 x + c$ (or $-\frac{1}{2}\cos 2x + c$)

b $-\frac{1}{2}\cos 4x + c$

c $-\frac{1}{12}\cos 6x + c$

d $\frac{1}{8}\cos 8x + c$

e $\frac{1}{8}x - \frac{1}{32}\sin 4x + c$

f $\frac{1}{8}x - \frac{1}{64}\sin 8x + c$

g $\frac{1}{4}x - \frac{1}{64}\sin 16x + c$

h $\frac{1}{4}x - \frac{1}{48}\sin 12x + c$

i $\frac{3}{4}x - \frac{3}{8}\sin 2x + c$

j $\frac{1}{2}x - \frac{3}{8}\sin\frac{4x}{3} + c$

k $\frac{1}{8}x - \frac{1}{80}\sin 10x + c$

l $-\frac{1}{4}x + \frac{3}{64}\sin\frac{16x}{3} + c$

3 a C **b** A **c** E

4 a $\frac{1}{3}\cos^3 x - \cos x$ **b** $\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x$

c $-\frac{3}{2}\cos 4x + \frac{1}{2}\cos^3 4x$ **d** $\frac{4}{3}\sin 3x - \frac{4}{9}\sin^3 3x$

e $\frac{1}{21}\cos^3 7x - \frac{1}{7}\cos 7x$ **f** $\frac{1}{6}\sin 6x - \frac{1}{18}\sin^3 6x$

g $2\cos^3\frac{x}{2} - 6\cos\frac{x}{2}$ **h** $6\sin\frac{x}{3} - 2\sin^3\frac{x}{3}$

i $\frac{2}{9}\cos^3\frac{3x}{2} - \frac{2}{3}\cos\frac{3x}{2}$

j $\frac{2}{5}\sin\frac{5x}{2} - \frac{2}{15}\sin^3\frac{5x}{2}$

k $\frac{4}{9}\cos^3\frac{3x}{4} - \frac{4}{3}\cos\frac{3x}{4}$

l $\frac{3}{4}\sin\frac{4x}{3} - \frac{1}{4}\sin^3\frac{4x}{3}$

5 a $\cos x - \frac{2}{3}\cos^3 x + c$ **b** $\frac{1}{2}\sin 2x - \frac{1}{3}\sin^3 2x + c$

c $\frac{1}{3}\cos 3x - \frac{2}{9}\cos^3 3x + c$ **d** $\frac{1}{4}\sin 4x - \frac{1}{6}\sin^3 4x + c$

e $2\cos\frac{x}{2} - \frac{4}{3}\cos^3\frac{x}{2} + c$

f $3\sin\frac{x}{3} - 2\sin^3\frac{x}{3} + c$

6 a $-\frac{1}{5}\cos^5 x + c$ **b** $-\frac{1}{8}\cos^4 2x + c$

c $-\frac{1}{3}\cos^6\frac{x}{2} + c$ **d** $\frac{1}{15}\sin^5 3x + c$

e $\frac{5}{7}\sin^7\frac{x}{5} + c$ **f** $\frac{3}{16}\sin^8\frac{2x}{3} + c$

7 a $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$ **b** $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

c $\frac{1}{10}\cos^5 2x - \frac{1}{6}\cos^3 2x + c$

d $\frac{1}{9}\sin^3 3x - \frac{1}{15}\sin^5 3x + c$

e $\frac{2}{5}\cos^5\frac{x}{2} - \frac{2}{3}\cos^3\frac{x}{2} + c$

f $\frac{2}{9}\sin^3\frac{3x}{2} - \frac{2}{15}\sin^5\frac{3x}{2} + c$

g $\frac{12}{5}\cos^5\frac{x}{3} - 4\cos^3\frac{x}{3} + c$

h $-\frac{8}{5}\sin^3\frac{5x}{4} + \frac{24}{25}\sin^5\frac{5x}{4} + c$

i $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + c$

j $\frac{1}{10}\sin^5 2x - \frac{1}{14}\sin^7 2x + c$

k $\frac{1}{8}\cos^8 2x - \frac{1}{6}\cos^6 2x + c$

l $-\frac{2}{21}\sin^7 3x + \frac{2}{27}\sin^9 3x + c$

m $\frac{8}{9}\cos^9\frac{x}{2} - \frac{8}{7}\cos^7\frac{x}{2} + c$

n $\frac{1}{12}\sin^8\frac{3x}{2} - \frac{1}{15}\sin^{10}\frac{3x}{2} + c$

8 a $\frac{1}{2}\tan 2x$ **b** $3\tan\frac{x}{3}$

c $\frac{1}{3}\tan^3 x$ **d** $\frac{1}{4}\tan^4 x$

e $\frac{1}{3}\tan^6 2x$ **f** $\frac{16}{5}\tan^5\frac{x}{2}$

g $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x$ **h** $\tan^3 2x + \frac{3}{5}\tan^5 2x$

i $\frac{4}{3}\tan^3\frac{x}{2} + \frac{4}{5}\tan^5\frac{x}{2}$ **j** $\frac{1}{4}\tan^4 3x + \frac{1}{6}\tan^6 3x$

k $\tan^5\frac{x}{5} + \frac{5}{7}\tan^7\frac{x}{5}$

l $\frac{1}{3}\tan^6 6x + \frac{1}{2}\tan^8 6x + \frac{1}{5}\tan^{10} 6x$

9 a $\frac{-\cos^{n+1}x}{n+1} + c$ **b** $\frac{\sin^{n+1}x}{n+1} + c$

c $\frac{\tan^{n+1}x}{n+1} + c$ **d** $\frac{\cos^{n+3}x}{n+3} - \frac{\cos^{n+1}x}{n+1} + c$

e $\frac{\sin^{n+1}x}{n+1} - \frac{\sin^{n+3}x}{n+3} + c$

10 $f(x) = -2\cos^3 x + \frac{1}{4}$

11 $f(x) = \frac{x}{2} - \frac{1}{16}\sin 8x + \frac{7\pi}{8}$

12 $g(x) = \frac{2}{7}\cos^7\frac{x}{2} - \frac{2}{5}\cos^5\frac{x}{2}$

Exercise 4E — Antidifferentiation using partial fractions

1 **a** $a = 1, b = 2$ **b** $a = -1, b = 2$

c $a = 3, b = 10$ **d** $a = 2, b = -1$

e $a = -2, b = 3$ **f** $a = -5, b = 7$

g $a = -6, b = -1, c = 1$ **h** $a = -1, b = 2, c = 2$

2 **a** $\frac{1}{x+1} - \frac{1}{x+2}$ **b** $\frac{3}{x-2} - \frac{3}{x+2}$

c $\frac{9}{2(x+3)} + \frac{3}{2(x-1)}$ **d** $\frac{2}{x-2} + \frac{1}{x+1}$

e $\frac{1}{x+2}$ **f** $\frac{3}{x-4} - \frac{2}{x+4}$

g $\frac{4}{x+2} - \frac{3}{(x+2)^2}$ **h** $\frac{5}{x-5} - \frac{1}{(x-5)^2}$

i $\frac{-2}{x} + \frac{3}{x-2}$ **j** $\frac{2}{x-2} + \frac{5}{x+3}$

k $\frac{4}{2x+1} + \frac{2}{x-3}$ **l** $\frac{-3}{3x-2} + \frac{4}{x+1}$

m $\frac{1}{2-x} + \frac{4}{x+3}$ **n** $\frac{5}{1-x} - \frac{3}{3-x}$

3 **a** $\log_e\left(\frac{x+1}{x+2}\right) + c$ **b** $3\log_e\left(\frac{x-2}{x+2}\right) + c$

c $\frac{1}{2}\log_e[(x+3)^9(x-1)^3] + c, x > 1$

d $2\log_e(x-2) + \log_e(x+1) + c$

e $\log_e(x+2) + c$

f $3\log_e(x-4) - 2\log_e(x+4) + c$

g $4\log_e(x+2) + \frac{3}{x+2} + c$

h $5\log_e(x-5) + \frac{1}{x-5} + c$

i $3\log_e(x-2) - 2\log_e x + c$

j $2\log_e(x-2) + 5\log_e(x+3) + c$

k $2\log_e(2x+1) + 2\log_e(x-3) + c$

l $4\log_e(x+1) - \log_e(3x-2) + c$

m $4\log_e(x+3) - \log_e(2-x) + c$

n $3\log_e(3-x) - 5\log_e(1-x) + c$

4 **a** D **b** B

5 C

6 **a** $5\log_e x - 2\log_e(x+2) + c$

b $2\log_e(x-2) + 3\log_e(x+1) + c$

c $2\log_e(x+1) - \log_e(x+2) + c$

d $5\log_e(x-6) + \log_e(x+1) + c$

e $4\log_e(x-3) + \log_e(x-1) + c$

f $3\log_e(x+1) - 2\log_e(x+6) + c$

g $2\log_e(x+3) + 5\log_e(x-3) + c$

h $3\log_e(x+1) + 4\log_e(x-1) + c$

i $\frac{1}{2}\log_e(2x+1) + 2\log_e(x-2) + c$

j $\frac{4}{3}\log_e(3x-2) - 2\log_e(x+3) + c$

k $2\log_e(x-2) - \frac{3}{2}\log_e(2x-1) + c$

l $\log_e\frac{2+x}{2-x} + c, -2 < x < 2$

m $-\log_e(4-x) - 2\log_e(4+x) + c$

n $2\log_e(1+x) - 3\log_e(5-x) + c$

7 **a** $x - 6\log_e(x+5) + c$

b $x + 5\log_e(x-2) + c$

c $x - \frac{1}{3}\log_e x - \frac{8}{3}\log_e(x+3) + c$

d $x + 7\log_e(x-4) - \log_e x + c$

e $x + \log_e(x+1) - 6\log_e(x+3) + c$

f $x + 4\log_e(x-3) - \log_e(x+1) + c$

g $x + \log_e(x+2) + 2\log_e(x-2) + c$

h $\frac{1}{2}x^2 + x - 10\log_e(x+2) + 4\log_e(x+1) + c$

i $\frac{1}{2}x^2 + 4x + 22\log_e(x-5) + 3\log_e(x+1) + c$

j $x^2 + x + 3\log_e(x+1) - \log_e(x-1) + c$

k $x - 3\log_e(x+1) - \frac{4}{x+1} + c$

l $2x + 3\log_e(x-3) + \frac{2}{x-3} + c$

8 **a** $2\log_e x - 3\log_e(x+2) + c$

b $5\log_e(x-3) + 4\log_e(x+4) + c$

c $3\log_e(x-5) + 2\log_e(x+5) + c$

d $x + 2\log_e(x-3) - 2\log_e(x+3) + c$

e $x + 4\log_e(x-4) + \log_e(x+2) + c$

f $x - \frac{11}{4}\log_e(x+7) + \frac{3}{4}\log_e(x-1) + c$

g $\frac{1}{2}x^2 + 5x - \frac{4}{x-2} + 12\log_e(x-2) + c$

h $2x + \frac{4}{7}\log_e(2x+3) + \frac{24}{7}\log_e(x-2) + c$

i $\frac{1}{2}\log_e(x^2+4) + \frac{1}{2}\text{Tan}^{-1}\frac{x}{2} + c$

j $2\log_e(x^2+9) - \frac{2}{3}\text{Tan}^{-1}\frac{x}{3} + c$

k $-4\log_e(x-1) + \frac{33}{15}\log_e(x+2) + \frac{34}{5}\log_e(x-3) + c$

l $2\log_e(x+1) + 3\log_e(x-2) - 4\log_e(x+3) + c$

m $2\log_e(x-1) - \log_e(x+2) - \frac{1}{x+2} + c$

n $\log_e(x^2+1) - \text{Tan}^{-1}x - 3\log_e(2-x) + c$

9 **a** $f(x) = 3\log_e\left[\frac{6(x-1)}{x+1}\right]$ **b** $x > 1$

10 **a** $g(x) = x + \frac{1}{2}\log_e\left[\frac{(x-3)^5}{5(x+1)}\right]$

b $x > 3$

Exercise 4F — Integration by parts

1 **a** $\cos x + x\sin x + c$ **b** $(x-1)e^x + c$

c $\frac{1}{2}x^2\log_e x - \frac{1}{4}x^2 + c$

d $\frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x + c$

2 **a** $x^2\sin x + 2x\cos x - 2\sin x + c$

b $-x^2\cos x + 2x\sin x + 2\cos x + c$

c $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$

d $\frac{1}{6}(x+1)^6x^2 - \frac{1}{21}x(x+1)^7 + \frac{1}{168}(x+1)^8 + c$

3 **a** $x\log_e 2x - x + c$

b $x\,\mathrm{Sin}^{-1}\,x + \sqrt{1-x^2} + c$

4 **a** $\frac{1}{2}(e^x\cos x + e^x\sin x) + c$

b $\dfrac{e^{2x}}{5}(2\cos x + \sin x) + c$

c $\frac{1}{5}e^x(\sin 2x - 2\cos 2x) + c$

5 **a** $-x\cos x + \sin x + \cos x + c$

b $(x+1)e^x + c$

c $\frac{1}{4}x^4\log_e x - \frac{1}{16}x^4 + c$

d $\frac{1}{2}(x+1)^2\log_e x - \frac{1}{4}x^2 - x - \frac{1}{2}\log_e x + c$

e $\frac{1}{3}x^3\sin^{-1}x + \frac{1}{9}x^2\sqrt{1-x^2} + \frac{2}{9}\sqrt{1-x^2} + c$

f $(x^2+1)e^x + c$

g $e^x(x^3 - 3x^2 + 6x - 6) + c$

h $-e^{-x}(x+1)$

i $\dfrac{x^{n+1}}{n+1}\left(\log_e x - \dfrac{1}{n+1}\right) + c$

6 **b** $e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + c$

7 **b** $\sin x\cos^2 x + \frac{2}{3}\sin^3 x + c$

8 $\frac{1}{8}(x - 3\sin x + \frac{3}{2}\sin x\cos x +$

$\frac{3}{2}x - \sin x\cos^2 x - \frac{2}{3}\sin^3 x) + c$

Exercise 4G — Definite integrals

1 **a** **i** R\{–3, 3} **ii** The integral does exist.
b **i** (–2, 2) **ii** The integral does exist.
c **i** (–4, 4) **ii** The integral does not exist.
d **i** R **ii** The integral does exist.
e **i** R\{0} **ii** The integral does exist.
f **i** R\{–1, 0} **ii** The integral does exist.
g **i** R\{–2, –3} **ii** The integral does exist.
h **i** R\{1} **ii** The integral does not exist.
i **i** R\{2, 6} **ii** The integral does not exist.
j **i** R **ii** The integral does exist.
k **i** $(-\frac{1}{3}, \frac{1}{3})$ **ii** The integral does not exist.
l **i** $[\frac{1}{2}, \infty)$ **ii** The integral does exist.
m **i** R\{2} **ii** The integral does not exist.
n **i** R **ii** The integral does exist.

2 **a** $\frac{1}{6}\log_e\frac{5}{2}$ **b** $-\dfrac{\pi}{6}$ **d** Approx. 1.893

e $2\log_e 4$ **f** $\log_e\frac{4}{3}$ **g** $2\log_e 6$

j $\dfrac{\pi}{36}$ **l** $\frac{9\sqrt{3}}{5}$ **n** $6 + \frac{1}{2}e^6 - \frac{1}{2}e^{-6}$

3 **a** C **b** A

4 **a** E **b** D **c** B

5 **a** $\displaystyle\int_2^{10}\frac{u^3}{3}\,du$ **b** $\displaystyle\int_{-2}^{-1}2u^{-2}\,du$

c $\displaystyle\int_1^2\frac{u^{\frac{1}{2}}}{2}\,du$ **d** $\displaystyle\int_0^8\frac{u^{\frac{1}{2}}}{2}\,du$

e $\displaystyle\int_0^1\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$ **f** $\displaystyle\int_1^4\left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$

g $\displaystyle\int_0^{\log_e 3}u\,du$ **h** $\displaystyle\int_{\frac{1}{2}}^0 -e^u\,du$

i $\displaystyle\int_1^0(u^{11} - u^{10})\,du$ **j** $\displaystyle\int_0^1 u^{\frac{1}{2}}\,du$

k $\displaystyle\int_0^1 u^3\,du$ **l** $\displaystyle\int_0^{\frac{\pi^2}{4}}\frac{\sin u}{2}\,du$

m $\displaystyle\int_1^0(1-u^2)\,du$ **n** $\displaystyle\int_2^{e+1}u^{-\frac{1}{2}}\,du$

6 **a** 832 **b** 1 **c** $\dfrac{2\sqrt{2}-1}{3}$

d $\dfrac{16\sqrt{2}}{3}$ **e** $1\frac{1}{15}$ **f** $5\frac{1}{15}$

g $\dfrac{(\log_e 3)^2}{2}$ **h** $e^{\frac{1}{2}} - 1$ **i** $\frac{1}{132}$

j $\frac{2}{3}$ **k** $\frac{1}{4}$ **l** Approx. 0.89

m $-\frac{2}{3}$ **n** $2\sqrt{e+1} - 2\sqrt{2}$ or approx. 1.028

7 **a** $2 - 2e^4$ **b** 18 **c** $\frac{4}{3}$

d $\dfrac{8\sqrt{2}-10}{3}$ **e** $\frac{31}{160}$ **f** $\log_e\frac{27}{5}$

g $\dfrac{\pi}{3}$ **h** 1.249 (or $\mathrm{Tan}^{-1}3$ or $-\mathrm{Tan}^{-1}(-3)$)

i $-\dfrac{\pi}{2}$ **j** $\frac{4}{15}$ **k** $-\log_e\dfrac{\sqrt{2}}{2}$

l $\log_e 6 \approx 1.79$ **m** $4 - \pi$

n $\frac{4}{3}$ **o** $e^2 - 1$ **p** $\dfrac{\pi}{12} + \sqrt{3} - 1$

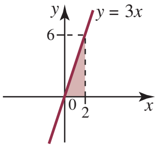
q $23\frac{11}{15}$ **r** $8 + 3\log_e\frac{12}{5}$ (or 10.626)

8 $\dfrac{\pi}{4}$ **9** $\dfrac{2\pi}{3} + \dfrac{\sqrt{3}}{2}$ **10** $\dfrac{\pi}{8} + \dfrac{1}{4}$

11 $a = 1$ **12** $a = -1$ **13** $a = 2$ **14** $a = \sqrt{2}$

Exercise 4H — Volumes of solids of revolution

1 **a**

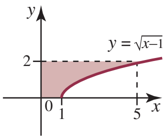


b 24π cubic units

c Answers will vary.

2 **a** $256\frac{\pi}{3}$ cubic units **b** Answers will vary.

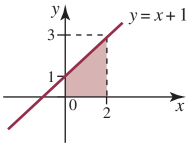
3 **a**



b $\frac{206\pi}{15}$ cubic units

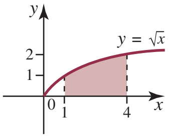
4 **a** $\frac{16\pi}{15}$ cubic units **b** $\frac{\pi}{2}$ cubic units

5 **a** **i**



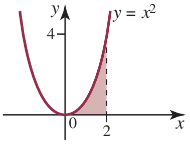
ii $\frac{26\pi}{3}$ cubic units

b **i**



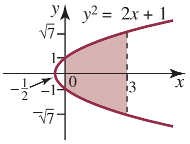
ii $\frac{15\pi}{2}$ cubic units

c **i**



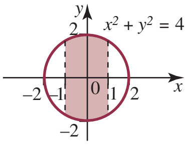
ii $\frac{32\pi}{5}$ cubic units

d **i**



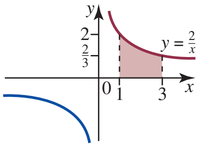
ii 12π cubic units

e **i**



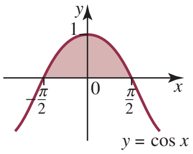
ii $\frac{22\pi}{3}$ cubic units

f **i**



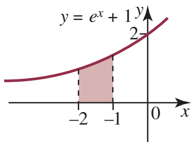
ii $\frac{8\pi}{3}$ cubic units

g **i**



ii $\frac{\pi^2}{2}$ cubic units

h **i**



ii Approx. 4.787 cubic units

6 **a** $\frac{28\pi}{3}$ cubic units **b** $\frac{124\pi}{5}$ cubic units

c ≈11.745 cubic units

d $\frac{(224\sqrt{7}+4)\pi}{15}$ or 124.96 cubic units

e $\frac{(32-12\sqrt{3})\pi}{3}$ or 11.744 cubic units

f 8π cubic units

7 **a** B **b** D **c** A

8 **a** $\frac{2\pi}{15}$ cubic units **b** $\frac{\pi}{6}$ cubic units

9 π cubic units

10 $2\pi(\log_e 2 - 1)^2 \approx 0.592$ cubic units

11 $\left(\frac{\pi^2}{2}-\pi\right)$ or approx. 1.793 cubic units

12 $\frac{227\pi}{30}$ cubic units

13 **a** $\left(1, \frac{1}{\sqrt{3}}\right)$ **b** $\frac{\pi}{4}\log_e 3$ cubic units

c $(4-2\sqrt{3})\pi$ cubic units

14 **a** 24π cubic units **b** 16π cubic units

15 **a** $\frac{48\pi}{5}$ cubic units **b** 4.8π cubic units

16 **a** $\frac{2\pi}{35}$ cubic units **b** $\frac{\pi}{10}$ cubic units

Exercise 4I — Approximation using Simpson's rule

1 **a** 0.828 **b** 0.828
2 **a** 0.810 **b** 0.810
3 **a** 1.105 **b** 1.107
4 **a** 1.107 **b** Very close
5 **a** **i** 14.137 **ii** 13.734
b About 3% error
6 6300 m^3
7 793.3 m^2
8 11.2 cm^2
9 **a** 20.53 cm^2
b Exact answer = 21.1 cm^2
10 2980 m^2

Chapter review

1 **a** $16 + 32x$ **b** 16.96 **c** 0.13%

2 $7\frac{1}{14}$

3 $4\frac{1}{16}$

4 B

5 $\dfrac{1}{2\cos^2 x}$

6 A

7 **a** $e^{\sin x} + c$ **b** $\frac{1}{3}(\log_e x)^3 + c$

8 $\dfrac{(x+2)^{11}(11x-2)}{132} + c$

9 $-\frac{2}{15}(2-x)^{\frac{3}{2}}(3x+4) + c$

10 A

11 $\frac{2}{3}(x-2)\sqrt{x+1} + c$

12 E **13** E **14** C **15** D

16 **a** $\dfrac{x}{2} + \frac{1}{8}\sin 4x + c$ **b** $\dfrac{x - \sin x}{8} + c$

17 $f(x) = -\frac{2}{3}\cos^3 x + \frac{1}{3}$

18 $f(x) = \text{Sin}^{-1} x + x\sqrt{1-x^2} - 3$

19 A

20 $2\log_e(x+1) - \dfrac{1}{x+1} + c$

21 $x + 4\log_e(x-8) + \log_e(x+1)$

22 $e^x(x^2 - 2x + 2) + c$

23 $\frac{1}{5}e^{2x}(2\sin x - \cos x) + c$

24 E

25 **a** Unable to be calculated
b $\sin(\pi^2)$

26 **a** $\dfrac{\pi}{8}$ **b** $-\frac{2}{3}$

27 D **28** B

29 $(1 - e^{-2})\dfrac{\pi}{2}$ cubic units

30 63π cm^3

31 1.59

32 **a** **i** $9\pi \approx 28.27$ **ii** 27.99
b Approximately 1% error when using Simpson's rule

Modelling and problem solving

1 $1200I\,\Delta I$

2 $\dfrac{625\pi}{3}$ cm^3 **3** $\dfrac{140\sqrt{35}}{3}\pi$ cm^3

4 130π cm^3

5 **a** $x^2 + (y-10)^2 = 16$ **b** $320\pi^2$ cm^3

6 **a** 7.21 cm **b** $\pi(e^2 - 1)$ or 20.07 cm^3
c 318.27 cm^3

7 **a** 3.75 m **b** 5.96 m^2 **c** 13.39 m^3 **d** 119.2 m^3

CHAPTER 5 Differential equations

Exercise 5A — Differential equations: related rates

1 **a** $\dfrac{dA}{dL}$ **b** $\dfrac{dV}{dr}$ **c** $\dfrac{dP}{dh}$ **d** $\dfrac{da}{dt}$

2 **a** $4\pi r^2$ **b** $0.9\sqrt{A}$ **c** $0.04te^{0.01t^2}$
d $-\dfrac{2}{5x^2}$ **e** $4\pi r + 20\pi$

3 **a** $6r$ **b** $4.5L^2$ **c** 3.2π **d** $12 + 8r$
e $\sqrt{h}$ **f** 0.2 **g** $\dfrac{2}{\pi r}$ **h** $\dfrac{1}{16L}$

4 **a** 24 **b** 14.4 **c** $\frac{1}{8}e^3$ **d** $-\dfrac{8}{3}$

5 **a** B **b** E

6 C **7** 100π m^2/h **8** 80π cm^2/min

9 0.1 cm/s **10** 0.9 cm/s **11** 7.5 beats/min/s

12 **a** 2.4π cm^2/min **b** 27π cm^3/min

13 Approx. −0.000 44°C/s **14** −0.125 amp/s

15 Approx. 0.0025 cm/s **16** $\frac{2}{15}$ m/s

17 **a** $D = \sqrt{9 + x^2}$ **b** −16 km/h

Exercise 5B — Differential equations of the form $\dfrac{dy}{dx} = f(x)$

1 **a** $y = x^2 + x + c$ **b** $y = \frac{1}{4}x^4 - 2x^2 + c$
c $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + c$ **d** $y = \frac{2}{3}e^{3x} + c$
e $y = \frac{1}{3}x^3 + 3\log_e x + c$ **f** $y = \frac{1}{3}(x^2 - 2)^{\frac{3}{2}} + c$
g $y = 2\tan 2x + c$ **h** $f(x) = -\dfrac{1}{2x^2} + \dfrac{2x^{\frac{3}{2}}}{3} + c$
i $f(x) = x^2 - 7x + 20\log_e(x+3) + c$
j $y = \text{Cos}^{-1}\dfrac{x}{3} + c$ **k** $y = \frac{1}{2}\text{Tan}^{-1}\dfrac{x}{2} + c$
l $y = -\frac{1}{4}\cos 4x - 2\sin x + c$ **m** $h = -\frac{1}{5}(5 - 2t)^5 + c$
n $x = \frac{1}{4}\sin^4\sigma + c$

2 D **3** C **4** A

5 **a** $y = x^3 + x + 3$ **b** $y = -\frac{1}{2}e^{-2x} + \frac{1}{2}$
c $y = 1 + 2\sin 3x$ **d** $y = \frac{1}{3}(2x - 1)^{\frac{3}{2}} - 5$
e $h = \text{Sin}^{-1}\dfrac{t}{4} + 3$
f $h = \log_e\dfrac{(t+3)^2}{t+1}, t > -1$
g $P = \sqrt{n^2 - 7} + 2$ **h** $L = 2\sigma - \sin 2\sigma - \pi$
i $M = \sigma + \tan\sigma - 3$
j $u = x - 1 + \dfrac{\pi}{2} - 2\,\text{Tan}^{-1}\dfrac{x}{2}$
k $V = 2e^{t^2+2} + e^2$ **l** $y = 2\log_e(x^2 + 1) + \text{Tan}^{-1}x + 7$
m $L = 2(1 + \sin\sigma)^{\frac{3}{2}} - 2$ **n** $y = -\log_e(2 - e^x) + 3$

6 $y = 2x^2 + 1$ **7** $y = 3\,\text{Cos}^{-1}x - \pi$ or $\dfrac{\pi}{2} - 3\,\text{Sin}^{-1}x$

8 $h = 7.5t + 6 - 3\sin 0.5t$

9 **a** $V = 10\,000(e^{0.1t} - 1)$ **b** 6487.2 kL
c 2321.1 kL/h

10 **a** $V = (t^2 + 4)^{\frac{3}{2}} - 8$ **b** 3.46 s

Exercise 5C — Differential equations of the form $\frac{dy}{dx} = g(y)$

1 **a** $y = \pm\sqrt{x-c}$

b $y = e^{\frac{1}{3}(x-c)}$ or $Ae^{\frac{x}{3}}$ where $A = e^{-\frac{c}{3}}$

c $y = e^{x-c} - 3$ or $Ae^{x} - 3$ where $A = e^{-c}$

d $y = \frac{1}{3}e^{3x-c} + \frac{5}{3}$ or $Ae^{3x} + \frac{5}{3}$ where $A = \frac{1}{3}\ e^{-c}$

e $y = \pm\sqrt{x+k} - \frac{7}{2}$ **f** $y = \frac{7}{4} \pm \sqrt{k-x}$

g $y = 40 - e^{\frac{1}{5}(c-x)}$ **h** $y = -\frac{2}{x-c}$

i $y = \text{Tan}^{-1}(x-c)$ **j** $y = \text{Cos}^{-1}(e^{c-x})$

2 B

3 **a** D **b** C

4 **a** **i** $y = -3 \pm 2x^{\frac{1}{4}}$ **ii** $(0, \infty)$

b **i** $y = \frac{1}{2}\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)$ **ii** $R\backslash\left(-\frac{5\pi}{4}, \frac{3\pi}{4}\right)$

c **i** $y = 3 + \sqrt{2x}$ **ii** $(0, \infty)$

d **i** $y = \sqrt[3]{3x+15} - 1$ **ii** R

e **i** $y = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ **ii** $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

f **i** $h = \frac{3e^{6t}}{e^{6t}+2}$ **ii** R

g **i** $V = \frac{1}{2}\log_e(2x-1)$ **ii** $(\frac{1}{2}, \infty)$

h **i** $y = \log_e(2e^{x} - 1)$ **ii** $(\log_e\frac{1}{2}, \infty)$

i **i** $y = \frac{2}{2-e^{x}}$ **ii** $R\backslash\{\log_e 2\}$

j **i** $u = \frac{6e^{4x}-2}{3e^{4x}+1}$ **ii** R

5 $y = \frac{3}{13-3x} - 1$ or $y = \frac{3x-10}{13-3x}$

6 **a** $P = 10e^{0.1t}$ **b** Approx. 46 weeks

7 **a** $V = 10\sqrt{3t+1}$ **b** $V = 50\ \text{cm}^3$

8 46.44 years

Exercise 5D — Setting up and solving differential equations

1 **a** $\frac{dy}{dx} = x^2$ **b** $\frac{dy}{dx} = ky^{\frac{1}{3}}$ **c** $\frac{dP}{dt} = kP$

d $\frac{dv}{dt} = \frac{k}{x}$ **e** $\frac{dh}{dt} = kh$ **f** $\frac{dv}{dt} = k\sqrt{t^3}$

2 65 726 **3** 0.693 years **4** 3 hours 19 minutes

5 **a** $\frac{dp}{dt} = 0.02P + 50$ **b** 1949

6 17.39 weeks **7** 46°C

8 **a** 14 minutes 22 seconds **b** 86.3°C

9 **a** 4 minutes 22 seconds **b** −2°C

10 5 minutes 21 seconds

Investigation — Double decay

Spreadsheet indicates 198.4 g of B after 100 s.
Differential equation shows 200.7 g of B.

History of mathematics

1 Calculus

2 Very few women received more than a basic education.

3 The University of Bologna

4 The witch of Agnesi

Chapter review

1 E **2** B **3** A

4 $5.4\ \text{cm}^3/\text{s}$ **5** A **6** E **7** B

8 $y = 2x^2 - 1$ **9** E **10** A

11 B **12** D

13 **a** $y = \text{Sin}^{-1}(e^{x-2})$ **b** $(-\infty, 2]$

14 C **15** E **16** A **17** 32.3°C

18 71 days

Modelling and problem solving

1 **a** $V = \frac{12h - 3h^2}{2}$ **b** **i** −10 mm/min **ii** −7 mm/min

2 **a** $N = 121\ T^{0.4}$ **b** 357

c 23.1°C **d** 40.2°C

3 **a** 10 minutes 8 seconds **b** $x = 10\sqrt{t+1} - 10$

c 9 minutes 20 seconds **d** Yes, by 48 seconds

4 **a** $D = 50\ e^{-0.1733t}$ **b** 17.68 mL

c 9 hours 17 minutes **d** 10 hours 20 minutes

5 4.41 days

CHAPTER 6 Structures and patterns

Exercise 6A — Sequences and series

1 **a** **i** $2 + 3(n-1)$ **ii** 149

b **i** $3 + 2(n-1)$ **ii** 101

c **i** $2 - 6(n-1)$ **ii** −292

d **i** $16 - 3(n-1)$ **ii** −131

e **i** $1 + 0.75(n-1)$ **ii** 37.75

f **i** $8 - 1.8(n-1)$ **ii** −80.2

g **i** $1000 - 70(n-1)$ **ii** −2430

h **i** $220 + 20(n-1)$ **ii** 1200

i **i** $100 - d(n-1)$ **ii** $100 - 49d$

j **i** $N + 2.5(n-1)$ **ii** $N + 122.5$

2 **a** 610 **b** 440 **c** −1100 **d** −250

e 162.5 **f** −182 **g** 6700 **h** 8200

i $2000 - 190d$ **j** $20N + 475$

3 **a** **i** $3 \times 2^{n-1}$ **ii** 3×2^{19}

b **i** $2 \times 3^{n-1}$ **ii** 2×3^{19}

c **i** $2 \times (-2)^{n-1}$ **ii** -2^{20}

d **i** $-4 \times 2^{1-n}$ **ii** -2^{-17}

e **i** $3 \times 2^{1-n}$ **ii** 3×2^{-19}

f **i** $256 \times (-0.5)^{n-1}$ **ii** $(-2)^{-11}$

g **i** $100 \times 1.1^{n-1}$ **ii** 100×1.1^{19}

h **i** $240 \times (-0.75)^{n-1}$ **ii** $240 \times (-0.75)^{19}$

i **i** $N \times 0.75^{n-1}$ **ii** $N \times 0.75^{19}$

j **i** $100 \times \left(-\frac{1}{d}\right)^{n-1}$ **ii** $100 \times \left(-\frac{1}{d}\right)^{19}$

4 **a** $3(2^{n} - 1)$ **b** $3^{n} - 1$

c $\frac{-2[(-2)^{n} - 1]}{3}$ **d** $8(2^{-n} - 1)$

e $-6(2^{-n} - 1)$ **f** $\dfrac{-512[(-2)^{-n} - 1]}{3}$

g $1000(1.1^n - 1)$ **h** $\dfrac{960[1 - (-0.75)^n]}{7}$

i $4N[1 - (\frac{3}{4})^n]$ **j** $\dfrac{100d}{d+1}[1 - (-d)^{-n}]$

5 **a** **i** 2^{2-n} **ii** 2^{-18} **iii** $4(1 - 2^{-20})$
b **i** $3 - n$ **ii** -17 **iii** -150
c **i** 4 **ii** 4 **iii** 80
d **i** $8 - 3n$ **ii** -52 **iii** -470
e **i** 2^{n-3} **ii** 2^{17} **iii** $0.25(2^{20} - 1)$
f **i** $17.1(\frac{1}{3})^{n-1}$ **ii** $17.1(\frac{1}{3})^{19}$
iii 25.65
g **i** $21.9 + 32.2n$ **ii** 665.9
iii 7200
h **i** $n + 1.5 + 0.5 \times (-1)^{n+1}$ **ii** 21
iii 240
i **i** ${}^{T}n = 3 \times 2^{\frac{n}{2} - 1}$, n is even
ii 3×2^9 **iii** $6(2^{10} - 1)$
j **i** $6a - 2an$ **ii** $-34a$ **iii** $-300a$

6 **a** \$63 000 **b** \$682 500

7 **a** \$48 487 **b** \$560 660

8 **a** 2, 4, 8, 16, 32 **b** 0, –2, –4, –6, –8
c 1, –2, 4, –8, 16 **d** 2, –1, 5, –7, 17
e 2, 7, 17, 37, 77

9 **a** $T_{n+1} = T_n + 2,\ T_1 = 3$
b $T_{n+1} = \frac{5}{4}T_n,\ T_1 = 4$
c $T_{n+1} = T_n - 3,\ T_1 = 8$
d $T_{n+1} = 1.5T_n,\ T_1 = 5$
e $T_{n+1} = -0.5T_n,\ T_1 = 4$

10 **a** $2 + 4 + 6 + 8 + 10$
b $6 + 11 + 18 + 27$
c $2 + 1 + \frac{2}{3} + \frac{1}{2}$
d $4 + 9 + 16 + 25 + 36$
e $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6}$

11 **a** 299 **b** 15 050

12 **a** 31.73 **b** 329

13 **a** 511 334 **b** 1.5×10^6

14 \$738 591.82

Exercise 6B — Mathematical induction

1 **a** $1 = \frac{1}{2}(1 + 1)$
b $\sum_{1}^{5} r = 1 + 2 + 3 + 4 + 5 = 15 = \frac{5}{2}(5 + 1)$
c $\sum_{1}^{k} r = \dfrac{k}{2}(k + 1)$
d $\sum_{1}^{k+1} r = \dfrac{k+1}{2}(k + 2)$
e Add $k + 1$ to both sides.

2 **a** $2 = 1(1 + 1)$
b $\sum_{1}^{5} 2r = 2 + 4 + 6 + 8 + 10 = 5(5 + 1)$
c $\sum_{1}^{k} 2r = k(k + 1)$
d $\sum_{1}^{k+1} 2r = (k + 1)(k + 2)$
e Add $2(k + 1)$ to both sides.

3 **a** $2 - 1 = 1^2$
b $\sum_{1}^{5} 2r - 1 = 1 + 3 + 5 + 7 + 9 = 5^2$
c $\sum_{1}^{k} 2r - 1 = k^2$
d $\sum_{1}^{k+1} 2r - 1 = (k + 1)^2$
e Add $2(k + 1) - 1$ to both sides.

4 **a** $2^1 = 2^2 - 2$
b $\sum_{1}^{5} 2^r = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2^{5+1} - 2$
c $\sum_{1}^{k} 2^r = 2^{k+1} - 2$
d $\sum_{1}^{k+1} 2^r = 2^{k+2} - 2$
e Add 2^{k+1} to both sides.

5 **a** $1^2 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6}$
b $\sum_{1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{1}{3}5^3 + \frac{1}{2}5^2 + \frac{1}{6}5$
c $\sum_{1}^{k} r^2 = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k$
d $\sum_{1}^{k+1} r^2 = \frac{1}{3}(k + 1)^3 + \frac{1}{2}(k + 1)^2 + \frac{1}{6}(k + 1)$
e Add $(k + 1)^2$ to both sides.

6 **a** $1^3 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4}$
b $\sum_{1}^{5} r^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \frac{1}{4}5^4 + \frac{1}{2}5^3 + \frac{1}{4}5^2$
c $\sum_{1}^{k} r^3 = \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2$

d $\sum_{1}^{k+1} r^3 = \frac{1}{4}(k+1)^4 + \frac{1}{2}(k+1)^3 + \frac{1}{4}(k+1)^2$

e Add $(k+1)^3$ to both sides.

7 **a** $a = \frac{1}{2}(2a+0)$

b $a + (a+d) + (a+2d) + (a+3d) + (a+4d) = \frac{5(2a+4d)}{2}$

c $a + (a+d) + (a+2d) + \ldots (a + [k-1]d) = \frac{k(2a+[k-1]d)}{2}$

d $a + (a+d) + (a+2d) + \ldots (a+kd) = \frac{k+1}{2}(2a+kd)$

e Add $a + kd$ to both sides.

8 **a** $a = \frac{a(r^1-1)}{(r-1)}$

b $a + ar + ar^2 + ar^3 + ar^4 = \frac{a(r^5-1)}{(r-1)}$

c $a + ar + ar^2 + \ldots ar^{k-1} = \frac{a(r^k-1)}{(r-1)}$

d $a + ar + ar^2 + \ldots ar^k = \frac{a(r^{k+1}-1)}{(r-1)}$

e Add ar^k to both sides.

9 **a** $3 = \frac{3(1+1)}{2}$

b $3 + 6 + 9 + 12 + 15 = 3 \times \frac{5(1+5)}{2}$

c $3 + 6 + 9 + \ldots 3k = \frac{3k(1+k)}{2}$

d $3 + 6 + 9 + \ldots 3(k+1) = \frac{3(k+1)(k+2)}{2}$

e Add $3(k+1)$ to both sides.

10 **a** $2 = 2^2 - 2$

b $2 + 4 + 8 + 16 + 32 = 2^6 - 2$

c $2 + 4 + 8 + \ldots 2^k = 2^{k+1} - 2$

d $2 + 4 + 8 + \ldots 2^{k+1} = 2^{k+2} - 2$

e Add 2^{k+1} to both sides.

11 **a** $-2 = \frac{2}{3}[(-2)^1 - 1]$

b $-2 + 4 - 8 + 16 - 32 = \frac{2}{3}[(-2)^5 - 1]$

c $-2 + 4 - 8 + \ldots (-2)^k = \frac{2}{3}[(-2)^k - 1]$

d $-2 + 4 - 8 + \ldots (-2)^{k+1} = \frac{2}{3}[(-2)^{k+1} - 1]$

e Add $(-2)^{k+1}$ to both sides.

12 **a** $\frac{1}{1\times 2} = 1 - \frac{1}{2}$

b $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \frac{1}{5\times 6} = 1 - \frac{1}{5+1}$

c $\sum_{1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$

d $\sum_{1}^{k+1} \frac{1}{r(r+1)} = 1 - \frac{1}{k+2}$

e Add $\frac{1}{(k+1)(k+2)}$ to both sides.

13 **a** $1 = \frac{1}{3} \times 1 \times 1 \times 3$

b $1^2 + 3^2 + 5^2 + 7^2 + 9^2 = \frac{1}{3} \times 5 \times (10-1)(10+1)$

c $1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

d $1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 = \frac{1}{3}(k+1)(2k+1)(2k+3)$

e Add $(2k+1)^2$ to both sides.

14 **a** $1 = \frac{1}{5} + \frac{1}{2} + \frac{1}{3} - \frac{1}{30}$

b $1^4 + 2^4 + 3^4 + 4^4 + 5^4 = \frac{1}{5} \times 5^5 + \frac{1}{2} \times 5^4 + \frac{1}{3} \times 5^3 - \frac{1}{30} \times 5$

c $\sum_{1}^{k} r^4 = \frac{1}{5}k^5 + \frac{1}{2}k^4 + \frac{1}{3}k^3 - \frac{1}{30}k$

d $\sum_{1}^{k+1} r^4 = \frac{1}{5}(k+1)^5 + \frac{1}{2}(k+1)^4 + \frac{1}{3}(k+1)^3 - \frac{1}{30}(k+1)$

e Add $(k+1)^4$ to both sides.

15 **a** $4! > 2$ **b** $8! > 2^5$ **c** $(k+3)! > 2^k$

d $(k+4)! > 2^{k+1}$

e Multiply both sides by $k+4$

16 **a** $3^4 - 1$ is divisible by 80

b $3^{20} - 1$ is divisible by 80

c $3^{4k} - 1$ is divisible by 80

d $3^{4k+4} - 1$ is divisible by 80

e Add 1, multiply by 3^4 and take 1.

17 **a** $5^1 + 2 \times 11^1$ is divisible by 3

b $5^5 + 2 \times 11^5$ is divisible by 3

c $5^k + 2 \times 11^k$ is divisible by 3

d $5^{k+1} + 2 \times 11^{k+1}$ is divisible by 3

e $5^{k+1} + 2 \times 11^{k+1} = 5 \times 5^k + 22 \times 11^k = 5(5^k + 2 \times 11^k) + 12 \times 11^k$

Investigation — Fibonacci sequence

1 **a** F_{19} is not prime. 4181 = 37 × 113

c $F_{12} = 144$

2 5896 Lunx

Exercise 6C — Finite differences

1 **a** **i** Quadratic **ii** $t_n = 1.5n^2$

b **i** Quadratic **ii** $t_n = 3n^2 + 2n$

c **i** Quadratic **ii** $t_n = 2n^2 - 3n$

d i Quadratic **ii** $t_n = 2.5n^2 - 2n$
e i Cubic **ii** $t_n = n^3 - 2n^2 + 2n$
f i Cubic **ii** $t_n = 3n^3 - 4n^2 + 2n$
g i Quadratic **ii** $t_n = 8n^2 - 12n + 20$
h i Not linear, quadratic or cubic
i i Cubic **ii** $t_n = 2.5n^3 - 6n^2 + 4n$
j i Cubic **ii** $t_n = -2.5n^3 + 10n^2 - 4n$
k i Cubic **ii** $t_n = 1.5n^3 + 3n - 3$

2 a $t_n = 2n^2 + 1.5n$
b $t_n = -2n^2 + 8n$
c $t_n = 1.5n^2 + 6n - 5$

3 Check with your teacher.

Investigation — Pascal's triangle

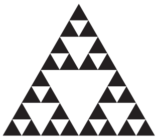
1 a Each row of Pascal's triangle is symmetric about the centre.

b $$\sum_{r=0}^{6} {}^6C_r = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$$

c Sum of row 0 is 2^0
Sum of row 1 is 2^1
Sum of row 2 is 2^2 and so on.

2
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
Triangular numbers

3 There are many examples.
This is one.



Chapter review

1 a $8 + 12n$ **b** 1208 **c** 5820

2 a $16 \times (-\frac{1}{2})^{n-1}$ **b** $16 \times (-\frac{1}{2})^{14}$
c $10\frac{2}{3}$

3 a $6 + 8 + 10 + 12 + 14$ **b** 50

4 10 000

5 29 918

6 a 100, –46, 27, –9.5, 8.75 **b** 131.56

7 a 100, 14, 5.4, 4.54, 4.454 **b** 217.3

10 a $t_n = n^2 - 4n$ **b** $t_n = 0.5n^3 - 4n + 3$

Modelling and problem solving

1 a 36 mg, 92 mg, $20 + 8(n - 1)$ mg
b Day 24
c 39.2 mg, 413.2 mg, $20 \times 1.4^{n-1}$ mg
d Day 8

2 b $2200 + 3.6(n - 1)$ **c** 251
d 665.15 m

3 a i \$5500 **ii** \$8052.55 **iii** \$87 247.01
b i \$5500 **ii** \$79 315.46
c \$904 717

4 a 64 079 **b** 103 678 **c** 1.618

CHAPTER 7 Advanced periodic functions

Exercise 7A — Circular and reciprocal functions

1 a $\cos\theta$ **b** $-\cos\theta$ **c** $\sin\theta$
d $-\cot\theta$ **e** $-\tan\theta$

3 a 0 **b** –1 **c** $\operatorname{cosec}\theta$
d $-2\tan\theta$ **e** 1
f $-\cos\theta(\cos\theta + \sin\theta)$

Exercise 7B — Exact values and the Pythagorean relationships

1 a $\frac{1}{2}$ **b** 2 **c** $\frac{-1}{\sqrt{2}}$
d $-\sqrt{2}$ **e** $\sqrt{3}$ **f** $\frac{1}{\sqrt{3}}$
g $\frac{-2}{\sqrt{3}}$ **h** $\frac{-2}{\sqrt{3}}$ **i** $\sqrt{2} + \frac{2}{\sqrt{3}}$
j 8

2 a $\frac{-4}{3}$ **b** $\frac{-5}{4}$ **c** $8\frac{73}{144}$ **d** $3\frac{11}{48}$

3 a $-1\frac{59}{60}$ **b** $3\frac{41}{60}$ **c** $\frac{7}{17}$

4 $-(\sqrt{2} + \sqrt{3})$

5 a $\frac{-5}{4}$ **b** $\frac{5}{3}$ **c** $\frac{-3}{5}$
d $\frac{4}{3}$ **e** $\frac{5}{3}$ **f** $\frac{-5}{3}$

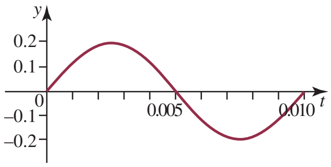
6 a –1 **b** 1
c $2\cot\theta\operatorname{cosec}\theta$ **d** $2\sec\theta$
e $|\tan x|$ **f** 1
g $-\tan\theta$ **h** $-\cot\theta$

8 $\pm\sqrt{2}$

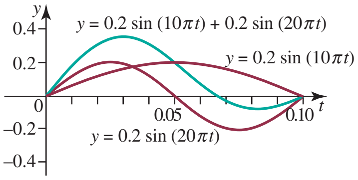
9 a i $\overline{DA} = 3$ **ii** $\overline{DC} = 3(2 - \sqrt{3})$
c $-2 + \sqrt{3}$

Exercise 7C — Modelling soundwaves

1 a 200π radians/s **b** $y = 0.2\sin(200\pi t)$
c

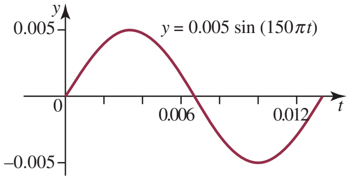


2 a 20π radians/s, 10π radians/s
b $y_1 = 0.2\sin(20\pi t)$
$y_2 = 0.2\sin(10\pi t)$
$y = 0.2\sin(20\pi t) + 0.2\sin(10\pi t)$
c



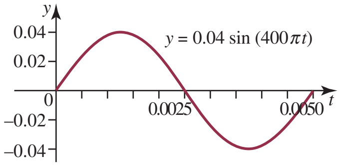
d Loudness is almost doubled.

3 a Amplitude = 0.005, period = $\frac{1}{75}$ seconds, frequency = 75 Hz



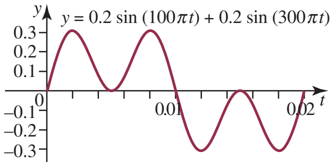
b Amplitude = 0.04, period = $\frac{1}{200}$ seconds,

frequency = 200 Hz



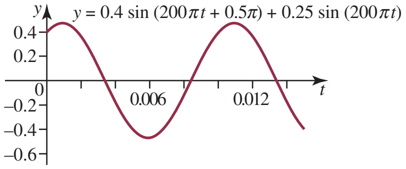
c Amplitude = 0.31, period = $\frac{1}{50}$ seconds,

frequency = 50 Hz



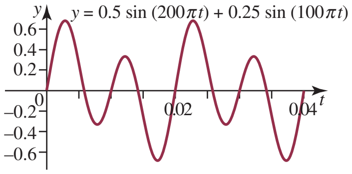
d Amplitude = 0.47, period = $\frac{1}{100}$ seconds,

frequency = 100 Hz

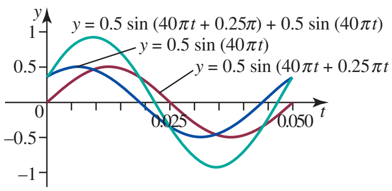


e Amplitude = 0.68, period = $\frac{1}{50}$ seconds,

frequency = 50 Hz



4 a

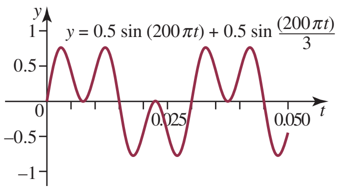


b Same frequency, 20 Hz

c Composite amplitude 0.92; that is, nearly doubled

5 a $y = 0.5 \sin(200\pi t) + 0.5 \sin\left(\frac{200\pi}{3}t\right)$

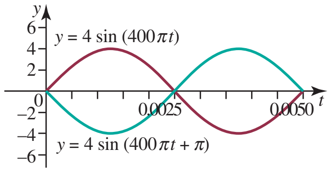
b



c Amplitude of composite is 0.77; that is, about $1\frac{1}{2}$ times the partials. The frequency of the composite is the same as the overtone; that is, $\frac{100}{3}$ Hz.

6 a Same frequency, 200 Hz

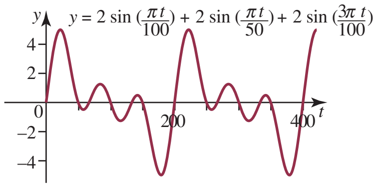
b



c Sound is inaudible as the waveforms cancel each other out.

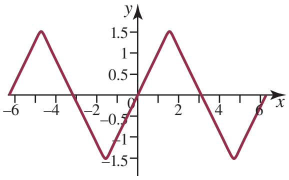
7 a $y = 2 \sin\left(\frac{\pi}{100}t\right) + 2 \sin\left(\frac{\pi}{50}t\right) + 2 \sin\left(\frac{3\pi}{100}t\right)$

b



Frequency of the composite is the same as the fundamental; that is, $\frac{1}{200}$ Hz. The amplitude of the composite is $2\frac{1}{2}$ times the fundamental.

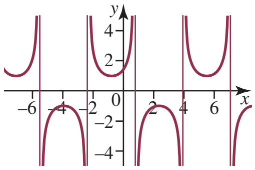
8



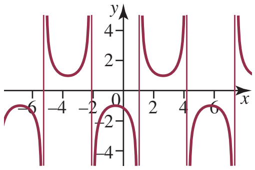
'Sawtooth' waveform which has an amplitude of $1\frac{1}{2}$

Exercise 7D — Graphing the reciprocal trigonometric functions

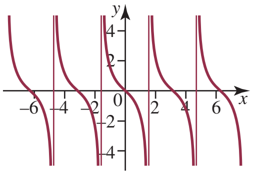
1 a



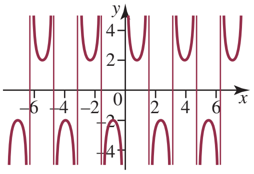
b



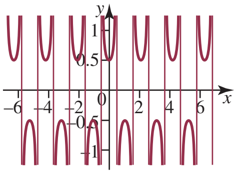
c



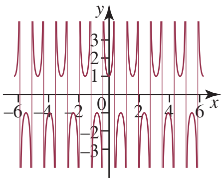
d



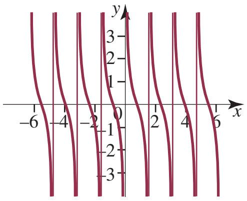
e



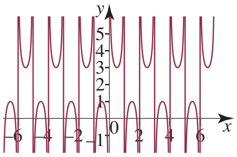
f



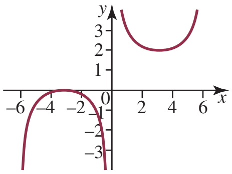
g



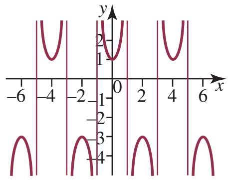
h



i



j



2 **a** Three solutions. The least is 0.349.
b Four solutions. The least is −1.386.
c −0.524 and 0.524
d Two solutions. Greatest is 45°. Solution domain: 22°30′ ≤ θ ≤ 45°

3 **b** $\theta = 2.310$

4 **a** Area $= x^2 \sin\theta(1+\cos\theta)$
b *Hint*: Maximise $\sin\theta(1+\cos\theta)$, $\theta_{\max} = 60°$

Exercise 7E — Addition identities for sin (*x* ± *y*) and cos (*x* ± *y*)

1 **a** $\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$
b $\frac{1}{2}(\sqrt{3}\cos x + \sin x)$
c $\sin(2x) - \sqrt{3}\cos(2x)$
d $-\sin(\frac{1}{2}x)$

2 **a** $\sin(\alpha - 2\beta)$ **b** $\cos(5°)$
c $\cos\left(2\theta + \frac{\pi}{4}\right)$ **d** $\frac{\sqrt{3}\sin\theta + \cos\theta}{\sqrt{3}\cos\theta + \sin\theta}$

4 **a** $\frac{\sqrt{2}+\sqrt{6}}{4}$ **b** $\frac{\sqrt{6}-\sqrt{2}}{4}$

5 **a** $\frac{1}{2}\sqrt{\sqrt{2}+2}$ **b** $\frac{1}{2}\sqrt{\sqrt{2}+2}$

8 **b** **i** $\frac{a}{b} = \frac{n-1}{n+1}$ **ii** $n \to \infty, a \to b$

Exercise 7F — Addition identities for tan (*x* ± *y*)

1 **a** $\frac{\tan x + \tan a}{1 - \tan x \tan a}$ **b** $\frac{1-\tan x}{1+\tan x}$

2 **a** 1 **b** $\tan(4B)$

3 **a** $-(2+\sqrt{3})$ **b** $\frac{-7}{23}$

4 **b** $\sqrt{2} - 1$

Exercise 7G — The Werner or factorisation identities

1 $\sin(4x) + \sin(2x)$ **2** $\sin(5x) - \sin(3x)$
3 $\cos(3x) + \cos x$ **4** $\cos(5x) - \cos x$
5 $2[\sin(7x) - \sin x]$
6 $\frac{1}{2}[\cos(x+60°) + \cos x]$ **7** $\frac{1}{2}[\cos(8x) - \cos(2x)]$
8 $\sin(2a) + \sin(2b)$ **9** $\frac{a}{2}[\cos(6b) + 1]$
10 $\frac{5}{2}[\sin(3x+y) - \sin(x-3y)]$
11 $\frac{1}{2}(\sin x + \sin y)$
12 $-\frac{1}{2}[\cos(2x) - \cos(2y)]$

Exercise 7H — The Simpson or half-sum/half-difference identities

1 **a** $2\sin(3x)\cos x$ **b** $2\cos(3x)\sin x$
c $2\cos(2y)\cos y$ **d** $-2\sin(2\theta)\sin\theta$
e $2\sin(4\alpha)\cos\alpha$ **f** $2\cos\left(\frac{5\beta}{2}\right)\sin\left(\frac{3\beta}{2}\right)$
g $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{x}{2}\right)$ **h** $2\sin\left(\frac{7x}{2}\right)\sin\left(\frac{x}{2}\right)$
i $\sin\left(\frac{7x}{2}\right)\cos\left(\frac{x}{2}\right)$ **j** $-\cos(4x)\sin(2x)$
k $\sin(\theta+\alpha)\sin(\theta-\alpha)$ **l** $8\cos x\cos y$

m $\sin(2\alpha)(2\cos\alpha + 1)$
n $\cos(3y)[2\cos(2y) + 1]$

2 a $-\cot\dfrac{\theta+\alpha}{2}$ **b** $\cot\dfrac{x+\alpha}{2}$
c $\tan(\theta - \beta)$ **d** $\frac{1}{2}$ cosec $(3x)$
e cosec x **f** $\tan(2y)$

4 a $\left\{0, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, 2\pi\right\}$ **b** $\left\{\dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}\right\}$

5 a $\frac{1}{2}n\pi, 2n\pi \pm \dfrac{2\pi}{3}$ for integral n.
b $\frac{2}{3}n\pi \pm \dfrac{\pi}{6}, n\pi \pm \dfrac{\pi}{3}$ for integral n.

Exercise 7I — Modelling and problem solving

1 a $\sqrt{2}\cos(x - 315°)$ **b** $\sqrt{2}\sin(x + 45°)$
c $5\sin(x - 0.644)$ **d** $\sqrt{5}\cos(x + 1.107)$
e $2\sin(x - 210°)$ **f** $\sqrt{3}\cos(x - 35°16')$

2 a i $\sqrt{7}\sin(x - 3.855)$ **ii** $\sqrt{7}\sin(x + 2.428)$
iii $\sqrt{7}\cos(x - 5.426)$ **iv** $\sqrt{7}\cos(x + 0.857)$
b In each case the maximum is $\sqrt{7}$ and the minimum $-\sqrt{7}$.
c In each case the maximum is at $x = 5.426$ and the minimum is at $x = 2.285$.

3 a $r = 17$ and $\alpha = 61°56'$
b $x = 240°2'$

4 a $-\frac{1}{3}(\cos 3x + 3\cos x) + C$
b $-\frac{1}{4}(\cos 4x - 2\cos 2x) + C$
c $\frac{1}{10}(3\sin 5x + 5\sin 3x) + C$
d $\frac{1}{8}(\sin 4x - 2\sin 2x) + C$
e $-\cos 2x + C$
f $\frac{1}{8}(2x - \sin 2x) + C$
g $\frac{1}{10}(\cos 5x + 5\cos x) + C$
h $\dfrac{1}{4\pi}(2\cos\pi x - \cos 2\pi x) + C$

5 a i $-1\frac{1}{2}$ **ii** $\frac{3}{20}$ **iii** 0 **iv** 0

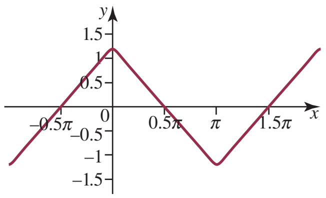
6 b 0.928

7 b $1\frac{1}{3}$

8 a $3\frac{3}{8}$

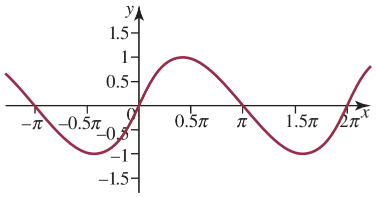
10 a i 0 **ii** 0 **iii** $\dfrac{-2n}{m^2 - n^2}$
b i $\frac{1}{2}\pi$ **ii** 0 **iii** not possible.

11 a



Period is 2π, amplitude is 1.17.

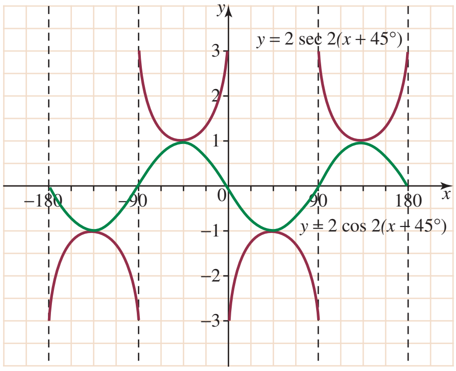
b



Period is 2π, amplitude is 1.

Chapter review

1 B **2** A
3 sec θ
5 D **6** C **7** D
8 a $-2\frac{3}{7}$ **b** $-2\frac{49}{60}$
10 A
11 a Amplitude 1.04, period 0.04, frequency 25
b Frequency and period remain the same; loudness is nearly doubled.
12 B **13** E
14



15 24°23′ only
16 A **17** E **18** B
20 b i $\frac{1}{2}\sqrt{\sqrt{3} + 2}$ **ii** $-\frac{1}{2}\sqrt{\sqrt{3} + 2}$
21 C
22 a $2 - \sqrt{3}$ **b** $-2 + \sqrt{3}$
23 a $\sin(4x + 2y) + \sin(4x - 2y)$
b $\frac{a}{4}(\cos\pi x - \cos 3\pi x)$
24 a i $\tan 3x$ **ii** $-\tan 2x$
b $\frac{1}{2}n\pi$ for integral n.
25 a i $\frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C$ **ii** $\frac{1}{3}$
b $\frac{2}{3}$ units squared
c $\frac{2}{3}$ units squared

Modelling and problem solving

1 **b** $b = \frac{\pi}{2}$; $a = 2\tan\frac{\pi}{4}$

c $f(x) = 2\tan\frac{\pi}{4}\ \cot\frac{\pi}{7}$; 0.22 m

d –4.14; –0.20

2 **a** **i** –28.64 **ii** 4.775

3 **a** 0.429

4 **b** The antinodes occur at $x = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda, \ldots$

5 **a** $d = 0.981\sin(0.521t - 0.202) + 1.580$

b Between 7.15 pm and 12:20 am the next day.

6 $\theta = 0, \pi, 2\pi$ and $\alpha = \frac{2\pi}{3}$.

The engineers are incorrect.

CHAPTER 8 Advanced exponential functions

Exercise 8A — Trigonometric links to complex numbers

These are all proofs.

Exercise 8B — Euler's formula and calculus applications

1 **a** 1 **b** 1 **c** i

d $\frac{3}{2}(1 + i\sqrt{3})$ **e** $-1 + i\sqrt{3}$ **f** $\frac{5}{2}(\sqrt{3} - i)$

g $-0.401 - 0.299i$ **h** $1.261 - 3.796i$

i $1.645 + 2.803i$ **j** $0.768 - 0.716i$

k $0.225 - 0.109i$ **l** $93.7 - 35.1i$

2 **a** $\sqrt{2}e^{\frac{3\pi i}{4}}$ **b** $\sqrt{2}e^{-\frac{3\pi i}{4}}$ **c** $2e^{\frac{\pi i}{6}}$

d $2e^{-\frac{\pi i}{6}}$ **e** $2e^{-\frac{2\pi i}{3}}$ **f** $e^{-\frac{\pi i}{6}}$

g $5e^{-0.644i}$ **h** $5e^{-2.214i}$ **i** $13e^{-1.966i}$

j $13e^{1.966i}$ **k** $25e^{-0.284i}$ **l** $7.550e^{1.094i}$

3 **a** $319 + 9442i$ **b** $-3.0205 - 0.1274i$

c $2.3820 + 4.6180i$

d $-1211.2629 - 684.7205i$

e $-64i$

f $-0.0034 - 0.0004i$

g $-1.1280 + 1.0242i$ **h** $59.7500 + 7140i$

4 **a** $\frac{1}{2}e^{x}(\sin x - \cos x) + c$ **b** $\frac{1}{2}e^{-x}(\sin x - \cos x) + c$

c $\frac{1}{5}e^{2x}(2\sin x - \cos x) + c$

d $\frac{1}{5}e^{-x}(2\sin 2x - \cos 2x) + c$

e $\frac{1}{4}e^{2x}(\sin 2x - \cos 2x) + c$

f $-\frac{1}{2}(e^{\pi} + 1)$ **g** $\frac{\pi}{1 + \pi^2}(e + 1)$

h $\frac{1}{10}(e^{-\frac{\pi}{3}} + 1)$ **i** $\frac{1}{4}(e^{-\pi} - e^{\pi})$

Exercise 8C — Applications of Euler's formula

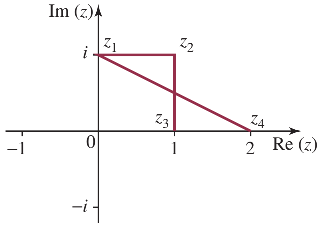
1 **a** **i** $\{z_4\} = \{i, 1 + i, 1, 2\}$

ii $\{z_6\} = \{\frac{1}{2} + i\frac{\sqrt{3}}{2}, i\sqrt{3}, -1 + i\sqrt{3}, -\frac{1}{2}(3 - i\sqrt{3}), -1, 0\}$

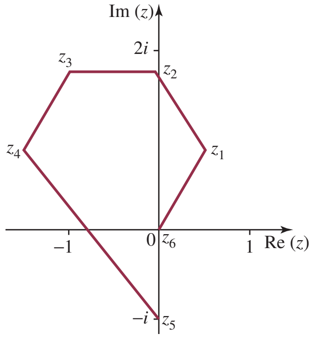
iii $\{z_8\} = \{\frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(1 + \sqrt{2} + i), 1 + i\sqrt{2}, 2 + i\sqrt{2}, (2 - \frac{1}{\sqrt{2}}) + \frac{i}{\sqrt{2}}, (3 - \frac{1}{\sqrt{2}}) + i(\sqrt{2}), 3, 4\}$

iv $\{z_{12}\} = \{0.8660 + 0.5i, 0.3660 - 0.3660i, 0.3660 + 0.6340i, -0.1340 + 1.5i, -1 + 2i, 2i, -0.8660 + 1.5i, -1.366 + 0.6340i, -1.366 - 0.366i, -1.866 + 0.5i, -1, 0\}$

b **i**



ii



$z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$z_2 = i\sqrt{3}$

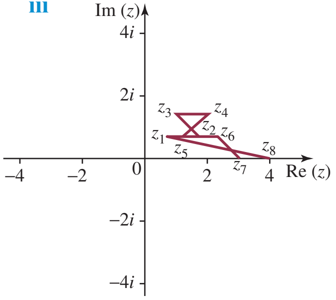
$z_3 = -1 + i\sqrt{3}$

$z_4 = -\frac{3}{2} + \frac{i\sqrt{3}}{2}$

$z_5 = -1$

$z_6 = 0$

iii



$z_1 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$z_2 = \frac{1}{\sqrt{2}}(\sqrt{2} + 1 + i)$

$z_3 = 1 + \sqrt{2}i$

$z_4 = 2 + \sqrt{2}i$

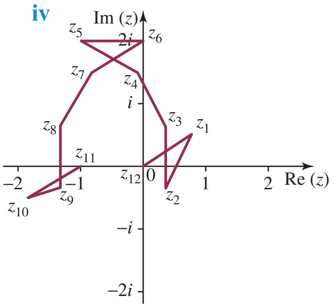
$z_5 = \frac{1}{\sqrt{2}}(2\sqrt{2} - 1 + i)$

$z_6 = \frac{1}{\sqrt{2}}(3\sqrt{2} - 1 + i)$

$z_7 = 3$

$z_8 = 4$

iv



$z_1 = 0.8660 + 0.5i$
$z_2 = 0.3660 - 0.3660i$
$z_3 = 0.3660 + 0.6340i$
$z_4 = -0.1340 + 1.5i$
$z_5 = -1 + 2i$
$z_6 = 2i$
$z_7 = -0.8660 + 1.5i$
$z_8 = -1.3660 + 0.6340i$
$z_9 = -1.3660 - 0.3660i$
$z_{10} = -1.8660 + 0.5i$
$z_{11} = -1$
$z_{12} = 0$

2 **a** $\text{Re}\,(z_n) = \sum_{k=1}^{n} \cos\left(\frac{2\pi k^3}{M}\right)$ and

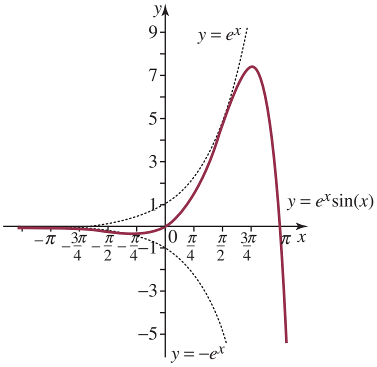
$\text{Im}\,(z_n) = \sum_{k=1}^{n} \sin\left(\frac{2\pi k^3}{M}\right)$

b $z_n = z_{n-1} + \exp\left(\frac{2\pi i n^3}{M}\right)$

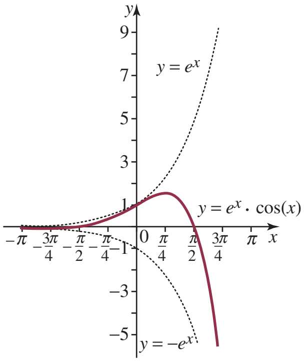
5 2.957×10^{139} years **6** $\frac{3\pi}{4}$

Exercise 8D — Damped functions

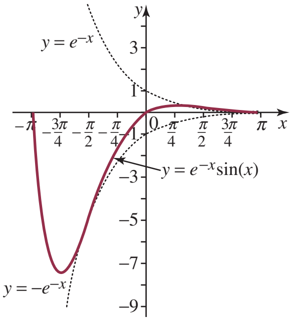
1 **a** **i**



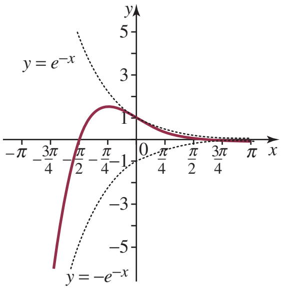
ii



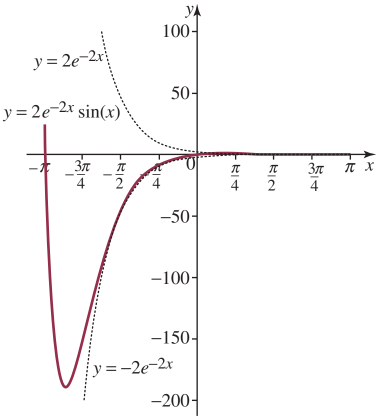
iii



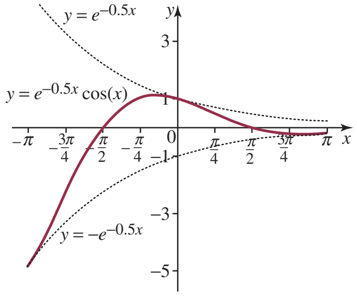
iv



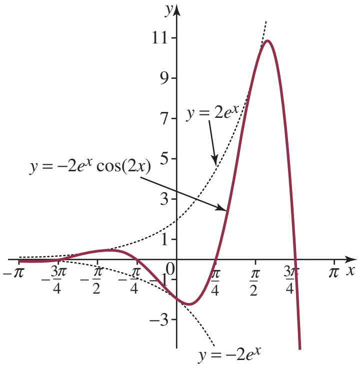
v



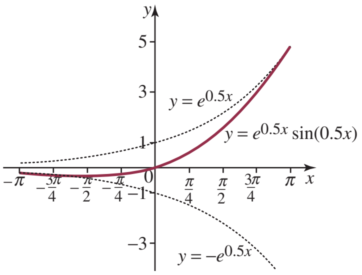
vi



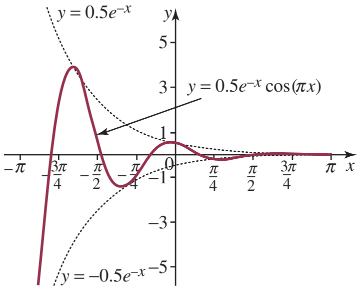
vii



viii



ix



2 **a** **i** $y \to \pm\infty, y \to 0$ **ii** $y \to \pm\infty, y \to 0$
iii $y \to 0, y \to \pm\infty$ **iv** $y \to 0, y \to \pm\infty$
v $y \to 0, y \to \pm\infty$ **vi** $y \to 0, y \to \pm\infty$
vii $y \to \pm\infty, y \to 0$ **viii** $y \to \pm\infty, y \to 0$
ix $y \to 0, y \to \pm\infty$

b For $a < 0$, the period is $\frac{2\pi}{b}$; and the amplitude increases as x decreases and decreases (approaches zero) as x increases.

For $a = 0$, the period is $\frac{2\pi}{b}$ and the amplitude is 1.

For $a > 0$, the period is $\frac{2\pi}{b}$; and the amplitude increases as x increases and decreases (approaches zero) as x decreases.

3 **a** **i** $\frac{1}{2}[1 - e^{-c}(\sin c + \cos c)]$

ii $\frac{1}{2}[1 + e^{-c}(\sin c - \cos c)]$

iii $\frac{1}{5}[2 - e^{-c}(\sin 2c + 2\cos 2c)]$

iv $\frac{1}{4+\pi^2}[2 + e^{-2c}(\pi \sin \pi c - 2\cos \pi c)]$

v $\frac{1}{5}[1 - e^{-2c}(2\sin c + \cos c)]$

vi $\frac{1}{5}[2 + e^{-2c}(\sin c - 2\cos c)]$

vii $\frac{1}{1+\pi^2}[\pi - e^{-c}(\sin \pi c + \pi \cos \pi c)]$

viii $\frac{1}{a^2+4\pi^2}[2\pi - e^{-ac}(a\sin 2\pi c + 2\pi\cos 2\pi c)]$

ix $\frac{2}{1+4a^2}[1 + 2e^{-\frac{1}{2}c}(a\sin ac - \frac{1}{2}\cos ac)]$

b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{2}{5}$

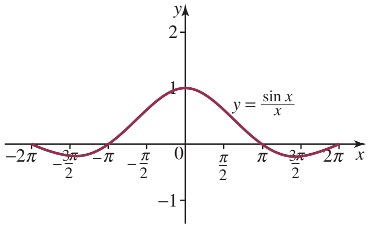
iv $\frac{2}{4+\pi^2}$ **v** $\frac{1}{5}$ **vi** $\frac{2}{5}$

vii $\frac{\pi}{1+\pi^2}$ **viii** $\frac{2\pi}{a^2+4\pi^2}$ **ix** $\frac{2}{1+4a^2}$

5 **b** **i** $\frac{1}{5}e^{-2t}(\sin t - 2\cos t) + c$

ii $-\frac{1}{5}e^{-2t}(\cos t + 2\sin t) + c$

7 **a**



b **i** $y \to 1$ **ii** $y \to 0$

Investigation — Approximating the sine integral

1 **a** 1.6054 **b** 1.8487 **c** 1.6583 **d** 1.4182

Exercise 8E — Modelling real-life damped systems

1 **a** $\frac{\sqrt{2}}{5}e^{-\frac{\pi}{20}}$ **b** $0.36\sqrt{2}e^{-\frac{\pi}{20}}$

c $-0.952\sqrt{2}e^{-\frac{\pi}{20}}$

2 **a** -1.263
b **i** $-4.404, -0.340$ **ii** $11.455, -28.933$
c 0.3829

3 **a** $\frac{5\sqrt{3}}{2}e^{-0.05}$ **b** -4.329 **c** 2.909

4 **a** $10\sqrt{2}\,e^{-0.6}$ **b** 131.72
c 123.91 **d** 23.91%

5 **a** $5\sqrt{2}$ **b** $-5\sqrt{2}e^{-\frac{\pi}{20}}$

c **i** 2.159 **ii** $\frac{\pi}{4}$

d $\frac{24}{5}\sqrt{2}e^{-\frac{3\pi}{20}}$

6 **a** $A = \frac{2}{\sqrt{3}}, k = 0.1994$ **b** -0.9823
c 1.513 **d** 2.027
7 **b** **i** $A = 20, k = -1$ **ii** 0.704 **iii** 0.0234
c **ii** 0.208 **d** 0.1516
8 **a** $A = 4, k = -\frac{1}{4}$ **b** 0.0125
c 3.975 **e** 5.0253

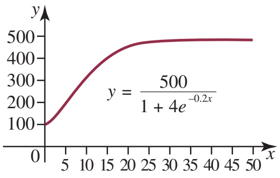
Exercise 8F — Malthusian models and limited growth

1 8.01×10^9
2 **a** 59.8 **b** 28.3 days
3 **b** 338.4
c **i** 53.3 and 135.5 **ii** $53.3 \le t \le 135.5$
iii 100 **iv** $16.8 \le t \le 153.2$
4 After 50.0 years
5 61.7 thousand
6 **a** **i** 32.85 **ii** 49.3 **b** 2 h 9 min
7 $21.0 < t < 66.1$
8 **b** $N(t) = L(1 - e^{-0.051t})$ **c** 27.1
d $N \to L$ as $t \to \infty$
9 **a** 3.45 words/min/h **b** 147%
10 1191

Exercise 8G — Logistic model and population growth

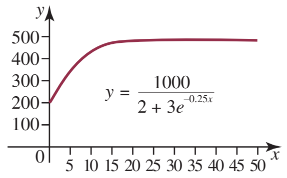
1 **a** $N(t) = \frac{500}{1 + 4e^{-0.2t}}$

d 6.93



2 **a** $P(t) = \frac{1000}{2 + 3e^{-0.25t}}$

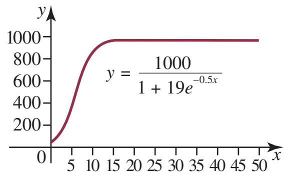
d



Maximal when $t = 1.622$

3 **a** $F(t) = \dfrac{1000}{1 - 19e^{-0.5t}}$

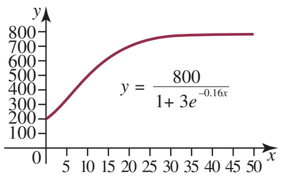
d



Maximal when $t = 5.889$

4 **a** **i** 310 **ii** 436 **iii** 31.5 **iv** 32.0

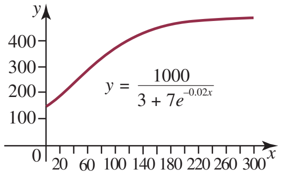
b



c 6.87 years

5 **a** **i** 219 **ii** 269 **iii** 2.49 **iv** 2.50

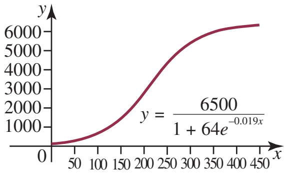
b



c 152.2 hours

6 **a** **i** 1467 **ii** 2832 **iii** 27.32 **iv** 16.66

b



c 237.38 days

7 **a** $k = 0.001$ **b** $x(t) = \dfrac{2000}{(1 + 1999e^{-2t})}$

c $x(5) = 1833.6$ **d** 36 days **e** 2000

8 **b** 213.03 **c** 20.01 days

9 **a** $x(t) = \dfrac{45\,000}{(1 + 224e^{-0.9t})}$ **b** $x(8) = 38\,553$

c Everyone

Chapter review

1 D **4** E **5** B **6** A

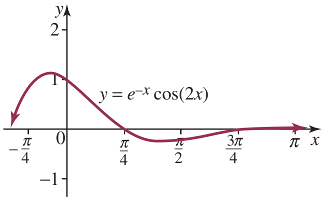
7 **a** $2e^{-\frac{\pi i}{6}}, \sqrt{2}e^{\frac{3\pi i}{4}}$ **b** $4e^{\frac{2\pi i}{3}}$

8 **a** $\frac{1}{2}e^{x}(\cos x + \sin x) + C$ **b** $\frac{1}{4}[2 - (1 + \sqrt{3})e^{-\frac{\pi}{3}}]$

9 $x = 0.2(3 + 2\sqrt{3})$ and $y = 1.3(1 - \sqrt{3})$

10 C **11** E **12** B

13 **a**



b $\frac{1}{5}[1 + e^{-c}\{2 \sin(2c) - \cos(2c)\}]$

c $\frac{1}{5}$

14 A **15** C

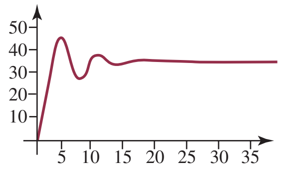
16 **a** 8 **b** $\dfrac{\pi}{12}$ s, $-16e^{-\frac{\pi}{40}}$ cm/s

c $v = 0$ at 0.973 s

17 $x(t) = \dfrac{-e^{-2t}}{25}[4 \sin(t) - 3 \cos(t)]$

18 **a** $\dfrac{25e^{-0.6}}{\sqrt{2}}$

c



d 39.6 **e** 29.9

19 E

20 **b** 519

c **i** 73.5 and 186.9 days
  ii $73.5 \le t \le 186.9$ days
  iii 150 days
  iv 87.9 – 108.3 and 176.6 – 183.4 days.

21 **a** $x(t) = \dfrac{800}{1 + 799e^{-3.2t}}$, 344

b 2 hours and 15 minutes

c 800

22 **a** 12 dogs **b** 3.66 years

c 7.3 years **d** 0.65 dogs per annum

Modelling and problem solving

1 **a** 100% **b** $\dfrac{100a}{a + b}$%

2 **b** $\dfrac{500}{N_0}$

4 1 h 50 min

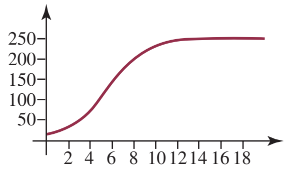
5 **a** $\dfrac{1}{a}\ln\left(\dfrac{a - bN_0}{bN_0}\right)$ **b** $\dfrac{1}{a}\ln\left(\dfrac{2a - 2bN_0}{a - 2bN_0}\right)$

6 $m = 1.2 - 1.2k$ and $n = -1.6 - 2.4k$, for integer k.

7 **a** $\dfrac{dN}{dt} = 0.5N - 0.002N^2$

b **i** $N(t) = \dfrac{250}{1 + 24e^{-0.5t}}$

ii



c Move during 2009
d 31.3 when $t = 6.36$ years; that is, early 2002
e 22.8

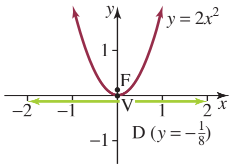
CHAPTER 9 Conics on the Cartesian and complex planes

Exercise 9A — Cartesian form of the straight line and the circle

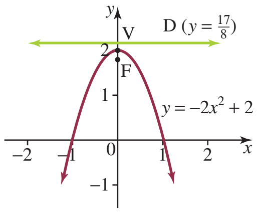
1 **a** $y = 5$ **b** $y = -1$
c $3x + y - 7 = 0$ **d** $3y - x - 5 = 0$
e $2x - 15 = 0$ **f** $x - y = 0$
2 **a** $x \pm 3y = 0$
b $3x - 4y - 26 = 0$ and $3x + 4y + 14 = 0$
3 $3x + 5y - 3 = 0$
4 **a** $x^2 + y^2 + 2x + 8y + 8 = 0$
b $x^2 + y^2 - 6x + 6y + 2 = 0$
c $x^2 + y^2 - 6x - 4y - 40 = 0$
d $x^2 + y^2 + 2x - 8y = 0$
5 **a** $(\frac{5}{2}, -\frac{3}{2})$, $\frac{\sqrt{38}}{2}$ units **b** $(-3, 0)$, $\sqrt{3}$ units
c $(2, -\frac{5}{4})$, $\frac{\sqrt{65}}{4}$ units **d** $(-2, -1)$, $\sqrt{10}$ units
e $(-\frac{3}{2}, \frac{2}{3})$, $\frac{\sqrt{385}}{6}$ units
6 $3x^2 + 3y^2 + 14x + 6y - 2 = 0$
7 $x^2 + y^2 - 4x - 4y = 0$
8 **a** $x^2 + y^2 + 7x - 5y - 44 = 0$
b $x^2 + y^2 - 6x + 4y - 12 = 0$
c $8x^2 + 8y^2 - 79x - 32y + 95 = 0$
d $3x^2 + 3y^2 + 14x + 6y - 2 = 0$
9 $(-8, 1)$
10 $(2, 1)$, $\sqrt{41}$ units

Exercise 9B — Cartesian form of the parabola

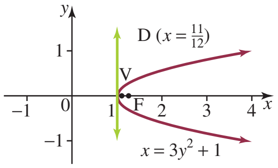
1 **a** $(\frac{3}{2}, 0)$, $x = -\frac{3}{2}$ **b** $(0, 2)$, $y = -2$
c $(-\frac{1}{3}, 0)$, $x = \frac{1}{3}$
2 **a** V(0, 0), $a = \frac{1}{8}$, F(0, $\frac{1}{8}$), D $y = -\frac{1}{8}$



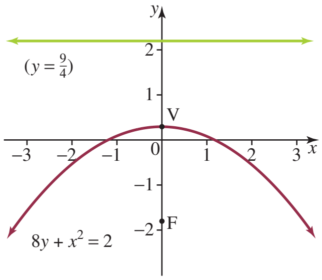
b V(0, 2), $a = -\frac{1}{8}$, F(0, $\frac{15}{8}$), D $y = \frac{17}{8}$



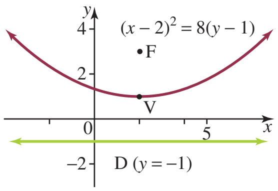
c V(1, 0), $a = \frac{1}{12}$, F($\frac{13}{12}$, 0), D $x = \frac{11}{12}$



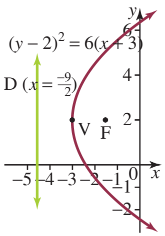
d V(0, $\frac{1}{4}$), $a = 2$, F(0, $-1\frac{3}{4}$), D $y = \frac{9}{4}$



e V(2, 1), $a = 2$, F(2, 3), D $y = -1$



f V(−3, 2), $a = \frac{3}{2}$, F($-\frac{3}{2}$, 2), D $x = -\frac{9}{2}$



3 **a** $y^2 = 12x$ **b** $x^2 = 12(y - 3)$
c $y^2 = -12x$ **d** $(y - 3)^2 = 12(x + 2)$
4 $y^2 - 6y - 8x - 23 = 0$
5 **a** $(y - 2)^2 = -6(x - 2)$, V(2, 2), F($\frac{1}{2}$, 2), $x = \frac{7}{2}$
b $(x - \frac{3}{2})^2 = \frac{5}{3}(y + \frac{7}{4})$, V($\frac{3}{2}$, $-\frac{7}{4}$), F($\frac{3}{2}$, $-\frac{4}{3}$), $y = -\frac{13}{6}$
c $(y - 2)^2 = 6(x - \frac{3}{2})$, V($\frac{3}{2}$, 2), F(3, 2), $x = 0$
6 $y^2 - 2y - 4x + 9 = 0$
7 $x^2 - 4x - 2y + 10 = 0$

Exercise 9C — Cartesian form of the ellipse

1 **a** **i** $e = \frac{3}{5}$, C(0, 0), $V_1(-5, 0)$, $V_2(5, 0)$, $V_3(0, 4)$, $V_4(0, -4)$, $F_1(-3, 0)$, $F_2(3, 0)$, $x = \pm 8\frac{1}{3}$, Major = 10, Minor = 8
ii $e = \frac{5}{13}$, C(0, 0), $V_1(-13, 0)$, $V_2(13, 0)$, $V_3(0, 12)$, $V_4(0, -12)$ $F_1(-5, 0)$, $F_2(5, 0)$, $x = \pm 33\frac{4}{5}$, Major = 26, Minor = 24

iii $e = \frac{\sqrt{17}}{9}$, C(0, 0), $V_1(0, -9)$, $V_2(0, 9)$,

$V_3(8, 0)$, $V_4(-8, 0)$, $F_1(0, \sqrt{17})$, $F_2(0, -\sqrt{17})$,

$y = \pm\frac{81}{\sqrt{17}}$, Major = 18, Minor = 16

iv $e = \frac{2\sqrt{6}}{5}$, C(0, 0), $V_1(0, -5)$, $V_2(0, 5)$,

$V_3(1, 0)$, $V_4(-1, 0)$, $F_1(0, -2\sqrt{6})$,

$F_2(0, 2\sqrt{6})$, $y = \pm\frac{25}{2\sqrt{6}}$, Major = 10, Minor = 2

v $e = \frac{3}{5}$, C(4, 3), $V_1(-1, 3)$, $V_2(9, 3)$, $V_3(1, 10)$,

$V_4(1, -14,)$, $F_1(1, 3)$, $F_2(7, 3)$, $x = 12\frac{1}{2}$,

$x = -4\frac{1}{3}$, Major = 10, Minor = 8

vi $e = \frac{5}{13}$, C(1, −2), $V_1(-12, -2)$, $V_2(14, -2)$,

$V_3(1, 10)$, $V_4(1, -14,)$, $F_1(-4, -2)$, $F_2(6, -2)$,

$x = 34\frac{4}{5}$, $x = -32\frac{4}{5}$, Major = 26, Minor = 24

vii $e = \frac{4}{5}$, C(−3, 2), $V_1(-3, -3)$, $V_2(-3, 7)$,

$V_3(0, 2)$, $V_4(-6, 2)$, $F_1(-3, -2)$, $F_2(-3, 6)$,

$y = 8\frac{1}{4}$ or $y = -4\frac{1}{4}$, Major = 10, Minor = 6

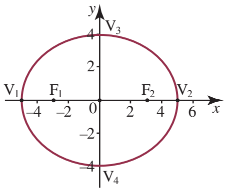
viii $e = \frac{\sqrt{5}}{3}$, C(−5, −1), $V_1(-5, -4)$, $V_2(-5, 2)$,

$V_3(-3, -1)$, $V_4(-7, -1)$, $F_1(-5, -1 - \sqrt{5})$,

$F_2(-5, -1 + \sqrt{5})$, $y = -1 \pm \frac{9}{\sqrt{5}}$, Major = 6,

Minor = 4

b **i**



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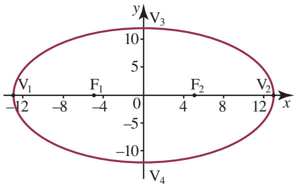

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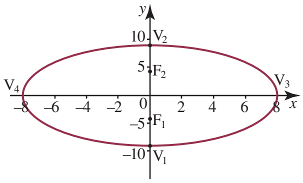

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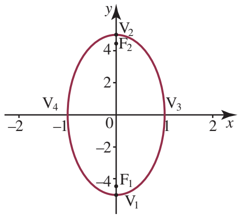

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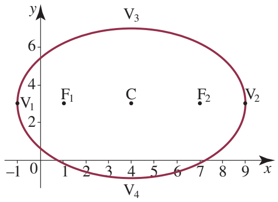

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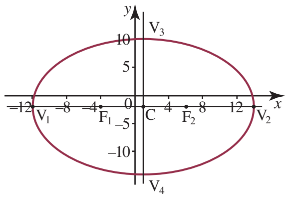

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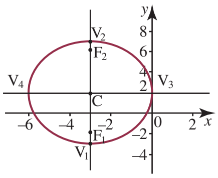

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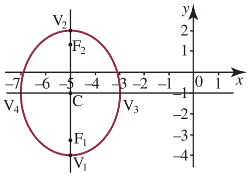
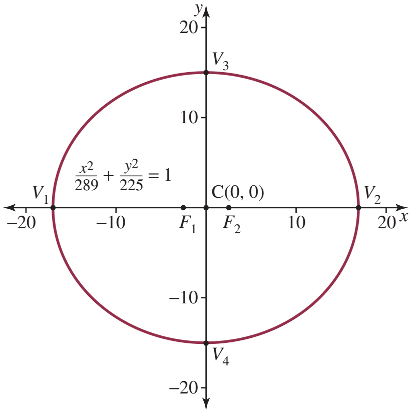
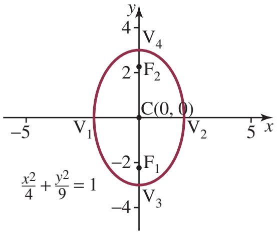


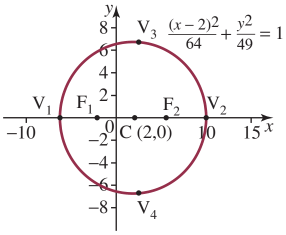
2 **a**



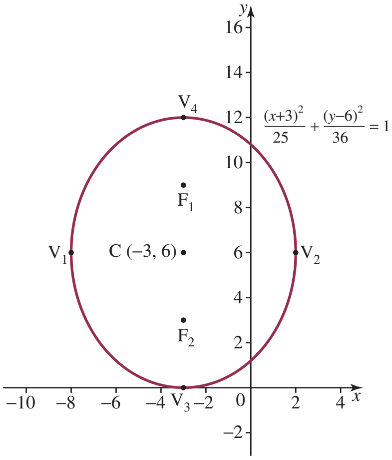
b



c



d



3 $e = \frac{\sqrt{7}}{4}$, C(2, −3), F(2 ± $\sqrt{7}$, −3),

Vertices (2 ± 4, −3) (2, −3 ± 3), D: $x = 2 \pm \frac{16}{\sqrt{7}}$

4 **a** $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **b** $\frac{x^2}{225} + \frac{y^2}{289} = 1$

c $\frac{x^2}{64} + \frac{y^2}{100} = 1$ **d** $\frac{x^2}{64} + \frac{y^2}{39} = 1$

5 **a** $\frac{(x-4)^2}{18} + \frac{(y+1)^2}{9} = 1$

b $\frac{(x-3)^2}{8} + \frac{(y-1)^2}{9} = 1$

c $\frac{(x+2)^2}{2} + \frac{(y+1)^2}{1} = 1$

d $\frac{x^2}{100} + \frac{y^2}{36} = 1$

6 $\frac{x^2}{36} + \frac{y^2}{16} = 1$

7 $36x^2 + 11y^2 - 144x - 44y - 208 = 0$

8 $9x^2 + 25y^2 + 18x - 50y - 191 = 0$

9 $\frac{15\sqrt{55}}{4}$ m

10 Orbit is elliptical with an eccentricity of approximately 0.997.

11 As the second comet has a smaller eccentricity, it has a more circular orbit than the first comet.

Exercise 9D — Cartesian form of the hyperbola

1 **a** **i** $e = \frac{\sqrt{13}}{2}$, C(0, 0), $V_1(-3, 0)$, $V_2(3, 0)$,

$F_1(-\sqrt{13}, 0)$, $F_2(\sqrt{13}, 0)$, $x = \pm\frac{9}{\sqrt{13}}$,

$y = \pm\frac{2}{3}x$

ii $e = \frac{\sqrt{5}}{2}$, C(−2, 1), $V_1(-6, 1)$, $V_2(2, 1)$,

$F_1(-2\sqrt{5} - 2, 1)$, $F_2(-2\sqrt{5} + 2, 1)$,

$x = -2 \pm \frac{8}{\sqrt{5}}$, $y = 1 \pm \frac{1}{2}(x + 2)$

iii $e = \frac{7}{3\sqrt{5}}$, C(0, 0), $V_1(-3\sqrt{5}, 0)$, $V_2(3\sqrt{5}, 0)$,

$F_1(-7, 0)$, $F_2(7, 0)$, $x = \pm 6\frac{3}{7}$, $y = \pm\frac{2}{3\sqrt{5}}x$

iv $e = \frac{5}{3}$, C(0, 0), $V_1(0, -3)$, $V_2(0, 3)$, $F_1(0, -5)$,

$F_2(0, 5)$, $y = \pm 1\frac{4}{5}$, $y = \pm\frac{3}{4}x$

v $e = \frac{\sqrt{41}}{4}$, C(3, −1), $V_1(3, -5)$, $V_2(3, 3)$,

$F_1(3, -1 - \sqrt{41})$, $F_2(3, -1 + \sqrt{41})$,

$y = -1 \pm \frac{16}{\sqrt{41}}$, $y = -1 \pm \frac{4}{5}(x - 3)$

vi $e = \frac{\sqrt{65}}{4}$, C(0, 0), V(0, ±4), $F_1(0, \pm\sqrt{65})$,

$y = \pm\frac{16}{\sqrt{65}}$, $y = \pm\frac{4}{7}x$

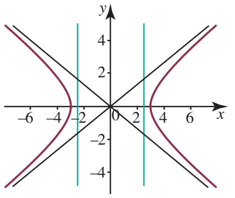
vii $e = \sqrt{2}$, C(0, 0), $V_1(-3, 0)$, $V_2(3, 0)$,

$F_1(-3\sqrt{2}, 0)$, $F_2(3\sqrt{2}, 0)$, $x = \pm\frac{3}{\sqrt{2}}$, $y = \pm x$

viii $e = \sqrt{2}$, C(0, 0), $V_1(-5, 0)$, $V_2(5, 0)$,

$F_1(-5\sqrt{2}, 0)$, $F_2(5\sqrt{2}, 0)$, $x = \pm\frac{5}{\sqrt{2}}$, $y = \pm x$

b **i**



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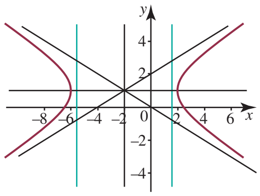

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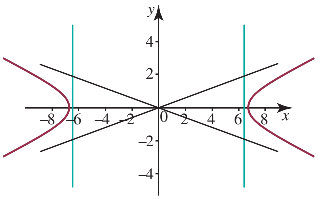

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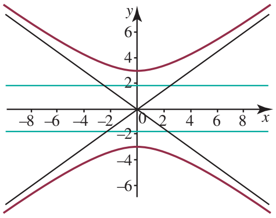

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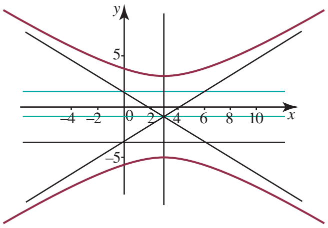

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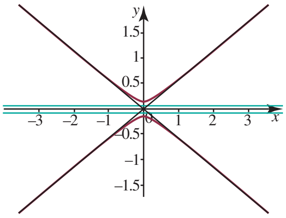

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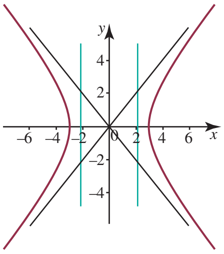

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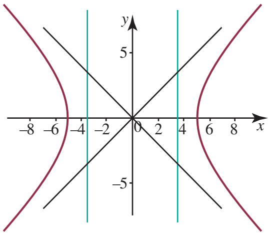


2 **a** $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **b** $\frac{x^2}{36} - \frac{y^2}{28} = 1$

c $\frac{y^2}{25} - \frac{x^2}{144} = 1$

3 $\frac{x^2}{6} - \frac{y^2}{2} = 1$

4 $\frac{x^2}{36} - \frac{y^2}{49} = 1$

5 C(2, −1), $F_1(7, -1)$, $F_2(-3, -1)$, $V_1(6, -1)$, $V_2(-2, -1)$, $y + 1 = \pm\frac{3}{4}(x - 2)$

6 $\frac{y^2}{9} - \frac{x^2}{25} = 1$

$F_1(-\sqrt{34}, 0)$, $F_2(\sqrt{34}, 0)$ and $F_1(0, -\sqrt{34})$, $F_2(0, \sqrt{34})$

$y = \pm\frac{3}{5}x$ for both

7 $\frac{x^2}{9} - \frac{y^2}{4} = 1$, $\frac{y^2}{4} - \frac{x^2}{9} = 1$

8 $xy = 8$

9 $44y^2 - 100x^2 = 275$

10 $\frac{(x+2)^2}{9} - \frac{(y+4)^2}{7} = 1$ or

$7x^2 - 9y^2 + 28x - 72y - 179 = 0$

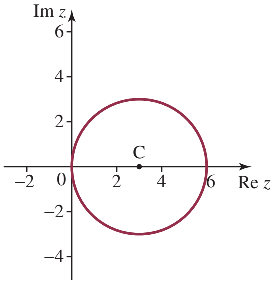
11 $4x^2 - y^2 - 4x + 3y = 26$

Exercise 9E — Lines, circles and regions on the complex plane

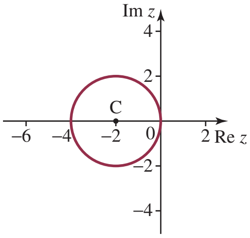
1 **a** $6x - 8y - 13 = 0$ **b** $14x - 8y - 13 = 0$

c $x = -1.5$

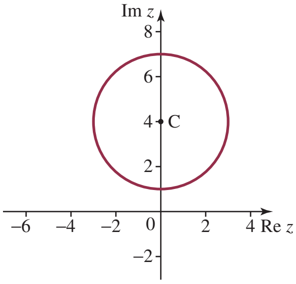
2 **a** $(x - 3)^2 + y^2 = 9$



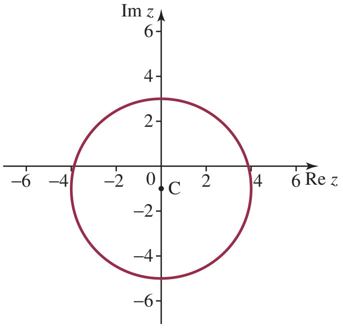
b $(x + 2)^2 + y^2 = 4$



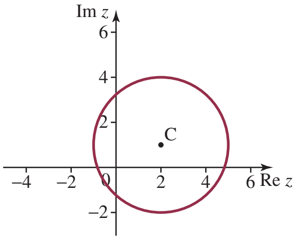
c $x^2 + (y - 4)^2 = 9$



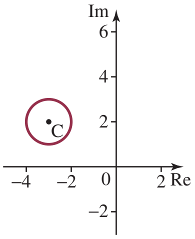
d $x^2 + (y + 1)^2 = 16$



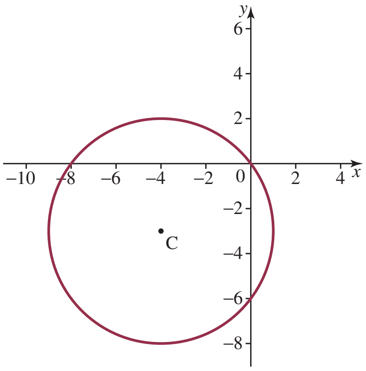
e $(x - 2)^2 + (y - 1)^2 = 9$



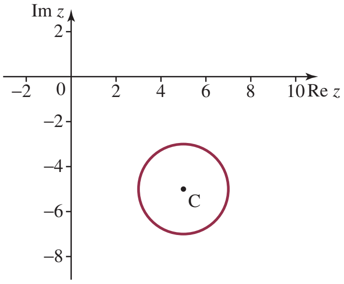
f $(x + 3)^2 + (y - 2)^2 = 1$



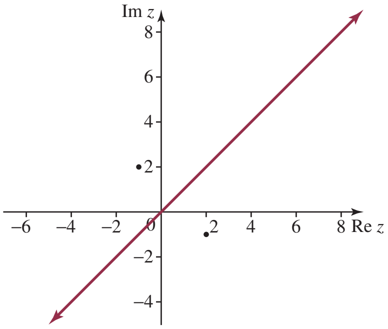
g $(x + 4)^2 + (y + 3)^2 = 25$



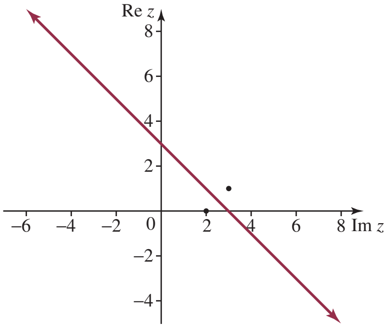
h $(x - 5)^2 + (y + 5)^2 = 4$



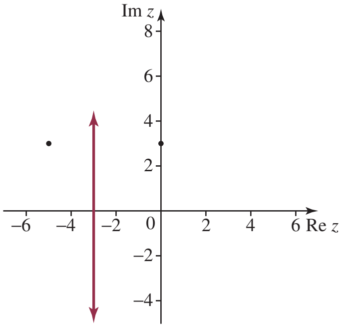
i $x = y$



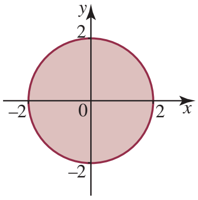
j $x + y - 3 = 0$



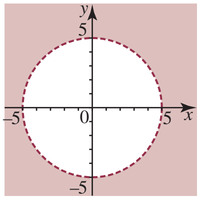
k $2x + 5 = 0$



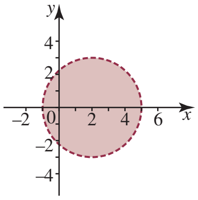
3 **a** Interior of a circular region including circumference. Centred at the origin with a radius of 2.



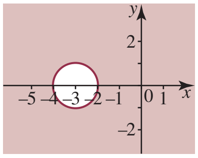
b Exterior of a circular region excluding circumference. Centred at the origin with a radius of 5.



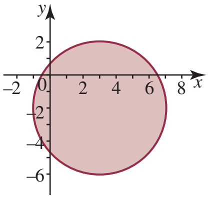
c Interior of a circular region excluding circumference. Centred at (2, 0) and with a radius of 3.



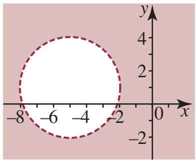
d Exterior of a circular region including circumference. Centred at (–3, 0) and with a radius of 1.



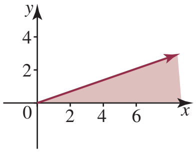
e Interior of a circular region including circumference. Centred at (3, –2) and with a radius of 4.



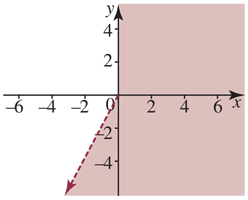
f Exterior of a circular region excluding circumference. Centred at (–5, 1) and with a radius of 3.



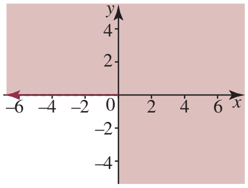
4 a Region is an open-ended sector anchored at the origin with radial edges arg $(z) = 0$ excluded and arg $(z) = \frac{\pi}{6}$ included.



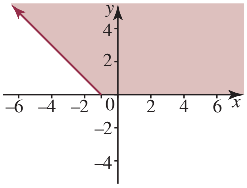
b Region is an open-ended sector anchored at the origin with radial edges arg $(z) = \frac{2\pi}{3}$ excluded and arg $(z) = \frac{\pi}{2}$ included.



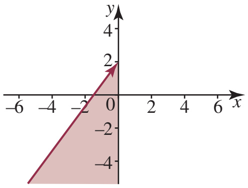
c Region is an open-ended sector anchored at the origin with radial edges arg $(z) = \frac{\pi}{2}$ included and arg $(z) = \pi$ excluded.



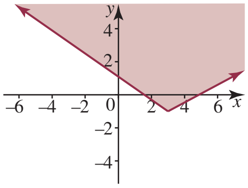
d Region is an open-ended sector anchored at (–1, 0) with radial edges arg $(z + 1) = 0$ included and arg $(z) = \frac{2\pi}{3}$ included.



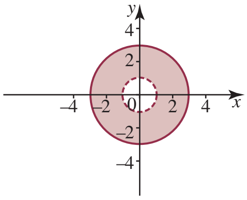
e Region is an open-ended sector anchored at (0, 2) with radial edges arg $(z - 2i) = \frac{3\pi}{4}$ included and arg $(z) = \frac{\pi}{2}$ included.



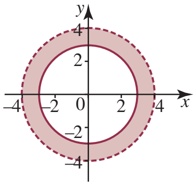
f Region is an open-ended sector anchored at (3, –1) with radial edges arg $(z - 3 + i) = \frac{\pi}{6}$ excluded and arg $(z - 3 + i) = \frac{5\pi}{6}$ excluded.



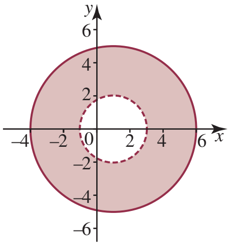
5 a Annulus centred origin.
Inner radius 1 (excluded)
Outer radius 3 (included)



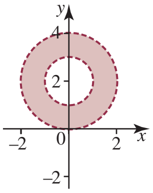
b Annulus centred origin.
Inner radius 3 (included)
Outer radius 4 (excluded)



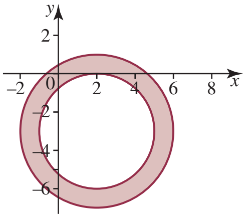
c Annulus centred (1, 0).
Inner radius 2 (excluded)
Outer radius 5 (included)



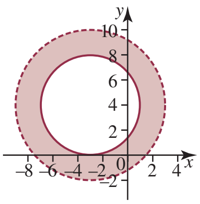
d Annulus centred (0, 2).
Inner radius 1 (excluded)
Outer radius 2 (excluded)



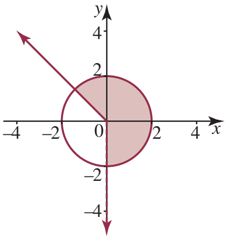
e Annulus centred (2, –3).
Inner radius 3 (included)
Outer radius 4 (included)



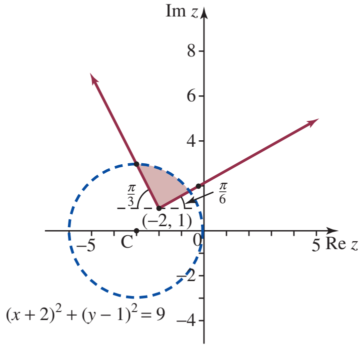
f Annulus centred (–3, 4).
Inner radius 4 (included)
Outer radius 6 (excluded)



6 a The interior region of the circle centred at the origin and with radius 2 between the radial edges arg $(z) = \frac{\pi}{2}$ excluded and arg $(z) = \frac{3\pi}{4}$ included.

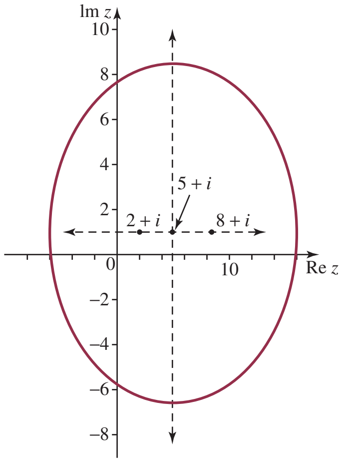


b The exterior region of the circle centred at (–3, 0) and with radius 2 between the radial edges arg $(z + 2) = \frac{\pi}{6}$ excluded and arg $(z + 2) = \frac{2\pi}{3}$ included.

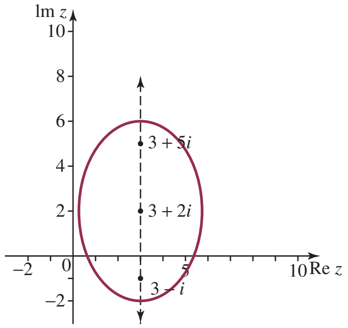


Exercise 9F — The ellipse and hyperbola on the complex plane

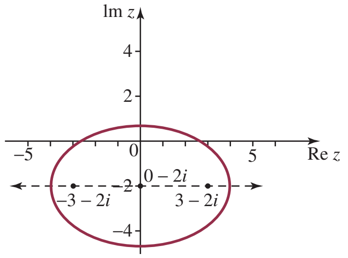
1 a Locus is an ellipse with foci $2 + i$ and $8 + i$; major axis is 10 units; minor axis is 8 units; centre is $5 + i$.



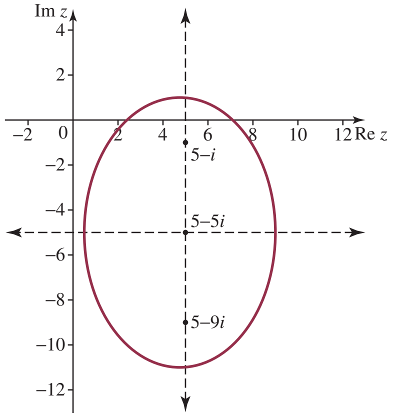
b Locus is an ellipse with foci $3 - i$ and $3 + 5i$; major axis is 8 units; minor axis is $2\sqrt{7}$ units; centre is $3 + 2i$.



c Locus is an ellipse with foci $-3 - 2i$ and $3 - 2i$; major axis is 8 units; minor axis is $2\sqrt{7}$ units; centre is $0 - 2i$.



d Locus is an ellipse with foci $5 - i$ and $5 - 9i$; major axis is 12 units; minor axis is $4\sqrt{5}$ units; centre is $5 - 5i$.



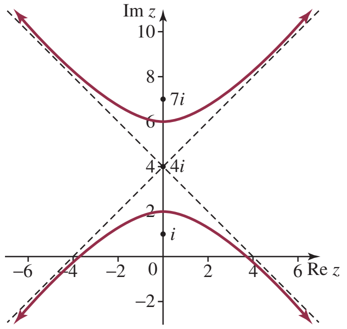
2 a $\dfrac{x^2}{16} + \dfrac{(y-4)^2}{25} = 1$

b $\dfrac{(x-2)^2}{49} + \dfrac{y^2}{24} = 1$

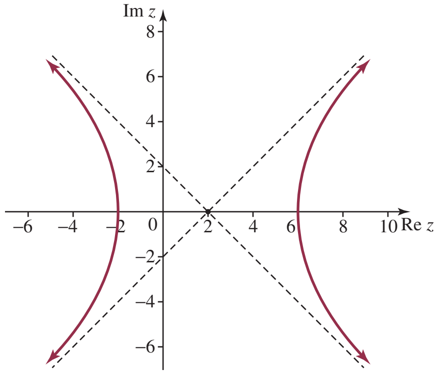
c $\dfrac{(x-7)^2}{64} + \dfrac{(y-4)^2}{48} = 1$

d $\dfrac{(x-5)^2}{49} + \dfrac{(y-7)^2}{13} = 1$

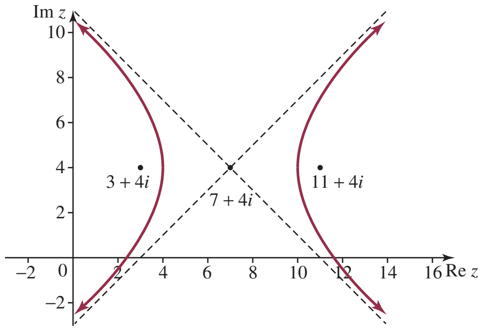
3 a Full hyperbola with foci at i and $7i$; transverse axis is 4 units; centre is at $4i$



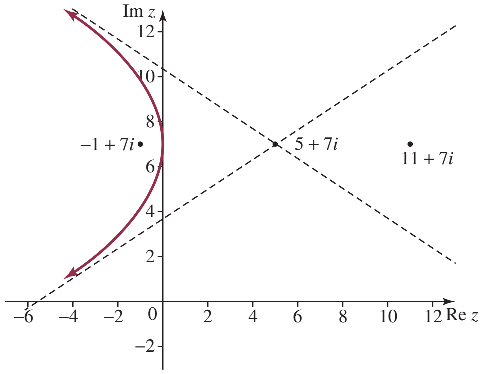
b Full hyperbola with foci at $-3 + 0i$ and $7 + 0i$; transverse axis is 8 units; centre is at $2 + 0i$



c Full hyperbola with foci at $3 + 4i$ and $11 + 4i$; transverse axis is 6 units; centre is at $7 + 4i$



d Left-hand branch of hyperbola with foci at $-1 + 7i$ and $11 + 7i$; transverse axis is 10 units; centre is at $5 + 7i$



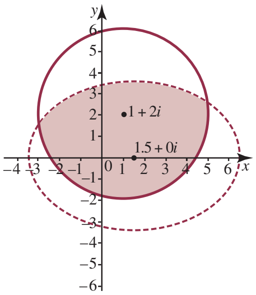
4 **a** $\frac{4(x-5)^2}{25} - \frac{4(y-1)^2}{11} = 1$

b $\frac{4(y-2)^2}{25} - \frac{4(x-3)^2}{11} = 1$

c $\frac{x^2}{4} - \frac{(y+2)^2}{5} = 1$

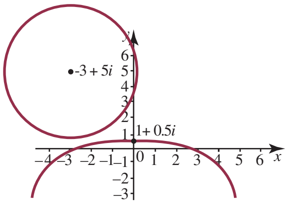
d $\frac{4(y+5)^2}{49} - \frac{4(x-5)^2}{15} = 1$

5 **a** Circle centre is at $1 + 2i$, radius is 4. Ellipse centre is $1.5 + 0i$, foci are at $-2 + 0i$ and $5 + 0i$, axes are 10 and $\sqrt{51}$.



b $z = 0.36 + 2.61i$ and $z = 0.36 - 2.61i$

c There are no solutions. Circle centre is at $-3 + 5i$, radius is 4. One branch of hyperbola only, foci are $6i$ and $-2i$, centre is $0 + 2i$ and transverse axis is 3 units.



Chapter review

1 B **2** D **3** A

4 A

5 **a** V(0, 0), F$\left(0, \frac{1}{32}\right)$, D $y = -\frac{1}{32}$

b V(0, 16), F $\left(0, 15\frac{7}{8}\right)$, D $x = 16\frac{1}{8}$

c $V\left(\frac{1}{3}, 1\right)$, $F\left(-\frac{5}{12}, 1\right)$, D $x = \frac{13}{12}$

d $V\left(-\frac{1}{2}, -1\right)$, $F\left(-\frac{1}{6}, -1\right)$, D $x = -\frac{5}{6}$

e $V\left(-1, \frac{1}{3}\right)$, $F\left(-1, -\frac{5}{12}\right)$, D $y = 1\frac{1}{12}$

f V, F, D

6 **a** $y^2 = 12(x + 1)$

b $x^2 = 8(y - 3)$

c $x^2 = -\frac{1}{3}y$

7 C

8 **a** $e = \frac{\sqrt{17}}{9}$, C(0, 0), $V_1(-9, 0)$, $V_2(9, 0)$, $V_3(0, 8)$, $V_4(0, -8)$, $F_1(\sqrt{17}, 0)$, $F_2(-\sqrt{17}, 0)$, $x = \pm\frac{81}{\sqrt{17}}$, Major = 18, Minor = 16

b $e = \frac{3}{5}$, C(0, 0), $V_1(0, -5)$, $V_2(0, 5)$, $V_3(4, 0)$, $V_4(-4, 0)$, $F_1(0, -3)$, $F_2(0, 3)$, $y = \pm 8\frac{1}{3}$, Major = 10, Minor = 8

c $e = \frac{\sqrt{5}}{3}$, C(−1, −5), $V_1(-4, -5)$, $V_2(2, -5)$, $V_3(-1, -3)$, $V_4(-1, -7)$, $F_1(-1-\sqrt{5}, -5)$, $F_2(-1+\sqrt{5}, -5)$, $x = -1 \pm \frac{9}{\sqrt{5}}$, Major = 6, Minor = 4

d $e = \frac{5}{13}$, C(−2, 1), $V_1(-2, -12)$, $V_2(-2, 14)$, $V_3(10, 1)$, $V_4(-14, 1)$, $F_1(-2, -4)$, $F_2(-2, 6)$, $y = 34\frac{4}{5}$, $y = -32\frac{4}{5}$, Major = 26, Minor = 24

9 $e = \frac{\sqrt{7}}{4}$, C(−3, 2), $F\left(-3, 2 \pm \sqrt{7}\right)$,

Vertices (−3, 2 ± 4) (−3 ± 3, 2), D $y = 2 \pm \frac{16}{\sqrt{7}}$

10 **a** $\frac{x^2}{9} + \frac{y^2}{25} = 1$

b $\frac{x^2}{9} + \frac{y^2}{25} = 1$

c $\frac{(x-6)^2}{18} + \frac{(y-1)^2}{9} = 1$

d $\frac{(x-5)^2}{8} + \frac{(y-3)^2}{9} = 1$

11 D **12** D **13** A **14** D

15 **a** $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b $\frac{(x-1)^2}{36} - \frac{(y-1)^2}{28} = 1$

c $\frac{y^2}{25} - \frac{(x-1)^2}{144} = 1$

d $\frac{y^2}{36} - \frac{x^2}{49} = 1$

16 C

17 **a** $(x - 2)^2 + (y + 3)^2 = 9$

b $y = x + 6$

18 **a** $\frac{(x-5)^2}{25} + \frac{(y-1)^2}{16} = 1$

b $\frac{(x-5)^2}{20} + \frac{(y+5)^2}{36} = 1$

19 **a** $\frac{(x-4)^2}{4} - \frac{x^2}{5} = 1$

b $\frac{(x-7)^2}{9} - \frac{(y-4)^2}{7} = 1$

Modelling and problem solving

1 $x^2 + y^2 = 9,\ 0 \le x \le 3,\ 0 \le y \le 3$
2 $x^2 + y^2 - 4x = 0$ and $x^2 + y^2 + 4x = 0$
3 $x^2 + y^2 - 7x + 5y - 14 = 0$
4 $x^2 + y^2 = 108\,790$
5 $x^2 + y^2 = 4.5 \times 10^{-4}$
6 $y^2 - 6y - 4x + 17 = 0,\ 11y^2 - 98y - 108x + 539 = 0$
7 Less than 5 cm
8 **a** $\frac{x^2}{100} + \frac{y^2}{25} = 1$
b $\frac{x^2}{9} + \frac{y^2}{16} = 1$
9 $3x^2 + 4y^2 - 28x - 16y + 48 = 0$
10 $4x^2 + y^2 - 16x - 6y - 43 = 0,\ (2, 3)$
11 $x^2 + 4y^2 = 144$
12 **a** $\frac{(x-17.35)^2}{(17.94)^2} + \frac{y^2}{(4.56)^2} = 1$
b 0.59 AU **c** 35.3 AU
13 $a = 0,\ b = -4,\ c = 0$
14 $(x-2)(y-1) = 1$
15 $y^2 + 2\sqrt{3}\,xy - x^2 - 20 = 0$
16 $xy = 3$
17 $xy = 3$
18 $\frac{7}{2},\ 3\sqrt{7}$
19 $y^2 = 12\sqrt{2}\,x$
21 The sum of distances between the foci and the outside of the ellipse is constant. The water travels from the first focus, is reflected in the outside and travels back. This means that the reflected waves will pass through the second focus at the same point in time and therefore a drop of water will shoot up.
22 $\frac{x^2}{244} + \frac{25y^2}{2196} = 1$; if $e < 0.8$, more circular and more costly.
23 $3x^2 - y^2 + 2cx - c^2 = 0$
24 31
25 46

CHAPTER 10 Parametric and polar equations of conic sections

Exercise 10A — Parametric equations

1 **a** $y = x - 7$
b $y = x^2 - 2x + 5$
c $y = x^{\frac{2}{3}}$
d $y = x(x-6)$
e $(x-2)(y-2) = -1$
f $(x-y)^2 = 2(x+y)$
2 **a** $x^2 + y^2 = 1$
b $\frac{x^2}{16} + \frac{y^2}{4} = 1$
c $x^2 - y^2 = 1$
d $x = 1 - 2y^2$
e $x^2 - y^2 = 1$
f $x = \cos^{-1}\left(\frac{3-y}{2}\right) - \frac{\sqrt{4-(3-y)^2}}{2}$
g $\frac{x^2}{16} + \frac{y^2}{25} = 1$
h $y = 2x\sqrt{1-x^2}$
i $y^2 = x$
j $x^2 + y^2 = 1$
k $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{2}{3}}$

Exercise 10B — Tangents and normals

1 **a** $3x + 4y - 25 = 0,\ 4x - 3y = 0$
b $x + y - 2 = 0,\ x - y - 2 = 0$
c $x - y + 3 = 0,\ x + y + 1 = 0$
d $x - y + 5 = 0,\ x + y + 1 = 0$
e $3x + y + 11 = 0,\ x - 3y - 3 = 0$
f $x - y - 1 = 0,\ x + y - 3 = 0$
g $20x + 33y - 46 = 0,\ 33x - 20y + 73 = 0$
h $x + y - 4 = 0,\ x - y = 0$
i $x + y + 4 = 0,\ x - y + 2 = 0$
2 **a** $2x - y + 1 = 0,\ x + 2y - 12 = 0$
b $x - 2y + 12 = 0,\ 2x + y + 4 = 0$
c $x + 4y - 3 - 0,\ 4x - y + 5 = 0$
d $x + 2y - 4\sqrt{2} = 0,\ 2x - y - 3\sqrt{2} = 0$
e $\sqrt{3}\,x - 2y + 4 = 0,\ 4\sqrt{3}\,x + 6y + 9 = 0$
f $y = 1,\ x = 1$
3 **a** $5x - 2y - 4 = 0$
b $2x + y \pm 4 = 0$
c $x - 2y \pm \frac{2\sqrt{31}}{3} = 0$
d $4x - y \pm 8\sqrt{2} = 0$
e $9x + 6y - 2 = 0$
4 $\left(\pm\frac{10\sqrt{34}}{17}, \pm\frac{4\sqrt{34}}{17}\right)$
5 $6x - 5y - 9 = 0,\ 25x + 30y - 129 = 0$
9 $b = -am^2,\ 2x - 2y + 1 = 0$

Exercise 10C — Parametric equations of conics

1 **a** $y^2 = -8x,\ (0, 0),\ (-2, 0),\ x = 2$
b $y^2 = -2(x+2)$
2 **a** $y^2 = 9x$
b $T(4 + 2t^2, 0)$ Locus is $x = 0$
3 $9y = 2x^2$
4 **d** $y^2 = 4a(x - 4a)$
5 **a** $8x = y^2$
b $ty - x - 2t^2 = 0,\ (0, 2t),\ (-2t^2, 0)$
6 $4(ay - x^2) = a^2$
7 $x^2 = \frac{2(3y-4)^2}{2-y},\ y \ne 2$

Exercise 10D — Parametric equations of circles and ellipses

1 $x^2 + y^2 = 32$
2 $4x^2 + 9y^2 = 72$
3 $\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$

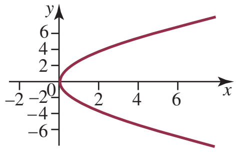
Exercise 10E — Parametric equations of hyperbolas

1 $\frac{x}{a}\left(t+\frac{1}{t}\right)-\frac{y}{b}\left(t-\frac{1}{t}\right) = 2$

3 a 2 units squared

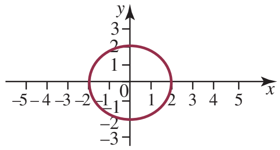
Exercise 10F — Applying technology to parametric equations

1 a Parabola $y^2 = 8x$

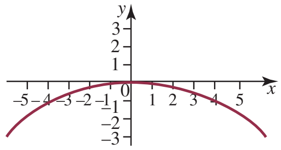


b Circle, centre origin and radius 2

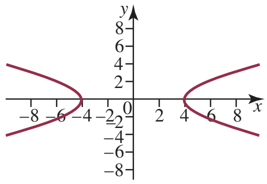
$x^2 + y^2 = 4$



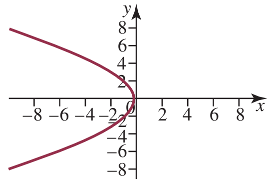
c Parabola $x^2 = -12y$



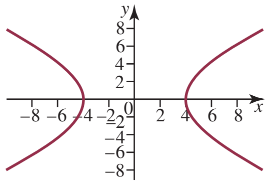
d Hyperbola $x^2 - 4y^2 = 16$



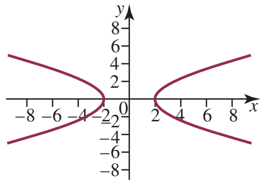
e Parabola $y^2 = -8x$



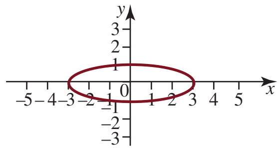
f Hyperbola $x^2 - y^2 = 16$



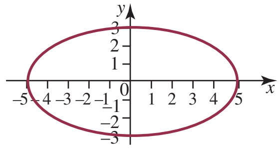
g Hyperbola $x^2 - 4y^2 = 4$



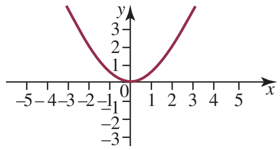
h Ellipse $x^2 + 9y^2 = 9$



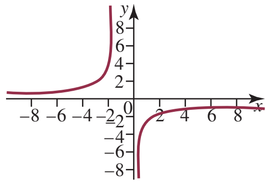
i Ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$



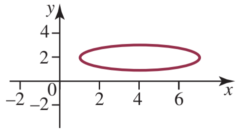
j Parabola $x^2 = 4y$



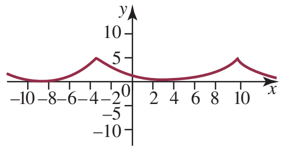
k Rectangular hyperbola $xy = -4$



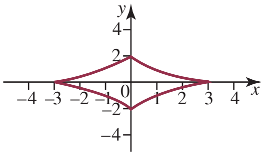
l Ellipse $(x - 4)^2 + 9(y - 2)^2 = 9$



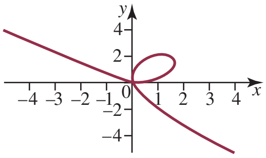
2 a Cycloid family



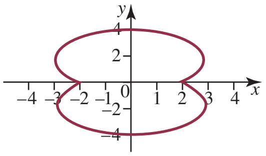
b Evolute of an ellipse



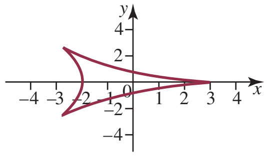
c Folium of Descartes



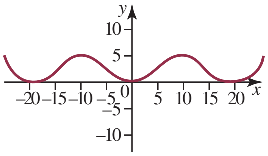
d Nephroid family



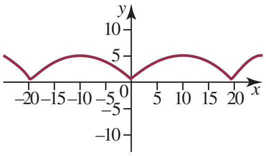
e Hypocycloid family



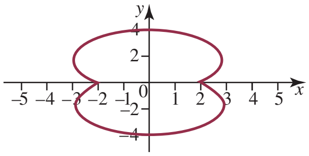
f Sinusoidal family



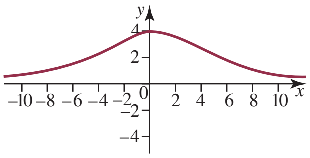
g Curtate cycloid family



h Nephroid family

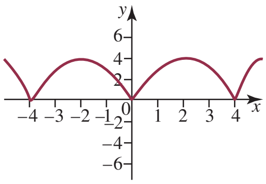


i Witch of Agnesi



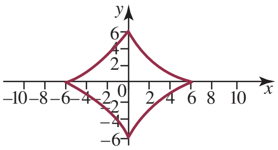
3 **a** 5.936 **b** 25.527
c 41.411 **d** ∞
e 2.264 **f** 17.617
g 18.887 **h** 7.6

4 The curves belong to the cycloid family. Changing the value of the constant a varies the length of the circumference.
The ± sign has the effect of inverting the curve as indicated below.



5 The value of the parameter t is the gradient of the tangent to the curve at the given point.

6 **a** Changing the value of the constant a changes the radius of the circle which circumscribes the astroid.



b

A	2	3	4	5	6
L	12	18	24	30	36

c $L = 6A$

d $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Exercise 10G — Polar equations

1 **a** $2r\cos\theta + 3r\sin\theta = 4$ **b** $\tan\theta = \frac{3}{4}$
c $r = 4\sec\theta$ **d** $r = -3\operatorname{cosec}\theta$
e $r = 4$ **f** $r = 2(\sin\theta + \cos\theta)$
g $r = a\cos\theta$ **h** $r = 2\cot\theta\operatorname{cosec}\theta$
i $r = 4\tan\theta\sec\theta$
j $16r^2\cos^2\theta + 9r^2\sin^2\theta = 144$
k $25r^2\cos 2\theta + 16r^2\sin\theta = 400$
l $4r^2\cos^2\theta - 9r^2\sin\theta = 36$
m $r^2\sin\theta\cos\theta = 2,\ r^2\sin 2\theta = 4$
n $r = 2\sin\theta$

2 **a** $x = 3$ **b** $y = x$
c $x^2 + y^2 - 3x = 0$ **d** $y = 3$
e $x^2 + y^2 = 16$ **f** $y = x^2$
g $y^2 = 4(x + 1)$ **h** $x^2 + y^2 - 4y = 0$

3 **a** $r = \pm 3$ **b** $r = \pm 5$
c $r = 2\cos\theta$ **d** $r = 4\sin\theta$
e $r = -8\sin\theta$ **f** $r = 6\cos\left(\theta - \frac{\pi}{3}\right)$
g $r = 10\cos\left(\theta - \frac{2\pi}{3}\right)$ **h** $r^2 - 4r\sin\theta = 5$
i $r^2 - 6r\cos\left(\theta - \frac{2\pi}{3}\right) = 7$

4 **a** $\left(4, \frac{\pi}{6}\right), \sqrt{31}$ **b** $\left(5, \frac{-\pi}{3}\right), 5$

Exercise 10H — Conics in polar form

1 **a** Parabola, $e = 1$, directric 4 units to the right of the pole

b Ellipse, $e = \frac{1}{2}$, directrix 5 units to the right of the pole

c Hyperbola, $e = 1\frac{1}{2}$, directric $1\frac{1}{3}$ units to the right of the pole

d Ellipse, $e = \frac{2}{3}$, directrix $1\frac{1}{2}$ units to the left of the pole

e Parabola, $e = 1$, directrix $3\frac{1}{3}$ units to the left of the pole

f Ellipse, $e = \frac{1}{2}$, directrix 6 units below the pole

2 **a** $r = \dfrac{5}{1 + \cos\theta}$ **b** $r = \dfrac{a}{2 + \cos\theta}$ **c** $r = \dfrac{2p}{1 - 2\cos\theta}$

3 **a** $y^2 = 6x + 9$ **b** $y^2 = -2x + 1$

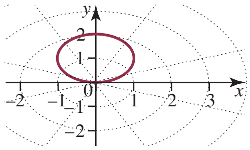
c $y^2 = -4a(x - a)$ **d** $\dfrac{(x+3)^2}{4} - \dfrac{y^2}{5} = 1$

e $\dfrac{4\left(x + \frac{5}{2}\right)^2}{9} - \dfrac{y^2}{4} = 1$ **f** $\dfrac{(x-5)^2}{16} - \dfrac{y^2}{9} = 1$

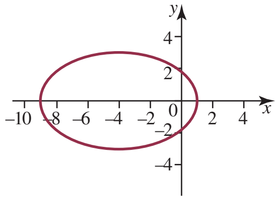
g $\dfrac{(x-4)^2}{25} + \dfrac{y^2}{9} = 1$ **h** $9\left(y + \frac{2}{3}\right)^2 - 3x^2 = 1$

i $\dfrac{(y-3)^2}{4} - \dfrac{x^2}{5} = 1$

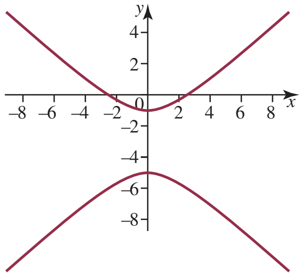
4 **a** A circle centred at (0, 1) and with radius 1.



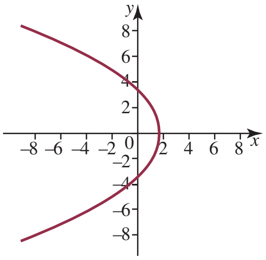
b Ellipse lengthwise horizontal, focus at pole and directrix to the right of the pole. Vertex at (1, 0).



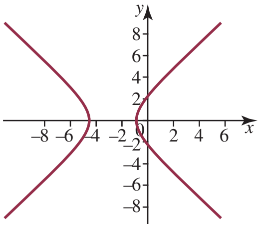
c Hyperbola lengthwise vertical, top branch has focus at pole and directrix below pole. Vertex at $\left(1, -\frac{\pi}{2}\right)$.



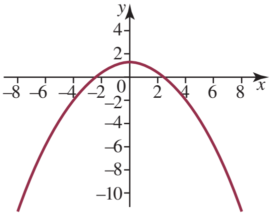
d Parabola lengthwise horizontal with focus at pole and directrix to the right of the pole. Vertex at $(1\frac{2}{3}, 0)$.



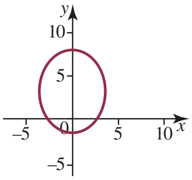
e Hyperbola lengthwise horizontal, right branch has focus at pole and directrix left of pole. Vertex at $(0.9, \pi)$.



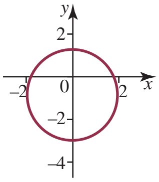
f Parabola lengthwise vertical with focus at pole and directrix above the pole. Vertex at $\left(1\frac{1}{4}, \frac{\pi}{2}\right)$.



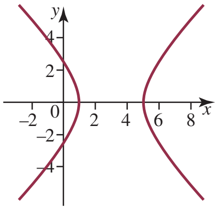
g Ellipse lengthwise vertical, focus at pole and directrix below the pole. Vertex at $\left(1\frac{3}{5}, -\frac{\pi}{2}\right)$.



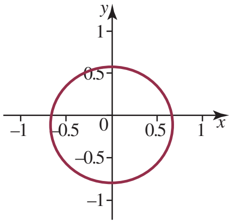
h Ellipse lengthwise vertical, focus at pole and directrix above the pole. Vertex at $\left(1\frac{2}{7}, \frac{\pi}{2}\right)$.



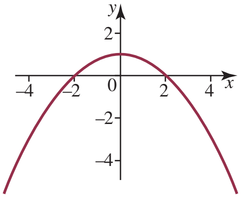
i Hyperbola lengthwise horizontal, left branch has focus at pole and directrix right of pole. Vertex at (1, 0).



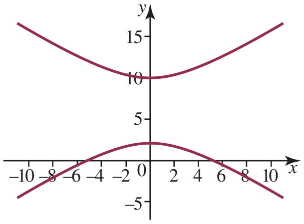
j Ellipse lengthwise vertical, focus at pole and directrix above the pole. Vertex at $\left(\frac{4}{7}, \frac{\pi}{2}\right)$.



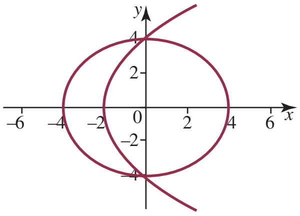
k Parabola lengthwise vertical with focus at pole and directrix above the pole. Vertex at $\left(1, \frac{\pi}{2}\right)$.



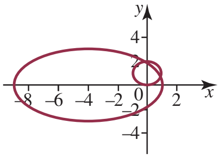
l Hyperbola lengthwise vertical, bottom branch has focus at pole and directrix above the pole. Vertex at $\left(2, \frac{\pi}{2}\right)$.



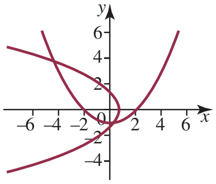
5 **a** (0, ±4)



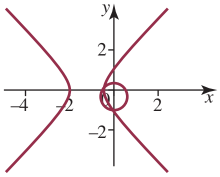
b (0.91, 0.58) and (–0.23, 1.97)



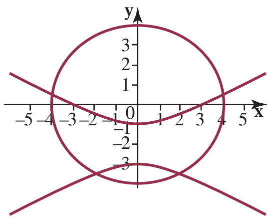
c (–4.31, 3.64) and (0.31, –0.98)



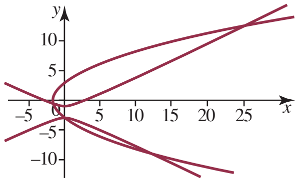
d (0, –1) and (–0.39, 0.16)



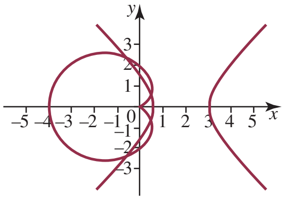
e (±3.97, 0.50) and (±1.94, –3.5)



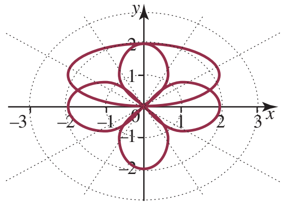
f (–1.42, –0.71), (0, –3), (12.1, –9.0), and (25.3, 12.7)



g (0.47, ±0.70) and (–0.70, ±2.45)



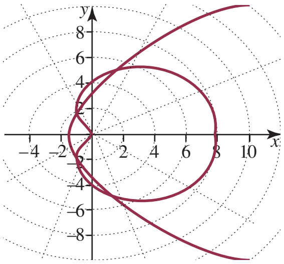
h (0, 0), (±0.87, 0.50) and (0, 2)



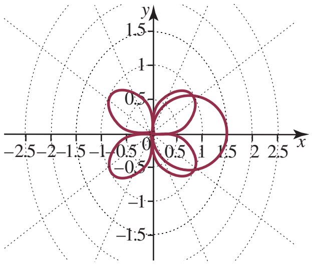
Exercise 10I — Some special curves and interesting problems

1 $r = a\theta$, Archimedes Spiral

2 **a** Points of intersection are $(-1, \pm\sqrt{3})$ and $(3, \pm 3\sqrt{3})$.



b Points of intersection are (0, 0) and $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$.



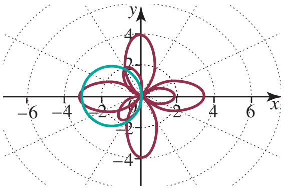
3 **a** $\frac{3\pi}{4}$ **b** π **c** $\frac{9\pi}{4} + 4$

d $\frac{33\pi}{8} + 4\sqrt{2} + \frac{1}{4}$

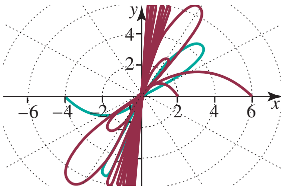
4 **a** π **b** $\frac{\pi a^2}{2}$ **c** $\frac{\pi a^2}{2}$

5 $r = a \sin 2\theta$, 4-leaved rose

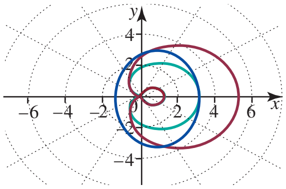
6 **a** Varying n changes the number of leaves and varying a adjusts the radius of the circle which circumscribes the leaves.



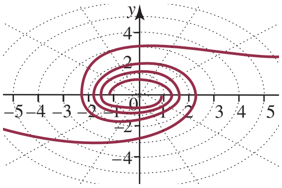
b Varying a alters the starting point of the strophoid. It also alters the size of the loops.



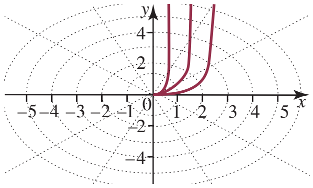
c For values of $a < b$, a determines where the nephroid cuts the y-intercept. The value of b is the length of a semi-secant which passes across the nephroid. For negative values of a, the nephroid is reflected in the y-axis.



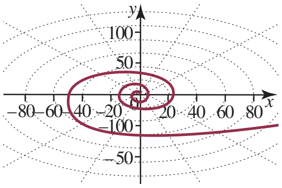
d The Lituus family of curves is spiral-like. As the value of the constant a changes so does the compactness of the spiralling at the origin change.



e The cissoid family converge towards the line parallel to the y-axis a units from the origin. If a is positive, the branches converge to the right and to the left if a is negative.



f The logarithmic spiral changes according to the value assigned to the constant a. For smallish values, the spirals can be seen easily, for large positive and negative values of a the spirals are difficult to observe. Changing the base would add another interesting variation to the spiral pattern.



Chapter review

1 B **2** C **3** B **4** A **7** D

9 **a** $y = -x^2 + 8x - 8, y = x + 2$

b Yes at (2, 4) and (5, 7)

c The particles collide at $t = -2$ and $t = 1$; that is, at the points (2, 4) and (5, 7).

10 **a** $x^2 + y^2 = 50$

b **i** $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **ii** $\frac{x^2}{18} + \frac{y^2}{32} = 1$

11 C **12** E **13** A **14** D **15** B

Modelling and problem solving

1 Answers will vary.

2 $x - y - 3a = 0, 2x - y - 12a = 0, 3x + y - 33a = 0$

3 Answers will vary.
4 Answers will vary.
5 Answers will vary.
6 Answers will vary.
7 Answers will vary.
8 (3, –2)
9 a $9xx_1 - 16yy_1 = 144$
10 $x = 8t - 2\sin 2t,\ y = 4 - 2\cos 2t$

CHAPTER 11 Dynamics I

Exercise 11A — Displacement, velocity and acceleration

1 a $\underset{\sim}{v}(t) = 6\underset{\sim}{i} + (3 - 14t)\underset{\sim}{j},\ \underset{\sim}{a}(t) = -14\underset{\sim}{j}$
b $\underset{\sim}{r}(0) = \underset{\sim}{0}$ m, $v(0) = 3\sqrt{5}$ m/s
c i $\underset{\sim}{r}(2) = 12\underset{\sim}{i} - 22\underset{\sim}{j}$ m
ii 25.7 m/s downwards at an angle of 13°30′ to the vertical
iii 15°7′
d Yes, 14 m/s^2 downwards
2 a, b Check with your teacher.
c $y = \frac{1}{25}(1225 - x^2)$
3 a i $\underset{\sim}{v}(t) = 2\underset{\sim}{i} - \underset{\sim}{j},\ \underset{\sim}{a}(t) = \underset{\sim}{0}$
ii $\underset{\sim}{r}(0) = 5\underset{\sim}{j}$ m, $v(0) = \sqrt{5}$ m/s
iii $y = 5 - \frac{1}{2}x$
b i $\underset{\sim}{v}(t) = \underset{\sim}{i} + (6 - 2t)\underset{\sim}{j},\ \underset{\sim}{a}(t) = -2\underset{\sim}{j}$
ii $\underset{\sim}{r}(0) = \underset{\sim}{0}$ m, $v(0) = \sqrt{37}$ m/s
iii $y = 6x - x^2$
c i $\underset{\sim}{v}(t) = -\underset{\sim}{i} - 6t\underset{\sim}{j},\ \underset{\sim}{a}(t) = -6\underset{\sim}{j}$
ii $\underset{\sim}{r}(0) = 10\underset{\sim}{j}$ m, $v(0) = 1$ m/s
iii $y = 10 - 3x^2$
d i $\underset{\sim}{v}(t) = 3\underset{\sim}{i} + (5 - 4t)\underset{\sim}{j},\ \underset{\sim}{a}(t) = -4\underset{\sim}{j}$
ii $\underset{\sim}{r}(0) = \underset{\sim}{0}$ m, $v(0) = \sqrt{34}$ m/s
iii $y = \frac{x}{9}(15 - 2x)$
e i $\underset{\sim}{v}(t) = \underset{\sim}{i} + 6t\underset{\sim}{j},\ \underset{\sim}{a}(t) = 6\underset{\sim}{j}$
ii $\underset{\sim}{r}(0) = 4\underset{\sim}{i}$ m, $v(0) = 1$ m/s
iii $y = 3(x - 4)^2$
f i $\underset{\sim}{v}(t) = 2t\underset{\sim}{i} - \underset{\sim}{j},\ \underset{\sim}{a}(t) = 2\underset{\sim}{i}$
ii $\underset{\sim}{r}(0) = \underset{\sim}{0}$ m, $v(0) = 1$ m/s
iii $y^2 = x$
g i $\underset{\sim}{v}(t) = 2t\underset{\sim}{i} - 4\underset{\sim}{j},\ \underset{\sim}{a}(t) = 2\underset{\sim}{i}$
ii $\underset{\sim}{r}(0) = \underset{\sim}{0}$ m, $v(0) = 4$ m/s
iii $y^2 = 16x$
h i $\underset{\sim}{v}(t) = \frac{1}{2\sqrt{t}}\underset{\sim}{i} - 6t\underset{\sim}{j},\ t \neq 0;$
$\underset{\sim}{a}(t) = -\frac{1}{4t\sqrt{t}}\underset{\sim}{i} - 6\underset{\sim}{j},\ t \neq 0$
ii $\underset{\sim}{r}(0) = \underset{\sim}{0}$ m, $v(0)$ is undefined
iii $y = -3x^4$
4 a $t = 5$ seconds **b** (–3, 18)
c $y_A = (x - 1)^2 + 2,\ y_B = (x + 8.5)^2 - 12.25$
5 a $\underset{\sim}{v}(t) = 2\underset{\sim}{i} + 2(1 - t)\underset{\sim}{j},\ \underset{\sim}{a}(t) = -2\underset{\sim}{j}$
b $2\sqrt{2}$ m/s upwards at an angle of 45° to the vertical
c i $\sqrt{37}$ m
ii $2\sqrt{5}$ m/s downwards at an angle of 26°34′ to the vertical
iii 26°34′
6 a, b Check with your teacher. **c** $t = 5$ seconds
7 a P and Q do not collide. **c** $y = \frac{12}{x} - 1$
8 a Check with your teacher. **b** $-\frac{1}{5}\underset{\sim}{i} + 3\underset{\sim}{j}$ m/s
c $\underset{\sim}{v}(5) = -\frac{2}{25}\underset{\sim}{i} + 3\underset{\sim}{j},\ \underset{\sim}{a}(5) = \frac{4}{125}\underset{\sim}{i}$
d Approaches zero
9 a $2\underset{\sim}{i} + 3\underset{\sim}{j}$ **b** $\sqrt{145}$ m/s **c** 85°14′
d i $\sqrt{65}$ m/s downwards at an angle of 7°08′ to the vertical
ii $4\underset{\sim}{i} + 7\underset{\sim}{j}$
10 a $\underset{\sim}{r}_1(0) = \underset{\sim}{0},\ \underset{\sim}{r}_2(0) = 80\underset{\sim}{i} + a\underset{\sim}{j}$ **b** $a = 15$
c (50, 6.25) **d** 100°18′
11 a $b = 25,\ c = -5$ **b** $y = -\frac{1}{80}(x - 50)^2 + 26\frac{1}{4}$
c 81.4 m
12 d $R_{\max} = 76\frac{9}{16}$ m when $\alpha \approx 63°26'$

Exercise 11B — Projectile motion

1 a $\underset{\sim}{r}(t) = (t + 3)\underset{\sim}{i} + 2\underset{\sim}{j}$
b $\underset{\sim}{r}(t) = (t^2 + 1)\underset{\sim}{i} + (1 - 3t)\underset{\sim}{j}$
c $\underset{\sim}{r}(t) = 2(2t - 1)\underset{\sim}{i} + \frac{1}{2}(7t^2 - 9)\underset{\sim}{j}$
d $\underset{\sim}{r}(t) = (3t + \frac{1}{2}t - 7)\underset{\sim}{i} + (-2t + 9)\underset{\sim}{j}$
e $\underset{\sim}{r}(t) = (\frac{3}{2}t^2 - 1)\underset{\sim}{i} + (4t + \frac{1}{2}t^2 + 1)\underset{\sim}{j}$
f $\underset{\sim}{r}(t) = -\frac{1}{2}t(t - 4)\underset{\sim}{i} - \frac{1}{3}t(t^2 + 18)\underset{\sim}{j}$
g $\underset{\sim}{r}(t) = (15t)\underset{\sim}{i} + (10t - \frac{3}{2}t^2 + 20)\underset{\sim}{j}$
h $\underset{\sim}{r}(t) = (-t^2 + 3)\underset{\sim}{i} + (5t - \frac{1}{t} - 11)\underset{\sim}{j}$
2 a $\underset{\sim}{v}(t) = (2t + 1)\underset{\sim}{i} - 3\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (t^2 + t - 1)\underset{\sim}{i} + (-3t + 5)\underset{\sim}{j}$
b $\underset{\sim}{v}(t) = (3t + 2)\underset{\sim}{i} + (-5t - 1)\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (\frac{3}{2}t^2 + 2t)\underset{\sim}{i} + (-\frac{5}{2}t^2 - t)\underset{\sim}{j}$
c $\underset{\sim}{v}(t) = (-t + 3)\underset{\sim}{i} + (\frac{1}{2}t^2 - 4)\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (-\frac{1}{2}t^2 + 3t + 2)\underset{\sim}{i} + (\frac{1}{6}t^3 - 4t - 3)\underset{\sim}{j}$
d $\underset{\sim}{v}(t) = (t^2 - 1)\underset{\sim}{i} + (-3t - 1)\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (\frac{1}{3}t^3 - t + 1)\underset{\sim}{i} + (-\frac{3}{2}t^2 - t + 1)\underset{\sim}{j}$
e $\underset{\sim}{v}(t) = (-5t + \frac{15}{2})\underset{\sim}{i} + (\frac{1}{2}t^2 + t - \frac{13}{6})\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (-\frac{5}{2}t^2 + \frac{15}{2}t + 3)\underset{\sim}{i} + (\frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{13}{6}t + 4)\underset{\sim}{j}$
f $\underset{\sim}{v}(t) = (t + \frac{7}{2})\underset{\sim}{i} + (-\frac{1}{2}t^2 + 6t - \frac{65}{6})\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (\frac{1}{2}t^2 + \frac{7}{2}t - 5)\underset{\sim}{i} + (-\frac{1}{6}t^3 + 3t^2 - \frac{65}{6}t + 12)\underset{\sim}{j}$
g $\underset{\sim}{v}(t) = (t^2 + t - \frac{23}{6})\underset{\sim}{i} + (\frac{5}{2}t^2 - 3t + \frac{5}{3})\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (\frac{1}{3}t^3 + \frac{1}{2}t^2 - \frac{23}{6}t)\underset{\sim}{i} + (\frac{5}{6}t^3 - \frac{3}{2}t^2 + \frac{5}{3}t + 8)\underset{\sim}{j}$
h $\underset{\sim}{v}(t) = (3t - 6)\underset{\sim}{i} + (-\frac{5}{2}t^2 + 12t - \frac{35}{2})\underset{\sim}{j},$
$\underset{\sim}{r}(t) = (\frac{3}{2}t^2 - 6t - \frac{11}{2})\underset{\sim}{i} + (-\frac{5}{6}t^3 + 6t^2 - \frac{35}{2}t + \frac{116}{3})\underset{\sim}{j}$

3 **a** $\underset{\sim}{v}(t) = 12\underset{\sim}{i} + (15 - gt)\underset{\sim}{j}$
b 19.2 m/s at an angle of 51°20′ above the horizontal.
c 11.5 m 3.06 seconds
d 19.2 m/s downwards at an angle of 51°20′ to the horizontal

4 **a** $\underset{\sim}{v}(t) = 15\underset{\sim}{i} + (30 - gt)\underset{\sim}{j}$
b 33.5 m/s at an angle of 63°26′ above the horizontal
c 45.9 m
d No, as it falls short by about 40 cm.

5 **b** $\underset{\sim}{r}(t) = 18\sqrt{3}t\underset{\sim}{i} + (50 + 18t - \frac{1}{2}gt^2)\underset{\sim}{j}$
c Maximum height is 66.5 m which is 16.5 m above the launch platform.
d 47.7 m/s downwards at an angle of 49°12′ to the horizontal
e 172.1 m

6 **a** $\underset{\sim}{r}_A(t) = 4(t - 1)\underset{\sim}{i} + (3 + 5t - t^2)\underset{\sim}{j}$,
$\underset{\sim}{r}_B(t) = 3(8 - t)\underset{\sim}{i} + (6 - 7t + 2t^2)\underset{\sim}{j}$
b Particles don't collide but they do pass through two common points at different times. At (13.70, 5.55), one particle arrives when $t = 4.43$ s and the other when $t = 3.43$ s. At the point (18.20, –0.06), one particle arrives when $t = 5.55$ s and the other when $t = 1.93$ s.
c $y_A = 3 + \frac{1}{16}(x + 4)(16 - x)$,
$y_B = 6 + \frac{1}{9}(24 - x)(27 - 2x)$

7 **a** $\underset{\sim}{r}_\lambda(t) = (5t - 14)\underset{\sim}{i} + (32t - 45 - 4t^2)\underset{\sim}{j}$
b When $t = 5$ s, the particles do collide at (11, 15). Particles appear to collide at (9.15, 17.4) but actually pass through this point at different times; so they don't collide. Particle λ at $t = 4.63$ s and particle ξ at time $t = 3.15$ s.

8 **b** 0.618 when $\theta \approx 58°17'$
c 35.6 m/s

9 $\theta \approx 17°46'$ and $\theta \approx 77°56'$

10 21.429 m

11 100 m/s, 192 m

12 Yes, they meet at (38.4, 9.2) when $t = 2$ s.

13 20 m/s at an angle of approx. 36°52′

14 75.431 m, approx. 4.64 s

15 **b** $R_{max} = \frac{V^2}{(1+\sin\theta)g}$ when $\phi = 45° + \frac{\theta}{2}$

Exercise 11C — Motion under constant acceleration

1 **a** 4 m/s^2 **b** 32 m
2 **a** 6.5 m/s^2 **b** Approx. 1.54 s
3 **a** 80 m/s **b** Approx. 8.94 s
4 **a** 18 m/s **b** –0.6 m/s^2
5 **a** 58.8 m/s **b** 176.4 m
6 **a** 19.6 m **b** 4 s
7 **a** 2.83 s **b** 27.72 m/s
8 **a** 78.4 m **b** 39.2 m/s
9 **a** 90 m **b** 7.14 s **c** 42 m/s
10 21.08 m/s or $\frac{20\sqrt{10}}{3}$
11 **a** 2 m/s^2 **b** 144 m **c** 7.5 s
12 **a** 12.5 m/s **b** 4.8 s
13 **a** –5 m/s^2 **b** 22 m **c** 3.2 s **d** 25.6 m
14 12 m/s
15 **a** 24.5 m **b** 17.5 s
16 **a** 8.04 m/s **b** 1.1 m **c** 1.77 s
17 **a** 16.1 m/s **b** 27.23 m
18 **a** **i** –19.6 m/s **ii** –27.7 m/s **iii** $t = \sqrt{8}$ s
b **i** –20.2 m/s **ii** –28.15 m/s **iii** $t = 2.4$ s
19 **a** 78.4 m **b** 2.53 s longer
20 0.1044 s
21 4 s, 78.4 m
22 5.83 s
23 195 m
24 8.9 s

Chapter review

1 B **2** A **3** D **4** B
5 C

6 **a** $\underset{\sim}{v}(t) = 6\underset{\sim}{i} + (12 - 6t)\underset{\sim}{j}$ m/s, $\underset{\sim}{a}(t) = -6\underset{\sim}{j}$ m/s^2
b Origin, $6\sqrt{5}$ m/s
c **i** $12\sqrt{2}$ m **ii** 6 m/s to the right **iii** 45°

7 **a** 5 seconds **b** (5, –5)
c $y_A = 4x - x^2$, $y_B = x - 10$

8 **c** $y = 25 - (x - 1)^2$

9 D **10** C **11** A

12 $\underset{\sim}{v}(t) = t\underset{\sim}{i} + (3t - t^2)\underset{\sim}{j}$ m/s,
$\underset{\sim}{r}(t) = \frac{1}{2}t^2\underset{\sim}{i} + \frac{t^2}{6}(9 - 2t)\underset{\sim}{j}$ m

13 **a** $\underset{\sim}{v}(t) = 6\underset{\sim}{i} + (45 - gt)\underset{\sim}{j}$ m/s
b $3\sqrt{229}$ m/s at an angle of elevation of 82°24′
c $101\frac{1}{4}$ m, 9 s
d $3\sqrt{229}$ m/s downwards at an angle of 7°36′ to the vertical

14 **b** $\underset{\sim}{r}(t) = 12t\sqrt{3}\underset{\sim}{i} + (60 + 12t - \frac{1}{2}gt^2)\underset{\sim}{j}$
c Clears by about 5 cm

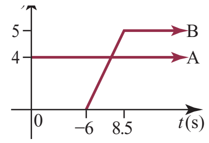
15 **b** Expression is positive for $26°34' \leq \theta < 90°$. Maximum is 0.618 when $\theta \approx 58°17'$.
c 30.845 m/s

16 **a** 30.625 m **b** $t \approx 1$ s and 4 s

17 E **18** C **19** B

20 **a** 2 min 14 s **b** 16.2 s

21 **a**



b 36.25 s
c 145 m
d 16 m

Modelling and problem solving

1 **a** Car A **b** 8.4 m
c **i** –2 m/s^2 **ii** 3.58 m/s^2
d **i** 23.62 m/s **ii** 28.33 m/s

2 **a** $\frac{x}{s} = \frac{1}{4}$ **b** $x = \frac{u\Delta t}{2} + \frac{1}{8}a(\Delta t)^2 \quad s = \frac{1}{2}a(\Delta t)^2 + u\Delta t$
c When $a = 0$

3 **a** $k = 4.96 \times 10^{12}$
b $v(r) = \sqrt{2\left(\frac{4.96 \times 10^{12}}{r} - 2.69 \times 10^6\right)}$
c 1.84×10^6 m

CHAPTER 12 Dynamics II

Exercise 12A — Position, velocity and acceleration

1 a i $3t^2 + 4t - 5$ **ii** $6t + 4$

b i $4 - \dfrac{1}{2\sqrt{t+2}}$ **ii** $\dfrac{1}{4(t+2)^{\frac{3}{2}}}$

c i $e^t - 2e^{-t}$ **ii** $e^t + 2e^{-t}$

d i $2t - \dfrac{1}{t+3}$ **ii** $2 + \dfrac{1}{(t+3)^2}$

e i $\dfrac{1}{\sqrt{9-t^2}}$ **ii** $\dfrac{t}{(9-t^2)^{\frac{3}{2}}}$

f i $10 + 6t - 3t^2$ **ii** $6 - 6t$

2 a $t = 1$, 2 and 3 **b** $v = 3t^2 - 12t + 11$

c $t = \dfrac{6-\sqrt{3}}{3}$ and $\dfrac{6+\sqrt{3}}{3}$ **d** −1 cm/s

e 6 cm

3 a Min. $x = 1$, max. $x = 5$

b $t = \frac{1}{3}$ **c** $t = \dfrac{\pi}{3}$ **d** $a = -9x + 27$

4 a $x = 2t^3 - t^2 - 8t + 2$ **b** $x = 12t - 2$

c $t = \frac{4}{3}$ **d** 7 m

e 14 m/s^2

5 a $x = \frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t - 1$ **b** $\ddot{x} = 4t - 9$

c $t = \frac{1}{2}, t = 4$ **d** 7 cm

e 3.5 cm/s **f** $\frac{2}{3}$ cm/s

6 a $v = 6t - 6t^2$ **b** $x = 3t^2 - 2t^3 + 4$

c $t = 0, x = 4$ and $t = 1, x = 5$ **d** 29 m

7 a $v = 2e^{2t} - t^2 + 3t$

b $x = e^{2t} - \frac{1}{3}t^3 + \frac{3}{2}t^2 - 1$

c 406.93 cm

8 a E **b** B

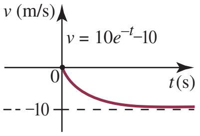
9 C

10 a 0 m/s **b** $x = \frac{1}{2}\log_e(t^2 + 1)$

c $a = \dfrac{1-t^2}{(t^2+1)^2}$ **d** 0.5 m/s

e $-\frac{1}{8}$ m/s^2

11 a $x = 10e^{-t} - 10$ **b**



c 11 s

12 a $x = 4\ \mathrm{Tan}^{-1}\ t + 1$ **b** $v = 4$ m/s, $t = 0$

c $a = \dfrac{-8t}{(1+t^2)^2}$ **d** −2 m/s^2

13 a $v = -2\cos 2t$ **b** $x = -\sin 2t$

c $a = -4x$ **d** $a = 2\sqrt{4 - v^2}$

Exercise 12B — Motion under constant acceleration

1

	v_0	v_t	a	x_t	t
a	0	10	2	25	5
b	30	45	2	281.25	7.5
c	−6	8	0.70	20	20
d	18.65	18	−1	12	0.65
e	4	10	1.5	28	4
f i	50	48.54	−3	24	0.49
f ii	50	−48.54	−3	24	32.85

2 a B **b** D **c** A

3 a 4 m/s **b** 50 m

4 a −1.6 m/s^2 **b** 32 m/s

5 a 8.16 s **b** 97.98 m/s

6 a ±53.67 m/s **b** 12.6 s or 0.7 s

7 a 78.4 m **b** 8 s

8 a 3.2 s **b** 31.3 m/s

9 a 78.4 m **b** −39.2 m/s

Exercise 12C — Applying differential equations to rectilinear motion

1 a $x = 3 + \sqrt{t+1}$ **b** $x = 7$ **c** 3 cm

2 a $x = \frac{1}{2}e^{2t} - \frac{1}{2}$ **b** $v = e^{2t}$ **c** 2 cm/s^2

3 a $x = e^{\sqrt{t+1}}$ **b** $v = \dfrac{e^{\sqrt{t+1}}}{2\sqrt{t+1}}$

c $a = \dfrac{e^{\sqrt{t+1}}\,(\sqrt{t+1} - 1)}{4(t+1)^{\frac{3}{2}}}$

4 2 m/s^2

5 a D **b** C **c** E

6 a $a = -16x$ **b** $x = \sin\left(4t + \dfrac{\pi}{2}\right)$
or $x = \cos(4t)$

7 a $a = 8x + 2x^3$ **b** $x = 2\tan\left(2t + \dfrac{\pi}{4}\right)$

c 16 cm/s

8 a $v = 2e^t - 2$ **b** $a = 2e^t$ **c** $x = 2e^t - 2t - 2$

9 a $v = 4 - e^{2-t}$ **b** e^{-1}

10 a $v = \dfrac{30(e^t - 1)}{e^t + 1}$ **b** $a = \dfrac{60e^t}{(e^t+1)^2}$

c $x = 60\log_e\left(\dfrac{e^t + 1}{2}\right) - 30t$

11 a The velocity is always increasing since $a > 0$. Since initially the velocity is zero and the velocity is always increasing, the velocity is always positive.

b $v = \sqrt{\dfrac{x}{2-x}}$ **c** $x = 1.8$

12 a $x = -1$

b Whenever $x > -1$, the acceleration is positive, which means the velocity is increasing.

c $v = x + 1$ **d** 1 m/s

13 3 cm/s

14 a Answers will vary. **b** $v = 5g(1 - e^{-\frac{t}{5}})$

c $5g$ m/s **d** 90 m

15 a Answers will vary. **b** $v = \sqrt{\dfrac{(g + ku^2)e^{-2kx} - g}{k}}$

c $\frac{1}{2k}\log_e\left(\dfrac{g + ku^2}{g}\right)$ **d** Approx. 42.2 m

Investigation — The falling ball bearing

a $x = \dfrac{1}{k^2}\left[g \log\left(\dfrac{g}{g - kv}\right) - kv\right]$

b $v(t) = \dfrac{g}{k}(1 - e^{-kt})$

c $x(t) = \dfrac{g}{k}\left(\dfrac{tk + e^{-kt} - 1}{k}\right)$

f Terminal velocity = $\dfrac{g}{k}$ which is the equation of the horizontal asymptote for curves $v(t)$ plotted in part **e** and the gradient of the oblique asymptote for curves $x(t)$ plotted in part **d**.

g The graph is $v(t) = 0.1(1 - e^{-kt})$, $t \approx 7.1 \times 10^{-3}$ s, $x \approx 0.2$ millimetres. The ball bearings achieve terminal velocity almost immediately.

Exercise 12D — Vector calculus

1 a 5 **b** $\sqrt{2426}$ **c** 1

d $\sqrt{5 + (\log_e 11)^2} \approx 3.28$

2 a $2\sqrt{2}\underset{\sim}{j}$ **b** $\underset{\sim}{i} - 16\underset{\sim}{j}$ **c** 0

d $2\underset{\sim}{i} - (\log_e 11)\underset{\sim}{j} - 2\underset{\sim}{k}$

3 a $-2\underset{\sim}{j}$ **b** $2t\underset{\sim}{i} - 2\underset{\sim}{j} + 3t^2\underset{\sim}{k}$

c $3\cos 3t\underset{\sim}{i} + 8\sin(-2t)\underset{\sim}{j}$ **d** $-8e^{-2t}\underset{\sim}{i} + 2t^{-2}\underset{\sim}{j}$

e $(t - 1)^{-1}\underset{\sim}{i} + (9t^2 + 2)\underset{\sim}{j} + 2t\underset{\sim}{k}$

f $(t^2 \cos t + 2t \sin t)\underset{\sim}{i} + (t + 1)e^t\underset{\sim}{j}$

g $-6\sin 2t\underset{\sim}{i} + 6\cos 2t\underset{\sim}{j}$

4 a i $2\sqrt{t^2 + 1}$ **ii** $2\sqrt{5}$ **iii** $2\underset{\sim}{i} - 2\underset{\sim}{j}$

b i $\sqrt{9t^4 - 2t^2 + 5}$ **ii** $\sqrt{141}$ **iii** $3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

c i $\sqrt{16 + \left(\dfrac{3}{3t + 1}\right)^2}$ **ii** 4.02 **iii** $0.973\underset{\sim}{i} - 4\underset{\sim}{j}$

d i 4 **ii** 4 **iii** $-0.573\underset{\sim}{i} - 0.495\underset{\sim}{j}$

5 a 0 **b** $2\underset{\sim}{i}$

c $-9\sin 3t\underset{\sim}{i} - 16\cos 2t\underset{\sim}{j} - 6t\underset{\sim}{k}$

d $16e^{-2t}\underset{\sim}{i} - 4t^{-3}\underset{\sim}{j}$ **e** $-(t - 1)^{-2}\underset{\sim}{i} + 18t\underset{\sim}{j} - 2t^{-3}\underset{\sim}{k}$

f $-\sin t\underset{\sim}{i} + (te^t + 2e^t)\underset{\sim}{j}$

6 a C **b** B **c** C

7 a $6t\underset{\sim}{i} + 3t^2\underset{\sim}{j} - 6\underset{\sim}{k}$ **b** $6\underset{\sim}{i} + 6t\underset{\sim}{j}$

c 45° **d** $t = 0$

8 a E **b** A

9 a $24\underset{\sim}{i} + \frac{2}{3}\underset{\sim}{j}$; 24.01 **b** $(3t^2 - 1)\underset{\sim}{i} - 2t^{-2}\underset{\sim}{j}$

c 11.01 **d** $t = \dfrac{1}{\sqrt{3}}$

10 a $3\underset{\sim}{j}$; 3 **b** $3\pi \cos 3\pi t\underset{\sim}{i} + \underset{\sim}{j}$

c $\sqrt{9\pi^2 + 1} = 9.48$ **d** $\sqrt{9\pi^2 + 1} = 9.48$

11 a $\frac{1}{3}\underset{\sim}{i} + 2\underset{\sim}{j}$; 2.03 **b** $\dfrac{-1}{(t + 1)^2}\underset{\sim}{i} + \underset{\sim}{j}$

c 1.01 **d** Various answers

12 a Various answers **b** 12 **c** $\dfrac{\pi}{4}$ **d** 90°

13 a 37.7 m/s **b** 710.6 m/s^2

14 a 0.5 **b** 12.5 m/s^2

Exercise 12E — Newton's Second Law of Motion

1 a -2.5 m/s^2 **b** 25 m **c** 11.25 m/s

2 a The resultant force acts in a direction opposite to the car's motion during that time. Without further information, nothing can be said about the magnitude of the force.

b The resultant force acts down the slope, in the direction of the skier's motion. Further information is needed to calculate the magnitude of the resultant force acting on the skier.

3

Force (N)	Mass (kg)	Acceleration (m/s^2)
10	2	**5**
5	**0.50**	10
2	2	1

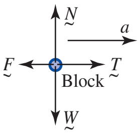
4 a i 1.6 m/s^2 **ii** 3.2 m/s **iii** 3.2 m

b i 3.2 m/s^2 **ii** 6.4 m/s **iii** 6.4 m

c i 4.8 m/s^2 **ii** 9.6 m/s **iii** 9.6 m

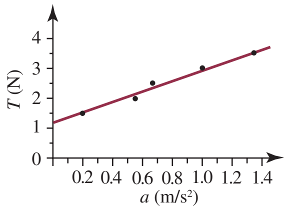
d i 6.4 m/s^2 **ii** 12.8 m/s **iii** 12.8 m

5 a



b Newton's second law states that the sum of the forces acting on a particle is equal to the product of its mass and its acceleration. The frictional force acts in the opposite direction to the applied force and the acceleration is in the same direction as the applied force. Therefore, $T - F = ma$.

c



d i The intercept

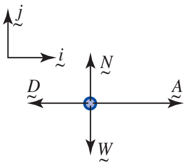
ii 1.2 N

e i The gradient

ii 1.8 kg

6 **a** B **b** E **c** D **d** C

7 **a**



b $\underset{\sim}{R} = (A - D)\underset{\sim}{i} + (N - W)\underset{\sim}{j}$

$= ma_x\underset{\sim}{i} + ma_y\underset{\sim}{j}$

c $\underset{\sim}{a} = 7.33\underset{\sim}{i} + 0\underset{\sim}{j}$

d $\underset{\sim}{N} = \underset{\sim}{W} = 588$ N

e 440 N

8 **a** 2.4 s **b** −9.3 m/s^2 **c** 11 111 N

9 **a** 5.0×10^{-5} s **b** -2.4×10^{7} m/s^2 **c** 7.2×10^{5} N

10 10.64 kg

11 **a** **i** $6\underset{\sim}{i} + 3\underset{\sim}{j}$ N **ii** $3\sqrt{5}$ N

iii $3.0\underset{\sim}{i} + 1.5\underset{\sim}{j}$ m/s^2

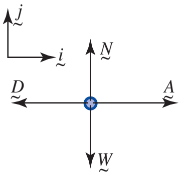
b **i** 339.4 N at N20.4°W **ii** 339.4 N

iii 4.53 m/s^2 at N20.4°W

c **i** $-4.3\underset{\sim}{i} - 22.1\underset{\sim}{j}$ N **ii** 22.5 N

iii $-1.8\underset{\sim}{i} - 9.2\underset{\sim}{j}$ m/s^2

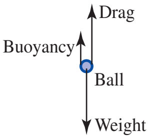
12 **a**



b $\underset{\sim}{R} = 388.9\underset{\sim}{i}$

c 588.9 N

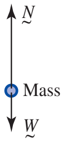
13 **a**



b **i** 0 N, 0.078 N

ii 4.5×10^{-4} N, 0.077 N

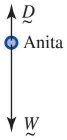
14 **a**



b 29.4 N upward

c 19.6 m/s^2

15 **a**



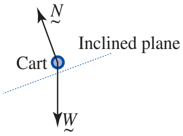
b 408 N

c 588 N

d 180 N

e 588 N upward

16 **a**



b 3.0 N acting down the plane

c 11.4 N acting into the plane

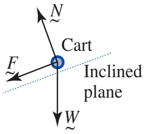
d 3.0 N acting down the plane

e The component of the weight force perpendicular to the plane is balanced by the normal force. The resultant force is therefore the component of the weight force acting parallel to the plane.

f −2.5 m/s^2

g 1.8 m

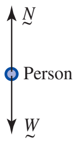
17 **a**



b 3.07 m/s^2

c 1.47 m

18 **a**



b $N = m(g + a)$ where $m = 80$ kg and N is the normal contact force.

c Various answers

d 160 N

e 784 N; 944 N

f The magnitude of the normal contact force is larger than the magnitude of the weight force because the lift is accelerating upwards.

g −160 N

h 784 N; 624 N

i The magnitude of the normal contact force is smaller than the magnitude of the weight force because the lift is accelerating downwards.

j The person will have a normal contact force equal to twice their weight when $\underset{\sim}{a} = 9.8\underset{\sim}{j}$ m/s^2.

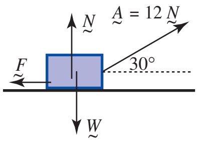
k The person will have a zero normal contact force when $\underset{\sim}{a} = -9.8\underset{\sim}{j}$ m/s^2.

Exercise 12F — Applications of Newton's Laws of Motion

1 **a** 19.6 N **b** 10 N **c** 0.51

2 7.84 N

3 **a**



b 13.6 N

c 10.4 N

d 0.76

4 **a** 0.082 **b** 10 m

5 **a** A **b** D

6 **a** B **b** E

7 **a** **i** 32 N acting upwards, perpendicular to the plane

ii 3.2 N acting up the plane

iii 2.7 m/s^2 down the plane

b **i** 26 N acting upwards, perpendicular to the plane

ii 2.6 N acting up the plane

iii 5.6 m/s^2 down the plane

c **i** 32 N acting upwards, perpendicular to the plane

ii 6.4 N acting up the plane

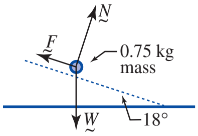
iii 1.5 m/s^2 down the plane

d **i** 64 N acting upwards, perpendicular to the plane

ii 6.4 N acting up the plane

iii 2.4 m/s^2 down the plane

8 **a**



b 2.27 N **c** 6.99 N

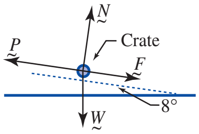
d 2.27 N **e** 0.32

f 6.37 N **g** 3.68 N

h 2.04 N **i** 2.2 m/s^2

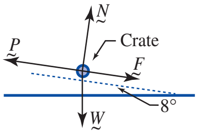
9 6.9 N

10 **a**



b Assuming that Sally applies her force parallel to the plane, she must apply a force of 70 N.

11 **a**

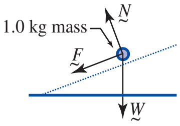


b 85 N

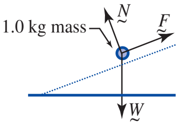
12 82 N

13 6.2 m

14 **a**



Moving up the plane



Moving down the plane

b 0.45 m
c 1.2 s
d 0.27

Investigation — Simple harmonic motion

1 Period = $\dfrac{2\pi}{n}$, amplitude = a

2 $\ddot{x} = -n^2 a \sin nt$

3 **i** $t = 0, \dfrac{\pi}{n}, \dfrac{2\pi}{n}, \dfrac{3\pi}{n}$ . . . etc. and $x = 0$

ii $t = \dfrac{\pi}{2n}, \dfrac{3\pi}{2n}, \dfrac{5\pi}{2n}$ . . . etc. and $x = \pm a$

4 Various answers

5 **i** $v = na \cos nt$

ii na when $t = 0, \dfrac{\pi}{n}, \dfrac{2\pi}{n}, \dfrac{3\pi}{n}$ . . . etc. and $x = 0$

6 $v^2 = n^2(a^2 - x^2)$

Exercise 12G — Variable forces

1 **a** $v = 2t^2 - t^3$ **b** $x = \frac{2}{3}t^3 - \dfrac{t^4}{4}$

2 **a** $v = \dfrac{t^2}{6} - \dfrac{2}{3}\sin\dfrac{t}{2}$ **b** $x = \dfrac{t^3}{18} + \dfrac{4}{3}\cos\dfrac{t}{2} - \dfrac{4}{3}$

3 **a** $2\frac{11}{12}$ m/s **b** $4\frac{11}{16}$ m

4 **a** $\dfrac{\pi + 2}{12}$ m/s **b** $\dfrac{\pi^2 + 2\pi}{48}$ m

5 **a** $t = 5$ s **b** $(20\log_e 2 - 10)$ or 3.86 m

6 **a** $2000 - 40t$ **b** **i** 9.5 m/s **ii** $24\frac{1}{6}$ m

7 3 m/s

8 $\dfrac{4\sqrt{3}}{3}$

9 **a** 5 m/s^2 **b** 20 m/s

10 $800(3 + 5e^{-0.2t})$

11 **a** $12\,500(9e^{-0.05t} + 1)$ **b** 44 s

12 **a** $60(1 - e^{-\frac{t}{3}})$ **b** 60 m/s **c** 204 m

Chapter review

1 E **2** C **3** B **4** D **5** A
6 A **7** –75 **8** $1\frac{1}{8}$ cm

9 **a** **i** $v = 3t^2 - 4t + 1$ **ii** $x = t^3 - 2t^2 + t$

b **i** $v = \dfrac{3}{2}t^2 + 4t + 6$ **ii** $x = \dfrac{1}{2}t^3 + 2t^2 + 6t + 0.8$

c **i** $v = 4e^{\frac{t}{2}} - 5$ **ii** $8e^{\frac{t}{2}} - 5t - 8$

d **i** $v = \dfrac{1}{\sqrt{4 - t^2}} + \dfrac{1}{2}$ **ii** $x = \text{Sin}^{-1}\dfrac{t}{2} + \dfrac{t}{2} - 2$

e **i** $v = \dfrac{3}{t + 1} - 3$ **ii** $a = 3\log_e(t + 1) - 3t + 5$

f **i** $v = \log_e\left(\dfrac{e^t + 1}{2}\right)$

10 **a** $20\frac{2}{3}$ m **b** $5\frac{1}{3}$ m **c** Down

11 **a** $x = \dfrac{3}{2(2t + 1)} + 3t - \dfrac{3}{2}$ **b** 4.8 cm

12 **a** 0.8 m/s^2 **b** 30 m **c** 17.5 s

13 **a** 6.8 s **b** 113.8 m

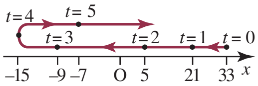
14 **a** 3.3 m **b** 5 s

15 17.6 m/s

16 C **17** E

18 **a** $t = 4, x = -15$ **b** $0 < t < 4$

c



19 **a** The acceleration is negative between $x = 0$ and $x = 4$, which means the velocity is decreasing between these positions.

b $v = 4$

20 $a = -1$

21 **a** $-\dfrac{5\pi}{4}$ **b** $10 + \dfrac{20}{\pi}$ or approx. 16.37

22 D

23 **a** $v = 3e^t - 3$ **b** $x = 3e^t - 3t - 3$

24 **a** $x = 2\cos\left(\dfrac{t}{3} - \dfrac{\pi}{2}\right)$ or $2\sin\dfrac{t}{3}$ **b** 2 m

25 D **26** C **27** A **28** E

29 18 m/s

30 C

31 **a** 1.5 m/s **b** 12 000 N **c** 108 m

32 D **33** D **34** C **35** A

36 **a** 0.364 **b** 1.82 m/s^2

37 B **38** C

39 $\sqrt{5}$

Modelling and problem solving

1 **a** 30 min **b** 305.6 m
c $v = 16$ m/min, $t = 15$ min

2 **a** 29.4 m/s **b** $v(t) = \dfrac{30}{12t + 1} - \dfrac{3}{5}$
c 7.08 s **d** 7.33 m

3 **a** **i** 7482 N **ii** 602 N
b 0.63 s **c** 12 people of mass 80 kg

4 **a** Various answers **b** $\beta = 0$

5 7°59′

6 **a** 168 N **b** 1106 N **c** 0.152
d 854 N
e Pull the roller. For the same applied force, the roller exerts less force on the ground and should be easier to move.

7 2.26 m

8 160.5 m

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