KAHNI BURROWS | SUE MICHELL | MILES FORD

JACARANDA MATHS QUEST
MATHEMATICAL METHODS 11
FOR QUEENSLAND

UNITS 1&2



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MATHEMATICAL METHODS 11

FOR QUEENSLAND

UNITS 1&2

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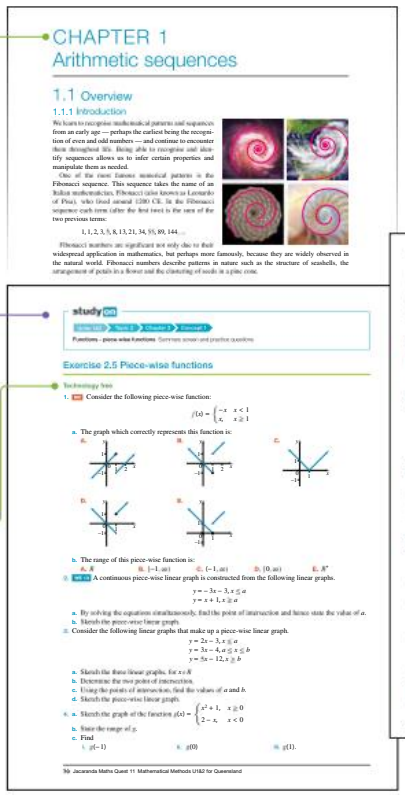
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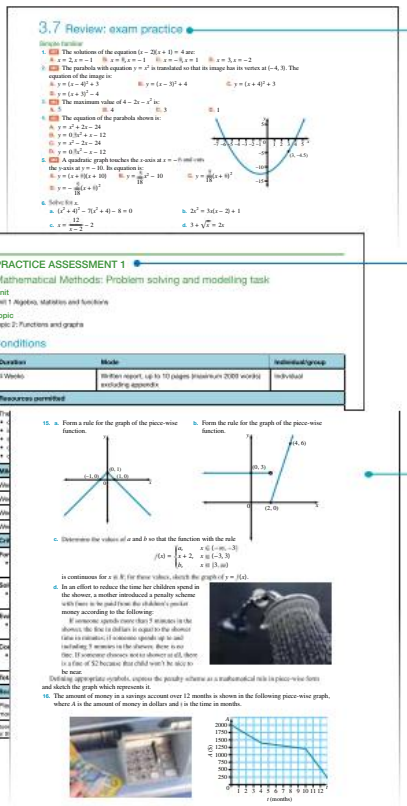
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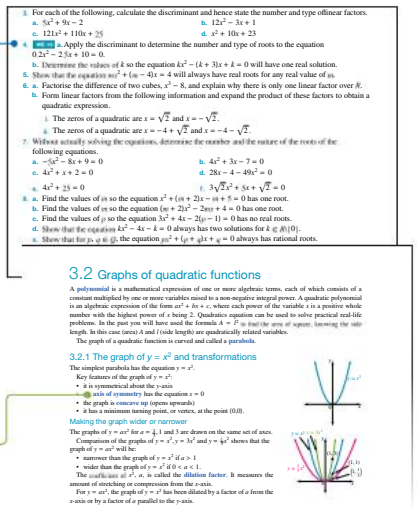
Chapter questions and activities are aligned with Marzano and Kendall's taxonomy of cognitive process — retrieval, comprehension, analysis and knowledge utilisation.

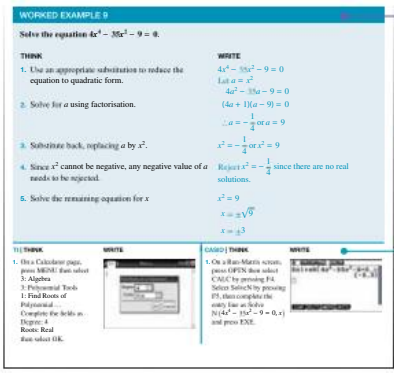
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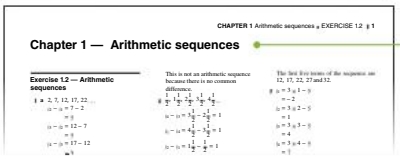
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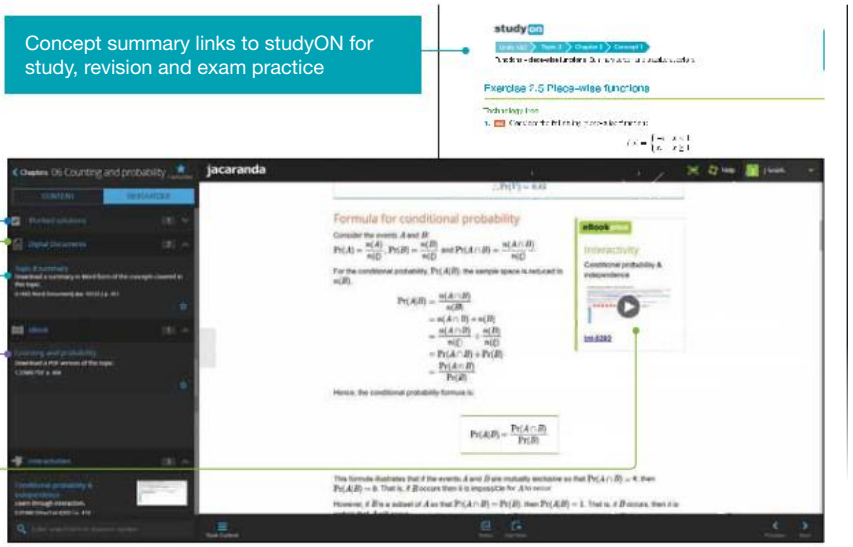
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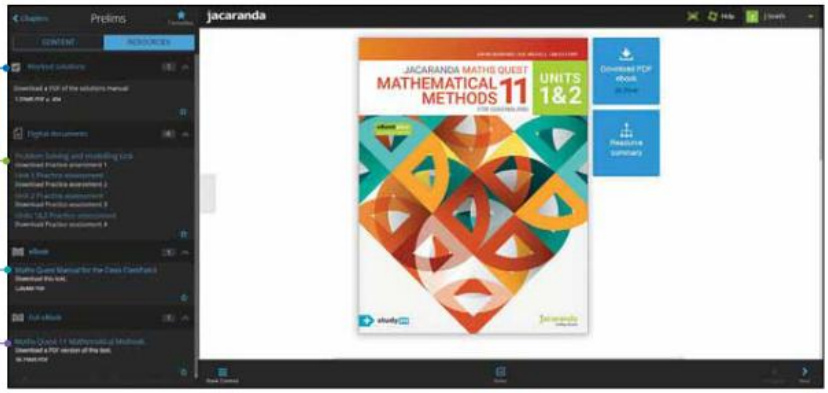


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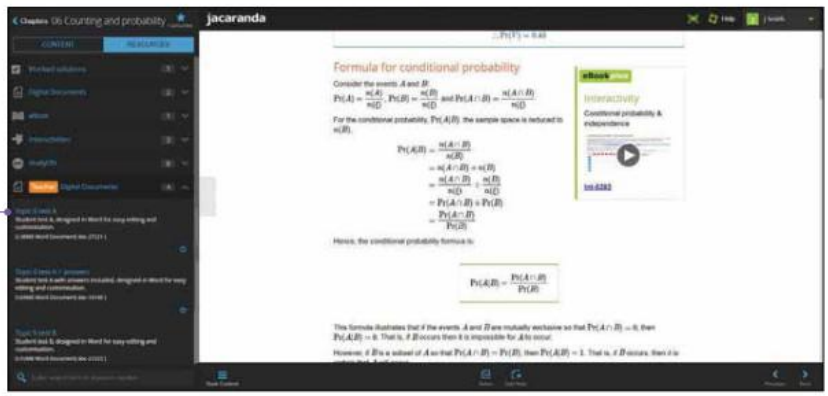
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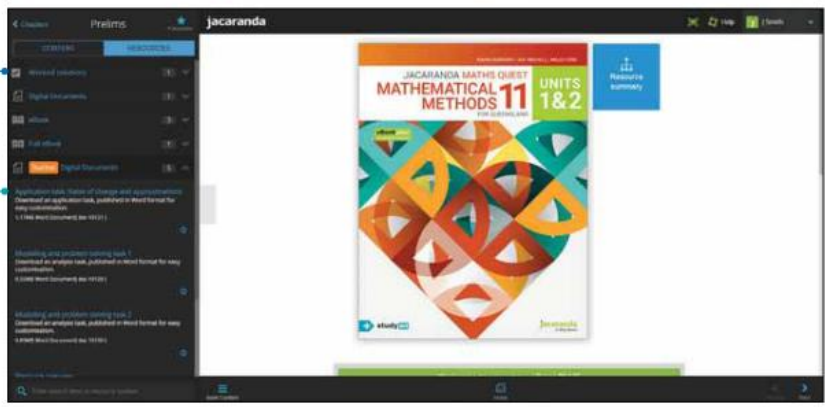
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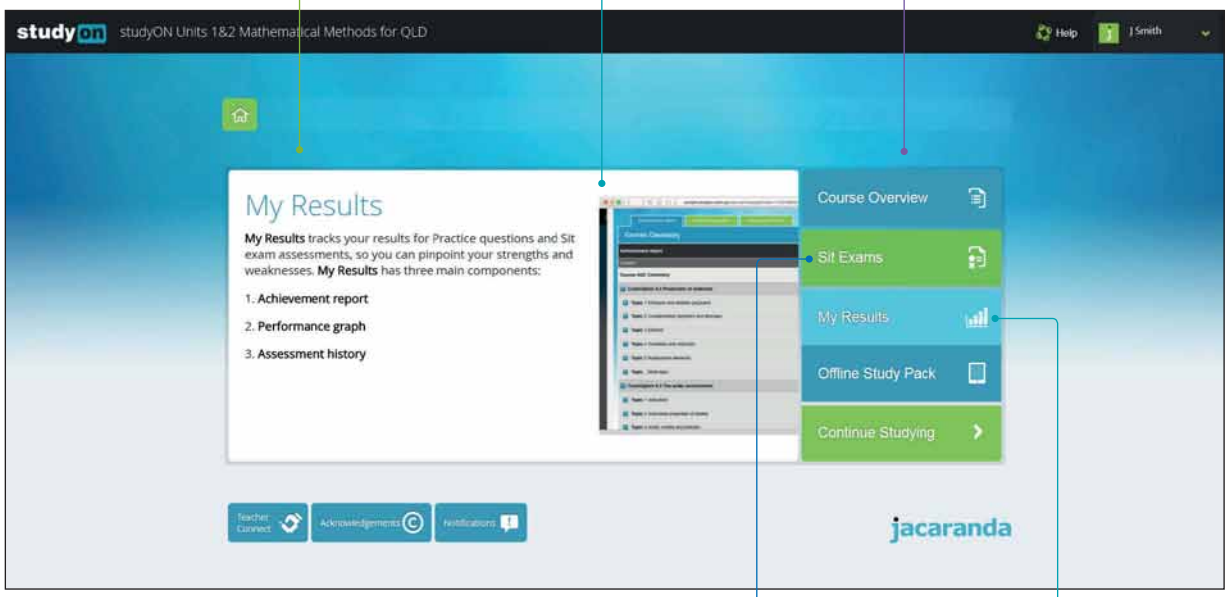
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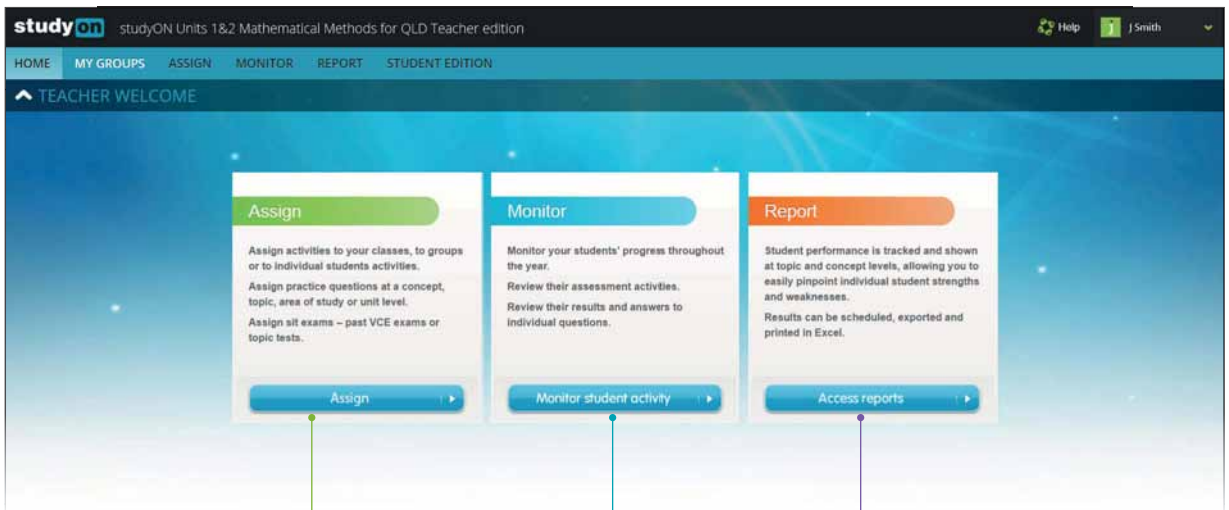


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CHAPTER 1
Arithmetic sequences

1.1 Overview

1.1.1 Introduction



We learn to recognise mathematical patterns and sequences from an early age — perhaps the earliest being the recognition of even and odd numbers — and continue to encounter them throughout life. Being able to recognise and identify sequences allows us to infer certain properties and manipulate them as needed.

One of the most famous numerical patterns is the Fibonacci sequence. This sequence takes the name of an Italian mathematician, Fibonacci (also known as Leonardo of Pisa), who lived around 1200 CE. In the Fibonacci sequence each term (after the first two) is the sum of the two previous terms:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Fibonacci numbers are significant not only due to their widespread application in mathematics, but perhaps more famously, because they are widely observed in the natural world. Fibonacci numbers describe patterns in nature such as the structure of seashells, the arrangement of petals in a flower and the clustering of seeds in a pine cone.

An interesting, if mysterious, aspect of the Fibonacci sequence is its significance in art and design. For centuries it has been recognised that some shapes are more appealing than others. The ratio of height to width (or width to height) of rectangles that appeal to the eye is $\frac{1+\sqrt{5}}{2} : 1$ (approximately $1.6 : 1$). This ratio is called the 'golden ratio'. As the number of terms in a Fibonacci sequence increases, we find that the ratio between two consecutive numbers approaches the golden ratio.

LEARNING SEQUENCE

1.1 Overview
1.2 Arithmetic sequences
1.3 The general form of an arithmetic sequence
1.4 The sum of an arithmetic sequence
1.5 Applications of arithmetic sequences
1.6 Review: exam practice

Fully worked solutions for this chapter are available in the resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Arithmetic sequences

1.2.1 Defining mathematical sequences

A **sequence** is a related set of objects or events that follow each other in a particular order. Sequences can be found in everyday life, with some examples being:

- the opening share price of a particular stock each day
- the daily minimum temperature readings in a particular city
- the lowest petrol prices each day
- the population of humans counted each year.

When data is collected in the order that the events occur, patterns often emerge. Some patterns can be complicated, whereas others are easy to define.



In mathematics, sequences are always ordered, and the links between different terms of sequences can be identified and expressed using mathematical equations.

You may already be familiar with some mathematical sequences, such as the multiples of whole numbers or the square numbers.

Multiples of 3: 3, 6, 9, 12, …

Multiples of 5: 5, 10, 15, 20, …

Square numbers: 1, 4, 9, 16, …

For each of these patterns there is a link between the numbers in the sequence (known as **terms**) and their position in the sequence (known as the **term number**).

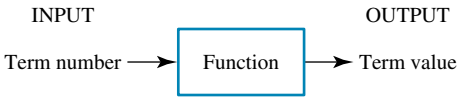
In general, mathematical sequences can be displayed as:

$$t_1,\ t_2,\ t_3,\ t_4,\ t_5 \ldots t_n$$

where t_1 is the first term, t_2 is the second term, and so on. The nth term is referred to as t_n where n represents the ordered position of the term in the sequence, for example 1st, 2nd, 3rd, …

1.2.2 Sequences expressed as functions

If we consider the term numbers in a sequence as the inputs of a **function** then the term values of that sequence are the outputs of that function.



If we are able to define a sequence as a function, then we can input term numbers into that function to determine any term value in the sequence.

WORKED EXAMPLE 1

Determine the first five terms of the sequence $t_n = 2n + 3$.

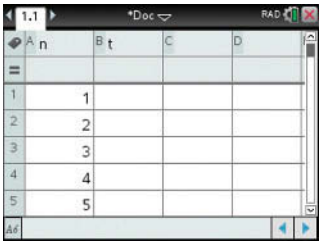
THINK	WRITE
1. Substitute $n = 1$ into the function.	$t_1 = 2 \times 1 + 3$ $= 5$
2. Substitute $n = 2$ into the function.	$t_2 = 2 \times 2 + 3$ $= 7$

3. Substitute $n = 3$ into the function.	$t_3 = 2 \times 3 + 3$ $= 9$
4. Substitute $n = 4$ into the function.	$t_4 = 2 \times 4 + 3$ $= 11$
5. Substitute $n = 5$ into the function.	$t_5 = 2 \times 5 + 3$ $= 13$
6. State the answer.	The first five terms of the sequence are 5, 7, 9, 11 and 13.

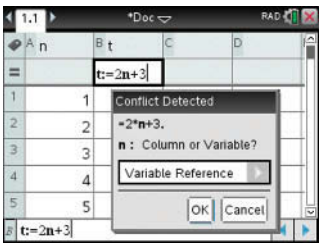
TI | THINK

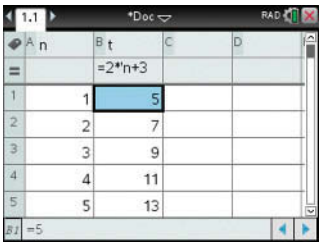
1. On a Lists & Spreadsheet page, label the first column as n and the second column as t. Enter the numbers 1 to 5 in the first column.

WRITE



2. In the function cell below the label t, complete the entry line as
$t := 2n + 3$
then press ENTER.
Select the variable reference for n when prompted.
Select OK.





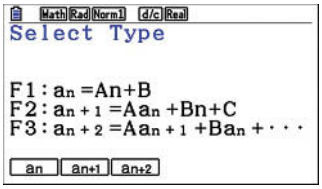
3. The first five terms can be read from the table.

The first five terms of the sequence are 5, 7, 9, 11 and 13.

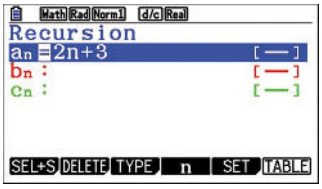
CASIO | THINK

1. On a Recursion screen, select TYPE by pressing F3, then select the first option by pressing F1.

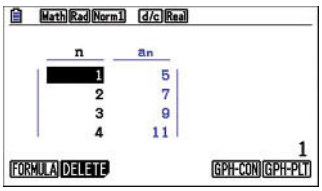
WRITE



Complete the entry line for a_n as
$a_n = 2n + 3$
then press EXE.



2. Select TABLE by pressing F6.



3. The first five terms can be read from the table.

The first five terms of the sequence are 5, 7, 9, 11 and 13.

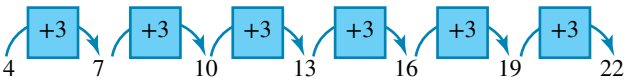
1.2.3 Arithmetic sequences

An **arithmetic sequence** is a sequence in which the difference between any two successive terms in the sequence is the same. The next term in an arithmetic sequence can be found by adding or subtracting a fixed value. This fixed value is known as the **common difference** (represented in sequence equations as d).

Consider the arithmetic sequence:

4, 7, 10, 13, 16, 19, 22.

The difference between each successive term is +3, or similarly, the next term is found by adding 3 to the previous term. We can see that a *positive common difference* gives a sequence that is *increasing*. We say that the common difference is +3, stated as $d = +3$.



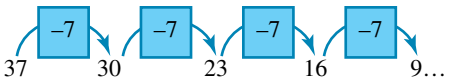
The first term of the sequence is 4. We refer to the first term of a sequence as 't_1'. So in this example, $t_1 = 4$. In the arithmetic sequence above, the first term is 4, the second term is 7, the third term is 10, and so on.

Another way of writing this is:

$$t_1 = 4,\ t_2 = 7 \text{ and } t_3 = 10.$$

There are 7 terms in this sequence. Because there are a countable number of terms in the sequence, it is referred to as a *finite sequence*.

The arithmetic sequence below



is an infinite sequence since it continues endlessly as indicated by the dots. The first term, t_1, is 37 and the common difference, d, is -7. We can see that a *negative common difference* gives a sequence that is *decreasing*.

Now consider the sequence 1, 3, 6, 10, 15. This is not an arithmetic sequence, as each term does not increase by the same constant value.

on Resources

Interactivity: Terms of an arithmetic sequence (int-6261)

WORKED EXAMPLE 2

Determine which of the following sequences are arithmetic sequences, and for those sequences which are arithmetic, state the values of t_1 and d.

a. **2, 5, 8, 11, 14, …**

b. **4, −1, −6, −11, −16, …**

c. **3, 5, 9, 17, 33, …**

THINK	WRITE
a. 1. Calculate the difference between consecutive terms of the sequence.	**a.** $t_2 - t_1 = 5 - 2$ $= 3$ $t_3 - t_2 = 8 - 5$ $= 3$ $t_4 - t_3 = 11 - 8$ $= 3$ $t_5 - t_4 = 14 - 11$ $= 3$
2. If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is t_1 and the common difference is d.	The common differences are constant, so the sequence is arithmetic. $t_1 = 2$ and $d = 3$

b. 1. Calculate the difference between consecutive terms of the sequence.

b. $t_2 - t_1 = -1 - 4$
$= -5$
$t_3 - t_2 = -6 - -1$
$= -6 + 1$
$= -5$
$t_4 - t_3 = -11 - -6$
$= -11 + 6$
$= -5$
$t_5 - t_4 = -16 - -11$
$= -16 + 11$
$= -5$

2. If the differences between consecutive terms are constant, then the sequence is arithmetic. The first term of the sequence is t_1 and the common difference is d.

The common differences are constant, so the sequence is arithmetic.
$t_1 = 4$ and $d = -5$

c. 1. Calculate the difference between consecutive terms of the sequence.

c. $t_2 - t_1 = 5 - 3$
$= 2$
$t_3 - t_2 = 9 - 5$
$= 4$
$t_4 - t_3 = 17 - 9$
$= 8$
$t_5 - t_4 = 33 - 17$
$= 16$

2. If the differences between consecutive terms are constant, then the sequence is arithmetic.

The common differences are not constant, so the sequence is not arithmetic.

1.2.4 The recursive definition of arithmetic sequences

Each term of an arithmetic sequence is dependent on the previous terms. We can represent this relationship using a **recursive function** of the form:

$$t_{n+1} = t_n + d$$

where d is the common difference.

WORKED EXAMPLE 3

Determine the recursive function for the following arithmetic sequences.

a. $-5, -1, 3, 7, 11$

b. $81, 63, 45, 27, \ldots$

THINK	WRITE
a. 1. Find d by calculating the difference between any two terms of the sequence. (Here we use t_1 and t_2, however, any pair of consecutive terms can be used.)	$d = t_2 - t_1$ $= (-1) - (-5)$ $= -1 + 5$ $= 4$
2. Recall the recursive function formula, then substitute d and state the answer.	$t_{n+1} = t_n + d$ The recursive function is $t_{n+1} = t_n + 4$.
b. 1. Find d by calculating the difference between any two terms of the sequence. (Here we use t_1 and t_2, however, any pair of consecutive terms can be used.)	$d = t_2 - t_1$ $= 63 - 81$ $= -18$
2. Recall the recursive function formula, then substitute d into the recursive function, and state the answer.	$t_{n+1} = t_n + d$ The recursive function is $t_{n+1} = t_n - 18$.

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Units 1 & 2 > Area 1 > Sequence 1 > Concept 1

Arithmetic sequences Summary screen and practice questions

Exercise 1.2 Arithmetic sequences

Technology free

1. State which of the following are arithmetic sequences.
   - **a.** 2, 7, 12, 17, 22, ...
   - **b.** 3, 7, 11, 15, 20, ...
   - **c.** 0, 100, 200, 300, 400, ...
   - **d.** −123, −23, 77, 177, 277, ...
   - **e.** 1, 0, −1, −3, −5, ...
   - **f.** 6.2, 9.3, 12.4, 15.5, 16.6, ...
   - **g.** $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}, \ldots$
   - **h.** $\frac{1}{4}, \frac{3}{4}, 1\frac{1}{4}, 1\frac{3}{4}, 2\frac{1}{4}, \ldots$
2. State which of the following situations are arithmetic sequences.



   - **a.** A teacher hands out 2 lollies to the first student, 4 lollies to the second student, 6 lollies to the third student and 8 lollies to the fourth student.
   - **b.** The sequence of numbers after rolling a die 8 times.
   - **c.** The number of layers of paper after each folding in half of a large sheet of paper.
   - **d.** The house numbers on the same side of a street on a newspaper delivery route.
   - **e.** The cumulative total of the number of seats in the first ten rows in a regular cinema (for example, with 8 seats in each row, so there are 8 seats after the first row, 16 seats after the first 2 rows, and so on).
3. Determine the next five terms in the sequence, $-7, 2, 11, 20, 29$.
4. **WE1** Determine the first five terms of the sequence $t_n = 5n + 7$.
5. Determine the first five terms of the sequence $t_n = 3n - 5$.

6. Determine the first five terms of the following arithmetic sequences.

a. $t_n = 5 + 3(n - 1)$ **b.** $t_n = -1 - 7(n - 1)$

c. $t_n = \frac{1}{3} + \frac{2}{3}(n - 1)$ **d.** $t_n = 3.3 - 0.7(n - 1)$

7. **WE2** Determine which of the following sequences are arithmetic sequences, and for those sequences that are arithmetic, state the values of t_1 and d.

a. 23, 68, 113, 158, 203, ... **b.** 3, 8, 23, 68, 203, ... **c.** $\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots$

8. Consider the following sequence:

$$-3.6, -2.1, -0.6, 0.9, 2.4, \ldots$$

a. Is this a finite or an infinite sequence?

b. Is the sequence increasing or decreasing?

c. State the values of t_1 and d for the sequence.

9. **WE3** Determine the recursive function for the following arithmetic sequences.

a. $-1, 3, 7, 11, 15, \ldots$ **b.** $1.5, -2, -5.5, -8, -11.5, \ldots$

c. $\frac{7}{2}, \frac{11}{2}, \frac{15}{2}, \frac{19}{2}, \frac{23}{2}, \ldots$ **d.** $6.2, 4.3, 2.4, 0.5, \ldots$

10. Find the missing values in the following arithmetic sequences.

a. $13, -12, -37, f, -87, \ldots$ **b.** $2.5, j, 8.9, 12.1, k, \ldots$

c. $p, q, r, \frac{9}{2}, \frac{25}{4}, \ldots$ **d.** $\frac{1}{2}, s, t, 2, \ldots$

11. For each value of t_1 and the corresponding recursive function, determine the first four terms of the following arithmetic sequences.

a. $t_1 = 3$, $t_{n+1} = t_n - 5$ **b.** $t_1 = -0.6$, $t_{n+1} = t_n + 1.4$

c. $t_1 = -23$, $t_{n+1} = t_n + 32$ **d.** $t_1 = 10$, $t_n = t_{n-1} - 3$

12. A batsman made 23 runs in his first innings, 33 in his second and 43 in his third. If he continued to add 10 runs each innings, write down a rule for the number of runs he would have made in his nth innings.



13. In a vineyard, rows of wire fences are built to support the vines. The length of the fence in row 1 is 40 m, the length of the fence in row 2 is 43 m, and the length of the fence in row 3 is 46 m. If the lengths of the fences continue in this pattern, write down a rule for the length of a fence in row number n.



14. The first fence post in a fence is 12 m from the road, the next is 15.5 m from the road and the next is 19 m from the road. The rest of the fence posts are spaced in this pattern.

a. Write down a rule for the distance of fence post n from the road.

b. If 100 posts are to be erected, how far will the last post be from the road?

1.3 The general form of an arithmetic sequence

1.3.1 The general term of an arithmetic sequence

Consider the recursive function for arithmetic sequences for values up to $n = 3$:

$$n = 1 : t_2 = t_1 + d$$
$$n = 2 : t_3 = t_2 + d$$
$$n = 3 : t_4 = t_3 + d$$

What happens if we substitute the equation for $n = 1$ into the equation for $n = 2$, and then the result of that into the equation for $n = 3$?

$$t_2 = t_1 + d$$

$$\begin{aligned} t_3 &= t_2 + d \quad \text{Substitute } t_2 = t_1 + d \\ &= (t_1 + d) + d \\ &= t_1 + 2d \end{aligned}$$

$$\begin{aligned} t_4 &= t_3 + d \quad \text{Substitute } t_3 = t_1 + 2d \\ &= (t_1 + 2d) + d \\ &= t_1 + 3d \end{aligned}$$

What pattern can we see when we write out t_2, t_3, t_4 in terms of t_1?

$$t_2 = t_1 + d$$
$$t_3 = t_1 + 2d$$
$$t_4 = t_1 + 3d$$

We can see that for each t_n on the left-hand side, the coefficient of d on the right-hand side is equal to $n - 1$. This allows us to generalise the following formula for the general term of an arithmetic sequence.

$$\boldsymbol{t_n = t_1 + (n - 1)d}$$

Given sufficient information, we can use this formula to determine any term in a sequence, the common difference, or the term number of a given value.

on Resources

Interactivity: Arithmetic sequence (int-6258)

WORKED EXAMPLE 4

Determine the equations that represent the following arithmetic sequences.

a. **3, 6, 9, 12, 15, …**

b. **40, 33, 26, 19, 12, …**

THINK	WRITE
a. 1. Determine the values of t_1 and d.	**a.** $t_1 = 3$ $d = t_2 - t_1$ $= 6 - 3$ $= 3$
2. Substitute the values for t_1 and d into the formula for arithmetic sequences.	$t_n = t_1 + (n-1)d$ $= 3 + (n-1) \times 3$ $= 3 + 3(n-1)$ $= 3 + 3n - 3$ $= 3n$
b. 1. Determine the values of t_1 and d.	**b.** $t_1 = 40$ $d = t_2 - t_1$ $= 33 - 40$ $= -7$
2. Substitute the values for t_1 and d into the formula for arithmetic sequences.	$t_n = t_1 + (n-1)d$ $= 40 + (n-1) \times -7$ $= 40 - 7(n-1)$ $= 40 - 7n + 7$ $= 47 - 7n$

TI | THINK

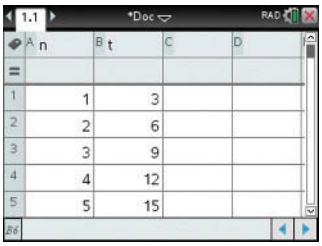
1. On a Lists & Spreadsheet page, label the first column as n and the second column as t. Enter the numbers 1 to 5 in the first column, and enter the given terms in the second column.

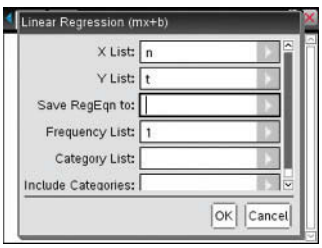
2. On a Calculator page, press MENU, then select:
6: Statistics
1: Stat Calculations
3: Linear Regression $(mx + b)$ ...
Complete the fields as:
X List: n
Y List: t
then select OK.

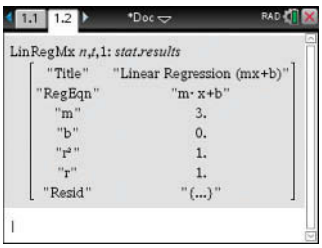
3. Interpret the output.

4. State the answer.

WRITE







The equation is given in the form $y = mx + b$
where
$y = t_n, m = 3, x = n$ and $b = 0$.

The equation is $t_n = 3n$.

CASIO | THINK

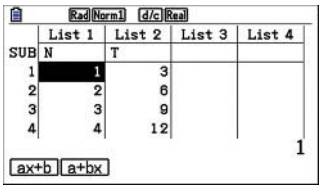
1. On a Statistics screen, label List 1 as N and List 2 as T.
Enter the numbers 1 to 5 in the first column, and enter the given terms in the second column.

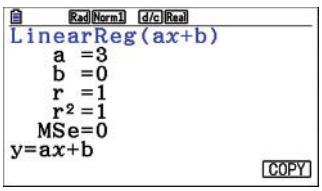
2. Select CALC by pressing F2, then select REG by pressing F3.
Select X by pressing F1, then select $ax + b$ by pressing F1.

3. Interpret the output.

4. State the answer.

WRITE





The equation is given in the form $y = ax + b$
where
$y = t_n, a = 3, x = n$ and $b = 0$.

The equation is $t_n = 3n$.

WORKED EXAMPLE 5

a. Find the 15th term of the sequence 2, 8, 14, 20, 26, …
b. Find the first term of the arithmetic sequence in which $t_{22} = 1008$ and $d = -8$.
c. Find the common difference of the arithmetic sequence which has a first term of 12 and an 11th term of 102.
d. An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?

THINK	WRITE
a. 1. As it has a common difference, this is an arithmetic sequence. State the known values.	**a.** $t_1 = 2, d = 6, n = 15$
2. Substitute the known values into the equation for an arithmetic sequence and solve.	$t_n = t_1 + (n-1)d$ $t_{15} = 2 + (15-1)6$ $= 2 + 14 \times 6$ $= 2 + 84$ $= 86$
3. State the answer.	The 15th term of the sequence is 86.
b. 1. State the known values of the arithmetic sequence.	**b.** $d = -8, n = 22, t_{22} = 1008$
2. Substitute the known values into the equation to determine the first term and solve.	$t_1 = t_n - (n-1)d$ $= 1008 - (22-1)(-8)$ $= 1008 - (21)(-8)$ $= 1008 - -168$ $= 1008 + 168$ $= 1176$
3. State the answer.	The first term of the sequence is 1176.
c. 1. State the known values of the arithmetic sequence.	**c.** $t_1 = 12, n = 11, t_{11} = 102$
2. Substitute the known values into the equation to determine the common difference and solve.	$d = \frac{t_n - t_1}{n-1}$ $= \frac{102 - 12}{11 - 1}$ $= \frac{90}{10}$ $= 9$
3. State the answer.	The common difference is 9.
d. 1. State the known values of the arithmetic sequence.	**d.** $t_1 = 40, d = 12, t_n = 196$
2. Substitute the known values into the equation to determine the term number and solve.	$n = \frac{t_n - t_1}{d} + 1$ $= \frac{196 - 40}{12} + 1$ $= 14$
3. State the answer.	The 14th term in the sequence has a value of 196.

1.3.2 Graphical display of sequences

We can plot graphs of mathematical sequences by treating the term number as our independent variable (x-values) and the term value as our dependent variable (y-values).

	t_1	t_2	t_3	t_4	t_5	...	t_n
Term number (x-values)	1	2	3	4	5	...	n
Term value (y-values)							

We will see in the following worked example that the graph will be a straight line, which is the case for all arithmetic sequences. The straight line indicates a linear relationship between the term number n and term value t_n. We can use this to find undetermined terms in the sequence.

WORKED EXAMPLE 6

An arithmetic sequence is given by the equation $t_n = 7 + 2(n - 1)$.

a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.

b. Plot the graph of the sequence.

c. Use your graph of the sequence to determine the 12th term of the sequence.

THINK

a. 1. Set up a table with the term number in the top row and the term value in the bottom row.

2. Substitute the first 5 values of n into the equation to determine the missing values.

3. Complete the table with the calculated values.

WRITE/DRAW

a.

Term number	1	2	3	4	5
Term value					

$t_1 = 7 + 2(1 - 1)$
$= 7 + 2 \times 0$
$= 7 + 0$
$= 7$

$t_2 = 7 + 2(2 - 1)$
$= 7 + 2 \times 1$
$= 7 + 2$
$= 9$

$t_3 = 7 + 2(3 - 1)$
$= 7 + 2 \times 2$
$= 7 + 4$
$= 11$

$t_4 = 7 + 2(4 - 1)$
$= 7 + 2 \times 3$
$= 7 + 6$
$= 13$

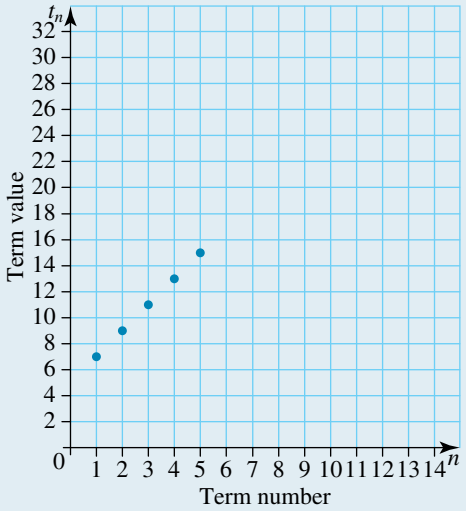
$t_5 = 7 + 2(5 - 1)$
$= 7 + 2 \times 4$
$= 7 + 8$
$= 15$

Term number	1	2	3	4	5
Term value	7	9	11	13	15

b. 1. Use the table of values to identify the points to be plotted.

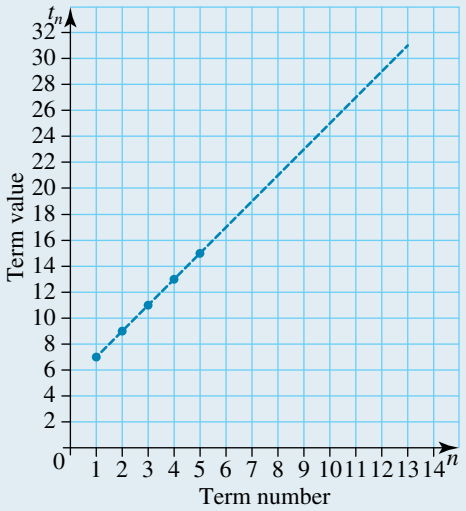
2. Plot the points on the graph.

b. The points to be plotted are (1, 7), (2, 9), (3, 11), (4, 13) and (5, 15).

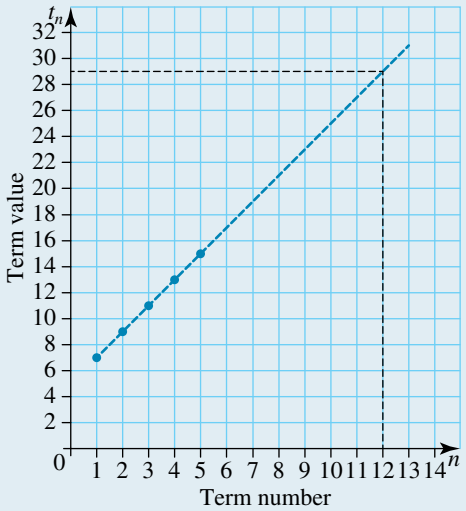


c. 1. Join the points with a dotted line (since the data values are discrete) and extend the line to cover future values of the sequence.

c.



2. Read the required value from the graph (when $n = 12$).



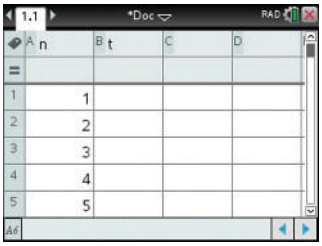
3. Write the answer.

The 12th term of the sequence is 29.

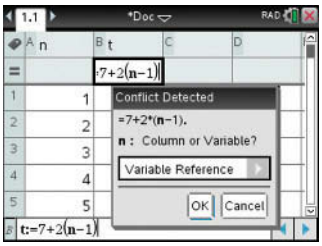
TI | THINK

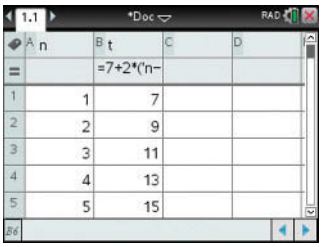
a.1. On a Lists & Spreadsheet page, label the first column as n and the second column as t. Enter the numbers 1 to 5 in the first column.

WRITE



2. In the function cell below the label t, complete the entry line as $t := 7 + 2(n - 1)$ then press ENTER. Select the variable reference for n when prompted. Select OK.

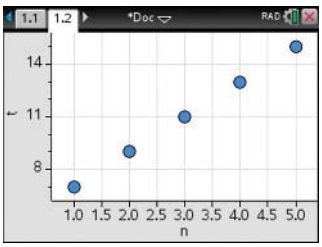




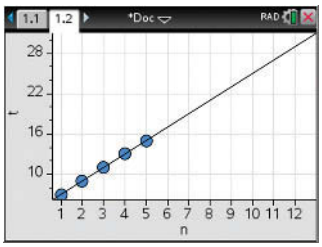
3. The table of values appears on the screen.

Term number	1	2	3	4	5
Term value	7	9	11	13	15

b.1. On a Data & Statistics page, click on the label of the horizontal axis and select n. Click on the label of the vertical axis and select t.



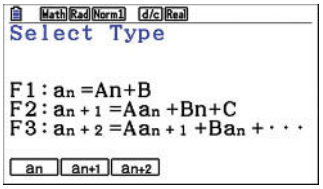
c.1. Press MENU, then select:
4: Analyze
6: Regression
1: Show Linear ($mx + b$).
Note: the window settings can be changed by pressing MENU, then selecting:
4: Window/Zoom
1: Window Settings.



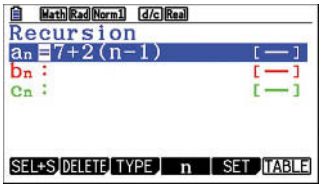
CASIO | THINK

a.1. On a Recursion screen, select TYPE by pressing F3, then select the first option by pressing F1.

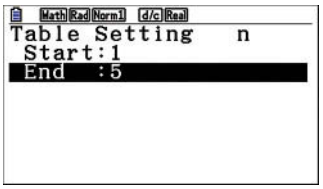
WRITE



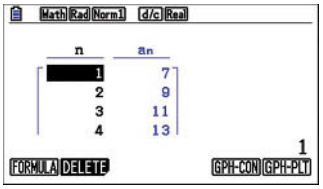
Complete the entry line for a_n as $a_n = 7 + 2(n - 1)$ then press EXE.



2. Select SET by pressing F5, then set the start value as 1 and the end value as 5. Press EXE.



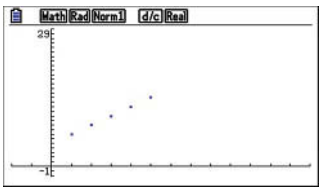
Select TABLE by pressing F6.



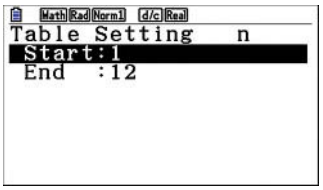
3. The table of values appears on the screen.

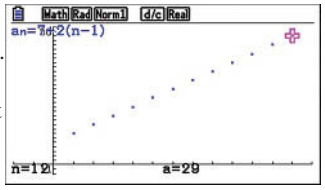
Term number	1	2	3	4	5
Term value	7	9	11	13	15

b.1. Select GPH-PLT by pressing F6.

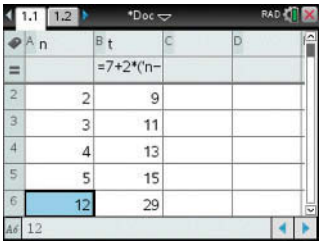


c.1. Select U by pressing F6, then select FORMULA by pressing F1. Select SET by pressing F5 and change the End value to 12. Press EXE. Select TABLE by pressing F6, then select GPH-PLT by pressing F6. Select Trace by pressing F1, then use the left/right arrows to move to the point where $n = 12$.





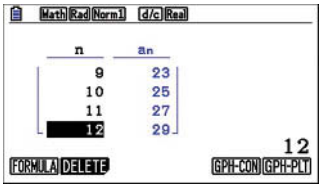
2. Alternatively, return to the Lists & Spreadsheet page, enter the value 12 into cell A6, then press ENTER.



3. The answer can be read from the screen.

The 12th term of the sequence is 29.

2. Alternatively, return to the table by pressing F6 then scroll down to find $n = 12$.



3. The answer can be read from the screen.

The 12th term of the sequence is 29.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 2

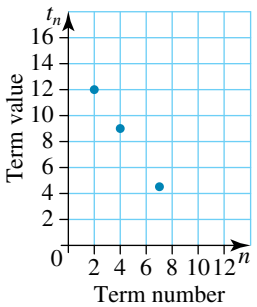
The recursive function and general term of an arithmetic sequence Summary screen and practice questions

Exercise 1.3 The general form of an arithmetic sequence

Technology free

1. **WE4** Determine the equations that represent the following arithmetic sequences.
   a. 4, 13, 22, 31, ... b. 9, 4.5, 0, ... c. −60, −49, −38, ... d. 100, 87, 74, ...
2. **WE5** a. Write the 20th term of the sequence 85, 72, 59, 46, 33, …
   b. Write the first term of the arithmetic sequence in which $t_{70} = 500$ and $d = -43$.
3. a. Determine the common difference of the arithmetic sequence that has a first term of −32 and an 8th term of 304.
   b. An arithmetic sequence has a first term of 5 and a common difference of 40. Which term number has a value of 85?
   c. An arithmetic sequence has a first term of 40 and a common difference of 12. Which term number has a value of 196?
4. **WE6** An arithmetic sequence is given by the equation $t_n = 5 + 10(n - 1)$.
   a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.
   b. Plot the graph of the sequence.
   c. Use your graph of the sequence to determine the 9th term of the sequence.
5. An arithmetic sequence is defined by the equation $t_n = 6.4 + 1.6(n - 1)$.
   a. Draw up a table of values showing the term number and term value for the first 5 terms of the sequence.
   b. Plot the graph of the sequence.
   c. Use your graph of the sequence to determine the 13th term of the sequence.
6. a. Determine the 15th term of the arithmetic sequence 6, 13, 20, 27, 34, …
   b. Determine the 20th term of the arithmetic sequence 9, 23, 37, 51, 65, …
   c. Determine the 30th term of the arithmetic sequence 56, 48, 40, 32, 24, …
   d. Determine the 55th term of the arithmetic sequence $\frac{72}{5}, \frac{551}{40}, \frac{263}{20}, \frac{501}{40}, \frac{119}{10}, \ldots$
7. For each of the arithmetic sequences given, determine:
   a. the 25th term of the sequence 2, 7, 12, 17, 22, ...
   b. the 30th term of the sequence 0, 100, 200, 300, 400, ...
   c. the 33rd term of the sequence 5, −2, −9, −16, −23, ...

8. Evaluate the following.
   a. The 2nd term of an arithmetic sequence is 13 and the 5th term is 31. What is the 17th term of this sequence?
   b. The 2nd term of an arithmetic sequence is −23 and the 5th term is 277. What is the 20th term of this sequence?
   c. The 2nd term of an arithmetic sequence is 0 and the 6th term is −8. What is the 32nd term of this sequence?
   d. The 3rd term of an arithmetic sequence is 5 and the 7th term is −19. What is the 40th term of this sequence?
   e. The 4th term of an arithmetic sequence is 2 and the 9th term is −33. What is the 26th term of this sequence?
9. a. Determine the first value of the arithmetic sequence which has a common difference of 6 and a 31st term of 904.
   b. Determine the first value of the arithmetic sequence which has a common difference of $\frac{2}{5}$ and a 40th term of −37.2.
   c. Determine the common difference of an arithmetic sequence which has a first value of 564 and a 51st term of 54.
   d. Determine the common difference of an arithmetic sequence which has a first value of −87 and a 61st term of 43.
10. a. An arithmetic sequence has a first value of 120 and a common difference of 16. Which term has a value of 712?
    b. An arithmetic sequence has a first value of 320 and a common difference of 4. Which term has a value of 1160?
11. Three consecutive terms of an arithmetic sequence are $x - 5$, $x + 4$ and $2x - 7$. Calculate the value of x.
12. The graph at right shows some points of an arithmetic sequence.
    a. What is the common difference between consecutive terms?
    b. What is the value of the first term of the sequence?
    c. What is the value of the 12th term of the sequence?



13. Sketch the graph of $t_n = t_1 + (n - 1)d$, where $t_1 = 15$ and $d = 25$, for the first 10 terms.
14. An employee starts a new job with a $60 000 salary in the first year and the promise of a pay rise of $2500 a year.
    a. How much will her salary be in their 6th year?
    b. How long will it take for her salary to reach $85 000?



1.4 The sum of an arithmetic sequence

1.4.1 Arithmetic sequences and series

When the terms of an arithmetic sequence are added together, an arithmetic series is formed.

So, 5, 9, 13, 17, 21, ... is an arithmetic sequence,
whereas 5 + 9 + 13 + 17 + 21 + ... is an arithmetic series.

The sum of n terms of an arithmetic sequence is given by S_n where

$$S_n = t_1 + t_2 + t_3 + ... + t_{n-1} + t_n$$

Consider the finite arithmetic sequence below.

$$5, 10, 15, 20, 25, 30, 35, 40, 45, 50$$

The sum of this arithmetic sequence is given by S_{10} since there are 10 terms in the sequence. So:

$$S_{10} = 5 + 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50 = 275.$$

Note that the sum of the first and last terms is 55. Also, the sum of the second and second-last terms is 55. Similarly, the sum of the third and third-last terms is 55. This pattern continues with the fourth and fourth-last terms as well as with the fifth and fifth-last terms. Thus, S_{10} is made up of five lots of 55.

We can formalise this pattern to obtain a rule that applies to all arithmetic sequences.

As $S_n = t_1 + t_2 + t_3 + ... + t_{n-1} + t_n$,

then $S_n = t_1 + (t_1 + d) + (t_1 + 2d) + ... + (t_n - 2d) + (t_n - d) + t_n$

where t_n is the last term of the sequence and where n represents the number of terms in the sequence.

By reversing the order of the series above, we obtain:

$$S_n = t_n + (t_n - d) + (t_n - 2d) + ... + (t_1 + 2d) + (t_1 + d) + t_1.$$

By adding these two equations, we obtain:

$$2S_n = (t_1 + t_n) + (t_1 + d + t_n - d) + (t_1 + 2d + t_n - 2d) + ... (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n)$$
$$2S_n = (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + ... (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n)$$
$$2S_n = n\,(t_1 + t_n)\,.$$

So, $S_n = \dfrac{n(t_n + t_1)}{2}$.

The sum of all arithmetic sequences is represented by:

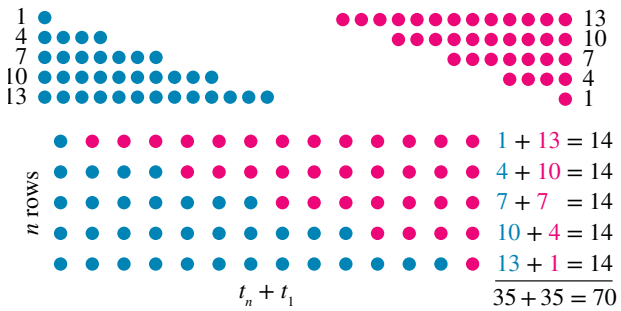
$$\boldsymbol{S_n = \frac{n\,(t_n + t_1)}{2}}$$

1.4.2 A visual explanation of S_n

Consider the arithmetic series below.

$$1 + 4 + 7 + 10 + 13 = 35$$

Let us represent two instances of this summation visually with coloured dots as given below, and join them together to make a rectangle.



Notice that the resulting rectangle has n rows and $t_n + t_1$ columns, and the total number of dots (representing the 'area' of the rectangle) can be found by multiplying them together:

Total number of dots $= n\,(t_n + t_1)$

As we know that the total number of dots is equal to twice the number of dots in the series:

$$2S_n = n(t_n + t_1)$$

$$\text{So, } S_n = \frac{n(t_n + t_1)}{2}$$

WORKED EXAMPLE 7

a. Find the sum of the first five terms of the arithmetic sequence represented by $t_n = 1 + 3(n - 1)$.
b. If the first term of a sequence is 10, the nth term is 1 and the sum of the first n terms is 22, what is n?
c. If the sum of the first six terms of an arithmetic sequence is 132 and $t_6 = 37$, what is the first term t_1?

THINK	WRITE
a. 1. State the known values.	**a.** $n = 5$
2. Calculate t_1 and t_n.	$t_1 = 1 + 3 \times 0 = 1$ $t_5 = 1 + 3 \times 4 = 1 + 12 = 13$
3. Substitute these values into the rule for S_n and solve for $n = 5$.	$S_5 = \frac{n(t_5 + t_1)}{2}$ $= \frac{5(13 + 1)}{2}$ $= \frac{5 \times 14}{2}$ $= \frac{70}{2}$ $= 35$
4. State the answer.	The sum of the first five terms of the arithmetic sequence represented by $t_n = 1 + 3(n - 1)$ is 35.
b. 1. State the known values.	**b.** $t_1 = 10,\ t_n = 1,\ S_n = 22$
2. Substitute the values into the formula for S_n and solve for n.	$S_n = \frac{n(t_n + t_2)}{2}$ $22 = \frac{n(1 + 10)}{2}$ $22 \times 2 = n(11)$ $11n = 44$ $n = 4$
3. State the answer.	The number of terms summed together is 4.
c. 1. State the known values.	**c.** $t_6 = 37, S_n = 132, n = 6$
2. Substitute the values into the formula for S_n and solve for t_1.	$S_n = \frac{n(t_n + t_1)}{2}$ $132 = \frac{6(37 + t_1)}{2}$ $264 = 6(37 + t_1)$ $44 = t_1 + 37$ $t_1 = 7$

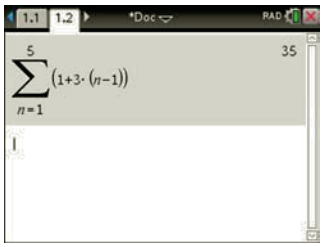
3. State the answer. — The first term of the arithmetic sequence is $t_1 = 7$.

TI | THINK

a.1. On a Calculator page, press the button and select $\sum$. Complete the entry line as $\sum_{n=1}^{5}(1 + 3 \times (n - 1))$ then press ENTER.

2. The answer appears on the screen.

WRITE



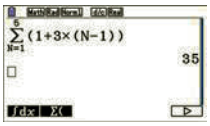
The sum of the first 5 terms is 35.

CASIO | THINK

a.1. On the Run-Matrix screen, select MATH by pressing F4, press F6 to scroll across to more menu options, then select $\sum($ by pressing F2. Complete the entry line as $\sum_{n=1}^{5}(1 + 3 \times (n - 1))$ then press EXE.

2. The answer appears on the screen.

WRITE



The sum of the first 5 terms is 35.

studyon

Unit 1 & 2 > Area 1 > Sequence 1 > Concept 3

The sum of an arithmetic sequence Summary screen and practice questions

Exercise 1.4 The sum of an arithmetic sequence

Technology free

1. Evaluate the sum of the sequence 5, 7, 9, 11 first by using regular addition then by using the formula.
2. **WE7a** Evaluate the sum of the first six terms of the arithmetic sequence represented by $t_n = 2 + 2(n - 1)$.
3. Evaluate the sum of the first four terms of the arithmetic sequence represented by $t_n = -4 - 5(n - 1)$.
4. **WE7b** If the first term of a sequence is 5, the nth term is -15 and the sum of the first n terms is -30, what is n?
5. If the first term of a sequence is -14, the nth term is 7 and the sum of the first n terms is -14, what is n?
6. **WE7c** If the sum of the first five terms of an arithmetic sequence is 75 and $t_5 = 33$, what is the first term t_1?
7. If the sum of the first seven terms of an arithmetic sequence is 0 and $t_1 = -27$, what is t_7?
8. Evaluate the sum of the first five terms of the arithmetic sequence with first term $t_1 = 20$ and common difference $d = 4$.
9. If the sum of the first five terms of an arithmetic sequence is 55, and the first term is 3, what is t_4?
10. An arithmetic sequence is such that $t_1 = -10$ and $t_4 = 29$. What is the sum of the first five terms of this sequence?
11. If the sum of the first three terms of an arithmetic sequence is 24, and the common difference is $d = 7$, what are the first three terms?

Technology active

12. Sam makes \$100 profit in his first week of business. If his profit increases by \$75 each week, what would his profit be in total by the end of week 15?
13. George's salary is to start at \$36 000 a year and increase by \$1200 each year after that. How much will George have earned in total after 10 years?
14. A staircase is designed so that the height of each step increases by 0.8 cm for each step. If the height of the first step is 15 cm, what is the total height of the first 17 steps?

15. Paula collects stamps. She bought 250 in the first month to start her collection and added 15 stamps to the collection each month thereafter. How many stamps will she have collected after 5 years?



16. Proceeds from the school fete were $3000 in 2000. In 2001 the proceeds were $3400 and in 2002 they were $3800. If they continued in this pattern:

a. what were the proceeds from the year 2019 fete?

b. how much in total would the proceeds from school fetes since 2000 have amounted to in the year 2019?



1.5 Applications of arithmetic sequences

1.5.1 Simple interest

If we have a practical situation involving linear growth or decay in discrete steps, this situation can be modelled by an arithmetic sequence.

Simple interest is calculated on the original amount of money invested. It is a fixed amount of interest paid at equal intervals, and as such it can be modelled by an arithmetic sequence.

Remember that simple interest is calculated by using the formula $I = Pin$, where I is the amount of simple interest, P is the principal, i is the percentage rate and n is the number of periods.



WORKED EXAMPLE 8

Jelena puts $1000 into an investment that earns simple interest at a rate of 0.5% per month.

a. Set up an equation that represents Jelena's situation as an arithmetic sequence, where t_n is the amount in Jelena's account after n months.

b. Use your equation from part a to determine the amount in Jelena's account at the end of each of the first 6 months.

c. Calculate the amount in Jelena's account at the end of 18 months.

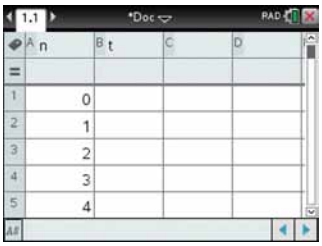


THINK	WRITE
a. 1. Use the simple interest formula to determine the amount of simple interest Jelena earns in one month.	**a.** $I = Pin$ $= 1000 \times 0.005 \times 1$ $= 5$
2. Calculate the amount in the account after the first month.	$t_1 = 1000 + 5$ $= 1005$
3. State the known values in the arithmetic sequence equation.	$t_1 = 1005,\ d = 5$
4. Substitute these values into the arithmetic sequence equation.	$t_n = 1005 + 5(n - 1)$
b. 1. Use the equation from part **a** to find the value of t_2, t_3, t_4, t_5 and t_6.	**b.** $t_2 = 1005 + 5(2 - 1)$ $= 1005 + 5 \times 1$ $= 1005 + 5$ $= 1010$ $t_3 = 1005 + 5(3 - 1)$ $= 1005 + 5 \times 2$ $= 1005 + 10$ $= 1015$ $t_4 = 1005 + 5(4 - 1)$ $= 1005 + 5 \times 3$ $= 1005 + 15$ $= 1020$ $t_5 = 1005 + 5(5 - 1)$ $= 1005 + 5 \times 4$ $= 1005 + 20$ $= 1025$ $t_6 = 1005 + 5(6 - 1)$ $= 1005 + 5 \times 5$ $= 1005 + 25$ $= 1030$
2. Write the answer.	The amounts in Jelena's account at the end of each of the first 6 months are \$1005, \$1010, \$1015, \$1020, \$1025 and \$1030.
c. 1. Use the equation from part **a** to find the value of t_{18}.	**c.** $t_{18} = 1005 + 5(18 - 1)$ $= 1005 + 5 \times 17$ $= 1005 + 85$ $= 1090$
2. Write the answer.	After 18 months Jelena has \$1090 in her account.

TI | THINK

b.1. On a Lists & Spreadsheet page, label the first column as n and the second column as t. Enter the numbers 0 to 6 in the first column.

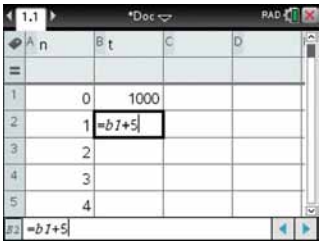
WRITE

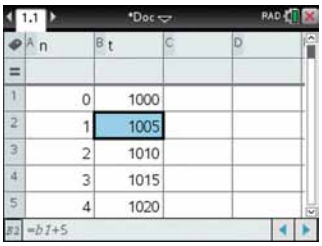


2. Calculate the amount of simple interest earned in 1 month.

$\$1000 \times 0.005 = \$5$

3. Enter 1000 into cell B1. In cell B2, complete the entry line as
$= b1 + 5$
then press ENTER.
Highlight cell B2, press MENU, and select:
3: Data
3: Fill.
Click and drag to select cells B3 to B7.

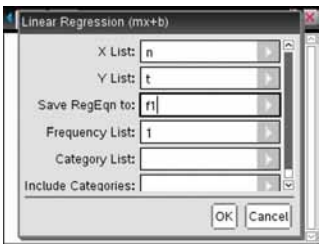


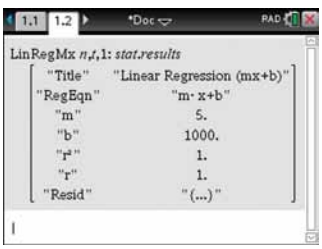


4. The answers can be read from the table.

The amounts in Jelena's account at the end of each of the first 6 months are \$1005, \$1010, \$1015, \$1020, \$1025 and \$1030.

a.1. On a Calculator page, press MENU, then select:
6: Statistics
1: Stat Calculations
3: Linear Regression($mx + b$).
Complete the fields as:
X List: n
Y List: t
Save RegEqn to: f1
then select OK.





2. Interpret the output.

The equation is given in the form $y = mx + b$, where $y = t_n$, $m = 5$, $x = n$ and $b = 1000$.

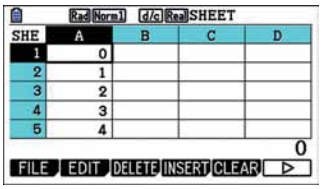
3. State the answer.

The equation is $t_n = 5n + 1000$.

CASIO | THINK

b.1. On a Spreadsheet page, enter the numbers 0 to 6 in the first column.

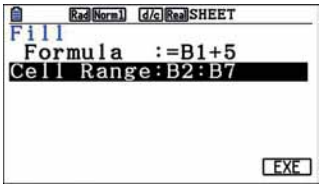
WRITE

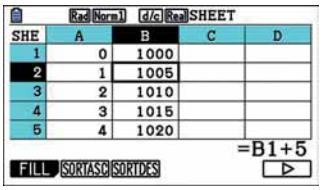


2. Calculate the amount of simple interest earned in 1 month.

$\$1000 \times 0.005 = \$5$

3. Enter 1000 into cell B1. Select EDIT by pressing F2, press F6 to scroll across to more menu options, then select FILL by pressing F1.
Complete the fields as
Formula: $= B1 + 5$
Cell Range: $B2 : B7$
then press EXE.

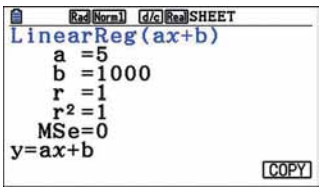




4. The answers can be read from the table.

The amounts in Jelena's account at the end of each of the first 6 months are \$1005, \$1010, \$1015, \$1020, \$1025 and \$1030.

a.1. Press EXIT to return to the original menu items, then press F6 to scroll across to more menu options. Select CALC by pressing F2, then select REG by pressing F3.
Select X by pressing F1, then select $ax + b$ by pressing F1.
Select COPY by pressing F6, then select Y1 by pressing EXE.



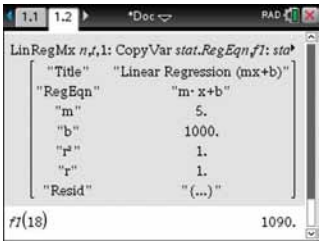
2. Interpret the output.

The equation is given in the form $y = mx + b$, where $y = t_n$, $a = 5$, $x = n$ and $b = 1000$.

3. State the answer.

The equation is $t_n = 5n + 1000$.

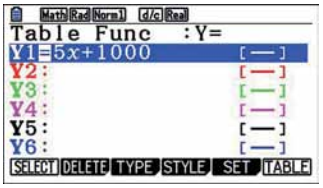
c.1. On the Calculator page, complete the entry line as *f*1 (18) then press ENTER.



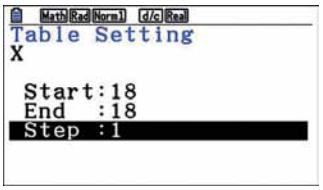
2. The answer appears on the screen.

After 18 months Jelena has $1090 in her account.

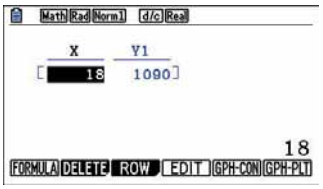
c.1. On a Table screen, highlight Y1 then select SELECT by pressing F1.



Select SET by pressing F5 and set both the Start and End values to18. Press EXE.



Select TABLE by pressing F6.



2. The answer appears on the screen.

After 18 months Jelena has $1090 in her account.

1.5.2 Depreciating assets

Many items, such as automobiles or electronic equipment, decrease in value over time as a result of wear and tear. At tax time individuals and companies use depreciation of their assets to offset expenses and to reduce the amount of tax they have to pay.

Unit cost depreciation

Unit cost depreciation is a way of depreciating an asset according to its use. For example, you can depreciate the value of a car based on how many kilometres it has driven. The unit cost is the amount of depreciation per unit of use, which would be 1 kilometre of use in the example of the car.



Future value and write-off value

When depreciating the values of assets, companies will often need to know the **future value** of an item. This is the value of that item at that specific time.

The **write-off value** or scrap value of an asset is the point at which the asset is effectively worthless (i.e. has a value of $0) due to depreciation.

WORKED EXAMPLE 9

Loni purchases a new car for $25 000 and decides to depreciate it at a rate of $0.20 per km.

a. Set up an equation to determine the value of the car after *n* km of use.

b. Use your equation from part a to determine the future value of the car after it has 7500 km on its clock.

THINK	WRITE
a. 1. Calculate the value of the car after 1 km of use.	**a.** $t_1 = 25\,000 - 0.2$ $= 24\,999.8$

2. State the known values in the arithmetic sequence equation.

$t_1 = 24\,999.8,\ d = -0.2$

3. Substitute these values into the arithmetic sequence equation.

$$\begin{aligned} t_n &= t_1 + (n-1)d \\ &= 24\,999.8 + (n-1) \times -0.2 \\ &= 24\,999.8 - 0.2(n-1) \end{aligned}$$

b. 1. Substitute $n = 7500$ into the equation determined in part **a**.

b.
$$\begin{aligned} t_n &= 24\,999.8 - 0.2(n-1) \\ t_{7500} &= 24\,999.8 - 0.2(7500-1) \\ &= 24\,999.8 - 0.2 \times 7499 \\ &= 24\,999.8 - 1499.8 \\ &= 23\,500 \end{aligned}$$

2. Write the answer.

After 7500 km the car will be worth $23 500.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 4

Modelling with arithmetic sequences Summary screen and practice questions

Exercise 1.5 Applications of arithmetic sequences

Technology active

1. **WE8** Grigor puts $1500 into an investment account that earns simple interest at a rate of 4.8% per year.
   a. Set up an equation that represents Grigor's situation as an arithmetic sequence, where t_n is the amount in Grigor's account after n months.
   b. Use your equation from part **a** to determine the amount in Grigor's account after each of the first 6 months.
   c. Calculate the amount in Grigor's account at the end of 18 months.
2. Justine sets up an equation to model the amount of her money in a simple interest investment account after n months. Her equation is $t_n = 8050 + 50(n-1)$, where t_n is the amount in Justine's account after n months.
   a. How much did Justine invest in the account?
   b. What is the annual interest rate of the investment?
3. **WE9** Phillipe purchases a new car for $24 000 and decides to depreciate it at a rate of $0.25 per km.
   a. Set up an equation to determine the value of the car after n km of use.
   b. Use your equation from part **a** to determine the future value of the car after it has 12 000 km on its clock.



4. Dougie is in charge of the equipment for his office. He decides to depreciate the value of a photocopier at the rate of x cents for every n copies made. Dougie's equation for the value of the photocopier after n copies is $t_n = 5399.999 - 0.001(n-1)$.
   a. How much did the photocopier cost?
   b. What is the rate of depreciation per copy made?

5. Nadia wants to invest her money and decided to place $90 000 into a credit union account earning simple interest at a rate of 6% per year.



   a. How much interest will Nadia receive after one year?
   b. What is the total amount Nadia has in the credit union after n years?
   c. For how long should Nadia keep her money invested if she wants a total of $154 800 returned?

6. Tom bought a car for $23 000, knowing it would depreciate in value by $210 per month.
   a. What is the value of the car after 18 months?
   b. By how much does the value of the car depreciate in 3 years?
   c. How many months will it take for the car to be valued at $6200?

7. Sanchia deposits an amount of money in a bank account that earns simple interest every year. After three years she has $2875 in the account and after five years she has $3125.
   a. Calculate the yearly interest that Sanchia receives on her deposit.
   b. Determine the amount of money Sanchia originally deposited.
   c. Determine the yearly interest rate as a percentage.

8. A confectionary manufacturer introduces a new sweet and produces 50 000 packets of the sweets in the first week. The stores sell them quickly, and in the following week there is demand for 30% more. In each subsequent week the increase in production is 30% of the original production.



   a. How many packs are manufactured in the 20th week?
   b. In which week will the confectionary manufacturer produce 5 540 000 packs?

9. A canning machine was purchased for a total of $250 000 and is expected to produce 500 000 000 cans before it is written off.
   a. By how much does the canning machine depreciate with each can made?
   b. If the canning machine were to make 40 200 000 cans each year, when will the machine theoretically be written off?
   c. When will the machine have a book value of $89 200?

10. Blood donations at a suburban location increased by 40 each year. If there were 520 donations in the first year:
   a. how many donations were made in the 15th year?
   b. what was the total number of donations made over those 15 years?

11. A rock dropped from the top of a high ridge on the Moon's surface falls a distance of 0.8 metres during the first second of its descent. During the next second it falls 2.4 metres; during the third second it falls 4 meters; during the fourth second it falls 5.6 metres. If this arithmetic pattern continues, how far will the rock have fallen in total at the end of the ninth second?

12. Lucca's parents are very pleased with his progress in the Chess club. As an incentive to do well in the local competition, they will give him $\$a$ for getting through the first round, and will increase the amount he gets by $\$d$ for each successive round he gets through. Lucca's total earnings are $32 after completing the fourth round and his total earnings are $77 after getting through the seventh round. If the average of the individual amounts he earns in the fourth and seventh rounds equals $14, what will be the total amount that Lucca's parents will have to pay him if he completes nine rounds?

13. The local rugby club wants to increase its membership. In the first year they had 5000 members, and so far they have managed to increase their membership by 1200 members per year.



a. If the increase in membership continues at the current rate, how many members will they have in 15 years' time?

Tickets for membership in the first year were $200, and each year the price has risen by a constant amount, with memberships in the 6th year costing $320.

b. How much will the tickets cost in 15 years' time?

c. What is the total membership income in both the first and 15th years?

14. A newly established quarry produces crushed rock for the building of roads and freeways. The amount of crushed rock, in tonnes, it produces increases by $3\frac{1}{2}$ tonnes each month and its production for the first 3 months of operation is shown in the table below.

Month	Crushed rock produced (tonnes)
1	8
2	11.5
3	15

a. Write down the amount of crushed rock produced in the 4th month.

b. Write down a rule for t_n, the amount of crushed rock produced in month n, expressed in terms of n, the nth month.

c. Write down the amount of crushed rock produced in the 60th month.

d. During which month will the amount of crushed rock coming from the quarry exceed 100 tonnes?

e. The local council has ordered that after a total of 3050 tonnes of crushed rock has been extracted from the quarry, an environmental impact survey must be completed. After how many months will that happen?

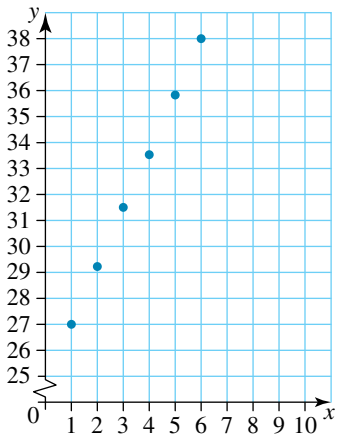


1.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Which of the following sequences are arithmetic sequences?
   a. 12, 16, 20, 24, 28
   b. 12, 4.8, 1.92, 0.77, 0.31
   c. 12, 15, 21, 30, 42
   d. 12, 6.2, 3.3, 1.88, 0.42
   e. 12, 14.8, 17.6, 20.4, 23.2
2. MC For the sequence $-16, -11.2, -6.4, -1.6, 3.2$, the correct values for t_1 and d are:
   A. $t_1 = -16, d = -11.2$
   B. $t_1 = 4.8, d = 3.2$
   C. $t_1 = -16, d = 4.8$
   D. $t_1 = -19.2, d = 3.2$
3. MC The missing value in the arithmetic sequence of $65, x, 58, 54.5, 51$ is:
   A. -3.5
   B. 68.5
   C. 60
   D. 61.5
4. MC The 41st term of the arithmetic sequence $-4.3, -2.1, 0.1, 2.3, 4.5, \ldots$ is:
   A. 83.7
   B. 85.9
   C. 92.3
   D. 172.4
5. MC The first term of an arithmetic sequence is 5.2 and the 2nd is 6. The sum of the first 22 terms of the sequence is:
   A. 598.4
   B. 299.2
   C. 242
   D. 137.2
6. Consider the sequence described by the equation $t_n = 2 + 5(n - 1)$.
   a. Determine the first five terms of the sequence.
   b. State the values of t_1 and d for the sequence.
   c. Which term of the sequence will have a value of 72?
7. Construct an equation to represent the following arithmetic sequence: 3, 0.5, -2, -4.5, -7, -9.5
8. The following graph shows points of an arithmetic sequence.
   a. What is the common difference between consecutive terms?
   b. What is the value of the first term of the sequence?
   c. Determine the equation for this sequence.
   d. What is the value of the 9th term?



9. Determine the sum S_n of the first eight terms in each of the following sequences. You may choose to use technology to answer the question.
   a. $4, 8, 12, \ldots$
   b. $2, -2.5, -7, -11.5, \ldots$
   c. $\frac{1}{2}, 2, 3\frac{1}{2}, \ldots$
10. Write a recursive formula for t_{n+1} in terms of t_n for the sequence $3, 5, 7, \ldots$.
11. List the first five terms of the sequence defined by $t_{n+1} = t_n - 2$ given that $t_1 = 0$.
12. Calculate the depreciated value of a car four years after it was bought new for \$30 000 if it depreciates at a rate of 16% of its original value per year. You may choose a technology of your choice to answer this question.

Complex familiar

13. Chris is saving for his first car. He put $900 into a simple interest savings account that earns 8.2% per year. You may choose to use technology to answer this question.

- **a.** Set up an equation that represents Chris's situation as an arithmetic sequence, where t_n is the amount in the account after n months.
- **b.** Use your equation from part **a** to determine the amount in Chris's account after each of the first 5 months.
- **c.** Calculate the amount in the savings account at the end of 20 months.
- **d.** At this interest rate, how many months will it take Chris to save $1200?

14. A teacher decides to give his 30 students lollies at an end-of-year party. He wishes to give different numbers of lollies to each student based on their attendance record.



He decides to give 1 lolly to the student with the highest number of absences, 3 lollies to the student with the next highest number of absences, 5 to the third highest, then 7, and so on. How many lollies in total will the teacher need to give out to the class?

You may choose to use a technology of your choice to answer questions **15–20**.

15. An accountant has been working with the same company for 15 years. She commenced on a salary of $28 000 and has received a $2500 increase each year. Calculate her average wage over this time period.

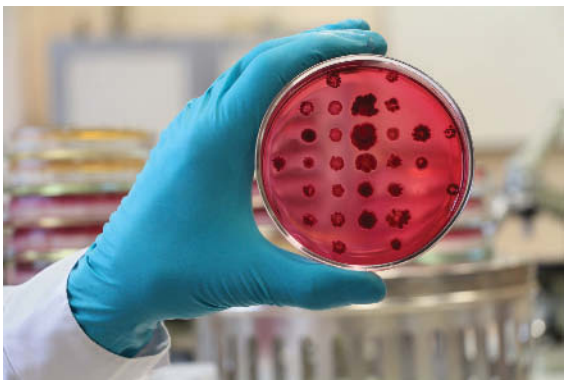
16. Two banks pay simple interest on short-term deposits. Hales Bank pays 8% p.a. over 5 years and Countrybank pays 12% p.a. for 3 years. Which bank will yield the highest return if $2000 was invested in each account?

Complex unfamiliar

17. Let an arithmetic sequence be such that $t_1 = 3$ and $S_n = 60$. What are the possible values of n that result in integer values for both t_n and d?

18. A biologist is growing a tissue culture in a Petri dish. Between the end of the second day and the end of the fourth day, the culture's mass had increased by 4 milligrams. By the sixth day the culture had a mass of 36 milligrams.

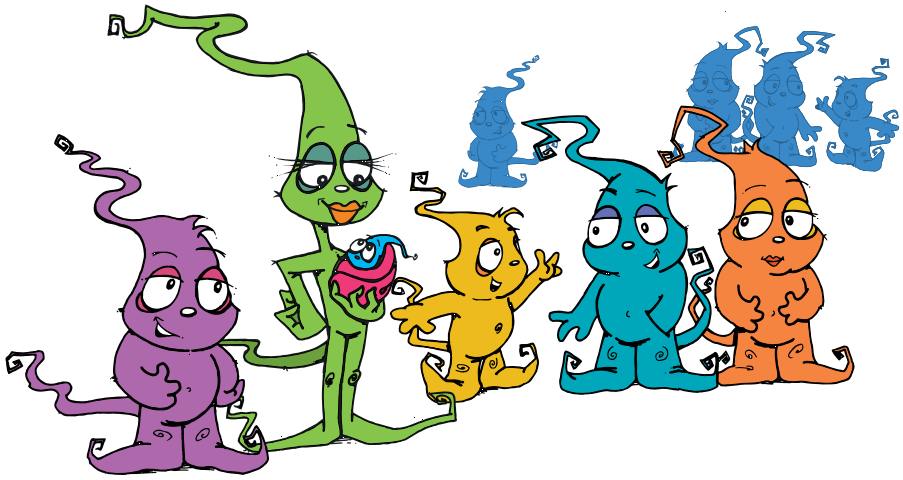
Assuming that the daily growth is arithmetic, on what day will the culture mass first exceed 200 milligrams?



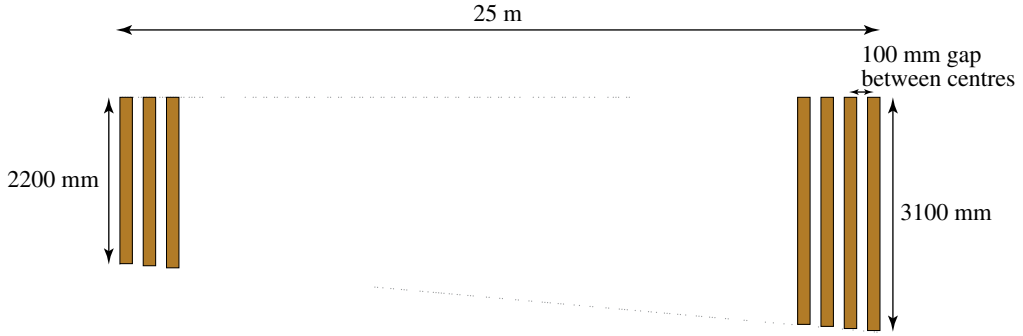
19. The Lunx is a creature that reproduces according to the following rules.

- All mature Lunx produce one baby each year.
- A Lunx becomes mature at 2 years and will produce an offspring at the beginning of its third year.
- Lunx die at the end of their 5th year.

At the end of year 1 the zoo has one Lunx aged 1 year. It will produce its first baby at the beginning of year 3. If the zoo population is left to grow uninhibited, what will the population be at the end of the 24th year?



20. Battens (thin strips of timber) are to be used to enclose the area underneath the house shown in the diagram. The centres of the strips are 100 mm apart. The block slopes so that at one end the battens are 220 mm in length while at the other end they are 3100 mm.



a. Prove that the length of successive battens differs by a constant amount.
b. If the first batten is 2200 mm long, give an expression for the length of the n th batten.
c. How many battens are needed?
d. Determine the combined length of all battens.

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Units 1 & 2 Sit chapter test

Answers

Chapter 1 Arithmetic sequences

Exercise 1.2 Arithmetic sequences

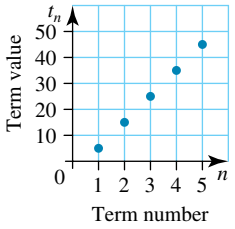
1. **a** is an arithmetic sequence where $d = 5$
   b is not an arithmetic sequence as there is no common difference
   c is an arithmetic sequence where $d = 100$
   d is an arithmetic sequence where $d = 100$
   e is not an arithmetic sequence as there is no common difference
   f is not an arithmetic sequence as there is no common difference
   g is an arithmetic sequence where $d = 1$
   h is an arithmetic sequence where $d = \frac{1}{2}$
2. **a** is an arithmetic sequence where $d = 2$
   b is not an arithmetic sequence as there is no common difference
   c is not an arithmetic sequence as there is no common difference
   d is an arithmetic sequence where $d = 2$
   e is an arithmetic sequence where $d = 8$
3. 38, 47, 56, 65, 74
4. 12, 17, 22, 27, 32
5. $-2, 1, 4, 7, 10$
6. **a.** 5, 8, 11, 14, 17 **b.** $-1, -8, -15, -22, -29$
   c. $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, 3$ **d.** $3.3, 2.6, 1.9, 1.2, 0.5$
7. **a.** Arithmetic sequence where $t_1 = 23$; $d = 45$
   b. is not an arithmetic sequence as there is no common difference
   c. Arithmetic sequence where $t_1 = \frac{1}{2}$; $d = \frac{1}{4}$
8. **a.** infinite
   b. increasing
   c. $t_1 = -3.6$; $d = 1.5$
9. **a.** $t_{n+1} = t_n + 4$ **b.** $t_{n+1} = t_n - 3.5$
   c. $t_{n+1} = t_n + 2$ **d.** $t_{n+1} = t_n + 1.9$
10. **a.** $f = -62$
    b. $j = 5.7$; $k = 15.3$
    c. $r = \frac{11}{4}$; $q = 1$; $p = -\frac{3}{4}$
    d. $s = 1$; $t = \frac{3}{2}$
11. **a.** 3, −2, −7, −12 **b.** −0.6, 0.8, 2.2, 3.6
    c. −23, 9, 41, 73 **d.** 10, 7, 4, 1
12. $t_n = 13 + 10n$
13. $t_n = 37 + 3n$
14. **a.** $t_n = 8.5 + 3.5n$
    b. 358.5 metres from the road

Exercise 1.3 The general form of an arithmetic sequence

1. **a.** $t_n = 4 + 9(n - 1)$ **b.** $t_n = 9 - 4.5(n - 1)$
   c. $t_n = -60 + 11(n - 1)$ **d.** $t_n = 100 - 13(n - 1)$
2. **a.** −162 **b.** 3467
3. **a.** $d = 48$ **b.** 3 **c.** 14
4. **a.**

Term number	1	2	3	4	5
Term value	5	15	25	35	45

b.

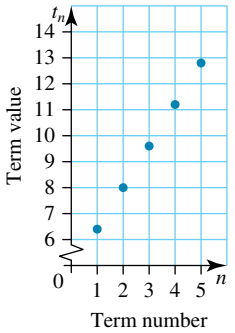


c. 85

5. **a.**

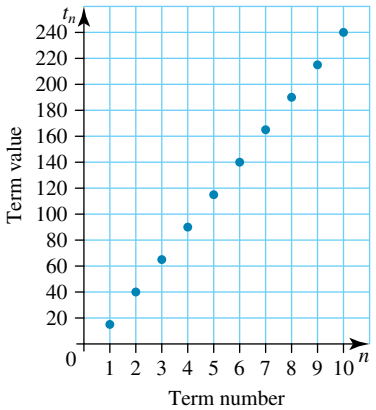
Term number	1	2	3	4	5
Term value	6.4	8	9.6	11.2	12.8

b.



c. 25.6

6. **a.** 104 **b.** 275
   c. −176 **d.** $-\frac{387}{20}$
7. **a.** 122 **b.** 2900 **c.** −219
8. **a.** 103 **b.** 1777 **c.** −60 **d.** −217 **e.** −152
9. **a.** 724 **b.** −52.8
   c. −10.2 **d.** $\frac{13}{6}$
10. **a.** 38 **b.** 211
11. $x = 20$
12. **a.** −1.5 **b.** 13.5 **c.** −3
13.



14. **a.** $72 500 **b.** 11 years

Exercise 1.4 The sum of an arithmetic sequence

1. 32
2. 42
3. −46
4. $n = 6$
5. $n = 4$

6. $t_1 = -3$
7. $t_7 = 27$
8. 140
9. $t_4 = 15$
10. 80
11. 1, 8, 15
12. $9375
13. $414 000
14. 363.8 cm
15. 1135 stamps
16. a. $10 600
    b. $136 000

Exercise 1.5 Applications of arithmetic sequences

1. a. $t_n = 1506 + 6(n-1)$
   b. $1506, $1512, $1518, $1524, $1530, $1536
   c. $1608
2. a. $8000 b. 7.5 %
3. a. $t_n = 23\,999.75 - 0.25(n-1)$
   b. $21 000
4. a. $5400 b. 0.1 cents
5. a. $5400
   b. $t_n = 95\,400 + 5400(n-1)$
   c. 12 years
6. a. $19 220 b. $7560 c. 80 months
7. a. $125 b. $2625 c. 4.76%
8. a. 335 000 packets b. 367th week
9. a. 0.05 cents per can
   b. 13th year
   c. after 8 years
10. a. 1080 b. 12 000
11. 98.4 m
12. $117
13. a. 21 800 b. $536 each c. $11 684 800
14. a. 18.5 tonnes
    b. $t_n = 4.5 + 3.5n$
    c. 214.5 tonnes
    d. 28th month
    e. after 40 months

1.6 Review: exam practice

Simple familiar

1. a is an arithmetic sequence where $d = 4$
   b is not an arithmetic sequence as there is no common difference
   c is not an arithmetic sequence as there is no common difference
   d is not an arithmetic sequence as there is no common difference
   e is an arithmetic sequence where $d = 2.8$
   a and e
2. C
3. D
4. A
5. B
6. a. 2, 7, 12, 17, 22
   b. $t_1 = 2$; $d = 5$
   c. 15
7. $t_n = 3 - 2.5(n-1)$
8. a. 2.2 b. 27
   c. $t_n = 27 + 2.2(n-1)$ d. 44.6
9. a. 144 b. −110 c. 46
10. $t_{n+1} = t_n + 2$ where $t_1 = 3$
11. 0, −2, −4, −6, −8
12. $10 800

Complex familiar

13. a. $t_n = 906.15 + 6.15(n-1)$
    b. $906.15, $912.30, $918.45, $924.60, $930.75
    c. $1023
    d. 49 months
14. 900 lollies
15. $45 500
16. Hales Bank
17. $n = 2$, 3 or 4 only
18. day 15
19. 2 Lunx
20. a. Sample responses can be found in the worked solutions in the online resources.
    b. $L_n = 2200 + 3.6(n-1)$
    c. 250
    d. 388.6 m

TOPIC 1 Arithmetic and geometric sequences and series 1

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



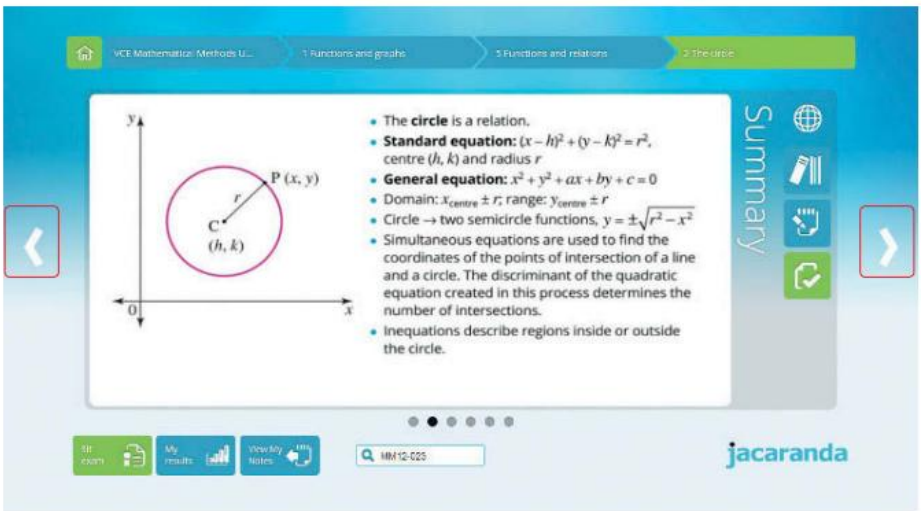
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 2
Functions

2.1 Overview

2.1.1 Introduction



Functions are one of the most fundamental ideas in modern mathematics. Concepts related to functions have been developed over centuries by many famous mathematicians, including Leibniz, Euler and Fourier. Defining a function on a basic level allows for analysis of situations that appear to be complex but can often be modelled by an equation or set of equations. More thorough investigation can occur by looking at derivatives and integrals of functions using methods of calculus, allowing for a deeper understanding of the model or optimisation of processes. These concepts and skills are particularly important in numerous careers including many engineering disciplines, medical research and computer science, where functions are used to develop safer structures, evaluate drug efficacy and design, and optimise programs, among many other uses. Functions are also used extensively in astrophysics to calculate trajectories for space travel! Calculating timing and direction is vital for successful launches and re-entries of space shuttles. When exploring further away from Earth, functions can be used to model gravitational slingshot manoeuvres around stars, planets and moons, allowing us to reach further into the cosmos.

LEARNING SEQUENCE

2.1 Overview
2.2 Functions and relations
2.3 Function notation
2.4 Transformations of functions
2.5 Piece-wise functions
2.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.2 Functions and relations

2.2.1 Set and interval notation

Set notation

A **set** is a collection of things, and in mathematics sets are usually used to represent a group of numbers. Each number within a set is called an **element**, and these elements can be listed individually or described by a rule. Elements within a set are separated by commas. Some important symbols and pre-defined common sets are listed below.

$\{...\}$ refers to a set of something.

$\in$ means 'is an element of'.

$\notin$ means 'is not an element of'.

$\subset$ means 'is a subset of'.

$\not\subset$ means 'is *not* a subset (or is not contained in)'.

$\cap$ means 'intersection with'.

$\cup$ means 'union with'.

$\setminus$ means 'excluding'.

$\varnothing$ refers to 'the null, or empty set'.

$\{a, b, c\}$ is a set of three letters.

$\{(a, b), (c, d), ...\}$ is an infinite set of ordered pairs.

A relation is a set of ordered pairs.

N refers to the set of Natural numbers.

J refers to the set of integers.

Q refers to the set of rational numbers.

R refers to the set of Real numbers.

R^+ refers to the set of positive Real numbers.

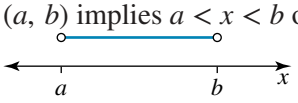
R^- refers to the set of negative Real numbers.

Interval notation

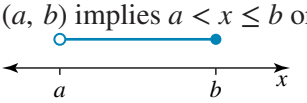
Interval notation is a convenient way to represent an **interval** using only the end values and indicating whether those end values are included or excluded. When using interval notation, a rounded bracket is used to indicate a value that is excluded and a square bracket is used to indicate a value that is included in the interval. Recall that on a number line and on a Cartesian plane, excluded values are represented by an open circle and included values by a closed circle.

If a and b are real numbers and $a < b$, then the following intervals are defined with an accompanying number line:

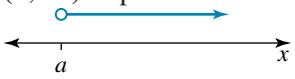
(a, b) implies $a < x < b$ or



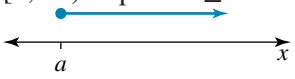
(a, b) implies $a < x \leq b$ or



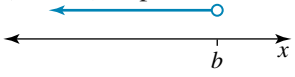
(a, ∞) implies $x > a$ or



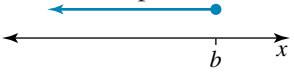
$[a, \infty)$ implies $x \geq a$ or



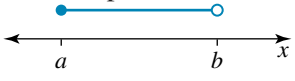
$(-\infty, b)$ implies $x < b$ or



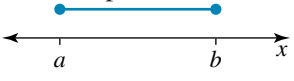
$(-\infty, b]$ implies $x \leq b$ or



$[a, b]$ implies $a \leq x < b$ or

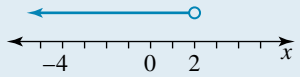


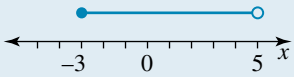
$[a, b]$ implies $a \leq x \leq b$ or



WORKED EXAMPLE 1

Describe each of the following subsets of the real numbers using interval notation.

a.


b.


c.


THINK

a. The interval is $x < 2$ (2 is not included).

b. The interval is $-3 \leq x < 5$ (-3 is included).

c. The interval is both $1 \leq x < 3$ and $x \geq 5$ (1 is included, 3 is not). The symbol $\cup$ indicates the combination of the two intervals.

WRITE

a. $(-\infty, 2)$

b. $[-3, 5)$

c. $[1, 3) \cup [5, \infty)$

WORKED EXAMPLE 2

Illustrate the following number intervals on a number line.

a. $(-2, 10]$ **b.** $[1, \infty)$

THINK

a. The interval is $-2 < x \leq 10$ (-2 is not included, 10 is).

b. The interval is $x \geq 1$ (1 is included).

WRITE

a.


b.


2.2.2 Relations

A mathematical **relation** is any set of ordered pairs.

The ordered pairs may be listed or described by a rule or presented as a graph. Examples of relations could include $A = \{(-2,4), (1,5), (3,4)\}$ where the ordered pairs have been listed; $B = \{(x, y) : y = 2x\}$ where the ordered pairs are described by a linear equation; and $C = \{(x, y) : y \leq 2x\}$ where the ordered pairs are described by a linear inequation. These relations could be presented visually by being graphed on coordinate axes. The graph of A would consist of three points, the graph of B would be a straight line and the graph of C would be a closed half-plane. Relations can be **continuous**, where all values of a variable are possible within a specified interval, or **discrete**, where only fixed values are permitted.

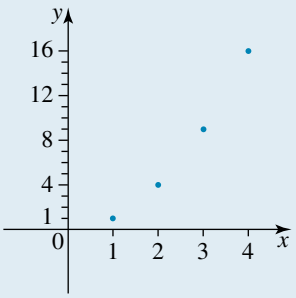
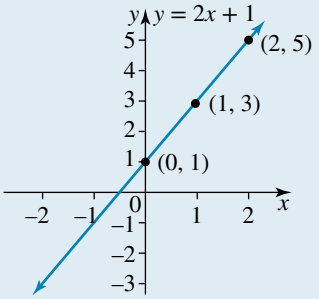
In a set of ordered pairs, the first value, or x-value, is referred to as the **independent variable** and the second value, or y-value, is called the **dependent variable**. The possible x-values are defined first, then the resulting y-values are found through substitution of these x-values into the rule that describes the relation. As such, the values of y are dependent on the given x-values.

WORKED EXAMPLE 3

Sketch the graph representing each of the following relations and state whether each is discrete or continuous.

a. $y = x^2$, **where** $x \in \{1, 2, 3, 4\}$

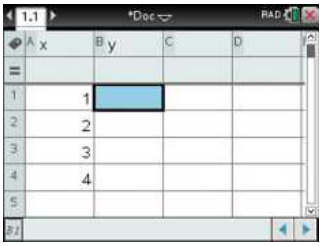
b. $y = 2x + 1$, **where** $x \in R$

THINK	WRITE
a. 1. Use the rule to calculate y and state the ordered pairs by letting $x = 1, 2, 3$ and 4.	**a.** When $x = 1$, $y = 1^2$ $= 1$ $(1, 1)$ $x = 2, y = 2^2$ $= 4$ $(2, 4)$ $x = 3, y = 3^2$ $= 9$ $(3, 9)$ $x = 4, y = 4^2$ $= 16$ $(4, 16)$
2. Plot the points $(1, 1), (2, 4), (3, 9)$ and $(4, 16)$ on a set of axes.	
3. Do not join the points as x is a discrete variable (whole numbers only).	It is a discrete relation as x can be only whole number values.
b. 1. Use the rule to calculate y. Select values of x, say $x = 0$, 1 and 2 (or find the intercepts). State the ordered pairs.	**b.** When $x = 0, y = 2(0) + 1$ $= 1$ $(0, 1)$ $x = 1, y = 2(1) + 1$ $= 3$ $(1, 3)$ $x = 2, y = 2(2) + 1$ $= 5$ $(2, 5)$
2. Plot the points $(0, 1)$, $(1, 3)$ and $(2, 5)$ on a set of axes.	
3. Join the points with a straight line, continuing in both directions as x is a continuous variable (any real number).	It is a continuous relation as x can be any real number.

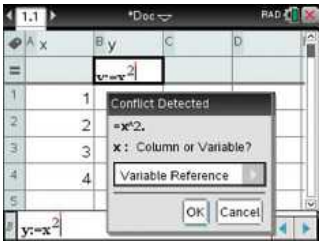
TI | THINK

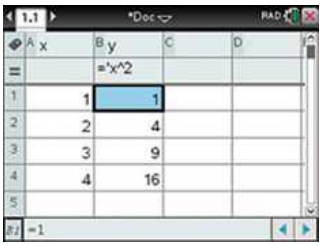
a.1. In a Lists & Spreadsheet page, label the first column as x and the second column as y. Enter the values 1–4 in the first column.

WRITE

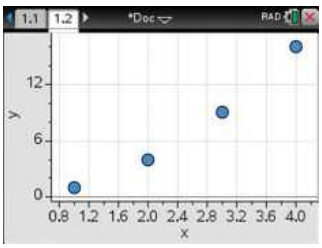


2. In the function cell below the label y, complete the entry line as $= x^2$ then press ENTER. Select the Variable Reference for x when prompted.



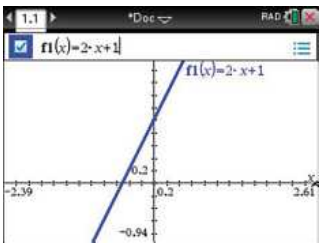


3. In a Data & Statistics page, click on the label of the horizontal axis and select x. Click on the label of the vertical axis and select y. The graph appears on the screen.



This relation is discrete.

b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 2x + 1$ then press ENTER. The graph appears on the screen.

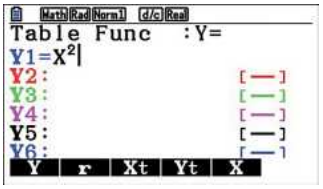


This relation is continuous.

CASIO | THINK

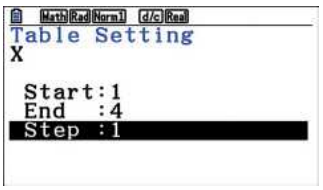
a.1. On a Table screen, complete the entry line for Y1 as: Y1 = x^2 then press EXE.

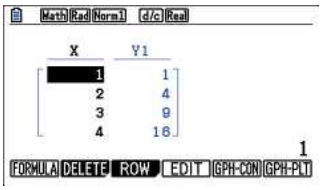
WRITE



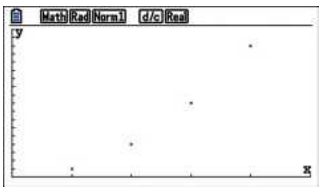
2. Select SET by pressing F5, then complete the fields as:
Start: 1
End: 4
Step: 1
then press EXE.

Select TABLE by pressing F6.



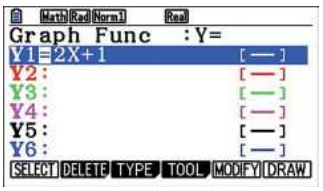


3. Select GPH-PLT by pressing F6.

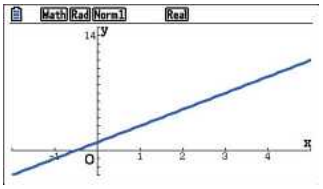


This relation is discrete.

b.1. On a Graph screen, complete the entry line for Y1 as Y1 = $2x + 1$ then press EXE.



2. Select DRAW by pressing F6.



This relation is continuous.

Types of relations

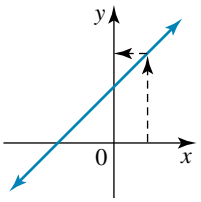
Relations are classified according to the correspondence between the coordinates of their ordered pairs. Note that the word *many* in this context means more than one, and the precise number is not considered.

One-to-one relations

A one-to-one relation exists if for any x-value there is only one corresponding y-value and vice versa. Examples are:

a. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

b.

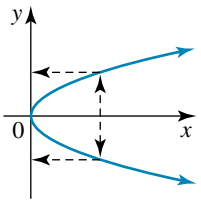


One-to-many relations

A one-to-many relation exists if for any x-value there is more than one y-value, but for any y-value there is only one x-value. Examples are:

a. $\{(1, 1), (1, 2), (2, 3), (3, 4)\}$

b.

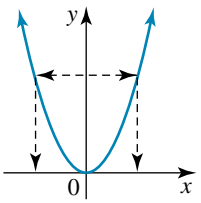


Many-to-one relations

A many-to-one relation exists if there is more than one x-value for any y-value but for any x-value there is only one y-value. Examples are:

a. $\{(-1, 1), (0, 1), (1, 2)\}$

b.

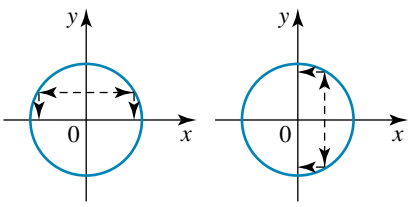


Many-to-many relations

A many-to-many relation exists if there is more than one x-value for any y-value and vice versa. Examples are:

a. $\{(0, -1), (0, 1), (1, 0), (-1, 0)\}$

b.



2.2.3 Functions

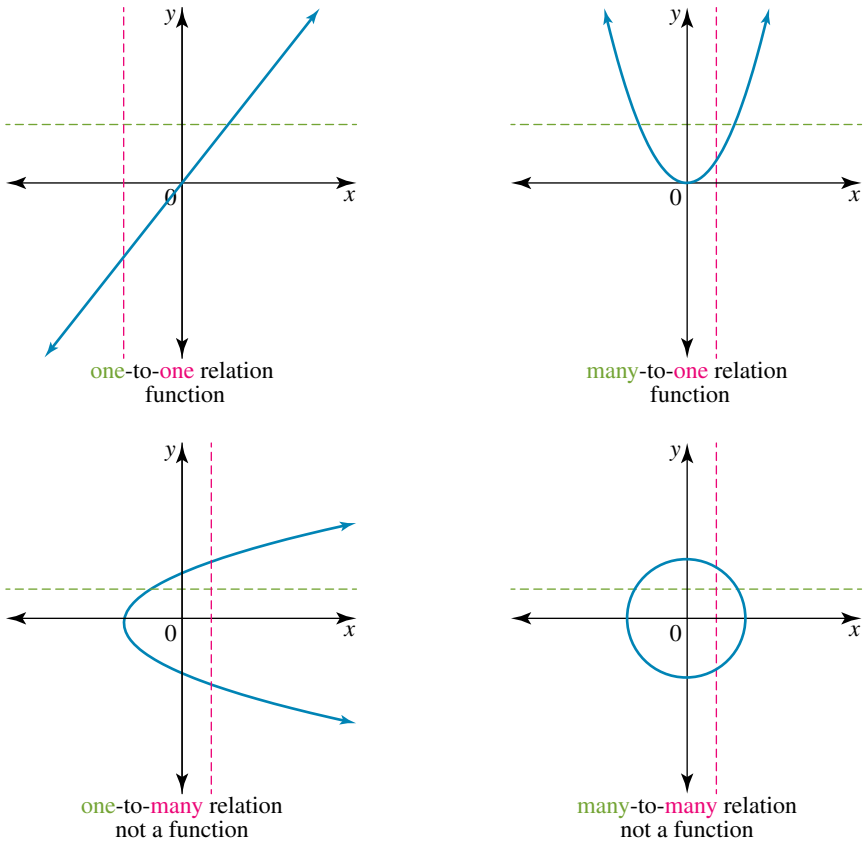
Relations which are one-to-one or many-to-one are called **functions**. That is, a function is a relation where for any x-value there is only one y-value.

The vertical line test

Line tests can be used to help classify functions and relations from a graph. A vertical line test is used to identify a function and can be applied by placing a vertical line (parallel to the y-axis) through the graph. If

there is only one intersection between this line and the graph for each possible x-value, then the graph is a function. If the line can be placed such that it intersects the graph more than once, while remaining parallel to the y-axis, then the graph does not represent a function. All polynomial relations are functions.

A horizontal line test, parallel to the x-axis, can be applied in a similar way to classify the type of the relation.



Notice that the first two graphs above pass the vertical line test (shown in pink), while the bottom two graphs do not. All four graphs are relations, but only the top two are functions. The horizontal line test (shown in green) has been applied to classify the type of relation.

on Resources

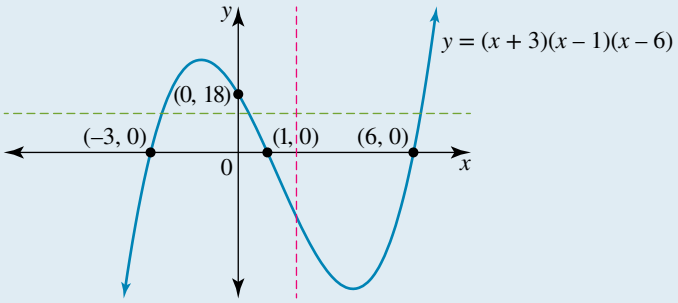
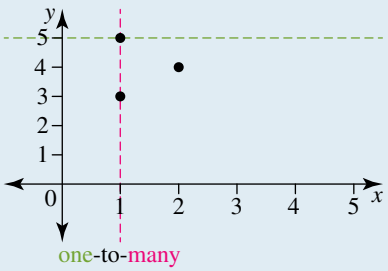
Interactivity: Vertical and horizontal line test (int-2570)

WORKED EXAMPLE 4

Classify each of the following relations as one-to-one, one-to-many, many-to-one, or many-to-many, or and state whether each relation is a function or not.

a. $y = (x + 3)(x - 1)(x - 6)$

b. $\{(1, 3), (2, 4), (1, 5)\}$

THINK	WRITE
a. 1. Draw the graph.	**a.** $y = (x + 3)(x - 1)(x - 6)$ x-intercepts: $(-3, 0)$, $(1, 0)$, $(6, 0)$ y-intercept: $(0, 18)$ The graph is a positive cubic. 
2. Use the horizontal line test and the vertical line test to determine the type of relation.	A horizontal line cuts the graph in more than one place. A vertical line cuts the graph exactly once. This is a many-to-one relation.
3. State whether the relation is a function.	$y = (x + 3)(x - 1)(x - 6)$ is a many-to-one relation which is a function.
b. 1. Look to see if there are points with the same x- or y-coordinates.	**b.** $\{(1, 3), (2, 4), (1, 5)\}$ $x = 1$ is paired to both $y = 3$ and $y = 5$. This is a one-to-many relation. It is not a function.
2. Alternatively, or as a check, plot the points and use the horizontal and vertical line tests.	 one-to-many A horizontal line cuts the graph exactly once. A vertical line cuts the graph in more than one place. This is a one-to-many relation.

studyon

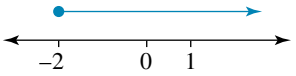
Units 1 & 2 > Area 2 > Sequence 1 > Concept 1

Functions and relations Summary screen and practice questions

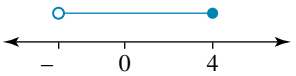
Exercise 2.2 Functions and relations

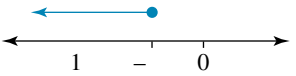
Technology free

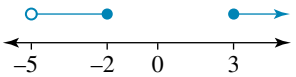
1. WE1 Describe each of the following subsets of the real numbers using interval notation.

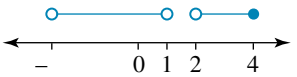
a.


b.


c.


d.


e.


f.


2. WE2 Illustrate each of the following number intervals on a number line.

a. $[-6, 2)$
b. $(-9, -3)$
c. $(-\infty, 2]$
d. $(1, 10]$
e. $(-\infty, -2) \cup [1, 3)$
f. $[-8, 0) \cup (2, 6]$

3. Describe each of the following sets using interval notation.

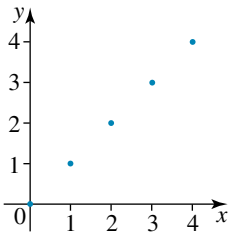
a. $\{x : -4 \leq x < 2\}$
b. $\{y : -1 < y < \sqrt{3}\}$
c. $\{x : x > 3\}$
d. $\{x : x \leq -3\}$
e. R
f. $R \setminus \{0\}$

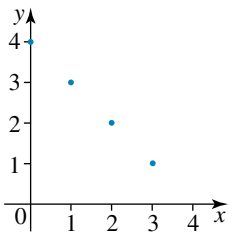
Note: Questions **4**, **5** and **6** relate to the following information.

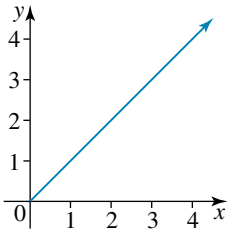
A particular relation is described by the following ordered pairs:

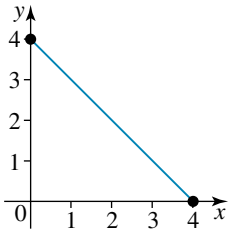
$$\{(0, 4), (1, 3), (2, 2), (3, 1)\}.$$

4. MC The graph of this relation is represented by:

A.


B.


C.


D.


5. MC The elements of the dependent variable are:

A. $\{1, 2, 3, 4\}$
B. $\{1, 2, 3\}$
C. $\{0, 1, 2, 3, 4\}$
D. $\{0, 1, 2, 3\}$

6. MC The rule for the relation is correctly described by:

A. $y = 4 - x,\ x \in R$ **B.** $y = x - 4,\ x \in N$

C. $y = 4 - x,\ x \in N$ **D.** $y = 4 - x,\ x \in \{0, 1, 2, 3\}$

7. WE3 Sketch the graph representing each of the following relations, and state whether each is discrete or continuous.

a.

Day	Mon	Tues	Wed	Thur	Fri	Sat	Sun
Cost of petrol (c/L)	168	167.1	166.5	164.9	167	168.5	170

b. $\{(0, 0), (1, 1), (2, 4), (3, 9)\}$

c. $y = -x^2$, where $x \in \{-2, -1, 0, 1, 2\}$

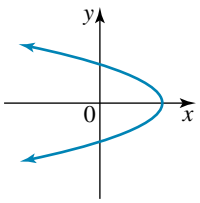
d. $y = x - 2$, where $x \in R$

e. $y = 2x + 3$, where $x \in J$

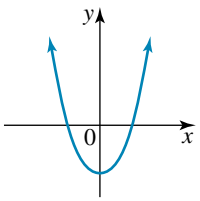
f. $y = x^2 + 2$, where $-2 \leq x \leq 2$ and $x \in R$

8. WE4 Classify each of the following relations as one-to-one, one-to-many, many-to-one or many-to-many, and state whether each relation is a function or not.

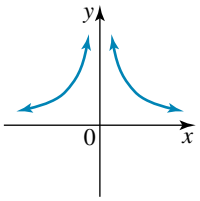
a.



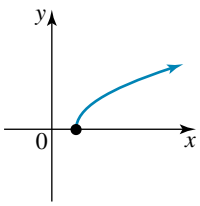
b.



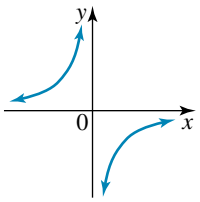
c.



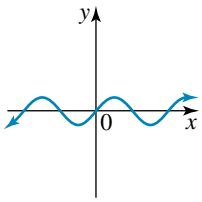
d.



e.

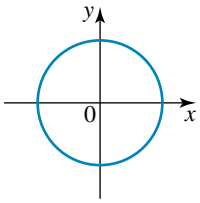


f.

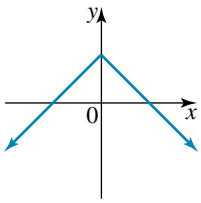


9. Classify each of the following relations as one-to-one, one-to-many, many-to-one or many-to-many, and state whether each relation is a function or not.

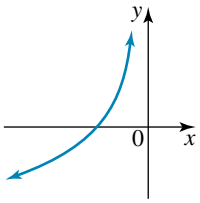
a.



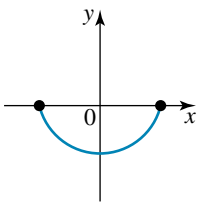
b.



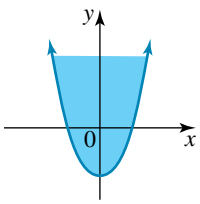
c.



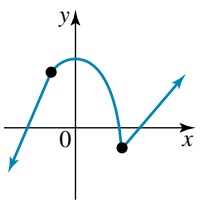
d.



e.



f.

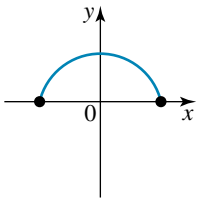


10. Consider the relations below and state:

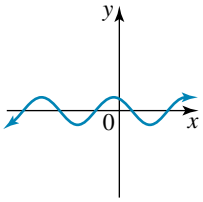
i. which are functions

ii. which are one-to-one functions.

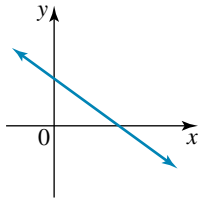
a.



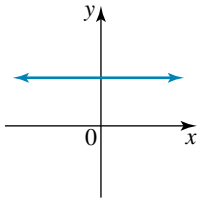
b.



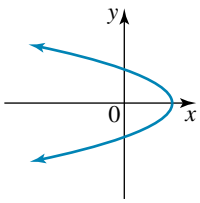
c.



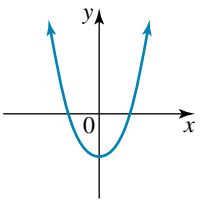
d.



e.



f.

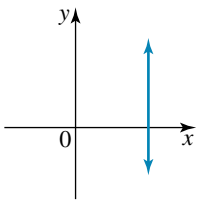


11. Consider the relations below and state:

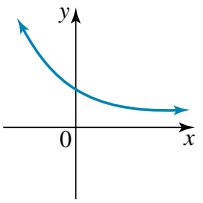
i. which are functions

ii. which are one-to-one functions.

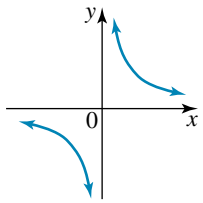
a.



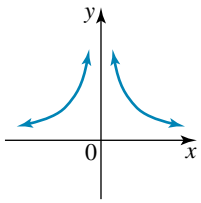
b.



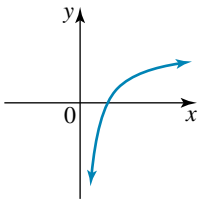
c.



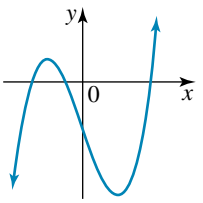
d.



e.



f.



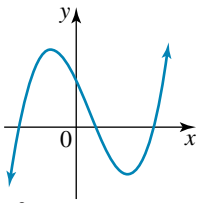
12. MC Which of the following rules does not describe a function?

A. $y = \frac{x}{5}$ **B.** $y = 2 - 7x$ **C.** $x = 5$ **D.** $y = 10x^2 + 3$

13. MC Which of the following relations is not a function?

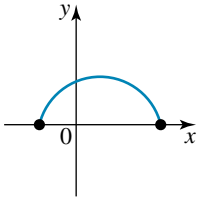
A. {(5, 8), (6, 9), (7, 9), (8, 10), (9, 12)}

B.



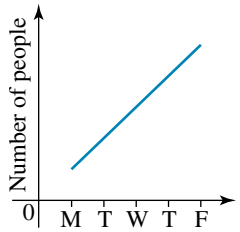
C. $y^2 = x$

D.

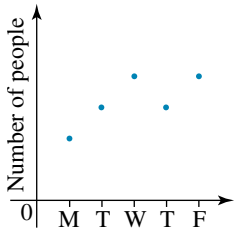


14. MC During one week, the number of people travelling on a particular train at a certain time progressively increases from Monday through to Friday. Which graph below best represents this information?

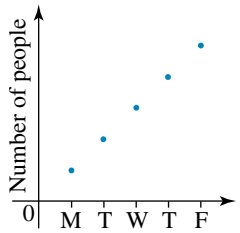
A.



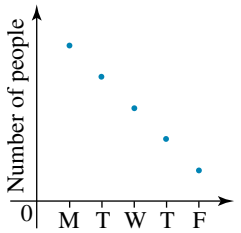
B.



C.



D.



15. The table below shows the temperature of a cup of coffee, T °C, t minutes after it is poured.



t (min)	0	2	4	6	8
T (°C)	80	64	54	48	44

a. Plot the points on a graph.

b. Join the points with a smooth curve.

c. Explain why this can be done.

d. Use the graph to determine how long it takes the coffee to reach half of its initial temperature.

16. A salesperson in a computer store is paid a base salary of $300 per week plus $40 commission for each computer she sells. If n is the number of computers she sells per week and P dollars is the total amount she earns per week, then:

a. copy and complete the following table

n	0	1	2	3	4	5	6
P							

b. plot the information on a graph

c. explain why the points cannot be joined together

d. write an equation in terms of P and n to represent this situation.

2.3 Function notation

2.3.1 Domain and range

A relation may be described by:

1. a listed set of ordered pairs
2. a graph
3. a rule or formula that defines one variable quantity in terms of another.

The set of all first elements of a set of ordered pairs is known as the **domain** and the set of all second elements of a set of ordered pairs is known as the **range**. Alternatively, the domain is the set of independent (x) values and the range is the set of dependent (y) values.

If a relation is described by a rule, it should also specify the domain. For example:

1. the relation $\{(x, y) : y = 2x, x \in \{1, 2, 3\}\}$ describes the set of ordered pairs $\{(1, 2), (2, 4), (3, 6)\}$
2. the domain is the set $X = \{1, 2, 3\}$, which is given
3. the range is the set $Y = \{2, 4, 6\}$, and can be found by applying the rule $y = 2x$ to the domain values.

If the domain of a relation is not specifically stated, it is assumed to consist of all real numbers for which the rule has meaning. This is referred to as the **implied domain** (or maximal domain) of a relation. For example:

1. $\{(x, y) : y = x^3\}$ has the implied domain R.
2. $\{(x, y) : y = \sqrt{x}\}$ has the implied domain $x \geq 0$, where $x \in R$, since the square root of a negative number is an imaginary value.

The graph of any polynomial relation normally has a domain of R. For some practical situations, restrictions have been placed on the values of the variables in some polynomial models. In these cases the polynomial relation has been defined on a **restricted domain**. A restricted domain usually affects the range.

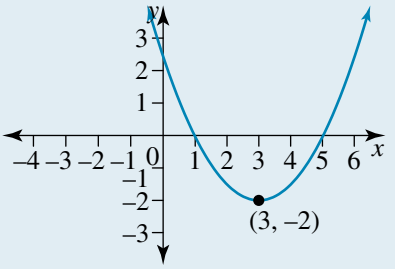
Set notation or interval notation should be used for domains and ranges.

WORKED EXAMPLE 5

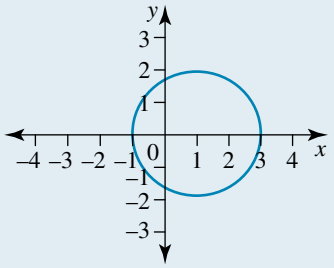
For each of the following, state the domain and range, and whether the relation is a function or not.

a. **$\{(1, 4), (2, 0), (2, 3), (5, -1)\}$**

b.



c.



d. **$\{(x, y) : y = 4 - x^3\}$**

THINK

a. 1. State the domain.

2. State the range.

WRITE

a. $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$

The domain is the set of x-values: $\{1, 2, 5\}$.

The range is the set of y-values: $\{-1, 0, 3, 4\}$.

3. Are there any ordered pairs which have the same x-coordinate?

The relation is not a function since there are two different points with the same x-coordinate: (2, 0) and (2, 3).

b. 1. Reading from left to right horizontally in the direction of the x-axis, the graph uses every possible x-value.

State the domain.

b.

The domain is $(-\infty, \infty)$ or R.

2. Reading from bottom to top vertically in the direction of the y-axis, the graph's y-values start at -2 and increase from there.

State the range.

The range is $[-2, \infty)$ or $\{y : y \geq -2\}$.

3. Use the vertical line test.

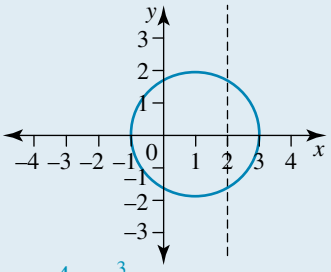
This is a function since any vertical line cuts the graph exactly once.

c. 1. State the domain and range.

c. The domain is $[-1, 3]$; the range is $[-2, 2]$.

2. Use the vertical line test.

This is not a function as a vertical line can cut the graph more than once.



d. 1. State the domain.

d. $y = 4 - x^3$

This is the equation of a polynomial so its domain is R.

2. It is the equation of a cubic polynomial with a negative coefficient of its leading term, so as $x \to \pm\infty$, $y \to \mp\infty$.

State the range.

The range is R.

3. Is the relation a function?

This is a function because all polynomial relations are functions, and it passes the vertical line test.

WORKED EXAMPLE 6

For each relation given, sketch its graph and state the domain and range using interval notation.

a. $\{(x, y) : y = \sqrt{x - 1}\}$ **b.** $\{(x, y) : y = x^2 - 4, x \in [0, 4]\}$

THINK

a. 1. The rule has meaning for $x \geq 1$ because if $x < 1, y = \sqrt{\text{negative number}}$.

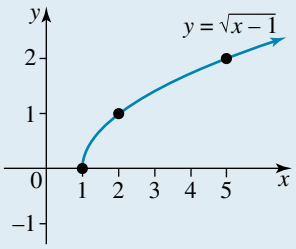
WRITE

a.

2. Therefore, calculate the value of y when $x = 1, 2, 3, 4$ and 5, and state the coordinate points.

When $x = 1$, $y = \sqrt{0}$
$= 0$ $(1, 0)$
$x = 2$, $y = \sqrt{1}$
$= 1$ $(2, 1)$
$x = 3$, $y = \sqrt{2}$ $(3, \sqrt{2})$
$x = 4$, $y = \sqrt{3}$ $(4, \sqrt{3})$
$x = 5$, $y = \sqrt{4}$
$= 2$ $(5, 2)$

3. Plot the points on a set of axes.
4. Join the points with a smooth curve starting from $x = 1$, extending it beyond the last point. Since no domain is given we can assume $x \in R$ (continuous).
5. Place a closed circle on the point $(1, 0)$ and put an arrow on the other end of the curve.



6. The domain is the set of values covered horizontally by the graph, or implied by the rule.

Domain $= [1, \infty)$

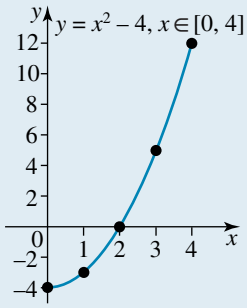
7. The range is the set of values covered vertically by the graph.

Range $= [0, \infty)$

b. 1. Calculate the value of y when $x = 0, 1, 2, 3$ and 4, as the domain is $[0, 4]$. State the coordinate points.

b. When $x = 0$, $y = 0^2 - 4$
$= -4$ $(0, -4)$
$x = 1$, $y = 1^2 - 4$
$= -3$ $(1, -3)$
$x = 2$, $y = 2^2 - 4$
$= 0$ $(2, 0)$
$x = 3$, $y = 3^2 - 4$
$= 5$ $(3, 5)$
$x = 4$, $y = 4^2 - 4$
$= 12$ $(4, 12)$

2. Plot these points on a set of axes.
3. Join the dots with a smooth curve from $x = 0$ to $x = 4$.
4. Place a closed circle on the points $(0, -4)$ and $(4, 12)$.



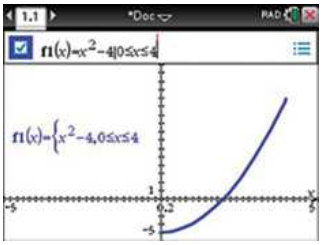
5. The domain is the set of values covered by the graph horizontally. Domain = [0, 4]
6. The range is the set of values covered by the graph vertically. Range = [−4, 12]

Technology can be used to check the graphs.

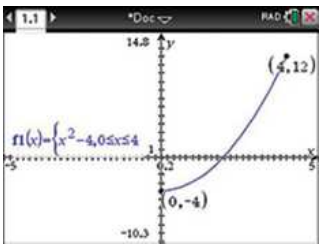
TI | THINK

b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = x^2 - 4 | 0 \leq x \leq 4$ then press ENTER. The graph appears on the screen.

WRITE



2. To find the endpoints of the graph, press MENU, then select:
5: Trace
1: Graph Trace.
Type '0' then press ENTER twice.
Type '4' then press ENTER twice.



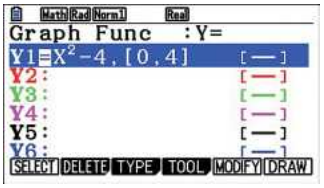
3. The domain and range can be read from the graph.

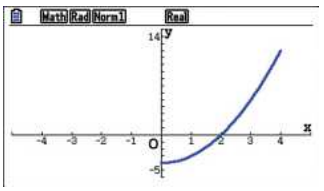
The domain is [0, 4] and the range is [–4, 12].

CASIO | THINK

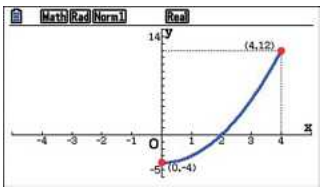
b1. On a Graph screen, complete the entry line for Y1 as $Y1 = x^2 - 4, [0, 4]$ then press EXE. Select DRAW by pressing F6.

WRITE





2. To find the endpoints of the graph, select Trace by pressing F1. Type '0' then press EXE twice. Type '4' then press EXE twice.



3. The domain and range can be read from the graph.

The domain is [0, 4] and the range is [–4, 12].

2.3.2 Function notation

The rule for a function such as $y = x^2$ will often be written as $f(x) = x^2$. This is read as 'f of x equals x^2'. We shall also refer to a function as $y = f(x)$, particularly when graphing a function as the set of ordered pairs (x, y) with x as the independent variable and y as the dependent variable.

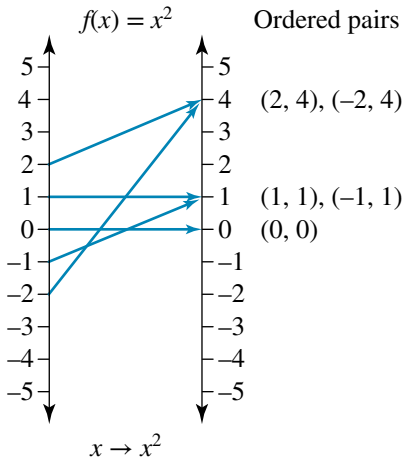
$f(x)$ is called the **image** of x under the function **mapping**, which means that if, for example, $x = 2$ then $f(2)$ is the y-value that $x = 2$ is paired with (mapped to), according to the function rule.

For $f(x) = x^2$, $f(2) = 2^2 = 4$. The image of 2 under the mapping f is 4; the ordered pair $(2, 4)$ lies on the graph of $y = f(x)$; 2 is mapped to 4 under f: these are all variations of the mathematical language that could be used for this function.

The ordered pairs that form the function with rule $f(x) = x^2$ could be illustrated on a mapping diagram. The mapping diagram shown uses two number lines, one for the x-values and one for the y-values, but there are varied ways to show mapping diagrams.

Under the mapping, every x-value in the domain is mapped to its square, $x \rightarrow x^2$. The range is the set of the images, or corresponding y-values, of each x-value in the domain. For this example, the polynomial function has a domain of R and a range of $[0, \infty)$, since squared numbers are not negative. Not all of the real numbers on the $y$number line are elements of the range in this example. The set of all the available y-values, whether used in the mapping or not, is called the **codomain**. Only the set of those y-values which are used for the mapping form the range. For this example, the codomain is R and the range is a subset of the codomain since $[0, \infty) \subset R$.

The mapping diagram also illustrates the many-to-one correspondence of the function defined by $y = x^2$.



WORKED EXAMPLE 7

If $f(x) = x^2 - 3$, find:

a. $f(-2)$ **b.** $f(a)$ **c.** $f(2a)$ **d.** $f(a+1)$.

THINK

a. 1. Write the rule.

2. Substitute $x = -2$ into the rule.

3. Simplify the expression if possible.

b. 1. Write the rule.

2. Substitute $x = a$ into the rule.

c. 1. Write the rule.

2. Substitute $x = 2a$ into the rule.

3. Simplify the expression if possible.

d. 1. Write the rule.

2. Substitute $x = 2a + 1$ into the rule.

3. Simplify the expression if possible.

WRITE

a. $f(x) = x^2 - 3$

$f(-2) = (-2)^2 - 3$

$= 4 - 3$

$= 1$

b. $f(x) = x^2 - 3$

$f(a) = a^2 - 3$

c. $f(x) = x^2 - 3$

$f(2a) = (2a)^2 - 3$

$= 2^2a^2 - 3$

$= 4a^2 - 3$

d. $f(x) = x^2 - 3$

$f(a+1) = (a+1)^2 - 3$

$f(a+1) = a^2 + 2a + 1 - 3$

$= a^2 + 2a - 2$

2.3.3 Formal mapping notation

The mapping $x \rightarrow x^2$ is written formally as:

$$\underset{\substack{\downarrow \\ \text{name of function}}}{f:} \quad \underset{\substack{\downarrow \\ \text{domain of } f}}{R} \quad \rightarrow \quad \underset{\substack{\downarrow \\ \text{codomain}}}{R}, \quad \underset{\substack{\downarrow \\ \text{rule for, or equation of, } f}}{f(x) = x^2}$$

The domain of the function must always be specified when writing functions formally.

We will always use R as the codomain. Mappings will be written as $f: D \rightarrow R$, where D is the domain. Usually a graph of the function is required in order to determine its range.

Note that f is a symbol for the name of the function or mapping, whereas $f(x)$ is an element of the range of the function: $f(x)$ gives the image of x under the mapping f. While f is the commonly used symbol for a function, other symbols may be used.

WORKED EXAMPLE 8

Consider $f: R \rightarrow R, f(x) = a + bx$, where $f(1) = 4$ and $f(-1) = 6$.

a. Calculate the values of a and b and state the function rule.

b. Evaluate $f(0)$.

c. Calculate the value of x for which $f(x) = 0$.

d. Find the image of -5.

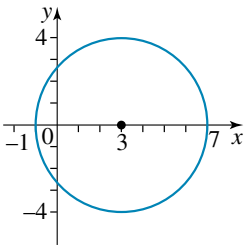
e. Write the mapping for a function g which has the same rule as f but a domain restricted to R^+.

THINK	WRITE
a. 1. Use the given information to set up a system of simultaneous equations.	**a.** $f(x) = a + bx$ $f(1) = 4 \Rightarrow 4 = a + b \times 1$ $\therefore\ a + b = 4$.....(1) $f(-1) = 6 \Rightarrow 6 = a + b \times -1$ $\therefore\ a - b = 6$.....(2)
2. Solve the system of simultaneous equations to obtain the values of a and b.	Equation (1) + equation (2) $2a = 10$ $a = 5$ Substitute $a = 5$ into equation (1) $\therefore\ b = -1$
3 State the answer.	$a = 5, b = -1$ $f(x) = 5 - x$
b. Substitute the given value of x.	**b.** $f(x) = 5 - x$ $f(0) = 5 - 0$ $= 5$
c. Substitute the rule for $f(x)$ and solve the equation for x.	**c.** $f(x) = 0$ $5 - x = 0$ $\therefore x = 5$
d. Write the expression for the image and then evaluate it.	**d.** The image of -5 is $f(-5)$. $f(x) = 5 - x$ $f(-5) = 5 - (-5)$ $= 10$ The image is 10.
e. Change the name of the function and change the domain.	**e.** $g: R^+ \rightarrow R, g(x) = 5 - x$

Exercise 2.3 Function notation

Technology free

1. MC The domain of the relation graphed at right is:



A. $[-4, 4]$
B. $(-4, 7)$
C. $[-1, 7]$
D. $(-4, 4)$

2. MC The range of the relation $\{(x, y) : y = 2x + 5, x \in [-1, 4]\}$ is:
A. $[7, 13]$
B. $[3, 13]$
C. $[3, \infty)$
D. R

3. MC A relation has the $y = x + 3$, where $x \in R^+$. The range of this relation is:
A. R^+
B. $[3, \infty)$
C. R
D. $(3, \infty)$

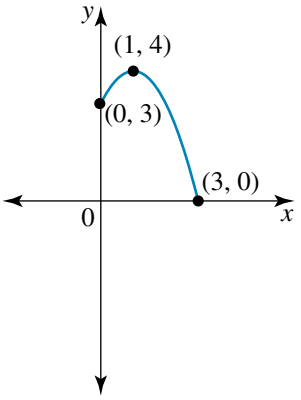
4. MC The function $f: \{x : x = 0, 1, 2\} \rightarrow R$, where $f(x) = x - 4$, may be expressed as:
A. $\{(0, -4), (1, -3), (2, -2)\}$
B. $\{0, 1, 2\}$
C. $\{(0, 4), (1, 3), (2, 2)\}$
D. $\{(-1, -5), (1, -3), (2, -2)\}$

5. WE5 For each of the following, state the domain and range and whether the relation is a function or not.

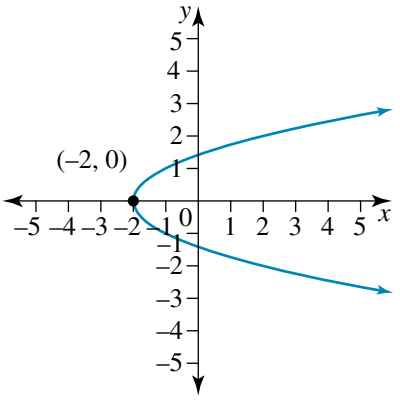
a. $\{(4, 4), (3, 0), (2, 3), (0, -1)\}$

b. $\{(x. y) : y = 4 - x^2\}$

c.

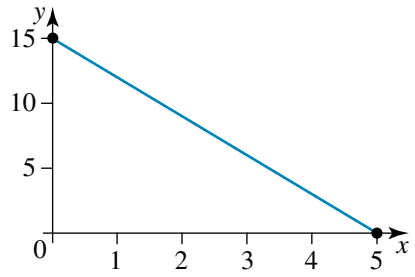


d.

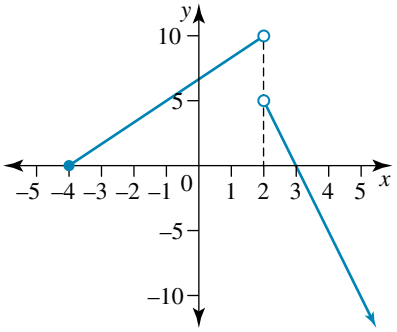


6. State the domain and range for each of the following relations.

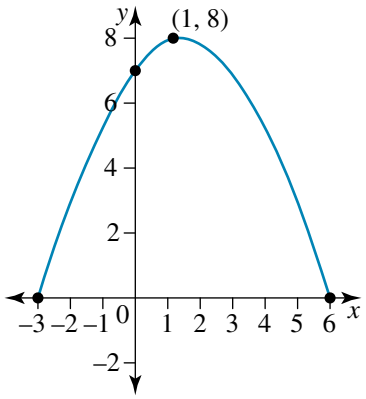
a.



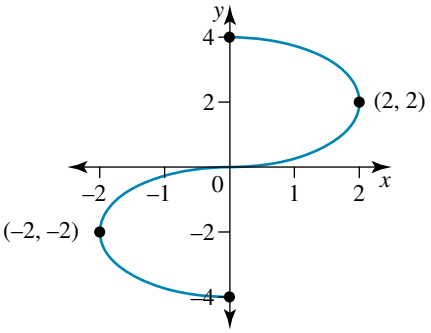
b.



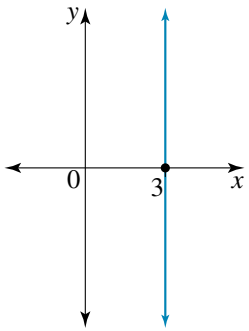
c.



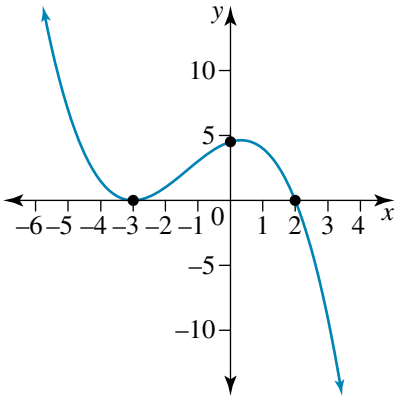
d.



e.



f.



7. Consider each of the graphs in question **6**.

a. Classify each relation as one-to-one, many-to-one, one-to-many or many-to-many.

b. Identify any of the relations that are not functions.

8. State

i. the domain and

ii. the range of each of the following relations.

a. {(3, 8), (4, 10), (5, 12), (6, 14), (7, 16)}

b. {(1.1, 2), (1.3, 1.8), (1.5, 1.6), (1.7, 1.4)}

c.

Time (min)	3	4	5	6
Distance (m)	110	130	150	170

d.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Cost($)	25	35	30	35	30

e. $y = 5x - 2$, where x is an integer greater than 2 and less than 6.

f. $y = x^2 - 1,\ x \in R$

9. a. Sketch the graph of $y = (x - 2)^2$, stating its domain, range and type of relation.

b. Restrict the domain of the function defined by $y = (x - 2)^2$ so that it will be a one-to-one and increasing function.

10. WE7 **a.** If $f(x) = 3x + 1$, determine

i. $f(0)$ **ii.** $f(2)$ **iii.** $f(-2)$ **iv.** $f(5)$

b. If $g(x) = \sqrt{x + 4}$, determine

i. $g(0)$ **ii.** $g(-3)$ **iii.** $g(5)$ **iv.** $g(-4)$

c. If $g(x) = 4 - \frac{1}{x}$, determine

i. $g(1)$ **ii.** $g\left(\frac{1}{2}\right)$ **iii.** $g\left(-\frac{1}{2}\right)$ **iv.** $g\left(-\frac{1}{5}\right)$

d. If $f(x) = (x + 3)^2$, determine

i. $f(0)$ **ii.** $f(-2)$ **iii.** $f(1)$ **iv.** $f(a)$

11. If $f(x) = x^2 + 2x - 3$, calculate the following.

a. i. $f(-2)$ **ii.** $f(9)$

b. i. $f(2a)$ **ii.** $f(1 - a)$

c. $f(x + h) - f(x)$

d. $\{x : f(x) > 0\}$

e. The values of x for which $f(x) = 12$

f. The values of x for which $f(x) = 1 - x$

12. Determine the value (or values) of x for which each function has the value given.

a. $f(x) = 3x - 4,\ f(x) = 5$ **b.** $g(x) = x^2 - 2,\ g(x) = 7$

c. $f(x) = \frac{1}{x},\ f(x) = 3$ **d.** $h(x) = x^2 - 5x + 6,\ h(x) = 0$

e. $g(x) = x^2 + 3x,\ g(x) = 4$ **f.** $f(x) = \sqrt{8 - x},\ f(x) = 3$

13. WE8 Consider $f: R \to R,\ f(x) = ax + b$, where $f(2) = 7$ and $f(3) = 9$.

a. Calculate the values of a and b and state the function rule.

b. Calculate the value of x for which $f(x) = 0$.

c. Evaluate the image of -3.

d. Write the mapping for a function g which has the same rule as f but a domain restricted to $(-\infty, 0]$.

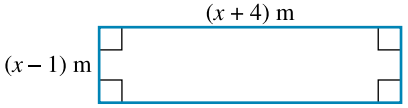
Technology active

14. MC The range of the function, $f(x) = 2\sqrt{4 - x}$ is:

A. R^+ **B.** R^- **C.** $[0, \infty)$ **D.** $(2, \infty)$

15. Express $y = x^2 - 6x + 10,\ 0 \leq x < 7$ in mapping notation and state its domain and range.

16. The maximum side length of the rectangle shown is 10 metres.



a. Write a function which gives the perimeter, P metres, of the rectangle.

b. State the domain and range of this function.

17. WE6 For each relation given, sketch its graph and state the domain and range using interval notation.

a. $\{(x, y) :\ y = 2 - x^2\}$ **b.** $\{(x, y) :\ y = x^3 + 1,\ x \in [-2, 2]\}$

c. $\{(x, y) :\ y = x^2 + 3x + 2\}$ **d.** $\{(x, y) :\ y = x^2 - 4,\ x \in [-2, 1]\}$

e. $\{(x, y) :\ y = 2x - 5,\ x \in [-1, 4)\}$ **f.** $\{(x, y) :\ y = 2x^2 - x - 6\}$

18. State the implied domain for each relation defined by the following rules.

a. $y = 10 - x$ **b.** $y = 3\sqrt{x}$ **c.** $y = -\sqrt{16 - x^2}$

d. $y = x^2 + 3$ **e.** $y = \dfrac{1}{x}$ **f.** $y = 10 - 7x^2$

19. The number of koalas remaining in a parkland t weeks after a virus strikes is given by the function $N(t) = 15 + \dfrac{96}{t+3}$ koalas per hectare.



a. How many koalas per hectare were there before the virus struck?

b. How many koalas per hectare were there 13 weeks after the virus strikes?

c. How long after the virus strikes are there 23 koalas per hectare?

d. Will the virus kill off all the koalas? Explain why.

20. Consider the functions f and g where $f(x) = a + bx + cx^2$ and $g(x) = f(x - 1)$.

a. Given $f(-2) = 0$, $f(5) = 0$ and $f(2) = 3$, determine the rule for the function f.

b. Express the rule for g as a polynomial in x.

c. Calculate any values of x for which $f(x) = g(x)$.

d. On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ and describe the relationship between the two graphs.

2.4 Transformations of functions

A graph has undergone one or more **transformations** if its position or shape has been altered. The transformations covered in this chapter are:

- **dilations** (stretching or compressing)
- **reflections** (flipping)
- **translations** (moving horizontally or vertically).

Once the basic shape of a function is known, its features can be identified after various transformations have been applied to it simply by interpreting the transformed equation of the image.

2.4.1 Dilations

A dilation from an axis either stretches or compresses a graph from that axis, depending on whether the dilation factor is greater than 1 or between 0 and 1, respectively.

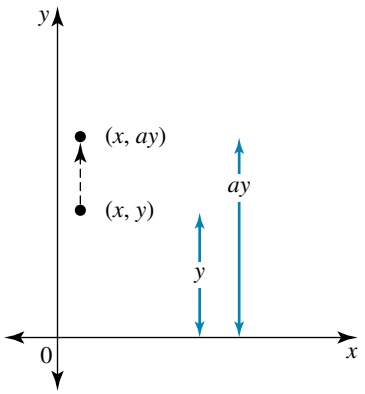
2.4.2 Dilation from the x-axis by factor a

A dilation from the x-axis acts parallel to the y-axis, or in the y-direction.

The point $(x, y) \rightarrow (x, ay)$ when dilated by a factor a from the x-axis.

A dilation of factor a from the x-axis transforms $y = x^2$ to $y = ax^2$ and, generalising, under a dilation of factor a from the x-axis, $y = f(x) \rightarrow y = af(x)$.

For any function:
$y = af(x)$ is the image of $y = f(x)$ under a dilation of factor a from the x-axis, parallel to the y-axis.

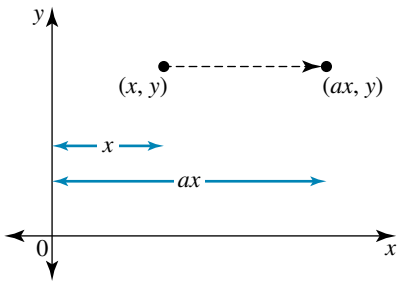


Dilation of factor a, $(a > 1)$, from the x-axis

2.4.3 Dilation from the *y*-axis by factor *b*

A dilation from the *y*-axis acts parallel to the *x*-axis, or in the *x*-direction.

The point $(x, y) \rightarrow (bx, y)$ when dilated by a factor b from the *y*-axis. To see the effect of this dilation, consider the graph of $y = x(x - 2)$ under a dilation of factor 2 from the *y*-axis. Choosing the key points, under this dilation:

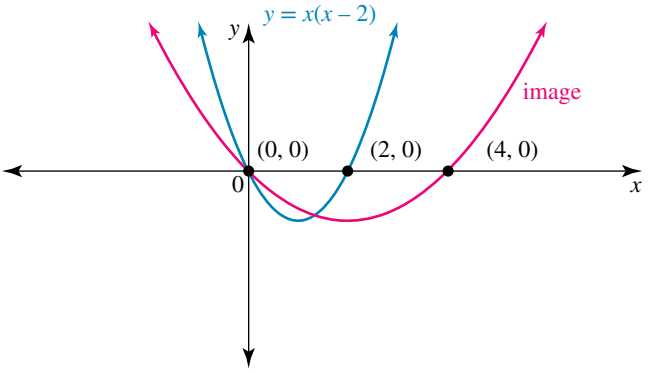


$$\begin{aligned} (0, 0) &\rightarrow (0, 0) \\ (1, -1) &\rightarrow (2, -1) \\ (2, 0) &\rightarrow (4, 0) \end{aligned}$$

and the transformed graph is as shown.

The equation of the image of $y = x(x - 2)$ under this dilation can be found by fitting the points to a quadratic equation. Its equation is $y = (0.5x)(0.5x - 2) \Rightarrow y = \left(\frac{x}{2}\right)\left(\left(\frac{x}{2}\right) - 2\right)$.

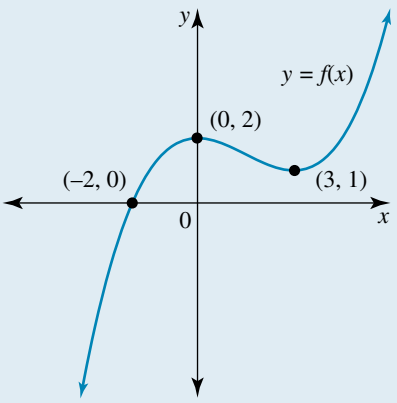
This illustrates that dilating $y = f(x)$ by a factor 2 from the *y*-axis gives the image $y = f\left(\frac{x}{2}\right)$.

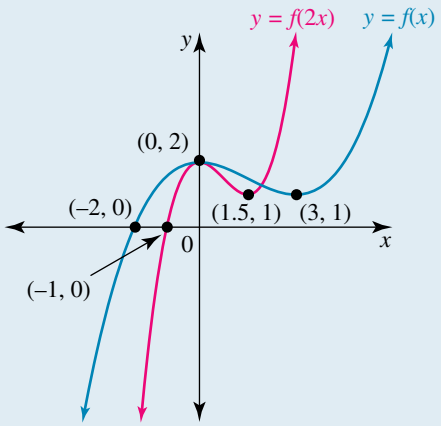


For any function:
$y = f(bx)$ is the image of $y = f(x)$ under a dilation of factor $\frac{1}{b}$ from the *y*-axis, parallel to the *x*-axis.

WORKED EXAMPLE 9

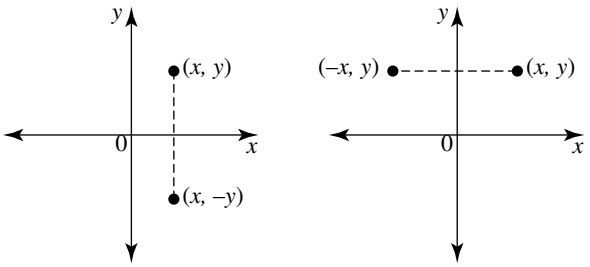
The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
Sketch the graph of $y = f(2x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.



THINK	WRITE
1. Identify the transformation.	$y = f(2x) \Rightarrow y = f\left(\frac{x}{\frac{1}{2}}\right)$ The transformation is a dilation from the y-axis of factor $\frac{1}{2}$. This dilation acts in the x-direction.
2. Find the image of each key point.	Under this dilation $(x, y) \rightarrow \left(\frac{x}{2}, y\right)$ $(-2, 0) \rightarrow (-1, 0)$ $(0, 2) \rightarrow (0, 2)$ $(3, 1) \rightarrow (1.5, 1)$
3. Sketch the image.	

2.4.4 Reflections

The point (x, y) becomes $(x, -y)$ when reflected in the x-axis and $(-x, y)$ when reflected in the y-axis.



Reflection in the x-axis Reflection in the y-axis

Reflecting the graph of $y = \sqrt{x}$ in the x-axis gives the graph of $y = -\sqrt{x}$, so under a reflection in the x-axis, $y = f(x) \rightarrow y = -f(x)$.

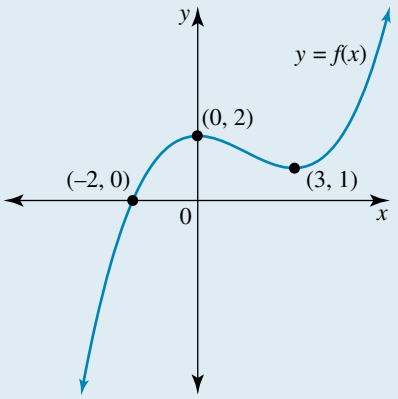
Reflecting the graph of $y = \sqrt{x}$ in the y-axis gives the graph of $y = \sqrt{-x}$, so under a reflection in the y-axis, $y = f(x) \rightarrow y = f(-x)$.

For any function:

- **$y = -f(x)$ is the image of $y = f(x)$ under a reflection in the x-axis**
- **$y = f(-x)$ is the image of $y = f(x)$ under a reflection in the y-axis**

WORKED EXAMPLE 10

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
Sketch the graph of $y = f(-x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.



THINK

1. Identify the transformation required.

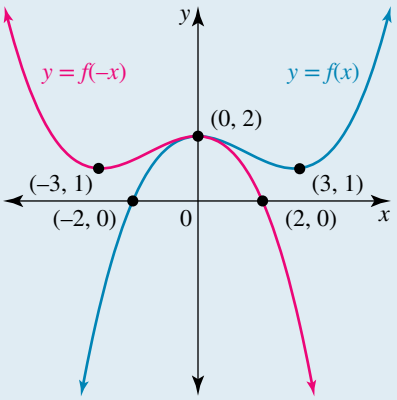
WRITE

$y = f(-x)$
This is a reflection in the y-axis of the graph of $y = f(x)$.

2. Find the image of each key point.

Under this transformation, $(x, y) \rightarrow (-x, y)$
$(-2, 0) \rightarrow (2, 0)$
$(0, 2) \rightarrow (0, 2)$
$(3, 1) \rightarrow (-3, 1)$

3. Sketch the image.



2.4.5 Translations

Translations parallel to the x- and y-axis move graphs horizontally to the left or right and vertically up or down, respectively.

Under a horizontal translation of c units to the left, the following effect is seen:

$$y = x^2 \rightarrow y = (x + c)^2;$$
$$y = \frac{1}{x} \rightarrow y = \frac{1}{x + c};$$
$$y = \sqrt{x} \rightarrow y = \sqrt{x + c};$$

and so, for any function, $y = f(x) \rightarrow y = f(x + c)$.

Under a vertical translation of d units upwards:

$$y = x^2 \rightarrow y = x^2 + d;$$
$$y = \frac{1}{x} \rightarrow y = \frac{1}{x} + d;$$
$$y = \sqrt{x} \rightarrow y = \sqrt{x} + d;$$

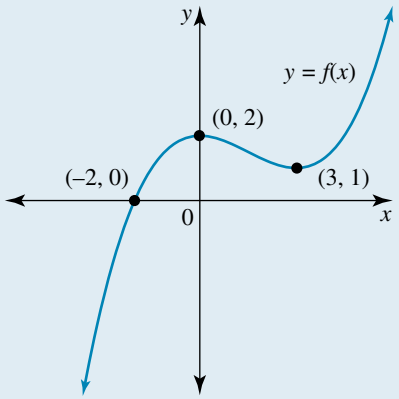
and so, for any function, $y = f(x) \rightarrow y = f(x) + d$.

For any function:

- **$y = f(x + c)$ is the image of $y = f(x)$ under a horizontal translation of c units to the left. A negative c-value will result in translation to the right.**
- **$y = f(x) + d$ is the image of $y = f(x)$ under a vertical translation of d units upwards. A negative d-value will result in translation downwards.**
- **Under the combined transformations of c units parallel to the x-axis and d units parallel to the y-axis, $y = f(x) \rightarrow y = f(x + c) + d$.**

WORKED EXAMPLE 11

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
Sketch the graph of $y = f(x + 1)$ using the images of these three points.



THINK

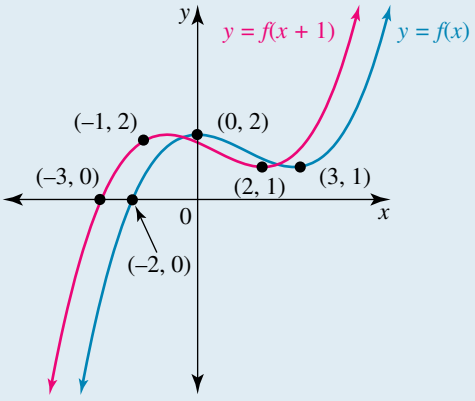
1 Identify the transformation required.

2 Find the image of each key point.

3 Sketch the image.

WRITE

$y = f(x + 1)$
This is a horizontal translation 1 unit to the left of the graph of $y = f(x)$.
Under this transformation:
$(-2, 0) \rightarrow (-3, 0)$
$(0, 2) \rightarrow (-1, 2)$
$(3, 1) \rightarrow (2, 1)$



2.4.6 Combinations of transformations

The graph of $y = af(b(x + c)) + d$ is the graph of $y = f(x)$ under a set of transformations which are identified as follows.

- **a gives the dilation factor $|a|$ from the x-axis, parallel to the y-axis.**
- **If $a < 0$, there is a reflection in the x-axis.**
- **b gives the dilation factor $\frac{1}{|b|}$ from the y-axis, parallel to the x-axis.**
- **If $b < 0$, there is a reflection in the y-axis.**
- **c gives the horizontal translation parallel to the x-axis.**
- **d gives the vertical translation parallel to the y-axis.**

When applying transformations to $y = f(x)$ to form the graph of $y = af(b(x+c))+d$, the order of operations can be important, so any dilation or reflection should be applied before any translation.

It is quite possible that more than one order or more than one set of transformations may achieve the same image. For example, $y = 4x^2$ could be considered a dilation of $y = x^2$ by factor 4 from the x-axis or, as $y = (2x)^2$, it's also a dilation of $y = x^2$ by a factor $\frac{1}{2}$ from the y-axis.

WORKED EXAMPLE 12

a. Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4 - 2f(3x + 2)$.
b. Describe the transformations applied to the graph of $y = \sqrt[3]{x}$ to obtain $y = \sqrt[3]{6 - 2x}$.

THINK	WRITE
a. 1. Express the image equation in the summary form.	**a.** $y = 4 - 2f(3x + 2)$ $= -2f\left(3\left(x + \frac{2}{3}\right)\right) + 4$
2. State the values of $a, b, c,$ and d from the summary form.	$y = af(b(x + c)) + d$ $a = -2, b = 3, c = \frac{2}{3}, d = 4$
3. Interpret the transformations, leaving the translations to last.	Dilation of factor 2 from the x-axis, followed by a reflection in the x-axis; then, a dilation of factor $\frac{1}{3}$ from the y-axis; then, a horizontal translation $\frac{2}{3}$ units to the left; finally, a vertical translation upwards of 4 units
b. 1. Express the image equation in the summary form.	**b.** $y = \sqrt[3]{6 - 2x}$ $= \sqrt[3]{-2(x - 3)}$
2. Identify the transformations in the correct order.	Dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the y-axis; then, a horizontal translation 3 units to the right

on Resources

Interactivity: Graph plotter: transformations of functions (int-2576)

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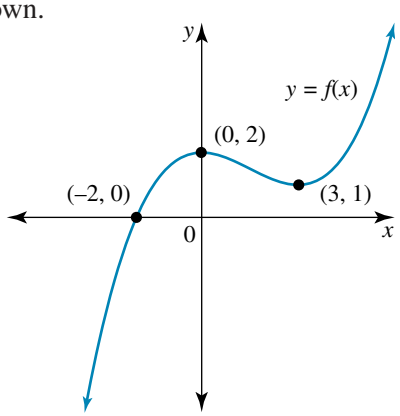
Units 1 & 2 > Area 2 > Sequence 1 > Concept 3

Transformations of functions Summary screen and practice questions

Exercise 2.4 Transformations of functions

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Note: Questions 1–5 relate to the following information.
Consider the function $y = f(x)$ as shown.



1. **WE9** For the graph of $y = f(x)$ shown above, sketch the graph of $y = f\left(\frac{x}{2}\right)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
2. For the graph of $y = f(x)$ shown above, sketch the graph of $y = \frac{1}{2}f(x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
3. **WE10** Consider again the graph given above. Sketch the graph of $y = -f(x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
4. **WE11** For the graph of $y = f(x)$ given above, sketch the graph of $y = f(x) - 2$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
5. For the graph of $y = f(x)$ given above, sketch the graph of $y = f(x - 2) + 1$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
6. **a.** The parabola with equation $y = (x - 1)^2$ is reflected in the x-axis followed by a vertical translation upwards of 3 units. What is the equation of its final image?
   b. Obtain the equation of the image if the order of the transformations in part **a** was reversed. Is the image the same as that in part **a**?
7. **WE12** **a.** Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4f\left(\frac{x}{2} - 1\right) + 3$.
   b. Describe the transformations applied to the graph of $y = \sqrt{x}$ to obtain $y = \sqrt{3 - \frac{x}{4}}$.

8. a. The graph of $y = \frac{1}{x}$ undergoes two transformations in the order: dilation of factor $\frac{1}{2}$ from the y-axis, followed by a horizontal translation of 3 units to the left.
What is the equation of its image?

b. Describe the sequence of transformations that need to be applied to the image to undo the effect of the transformations and revert to the graph of $y = \frac{1}{x}$.

9. Identify the transformations that would be applied to the graph of $y = x^2$ to obtain each of the following graphs.

a. $y = 3x^2$ **b.** $y = -x^2$ **c.** $y = x^2 + 5$ **d.** $y = (x + 5)^2$

10. Describe the transformations that have been applied to the graph of $y = x^3$ to obtain each of the following graphs.

a. $y = \left(\frac{x}{3}\right)^3$ **b.** $y = (2x)^3 + 1$ **c.** $y = (x - 4)^3 - 4$ **d.** $y = (1 + 2x)^3$

11. Give the equation of the image of

i. $y = \sqrt{x}$ and

ii. $y = x^4$ if their graphs are:

a. dilated by a factor 2 from the x-axis

b. dilated by a factor 2 from the y-axis

c. reflected in the x-axis and then translated 2 units vertically upwards

d. translated 2 units vertically upwards and then reflected in the x-axis

e. reflected in the y-axis and then translated 2 units to the right

f. translated 2 units to the right and then reflected in the y-axis.



12. Give the coordinates of the image of the point $(3, -4)$ if it is:

a. translated 2 units to the left and 4 units down

b. reflected in the y-axis and then reflected in the x-axis

c. dilated by a factor $\frac{1}{5}$ from the x-axis parallel to the y-axis

d. dilated by a factor $\frac{1}{5}$ from the y-axis parallel to the x-axis.

13. a. i. Give the equation of the image of $y = \frac{1}{x}$ after the two transformations are applied in the order given: dilation by a factor 3 from the y-axis, then reflection in the y-axis.

ii. Reverse the order of the transformations and give the equation of the image.

b. i. Give the equation of the image of $y = \frac{1}{x^2}$ after the two transformations are applied in the order given: dilation by a factor 3 from the x-axis, then vertical translation 6 units up.

ii. Reverse the order of the transformations and give the equation of the image.

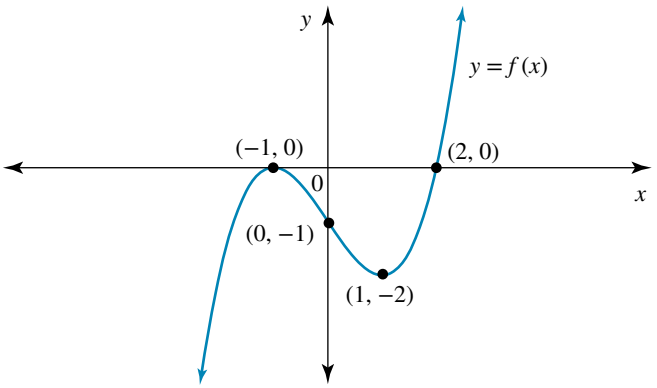
14. The graph of $y = f(x)$ is shown.

On separate diagrams sketch the graphs of the following.

a. $y = f(x - 1)$ **b.** $y = -f(x)$

c. $y = 2f(x)$ **d.** $y = f(-x)$

e. $y = f\left(\frac{x}{2}\right)$ **f.** $y = f(x) + 2$



15. Describe the transformations applied to $y = f(x)$ if its image is:

a. $y = 2f(x + 3)$ **b.** $y = 6f(x - 2) + 1$ **c.** $y = f(2x + 2)$

d. $y = f(-x + 3)$ **e.** $y = 1 - f(4x)$ **f.** $y = \frac{1}{9} f\left(\frac{x - 3}{9}\right)$.

16. Form the equation of the image after the given functions have been subjected to the set of transformations in the order specified.

a. $y = \frac{1}{x^2}$ undergoes a dilation of factor $\frac{1}{3}$ from the x-axis followed by a horizontal translation of 3 units to the left.

b. $y = x^5$ undergoes a vertical translation of 3 units down followed by reflection in the x-axis.

c. $y = \frac{1}{x}$ undergoes a reflection in the y-axis followed by a horizontal translation of 1 unit to the right.

d. $y = \sqrt[3]{x}$ undergoes a horizontal translation of 1 unit to the right followed by a dilation of factor 0.5 from the y-axis.

e. $y = (x + 9)(x + 3)(x - 1)$ undergoes a horizontal translation of 6 units to the right followed by a reflection in the x-axis.

f. $y = x^2(x + 2)(x - 2)$ undergoes a dilation of factor 2 from both the x- and y-axis.

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17. a. The function $g : R \to R, g(x) = x^2 - 4$ is reflected in the y-axis. Describe its image.

b. Show that the image of the function $f : R \to R, f(x) = x^{\frac{1}{3}}$ when it is reflected in the y-axis is the same as when it is reflected in the x-axis.

c. The function $h : [-3, 3] \to R, h(x) = -\sqrt{9 - x^2}$ is reflected in the x-axis. Describe its image. What single transformation when applied to the image would return the curve back to its original position?

d. The graph of $y = (x - 2)^2 + 5$ is reflected in both the x- and y-axis. What is the nature, and the coordinates, of the turning point of its image?

e. The graph of a relation is shifted vertically down 2 units, then reflected in the y-axis. If the equation of its image is $y^2 = (x - 3)$, undo the transformations to obtain the equation of the original graph.

f. A curve $y = f(x)$ is dilated by a factor 2 from the x-axis, then vertically translated 1 unit up, then reflected in the x-axis. After these three transformations have been applied, the equation of its image is $y = 6(x - 2)^3 - 1$. Determine the equation of $y = f(x)$.

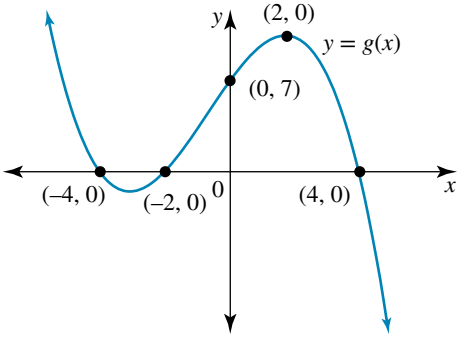
18. The graph of the function $y = g(x)$ is given.

a. Sketch the graph of $y = -g(2x)$.

b. Sketch the graph of $y = g(2 - x)$.

c. For what values of c will all the x-intercepts of the graph of $y = g(x + c)$ be negative?

d. Give a possible equation for the graph of $y = g(x)$ and hence find an expression for $g(2x)$.



2.5 Piece-wise functions

2.5.1 Piece-wise functions

A **piece-wise function** is one in which the rule may take a different form over different sections of its domain. An example of a simple piece-wise function is one defined by the rule:

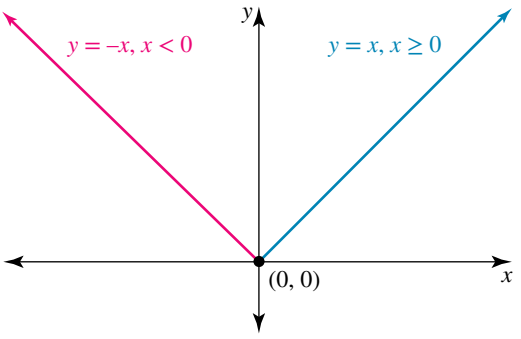
$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Graphing this function would give a line with positive gradient to the right of the y-axis and a line with negative gradient to the left of the y-axis.

This piece-wise function is continuous at $x = 0$ since both of its branches join, but that may not be the case for all piece-wise functions. If the branches do not join, the function is not continuous for that value of x: it is discontinuous at that point of its domain.

Sketching a piece-wise function is like sketching a set of functions with restricted domains all on the same graph. Each branch of the rule is valid only for part of the domain and, if the branches do not join, it is important to indicate which endpoints are included and which are excluded through the use of open and closed circles.

As for any function, each x-value can only be paired to exactly one y-value in a piece-wise function. To calculate the corresponding y-value for a given value of x, the choice of which branch of the rule to use depends on which section of the domain the x-value belongs to.



WORKED EXAMPLE 13

A continuous piece-wise linear graph is constructed from the following linear graphs.

$$y = 2x + 1, \; x \leq a$$
$$y = 4x - 1, \; x > a$$

a. By solving the equations simultaneously, find the point of intersection and hence state the value of a.

b. Sketch the piece-wise linear graph.

THINK

a. 1. Find the intersection point of the two graphs by solving the equations simultaneously.

WRITE/DRAW

a. $y = 2x + 1$
$y = 4x - 1$
Solve by substitution:

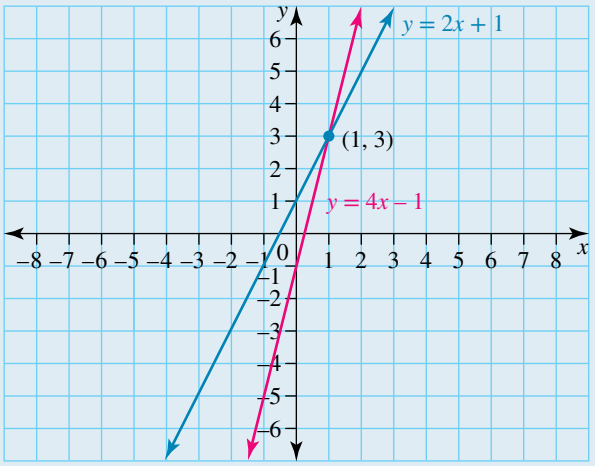
$$2x + 1 = 4x - 1$$
$$1 = 4x - 2x - 1$$
$$1 = 2x - 1$$
$$1 + 1 = 2x$$
$$2 = 2x$$
$$x = 1$$

Substitute $x = 1$ to find y:
$y = 2(1) + 1$
$= 3$
The point of intersection is $(1, 3)$.
$x = 1$ and $y = 3$
$x = 1$, therefore $a = 1$.

2. The x-value of the point of intersection determines the x-intervals for where the linear graphs meet.

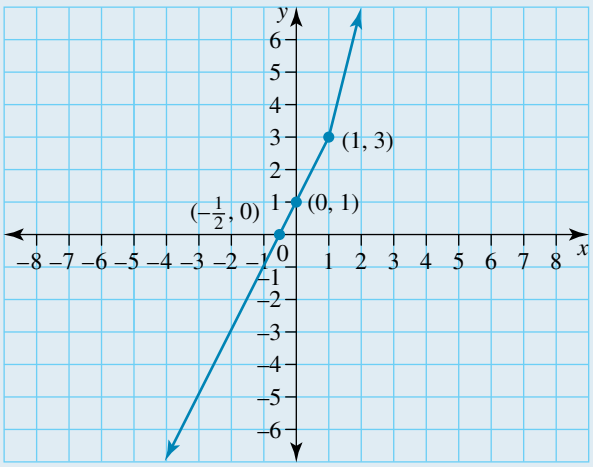
b. 1. Sketch the two graphs without taking into account the intervals.

b.



2. Identify which graph exists within the stated x-intervals to sketch the piece-wise linear graph.

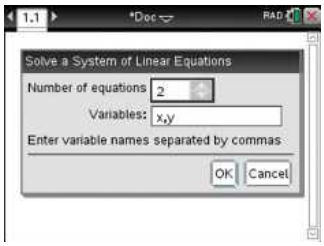
$y = 2x + 1$ exists for $x \leq 1$.
$y = 4x - 1$ exists for $x > 1$.
Remove the sections of each graph that do not exist for these values of x.

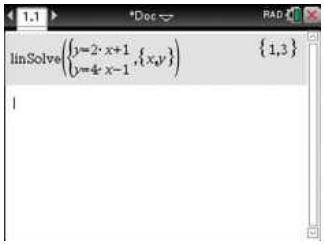


TI | THINK

a.1. On a Calculator page, press MENU, then select:
3: Algebra
2: Solve System of Linear Equations.
Complete the fields as:
Number of equations: 2
Variables: x, y
then select OK.
Complete the entry line as linSolve $\left(\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}, \{x, y\}\right)$
then press ENTER.

WRITE

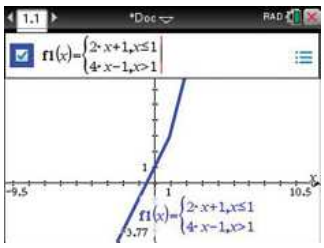




2. The answer appears on the screen.

The point of intersection is (1, 3), so $a = 1$.

b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = \begin{cases} 2x + 1, x \leq 1 \\ 4x - 1, x > 1 \end{cases}$
then press ENTER.
Note: The piece-wise function template can be found by pressing the button.



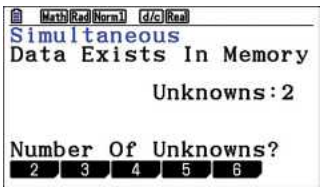
CASIO | THINK

a.1. On an Equation screen, select Simultaneous by pressing F1.

Select 2 unknowns by pressing F1.

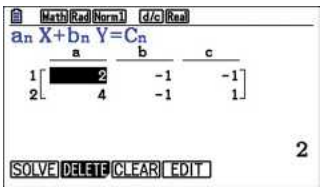
WRITE



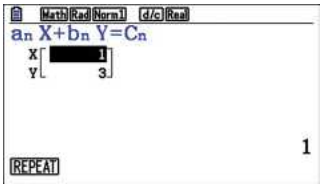


2. Rearrange the given equations into the form $ax + by = c$.
Enter the coefficients for x and y, and the constant term, into the matrix on the screen.

$y = 2x + 1 \Rightarrow 2x - y = -1$
$y = 4x - 1 \Rightarrow 4x - y = 1$



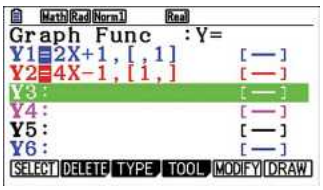
3. Select SOLVE by pressing F1.

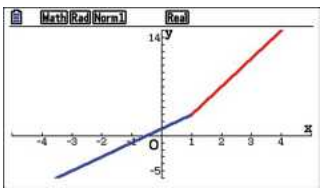


4. The answer appears on the screen.

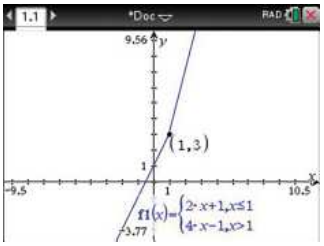
The point of intersection is (1, 3), so $a = 1$.

b.1. On a Graph screen, complete the entry lines for Y1 and Y2 as
$Y1 = 2x + 1, [, 1]$
$Y2 = 4x - 1, [1,]$
then press EXE.
Select DRAW by pressing F6.
Note: When restricting the domain of a function, use interval notation, leaving the upper or lower bound blank to represent $\pm\infty$.

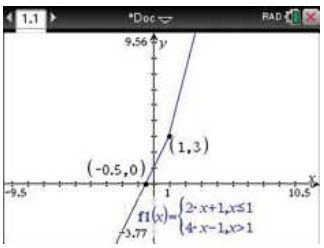




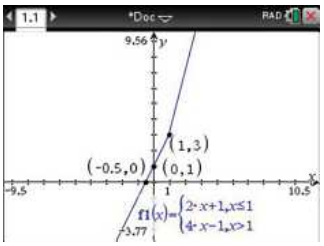
2. To mark the point where the branches join, press MENU then select
5: Trace
1: Graph Trace.
Type '1' then press ENTER twice.



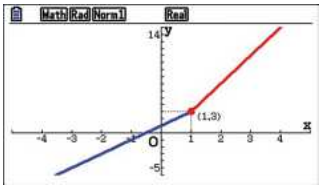
3. To find the x-intercept, press MENU, then select:
6: Analyze Graph
1: Zero.
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.



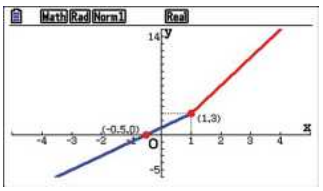
4. To find the y-intercept, press MENU, then select:
5: Trace
1: Graph Trace.
Type '0' then press ENTER twice.



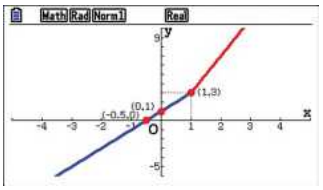
2. To mark the point where the branches join, select Trace by pressing F1.
Type '1' then press EXE twice.



3. To find the x-intercept, select G-Solv by pressing F5, then select ROOT by pressing F1. Use the up/down arrows to select the graph of Y1, then press EXE twice.



4. To find the y-intercept, select G-Solv by pressing F5, then select Y-ICEPT by pressing F4. Use the up/down arrows to select the graph of Y1, then press EXE twice.



WORKED EXAMPLE 14

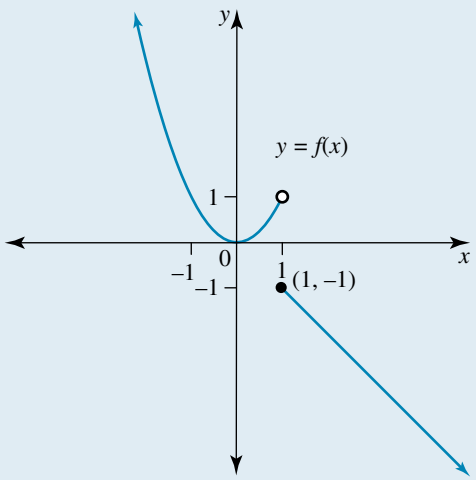
Consider the function:

$$f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$$

a. Evaluate:
- **i.** $f(-2)$
- **ii.** $f(1)$
- **iii.** $f(2)$.

b. Sketch the graph of $y = f(x)$ and state the domain and range.

c. State any value of x for which the function is not continuous.

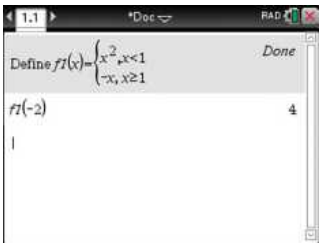
THINK	WRITE
a. Decide for each x-value which section of the domain it is in and calculate its image using the branch of the piece-wise function's rule applicable to that section of the domain.	**a.** $f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$ **i.** $f(-2)$: Since $x = -2$ lies in the domain section $x < 1$, use the rule $f(x) = x^2$. $f(-2) = (-2)^2$ $\therefore f(-2) = 4$ **ii.** $f(1)$: Since $x = 1$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$. $\therefore f(1) = -1$ **iii.** $f(2)$: Since $x = 2$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$. $\therefore f(2) = -2$
b. 1. Sketch each branch over its restricted domain to form the graph of the piece-wise function.	**b.** Sketch $y = x^2, x < 1$ Parabola, turning point (0, 0) open endpoint (1, 1) Sketch $y = -x, x \geq 1$ Line, closed endpoint (1, −1) Point $x = 2 \Rightarrow (2, -2)$
	
2. State the domain and range.	The domain is R. The range is $(-\infty, -1] \cup [0, \infty)$.
c. State any value of x where the branches of the graph do not join.	**c.** The function is not continuous at $x = 1$ because there is a break in the graph.

TI | THINK **WRITE**

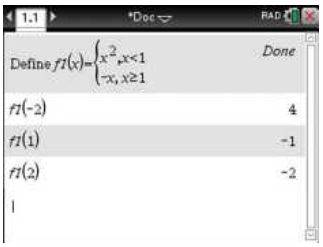
a.1. On a Calculator page, press MENU, then select:
1: Actions
1: Define.
Complete the entry line as Define $f1(x) = \begin{cases} x^2, x < 1 \\ -x, x \geq 1 \end{cases}$
then press ENTER.



2. Complete the next entry line as
$f1(-2)$
then press ENTER.



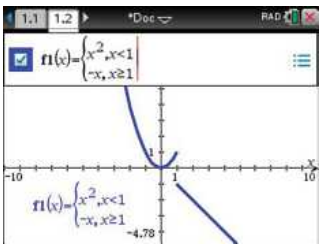
3. Complete the next entry line as
$f1(1)$ then press ENTER.
Complete the next entry line as
$f1(2)$ then press ENTER.



4. The answers appear on the screen.

$f(-2) = 4$, $f(1) = -1$ and $f(2) = -2$

b.1. On a Graphs page, the entry line for function 1 already contains the piece-wise function defined as $f1(x)$ on the Calculator page.
Click the tick box then press ENTER to view the graph.



2. Draw the graph.

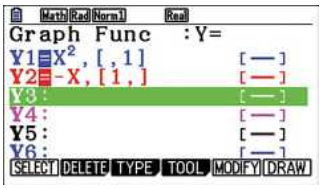
When copying the graph from the screen, remember to use an open circle at the point $(1, 1)$ and a closed circle at $(1, -1)$.

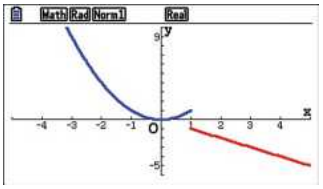
3. The domain and range can be read from the graph.

The domain is R and the range is $(-\infty, -1] \cup [0, \infty)$.

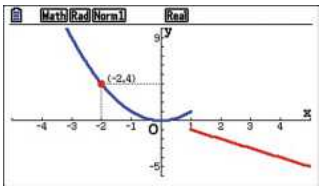
CASIO | THINK **WRITE**

a.1. On a Graph screen, complete the entry lines for Y1 and Y2 as
$Y1 = x^2, [, 1]$
$Y2 = -x, [1,]$
then press EXE.
Select DRAW by pressing F6.
Note: When restricting the domain of a function, use interval notation, leaving the upper or lower bound blank to represent $\pm\infty$.

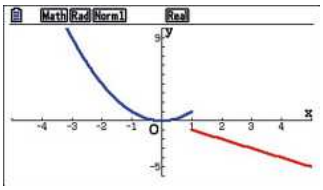




2. Select Trace by pressing F1, then, with the cursor on the graph of Y1, type '–2' and press EXE twice.



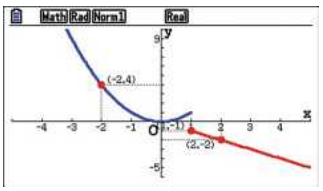
3. Select Trace by pressing F1, then use the up/down arrows to move the cursor to the graph of Y2. Type '1' and press EXE twice. Type '2' then press EXE twice.



4. The answers appear on the screen.

$f(-2) = 4$, $f(1) = -1$ and $f(2) = -2$

b.1. See **a.1.**



2. Draw the graph.

When copying the graph from the screen, remember to use an open circle at the point $(1, 1)$ and a closed circle at $(1, -1)$.

3. The domain and range can be read from the graph.

The domain is R and the range is $(-\infty, -1] \cup [0, \infty)$.

2.5.2 Modelling with piece-wise functions

Mathematical **modelling** is the process by which a real-life situation or system is represented using mathematical concepts, often in the form of a rule or equation. Sometimes scenarios must be simplified in order to apply a rule at this level of study. In these scenarios, if the values of one variable are influenced by the values of another variable, then the former is the *dependent* variable. An *independent* variable is a factor that influences the dependent variable.

When applying modelling techniques to practical problems, the following process should be considered:

1. Consider if it is suitable to apply a mathematical model to the problem.
2. Identify the key variables and:
   - i. identify which is independent and which is dependent
   - ii. consider the natural restrictions that are placed on both in the situation given (e.g. time cannot be negative in most cases).
3. Determine the formula or formulae that govern the relationship between key variables.
4. Sketch a graph if possible, considering any natural restrictions on the variables.
5. Use the known information to directly answer the questions being asked. Reflect the language from the question in your worded responses where possible.

When using piece-wise functions to model practical problems, the domain of each function branch must be stated, remembering that each x-value may only have one associated y-value across the whole piece-wise function.

WORKED EXAMPLE 15

The following two equations represent the distance travelled by a group of students over 5 hours. Equation 1 represents the first section of the hike, when the students are walking at a pace of 4 km/h. Equation 2 represents the second section of the hike, when the students change their walking pace.

Equation 1: $d = 4t$, $0 \leq t \leq 2$

Equation 2: $d = 2t + 4$, $2 \leq t \leq 5$

The variable d is the distance in km from the campsite, and t is the time in hours.



a. Identify the dependent variable.

b. Determine the time, in hours, for which the group travelled in the first section of the hike.

c. i. What was their walking pace in the second section of their hike?

ii. For how long, in hours, did they walk at this pace?

d. Sketch a piece-wise linear graph to represent the distance travelled by the group of students over the 5-hour hike.

THINK

a. The distance travelled depends on the time.

b. 1. Determine which equation the question applies to.

2. Look at the time interval for this equation.

3. Interpret the information.

WRITE

a. Distance is the dependent variable.

b. This question applies to Equation 1.

$0 \leq t \leq 2$

The group travelled for 2 hours.

c. i. 1. Determine which equation the question applies to.

2. Interpret the equation. The walking pace is found by the coefficient of t, as this represents the gradient.

3. Answer the question.

ii. 1. Look at the time interval shown.

2. Interpret the information and answer the question.

d. 1. Find the distance travelled before the change of pace.

2. Sketch the graph $d = 4t$ between $t = 0$ and $t = 2$.

3. Solve the simultaneous equations to find the point of intersection.

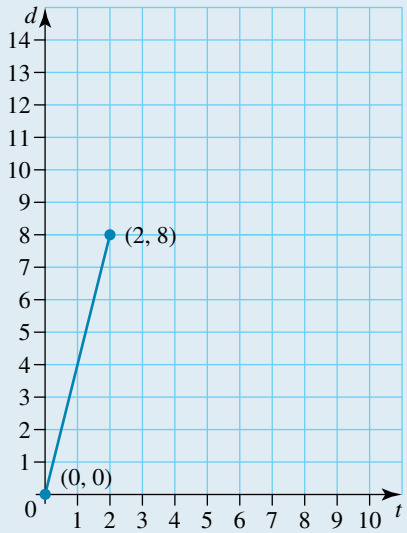
4. Sketch the graph of $d = 2t + 4$ between $t = 2$ and $t = 5$.
Ensure all points of interest including end points, intercepts and intersections are labelled with their co-ordinates.

c. i. This question applies to Equation 2.
$d = 2t + 4,\ 2 \leq t \leq 5$
The coefficient of t is 2.

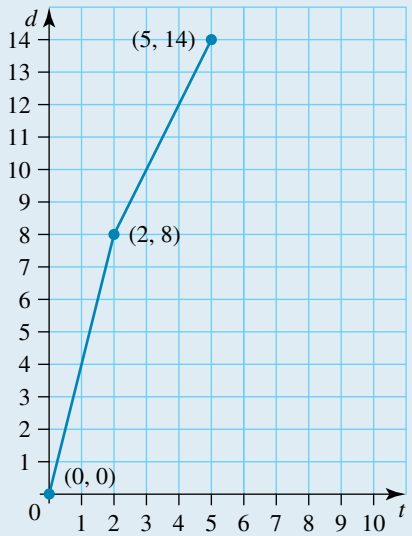
The walking pace is 2 km/h.

ii. $2 \leq t \leq 5$
They walked at this pace for 3 hours.

d. Change after $t = 2$ hours:
$d = 4t$
$d = 4 \times 2$
$d = 8$ km



$4t = 2t + 4$
$4t - 2t = 2t - 2t + 4$
$2t = 4$
$t = 2$
Substitute $t = 2$ into $d = 4t$:
$d = 4 \times 2$
$= 8$



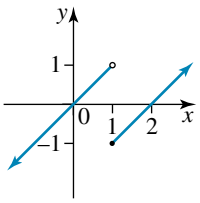
Exercise 2.5 Piece-wise functions

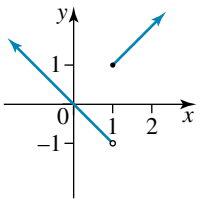
Technology free

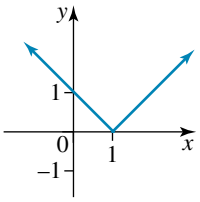
1. MC Consider the following piece-wise function:

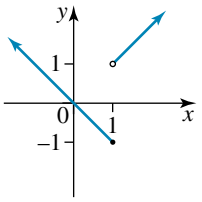
$$f(x) = \begin{cases} -x & x < 1 \\ x, & x \geq 1 \end{cases}$$

a. The graph which correctly represents this function is:

A.


B.


C.


D.


b. The range of this piece-wise function is:

A. R **B.** $[-1, \infty)$ **C.** $(-1, \infty)$ **D.** $[0, \infty)$

2. WE 13 A continuous piece-wise linear graph is constructed from the following linear graphs.

$$y = -3x - 3, x \leq a$$
$$y = x + 1, x \geq a$$

a. By solving the equations simultaneously, find the point of intersection and hence state the value of a.

b. Sketch the piece-wise linear graph.

3. Consider the following linear graphs that make up a piece-wise linear graph.

$$y = 2x - 3, x \leq a$$
$$y = 3x - 4, a \leq x \leq b$$
$$y = 5x - 12, x \geq b$$

a. Sketch the three linear graphs, for $x \in R$.

b. Determine the two points of intersection.

c. Using the points of intersection, find the values of a and b.

d. Sketch the piece-wise linear graph.

4. a. Sketch the graph of the function $g(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 2 - x, & x < 0 \end{cases}$

b. State the range of g.

c. Evaluate:

i. $g(-1)$ **ii.** $g(0)$ **iii.** $g(1)$.

5. **a.** Sketch the graph of the function $f(x) = \begin{cases} x - 2, & x < -2 \\ x^2 - 4, & -2 \leq x \leq 2 \\ x + 2, & x > 2 \end{cases}$

b. State any value of x for which the function is not continuous.

c. State the range of f.

d. Evaluate:

i. $f(-3)$ **ii.** $f(-2)$ **iii.** $f(1)$ **iv.** $f(2)$

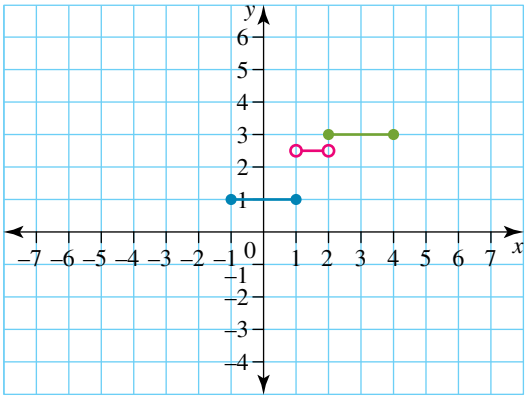
v. $f(5)$.

6. Consider the function defined by $f(x) = \begin{cases} 4x + a, & x < 1 \\ \dfrac{2}{x}, & 1 \leq x \leq 4 \end{cases}$

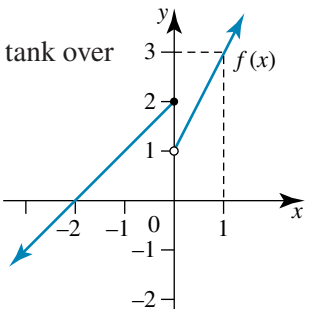
a. Determine the value of a so the function will be continuous as $x = 1$.

b. Explain whether the function is continuous $x = 0$.

7. A step graph is a special type of piece-wise function consisting of a series of horizontal line segments. Write the equations that make up the step graph shown below.



8. Specify the rule for the function represented by the graph on the right.



9. **WE15** The following two equations represent water being added to a water tank over 15 hours, where w is the water in litres and t is the time in hours.

Equation 1: $w = 25t, 0 \leq t \leq 5$

Equation 2: $w = 30t - 25, 5 \leq t \leq 15$

a. Identify the dependent variable.

b. Determine how many litres of water are in the tank after 5 hours.

c. **i.** At what rate is the water being added to the tank after 5 hours?

ii. For how long is the water added to the tank at this rate?

d. Sketch a piece-wise graph to represent the water in the tank at any time, t, over the 15-hour period.

10. A car hire company charges a flat rate of \$50 plus 75 cents per kilometre up to and including 150 kilometres. An equation to represent this cost, C, in dollars is given as $C = 50 + ak$, $0 \leq k \leq b$, where k is the distance travelled in kilometres.

a. Identify the independent variable.

b. Write the values of a and b.

The cost charged for distances over 150 kilometres is given by the equation $C = 87.50 + 0.5k$.

c. By solving the two equations simultaneously, find the point of intersection and hence show that the graph will be continuous.

d. Sketch both equations on the same set of axes.

11. The postage costs to send parcels from the Northern Territory to Sydney are shown in the following table:

Weight of parcel (kg)	Cost ($)
0 – 0.5	6.60
0.5 – 1	16.15
1 – 2	21.35
2 – 3	26.55
3 – 4	31.75
4 – 5	36.95

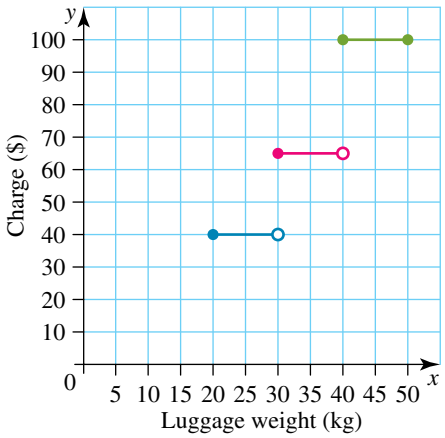


Where a parcel weight appears next to more than one cost, the higher price is applied.

a. Represent this information in a graph.

b. Pammie has two parcels to post to Sydney from the Northen Territory. One parcel weighs 450 g and the other weighs 525 g. Is it cheaper to send the parcels individually or together? Justify your answer using calculations.

12. Airline passengers are charged an excess for any luggage that weighs 20 kg or over. The following graph shows these charges for luggage weighing over 20 kg.





a. How much excess would a passenger be charged for luggage that weighs 31 kg?

b. How much excess would a passenger be charged for luggage that weighs 40 kg?

c. Nerada checks in her luggage and is charged $40. What is the maximum excess luggage she could have without having to pay any more?

d. Hilda and Hanz have two pieces of luggage between them. One piece weighs 32 kg and the other piece weighs 25 kg. Explain how they could minimise their excess luggage charges.

13. The growth of a small tree was recorded over 6 months. It was found that the tree's growth could be represented by three linear equations, where h is the height in centimetres and t is the time in months.

Equation 1: $h = 2t + 20, 0 \leq t \leq a$
Equation 2: $h = t + 22, a \leq t \leq b$
Equation 3: $h = 3t + 12, b \leq t \leq c$



a. i. By solving equations 1 and 2 simultaneously, determine the value of a.
   ii. By solving equations 2 and 3 simultaneously, determine the value of b.
b. Explain why $c = 6$.
c. During which time interval did the tree grow the fastest?
d. Sketch the piece-wise linear graph that shows the height of the tree over the 6-month period.



14. The temperature of a wood-fired oven, T °C, steadily increases until it reaches 200 °C. Initially the oven has a temperature of 18 °C and it reaches the temperature of 200 °C in 10 minutes.

a. Construct an equation that finds the temperature of the oven during the first 10 minutes. Include the time interval, t, in your answer.

Once the oven has heated up for 10 minutes, a loaf of bread is placed in the oven to cook for 20 minutes. An equation that represents the temperature of the oven during the cooking of the bread is $T = 200, a \leq t \leq b$.

b. i. Write the values of a and b.
   ii. In the context of this problem, what do a and b represent?

After the 20 minutes of cooking, the oven's temperature is lowered. The temperature decreases steadily, and after 30 minutes the oven's temperature reaches 60 °C. An equation that determines the temperature of the oven during the last 30 minute is $T = mt + 340$, $d \leq t \leq e$.

c. Determine the values of m, d and e. What does m represent in this equation?
d. Using your values from the previous parts, sketch the graph that shows the changing temperature of the wood-fired oven during the 60-minute interval.

15. Determine the points of intersection for the following four linear graphs, and hence complete the intervals for x by finding the values of a, b and c.

a. i. $y = x + 4, x \leq a$
   ii. $y = 2x + 3, a \leq x \leq b$
   iii. $y = x + 6, b \leq x \leq c$
   iv. $y = 3x + 1, x \geq c$
b. What problem do you encounter when trying to sketch a piece-wise linear graph formed by these four linear graphs?

Technology active

16. WE14 Consider the function for which $f(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \geq 1 \end{cases}$

a. Evaluate the following.
   i. $f(-2)$
   ii. $f(1)$
   iii. $f(2)$
b. Sketch the graph of $y = f(x)$ and state the domain and range.

17. a. Sketch the graph of the function $f(x) = \begin{cases} \frac{1}{x}, & x < 0 \\ x + 1, & x \geq 0 \end{cases}$

b. State the range of f.

18. Stamp duty is a government tax on the purchase of items such as cars and houses. The table shows the range of stamp duty charges for purchasing a car in South Australia.

Car price ($\$P$)	Stamp duty (S)
0–1000	1%
1000–2000	$\$10 + 2\%(P - 1000)$
2000–3000	$\$30 + 3\%(P - 2000)$
3000+	$\$60 + 4\%(P - 3000)$



a. Explain why the stamp duty costs for cars can be modelled by a piece-wise linear graph.
The stamp duty charge for a car purchased for $1000 or less can be expressed by the equation $S = 0.01P$, where S is the stamp duty charge and P is the purchase price of the car for $0 \leq P \leq 1000$. Similar equations can be used to express the charges for cars with higher prices.

Equation 1: $S = 0.01P, 0 \leq P \leq 1000$
Equation 2: $S = 0.02P - 10, a < P \leq b$
Equation 3: $S = 0.03P - c, 2000 < P \leq d$
Equation 4: $S = fP - e, P > 3000$

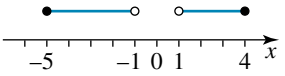
b. For equations 2, 3 and 4, determine the values of a, b, c, d, e and f.
c. Determine the points of intersections for the equations in part **b**.
d. Suki and Boris purchase a car and pay $45 in stamp duty. What price did they pay for their car?

2.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

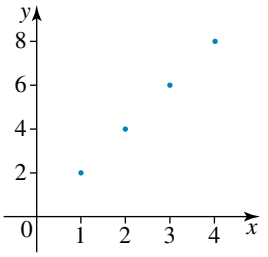
Simple familiar

1. MC The interval shown below is:



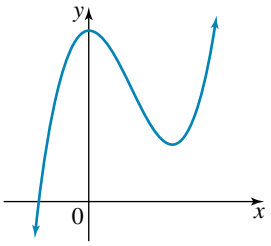
A. $[-5, -1] \cup [0, 4]$
B. $[-5, -1) \cup [0, 4]$
C. $(-5, -1) \cup (1, 4]$
D. $[-5, -1) \cup (1, 4]$

2. MC The rule describing the relation shown in the graph is:

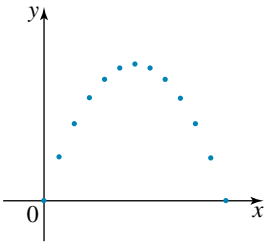


A. $y = 2x$
B. $y = 2x, x \in \{1, 2, 3, 4\}$
C. $y = 2x, x \in N$
D. $y = \frac{x}{2}$

3. State whether each of the following relations has discrete or continuous variables.
a. $\{(-4, 4), (-3, 2), (-2, 0), (-1, -2), (0, 0), (1, 2), (2, 4)\}$
b. The relation which shows the air pressure at any time of the day.
c.



d.



e. The relation which shows the number of student absences per day during term 3 at your school.

f. The relation describing the weight of a child from age 3 months to one year.

4. Which of the following relations are functions? State the domain and range for each function. *Hint:* It may be helpful to view the graphs of the relations in parts **e** and **f** using technology.

a. $\{(0, 2), (0, 3), (1, 3), (2, 4), (3, 5)\}$

b. $\{(-3, -2), (-1, -1), (0, 1), (1, 3), (2, -2)\}$

c. $\{(x, y) : y = 2, x \in R\}$

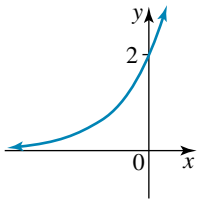
d. $\{(x, y) : x = -3, y \in J\}$

e. $y = x^3 + x$

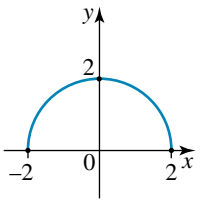
f. $x = y^2 + 1$

5. State the domain and range of each of the following relations.

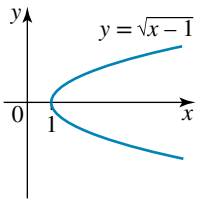
a.



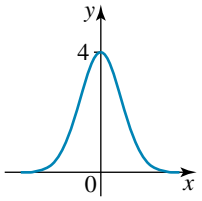
b.



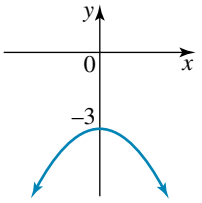
c.



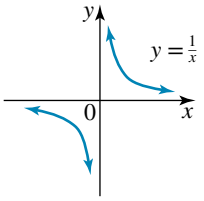
d.



e.



f.



6. Identify the implied domains of the rational functions with the following rules.
Hint: It may be helpful to view the graphs of these functions using technology.

a. $y = \dfrac{1}{16 - x^2}$

b. $y = \dfrac{2 - x}{x^2 + 3}$

7. A function f is defined as follows: f: $[-2, a] \rightarrow R$, where $f(x) = (x - 1)^2 - 4$.

a. Determine $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(3)$.

b. If $f(a) = 12$, find the value of a.

c. Sketch the function f, labelling the graph appropriately.

d. From the graph or otherwise, state the:

i. domain of $f(x)$

ii. range of $f(x)$.

8. Consider $f : R \rightarrow R$, $f(x) = x^3 - x^2$.

a. Determine the image of 2.

b. Use technology to sketch the graph of $y = f(x)$.

c. State the domain and range of the function f.

d. What is the type of relation?

e. Give a restricted domain so that f is one-to-one and increasing.

9. MC For the function $f : [-2, 4] \rightarrow R$, $f(x) = ax + b$, $f(0) = 1$ and $f(1) = 0$. The image of -2 under the mapping is:

A. -2 **B.** -1 **C.** 1 **D.** 3

10. MC The graph of $y = \sqrt{x}$ is translated 4 units upwards and then reflected in the y-axis. The equation of its image is:

A. $y = -\sqrt{x} + 4$ **B.** $y = \sqrt{-x} + 4$ **C.** $y = -\sqrt{x} - 4$ **D.** $y = \sqrt{-x} - 4$

11. MC The graph of a function $y = f(x)$ is shown in diagram (i) and is transformed in diagram (ii).

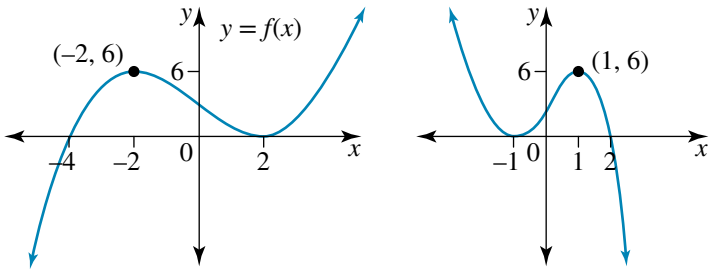


Diagram (i) Diagram (ii)

A possible equation for the graph in diagram (ii) is:

A. $y = f(-2x)$

B. $y = f\left(-\frac{1}{2}x\right)$

C. $y = -2f(x)$

D. $y = -\frac{1}{2}f(x)$

12. Consider $f: R \to R, f(x) = \begin{cases} -x - 1, & x < -1 \\ \sqrt{1 - x^2}, & -1 \leq x \leq 1 \\ x + 1, & x > 1 \end{cases}$

a. Calculate the value of:

i. $f(0)$ **ii.** $f(3)$ **iii.** $f(-2)$ **iv.** $f(1)$

b. Show the function is not continuous at $x = 1$.

c. Use technology to sketch the graph of $y = f(x)$ and state the type of relation.

d. Determine the value of a such that $f(a) = a$.

Complex familiar

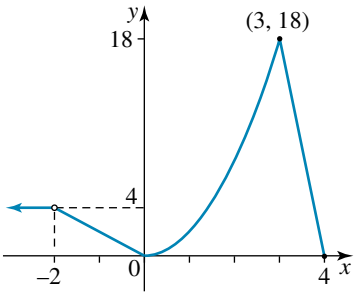
You may choose to use technology to answer questions **17–20**.

13. Define $f(x) = x^3 + lx^2 + mx + n$. Given $f(3) = -25, f(5) = 49, f(7) = 243$, use technology to answer the following questions.

a. Calculate the constants l, m and n and hence state the rule for $f(x)$.

b. What is the image of 1.2?

14. For the graph shown:



a. state the domain and range

b. find the rule for $x \in (-\infty, -2)$

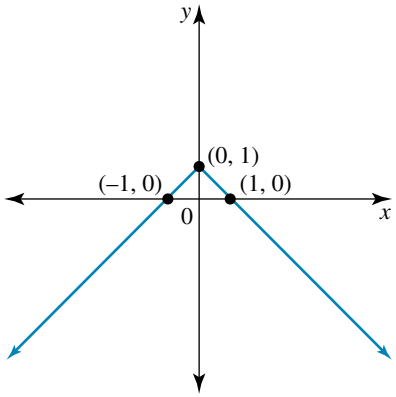
c. find the rule for $x \in (-2, 0)$

d. find the rule for $x \in [0, 3]$, given it is of the form $y = ax^2$

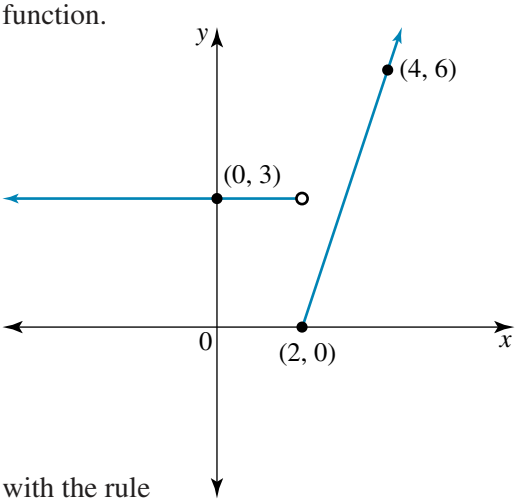
e. determine the rule when $x \geq 3$

f. describe the relation using piece-wise function notation of the form $f(x) = \begin{cases} \ldots, & \ldots \\ \ldots, & \ldots \\ \ldots, & \ldots \end{cases}$

15. **a.** Form a rule for the graph of the piece-wise function.



b. Form the rule for the graph of the piece-wise function.



c. Determine the values of a and b so that the function with the rule

$$f(x) = \begin{cases} a, & x \in (-\infty, -3] \\ x + 2, & x \in (-3, 3) \\ b, & x \in [3, \infty) \end{cases}$$

is continuous for $x \in R$; for these values, sketch the graph of $y = f(x)$.

d. In an effort to reduce the time her children spend in the shower, a mother introduced a penalty scheme with fines to be paid from the children's pocket money according to the following:

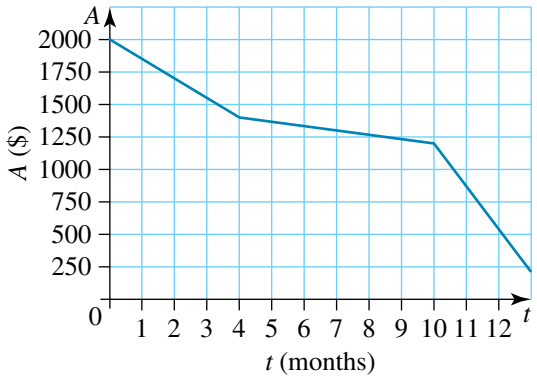
If someone spends more than 5 minutes in the shower, the fine in dollars is equal to the shower time in minutes; if someone spends up to and including 5 minutes in the shower, there is no fine. If someone chooses not to shower at all, there is a fine of $2 because that child won't be nice to be near.



Defining appropriate symbols, express the penalty scheme as a mathematical rule in piece-wise form and sketch the graph which represents it.

16. The amount of money in a savings account over 12 months is shown in the following piece-wise graph, where A is the amount of money in dollars and t is the time in months.





One of the linear graphs that make up the piece-wise linear graph is $A = 2000 - 150t$, $0 \leq t \leq a$.

a. Determine the value of a.

b. The equation that intersects with $A = 2000 - 150t$ is given by $A = b - 50t$. If the two equations intersect at the point $(4, 1400)$, show that $b = 1600$.

c. The third equation is given by the rule $A = 4100 - 300t$. By solving a pair of simultaneous equations, determine the time interval for this equation.

d. Using an appropriate equation, determine the amount of money in the account at the end of the 12 months.

Complex unfamiliar

You may choose to use technology to answer questions **17–20**.

17. a. A hat is thrown vertically into the air and at time t seconds its height above the ground is given by the function $h(t) = 10t - 5t^2$. Calculate how long it takes the hat to return to the ground and hence state the domain and range of this function.



b. For part of its growth over a two-week period, the length of a leaf at time t weeks is given by the function $l(t) = 0.5 + 0.2t^3$, $0 \leq t \leq 2$.

i. State the domain and determine the range of this function.

ii. Calculate how long it takes for the leaf to reach half the length that it reaches by the end of the time period.



18. Jerri and Samantha have both entered a 10-km fun run for charity. The distance travelled by Jerri can be modelled by the linear equation $d = 6t - 0.1$, where d is the distance in km from the starting point and t is time in hours.

a. Determine the time taken for Jerri to run the 10 kilometres. Give your answer correct to the nearest minute.

b. In the context of this problem, explain the meaning of the d-intercept (y-intercept).
The distance Samantha is from the starting point at any time, t hours, can be modelled by the piece-wise linear graph:

$$d = 4t, 0 \leq t \leq \frac{1}{2}$$
$$d = 8t - 2, \frac{1}{2} \leq t \leq b$$

c. How far, in kilometres, did Samantha travel in the first 30 minutes? What was her speed during this time?

d. i. Determine the value of b.

ii. Hence, show that Samantha crossed the finish line ahead of Jerri by 11 minutes.

e. By solving a pair of simultaneous equations, determine:

i. the time when Samantha passed Jerri on the run

ii. the distance from the starting point at which Samantha passed Jerri.

f. Construct two graphs on the same set of axes to represent the distances travelled by Jerri and Samantha for the 10-km race.

19. Consider the function given by $f(x) = x^2 - 10x + 21$.

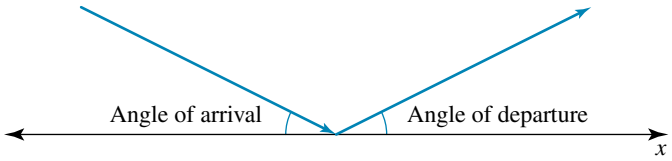
a. Sketch the following transformations of the graph of $y = f(x)$, showing the images of its turning point and intersections with the coordinate axes.

i. $y = f(x + 3)$ **ii.** $y = f(-x)$ **iii.** $y = f(2x)$

iv. $y = f\left(4 + \frac{2x}{3}\right)$ **v.** $y = -0.5f(x) - 2$

b. Determine the values of h so that the roots of the equation $f(x - h) = 0$ will always be negative.

20. A ray of light comes in along the line $x + y = 2$ above the x -axis and is reflected off the axis so that the angle of departure (the angle of reflection) is equal to the angle of arrival (the angle of incidence).

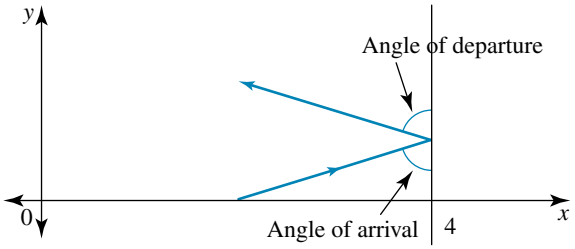


a. **i.** Calculate the magnitude of the angle of departure.

ii. Form the equation of the line along which the departing light travels.

iii. Express the path of the incoming and departing rays in terms of a piece-wise function.

b. The reflected ray of light strikes the vertical line $x = 4$ and is reflected off this line in the same way so that the angle of departure is equal to the angle of arrival.



i. Give a reason why this section of the path of the ray of light is not a function.

ii. Form a piece-wise rule with x in terms of y, which describes both the incoming and departing paths of the ray of light for this section of its path.

studyon

Units 1 & 2 Sit chapter test

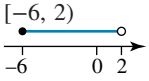
Answers

Chapter 2 Functions

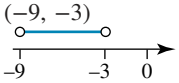
Exercise 2.2 Functions and relations

1. **a.** $[-2, \infty)$ **b.** $(-\infty, 5)$
c. $(-3, 4]$ **d.** $(-\infty, -1]$
e. $(-5, -2] \cup [3, \infty)$ **f.** $(-3, 1) \cup (2, 4]$

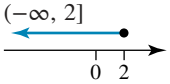
2. **a.** $[-6, 2)$



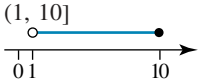
b. $(-9, -3)$



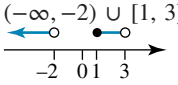
c. $(-\infty, 2]$



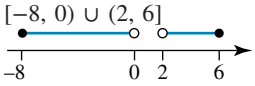
d. $(1, 10]$



e. $(-\infty, -2) \cup [1, 3)$



f. $[-8, 0) \cup (2, 6]$



3. **a.** $\{x : -4 \leq x < 2\}$
$= [-4, 2)$

b. $\left\{y : -1 < y < \sqrt{3}\right\}$
$\left(-1, \sqrt{3}\right)$

c. $\{x : x > 3\}$
$(3, \infty)$

d. $\{x : x \leq -3\}$
$(-\infty, -3]$

e. R or $(-\infty, \infty)$

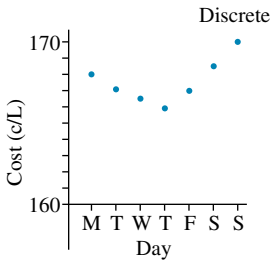
f. $(-\infty, 0) \cup (0, \infty)$

4. B

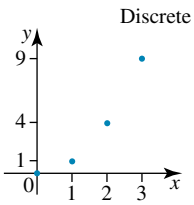
5. A

6. D

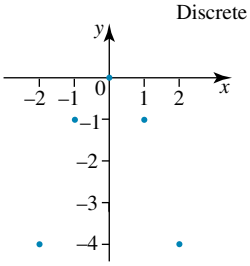
7. **a.**



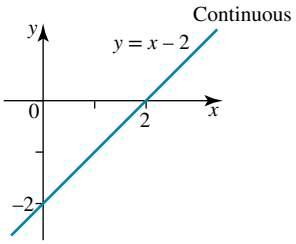
b.



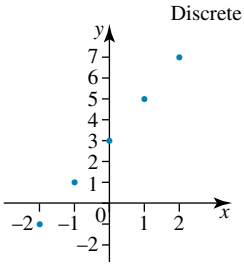
c.



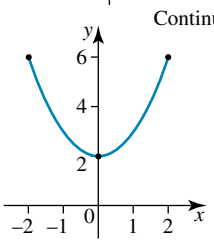
d.



e.



f.



8. **a.** one-to-many, not a function
b. many-to-one, function
c. many-to-one, function
d. one-to-one, function
e. one-to-one, function
f. many-to-one, function

9. **a.** many-to-many, not a function
b. many-to-one, function
c. one-to-one, function
d. many-to-one, function
e. many-to-many, not a function
f. many-to-one, function

10. **i.** The functions are a, b, c, d, f.
ii. The one-to-one function is c.

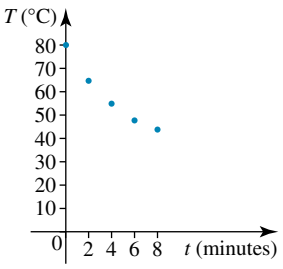
11. **i.** The functions are b, c, d, e, f.
ii. The one-to-one functions are b, c, e.

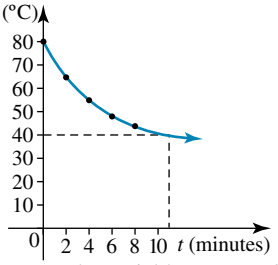
12. C

13. C

14. C

15. **a.**



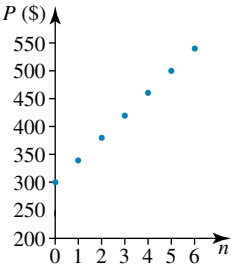
b. 

c. Because the variables are continuous, measurements can be taken in between the given values.

d. Half of the initial temperature is 40 °C. It takes approximately 11 minutes.

16. a.

n	0	1	2	3	4	5	6
p	300	340	380	420	460	500	540

b. 

c. The variables are discrete.
Only whole numbers of computers can be sold.

d. $P = 300 + 40n$

Exercise 2.3 Function notation

1. C

2. B

3. D

4. A

5. a. Domain is $\{0, 2, 3, 4\}$, range is $\{-1, 0, 3, 4\}$. This is a function.

b. The domain is R and range is $(-\infty, 4]$. It is a function.

c. Domain is $[0, 3]$, range is $[0, 4]$. This is a function.

d. Domain is $[-2, \infty)$, range is R. This is not a function.

6. a. Domain $[0, 5]$, range $[0, 15]$

b. Domain $[-4, 2) \cup (2, \infty)$, range $(-\infty, 10)$

c. Domain $[-3, 6]$, range $[0, 8]$

d. Domain $[-2, 2]$, range $[-4, 4]$

e. Domain $\{3\}$, range R

f. Domain R, range R

7. a. The relation in part **a** is one-to-one, part **b** is many-to-one, part **c** is many-to-one, part **d** is one-to-many, part **e** is one-to-many and part **f** is many-to-one.

b. The graphs of **d** and **e** are not of functions.

8. a. i. Domain $= \{3, 4, 5, 6, 7\}$

ii. Range $= \{8, 10, 12, 14, 16\}$

b. i. Domain $= \{1.1, 1.3, 1.5, 1.7\}$

ii. Range $= \{2, 1.8, 1.6, 1.4\}$
or $= \{1.4, 1.6, 1.8, 2\}$

c. i. Domain $= \{3, 4, 5, 6\}$

ii. Range $= \{110, 130, 150, 170\}$

d. i. Domain $= \{\text{M, Tu, W, Th, Fr}\}$

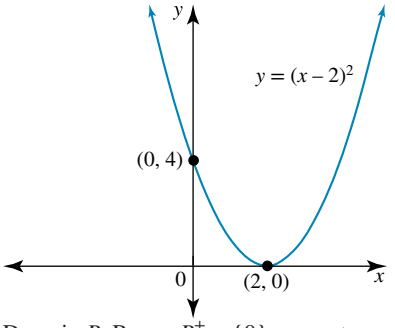
ii. Range $= \{25, 30, 35\}$

e. i. Domain $= \{3, 4, 5\}$

ii. Range $= \{13, 18, 23\}$

f. i. Domain $= R$

ii. Range $= [-1, \infty)$

9. a. 

Domain R, Range $R^+ \cup \{0\}$, many-to-one

b. An answer is $[2, \infty)$

10. a. i. $f(0) = 1$ **ii.** $f(2) = 7$

iii. $f(-2) = -5$ **iv.** $f(5) = 16$

b. i. $g(0) = 2$ **ii.** $g(-3) = 1$

iii. $g(5) = 3$ **iv.** $g(-4) = 0$

c. i. $g(1) = 3$ **ii.** $g\left(\frac{1}{2}\right) = 2$

iii. $g\left(-\frac{1}{2}\right) = 6$ **iv.** $g\left(-\frac{1}{5}\right) = 9$

d. i. $f(0) = 9$ **ii.** $f(-2) = 1$

iii. $f(1) = 16$ **iv.** $f(a) = a^2 + 6a + 9$

11. $f(x) = x^2 + 2x - 3$

a. i. $f(-2) = -3$ **ii.** $f(9) = 96$

b. i. $f(2a) = 4a^2 + 4a - 3$ **ii.** $f(1 - a) = a^2 - 4a$

c. $f(x + h) - f(x) = 2xh + h^2 + 2h$

d. $\{x : x < -3\} \cup \{x : x > 1\}$

e. $x = -5, x = 3$

f. $x = -4, x = 1$

12. a. $x = 3$ **b.** $x = \pm 3$

c. $x = \frac{1}{3}$ **d.** $x = 3$ or $x = 2$

e. $x = -4$ or $x = 1$ **f.** $x = -1$

13. a. $a = 2$
$b = 3$
$\Rightarrow f(x) = 2x + 3$

b. $x = -\frac{3}{2}$

c. $f(-3) = -3$

d. $g : (-\infty, 0] \to R, g(x) = 2x + 3$

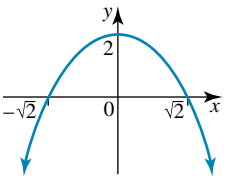
14. C

15. $f : [0, 7) \to R, f(x) = x^2 - 6x + 10$

Domain is $[0, 7)$, range is $[1, 17)$.

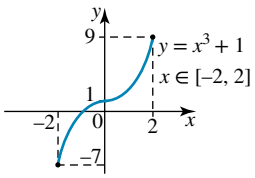
16. a. $P = 4x + 6$

b. Domain $(1, 6]$, range $(10, 30]$

17. a. 

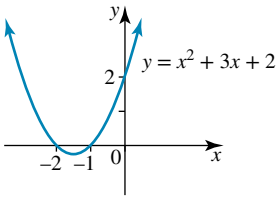
Domain $= (-\infty, \infty)$
Range $= (-\infty, 2]$

b.



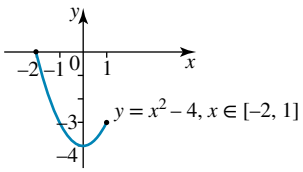
Domain = [−2, 2]
Range = [−7, 9]

c.



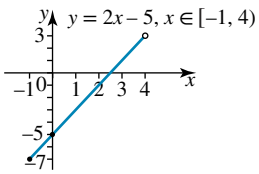
Domain = $(-\infty, \infty)$
Range = $[-\frac{1}{4}, \infty)$

d.



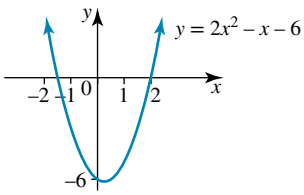
Domain = [−2, 1]
Range = [−4, 0]

e.



Domain = [−1, 4)
Range = [−7, 3)

f.



Domain = $(-\infty, \infty)$
Range = $\left[\frac{-49}{8}, \infty\right)$
or = $\left[-6\frac{1}{8}, \infty\right)$

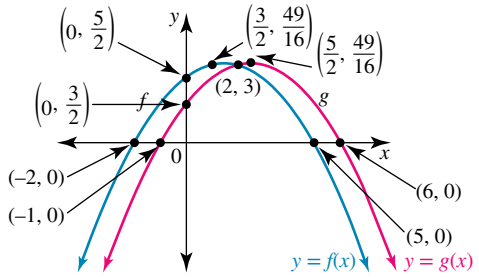
18. **a.** Domain = R **b.** Domain = $[0, \infty)$
c. Domain = $[-4, 4]$ **d.** Domain = R
e. Domain = $R\backslash\{0\}$ **f.** Domain = R

19. **a.** 47
b. 21
c. 9 weeks
d. As t increases $\frac{96}{t+3}$ gets smaller and approaches zero.
$N(t) \to 15$, so no.

20. **a.** $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$
b. $g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$

c. $x = 2$

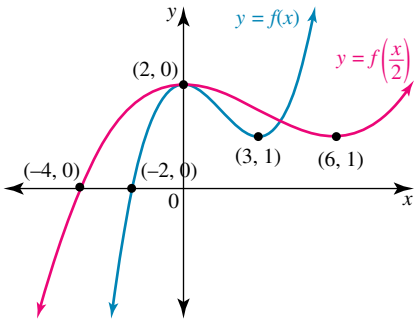
d.



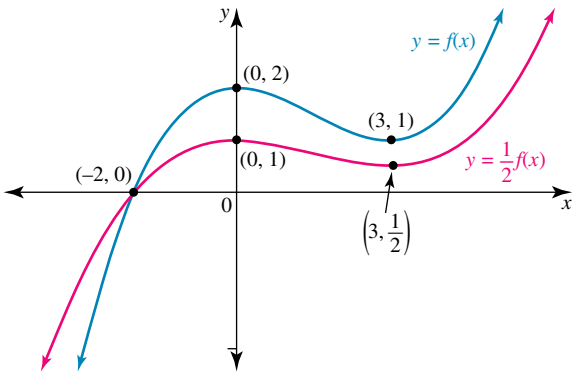
The graph of function g has the same shape as the graph of function f but g has horizontally translated 1 unit to the right.

Exercise 2.4 Transformations of functions

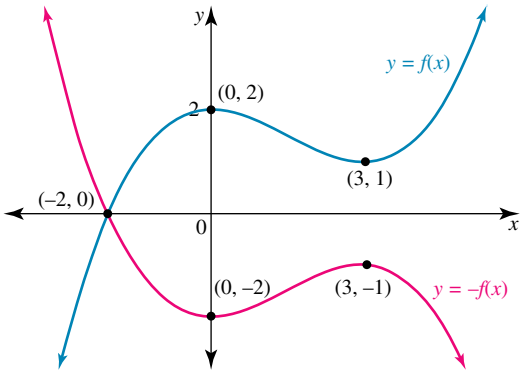
1.



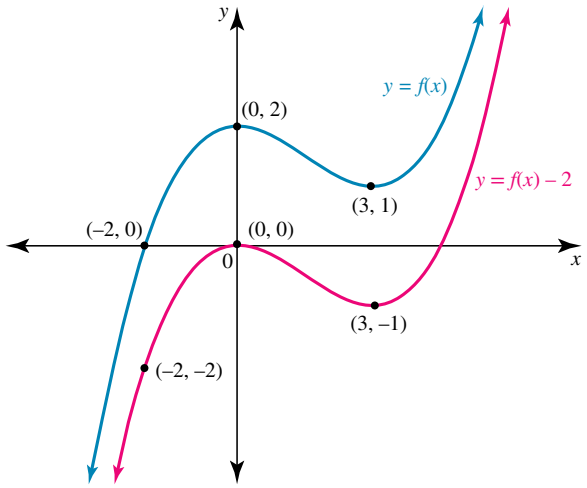
2.



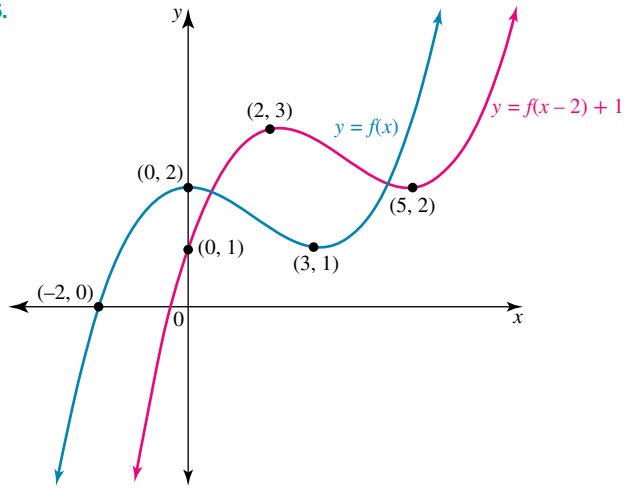
3.



4.



5.



6. a. $y = -(x-1)^2 + 3$

b. $y = -(x-1)^2 - 3$, not the same as in part **a**.

7. a. Dilation of factor 4 from the x-axis, dilation of factor 2 from the y-axis, horizontal translation 2 units to the right and vertical translation 3 units upwards.

b. Reflection in y-axis, dilation of factor 4 from the y-axis followed by horizontal translation 12 units to the right
or
Reflection in y-axis, dilation of factor $\frac{1}{2}$ from the x-axis followed by horizontal translation 12 units to the right.

8. a. $y = \dfrac{1}{2(x+3)}$

b. Undoing the transformations requires the image to undergo a horizontal translation 3 units to the right followed by dilation of factor 2 from the y-axis.

9. a. dilation of factor 3 from the x-axis

b. reflection in the x-axis

c. vertical translation of 5 units upwards

d. horizontal translation of 5 units to the left

10. a. dilation of factor 3 from the y-axis

b. dilation of factor $\frac{1}{2}$ from the y-axis followed by a vertical translation of 1 unit upwards

c. horizontal translation of 4 units to the right and a vertical translation of 4 units downwards

d. dilation of factor $\frac{1}{2}$ from the y-axis followed by a horizontal translation of $\frac{1}{2}$ unit to the left

11. i. a. $y = 2\sqrt{x}$ **b.** $y = \sqrt{\frac{x}{2}}$

c. $y = -\sqrt{x} + 2$ **d.** $y = -\sqrt{x} - 2$

e. $y = \sqrt{2 - x}$ **f.** $y = \sqrt{-x - 2}$

ii. a. $y = 2x^4$ **b.** $y = \left(\frac{x}{2}\right)^4 = \frac{x^4}{16}$

c. $y = -x^4 + 2$ **d.** $y = -x^4 - 2$

e. $y = (x-2)^4$ **f.** $y = (x+2)^4$

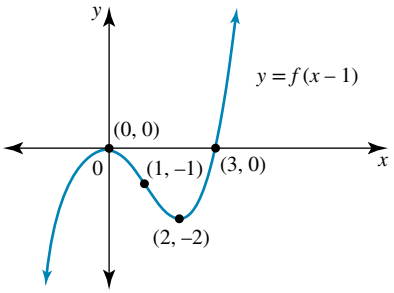
12. a. $(1, -8)$ **b.** $(-3, 4)$

c. $\left(-3, -\frac{4}{5}\right)$ **d.** $\left(\frac{3}{5}, -4\right)$

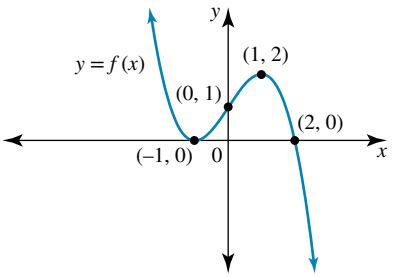
13. a. i. $y = -\frac{3}{x}$ **ii.** $y = -\frac{3}{x}$

b. i. $y = \frac{3}{x^2} + 6$ **ii.** $y = \frac{3}{x^2} + 18$

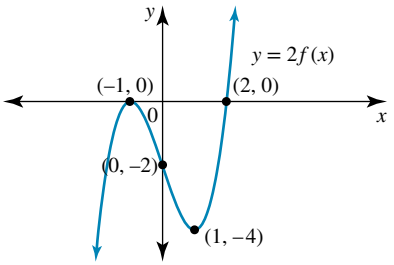
14. a.



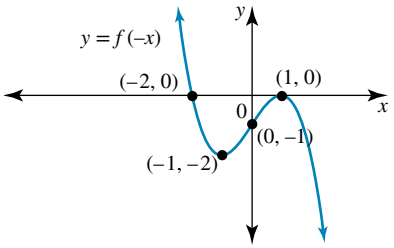
b.



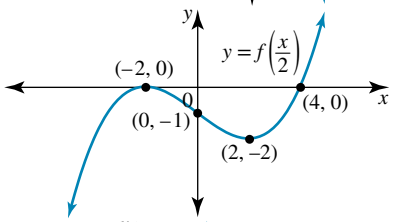
c.



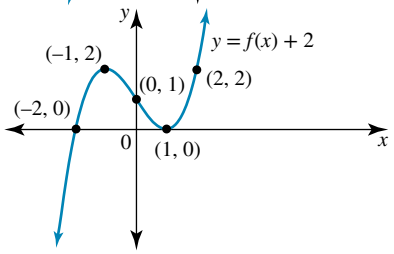
d.



e.



f.



15. a. a dilation of factor 2 from the x-axis followed by a horizontal translation of 3 units to the left

b. a dilation of factor 6 from the x-axis followed by a horizontal translation of 2 units to the right and a vertical translation of 1 unit upwards

c. a dilation of factor $\frac{1}{2}$ from the y-axis followed by a horizontal translation of 1 unit to the left

d. a reflection in the y-axis followed by a horizontal translation of 3 units to the right

e. a reflection in the x-axis, a dilation of factor $\frac{1}{4}$ from the y-axis and then a vertical translation of 1 unit upwards

f. a dilation of factor $\frac{1}{9}$ from the x-axis, a dilation of factor 9 from the y-axis and then a horizontal translation of 3 units to the right

16. a. $y = \dfrac{1}{3(x+3)^2}$ **b.** $y = -x^5 + 3$

c. $y = \dfrac{1}{1-x}$ **d.** $y = \sqrt[3]{2x-1}$

e. $y = -(x+3)(x-3)(x-7)$ **f.** $y = \dfrac{1}{8}x^2(x+4)(x-4)$

17. a. The function g is its own image under this reflection.

b. $f: R \to R, f(x) = x^{\frac{1}{3}}$

Under a reflection in the x-axis, $y = f(x) \to y = -f(x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = (-x)^{\frac{1}{3}}$.

Under a reflection in the y-axis, $y = f(x) \to y = f(-x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = (-x)^{\frac{1}{3}}$

$$\begin{aligned} y &= (-x)^{\frac{1}{3}} \\ &= (-1)^{\frac{1}{3}} x^{\frac{1}{3}} \\ &= -x^{\frac{1}{3}} \end{aligned}$$

The image under reflection in either axis is the same, $y = -x^{\frac{1}{3}}$.

c. The function h is the lower semicircle, centre $(0, 0)$, radius 3. After reflection in the x-axis its image is the upper semicircle.

To return the curve back to its original position, reflect in the x-axis again.

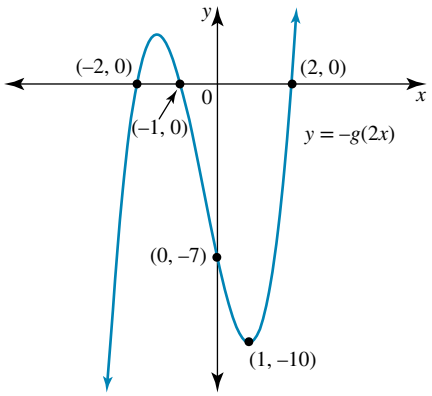
d. The image has a maximum turning point with coordinates $(-2, -5)$.

e. $y^2 = (x-3) \to y^2 = (-x-3) \to (y-2)^2 = (-x-3)$.

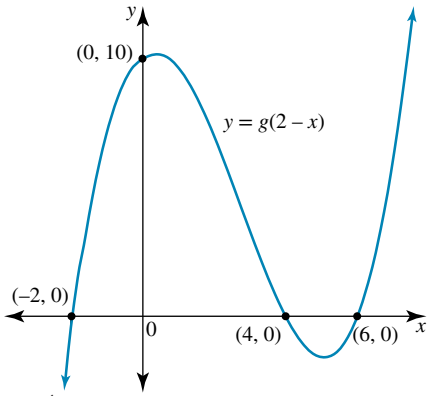
The original equation was $(y-2)^2 = -(x+3)$.

f. $y = -3(x-2)^3$

18. a.



b.



c. $c > 4$.

d. $g(x) = -\dfrac{7}{32}(x+4)(x+2)(x-4)$

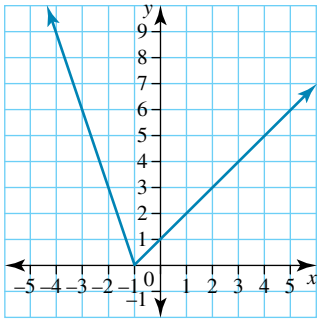
$g(2x) = -\dfrac{7}{4}(x+2)(x+1)(x-2)$

Exercise 2.5 Piece-wise functions

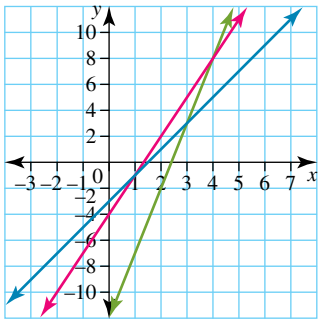
1. a. B **b.** C

2. a. Point of intersection $= (-1, 0)$, therefore $a = -1$.

b.



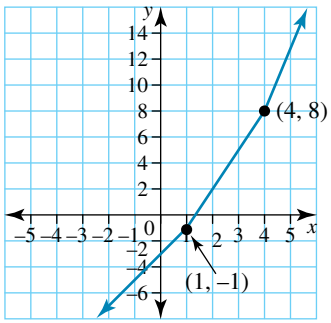
3. a.



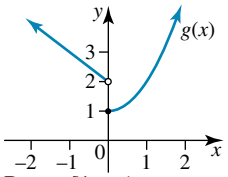
b. $(1, -1)$ and $(4, 8)$

c. $a = 1$ and $b = 4$

d.



4. a.



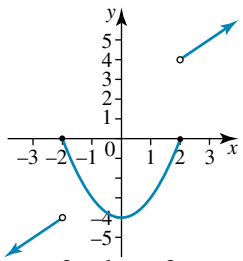
b. Range $[1, \infty)$

c. **i.** $g(-1) = 3$

ii. $g(0) = 1$

iii. $g(1) = 2$

5. a.



b. $x = -2$ and $x = 2$

c. $(-\infty, 0] \cup (4, \infty)$

d. **i.** $f(-3) = -5$ **ii.** $f(-2) = 0$

iii. $f(1) = -3$ **iv.** $f(2) = 0$

v. $f(5) = 7$

6. a. $a = -2$

b. $x = 0$ lies in the domain for which the rule is the linear function $y = 4x + a$ so the graph will be continuous at this point.

7. $y = 1, \quad 1 \le x \le 1$

$y = 2.5, \quad 1 < x < 2$

$y = 3, \quad 2 \le x \le 4$

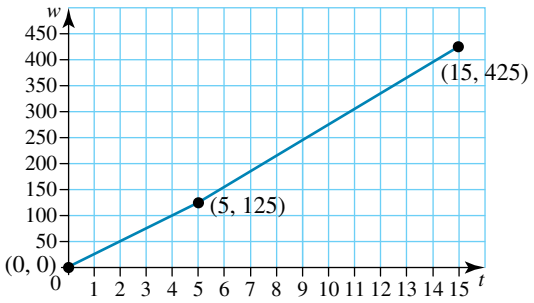
8. $f(x) = \begin{cases} x + 2 & x \le 0 \\ 2x + 1 & x > 0 \end{cases}$

9. a. water

b. 125 L

c. **i.** 30 L/h **ii.** 10 hours

d.

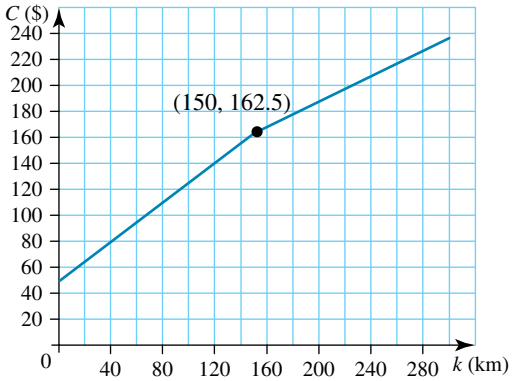


10. a. Distance

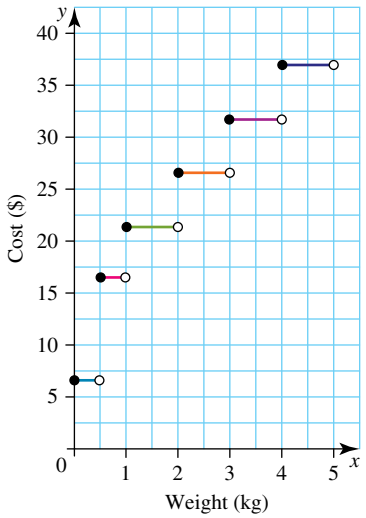
b. $a = 0.75$, $b = 150$

c. (150, 162.5) at this point both equations will have the same value and therefore the graph will be continuous.

d.



11. a.



b. Individually: 450 g costs \$6.60, 525 g costs \$16.15, total cost = \$22.75 together total weight = 450 + 525 = 975 costs \$16.15 to send

It is cheaper to post them together (\$16.15 together vs \$22.75 individually).

12. a. \$65

b. \$100

c. 10 kg

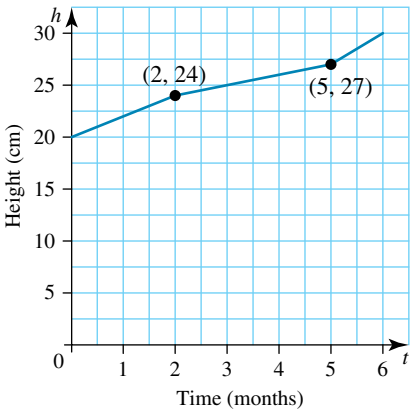
d. 32-kg charge = \$65, 25-kg charge = \$40, total = \$105. Place 2–3 kg from the 32-kg bag in the 25-kg bag, $32 - 3 = 29$ kg. $25 + 3 = 28$ kg, charge for each is \$40, total= \$80.

13. a. i. $a = 2$ ii. $b = 5$

b. The data is only recorded over 6 months.

c. $5 \leq t \leq 6$ (between 5 and 6 months)

d.



14. a. $T = 18 + 18.2t,\ 0 \leq t \leq 10$

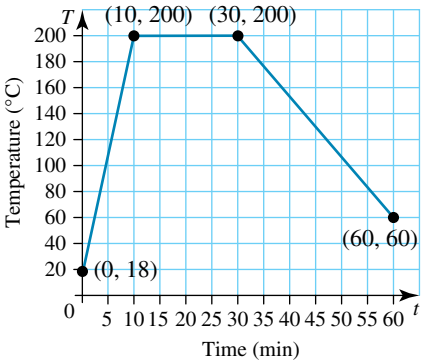
b. i. $a = 10,\ b = 30$

ii. a is the time the oven first reaches 200 °C and b is the time at which the bread stops being cooked.

c. $m = \frac{-14}{3},\ d = 30,\ e = 60$

m is the change in temperature for each minute in the oven.

d.



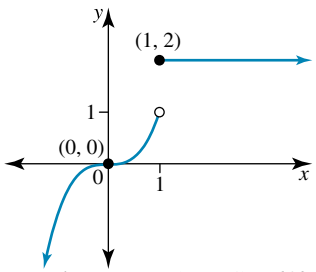
15. a. (1, 5), (3, 9) and (2.5, 8.5)

$a = 1,\ b = 3,\ c = 2.5$

b. $b > c$, which means that graph iii is not valid and the piece-wise linear graph cannot be sketched.

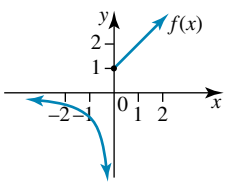
16. a. i. $f(-2) = -8$ ii. $f(1) = 2$ iii. $f(2) = 2$

b.



Domain R; range $(-\infty, 1) \cup \{2\}$

17. a.



b. Range of $f = (-\infty, 0) \cup [1, \infty)$

18. a. There is a change in the rate for different x-values (i.e. car prices).

b. $a = 1000,\ b = 2000,\ c = 30,\ d = 3000,\ e = 60, f = 0.04$

c. (1000, 10), (2000, 30) and (3000, 60)

d. $2500

2.6 Review: exam practice

Simple familiar

1. D

2. B

3. a. Discrete **b.** Continuous **c.** Continuous

d. Discrete **e.** Discrete **f.** Continuous

4. a. Not a function

b. Function
Domain $= \{-3, -1, 0, 1, 2\}$
Range $= \{-2, -1, 1, 3\}$

c. Function
Domain $= R$
Range$=\{2\}$

d. Not a function

e. Function
Domain $= R$
Range $= R$

f. Not a function

5. a. Domain $= R$
Range $= (0, \infty)$ or $R^{\pm}$

b. Domain $= [-2, 2]$
Range $= [0, 2]$

c. Domain $= [1, \infty)$
Range $= R$

d. Domain $= R$
Range $= (0, 4]$

e. Domain $= R$
Range $= (-\infty, -3]$

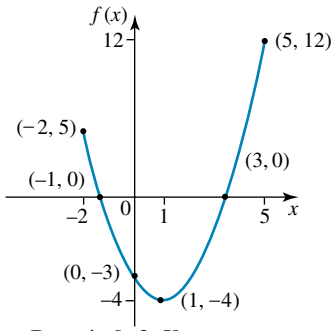
f. Domain $= R\backslash\{0\}$
Range $= R\backslash\{0\}$

6. a. $R\backslash\{\pm 4\}$ **b.** R

7. a. $f(-2) = 5$
$f(-1) = 0$
$f(0) = -3$
$f(1) = -4$
$f(3) = 0$

b. $a = 5$

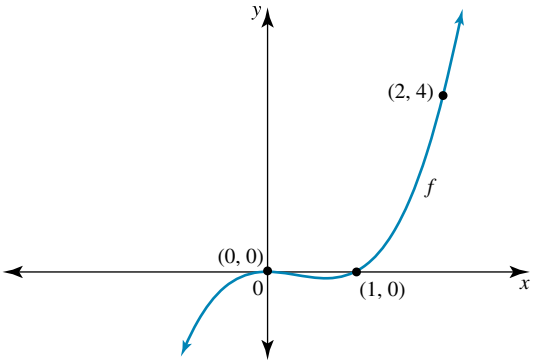
c.



d. i. Domain $[-2, 5]$ ii. Range $[-4, 12]$

8. a. The image of 2 is 4.

b.



c. Domain R, range R.

d. many-to-one

e. Many answers are possible. One answer is to restrict the domain to $(1, \infty)$ and another is to restrict the domain to R^-.

9. D

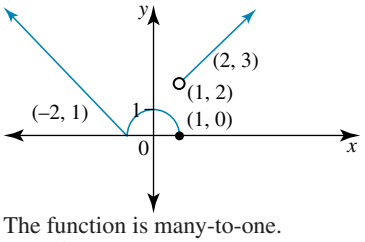
10. B

11. A

12. a. **i.** $f(0) = 1$ **ii.** $f(3) = 4$
iii. $f(-2) = 1$ **iv.** $f(1) = 0$

b. For $y = \sqrt{1 - x^2}$, there is a closed endpoint $(1, 0)$ but for $y = x + 1$ there is an open endpoint $(1, 2)$. The two branches do not join. Hence, the function is not continuous at $x = 1$ as there will be a break in its graph.

c.



The function is many-to-one.

d. $a = \frac{\sqrt{2}}{2}$

Complex familiar

13. a. $l = 0, m = -12, n = -16$,
$f(x) = x^3 - 12x - 16$

b. $f(1.2) = -28.672$

14. a. Domain $= x \in (-\infty, 4] \backslash \{-2\}$
Range $= [0, 18]$

b. $y = 4$

c. $y = -2x$

d. $y = 2x^2$

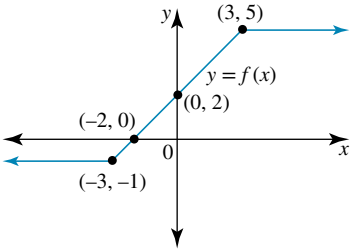
e. $y = -18x + 72$

f. $f(x) = \begin{cases} 4, & x \in (-\infty, -2) \\ -2x, & x \in (-2, 0] \\ 2x^2, & x \in [0, 3] \\ -18x + 72, & x \in [3, 4] \end{cases}$

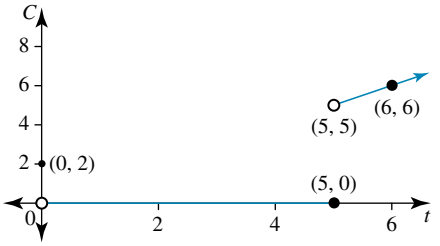
15. a. $y = \begin{cases} x + 1, & x \leq 0 \\ -x + 1, & x > 0 \end{cases}$

b. $y = \begin{cases} 3, & x < 2 \\ 3x - 6, & x \geq 2 \end{cases}$

c. $a = -1$
$b = 5$



d. Let the time in the showers be t minutes and the dollar amount of the fine be C.
The rule is $C = \begin{cases} 2, & t = 0 \\ 0, & 0 < t \leq 5 \\ t, & t > 5 \end{cases}$



16. a. $a = 4$

b. $A = 2000 - 150t$ and $A = b - 50t$ intersect at $(4, 1400)$
$2000 - 150(4) = b - 50(4)$
$1400 = b - 200$
$1600 = b$

c. $10 \leq t \leq 12$

d. $A = \$500$

Complex unfamiliar

17. a. It takes the hat 2 seconds to return to the ground.
The domain is $[0, 2]$.
The range is $[0, 5]$.

b. **i.** Domain is $[0, 2]$.
Range is $[0.5, 2.1]$.

ii. 1.4 weeks

18. a. 1 hour, 41 minutes

b. Jerri started 0.1 km (100 metres) behind the starting line.

c. 2 km
4 km/h

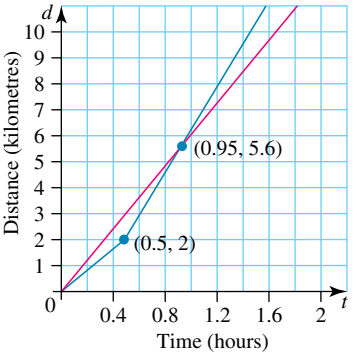
d. **i.** $b = 1.5$

ii. Samantha took 1 hour 30 minutes hours to run 10 km, Jerri took 1 hour 41 minutes. Difference: $41 - 30$ minutes $= 11$ minutes

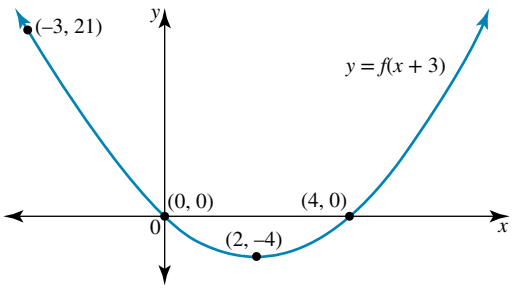
e. **i.** 0.95 hours (57 minutes)

ii. 5.6 km

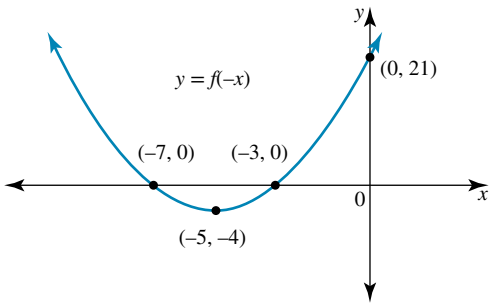
f.



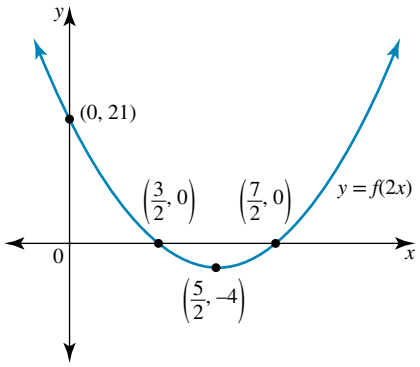
19. a. i.



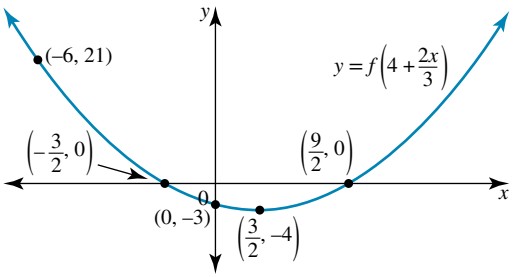
ii.



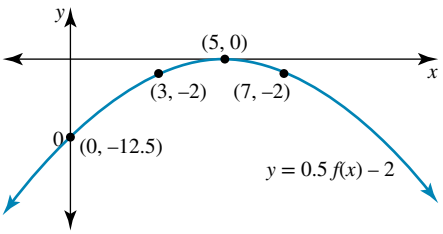
iii.



iv.



v.



b. $h < -7$

20. a. i. 45°

ii. $y = x - 2$

iii. $y = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$

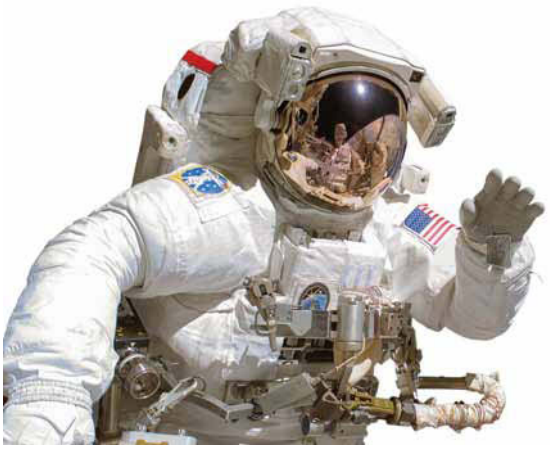
b. i. As a vertical line can cut the graph in more than one place, this section of the path of the ray of light is not a function. It is one-to-many.

ii. $x = \begin{cases} y + 2, & 0 < y < 2 \\ 6 - y, & y \geq 2 \end{cases}$

CHAPTER 3
Quadratic relationships

3.1 Overview

3.1.1 Introduction



Quadratic relationships have been of interest to mathematicians throughout history. In 400 BCE the Babylonians and Chinese were solving problems using a method we still use today. Greek mathematician Pythagoras and Egyptian mathematician Euclid had a long-lasting impact on many areas of mathematics, and in 300 BCE they found a way of solving quadratics using geometry. Over the centuries, mathematicians continued to be fascinated by quadratics and worked towards what is now our current day understanding of quadratics. By 700 CE the Indians found a way to write a quadratic equation. By 1100 CE the Indian mathematician Bhāskara refined the ideas of his fellow countryman, Brahmagupta, and Persian mathematician al-Khwarizmi. Further work led to a breakthrough by Italian mathematician Girolamo Cardano, who refined ideas relating right back to Euclid's work. Furthermore, french mathematician René Descartes was credited with creating the modern form of the quadratic equation in 1637 CE.

Quadratic functions are much more than algebraic curiosities, they are widely applied in real world situations to solve scientific, engineering and business problems. Objects, such as mobile phones, would not exist if someone, hadn't applied quadratic funtions to their design. Humans would not have travelled to the moon if quadratic equations weren't used. Police use them to work out the trajectories of bullets and the military use them to identify where shells will land. Engineers use them to work out the designs of any curved objects, including components of sound systems. Agriculturalists use them to help work out the maximum size for animal pens and paddocks or to help in decision making about crop planting. Needless to say, today's world would not be possible without quadratic equations.

LEARNING SEQUENCE

3.1 Overview
3.2 Graphs of quadratic functions
3.3 Solving quadratic equations with rational roots
3.4 Factorising and solving quadratics over *R*
3.5 The discriminant
3.6 Modelling with quadratic functions
3.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.2 Graphs of quadratic functions

A **polynomial** is a mathematical expression of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a non-negative integral power. A quadratic polynomial is an algebraic expression of the form $ax^2 + bx + c$, where each power of the variable x is a positive whole number with the highest power of x being 2. Quadratic equations can be used to solve practical real-life problems. In the past you will have used the formula $A = l^2$ to find the area of square, knowing the side length. In this case A (area) and l (side length) are quadratically related variables.

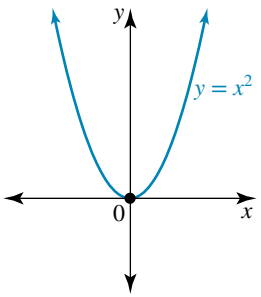
The graph of a quadratic function is curved and called a **parabola**.

3.2.1 The graph of $y = x^2$ and transformations

The simplest parabola has the equation $y = x^2$.

Key features of the graph of $y = x^2$:

- it is symmetrical about the y-axis
- the **axis of symmetry** has the equation $x = 0$
- the graph is **concave up** (opens upwards)
- it has a minimum turning point, or vertex, at the point (0,0).



Making the graph wider or narrower

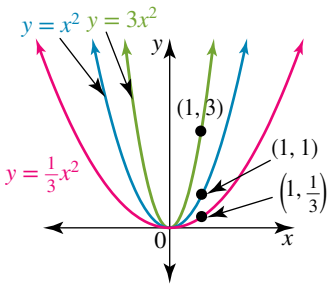
The graphs of $y = ax^2$ for $a = \frac{1}{3}$, 1 and 3 are drawn on the same set of axes.

Comparison of the graphs of $y = x^2, y = 3x^2$ and $y = \frac{1}{3}x^2$ shows that the graph of $y = ax^2$ will be:

- narrower than the graph of $y = x^2$ if $a > 1$
- wider than the graph of $y = x^2$ if $0 < a < 1$.

The coefficient of x^2, a, is called the **dilation factor**. It measures the amount of stretching or compression from the x-axis.

For $y = ax^2$, the graph of $y = x^2$ has been dilated by a factor of a from the x-axis or by a factor of a parallel to the y-axis.



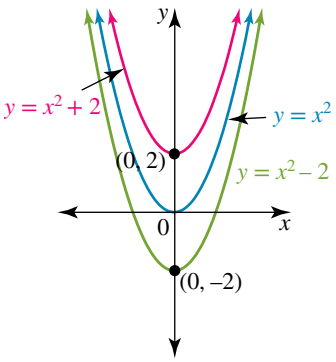
Translating the graph up or down

The graphs of $y = x^2 + k$ for $k = -2$, 0 and 2 are drawn on the same set of axes.

Comparison of the graphs of $y = x^2, y = x^2 + 2$ and $y = x^2 - 2$ shows that the graph of $y = x^2 + k$ will:

- have a turning point at $(0, k)$
- move the graph of $y = x^2$ vertically upwards by k units if $k > 0$
- move the graph of $y = x^2$ vertically downwards by k units if $k < 0$.

The value of k gives the vertical translation. For the graph of $y = x^2 + k$, the graph of $y = x^2$ has been translated vertically by k units from the x-axis.

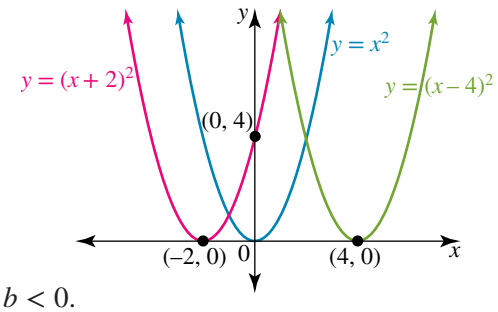


Translating the graph left or right

The graphs of $y = (x - b)^2$ for $b = -2$, 0 and 4 are drawn on the same set of axes.

Comparison of the graphs of $y = x^2$, $y = (x + 2)^2$ and $y = (x - 4)^2$ shows that the graph of $y = (x - b)^2$ will:

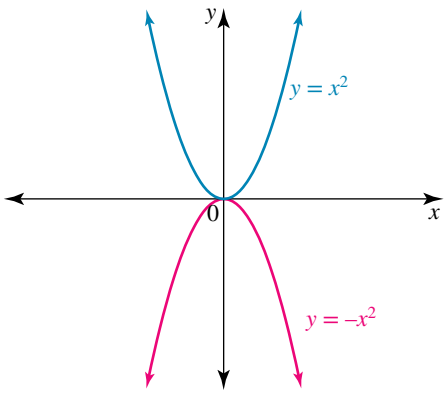
- have a turning point at $(b, 0)$
- move the graph of $y = x^2$ horizontally to the right by b units if $b > 0$
- move the graph of $y = x^2$ horizontally to the left by b units if $b < 0$.



The value of b gives the horizontal translation. For the graph of $y = (x - b)^2$, the graph of $y = x^2$ has been translated horizontally by b units from the y-axis.

Reflecting the graph in the x-axis

The graph of $y = -x^2$ is obtained by reflecting the graph of $y = x^2$ in the x-axis.



Key features of the graph of $y = -x^2$:

- it is symmetrical about the y-axis
- the axis of symmetry has the equation $x = 0$
- the graph is **concave down** (opens downwards)
- it has a maximum turning point, or vertex, at the point $(0, 0)$.

A negative coefficient of x^2 indicates the graph of a parabola is concave down.

If the region is closed, the points on the boundary parabola are included in the region. If the region is open, the points on the boundary parabola are not included in the region.

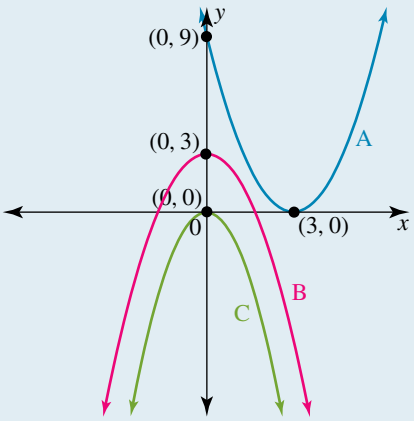
WORKED EXAMPLE 1

Match the graphs of the parabolas A, B, C with the following equations.

a. $y = -x^2 + 3$

b. $y = -3x^2$

c. $y = (x - 3)^2$



THINK

1. Compare graph A with the basic graph $y = x^2$ to identify the transformations.

2. Compare graph B with the basic graph $y = x^2$ to identify the transformations.

3. Check graph C for transformations.

WRITE

Graph A opens upwards and has been moved horizontally to the right.
Graph A matches with equation **c**: $y = (x - 3)^2$.

Graph B opens downwards and has been moved vertically upwards.
Graph B matches with equation **a**: $y = -x^2 + 3$.

Graph C opens downwards. It is narrower than both graphs A and B.
Graph C matches with equation **b**: $y = -3x^2$.

on Resources

Interactivity: Graph plotter: Quadratic polynomials (int-2562)

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concept 2

Graphs of quadratic functions in general, or polynomial form Summary screen and practice questions

3.2.2 Sketching parabolas from their equations

The key points required when sketching a parabola are:

- the turning point
- the y-intercept
- any x-intercepts.

The axis of symmetry is also a key feature of the graph.

The equation of a parabola allows this information to be obtained but in differing ways, depending on the form of the equation.

We shall consider three forms for the equation of a parabola:

- general form
- turning point form
- x-intercept form.

3.2.3 The general, or polynomial form, $y = ax^2 + bx + c$

If $a > 0$ then the parabola is concave up and has a minimum turning point.

If $a < 0$ then the parabola is concave down and has a maximum turning point.

The dilation factor a, $a > 0$, determines the width of the parabola. The dilation factor is always a positive number (so it could be expressed as $|a|$).

The methods to determine the key features of the graph are as follows.

- Substitute $x = 0$ to obtain the y-intercept (alternatively, the y-intercept is obvious from the equation).
- Substitute $y = 0$ and solve the quadratic equation $ax^2 + bx + c = 0$ to obtain the x-intercepts. There may be 0, 1 or 2 x-intercepts, and these are symmetrical about their midpoint $x = -\frac{b}{2a}$.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
- the turning point lies on the axis of symmetry so its x-coordinate is $x = -\frac{b}{2a}$. Substitute this value into the parabola's equation to calculate the y-coordinate of the turning point.

WORKED EXAMPLE 2

Sketch the graph of $y = \frac{1}{2}x^2 - x - 4$ and label the key points with their coordinates.

THINK	WRITE
1. Write down the y-intercept.	$y = \frac{1}{2}x^2 - x - 4$ y-intercept: if $x = 0$ then $y = -4 \Rightarrow (0, -4)$

2. Obtain any x-intercepts.

x-intercepts: let $y = 0$

$$\frac{1}{2}x^2 - x - 4 = 0$$

$$x^2 - 2x - 8 = 0$$

$$(x + 2)(x - 4) = 0$$

$$\therefore x = -2, 4$$

$$\Rightarrow (-2, 0), (4, 0)$$

3. Find the equation of the axis of symmetry.

Axis of symmetry formula $x = -\frac{b}{2a}$,

$a = \frac{1}{2}$, $b = -1$

$$x = -\frac{-1}{\left(2 \times \frac{1}{2}\right)}$$

$$= 1$$

4. Find the coordinates of the turning point.

Turning point: when $x = 1$,

$$y = \frac{1}{2} - 1 - 4 \Rightarrow$$

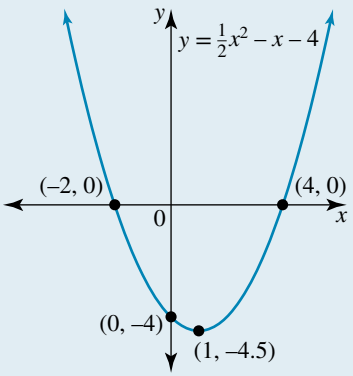
$$= -4\frac{1}{2}$$

$\left(1, -4\frac{1}{2}\right)$ is the turning point.

5. Identify the type of turning point.

Since $a > 0$, the turning point is a minimum turning point.

6. Sketch the graph using the information obtained in previous steps. Label the key points with their coordinates.



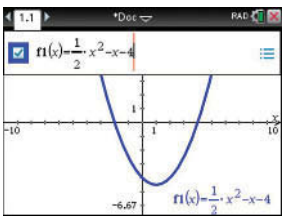
TI | THINK

1. On a Graphs page, complete the entry line for function 1 as

$f1(x) = \frac{1}{2}x^2 - x - 4$

then press ENTER.

WRITE



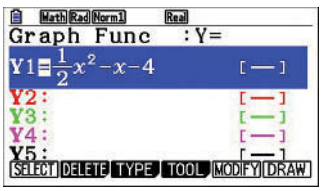
CASIO | THINK

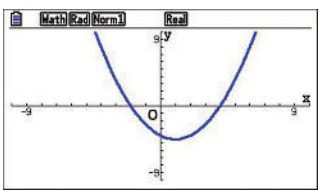
1. On a Graph screen, complete the entry line for y1 as

$y1 = \frac{1}{2}x^2 - x - 4$

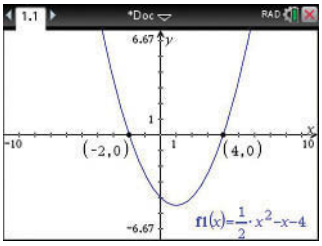
then press EXE. Select DRAW by pressing F6.

WRITE

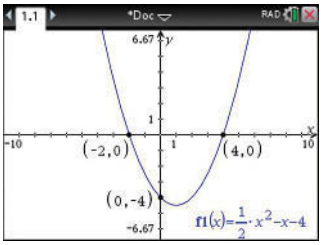




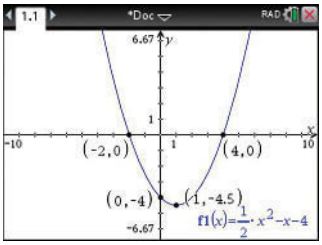
2. To find the x-intercepts, press MENU then select
   6: Analyze Graph
   1: Zero
   Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER. Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.
   Repeat this step to find the other x-intercept.



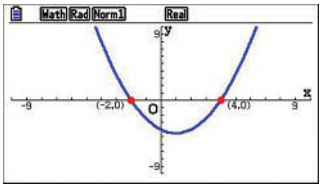
3. To find the y-intercept, press MENU then select
   5: Trace
   1: Graph Trace
   Type '0' then press ENTER twice.



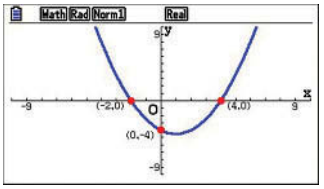
4. To find the minimum, press MENU then select
   6: Analyze Graph
   2: Minimum
   Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.



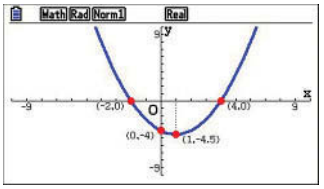
2. To find the x-intercepts, select G-Solv by pressing SHIFT F5, then select ROOT by pressing F1. Press EXE. Use the left/right arrows to move across to the next x-intercept, then press EXE.



3. To find the y-intercept, select G-Solv by pressing SHIFT F5, then select Y-ICEPT by pressing F4. Press EXE.



4. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3. Press EXE.



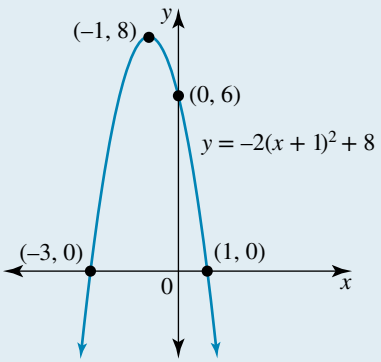
3.2.4 Turning point form, $y = a(x - b)^2 + c$

Since b represents the horizontal translation and c the vertical translation, this form of the equation readily provides the coordinates of the turning point.

- The turning point has coordinates (b, c).
  If $a > 0$, the turning point is a minimum and if $a < 0$ it will be a maximum. Depending on the nature of the turning point the y-coordinate of the turning point gives the minimum or maximum value of the quadratic.
- Find the y-intercept by substituting $x = 0$.
- Find the x-intercepts by substituting $y = 0$ and solving the equation $a(x - b)^2 + c = 0$. However, before attempting to find x-intercepts, consider the type of turning point and its y-coordinate as this will indicate whether there are any x-intercepts.

WORKED EXAMPLE 3

Sketch the graph of $y = -2(x + 1)^2 + 8$ and label the key points with their coordinates.

THINK	WRITE
1. Obtain the coordinates and the type of turning point from the given equation. *Note:* The x-coordinate of the turning point could also be obtained by letting $(x + 1) = 0$ and solving this for x.	$y = -2(x + 1)^2 + 8$ $\therefore y = -2(x - (-1))^2 + 8$ Maximum turning point at $(-1, 8)$
2. Calculate the y-intercept.	Let $x = 0$ $\therefore y = -2(1)^2 + 8$ $= 6$ $\Rightarrow (0, 6)$
3. Calculate any x-intercepts. *Note:* The graph is concave down with maximum y-value of 8, so there will be x-intercepts.	x-intercepts: let $y = 0$ $0 = -2(x + 1)^2 + 8$ $2(x + 1)^2 = 8$ $(x + 1)^2 = 4$ $(x + 1) = \pm\sqrt{4}$ $x = \pm 2 - 1$ $x = -3, 1$ $\Rightarrow (-3, 0), (1, 0)$
4. Sketch the graph, remembering to label the key points with their coordinates.	

3.2.5 Factorised, or x-intercept, form $y = a(x - b)(x - c)$

This form of the equation readily provides the x-intercepts.

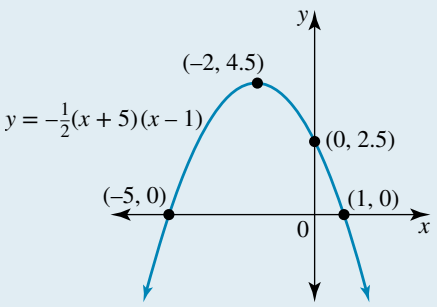
- The x-intercepts occur at $x = b$ and $x = c$.
- The axis of symmetry lies halfway between the x-intercepts and its equation, $x = \frac{b + c}{2}$, gives the x-coordinate of the turning point.
- The turning point is obtained by substituting $x = \frac{b + c}{2}$ into the equation of the parabola and calculating the y-coordinate.
- The y-intercept is obtained by substituting $x = 0$.

If the linear factors are distinct, the graph cuts through the x-axis at each x-intercept.

If the linear factors are identical making the quadratic a perfect square, the graph touches the x-axis at its turning point.

WORKED EXAMPLE 4

Sketch the graph of $y = -\frac{1}{2}(x + 5)(x - 1)$.

THINK	WRITE
1. Identify the x-intercepts.	$y = -\frac{1}{2}(x + 5)(x - 1)$ x-intercepts: let $y = 0$ $\frac{1}{2}(x + 5)(x - 1) = 0$ $x + 5 = 0$ or $x - 1 = 0$ $x = -5$ or $x = 1$ x-intercepts are $(-5, 0)$, $(1, 0)$.
2. Calculate the equation of the axis of symmetry.	Axis of symmetry has equation $x = \frac{-5 + 1}{2}$ $\therefore x = -2$
3. Obtain the coordinates of the turning point.	Turning point: substitute $x = -2$ in to the equation $y = -\frac{1}{2}(x + 5)(x - 1)$ $= -\frac{1}{2}(3)(-3)$ $= \frac{9}{2}$ Turning point is $\left(-2, \frac{9}{2}\right)$.
4. Calculate the y-intercept.	$y = -\frac{1}{2}(x + 5)(x - 1)$ y-intercept: let $x = 0$, $y = -\frac{1}{2}(5)(-1)$ $= \frac{5}{2}$ y-intercept is $\left(0, \frac{5}{2}\right)$.
5. Sketch the graph.	

3.2.6 Determining the rule of a quadratic polynomial from a graph

Whether the equation of the graph of a quadratic polynomial is expressed in $y = ax^2 + bx + c$ form, $y = a(x - b)^2 + c$ form or $y = a(x - b)(x - c)$ form, each equation contains 3 unknowns. Hence, 3 pieces of information are needed to fully determine the equation. This means that exactly one parabola can be drawn through 3 non-collinear points.

If the information given includes the turning point or the intercepts with the axes, one form of the equation may be preferable over another.

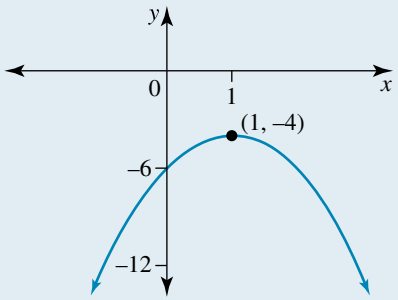
As a guide:

- **If the turning point is given, use the $y = a(x - b)^2 + c$ form.**
- **If the x-intercepts are given, use the $y = a(x - b)(x - c)$ form.**
- **If 3 points on the graph are given, use the $y = ax^2 + bx + c$ form.**

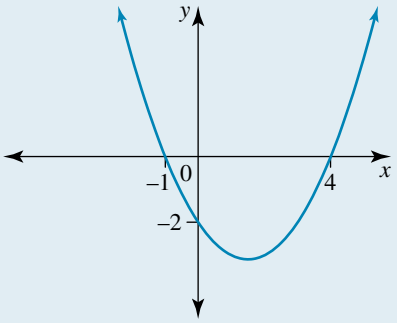
WORKED EXAMPLE 5

Determine the rules for the following parabolas.

a.



b.



THINK

a. 1. Consider the given information to choose the form of the equation for the graph.

2. Determine the value of a.

3. Is the sign of a appropriate?

4. Write the rule for the graph.
Note: Check if the question specifies whether the rule needs to be expanded into general form.

b. 1. Consider the given information to choose the form of the equation for the graph.

WRITE

a. Let the equation be $y = a(x - b)^2 + c$.
Turning point $(1, -4)$
$\therefore y = a(x - 1)^2 - 4$
Substitute the given point $(0, -6)$.
$-6 = a(0 - 1)^2 - 4$
$-6 = a - 4$
$\therefore \quad a = -2$
Check: graph is concave down so $a < 0$.
The equation of the parabola is $y = -2(x - 1)^2 - 4$.

b. Let the equation be $y = a(x - b)(x - c)$.
Given $b = -1$, $c = 4$
$\therefore y = a(x + 1)(x - 4)$

2. Determine the value of a.	Substitute the third given point $(0, -2)$. $-2 = a(0+1)(0-4)$ $-2 = a(1)(-4)$ $-2 = -4a$ $a = \frac{-2}{-4}$ $= \frac{1}{2}$
3. Is the sign of a appropriate?	Check: graph is concave up so $a > 0$.
4. Write the rule for the graph.	The equation of the parabola is $y = \frac{1}{2}(x+1)(x-4)$.

3.2.7 Using simultaneous equations

In **Worked example 5b** three points were available, but because two of them were key points, the x-intercepts, we chose to form the rule using the $y = a(x-b)(x-c)$ form. If the points were not key points, then simultaneous equations need to be created using the coordinates given.

WORKED EXAMPLE 6

Determine the equation of the parabola that passes through the points $(1, -4)$, $(-1, 10)$ and $(3, -2)$.

THINK	WRITE
1. Consider the given information to choose the form of the equation for the graph.	Let $y = ax^2 + bx + c$.
2. Substitute the first point to form the an equation in a, b and c.	First point $(1, -4) \Rightarrow -4 = a(1)^2 + b(1) + c$ $\therefore -4 = a + b + c$ [1]
3. Substitute the second point to form a second equation in a, b and c.	Second point $(-1, 10) \Rightarrow 10 = a(-1)^2 + b(-1) + c$ $\therefore 10 = a - b + c$ [2]
4. Substitute the third point to form a third equation in a, b and c.	Third point $(3, -2) \Rightarrow -2 = a(3)^2 + b(3) + c$ $\therefore -2 = 9a + 3b + c$ [3]
5. Write the equations as a system of 3×3 simultaneous equations.	$-4 = a + b + c$ [1] $10 = a - b + c$ [2] $-2 = 9a + 3b + c$ [3]

6. Solve the system of simultaneous equations.

Eliminate a and c from equations [1] and [2].
Equation [2] – equation [1]
$14 = -2b$
$b = -7$
Eliminate c from equations [1] and [3].
Equation [3] – equation [1]
$2 = 8a + 2b$ [4]
Substitute $b = -7$ in to equation [4].
$2 = 8a - 14$
$16 = 8a$
$a = 2$
Substitute $a = 2$, $b = -7$ in to equation [1].
$-4 = 2 - 7 + c$
$c = 1$

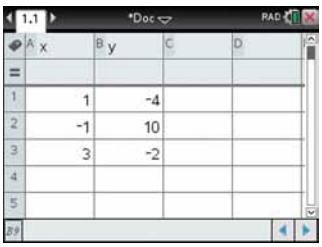
7. State the equation.

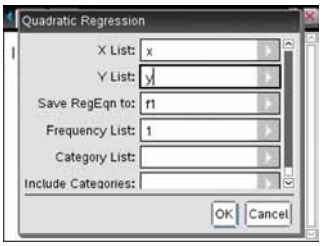
The equation of the parabola is $y = 2x^2 - 7x + 1$.

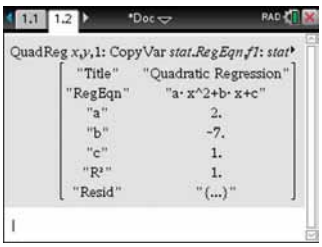
TI | THINK

1. On a Lists & Spreadsheet page, label the first column x and the second column y. Enter the x-coordinates of the given points in the first column and the corresponding y-coordinates in the second column.
2. On a Calculator page, press MENU then select
   6: Statistics
   1: Stat Calculations
   6: Quadratic Regression …
   Complete the fields as
   X List: x
   Y List: y
   then select OK.
3. The answer appears on the screen.

WRITE





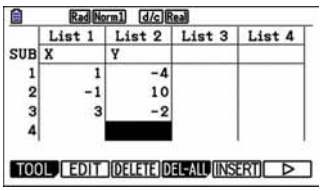


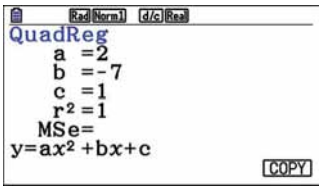
The equation of the parabola is $y = 2x^2 - 7x + 1$.

CASIO | THINK

1. On a Statistics screen, relabel List 1 as x and List 2 as y. Enter the x-coordinates of the given points in the first column and the corresponding y-coordinates in the second column.
2. Select CALC by pressing F2, select REG by pressing F3, then select X^2 by pressing F3.
3. The answer appears on the screen.

WRITE





The equation of the parabola is $y = 2x^2 - 7x + 1$.

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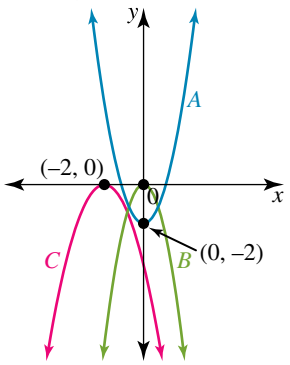
Units 1 & 2 > Area 2 > Sequence 2 > Concept 4

Determining the rule for the graph of a quadratic polynomial Summary screen and practice questions

Exercise 3.2 Graphs of quadratic functions

Technology free

1. Sketch the following parabolas on the same set of axes.
   - **a.** $y = 2x^2$
   - **b.** $y = -2x^2$
   - **c.** $y = 0.5x^2$
   - **d.** $y = -0.5x^2$
   - **e.** $y = (2x)^2$
   - **f.** $y = \left(-\frac{x}{2}\right)^2$
2. **WE1** Match the graphs of the parabolas A, B and C with the following equations.
   - **i.** $y = x^2 - 2$
   - **ii.** $y = -2x^2$
   - **iii.** $y = -(x + 2)^2$



3. State the coordinates of the turning points of the parabolas with the following equations.
   - **a.** $y = x^2 + 8$
   - **b.** $y = x^2 - 8$
   - **c.** $y = 1 - 5x^2$
   - **d.** $y = -\frac{x^2}{4} - 7$
   - **e.** $y = (x - 8)^2$
   - **f.** $y = (x + 8)^2$
   - **g.** $y = 7(x - 4)^2$
   - **h.** $y = -\frac{1}{2}(x + 12)^2$
4. **WE2** Sketch the graph of $y = \frac{1}{3}x^2 + x - 6$ and label the key points with their coordinates.
5. Sketch the graphs of the following parabolas, labelling their key points with their coordinates.
   - **a.** $y = 9x^2 + 18x + 8$
   - **b.** $y = -x^2 + 7x - 10$
   - **c.** $y = -x^2 - 2x - 3$
   - **d.** $y = x^2 - 4x + 2$
6. **WE3** Sketch the graph of $y = -2(x + 3)^2 + 2$ and label the key points with their coordinates.
7. State the nature and the coordinates of the turning point for each of the following parabolas.
   - **a.** $y = 4 - 3x^2$
   - **b.** $y = (4 - 3x)^2$
8. **MC** Select which of the following is the equation of a parabola with a turning point at $(-5, 2)$.
   - **A.** $y = -5x^2 + 2$
   - **B.** $y = 2 - (x - 5)^2$
   - **C.** $y = (x + 2)^2 - 5$
   - **D.** $y = -(x + 5)^2 + 2$
   - **E.** $y = (x + 5)^2 - 2$



9. **WE4** Sketch the graph of $y = 2x(4 - x)$.

10. Sketch the following graphs showing all intercepts with the coordinate axes and the turning point.

a. $y = (x + 1)(x - 3)$ **b.** $y = (x - 5)(2x + 1)$

c. $y = -\frac{1}{2}(2x - 7)(2x - 9)$ **d.** $y = (1 - 3x)(4 + x)$

For each of the parabolas in questions **11** to **12**, give the coordinates of:

i. the turning point

ii. the y-intercept

iii. any x-intercepts.

Then sketch each graph.

11. a. $y = x^2 - 9$ **b.** $y = (x - 9)^2$ **c.** $y = 6 - 3x^2$

d. $y = -3(x + 1)^2$ **e.** $y = \frac{1}{4}(1 - 2x)^2$ **f.** $y = -0.25(1 + 2x)^2$

12. a. $y = (x - 5)^2 + 2$ **b.** $y = 2(x + 1)^2 - 2$

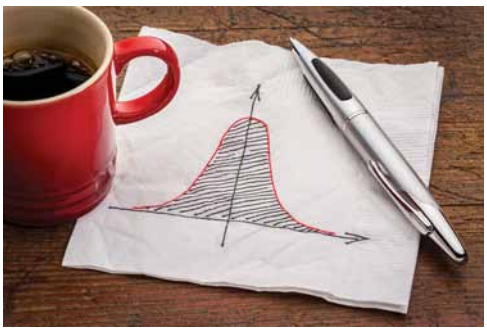
c. $y = -2(x - 3)^2 - 6$ **d.** $y = -(x - 4)^2 + 1$

e. $y + 2 = \frac{(x + 4)^2}{2}$ **f.** $9y = 1 - \frac{1}{3}(2x - 1)^2$

13. a. A parabola with equation $y = x^2 + c$ passes through the point $(1, 5)$. Determine the value of c and state the equation of the parabola.

b. A parabola with equation $y = ax^2$ passes through the point $(6, -2)$. Calculate the value of a and state the equation of the parabola.

c. A parabola with equation $y = a(x - 2)^2$ passes through the point $(0, -12)$. Calculate the value of a and state the equation of the parabola.

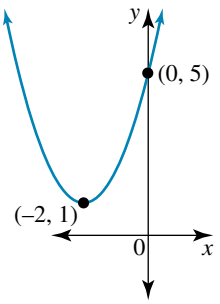


14. a. State the two linear factors of the equation of the parabola whose x-intercepts occur at $x = 3$ and at $x = 8$ and form a possible equation for this parabola.

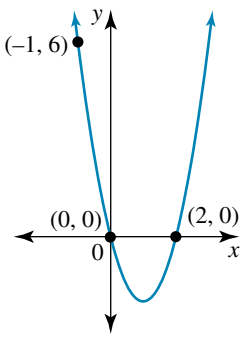
b. The x-intercepts of a parabola occur at $x = -11$ and $x = 2$. Form a possible equation for this parabola.

15. **WE5** Determine the rules for each of the following parabolas.

a.



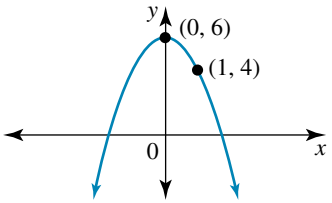
b.



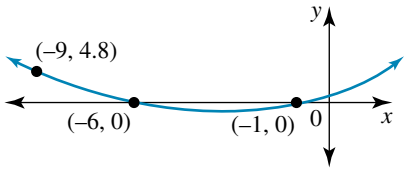
16. **WE6** Determine the equation of the parabola which passes through the points $(-1, -7)$, $(2, -10)$ and $(4, -32)$.

17. Determine the equation of each of the parabolas shown in the diagrams.

a.

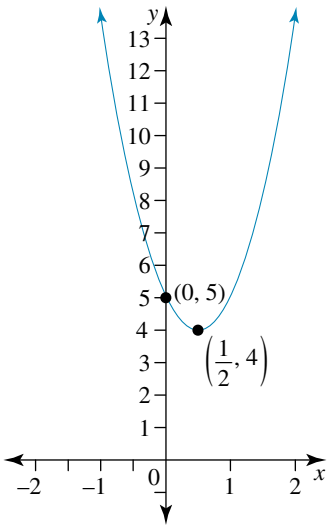


b.



18. For each of the following graphs two possible equation are given. Select the correct equation.

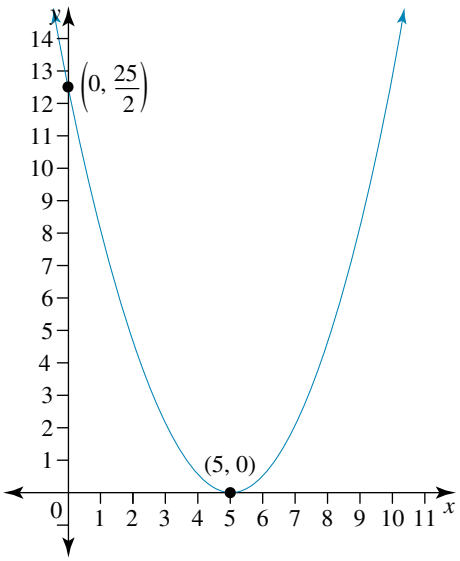
a.



Equation A: $y = (2x - 1)^2 + 4$

Equation B: $y = \frac{1}{4}\left(x - \frac{1}{2}\right)^2 + 4$

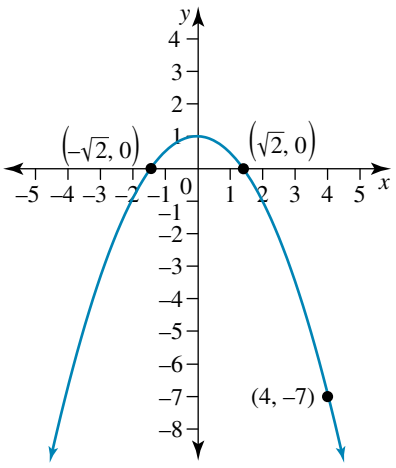
b.



Equation A: $y = \frac{1}{2}(5 - x)^2$

Equation B: $y = 2(x - 5)^2$

c.



Equation A: $y = -\frac{1}{2}(x^2 - 2)$

Equation B: $y = -\frac{7}{18}(x^2 + 2)$

3.3 Solving quadratic equations with rational roots

A **rational number** is any real number that can be expressed exactly as a fraction.

Rational roots of a quadratic equation are solutions that can be expressed as rational numbers.

3.3.1 Quadratic equations and the Null Factor Law

The general quadratic equation can be written as $ax^2+bx+c=0$, where a, b, c are real constants and $a \neq 0$. If the quadratic expression on the left-hand side of this equation can be factorised, the solutions to the quadratic equation may be obtained using the **Null Factor Law**.

The Null Factor Law states that, for any a and b, if the product $ab = 0$ then $a = 0$ or $b = 0$ or both a and $b = 0$.

Applying the Null Factor Law to a quadratic equation expressed in the factorised form as $(x-b)(x-c)=0$, would mean that

$$(x-b)=0 \text{ or } (x-c)=0$$

$$\therefore x = b \text{ or } x = c$$

To apply the Null Factor Law, one side of the equation must be zero and the other side must be in factorised form.

In the previous section the Null Factor Law was used to find the x-intercepts of the parabola.

Roots, zeros and factors

The solutions of an equation are also called the **roots** of the equation or the **zeros** of the quadratic expression. This terminology applies to all algebraic and not just quadratic equations. The quadratic equation $(x-1)(x-2)=0$ has roots $x = 1,\ x = 2$. These solutions are the zeros of the quadratic expression $(x-1)(x-2)$ since substituting either of $x = 1,\ x = 2$ in the quadratic expression $(x-1)(x-2)$ makes the expression equal zero.

As a converse of the Null Factor Law it follows that if the roots of a quadratic equation, or the zeros of a quadratic, are $x = b$ and $x = c$, then $(x-b)$ and $(x-c)$ are linear factors of the quadratic. The quadratic would be of the form $(x-b)(x-c)$ or any multiple of this form, $a(x-b)(x-c)$.

WORKED EXAMPLE 7

a. Solve the equation $5x^2 - 18x = 8$.

b. Given that $x = 2$ and $x = -2$ are zeros of a quadratic, form its linear factors and expand the product of these factors.

THINK	WRITE
a. 1. Rearrange the equation to make one side of the equation equal zero.	a. $5x^2 - 18x = 8$ Rearrange: $5x^2 - 18x - 8 = 0$
2. Factorise the quadratic trinomial.	$(5x+2)(x-4)=0$
3. Apply the Null Factor Law.	$5x+2=0$ or $x-4=0$
4. Solve these linear equations for x.	$5x=-2$ or $x=4$ $x=-\frac{2}{5}$ or $x=4$

b. 1. Use the converse of the Null Factor Law.

b. Since $x = 2$ is a zero, then $(x - 2)$ is a linear factor, and since $x = -2$ is a zero, then $(x - (-2)) = (x + 2)$ is a linear factor.
Therefore the quadratic has the linear factors $(x - 2)$ and $(x + 2)$.

2. Expand the product of the two linear factors.

The product $= (x - 2)(x + 2)$
Expanding, $(x - 2)(x + 2) = x^2 - 4$
The quadratic has the form $x^2 - 4$ or any multiple of this form $a(x^2 - 4)$.

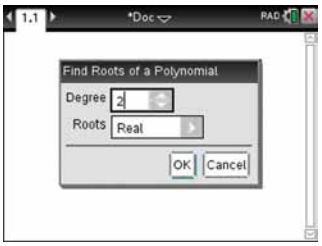
TI | THINK

a. 1. Rearrange the given equation so that all terms are on one side.

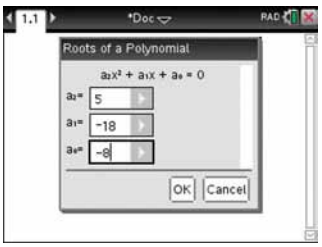
WRITE

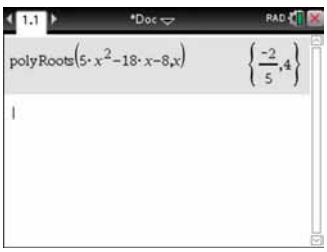
$5x^2 - 18x = 8$
Rearrange:
$5x^2 - 18x - 8 = 0$

2. On a Calculator page, press MENU then select
3: Algebra
3: Polynomial Tools
1: Find Roots of Polynomial …
Complete the fields as
Degree: 2
Roots: Real
then select OK.



3. Complete the fields for the coefficients as
$a_2 = 5$
$a_1 = -18$
$a_0 = -8$
then select OK.





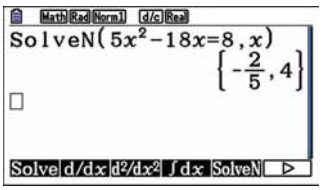
4. The answer appears on the screen.

$x = -\frac{2}{5}$ and $x = 4$

CASIO | THINK

a. 1. On a Run-Matrix screen, press OPTN then select CALC by pressing F4. Select SolveN by pressing F5, then complete the entry line as Solve N $(5x^2 - 18x = 8, x)$ and press EXE.

WRITE



2. The answer appears on the screen.

$x = -\frac{2}{5}$ and $x = 4$

Resources

Interactivity: Roots, zeros and factors (int-2557)

3.3.2 Using the perfect square form of a quadratic

As an alternative to solving a quadratic equation by using the Null Factor Law, if the quadratic is a perfect square, solutions to the equation can be found by taking square roots of both sides of the equation. A simple illustration is:

using the square root method

$x^2 = 9$

$x = \pm\sqrt{9}$

$= \pm 3$

or

using the Null Factor Law method

$x^2 = 9$

$x^2 - 9 = 0$

$(x - 3)(x + 3) = 0$

$x = \pm 3$

If the square root method is used, both the positive and negative square roots must be considered.

WORKED EXAMPLE 8

Solve the equation $(2x + 3)^2 - 25 = 0$.

THINK

1. Rearrange so that each side of the equation contains a perfect square.
2. Take the square roots of both sides.
3. Separate the linear equations and solve.
4. An alternative method uses the Null Factor Law.

WRITE

$(2x + 3)^2 - 25 = 0$

$(2x + 3)^2 = 25$

$2x + 3 = \pm 5$

$2x + 3 = 5$ or $2x + 3 = -5$

$2x = 2$ $\quad$ $2x = -8$

$x = 1$ or $x = -4$

Alternatively:

$(2x + 3)^2 - 25 = 0$

Factorise:

$((2x + 3) - 5)((2x + 3) + 5) = 0$

$(2x - 2)(2x + 8) = 0$

$2x = 2$ or $2x = -8$

$\therefore x = 1$ or $x = -4$

on Resources

Interactivity: The perfect square (int-2558)

3.3.3 Equations that reduce to quadratic form

Substitution techniques can be applied to the solution of equations such as those of the form $ax^4 + bx^2 + c = 0$. Once reduced to quadratic form, progress with the solution can be made.

The equation $ax^4 + bx^2 + c = 0$ can be expressed in the form $a(x^2)^2 + bx^2 + c = 0$. Letting $u = x^2$, this becomes $au^2 + bu + c = 0$, a quadratic equation in variable u.

By solving the quadratic equation for u, then substituting back x^2 for u, any possible solutions for x can be obtained. Since x^2 cannot be negative, it would be necessary to reject negative u values since $x^2 = u, u < 0$, would have no real solutions.

The quadratic form may be achieved from substitutions other than $u = x^2$, depending on the form of the original equation. The choice of symbol for the substitution is at the discretion of the solver. The symbol u

does not have to be used; a commonly chosen symbol is a. However, if the original equation involves variable x, do not use x for the substitution symbol.

WORKED EXAMPLE 9

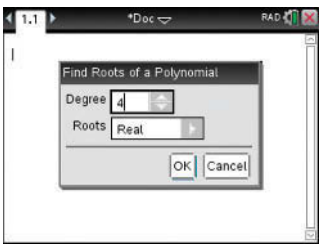
Solve the equation $4x^4 - 35x^2 - 9 = 0$.

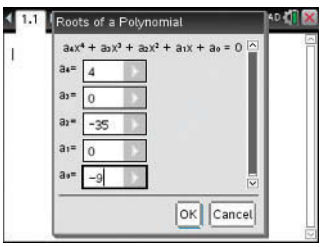
THINK	WRITE
1. Use an appropriate substitution to reduce the equation to quadratic form.	$4x^4 - 35x^2 - 9 = 0$ Let $a = x^2$ $4a^2 - 35a - 9 = 0$
2. Solve for a using factorisation.	$(4a + 1)(a - 9) = 0$ $\therefore a = -\frac{1}{4}$ or $a = 9$
3. Substitute back, replacing a by x^2.	$x^2 = -\frac{1}{4}$ or $x^2 = 9$
4. Since x^2 cannot be negative, any negative value of a needs to be rejected.	Reject $x^2 = -\frac{1}{4}$ since there are no real solutions.
5. Solve the remaining equation for x.	$x^2 = 9$ $x = \pm\sqrt{9}$ $x = \pm 3$

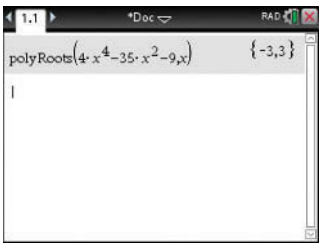
TI | THINK

1. On a Calculator page, press MENU then select
   3: Algebra
   3: Polynomial Tools
   1: Find Roots of Polynomial …
   Complete the fields as
   Degree: 4
   Roots: Real
   then select OK.
2. Complete the fields for the coefficients as
   $a_4 = 4$
   $a_3 = 0$
   $a_2 = -35$
   $a_1 = -0$
   $a_0 = -9$
   then select OK.
3. The answer appears on the screen.

WRITE





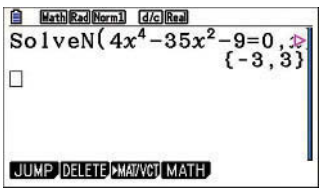


$x = \pm 3$

CASIO | THINK

1. On a Run-Matrix screen, press OPTN then select CALC by pressing F4. Select SolveN by pressing F5, then complete the entry line as Solve N $\left(4x^4 - 35x^2 - 9 = 0, x\right)$ and press EXE.
2. The answer appears on the screen.

WRITE



$x = \pm 3$

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concept 5

Solving quadratic equations with rational roots Summary screen and practice questions

Exercise 3.3 Solving quadratic equations with rational roots

Technology free

1. WE7 **a.** Solve the equation $10x^2 + 23x = 21$.
   b. Given that $x = -5$ and $x = 0$ are zeros of a quadratic, determine its linear factors and expand the product of these factors.
2. WE8 Solve the equation $(5x - 1)^2 - 16 = 0$.
3. Use the Null Factor Law to solve the following quadratic equations for x.
   a. $(3x - 4)(2x + 1) = 0$
   b. $x^2 - 7x + 12 = 0$
   c. $8x^2 + 26x + 21 = 0$
   d. $10x^2 = 2x$
   e. $12x^2 + 40x - 32 = 0$
   f. $\frac{1}{2}x^2 - 5x = 0$
4. Solve the following quadratic equations for x.
   a. $(x + 2)^2 = 9$
   b. $(x - 1)^2 - 25 = 0$
   c. $(x - 7)^2 + 4 = 0$
   d. $(2x + 11)^2 = 81$
   e. $(7 - x)^2 = 0$
   f. $8 - \frac{1}{2}(x - 4)^2 = 0$

For questions 5 and 6, solve each of the given equations using the Null Factor Law.

5. **a.** $3x(5 - x) = 0$
   b. $(3 - x)(7x - 1) = 0$
   c. $(x + 8)^2 = 0$
   d. $2(x + 4)(6 + x) = 0$
6. **a.** $6x^2 + 5x + 1 = 0$
   b. $12x^2 - 7x = 10$
   c. $49 = 14x - x^2$
   d. $5x + 25 - 30x^2 = 0$
7. Obtain the solutions to the following equations.
   a. $x^2 = 121$
   b. $9x^2 = 16$
   c. $(x - 5)^2 = 1$
   d. $(5 - 2x)^2 - 49 = 0$
   e. $2(3x - 1)^2 - 8 = 0$
   f. $(x^2 + 1)^2 = 100$
8. WE9 Solve the equation $9x^4 + 17x^2 - 2 = 0$.
9. Determine the roots (solutions) of the following equations.
   a. $18(x - 3)^2 + 9(x - 3) - 2 = 0$
   b. $5(x + 2)^2 + 23(x + 2) + 12 = 0$
   c. $x + 6 + \frac{8}{x} = 0$
   d. $2x + \frac{3}{x} = 7$
10. Use a substitution technique to solve the following equations.
    a. $(3x + 4)^2 + 9(3x + 4) - 10 = 0$
    b. $2(1 + 2x)^2 + 9(1 + 2x) = 18$
    c. $x^4 - 29x^2 + 100 = 0$
    d. $2x^4 = 31x^2 + 16$
    e. $36x^2 = \frac{9}{x^2} - 77$
    f. $(x^2 + 4x)^2 + 7(x^2 + 4x) + 12 = 0$

For questions 11 and 12, express each equation in quadratic form and hence solve the equations for x.

11. **a.** $x(x - 7) = 8$
    b. $4x(3x - 16) = 3(4x - 33)$
    c. $(x + 4)^2 + 2x = 0$
    d. $(2x + 5)(2x - 5) + 25 = 2x$
12. **a.** $2 - 3x = \frac{1}{3x}$
    b. $\frac{4x + 5}{x + 125} = \frac{5}{x}$
    c. $7x - \frac{2}{x} + \frac{11}{5} = 0$
    d. $\frac{12}{x + 1} - \frac{14}{x - 2} = 19$

13. Obtain the solutions to the following equations.

a. $x^4 = 81$

b. $(9x^2 - 16)^2 = 20(9x^2 - 16)$

c. $\left(x - \frac{2}{x}\right)^2 - 2\left(x - \frac{2}{x}\right) + 1 = 0$

d. $2\left(1 + \frac{3}{x}\right)^2 + 5\left(1 + \frac{3}{x}\right) + 3 = 0$

14. Solve the equation $\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) + 4 = 0$.

15. Solve the equation $(px + q)^2 = r^2$ for x in terms of p, q and r, $r > 0$.

16. Express the value of x in terms of the positive real numbers a and b.

a. $(x - 2b)(x + 3a) = 0$

b. $2x^2 - 13ax + 15a^2 = 0$

c. $(x - b)^4 - 5(x - b)^2 + 4 = 0$

d. $(x - a - b)^2 = 4b^2$

e. $(x + a)^2 - 3b(x + a) + 2b^2 = 0$

f. $ab\left(x + \frac{a}{b}\right)\left(x + \frac{b}{a}\right) = (a + b)^2 x$

17. Consider the quadratic equation $(x - \alpha)(x - \beta) = 0$.

a. If the roots of the equation are $x = 1$ and $x = 7$, form the equation.

b. If the roots of the equation are $x = -5$ and $x = 4$, form the equation.

c. If the roots of the equation are $x = 0$ and $x = 10$, form the equation.

d. If the only root of the equation is $x = 2$, form the equation.

18. a. If the zeros of the quadratic expression $4x^2 + bx + c$ are $x = -4$ and $x = \frac{3}{4}$, calculate the values of the integer constants b and c.

b. Express the roots of $px^2 + (p + q)x + q = 0$ in terms of p and q for $p, q \in Q$, $p \neq 0$ and hence solve $p(x - 1)^2 + (p + q)(x - 1) + q = 0$.

19. a. Solve $44 + 44x^2 = 250x$

b. The use of the symbol x for the variable is a conventional notation, although not obligatory. The Babylonians, who were the first to solve quadratic equations, just used words equivalent to 'length', 'breadth' and 'area', for example, for the unknown quantity and ignored their different dimensions. Write the following statement in contemporary form in terms of x and hence obtain the required quantity.

Obtain the side of a square if the 'area' less the 'side' is 870.



(The name first given to an unknown was 'shay', meaning 'thing', and it appeared in the work of al-Khwarizmi. De Nemore was the first European mathematician to use a symbol for an unknown. For reasons not understood, he used the symbol *abc* as the unknown.)

3.4 Factorising and solving quadratics over *R*

When a quadratic expression cannot be factorised into linear factors, **surds** need to be used and the resulting factorisation is over *R*.

When $x^2 - 4$ is expressed as $(x - 2)(x + 2)$ it has been factorised over *Q*, as both of the zeros are rational numbers. However, over *Q*, the quadratic expression $x^2 - 3$ cannot be factorised into linear factors. Surds need to be permitted for such an expression to be factorised.

3.4.1 Factorisation over *R*

The quadratic $x^2 - 3$ can be expressed as the difference of two squares $x^2 - 3 = x^2 - \left(\sqrt{3}\right)^2$ using surds. This can be factorised over *R* because it allows the factors to contain surds.

$$\begin{aligned} x^2 - 3 &= x^2 - \left(\sqrt{3}\right)^2 \\ &= \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) \end{aligned}$$

If a quadratic can be expressed as the difference of two squares, then it can be factorised over *R*. To express a quadratic trinomial as a difference of two squares a technique called '**completing the square**' is used.

'Completing the square' technique

Expressions of the form $x^2 \pm px + \left(\frac{p}{2}\right)^2 = \left(x \pm \frac{p}{2}\right)^2$ are perfect squares. For example, $x^2 + 4x + 4 = (x + 2)^2$.

To illustrate the 'completing the square' technique, consider the quadratic trinomial $x^2 + 4x + 1$.

If 4 is added to the first two terms $x^2 + 4x$ then this will form a perfect square $x^2 + 4x + 4$. However, 4 must also be subtracted in order not to alter the value of the expression.

$$x^2 + 4x + 1 = x^2 + 4x + 4 - 4 + 1$$

Grouping the first three terms together to form the perfect square and evaluating the last two terms,

$$\begin{aligned} &= (x^2 + 4x + 4) - 4 + 1 \\ &= (x + 2)^2 - 3 \end{aligned}$$

By writing this difference of two squares form using surds, factors over *R* can be found.

$$\begin{aligned} &= (x + 2)^2 - \left(\sqrt{3}\right)^2 \\ &= \left(x + 2 - \sqrt{3}\right)\left(x + 2 + \sqrt{3}\right) \end{aligned}$$

Thus: $x^2 + 4x + 1 = \left(x + 2 - \sqrt{3}\right)\left(x + 2 + \sqrt{3}\right)$.

'Completing the square' is the method used to factorise **monic** quadratics over *R*. A monic quadratic is one for which the coefficient of x^2 equals 1.

For a monic quadratic, to complete the square, add and then subtract the square of half the coefficient of *x*. This squaring will always produce a positive number regardless of the sign of the coefficient of *x*.

$$\begin{aligned} x^2 \pm px &= \left[x^2 \pm px + \left(\frac{p}{2}\right)^2\right] - \left(\frac{p}{2}\right)^2 \\ &= \left(x \pm \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 \end{aligned}$$

To complete the square on $ax^2 + bx + c$, the quadratic should first be written as $a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$ and the technique applied to the monic quadratic in the bracket. The common factor a is carried down through all the steps in the working.

The general form of the equation of a parabola can be converted to turning point form by the use of the completing the square technique: by expanding turning point form, the general form would be obtained.

WORKED EXAMPLE 10

Factorise the following over R.

a. $x^2 - 14x - 3$

b. $2x^2 + 7x + 4$

c. $4x^2 - 11$

d. i. **Express $y = 3x^2 - 12x + 18$ in the form $y = a(x - b)^2 + c$ and hence state the coordinates of its vertex (turning point).**

ii. **Sketch its graph.**

THINK	WRITE
a. 1. Add and subtract the square of half the coefficient of x. *Note:* The negative sign of the coefficient of x becomes positive when squared.	**a.** $x^2 - 14x - 3$ $= x^2 - 14x + 7^2 - 7^2 - 3$
2. Group the first three terms together to form a perfect square and evaluate the last two terms.	$= (x^2 - 14x + 49) - 49 - 3$ $= (x - 7)^2 - 52$
3. Factorise the difference of two squares expression.	$= (x - 7)^2 - (\sqrt{52})^2$ $= (x - 7 - \sqrt{52})(x - 7 + \sqrt{52})$
4. Express any surds in their simplest form.	$= (x - 7 - 2\sqrt{13})(x - 7 + 2\sqrt{13})$
5. State the answer.	Therefore: $x^2 - 14x - 3 = (x - 7 - 2\sqrt{13})(x - 7 + 2\sqrt{13})$
b. 1. First create a monic quadratic by taking the coefficient of x^2 out as a common factor. This may create fractions.	**b.** $2x^2 + 7x + 4$ $= 2\left(x^2 + \frac{7}{2}x + 2\right)$
2. Add and subtract the square of half the coefficient of x for the monic quadratic expression.	$= 2\left(x^2 + \frac{7}{2}x + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + 2\right)$

3. Within the bracket, group the first three terms together and evaluate the remaining terms.

$$= 2\left[\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) - \frac{49}{16} + 2\right]$$

$$= 2\left[\left(x + \tfrac{7}{4}\right)^2 - \frac{49}{16} + 2\right]$$

$$= 2\left[\left(x + \tfrac{7}{4}\right)^2 - \frac{49}{16} + \frac{32}{16}\right]$$

$$= 2\left[\left(x + \tfrac{7}{4}\right)^2 - \frac{17}{16}\right]$$

4. Factorise the difference of two squares that has been formed.

$$= 2\left[\left(x + \frac{7}{4}\right) - \sqrt{\frac{17}{16}}\right]\left[\left(x + \frac{7}{4}\right) + \sqrt{\frac{17}{16}}\right]$$

$$= 2\left(x + \frac{7}{4} - \frac{\sqrt{17}}{4}\right)\left(x + \frac{7}{4} + \frac{\sqrt{17}}{4}\right)$$

5. State the answer.

$$2x^2 + 7x + 4$$

$$= 2\left(x + \frac{7}{4} - \frac{\sqrt{17}}{4}\right)\left(x + \frac{7}{4} + \frac{\sqrt{17}}{4}\right)$$

$$= 2\left(x + \frac{7 - \sqrt{17}}{4}\right)\left(x + \frac{7 + \sqrt{17}}{4}\right)$$

c. The quadratic is a difference of two squares. Factorise it.

c. $4x^2 - 11$

$= (2x)^2 - (\sqrt{11})^2$

$= (2x - \sqrt{11})(2x + \sqrt{11})$

d. i. 1. Apply the completing the square technique to the general form of the equation.

d. $y = 3x^2 - 12x + 18$

$= 3(x^2 - 4x + 6)$

$= 3((x^2 - 4x + (2)^2) - (2)^2 + 6)$

$= 3((x - 2)^2 + 2)$

$= 3(x - 2)^2 + 6$

2. Expand to obtain the form $y = a(x - h)^2 + k$.

$\therefore y = 3(x - 2)^2 + 6$

3. State the coordinates of the vertex (turning point).

The vertex is $(2, 6)$.

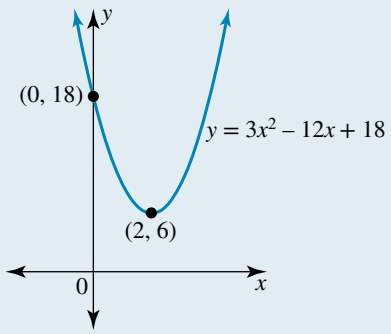
ii. 1. Obtain the y-intercept from the general form.

y-intercept is $(0, 18)$.

2. Will the graph have x-intercepts?

Since the graph is concave up with minimum y-value of 6, there are no x-intercepts.

3. Sketch the graph.



3.4.2 The quadratic formula

The **quadratic formula** is used for solving quadratic equations and is obtained by completing the square on the left-hand side of the equation $ax^2 + bx + c = 0$.

Using completing the square:

$$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right).$$

$$ax^2 + bx + c = 0$$

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Often the coefficients in the quadratic equation make the use of the formula less tedious than completing the square. Although the formula can also be used to solve a quadratic equation which factorises over Q, factorisation is usually simpler, making it the preferred method.

The x intercepts of a parabola may be determined by applying this formula.

Resources

Interactivity: The Quadratic formula (int-2561)

WORKED EXAMPLE 11a

Apply the quadratic formula to solve each of the following equations.
a. $3x^2 + 4x + 1 = 0$ **(exact answer)**
b. $-3x^2 - 6x - 1 = 0$ **(round to 2 decimal places)**

THINK	WRITE
a. 1. Write the equation.	**a.** $3x^2 + 4x + 1 = 0$
2. Write the quadratic formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. State the values for a, b and c.	Where $a = 3$, $b = 4$, $c = 1$
4. Substitute the values into the formula.	$x = \dfrac{-4 \pm \sqrt{(4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$
5. Simplify and solve for x.	$= \dfrac{-4 \pm \sqrt{4}}{6}$ $= \dfrac{-4 \pm 2}{6}$ $x = \dfrac{-4 + 2}{6}$ or $x = \dfrac{-4 - 2}{6}$
6. Write the two solutions.	$x = -\dfrac{1}{3}$ $\quad x = -1$
b. 1. Write the equation.	**b.** $-3x^2 - 6x - 1 = 0$
2. Write the quadratic formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. State the values for a, b and c.	Where $a = -3$, $b = -6$, $c = -1$
4. Substitute the values into the formula.	$x = \dfrac{-(-6) \pm \sqrt{36 - 4 \times -3 \times -1}}{2 \times -3}$
5. Simplify the fraction.	$= \dfrac{6 \pm \sqrt{24}}{-6}$ $= \dfrac{6 \pm 2\sqrt{6}}{-6}$ $= \dfrac{3 \pm \sqrt{6}}{-3}$ $x = \dfrac{3 + \sqrt{6}}{-3}$ or $\dfrac{3 - \sqrt{6}}{-3}$

6. Wrote the two solutions correct to two decimal places. $x \approx -1.82$ or $x \approx -0.18$

Note: When asked to given an answer in exact form, you should simplify any surds as necessary.

WORKED EXAMPLE 11b

Use the quadratic formula to solve the equation $x(9 - 5x) = 3$.

THINK	WRITE
1. The equation needs to be expressed in the general quadratic form $ax^2 + bx + c = 0$.	$x(9 - 5x) = 3$ $9x - 5x^2 = 3$ $5x^2 - 9x + 3 = 0$
2. State the values of a, b and c.	$a = 5,\ b = -9,\ c = 3$
3. State the formula for solving a quadratic equation.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4. Substitute the a, b, c values and evaluate.	$= \dfrac{-(-9) \pm \sqrt{(-9)^2 - 4 \times (5) \times (3)}}{2 \times (5)}$ $= \dfrac{9 \pm \sqrt{81 - 60}}{10}$ $= \dfrac{9 \pm \sqrt{21}}{10}$
5. Express the roots in simplest surd form and state the answers. *Note:* If the question asked for answers correct to 2 decimal places, use technology to find approximate answers of $x \simeq 0.44$ and $x \simeq 1.36$. Otherwise, do *not* approximate answers.	The solutions are: $x = \dfrac{9 - \sqrt{21}}{10},\ x = \dfrac{9 + \sqrt{21}}{10}$

Exercise 3.4 Factorising and solving quadratics over *R*

Technology free

1. Complete the following statements about perfect squares.
   a. $x^2 + 10x + ... = (x + ...)^2$
   b. $x^2 - 7x + ... = (x - ...)^2$
   c. $x^2 + x + ... = (x + ...)^2$
   d. $x^2 - \dfrac{4}{5}x + ... = (x - ...)^2$
2. **WE10** Factorise the following over R.
   a. $x^2 - 10x - 7$
   b. $3x^2 + 7x + 3$
   c. $5x^2 - 9$
3. Use the 'completing the square' method to factorise $-3x^2 + 8x - 5$ and check the answer by using another method of factorisation.

4. Use the 'competing the square' method to factorise, where possible, the following over R.
   a. $x^2 - 6x + 7$ b. $x^2 + 4x - 3$ c. $x^2 - 2x + 6$
   d. $2x^2 + 5x - 2$ e. $-x^2 + 8x - 8$ f. $3x^2 + 4x - 6$
5. Solve the equation $3(2x + 1)^4 - 16(2x + 1)^2 - 35 = 0$ for $x \in R$.
6. **WE10** Write the following in turning point form by completing the squre.
   a. $x^2 + 2x$ b. $x^2 + 7x$ c. $x^2 - 5x$ d. $x^2 + 4x - 2$
7. Factorise the following where possible.
   a. $3(x - 8)^2 - 6$ b. $(xy - 7)^2 + 9$
8. Factorise the following over R, where possible.
   a. $x^2 - 12$ b. $x^2 - 12x + 4$ c. $x^2 + 9x - 3$
   d. $2x^2 + 5x + 1$ e. $3x^2 + 4x + 3$ f. $1 + 40x - 5x^2$
9. Solve each of the following for x. Give exact answers.
   a. $x^2 - 10x + 23 = 0$ b. $x^2 - 5x + 5 = 0$
   c. $x^2 + 14x + 43 = 0$ d. $x^2 + 9x + 19 = 0$
10. Solve each of the following for x. Round answers to 2 decimal places.
    a. $x^2 - 3x - 5 = 0$ b. $x^2 - 6x + 4 = 0$
    c. $x^2 + 7x + 12 = 0$ d. $x^2 - 20x + 60 = 0$
11. State the quadratic formula and explain what it is used for.
12. State the values of a, b and c in each of the following quadratic expressions of the form $ax^2 + bx + c = 0$.
    a. $x^2 - 10x + 21 = 0$ b. $10x^2 - 93x + 68 = 0$
    c. $x^2 - 9x + 20 = 0$ d. $40x^2 + 32x + 6 = 0$
13. **WE11a** Apply the quadratic formula to solve each of the following quadratic equations. Give exact answers.
    a. $3x^2 - 2x - 4 = 0$ b. $2x^2 + 7x + 3 = 0$
    c. $-3x^2 - 6x + 4 = 0$ d. $12x^2 - 8x - 5 = 0$
14. Use the quadratic formula to solve each of the following quadratic equations. Round answers to 2 decimal places.
    a. $-2x^2 - 5x + 4 = 0$ b. $22x^2 - 11x - 20 = 0$
    c. $4x^2 - 29x + 19 = 0$ d. $-12x^2 + 2x + 15 = 0$
15. **MC** The solutions of the equation $15x^2 - 28x - 20 = 0$ are:
    A. 15, 20 **B.** $\frac{-28 + \sqrt{-416}}{30}, \frac{28 - \sqrt{-416}}{30}$
    C. $\frac{28 + \sqrt{1984}}{30}, \frac{28 - \sqrt{1984}}{30}$ **D.** $\frac{28 + \sqrt{1984}}{2}, \frac{28 - \sqrt{1984}}{2}$
16. **MC** The solutions of the equation $-6x^2 - 29x + 6 = 0$ are:
    A. $-5.03, 0.20$ **B.** $-0.22, -4.62$ **C.** $-6, 6$ **D.** $-31.62, -26.38$
17. **WE11b** Apply the quardratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the following equations, expressing solutions in simplest surd form.
    a. $3x^2 - 5x = -1$ b. $-5x^2 + x = -5$
    c. $2x^2 + 3x = -4$ d. $x(x + 6) = 8$
18. Use the quadratic formula to solve the equation $(2x + 1)(x + 5) - 1 = 0$.
19. a. i. Express $2x^2 - 12x + 9$ in the form $a(x + b)^2 + c$.
    ii. Hence state the coordinates of the turning point of the graph of $y = 2x^2 - 12x + 9$.
    iii. What is the minimum value of the polynomial $2x^2 - 12x + 9$?
    b. i. Express $-x^2 - 18x + 5$ in the form $a(x + b)^2 + c$.
    ii. Hence state the coordinates of the turning point of the graph of $y = -x^2 - 18x + 5$.
    iii. What is the maximum value of the polynomial $-x^2 - 18x + 5$?

20. Apply an appropriate substitution to reduce the following equations to quadratic form and hence obtain all solutions over R.

a. $(x^2 - 3)^2 - 4(x^2 - 3) + 4 = 0$

b. $5x^4 - 39x^2 - 8 = 0$

c. $x^2(x^2 - 12) + 11 = 0$

d. $\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) - 3 = 0$

e. $(x^2 - 7x - 8)^2 = 3(x^2 - 7x - 8)$

3.5 The discriminant

The choices of method to consider for solving the quadratic equation $ax^2 + bx + c = 0$ are:

- factorise over Q and use the Null Factor Law
- factorise over R by completing the square and use the Null Factor Law
- use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3.5.1 Defining the discriminant

Some quadratics factorise over Q and others factorise only over R. There are also some quadratics which cannot be factorised over R at all. This happens when the 'completing the square' technique does not create a difference of two squares but instead leads to a sum of two squares. In this case no further factorisation is possible over R. For example, completing the square on $x^2 - 2x + 6$ would give:

$$\begin{aligned} x^2 - 2x + 6 &= \left(x^2 - 2x + 1\right) - 1 + 6 \\ &= (x - 1)^2 + 5 \end{aligned}$$

As this is the sum of two squares, it cannot be factorised over R.

Completing the square can be a tedious process when fractions are involved so it can be useful to be able to determine in advance whether a quadratic factorises over Q or over R, or does not factorise over R. Evaluating what is called the **discriminant** will allow these three possibilities to be discriminated between. In order to define the discriminant, we need to complete the square on the general quadratic trinomial $ax^2 + bx + c$.

Previously it has been shown that a parabola may have 0, 1 or 2 x-intercepts, and the discriminant determines how many are possible.

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \end{aligned}$$

The sign of the term $b^2 - 4ac$ will determine whether a difference of two squares or a sum of two squares has been formed. If this term is positive, a difference of two squares is formed, but if the term is negative then a sum of two squares is formed. This term, $b^2 - 4ac$, is called the discriminant of the quadratic. It is denoted by the Greek letter delta, Δ.

$$\Delta = b^2 - 4ac$$

- If $\Delta < 0$ the quadratic has no real factors.
- If $\Delta \geq 0$ the quadratic has two real factors. The two factors are distinct (different) if $\Delta > 0$ and the two factors are identical if $\Delta = 0$.
  For a quadratic $ax^2 + bx + c$ with real factors and $a, b, c \in Q$:
- If Δ is a perfect square, the factors are rational; the quadratic factorises over Q.
- If $\Delta > 0$ but not a perfect square, the factors contain surds; the quadratic factorises over R. Completing the square will be required if $b \neq 0$.
- If $\Delta = 0$, the quadratic is a perfect square.

on Resources

Interactivity: Discriminant (int-2560)

WORKED EXAMPLE 12

For each of the following quadratics, calculate the discriminant and hence state the number and type of factors and whether the 'completing the square' method would be needed to obtain the factors.

a. $2x^2 + 15x + 13$

b. $5x^2 - 6x + 9$

c. $-3x^2 + 3x + 8$

d. $\frac{81}{4}x^2 - 12x + \frac{16}{9}$

THINK	WRITE
a. 1. State the values of a, b and c needed to calculate the discriminant.	**a.** $2x^2 + 15x + 13$, $a = 2, b = 15, c = 13$
2. State the formula for the discriminant.	$\Delta = b^2 - 4ac$
3. Substitute the values of a, b and c.	$\therefore \Delta = (15)^2 - 4 \times (2) \times (13)$ $= 225 - 104$ $= 121$
4. Interpret the value of the discriminant.	Since $\Delta > 0$ and is a perfect square, the quadratic has two rational factors. Completing the square is not essential as the quadratic factorises over Q. Check: $2x^2 + 15x + 13 = (2x + 13)(x + 1)$
b. 1. State a, b, c and calculate the discriminant.	**b.** $5x^2 - 6x + 9$, $a = 5$, $b = -6$, $c = 9$ $\Delta = b^2 - 4ac$ $= (-6)^2 - 4 \times (5) \times (9)$ $= 36 - 180$ $= -144$

2. Interpret the value of the discriminant.

Since $\Delta < 0$, the quadratic has no real factors.

c. 1. State a, b, c and calculate the discriminant.

c. $-3x^2 + 3x + 8$, $a = -3$, $b = 3$, $c = 8$

$\Delta = b^2 - 4ac$

$= (3)^2 - 4 \times (-3) \times (8)$

$= 9 + 96$

$= 105$

2. Interpret the value of the discriminant.

Since $\Delta > 0$ but is not a perfect square, there are two real factors. The quadratic factorises over R,
so completing the square would be needed to obtain the factors.

d. 1. State a, b, c and calculate the discriminant.

d. $\frac{81}{4}x^2 - 12x + \frac{16}{9}$

$a = \frac{81}{4}$, $b = -12$, $c = \frac{16}{9}$

$\Delta = b^2 - 4ac$

$\therefore \Delta = (-12)^2 - 4 \times \frac{81}{4} \times \frac{16}{9}$

$= 144 - 144$

$= 0$

2. Interpret the value of the discriminant.

Since $\Delta = 0$, there are two identical rational factors.
The quadratic is a perfect square. It factorises over Q, so completing the square is not essential.
Check: $\frac{81}{4}x^2 - 12x + \frac{16}{9} = \left(\frac{9}{2}x - \frac{4}{3}\right)^2$

3.5.2 The role of the discriminant in quadratic equations

The type of factor determines the type of solution to an equation, so it is no surprise that the discriminant determines the number and type of solutions as well as the number and type of factors.

The formula for the solution to the quadratic equation $ax^2 + bx + c = 0$ can be expressed as $x = \frac{-b \pm \sqrt{\Delta}}{2a}$, where the discriminant $\Delta = b^2 - 4ac$.

- If $\Delta < 0$, there are no real solutions to the equation.
- If $\Delta = 0$, there is one real solution (or two equal solutions) to the equation.
- If $\Delta > 0$, there are two distinct real solutions to the equation.

For a, b, $c \in Q$:

- If Δ is a perfect square, the roots are rational.
- If Δ is not a perfect square, the roots are irrational.

WORKED EXAMPLE 13

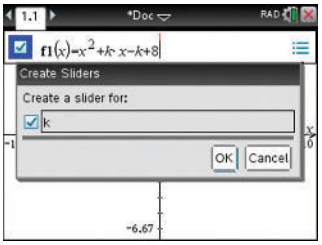
a. Use the discriminant to determine the number and type of roots to the equation $15x^2 + 8x - 5 = 0$.
b. Find the values of k so the equation $x^2 + kx - k + 8 = 0$ will have one real solution and check the answer.

THINK	WRITE
a. 1. Identify the values of a, b, c from the general $ax^2 + bx + c = 0$ form.	**a.** $15x^2 + 8x - 5 = 0$, $a = 15$, $b = 8$, $c = -5$
2. State the formula for the discriminant.	$\Delta = b^2 - 4ac$
3. Substitute the values of a, b, c and evaluate.	$= (8)^2 - 4 \times (15) \times (-5)$ $= 64 + 300$ $= 364$
4. Interpret the result.	Since the discriminant is positive but not a perfect square, the equation has two irrational roots.
b. 1. Express the equation in general form and identify the values of a, b and c.	**b.** $x^2 + kx - k + 8 = 0$ $\therefore x^2 + kx + (-k + 8) = 0$ $a = 1$, $b = k$, $c = (-k + 8)$
2. Substitute the values of a, b, c and obtain an algebraic expression for the discriminant.	$\Delta = b^2 - 4ac$ $= (k)^2 - 4 \times (1) \times (-k + 8)$ $= k^2 + 4k - 32$
3. State the condition on the discriminant for the equation to have one solution.	For one solution, $\Delta = 0$
4. Solve for k.	$k^2 + 4k - 32 = 0$ $(k + 8)(k - 4) = 0$ $k = -8, \ k = 4$
5. Check the solutions of the equation for each value of k.	If $k = -8$, the original equation becomes: $x^2 - 8x + 16 = 0$ $(x - 4)^2 = 0$ $\therefore \quad x = 4$ This equation has one solution. If $k = 4$, the original equation becomes: $x^2 + 4x + 4 = 0$ $(x + 2)^2 = 0$ $\therefore \quad x = -2$ This equation has one solution.

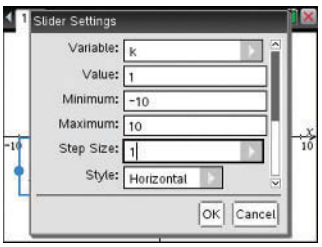
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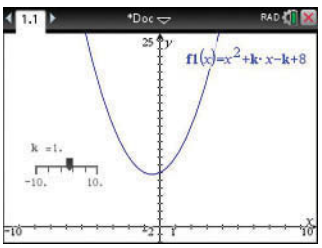
b. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = x^2 + k \times x - k + 8$, then press ENTER. Select the tick box for k then select OK.
Note: Be sure to include the multiplication operator between k and x.



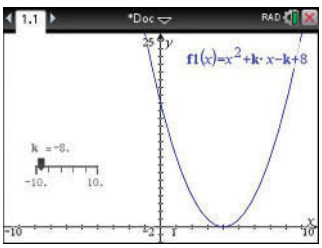
2. Press CTRL then MENU and select 1: Settings …
Complete the fields as
Variable: k
Value: 1
Minimum: −10
Maximum: 10
Step Size: 1
Style: Horizontal
then select OK.

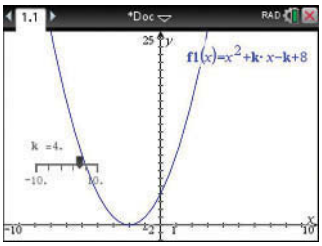


3. The graph appears on the screen.



4. Use the left/right arrows to change the value of k until the turning point of the graph lies on the x-axis. Note the values of k for which this occurs.





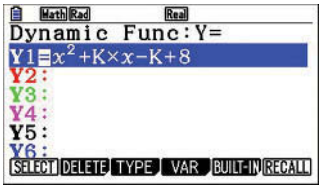
5. Answer the question.

There will be one real solution when $k = 4$ and $k = -8$.

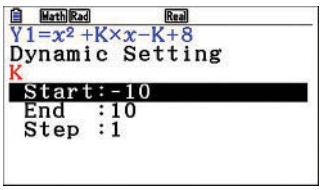
CASIO | THINK

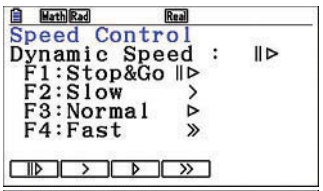
WRITE

b.1. On a Dyna Graph screen, complete the entry line for $y1$ as $y1 = x^2 + K \times x - K + 8$, then press EXE.
Note: Be sure to include the multiplication operator between k and x.

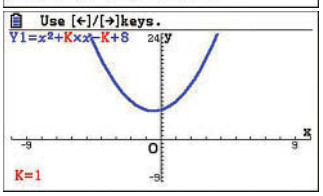


2. Select VAR by pressing by pressing F4, then select SET by pressing F2 and complete the fields as
Start: −10
End: 10
Step: 1
Press EXIT.
Select SPEED by pressing F3, then select Stop&Go by pressing F1 and press EXIT.

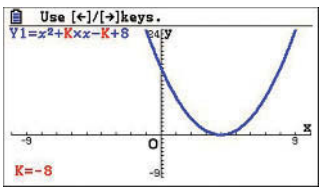


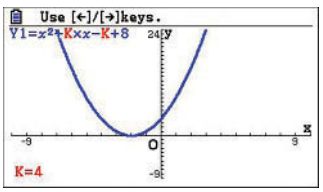


3. Select DYNA by pressing F6 to view the graph.



4. Use the left/right arrows to change the value of K until the turning point of the graph lies on the x-axis. Note the values of K for which this occurs.





5. Answer the question.

There will be one real solution when $k = 4$ and $k = -8$.

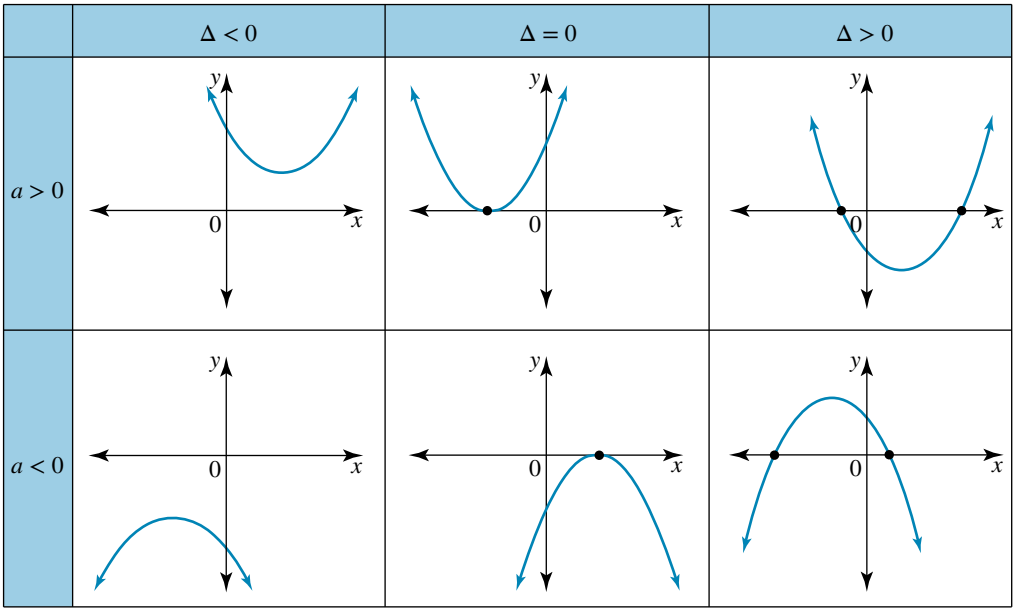
3.5.3 The discriminant and the x-intercepts

The zeros of the quadratic expression $ax^2 + bx + c$, the roots of the quadratic equation $ax^2 + bx + c = 0$ and the x-intercepts of the graph of a parabola with rule $y = ax^2 + bx + c$ all have the same x-values; and the discriminant determines the type and number of these values.

- If $\Delta > 0$, there are two x-intercepts. The graph *cuts* through the x-axis at two different places.
- If $\Delta = 0$, there is one x-intercept. The graph *touches* the x-axis at its turning point.
- If $\Delta < 0$, there are no x-intercepts. The graph does not intersect the x-axis and lies entirely above or entirely below the x-axis, depending on its concavity.

If $a > 0$, $\Delta < 0$, the graph lies entirely above the x-axis and every point on it has a positive y-coordinate. $ax^2 + bx + c$ is called **positive definite** in this case.

If $a < 0$, $\Delta < 0$, the graph lies entirely below the x-axis and every point on it has a negative y-coordinate. $ax^2 + bx + c$ is called **negative definite** in this case.



When $\Delta \geq 0$ and for $a, b, c \in Q$, the x intercepts are rational if Δ is a perfect square and irrational if Δ is not a perfect square.

WORKED EXAMPLE 14

Apply the discriminant to:

a. determine the number and type of x-intercepts of the graph defined by $y = 64x^2 + 48x + 9$

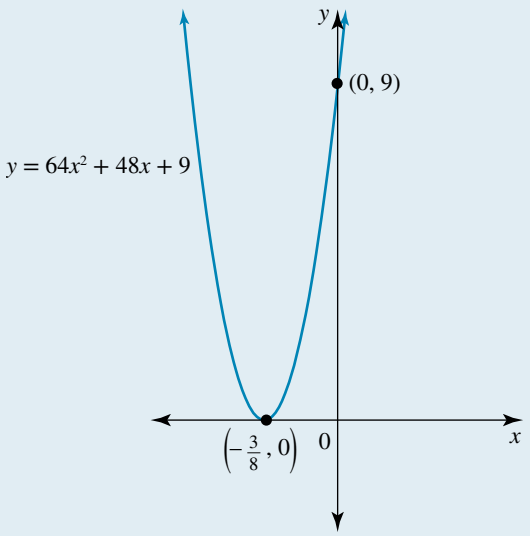
b. sketch the graph of $y = 64x^2 + 48x + 9$.

THINK	WRITE
a. 1. State the a, b, c values and evaluate the discriminant.	**a.** $y = 64x^2 + 48x + 9, a = 64, b = 48, c = 9$ $\Delta = b^2 - 4ac$ $= (48)^2 - 4 \times (64) \times (9)$ $= 2304 - 2304$ $= 0$
2. Interpret the result.	Since the discriminant is zero, the graph has one rational x-intercept.
b. 1. Interpret the implication of a zero discriminant for the factors.	**b.** The quadratic must be a perfect square. $y = 64x^2 + 48x + 9$ $= (8x + 3)^2$

2. Identify the key points.

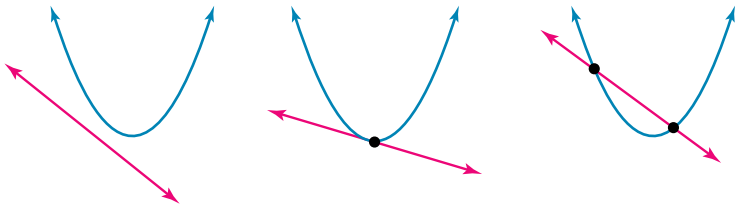
x-intercept: let $y = 0$.
$8x + 3 = 0$
$x = -\frac{3}{8}$
Therefore $\left(-\frac{3}{8}, 0\right)$ is both the x-intercept and the turning point.
y-intercept: let $x = 0$ in $y = 64x^2 + 48x + 9$.
$\therefore y = 9$
Therefore $(0, 9)$ is the y-intercept.

3. Sketch the graph.



3.5.4 Intersections of lines and parabolas

The possible number of points of intersection between a straight line and a parabola will be either 0, 1 or 2 points.



- If there is no point of intersection, the line makes no contact with the parabola.
- If there is 1 point of intersection, a non-vertical line is a **tangent line** to the parabola, touching the parabola at that one point of contact.
- If there are 2 points of intersection, the line cuts through the parabola at these points.

Simultaneous equations can be used to find any points of intersection and the discriminant can be used to predict the number of solutions. To solve a pair of linear-quadratic simultaneous equations, usually the method of substitution from the linear into the quadratic equation is used.

WORKED EXAMPLE 15

a. Calculate the coordinates of the points of intersection of the parabola $y = x^2 - 3x - 4$ and the line $y - x = 1$.

b. Determine the number of points of intersection there will be between the graphs of $y = 2x - 5$ and $y = 2x^2 + 5x + 6$.

THINK

a. 1. Set up the simultaneous equations.

2. Substitute from the linear equation into the quadratic equation.

3. Solve the newly created quadratic equation for the x-coordinates of the points of intersection of the line and parabola.

4. Find the matching y-coordinates using the simpler linear equation.

5. State the coordinates of the points of intersection.

b. 1. Set up the simultaneous equations.

2. Create the quadratic equation from which any solutions are generated.

3. The discriminant of this quadratic equation determines the number of solutions.

WRITE

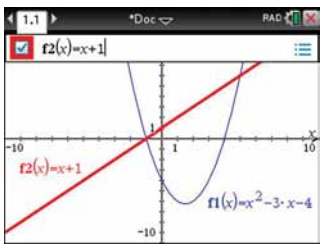
a. $y = x^2 - 3x - 4$ [1]
$y - x = 1$ [2]
From equation [2], $y = x + 1$.
Substitute this into equation [1].
$x + 1 = x^2 - 3x - 4$
$x^2 - 4x - 5 = 0$
$x^2 - 4x - 5 = 0$
$(x + 1)(x - 5) = 0$
$x = -1$ or $x = 5$
In equation [2]:
when $x = -1$, $y = 0$
when $x = 5$, $y = 6$
The points of intersection are $(-1, 0)$ and $(5, 6)$.

b. $y = 2x - 5$ [1]
$y = 2x^2 + 5x + 6$ [2]
Substitute equation [1] in equation [2].
$2x - 5 = 2x^2 + 5x + 6$
$2x^2 + 3x + 11 = 0$
$\Delta = b^2 - 4ac$, $a = 2$, $b = 3$, $c = 11$
$= (3)^2 - 4 \times (2) \times (11)$
$= -79$
$\therefore \Delta < 0$
There are no points of intersection between the two graphs.

TI | THINK

a. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = x^2 - 3x - 4$, then press ENTER. Rearrange the equation $y - x = 1$ as $y = x + 1$ and complete the entry line for function 2 as $f2(x) = x + 1$, then press ENTER.

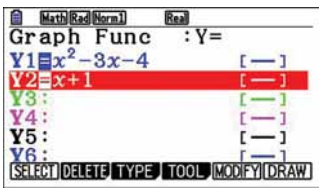
WRITE

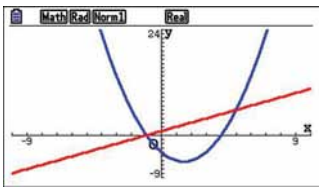


CASIO | THINK

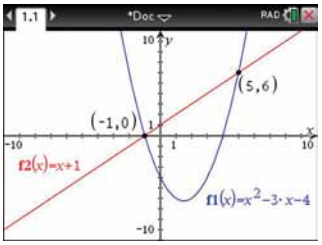
a. 1. On a Graph screen, complete the entry line for y1 as $y1 = x^2 - 3x - 4$, then press EXE. Rearrange the equation $y - x = 1$ as $y = x + 1$ and complete the entry line for y2 as $y2 = x + 1$, then press EXE. Select DRAW by pressing F6.

WRITE



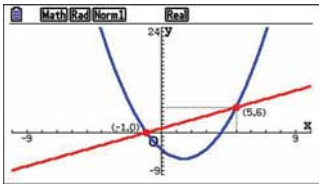


2. To find the points of intersection, press MENU then select 6: Analyze Graph 4: Intersection Move the cursor to the left of the point of intersection when prompted for the lower bound, then press ENTER. Move the cursor to the right of the point of intersection when prompted for the upper bound, then press ENTER. Repeat this step to find the other point of intersection.



3. The answer appears on the screen. The points of intersections are $(-1, 0)$ and $(5, 6)$.

2. To find the points of intersection, select G-Solv by pressing SHIFT then F5, then select INTSECT by pressing F5 With the cursor on the first point of intersection, press EXE. Use the left/right arrows to move to the other point of intersection, then press EXE.



3. The answer appears on the screen. The points of intersections are $(-1, 0)$ and $(5, 6)$.

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concept 8

The discriminant Summary screen and practice questions

Exercise 3.5 The discriminant

Technology free

1. **WE12** For each of the following quadratics, calculate the discriminant and hence state the number and type of factors and whether the 'completing the square' method would be needed to obtain the factors.
   a. $4x^2 + 5x + 10$
   b. $169x^2 - 78x + 9$
   c. $-3x^2 + 11x - 10$
   d. $\frac{1}{3}x^2 - \frac{8}{3}x + 2$
2. a. Calculate the discriminant for the equation $3x^2 - 4x + 1 = 0$.
   b. Use the result of a to determine the number and nature of the roots of the equations $3x^2 - 4x + 1 = 0$.
   In parts c to f, apply the discriminant to determine the number and type of solutions to the given equation.
   c. $-x^2 - 4x + 3 = 0$
   d. $2x^2 - 20x + 50 = 0$
   e. $x^2 + 4x + 7 = 0$
   f. $1 = x^2 + 5x$
3. For each of the following, calculate the discriminant and hence state the number and type of linear factors.
   a. $5x^2 + 9x - 2$
   b. $12x^2 - 3x + 1$
   c. $121x^2 + 110x + 25$
   d. $x^2 + 10x + 23$
4. **WE13** a. Apply the discriminant to determine the number and type of roots to the equation $0.2x^2 - 2.5x + 10 = 0$.
   b. Determine the values of k so the equation $kx^2 - (k + 3)x + k = 0$ will have one real solution.
5. Show that the equation $mx^2 + (m - 4)x = 4$ will always have real roots for any real value of m.

6. **a.** Factorise the difference of two cubes, $x^3 - 8$, and explain why there is only one linear factor over R.

b. Form linear factors from the following information and expand the product of these factors to obtain a quadratic expression.

i. The zeros of a quadratic are $x = \sqrt{2}$ and $x = -\sqrt{2}$.

ii. The zeros of a quadratic are $x = -4 + \sqrt{2}$ and $x = -4 - \sqrt{2}$.

7. Without actually solving the equations, determine the number and the nature of the roots of the following equations.

a. $-5x^2 - 8x + 9 = 0$ **b.** $4x^2 + 3x - 7 = 0$

c. $4x^2 + x + 2 = 0$ **d.** $28x - 4 - 49x^2 = 0$

e. $4x^2 + 25 = 0$ **f.** $3\sqrt{2}x^2 + 5x + \sqrt{2} = 0$

8. **a.** Determine the values of m so the equation $x^2 + (m + 2)x - m + 5 = 0$ has one root.

b. Determine the values of m so the equation $(m + 2)x^2 - 2mx + 4 = 0$ has one root.

c. Determine the values of p so the equation $3x^2 + 4x - 2(p - 1) = 0$ has no real roots.

d. Show that the equation $kx^2 - 4x - k = 0$ always has two solutions for $k \in R\backslash\{0\}$.

e. Show that for p, $q \in Q$, the equation $px^2 + (p + q)x + q = 0$ always has rational roots.

9. **WE14** Apply the discriminant to:

a. determine the number and type of x-intercepts of the graph defined by $y = 42x - 18x^2$

b. sketch the graph of $y = 42x - 18x^2$.

10. Use the discriminant to determine the number and type of intercepts each of the following graphs makes with the x-axis.

a. $y = 9x^2 + 17x - 12$ **b.** $y = -5x^2 + 20x - 21$

c. $y = -3x^2 - 30x - 75$ **d.** $y = 0.02x^2 + 0.5x + 2$

11. For what values of k does the graph of $y = 5x^2 + 10x - k$ have:

a. one x-intercept

b. two x-intercepts

c. no x-intercepts?

12. **a.** For what values of m is $mx^2 - 2x + 4$ positive definite?

b. **i.** Show that there is no real value of p for which $px^2 + 3x - 9$ is positive definite.

ii. If $p = 3$, find the equation of the axis of symmetry of the graph of $y = px^2 + 3x - 9$.

c. **i.** For what values of t does the turning point of $y = 2x^2 - 3tx + 12$ lie on the x-axis?

ii. For what values of t will the equation of the axis of symmetry of $y = 2x^2 - 3tx + 12$ be $x = 3t^2$?

13. **WE15** **a.** Calculate the coordinates of the points of intersection of the parabola $y = x^2 + 3x - 10$ and the line $y + x = 2$.

b. How many points of intersection will there be between the graphs of $y = 6x + 1$ and $y = -x^2 + 9x - 5$?

14. Show that the line $y = 4x$ is a tangent to the parabola $y = x^2 + 4$ and sketch the line and parabola on the same diagram, labelling the coordinates of the point of contact.

15. Solve each of the following pairs of simultaneous equations.

a. $y = 5x + 2$
$y = x^2 - 4$

b. $4x + y = 3$
$y = x^2 + 3x - 5$

c. $2y + x - 4 = 0$
$y = (x - 3)^2 + 4$

d. $\frac{x}{3} + \frac{y}{5} = 1$
$x^2 - y + 5 = 0$

16. Obtain the coordinates of the point(s) of intersection of:

a. the line $y = 2x + 5$ and the parabola $y = -5x^2 + 10x + 2$

b. the line $y = -5x - 13$ and the parabola $y = 2x^2 + 3x - 5$

c. the line $y = 10$ and the parabola $y = (5 - x)(6 + x)$

d. the line $19x - y = 46$ and the parabola $y = 3x^2 - 5x + 2$.

17. Use the discriminant to determine the number of intersections of:
    a. the line $y = 4 - 2x$ and the parabola $y = 3x^2 + 8$
    b. the line $y = 2x + 1$ and the parabola $y = -x^2 - x + 2$
    c. the line $y = 0$ and the parabola $y = -2x^2 + 3x - 2$.
18. Consider the line $2y - 3x = 6$ and the parabola $y = x^2$.
    a. Calculate the coordinates of their points of intersection, correct to 2 decimal places.
    b. Sketch the line and the parabola on the same diagram.

3.6 Modelling with quadratic functions

Quadratic equations may occur in problem solving and as mathematical models. In formulating a problem, variables should be defined and it is important to check whether mathematical solutions are feasible in the context of the problem.

Quadratic polynomials can be used to model a number of situations such as the motion of a falling object and the time of flight of a projectile. They can be used to model the shape of physical objects such as bridges, and they can also occur in economic models of cost and revenue.

WORKED EXAMPLE 16

The owner of a gift shop imported a certain number of paperweights for $900 and was pleased when all except 4 were sold for $10 more than what each paperweight had cost the owner to import. From the sale of the paperweights the gift shop owner received a total of $1400. How many paperweights were imported?



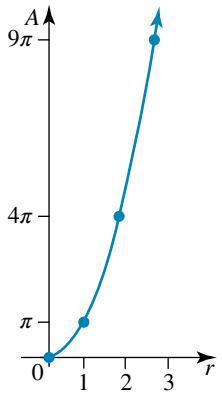
THINK	WRITE
1. Define the key variable.	Let x be the number of paperweights imported.
2. Find an expression for the cost of importing each paperweight.	The total cost of importing x paperweights is $900. Therefore the cost of each paperweight is $\left(\frac{900}{x}\right)$ dollars.
3. Find an expression for the selling price of each paperweight and identify how many are sold.	The number of paperweights sold is $(x - 4)$ and each is sold for $\left(\frac{900}{x} + 10\right)$ dollars.
4. Create the equation showing how the sales revenue of $1400 is formed.	$\left(\frac{900}{x} + 10\right) \times (x - 4) = 1400$
5. Now the equation has been formulated, solve it.	Expand: $900 - \frac{3600}{x} + 10x - 40 = 1400$ $-\frac{3600}{x} + 10x = 540$ $-3600 + 10x^2 = 540x$ $10x^2 - 540x - 3600 = 0$ $x^2 - 54x - 360 = 0$ $(x - 60)(x + 6) = 0$ $\therefore x = 60, x = -6$

6. Check the feasibility of the mathematical solutions.	Reject $x = -6$ since x must be a positive whole number.
7. Write the answer in context.	Therefore 60 paperweights were imported by the gift shop owner.

3.6.1 Quadratically related variables

The formula for the area, A, of a circle in terms of its radius, r, is $A = \pi r^2$. This is of the form $A = kr^2$ as π is a constant. The area varies directly as the square of its radius with the constant of proportionality $k = \pi$. This is a quadratic relationship between A and r.

r	0	1	2	3
A	0	π	4π	9π



WORKED EXAMPLE 17

The volume of a cone of fixed height is directly proportional to the square of the radius of its base. When the radius is 3 cm, the volume is 30π cm^3. Calculate the radius when the volume is 480π cm^3.

THINK	WRITE
1. Write the variation equation, defining the symbols used.	$V = kr^2$ where V is the volume of a cone of fixed height and radius r. k is the constant of proportionality.
2. Use the given information to find k.	$r = 3,\ V = 30\pi \Rightarrow 30\pi = 9k$ $\therefore \quad k = \dfrac{30\pi}{9}$ $= \dfrac{10\pi}{3}$
3. Write the rule connecting V and r.	$V = \dfrac{10\pi}{3}r^2$
4. Substitute $V = 480\pi$ and find r.	$480\pi = \dfrac{10\pi}{3}r^2$ $10\pi r^2 = 480\pi \times 3$ $r^2 = \dfrac{480\pi \times 3}{10\pi}$ $r^2 = 144$

	$r = \pm\sqrt{144}$
	$r = \pm 12$
5. Check the feasibility of the mathematical solutions.	Reject $r = -12$ since r must be positive. $\therefore r = 12$
6. Write the answer in context.	The radius of the cone is 12 cm.

3.6.2 Maximum and minimum values

The greatest or least value of the quadratic model is often of interest.

A quadratic reaches its local maximum or local minimum value at its turning point. The y-coordinate of the turning point represents the maximum or minimum value, depending on the nature of the turning point (and whether the graph is restricted).

If $a < 0$, $a(x-h)^2 + k \leq k$ so the maximum value of the quadratic is k.

If $a > 0$, $a(x-h)^2 + k \geq k$ so the minimum value of the quadratic is k.

WORKED EXAMPLE 18

A stone is thrown vertically into the air so that its height h metres above the ground after t seconds is given by $h = 1.5 + 5t - 0.5t^2$.

a. What is the greatest height the stone reaches?

b. After how many seconds does the stone reach its greatest height?

c. When is the stone 6 metres above the ground? Why are there two times?

d. Sketch the graph and give the time to return to the ground to 1 decimal place.

THINK	WRITE
a. 1. The turning point is required. Calculate the coordinates of the turning point and state its type.	**a.** $h = 1.5 + 5t - 0.5t^2$ $a = -0.5,\ b = 5,\ c = 1.5$ Turning point: Axis of symmetry has equation $t = -\frac{b}{2a}$. $t = -\frac{5}{2 \times (-0.5)}$ $= 5$ When $t = 5$, $h = 1.5 + 5(5) - 0.5(5)^2$ $= 14$ Turning point is $(5, 14)$. This is a maximum turning point as $a < 0$.
2. State the answer. *Note:* The turning point is in the form (t, h) as t is the independent variable and h the dependent variable. The greatest height is the h-coordinate.	Therefore the greatest height the stone reaches is 14 metres above the ground.

b. The required time is the t-coordinate of the turning point.

b. The stone reaches its greatest height after 5 seconds.

c. 1. Substitute the given height and solve for t.

c. $h = 1.5 + 5t - 0.5t^2$

When $h = 6$, $6 = 1.5 + 5t - 0.5t^2$

$0.5t^2 - 5t + 4.5 = 0$

$t^2 - 10t + 9 = 0$

$(t - 1)(t - 9) = 0$

$\therefore t = 1$ or $t = 9$

2. Interpret the answer.

Therefore the first time the stone is 6 metres above the ground is 1 second after it has been thrown into the air and is rising upwards. It is again 6 metres above the ground after 9 seconds when it is falling down.

d. 1. Calculate the time the stone returns to the ground.

d. Returns to ground when $h = 0$

$0 = 1.5 + 5t - 0.5t^2$

$t^2 - 10t - 3 = 0$

$t^2 - 10t = 3$

$t^2 - 10t + 25 = 3 + 25$

$(t - 5)^2 = 28$

$t = 5 \pm \sqrt{28}$

$t \simeq 10.3$ (reject negative value)

The stone reaches the ground after 10.3 seconds.

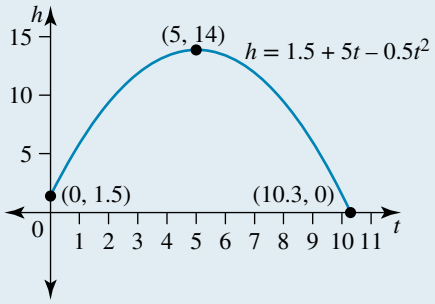
2. Sketch the graph, from its initial height to when the stone hits the ground. Label the axes appropriately.

When $t = 0$, $h = 1.5$ so stone is thrown from a height of 1.5 metres.

Initial point: (0, 1.5)

Maximum turning point: (5, 14)

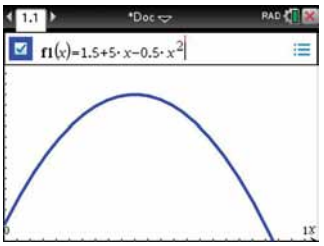
Endpoint: (10.3, 0)



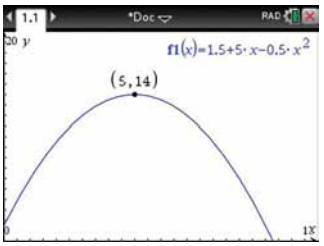
TI | THINK

a. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 1.5 + 5x - 0.5x^2$, then press ENTER.

WRITE



2. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



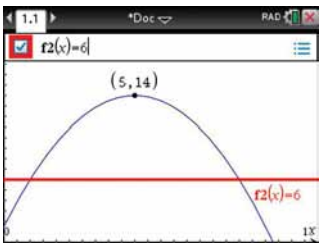
3. The answer appears on the screen.

The greatest height the stone reaches is 14 metres above the ground.

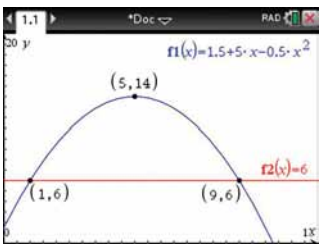
b. 1. See **a.2.**

The stone reaches its greatest height after 5 seconds.

c. 1. Press TAB to bring up the function entry line and complete the entry line for function 2 as $f2(x) = 6$, then press ENTER.



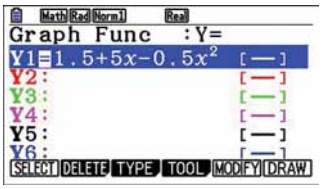
2. To find the points of intersection,
press MENU then select
6: Analyze Graph
4: Intersection
Move the cursor to the left of the point of intersection when prompted for the lower bound, then press ENTER. Move the cursor to the right of the point of intersection when prompted for the upper bound, then press ENTER. Repeat this step to find the other point of intersection. Press TAB and unselect the graph of function 2.

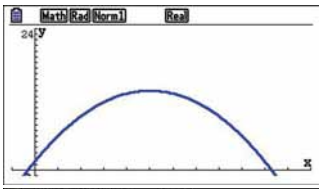


CASIO | THINK

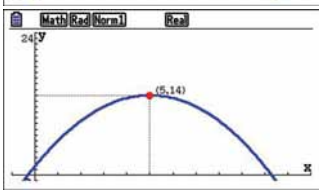
a. 1. On a Graph screen, complete the entry line for $y1$ as $y1 = 1.5 + 5x - 0.5x^2$, then press EXE. Select DRAW by pressing F6.

WRITE





2. To find the maximum, select G-Solv by pressing SHIFT then F5, then select MAX by pressing F2. Press EXE.



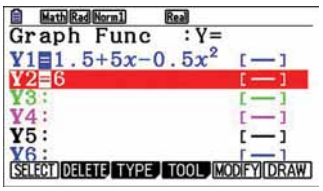
3. The answer appears on the screen.

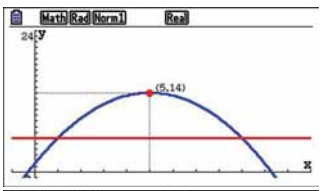
The greatest height the stone reaches is 14 metres above the ground.

b. 1. See **a.2.**

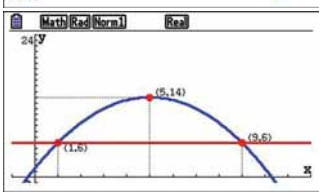
The stone reaches its greatest height after 5 seconds.

c. 1. Return to the function entry screen by pressing SHIFT then F6, then complete the entry line for y2 as $y2 = 6$ and press EXE. Select DRAW by pressing F6.





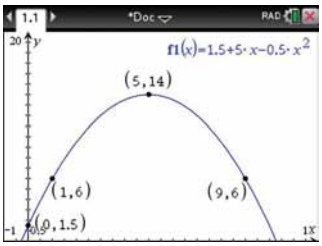
2. To find the points of intersection, select G-Solv by pressing SHIFT then F5, then select INTSECT by pressing F5.
With the cursor on the first point of intersection, press EXE. Use the left/right arrows to move to the other point of intersection, then press EXE.



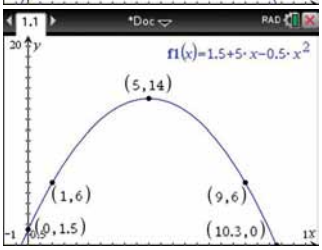
3. The answer appears on the screen.

The first time the stone is 6 metres above the ground is 1 second after it has been thrown into the air and is rising upwards. It is again 6 metres above the ground after 9 seconds when it is falling down.

d. 1. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



2. To find the x-intercept, press MENU then select
6: Analyze Graph
1: Zero
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.
Note: Only the positive x-intercept needs to be located due to the implied domain $0 \leq x \leq 10.3$.



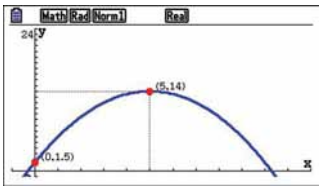
3. The answer appears on the screen.

The stone reaches the ground after 10.3 seconds.

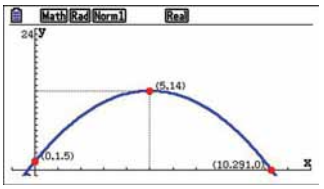
3. The answer appears on the screen.

The first time the stone is 6 metres above the ground is 1 second after it has been thrown into the air and is rising upwards. It is again 6 metres above the ground after 9 seconds when it is falling down.

d. 1. To find the y-intercept, select G-Solv by pressing SHIFT then F5, then select Y-ICEPT by pressing F4. Press EXE.



2. To find the x-intercept, select G-Solv by pressing SHIFT then F5, then select ROOT by pressing F1.
Use the left/right arrows to move to the positive x-intercept, then press EXE.
Note: Only the positive x-intercept needs to be located due to the implied domain $0 \leq x \leq 10.3$.



3. The answer appears on the screen.

The stone reaches the ground after 10.3 seconds.

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concept 10

Modelling with quadratic functions Summary screen and practice questions

Exercise 3.6 Modelling with quadratic functions

Technology active

1. WE16 The owner of a fish shop bought x kilograms of salmon for $400 from the wholesale market. At the end of the day all except for 2 kg of the fish were sold at a price per kg which was $10 more than what the owner paid at the market. From the sale of the fish, a total of $540 was made. How many kilograms of salmon did the fish-shop owner buy at the market?



2. WE17 The surface area of a sphere is directly proportional to the square of its radius. When the radius is 5 cm, the area is 100π cm^2. Calculate the radius when the area is 360π cm^2.

3. The cost of hiring a chainsaw is \$10 plus an amount that is proportional to the square of the number of hours for which the chainsaw is hired. If it costs \$32.50 to hire the chainsaw for 3 hours, find, to the nearest half hour, the length of time for which the chainsaw was hired if the cost of hire was \$60.



4. a. The area of an equilateral triangle varies directly as the square of its side length. A triangle of side length $2\sqrt{3}$ cm has an area of $3\sqrt{3}$ cm^2. Calculate the side length if the area is $12\sqrt{3}$ cm^2.

b. The distance a particle falls from rest is in direct proportion to the square of the time of fall. What is the effect on the distance fallen if the time of fall is doubled?

c. The number of calories of heat produced in a wire in a given time varies as the square of the voltage. If the voltage is reduced by 20%, what is the effect on the number of calories of heat produced?

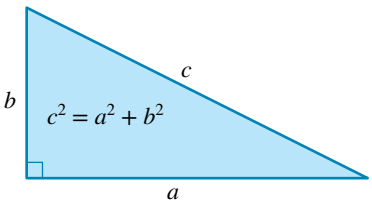
5. The cost of producing x hundred litres of olive oil is $20 + 5x$ dollars. If the revenue from the sale of x hundred litres of the oil is $1.5x^2$ dollars, calculate to the nearest litre, the number of litres that must be sold to make a profit of \$800.



6. The product of two consecutive even natural numbers is 440. What are the numbers?

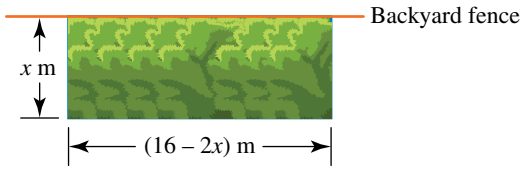
7. The sum of the squares of two consecutive natural numbers plus the square of their sums is 662. What are the numbers?

8. The hypotenuse of a right angled-triangle is $(3x + 3)$ cm and the other two sides are $3x$ cm and $(x - 3)$ cm. Determine the value of x and calculate the perimeter of the triangle.



9. A photograph, 17 cm by 13 cm, is placed in a rectangular frame. If the border around the photograph i s of uniform width and has an area of 260 cm^2, measured to the nearest cm^2, what are the dimensions of the frame measured to the nearest cm?

10. A gardener has 16 metres of edging to place around three sides of a rectangular garden bed, the fourth side of which is bounded by the backyard fence.



a. If the width of the garden bed is x metres, explain why its length is $(16 - 2x)$ metres.

b. If the area of the rectangular garden is k square metres, show that $2x^2 - 16x + k = 0$.

c. Determine the value of the discriminant and hence find the values of k for which this equation will have:

i. no solutions **ii.** one solution **iii.** two solutions.

d. What is the largest area the garden bed can be and what are its dimensions in this case?

e. The gardener decides the area of the garden bed is to be 15 square metres. Given that the gardener would also prefer to use as much of the backyard fence as possible as a boundary to the garden bed, calculate the dimensions of the rectangle in this case, correct to 1 decimal place.

Use the following information in questions **11** and **12**: The formula for the total surface area A of a cone of base radius r and slant height l is $A = \pi r^2 + \pi rl$.

11. Determine, correct to 3 decimal places, the radius of the base of a cone with slant height 5 metres and total surface area 20 m^2.

12. For any cone which has a surface area of 20 m^2, determine the value of r in terms of l and use this expression to check the answer to question **11**.

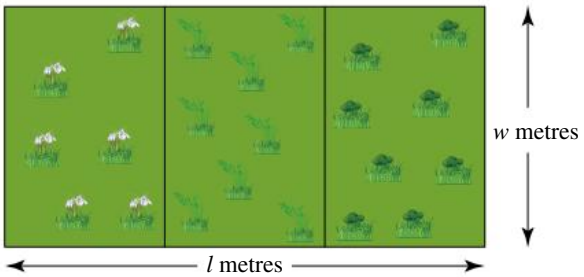
13. A gardener has 30 metres of edging to enclose a rectangular area using the back fence as one edge.

- **a.** Show the area function is $A = 30x - 2x^2$ where A square metres is the area of the garden bed of width x metres.
- **b.** Calculate the dimensions of the garden bed for maximum area.
- **c.** What is the maximum area that can be enclosed?

14. **WE18** A missile is fired vertically into the air from the top of a cliff so that its height h metres above the ground after t seconds is given by $h = 100 + 38t - \frac{19}{12}t^2$.

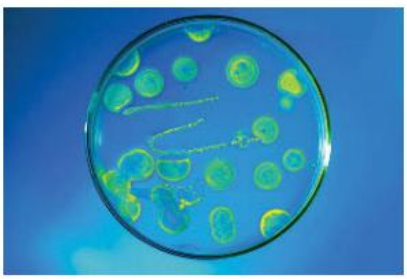
- **a.** What is the greatest height the missile reaches?
- **b.** After how many seconds does the missile reach its greatest height?
- **c.** Sketch the graph and give the time to return to the ground to 1 decimal place.

15. Georgie has a large rectangular garden area with dimensions l metres by w metres which she wishes to divide into three sections so she can grow different vegetables. She plans to put a watering system along the perimeter of each section. This will require a total of 120 metres of hosing.



- **a.** Show the total area of the three sections, A m^2 is given by $A = 60w - 2w^2$ and hence calculate the dimensions when the total area is a maximum.
- **b.** Using the maximum total area, Georgie decides she wants the areas of the three sections to be in the ratio 1 : 2 : 3. What is the length of hosing for the watering system that is required for each section?

16. The number of bacteria in a slowly growing culture at time t hours after 8.00 am is given by $N = 100 + 46t + 2t^2$.



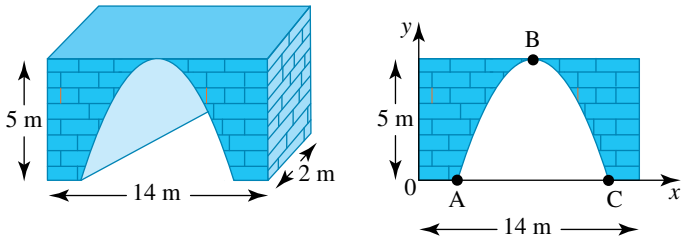
- **a.** How long does it take for the initial number of bacteria to double?
- **b.** How many bacteria are present at 1.00 pm?
- **c.** At 1.00 pm a virus is introduced that initially starts to destroy the bacteria so that t hours after 1.00 pm the number of bacteria is given by $N = 380 - 180t + 30t^2$. What is the minimum number the population of bacteria reaches and at what time does this occur?

17. The cost C dollars of manufacturing n dining tables is the sum of three parts. One part represents the fixed overhead costs c, another represents the cost of raw materials and is directly proportional to n and the third part represents the labour costs which are directly proportional to the square of n.



- **a.** If 5 tables cost \$195 to manufacture, 8 tables cost \$420 to manufacture and 10 tables cost \$620 to manufacture, determine the relationship between C and n.
- **b.** Determine the maximum number of dining tables that can be manufactured if costs are not to exceed \$1000.

18. The arch of a bridge over a small creek is parabolic in shape with its feet evenly spaced from the ends of the bridge. Relative to the coordinate axes, the points A, B and C lie on the parabola.



- **a.** If AC = 8 metres, write down the coordinates of the points A, B and C.
- **b.** Determine the equation of the parabola containing points A, B and C.
- **c.** Following heavy rainfall the creek floods and overflows its bank, causing the water level to reach 1.5 metres above AC. Calculate is the width of the water level to 1 decimal place?

19. **a.** If the sum of two numbers is 16, determine the numbers for which:

- **i.** their product is greatest
- **ii.** the sum of their squares is least.

- **b.** If the sum of two non-zero numbers is k:
  - **i.** express their greatest product in terms of k
  - **ii.** are there any values of k for which the sum of the squares of the numbers and their product are equal? If so, state the values; if not, explain why.

20. In a game of volleyball a player serves a 'sky-ball' serve from the back of a playing court of length 18 metres. The path of the ball can be considered to be part of the parabola $y = 1.2 + 2.2x - 0.2x^2$ where x (metres) is the horizontal distance travelled by the ball from where it was hit and y (metres) is the vertical height the ball reaches.

- **a.** Use completing the square technique to express the equation in the form $y = a(x - b)^2 + c$.
- **b.** How high does the volleyball reach?
- **c.** The net is 2.43 metres high and is placed in the centre of the playing court. Show that the ball clears the net and calculate by how much.

3.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC The solutions of the equation $(x - 2)(x + 1) = 4$ are:

A. $x = 2, x = -1$ **B.** $x = 6, x = -1$ **C.** $x = -6, x = 1$ **D.** $x = 3, x = -2$

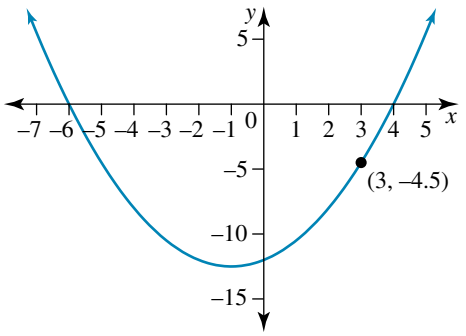
2. MC The parabola with equation $y = x^2$ is translated so that its image has its vertex at $(-4, 3)$. The equation of the image is:

A. $y = (x - 4)^2 + 3$ **B.** $y = (x - 3)^2 + 4$ **C.** $y = (x + 4)^2 + 3$
D. $y = (x + 3)^2 - 4$

3. MC The maximum value of $4 - 2x - x^2$ is:

A. 5 **B.** 4
C. 3 **D.** 1

4. MC The equation of the parabola shown is:



A. $y = x^2 + 2x - 24$
B. $y = 0.5x^2 + x - 12$
C. $y = x^2 - 2x - 24$
D. $y = 0.5x^2 - x - 12$

5. MC A quadratic graph touches the x-axis at $x = -6$ and cuts the y-axis at $y = -10$. Its equation is:

A. $y = (x + 6)(x + 10)$ **B.** $y = \frac{5}{18}x^2 - 10$

C. $y = \frac{5}{18}(x + 6)^2$ **D.** $y = -\frac{5}{18}(x + 6)^2$

6. Solve for x.

a. $(x^2 + 4)^2 - 7(x^2 + 4) - 8 = 0$ **b.** $2x^2 = 3x(x - 2) + 1$

c. $x = \frac{12}{x - 2} - 2$ **d.** $3 + \sqrt{x} = 2x$

7. Sketch the graphs of the following, showing all key points.

a. $y = 2(x - 3)(x + 1)$ **b.** $y = 1 - (x + 2)^2$ **c.** $y = x^2 + x + 9$

8. MC The x-coordinates of the points of intersection of the parabola $y = 3x^2 - 10x + 2$ with the line $2x - y = 1$ can be determined from the equation:

A. $3x^2 - 10x + 2 = 0$ **B.** $3x^2 - 12x + 3 = 0$

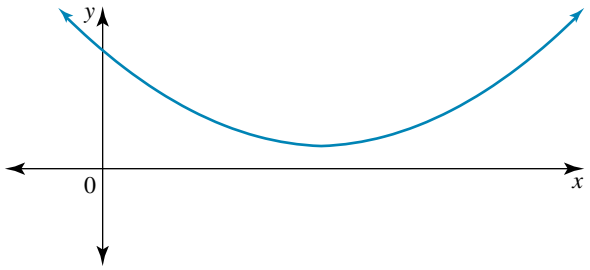
C. $x^2 - 6x + 1 = 0$ **D.** $3x^2 - 8x + 1 = 0$

9. MC If $x^2 + 4x - 6$ is expressed in the form $(x + b)^2 + c$ then the values of b and c would be:

A. $b = 2, c = -10$ **B.** $b = -2, c = -10$

C. $b = 4, c = -2$ **D.** $b = -4, c = -2$

10. MC For the graph of the parabola $y = ax^2 + bx + c$ shown, with $\Delta = b^2 - 4ac$, which statement is correct?



A. $a > 0$ and $\Delta > 0$

B. $a > 0$ and $\Delta < 0$

C. $a < 0$ and $\Delta < 0$

D. $a < 0$ and $\Delta > 0$

11. MC The solution set of $\{x : x^2 < 4x\}$ is:

A. $\{x : x < 4\}$

B. $\{x : -4 < x < 0\}$

C. $\{x : 0 < x < 4\}$

D. $\{x : x < 0\} \cup \{x : x > 4\}$

12. Solve the following quadratic inequations.

a. $2x^2 - 5x - 3 > 0$ **b.** $10 - x^2 \geq 0$ **c.** $20x^2 + 20x + 5 \geq 0$

Complex familiar

13. MC The values of x for which $-5x^2 + 8x + 3 = 0$ are closest to:

A. $-0.6, -1$ **B.** $0.6, -1$ **C.** $-0.3, 1.9$ **D.** $0.3, -1.9$

14. Factorise over R.

a. $-x^2 + 20x + 24$ **b.** $4x^2 - 2x - 9$

15. For what values of k does the equation $kx^2 - 4x(k + 2) + 36 = 0$ have no real roots?

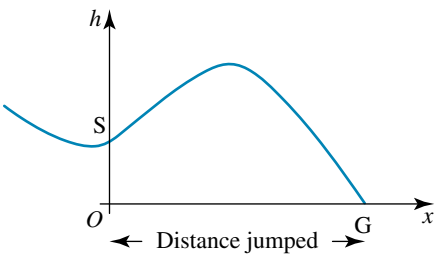
16. a. Use an algebraic method to find the coordinates of the points of intersection of the parabola $y = x^2 + 2x$ and the line $y = x + 2$

b. Sketch $y = x^2 + 2x$ and $y = x + 2$ on the same set of axes.

Complex unfamiliar

17. At a winter skiing championship, two competitors, one from Japan and the other from Canada, compete for the gold medal in one of the jump events.

Each competitor leaves the ski run at point S and travels through the air, landing back on the ground at some point G. The winner will be the competitor who covers the greater horizontal distance OG.



The Canadian skier jumps first and her height (h metres) above the ground is described by $h = -\frac{1}{35}(x^2 - 60x - 700)$, where x metres is the horizontal distance travelled.

a. Show that the point S is 20 metres above O.

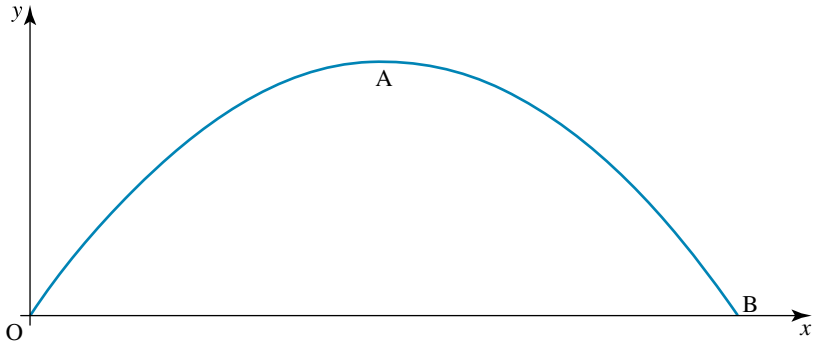
b. How far does the Canadian skier jump?

The Japanese skier jumps next and she reaches a maximum height of 35 metres above the ground after a horizontal distance of 30 metres has been covered.

c. Assuming the path is a parabola, form the equation for h in terms of x which describes this competitor's path.

d. Decide which competitor receives the gold medal. Your decision should be supported with appropriate mathematical reasoning.

18. The diagram shows the arch of a bridge where the shape of the curve, OAB, is a parabola. OB is the horizontal road level. Taking O as the origin, the equation of the curve OAB is $y = 2.5x - 0.3125x^2$. All measurements are in metres.

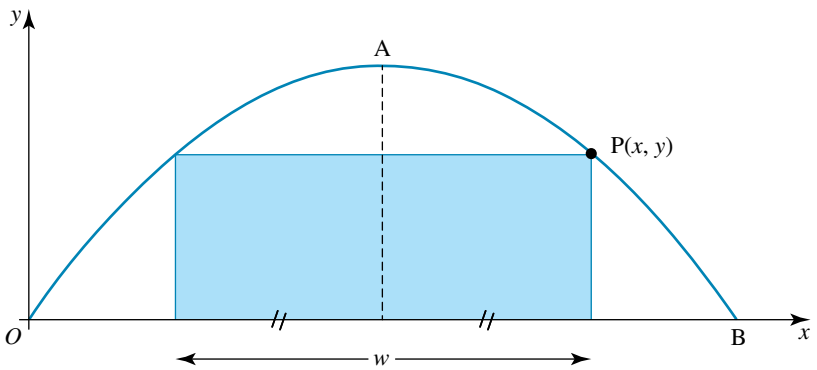


a. Calculate the length of OB, the span of the bridge.

b. How far above the road is the point A, the highest point on the curve?

c. A car towing a caravan needs to drive under the bridge. The caravan is 5 metres wide and has a height of 2 metres. Only one single lane of traffic can pass under the bridge. Explain clearly, using mathematical analysis, whether the caravan can be towed under this bridge.

To avoid accidents, the bridge engineers decide to place height and width limits. Only vehicles whose height and width fit into the greatest allowable dimensions are permitted to travel under the bridge.



P (x, y) lies on the curve and is a corner of the rectangle formed by the height and width restrictions.

d. Express the width w of the rectangle in terms of x.

e. If the height restriction is 3.2 m, calculate the x-coordinate of P.

f. Would the caravan be permitted to be towed under the bridge under these restrictions?

19. ABCD is a rectangle of length one more unit than its width. Point F lies on AB and divides AB in the ratio $x : 1$ so that AF is x units in length and FB is 1 unit in length. Point G lies on DC and divides DC in the same ratio, $x : 1$.

a. Draw a diagram showing this information.

b. What is the width of rectangle ABCD?

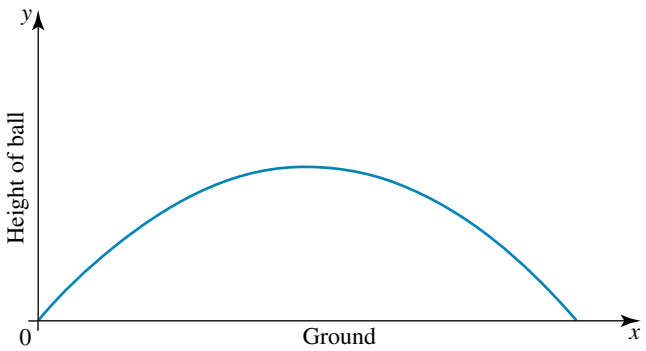
c. If the area of the square AFGD is one more square unit than the area of the rectangle FBCG, show that $x^2 - x - 1 = 0$.

d. Hence find the value of x in simplest surd form.

e. The value found for x is called the Golden ratio and usually given the symbol ϕ. Calculate $\frac{1}{\phi}$ and give its relationship to the other root of the equation $x^2 - x - 1 = 0$.

f. Show $\frac{1}{\phi} = \phi - 1$ and explain this relationship using the equation $x^2 - x - 1 = 0$.

20. Ignoring air resistance, the path of a cricket ball hit by a batsman can be considered to travel on a parabolic path which starts at the point $(0, 0)$ where the ball is struck by the batsman.
Let x metres measure the horizontal distance of the ball from the batsman in the direction the ball travels, and y metres measure the vertical height above the ground that the ball reaches.





A batsman hits a cricket ball towards a fielder who is 65 metres away. The ball is struck with a horizontal speed of 28 m/s, which is assumed to remain constant throughout the flight of the ball. On its way, the ball reaches a maximum height of 4.9 metres after 1 second.

a. Calculate the coordinates of the turning point of the quadratic path of the ball.

b. Form the equation of the path of the ball.
The fielder starts running forward at the instant the ball is hit and catches it at a height of 1.3 metres above the ground.

c. Calculate the time it takes the fielder to reach the ball.

d. Hence obtain the uniform speed in m/s with which the fielder runs in order to catch the ball.

studyon

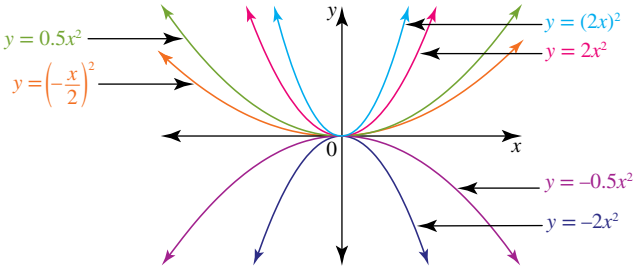
Units 1 & 2 Sit chapter test

Answers

Chapter 3 Quadratic relationships

Exercise 3.2 Graphs of quadratic functions

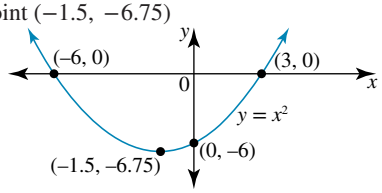
1.



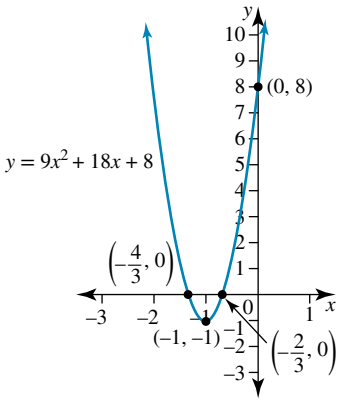
2. ***A*** **i.** $y = x^2 - 2$
B **ii.** $y = -2x^2$
C **iii.** $y = -(x + 2)^2$

3. a. $(0, 8)$ **b.** $(0, -8)$
c. $(0, 1)$ **d.** $(0, -7)$
e. $(8, 0)$ **f.** $(-8, 0)$
g. $(4, 0)$ **h.** $(-12, 0)$

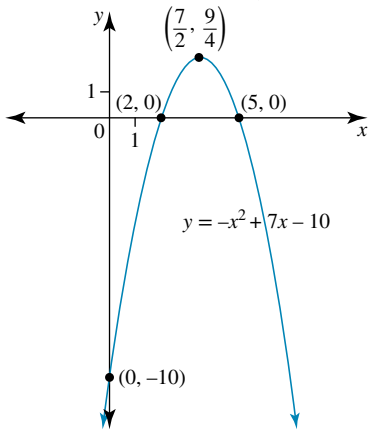
4. Axis intercepts $(-6, 0)$, $(3, 0)$, $(0, -6)$; minimum turning point $(-1.5, -6.75)$



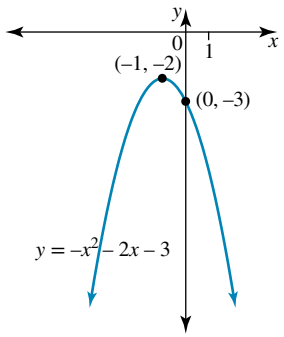
5. a.



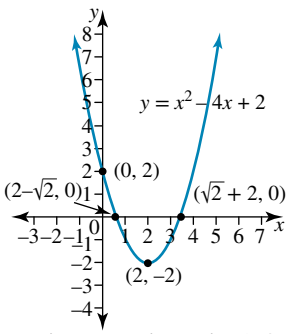
b.



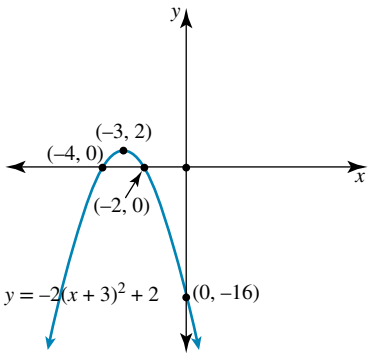
c.



d.



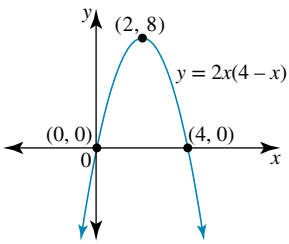
6. a. Maximum turning point $(-3, 2)$; axis intercepts $(0, -16)$, $(-4, 0)$, $(-2, 0)$



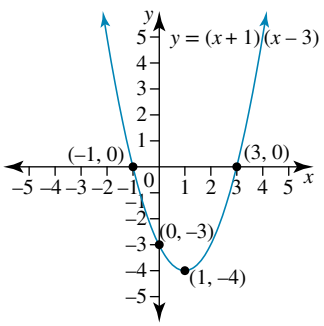
7. a. Maximum turning point at $(0, 4)$
b. Minimum turning point at $\left(\frac{4}{3}, 0\right)$

8. D

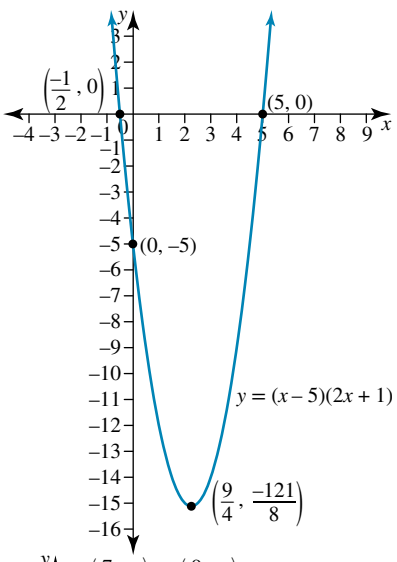
9. Axis intercepts $(0, 0)$, $(4, 0)$; maximum turning point $(2, 8)$



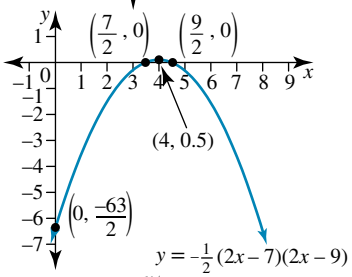
10. a.



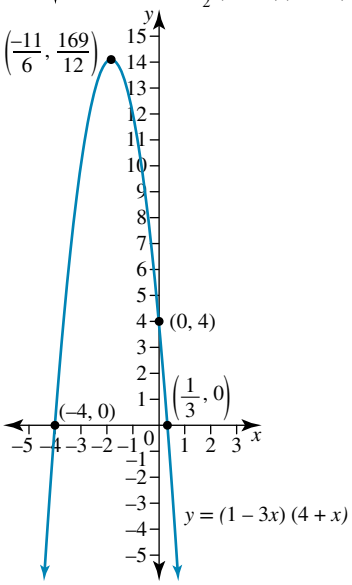
b.



c.



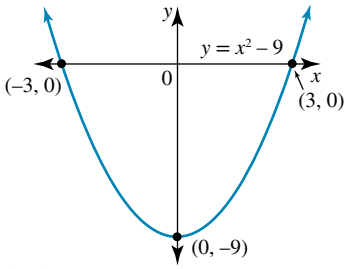
d.



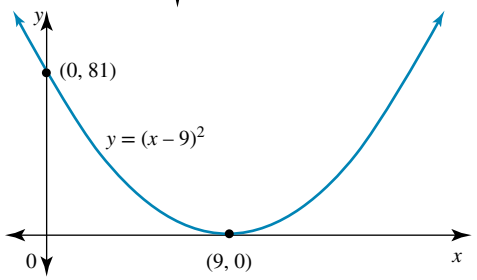
11.

	Turning point	*y*-intercept	*x*-intercepts
a	(0, –9)	(0, –9)	(±3, 0)
b	(9, 0)	(0, 81)	(9, 0)
c	(0, 6)	(0, 6)	$\left(\pm\sqrt{2}, 0\right)$
d	(–1, 0)	(0, –3)	(–1, 0)
e	$\left(\frac{1}{2}, 0\right)$	$\left(0, \frac{1}{4}\right)$	$\left(\frac{1}{2}, 0\right)$
f	$\left(-\frac{1}{2}, 0\right)$	$\left(0, -\frac{1}{4}\right)$	$\left(-\frac{1}{2}, 0\right)$

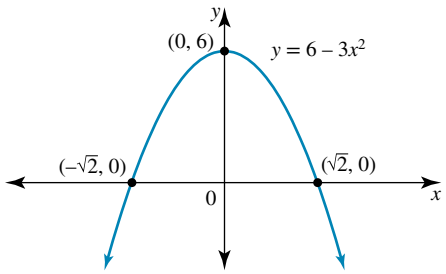
a.



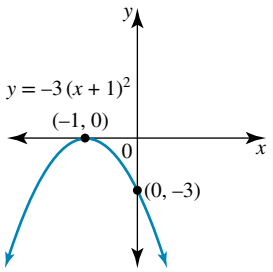
b.



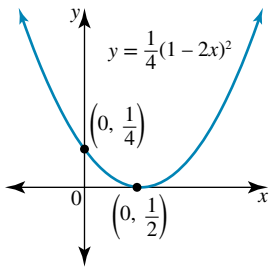
c.



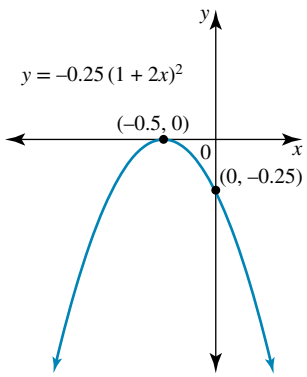
d.



e.



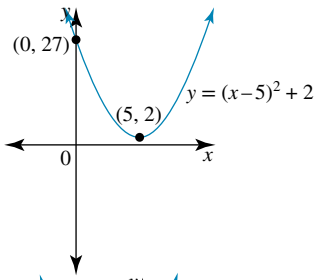
f.



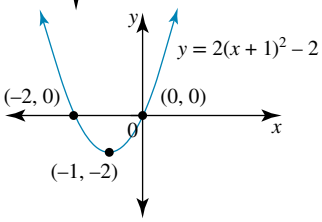
12.

	Turning point	y-intercept	x-intercepts
a	$(5, 2)$	$(0, 27)$	none
b	$(-1, -2)$	$(0, 0)$	$(0, 0), (-2, 0)$
c	$(3, -6)$	$(0, -24)$	none
d	$(4, 1)$	$(0, -15)$	$(3, 0), (5, 0)$
e	$(-4, -2)$	$(0, 6)$	$(-6, 0), (-2, 0)$
f	$\left(\frac{1}{2}, \frac{1}{9}\right)$	$\left(0, \frac{2}{27}\right)$	$\left(\frac{1 \pm \sqrt{3}}{2}, 0\right)$

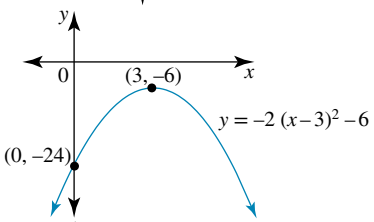
a.



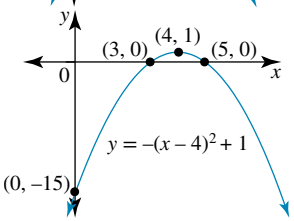
b.



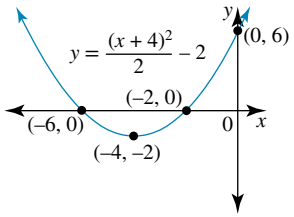
c.



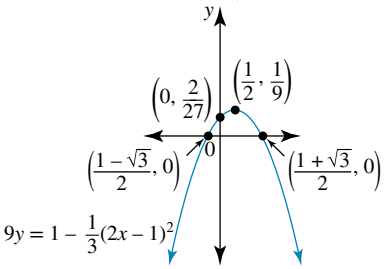
d.



e.



f.



13. a. $c = 4, y = x^2 + 4.$

b. $a = -\frac{1}{18}, y = -\frac{1}{18}x^2$

c. $a = -3, y = -3(x - 2)^2$

14. a. $x - 3, x - 8, y = (x - 3)(x - 8)$

b. $y = (x + 11)(x - 2)$

15. a. $y = (x + 2)^2 + 1$ **b.** $y = 2x(x - 2)$

16. $y = -2x^2 + x - 4$

17. a. $y = -2x^2 + 6$ **b.** $y = 0.2(x + 6)(x + 1)$

18. a. Equation A **b.** Equation A

c. Equation A

Exercise 3.3 Solving quadratic equations with rational roots

1. a. $x = -3, \frac{7}{10}$ **b.** $(x + 5)x = x^2 + 5x$

2. $x = -\frac{6}{10}, 1$

3. a. $x = \frac{4}{3}, -\frac{1}{2}$ **b.** $x = 4, 3$

c. $x = -\frac{3}{2}, x = -\frac{7}{4}$ **d.** $x = 0, \frac{1}{5}$

e. $x = \frac{2}{3}, -4$ **f.** $x = 0, 10$

4. a. $x = -5, 1$ **b.** $x = -4, 6$

c. No real solutions **d.** $x = -10, -1$

e. $x = 7$ **f.** $x = 0, 8$

5. a. $0, 5$ **b.** $\frac{1}{7}, 3$

c. -8 **d.** $-6, -4$

6. a. $-\frac{1}{2}, -\frac{1}{3}$ **b.** $-\frac{2}{3}, \frac{5}{4}$

c. 7 **d.** $-\frac{5}{6}, 1$

7. a. ± 11 **b.** $\pm\frac{4}{3}$ **c.** $4, 6$

d. $-1, 6$ **e.** $-\frac{1}{3}, 1$ **f.** ± 3

8. $x = \pm\frac{1}{3}$

9. a. $x = \frac{19}{6}, \frac{7}{3}$ **b.** $x = -6, -\frac{13}{5}$

c. $x = -2, -4$ **d.** $x = \frac{1}{2}, 3$

10. a. $-\frac{14}{3}, -1$ **b.** $-\frac{7}{2}, \frac{1}{4}$ **c.** $\pm 2, \pm 5$

d. ± 4 **e.** $\pm\frac{1}{3}$ **f.** $-3, -2, -1$

11. a. $-1, 8$ **b.** $\frac{11}{6}, \frac{9}{2}$

c. $-8, -2$ **d.** $0, \frac{1}{2}$

12. a. $\frac{1}{3}$ **b.** $\pm\frac{25}{2}$

c. $-\frac{5}{7}, \frac{2}{5}$ **d.** $0, \frac{17}{19}$

13. a. ± 3 **b.** $\pm\frac{4}{3}, \pm 2$

c. $-1, 2$ **d.** $-\frac{3}{2}, -\frac{6}{5}$

14. $x = 1$

15. $x = \frac{r - q}{p}, \ x = -\frac{(r + q)}{p}$

16. a. $x = -3a,\ x = 2b$

b. $x = \dfrac{3a}{2}, x = 5a$

c. $x = b - 1, x = b + 1, x = b - 2, x = b + 2$

d. $x = a - b, x = a + 3b$

e. $x = b - a, x = 2b - a$

f. $x = 1$

17. a. $(x - 1)(x - 7) = 0$ **b.** $(x + 5)(x - 4) = 0$

c. $x(x - 10) = 0$ **d.** $(x - 2)^2 = 0$

18. a. $b = 13,\ c = -12$

b. Roots are $-\dfrac{q}{p}$, -1, solutions $x = \dfrac{p - q}{p}, 0$

19. a. $\dfrac{2}{11}, \dfrac{11}{2}$

b. Side of square is 30 units.

Exercise 3.4 Factorising and solving quadratics over *R*

1. a. $x^2 + 10x + 25 = (x + 5)^2$

b. $x^2 - 7x + \dfrac{49}{4} = \left(x - \dfrac{7}{2}\right)^2$

c. $x^2 + x + \dfrac{1}{4} = \left(x + \dfrac{1}{2}\right)^2$

d. $x^2 - \dfrac{4}{5}x + \dfrac{4}{25} = \left(x - \dfrac{2}{5}\right)^2$

2. a. $\left(x - 5 - 4\sqrt{2}\right)\left(x - 5 + 4\sqrt{2}\right)$

b. $3\left(x + \dfrac{7 - \sqrt{13}}{6}\right)\left(x + \dfrac{7 + \sqrt{13}}{6}\right)$

c. $\left(\sqrt{5}x - 3\right)\left(\sqrt{5}x + 3\right)$

3. $(-3x + 5)(x - 1)$

4. a. $(x - 3 - \sqrt{2})(x - 3 + \sqrt{2})$

b. $(x + 2 - \sqrt{7})(x + 2 + \sqrt{7})$

c. No linear factors over R

d. $2\left(x + \dfrac{5}{4} - \dfrac{\sqrt{41}}{4}\right)\left(x + \dfrac{5}{4} + \dfrac{\sqrt{41}}{4}\right)$

e. $-(x - 4 - 2\sqrt{2})(x - 4 + 2\sqrt{2})$

f. $3\left(x + \dfrac{2}{3} - \dfrac{\sqrt{22}}{3}\right)\left(x + \dfrac{2}{3} + \dfrac{\sqrt{22}}{3}\right)$

5. $x = \dfrac{-1 \pm \sqrt{7}}{2}$

6. a. $x^2 + 2x$

$= x^2 + 2x + 1 - 1$

$= (x + 1)^2 - 1$

b. $x^2 + 7x$

$= x^2 + 7x + \dfrac{49}{4} - \dfrac{49}{4}$

$= \left(x + \dfrac{7}{2}\right)^2 - \dfrac{49}{4}$

c. $x^2 - 5x$

$= x^2 - 5x + \dfrac{25}{4} - \dfrac{25}{4}$

$= \left(x - \dfrac{5}{2}\right)^2 - \dfrac{25}{4}$

d. $x^2 + 4x - 2$

$= x^2 + 4x + 4 - 4 - 2$

$= (x + 2)^2 - 6$

7. a. $3\left(x - 8 - \sqrt{2}\right)\left(x - 8 + \sqrt{2}\right)$

b. No real factors

8. a. $(x - 2\sqrt{3})(x + 2\sqrt{3})$

b. $(x - 6 - 4\sqrt{2})(x - 6 + 4\sqrt{2})$

c. $\left(x + \dfrac{9 - \sqrt{93}}{2}\right)\left(x + \dfrac{9 + \sqrt{93}}{2}\right)$

d. $2\left(x + \dfrac{5 - \sqrt{17}}{4}\right)\left(x + \dfrac{5 + \sqrt{17}}{4}\right)$

e. $3\left(\left(x + \frac{2}{3}\right)^2 + \dfrac{5}{9}\right)$ no linear factors over R

f. $-5\left(x - 4 - \dfrac{9\sqrt{5}}{5}\right)\left(x - 4 + \dfrac{9\sqrt{5}}{5}\right)$

9. a. $x = 5 - \sqrt{2}$ or $5 + \sqrt{2}$

b. $x = \dfrac{5}{2} + \dfrac{\sqrt{5}}{2}$ or $\dfrac{5}{2} - \dfrac{\sqrt{5}}{2}$

c. $x = -7 + \sqrt{6}$ or $-7 - \sqrt{6}$

d. $x = \dfrac{-9}{2} + \dfrac{\sqrt{5}}{2}$ or $\dfrac{-9}{2} - \dfrac{\sqrt{5}}{2}$

10. a. -1.19 or 4.19 **b.** 0.76 or 5.24

c. -4 or -3 **d.** 3.68 or 16.32

11. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

It is used to find solutions for x when $ax^2 + bx + c = 0$.

12. a. $a = 1,\ b = -10,\ c = 21$

b. $a = 10,\ b = -93,\ c = 68$

c. $a = 1,\ b = -9,\ c = 20$

d. $a = 40,\ b = 32,\ c = 6$

13. a. $x = \dfrac{1}{3} + \dfrac{\sqrt{13}}{3}$ or $\dfrac{1}{3} - \dfrac{\sqrt{13}}{3}$

b. $x = -3$ or $-\dfrac{1}{2}$

c. $x = -1 + \dfrac{\sqrt{21}}{3}$ or $-1 - \dfrac{\sqrt{21}}{3}$

d. $x = \dfrac{1}{3} + \dfrac{\sqrt{19}}{6}$ or $\dfrac{1}{3} - \dfrac{\sqrt{19}}{6}$

14. a. $x = -3.14$ or 0.64 **b.** $x = -0.74$ or 1.24

c. $x = 0.73$ or 6.52 **d.** $x = -1.04$ or 1.20

15. C

16. A

17. a. $x = \dfrac{5 \pm \sqrt{13}}{6}$ **b.** $x = \dfrac{1 \pm \sqrt{101}}{10}$

c. no real solutions **d.** $-3 \pm \sqrt{17}$

18. $x = \dfrac{-11 \pm \sqrt{89}}{4}$

19. a. **i.** $2(x - 3)^2 - 9$ **ii.** $(3, -9)$ **iii.** -9

b. **i.** $-(x + 9)^2 + 86$ **ii.** $(-9, 86)$ **iii.** 86

20. a. $\pm\sqrt{5}$ **b.** $\pm 2\sqrt{2}$ **c.** $\pm\sqrt{11}, \pm 1$

d. $\dfrac{-3 \pm \sqrt{5}}{2}$ **e.** $-1, 8, \dfrac{7 \pm \sqrt{93}}{2}$

Exercise 3.5 The discriminant

1. a. $\Delta = -135$, no real factors
b. $\Delta = 0$, two identical rational factors
c. $\Delta = 1$, two rational factors
d. $\Delta = \frac{40}{9}$, two real factors, completing the square needed to obtain the factors

2. a. 4
b. two rational roots
c. two irrational solutions
d. one rational solution
e. no real solutions
f. two irrational solutions

3. a. $\Delta = 121$, 2 rational factors
b. $\Delta = -39$, no real factors
c. $\Delta = 0$, 1 repeated rational factor
d. $\Delta = 8$, 2 irrational factors

4. a. There are no real roots
b. $k = -1,\ k = 3$

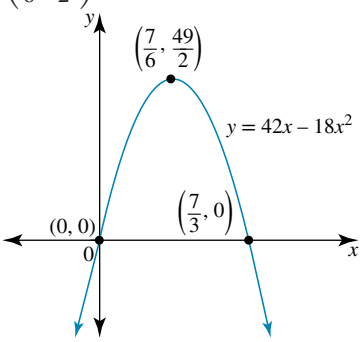
5. $\Delta = (m+4)^2 \Rightarrow \Delta \geq 0$

6. a. $(x-2)\left(x^2+2x+4\right)$, quadratic factor has a negative discriminant
b. i. $\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right) = x^2 - 2$
ii. $\left(x+4-\sqrt{2}\right)\left(x+4+\sqrt{2}\right) = x^2 + 8x + 14$

7. a. 2 irrational roots
b. 2 rational roots
c. no real roots
d. 1 rational root
e. no real roots
f. 2 irrational roots

8. a. $m = -4 \pm 4\sqrt{2}$
b. $m = 2 \pm 2\sqrt{3}$
c. $p < \frac{1}{3}$
d. $\Delta > 0$
e. Δ is a perfect square.

9. a. 2 rational x-intercepts
b. Axis intercepts $(0, 0)$, $\left(\frac{7}{3}, 0\right)$; maximum turning point $\left(\frac{7}{6}, \frac{49}{2}\right)$



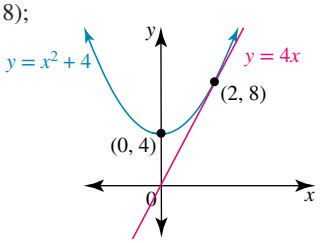
10. a. 2 irrational x-intercepts
b. No x-intercepts
c. 1 rational x-intercept
d. 2 rational x-intercepts

11. a. $k = -5$ **b.** $k > -5$ **c.** $k < -5$

12. a. $m > \frac{1}{4}$
b. i. Proof required – check with your teacher
ii. $x = -\frac{1}{2}$
c. i. $t = \pm\frac{4\sqrt{6}}{3}$
ii. $t = 0, \frac{1}{4}$

13. a. $(-6, 8)$ and $(2, 0)$
b. No intersections

14. $(2, 8)$;

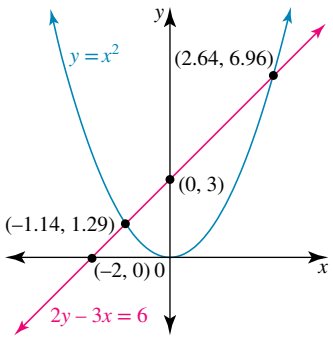


15. a. $x = 6, y = 32$ or $x = -1, y = -3$
b. $x = -8, y = 35$ or $x = 1, y = -1$
c. No solution
d. $x = 0, y = 5$ or $x = -\frac{5}{3}, y = \frac{70}{9}$

16. a. $\left(\frac{3}{5}, \frac{31}{5}\right)$, $(1, 7)$
b. $(-2, -3)$
c. $(-5, 10)$, $(4, 10)$
d. $(4, 30)$

17. a. No intersections
b. 2 intersections
c. No intersections

18. a. $(-1.14, 1.29)$, $(2.64, 6.96)$
b.



Exercise 3.6 Modelling with quadratic functions

1. 20 kg **2.** $3\sqrt{10}$ cm **3.** $4\frac{1}{2}$ hours

4. a. $4\sqrt{3}$ cm
b. Distance is quadrupled.
c. Heat is reduced by 36%.

5. 2511 litres

6. 20 and 22

7. 10, 11

8. $x = 24$; perimeter = 168 cm

9. Length 24 cm; width 20 cm

10. **a.** Sample responses can be found in the worked solutions in the online resources.

b. Sample responses can be found in the worked solutions in the online resources.

c. $\Delta = 256 - 8k$

i. $k > 32$ **ii.** $k = 32$ **iii.** $0 < k < 32$

d. $32\,m^2$; width 4 m, length 8 m

e. Width 1.1 metres and length 13.8 metres

11. 1.052 metres

12. $r = \dfrac{-\left(\pi l - \sqrt{\pi^2 l^2 + 80\pi}\right)}{2\pi}$

13. **a.** Sample responses can be found in the worked solutions in the online resources.

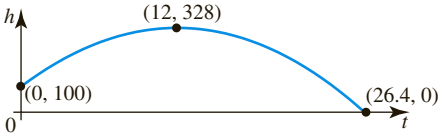
b. Width 7.5 metres; length 15 metres

c. 112.5 square metres

14. **a.** 328 metres

b. 12 seconds

c. Reaches ground after 26.4 seconds;



15. **a.** Sample responses can be found in the worked solutions in the online resources; length 30 metres, width 15 metres

b. 40 metres, 35 metres, 45 metres

16. **a.** 2 hours **b.** 380

c. 110 bacteria at 4 pm

17. **a.** $C = 20 + 10n + 5n^2$ **b.** 13

18. **a.** A(3, 0), B(7, 5), C (11, 0)

b. $y = -\dfrac{5}{16}(x - 7)^2 + 5$

c. 6.7 metres

19. **a.** **i.** Both numbers are 8.

ii. Both numbers are 8.

b. **i.** $\dfrac{k^2}{4}$

ii. No values possible

20. **a.** $y = -0.2(x - 5.5)^2 + 7.25$

b. 7.25 metres

c. 2.37 metres

3.7 Review: exam practice

1. D **2.** C **3.** A **4.** B **5.** D

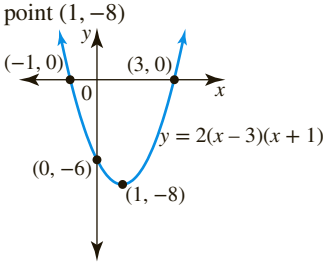
6. **a.** $x = \pm 2$

b. $x = 3 \pm 2\sqrt{2}$

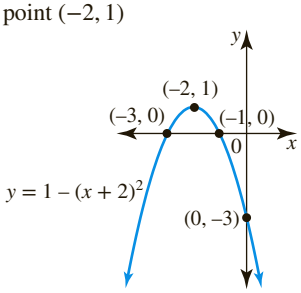
c. $x = \pm 4$

d. $x = \dfrac{9}{4}, 1$

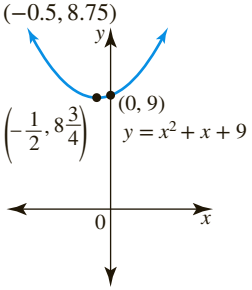
7. **a.** x-intercepts $(3, 0), (-1, 0)$; y-intercept $(0, -6)$; turning point $(1, -8)$



b. x-intercepts $(-3, 0)$, $(-1, 0)$; y-intercept $(0, -3)$; turning point $(-2, 1)$



c. No x-intercepts; y-intercept $(0, 9)$; turning point $(-0.5, 8.75)$



8. B **9.** A **10.** B **11.** C

12. **a.** $x < -\dfrac{1}{2}$ or $x > 3$

b. $-\sqrt{10} \leq x \leq \sqrt{10}$

c. $x \in R$

13. C

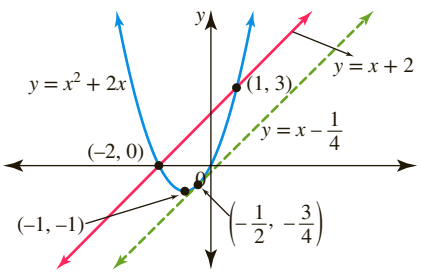
14. **a.** $-\left(x - 10 - 2\sqrt{31}\right)\left(x - 10 + 2\sqrt{31}\right)$

b. $4\left(x - \dfrac{1 + \sqrt{37}}{4}\right)\left(x - \dfrac{1 - \sqrt{37}}{4}\right)$

15. $1 < k < 4$

16. **a.** $(-2, 0), (1, 3)$

b. Parabola: x-intercepts $(-2, 0), (0, 0)$; turning point $(-1, -1)$ Line through (–2, 0), (0, 2). Both graphs meet at $(-2, 0), (1, 3)$.



17. a. $h = -\frac{1}{35}(x^2 - 60x - 700)$

When $x = 0$, $h = -\frac{1}{35}(-700)$

$\therefore h = 20$ and the point S is $(0, 20)$.

Therefore, S is 20 metres above O.

b. 70 metres

c. $h = -\frac{1}{60}(x - 30)^2 + 35$

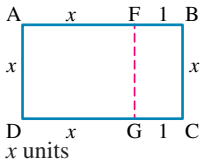
d. Japanese competitor wins

18. a. 8 metres **b.** 5 metres

c. The caravan can be towed under the bridge.

d. $2x - 8$ **e.** 6.4 **f.** No

19. a.



b. x units

c. Area measure of rectangle AFGD is x^2.

Area measure of rectangle FBCG is $1 \times x = x$.

$\therefore x^2 = x + 1$

$\therefore x^2 - x - 1 = 0$

d. $\frac{1 + \sqrt{5}}{2}$

e. $\frac{1}{\phi} = -\frac{1 - \sqrt{5}}{2}$

f. $\phi - 1 = \frac{1 + \sqrt{5}}{2} - 1$

$= \frac{1 + \sqrt{5} - 2}{2}$

$= \frac{\sqrt{5} - 1}{2}$

and:

$\frac{1}{\phi} = -\frac{1 - \sqrt{5}}{2}$

$= \frac{\sqrt{5} - 1}{2}$

$= \phi - 1$

As $x = \phi$ is a root of $x^2 - x - 1 = 0$,

$\phi^2 - \phi - 1 = 0$

$\phi(\phi - 1) = 1$

$\phi - 1 = \frac{1}{\phi}$

20. a. $(28, 4.9)$ **b.** $y = -\frac{1}{160}(x - 28)^2 + 4.9$

c. $\frac{13}{7}$ seconds **d.** 7 m/s

CHAPTER 4
Inverse proportions and graphs of relations

4.1 Overview

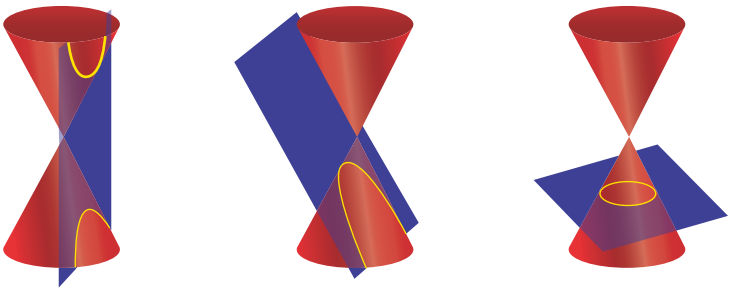
4.1.1 Introduction

The graphs examined in this chapter – the hyperbola, parabola and the circle – are members of a family of curves known as the **conic sections**. They are named as such because their shapes are the result of the intersection of a plane with a solid cone at different angles.

The motion of spacecraft, satellites and probes as they move through space can be modelled by the equations of the conic curves. The International Space Station traces out a nearly circular path as it orbits the Earth sixteen times a day at an altitude of 400 kilometres.

Interplanetary probes travel on hyperbolic trajectories. These probes use the gravity of nearby planets or moons to curve their path and to boost their velocity without expenditure of fuel. NASA probes Voyager 1 and Voyager 2 were able to explore the four gas giants, allowing detailed data to be collected, and to achieve enough speed to escape our solar system.





Hyperbola Parabola Circle

LEARNING SEQUENCE

4.1 Overview
4.2 The hyperbola
4.3 Inverse proportion
4.4 The circle
4.5 The sideways parabola
4.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.2 The hyperbola

The family of functions with rules $y = x^n, n \in N$ are polynomial functions. If n is a negative number, however, these equations are not those of polynomials.

4.2.1 The graph of $y = \frac{1}{x}$

With $n = -1$, the rule $y = x^{-1}$ can also be written as $y = \frac{1}{x}$. This is the rule for a **rational function** called a **hyperbola**. Two things can be immediately observed from the rule:

- $x = 0$ must be excluded from the domain, since division by zero is not defined.
- $y = 0$ must be excluded from the range, since there is no number whose reciprocal is zero.

The lines $x = 0$ and $y = 0$ are **asymptotes**. An asymptote is a line the graph will approach but never reach. As these two asymptotes $x = 0$ and $y = 0$ are a pair of perpendicular lines, the hyperbola is known as a **rectangular hyperbola**. The asymptotes are a key feature of the graph of a hyperbola.

Completing a table of values can give us a 'feel' for this graph.

x	-10	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{10}$	0	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	10
y	$-\frac{1}{10}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-10	no value possible	10	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{10}$

The values in the table illustrate that as $x \to \infty, y \to 0$ and as $x \to -\infty, y \to 0$.

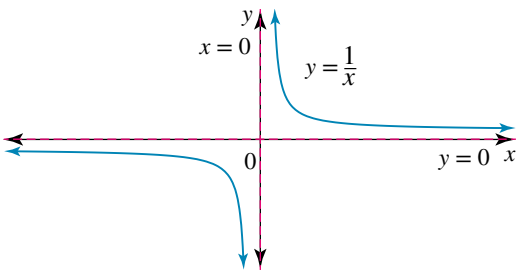
The table also illustrates that as $x \to 0$, either $y \to -\infty$ or $y \to \infty$.

These observations describe the asymptotic behaviour of the graph.

The graph of the basic hyperbola $y = \frac{1}{x}$ is shown.

Key features:

- Vertical asymptote has equation $x = 0$ (the y-axis).
- Horizontal asymptote has equation $y = 0$ (the x-axis).
- Domain is $R\backslash\{0\}$.
- Range is $R\backslash\{0\}$.
- As $x \to \infty$, $y \to 0$ from above and as $x \to -\infty$, $y \to 0$ from below. This can be written as: $x \to \infty$, $y \to 0^+$ and as $x \to -\infty$, $y \to 0^-$.
- As $x \to 0^-$, $y \to -\infty$ and as $x \to 0^+$, $y \to \infty$.
- The graph is that of a one-to-one function.
- The graph has two branches separated by the asymptotes.
- As the two branches do not join at $x = 0$, the function is said to be **discontinuous** at $x = 0$.
- The graph lies in **quadrants** 1 and 3 as defined by the asymptotes.



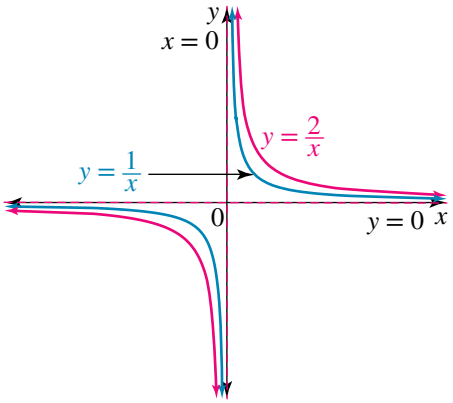
The asymptotes divide the Cartesian plane into four areas or quadrants. The quadrants formed by the asymptotes are numbered 1 to 4 anticlockwise.

With the basic shape of the hyperbola established, transformations of the graph of $y = \frac{1}{x}$ can be studied.

Dilation from the *x*-axis

The effect of a dilation factor on the graph can be illustrated by comparing $y = \frac{1}{x}$ and $y = \frac{2}{x}$.

For $x = 1$ the point (1, 1) lies on $y = \frac{1}{x}$ whereas the point (1, 2) lies on $y = \frac{2}{x}$. The dilation effect on $y = \frac{2}{x}$ is to move the graph further out from the x-axis. The graph has a dilation factor of 2 from the x-axis.

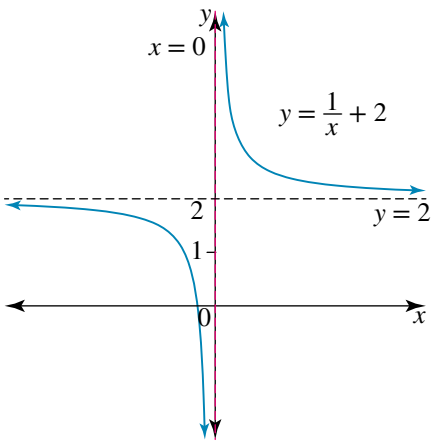


Vertical translation

The graph of $y = \frac{1}{x} + 2$ illustrates the effect of a vertical translation of 2 units upwards.

Key features:

- The horizontal asymptote has equation $y = 2$. This means that as $x \to \pm\infty$, $y \to 2$.
- The vertical asymptote is unaffected and remains $x = 0$.
- Domain is $R\backslash\{0\}$.
- Range is $R\backslash\{2\}$.



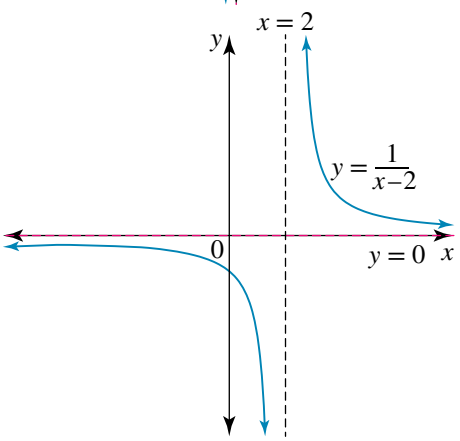
Horizontal translation

For $y = \frac{1}{x - 2}$ as the denominator cannot be zero, $x - 2 \neq 0 \Rightarrow x \neq 2$. The domain must exclude $x = 2$, so the line $x = 2$ is the vertical asymptote.

Key features:

- Vertical asymptote has equation $x = 2$.
- Horizontal asymptote is unaffected by the horizontal translation and still has the equation $y = 0$.
- Domain is $R\backslash\{2\}$.
- Range is $R\backslash\{0\}$.

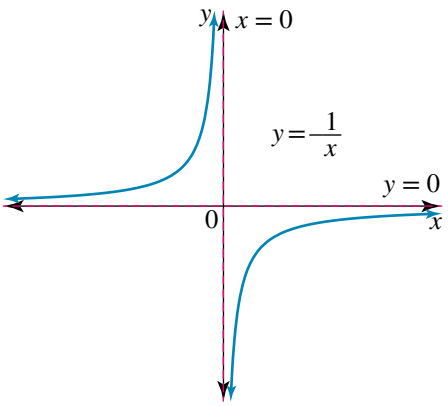
The graph of $y = \frac{1}{x - 2}$ demonstrates the same effect that we have seen with other graphs that are translated 2 units to the right.



Reflection in the *x*-axis

The graph of $y = -\frac{1}{x}$ illustrates the effect of inverting the graph by reflecting $y = \frac{1}{x}$ in the x-axis.

The graph of $y = -\frac{1}{x}$ lies in quadrants 2 and 4 as defined by the asymptotes.



on Resources

Interactivity: Graph plotter: The hyperbola (int-2573)

4.2.2 General equation of a hyperbola

The equation $y = \frac{a}{x-c} + d$ is that of a hyperbola with the following key features.

- Vertical asymptote has the equation $x = c$.
- Horizontal asymptote has the equation $y = d$.
- Domain is $R\backslash\{c\}$.
- Range is $R\backslash\{d\}$.
- Asymptotic behaviour: as $x \to \pm\infty$, $y \to d$ and as $x \to c$, $y \to \pm\infty$.
- There are two branches to the graph and the graph is discontinuous at $x = c$.
- If $a > 0$ the graph lies in the asymptote-formed quadrants 1 and 3.
- If $a < 0$ the graph lies in the asymptote-formed quadrants 2 and 4.
- $|a|$ gives the dilation factor from the x-axis.

If the equation of the hyperbola is in the form $y = \frac{a}{bx-c} + d$, then the vertical asymptote can be identified by finding the x-value for which the denominator term $bx - c = 0$. The horizontal asymptote is $y = d$ because as $x \to \pm\infty$, $\frac{a}{bx-c} \to 0$ and therefore $y \to d$.

WORKED EXAMPLE 1

State the changes that should be made to the graph of $y = \frac{1}{x}$ to obtain the graph of $y = \frac{-4}{x+2} - 1$.

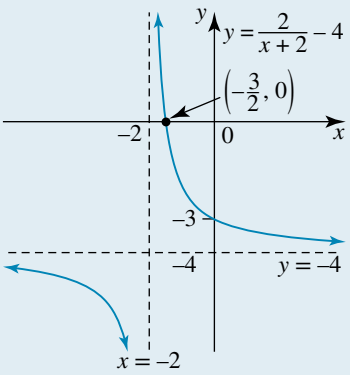
THINK	WRITE
1. Write the general equation of the hyperbola.	$y = \frac{a}{x-c} + d$
2. Identify the value of a.	$a = -4$
3. State the changes to $y = \frac{1}{x}$, caused by a.	The graph of $y = \frac{1}{x}$ is dilated by the factor of 4 in the y direction and reflected in the x-axis.
4. Identify the value of c.	$c = -2$
5. State the changes to the graph caused by c.	The graph is translated 2 units to the left.
6. Identify the value of d.	$d = -1$
7. State the changes to the graph caused by d.	The graph is translated 1 unit down.

Sketching the graph of the hyperbola by hand can be easily done by following these steps:

Step 1 Find the position of the asymptotes.
Step 2 Find the values of the intercepts with the axes.
Step 3 Decide whether the hyperbola is positive or negative.
Step 4 On the set of axes, draw the asymptotes (using dotted lines) and mark the intercepts with the axes.
Step 5 Treating the asymptotes as the new set of axes, sketch either the positive or negative hyperbola, making sure it passes through the intercepts that have been previously marked.

WORKED EXAMPLE 2

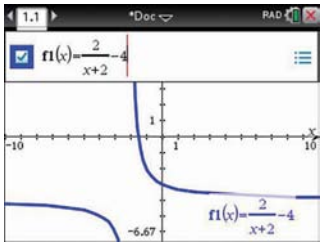
Sketch the graph of $y = \dfrac{2}{x+2} - 4$, clearly showing the intercepts with the axes and the position of the asymptotes.

THINK	WRITE
1. Compare the given equation with $y = \dfrac{a}{x-c} + d$ and state the values of a, c and d.	$a = 2,\ c = -2,\ d = -4$
2. Write a short statement about the effects of a, c and d on the graph of $y = \dfrac{1}{x}$.	The graph of $y = \dfrac{1}{x}$ is dilated by the factor of 2 in the y direction and translated 2 units to the left and 4 units down.
3. Write the equations of the asymptotes.	Asymptotes: $x = -2$; $y = -4$
4. Find the value of the y-intercept by letting $x = 0$.	y-intercept: when $x = 0$, $y = \dfrac{2}{0+2} - 4$ $= 1 - 4$ $= -3$
5. Find the value of the x-intercept by making $y = 0$.	x-intercept: when $y = 0$, $0 = \dfrac{2}{x+2} - 4$ $\dfrac{2}{x+2} = 4$ $2 = 4(x+2)$ $= 4x + 8$ $4x = 2 - 8$ $= -6$ $x = -\dfrac{6}{4}$ $= -\dfrac{3}{2}$
6. To sketch the graph: • Draw the set of axes and label them. • Use dotted lines to draw the asymptotes and label. • Mark the intercepts with the axes. • Treating the asymptotes as your new set of axes, sketch the graph of the hyperbola (as a is positive, the graph is not reflected); make sure the upper branch passes through the x- and y-intercepts previously marked.	

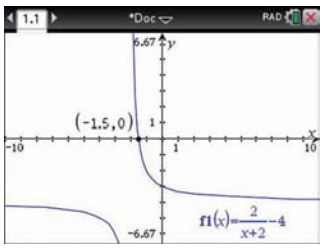
TI | THINK

WRITE

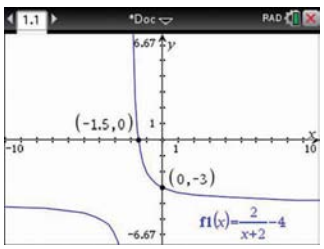
1. On a Graphs page, complete the entry line for function 1 as
$f1(x) = \frac{2}{x+2} - 4$
then press ENTER.



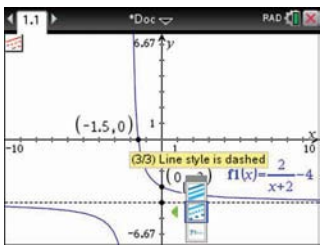
2. To find the x-intercept, press MENU then select
6: Analyze Graph
1: Zero
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER. Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.

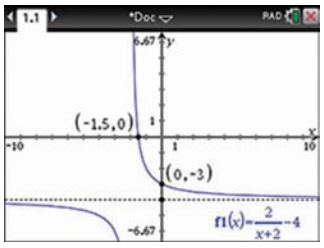


3. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



4. To draw the horizontal asymptote on the screen, press MENU then select:
8: Geometry
4: Construction
1: Perpendicular
Click on the y-axis then click on the point on the y-axis where $y = -4$. Press MENU then select:
1: Actions
4: Attributes
Click on the asymptote, press the down arrow and then the right arrow twice to change the line style to dashed. Press ENTER.

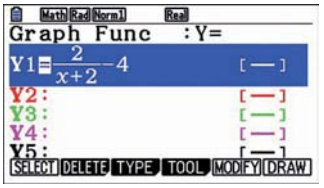


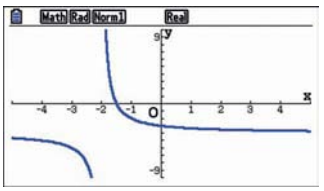


CASIO | THINK

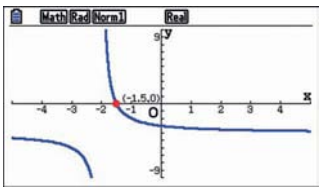
WRITE

1. On a Graph screen, complete the entry line for Y1 as
$Y1 = \frac{2}{x+2} - 4$
then press EXE. Select DRAW by pressing F6.

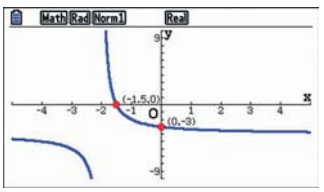




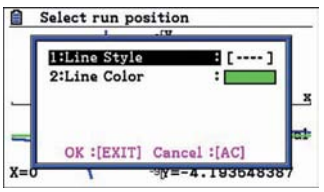
2. To find the x-intercept, select G-Solve by pressing F5, then select ROOT by pressing F1. Press EXE.

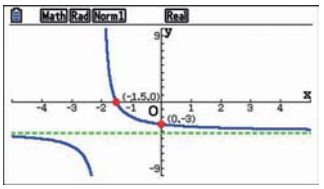


3. To find the y-intercept, select G-Solve by pressing F5, then select Y-ICEPT by pressing F4. Press EXE.



4. To draw the horizontal asymptote on the screen, select Sketch by pressing F4, press F6 to scroll across to more menu options, then select Horz by pressing F5. Use the up/down arrows to position the horizontal line, then select FORMAT by pressing SHIFT 5. Change the Line Style to Broken, then select OK by pressing EXIT. Press EXE.





Proper rational functions

The equation of the hyperbola $y = \frac{a}{x-c} + d$ is expressed in **proper rational function** form. This means the rational term, $\frac{a}{x-c}$, has a denominator of a higher degree than that of the numerator. For example, $y = \frac{x-1}{x-2}$

is not expressed as a proper rational function: the numerator has the same degree as the denominator. Using division, it can be converted to the proper form $y = \frac{1}{x-2} + 1$ which is recognisable as a hyperbola and from which the asymptotes can be obtained.

WORKED EXAMPLE 3

Identify the asymptotes of the hyperbola with equation $y = \frac{2x-3}{5-2x}$.

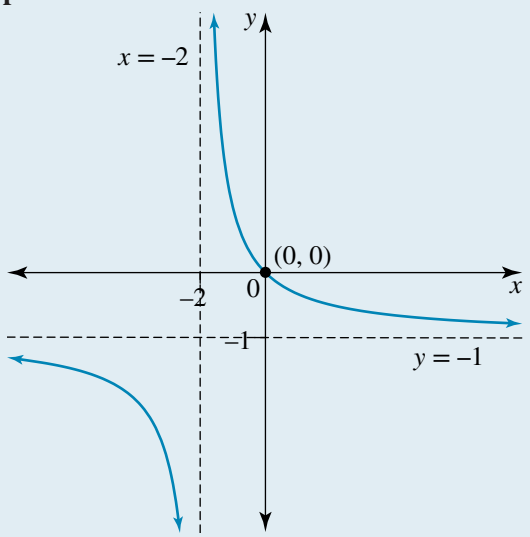
THINK	WRITE
1. The equation is in improper form so reduce it to proper form using division. *Note:* The long-division algorithm could also have been used to reduce the function to proper form.	$y = \frac{2x-3}{5-2x}$ $= \frac{-1(5-2x)+2}{5-2x}$ $= -1 + \frac{2}{5-2x}$ $y = \frac{2}{5-2x} - 1$ is the proper rational function form.
2. State the equations of the asymptotes.	Vertical asymptote when $5 - 2x = 0$ $\therefore x = \frac{5}{2}$ Horizontal asymptote is $y = -1$.

4.2.3 Finding the equation of a hyperbola

From the equation $y = \frac{a}{x-c} + d$ it can be seen that three pieces of information will be needed to form the equation of a hyperbola. These are usually the equations of the asymptotes and the coordinates of a point on the graph.

WORKED EXAMPLE 4

Form the equation of the hyperbola shown.



THINK	WRITE
1. Substitute the equations of the asymptotes shown on the graph into the general equation of a hyperbola.	Let equation of the graph be $y = \frac{a}{x-c} + d$. From the graph, asymptotes have equations $x = -2, y = -1$ $\therefore y = \frac{a}{x+2} - 1$
2. Use a known point on the graph to determine the remaining unknown constant.	Point $(0, 0)$ lies on the graph. $0 = \frac{a}{2} - 1$ $\therefore 1 = \frac{a}{2}$ $\therefore a = 2$
3. State the equation of the hyperbola.	The equation is $y = \frac{2}{x+2} - 1$.

4.2.4 Modelling with the hyperbola

Applications involving the use of the hyperbolic function for modelling and predicting data may need domain restrictions. Unlike many polynomial models, the hyperbola has neither maximum nor minimum turning points, so the asymptotes are often where the interest will lie. The horizontal asymptote is often of particular interest as it represents the limiting value of the model.

WORKED EXAMPLE 5

A relocation plan to reduce the number of bats in a public garden is formed and t months after the plan is introduced the number of bats N in the garden is thought to be modelled by $N = 250 + \frac{30}{t+1}$.



a. How many bats were removed from the garden in the first 9 months of the relocation plan?

b. Sketch the graph of the bat population over time using the given model and state its domain and range.

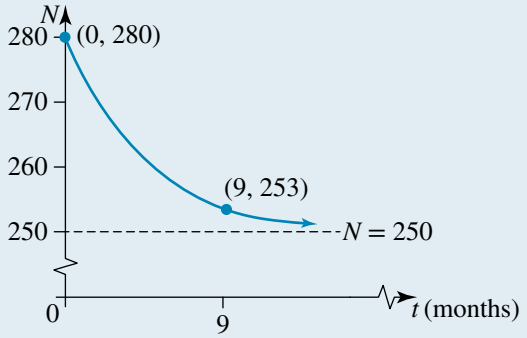
c. What is the maximum number of bats that will be relocated according to this model?

THINK	WRITE
a. Find the number of bats at the start of the plan and the number after 9 months and calculate the difference.	a. $N = 250 + \frac{30}{t+1}$ When $t = 0$, $N = 250 + \frac{30}{1}$. Therefore there were 280 bats when the plan was introduced. When $t = 9$, $N = 250 + \frac{30}{10}$. Therefore 9 months later there were 253 bats. Hence, over the first 9 months, 27 bats were removed.

b. 1. Identify the asymptotes and other key features which are appropriate for the restriction $t \geq 0$.

b. $N = 250 + \dfrac{30}{t+1}, t \geq 0$
Vertical asymptote $t = -1$ (not applicable)
Horizontal asymptote $N = 250$
Initial point is (0, 280).

2. Sketch the part of the graph of the hyperbola that is applicable and label axes appropriately.
Note: The vertical scale is broken in order to better display the graph.



3. State the domain and range for this model.

Domain $\{t : t \geq 0\}$
Range (250, 280]

c. 1. Interpret the meaning of the horizontal asymptote.

c. The horizontal asymptote shows that as $t \to \infty, N \to 250$. This means $N = 250$ gives the limiting population of the bats.

2. State the answer.

Since the population of bats cannot fall below 250 and there were 280 initially, the maximum number of bats that can be relocated is 30.

studyon

Units 1 & 2 > Area 2 > Sequence 3 > Concept 1

The hyperbola Summary screen and practice questions

Exercise 4.2 The hyperbola

Technology free

1. WE1 State the changes that should be made to the graph of $y = \dfrac{1}{x}$ to obtain the graph of each of the following.

a. $y = \dfrac{2}{x}$ **b.** $y = -\dfrac{3}{x}$ **c.** $y = \dfrac{1}{x-6}$

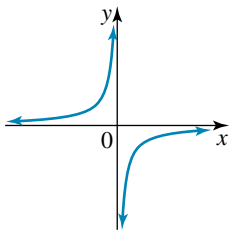
d. $y = \dfrac{2}{x+4}$ **e.** $y = \dfrac{1}{x} + 7$ **f.** $y = \dfrac{2}{x} - 5$

g. $y = \dfrac{1}{4+x} - 3$ **h.** $y = \dfrac{2}{x-3} + 6$ **i.** $y = -\dfrac{4}{x-1} - 4$

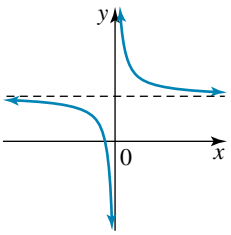
2. Identify which of the following transformations were applied to the graph of $y = \dfrac{1}{x}$ to obtain each of the graphs shown below.

i. translation to the right **ii.** translation to the left **iii.** translation up
iv. translation down **v.** reflection in the x-axis

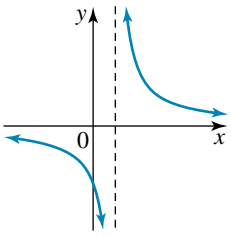
a.



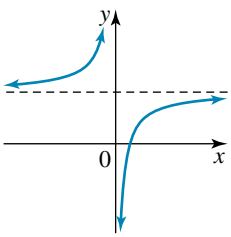
b.



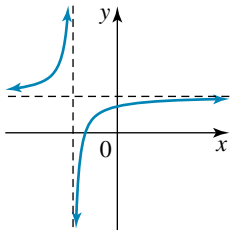
c.



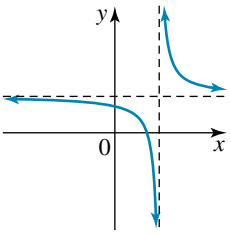
d.



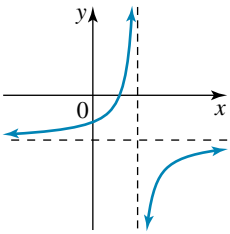
e.



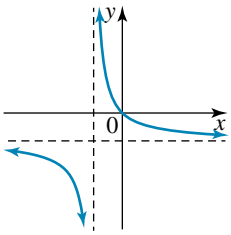
f.



g.



h.



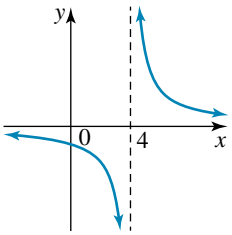
3. For each of the following graphs, state:

i. the equations of the asymptotes

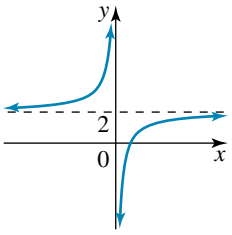
ii. the domain

iii. the range.

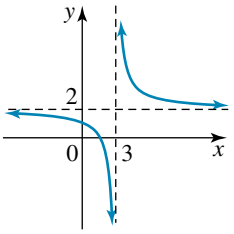
a.



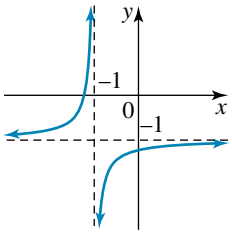
b.



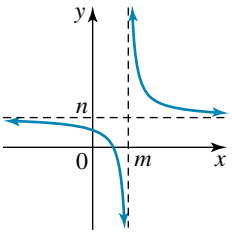
c.



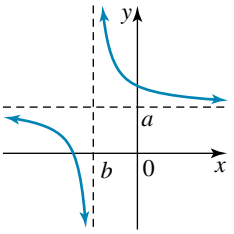
d.



e.



f.



4. **WE2** Sketch each of the following, clearly showing the position of the asymptotes and the intercepts with the axes.

a. $y = \dfrac{1}{x+3}$

b. $y = \dfrac{1}{x+2} - 1$

c. $y = \dfrac{3}{x-1} - \dfrac{3}{4}$

d. $y = -\dfrac{2}{x+5}$

e. $y = \dfrac{6}{1-x} - 3$

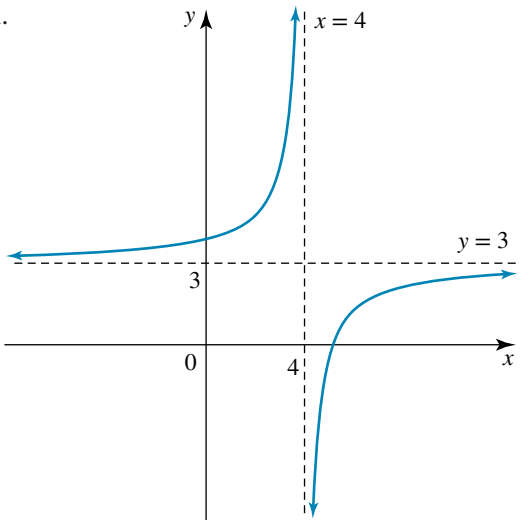
f. $y = -\dfrac{3}{x-2} + 6$

g. $y = 1 - \dfrac{1}{2-x}$

h. $y = \dfrac{2}{5} + \dfrac{4}{1+x}$

i. $y = \dfrac{1}{2x+3} + 4$

5. MC Identify the equation that represents the graph shown.



A. $y = 3 + \dfrac{1}{x-4}$

B. $y = 3 - \dfrac{1}{4-x}$

C. $y = \dfrac{1}{4-x} - 3$

D. $y = 3 - \dfrac{1}{x-4}$

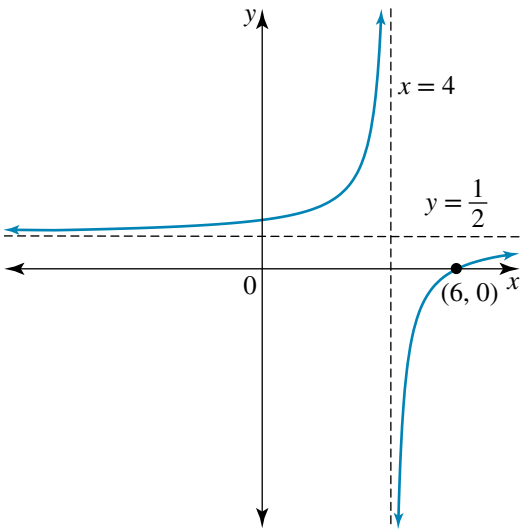
6. If a function is given by $f(x) = \dfrac{1}{x}$, sketch each of the following, labelling the asymptotes and the intercepts with the axes.

a. $f(x+2)$ **b.** $f(x)-1$ **c.** $-f(x)-2$

d. $f(1-x)+2$ **e.** $-f(x-1)-1$

7. WE3 **a.** Identify the asymptotes of the hyperbola with equation $y = \dfrac{6x}{3x+2}$.

WE4 **b.** Form the equation of the hyperbola shown.



8. State the equations of the asymptotes of the following hyperbolas.

a. $y = \dfrac{1}{x+5} + 2$ **b.** $y = \dfrac{8}{x} - 3$

c. $y = \dfrac{-3}{4x}$ **d.** $y = \dfrac{-3}{14+x} - \dfrac{3}{4}$

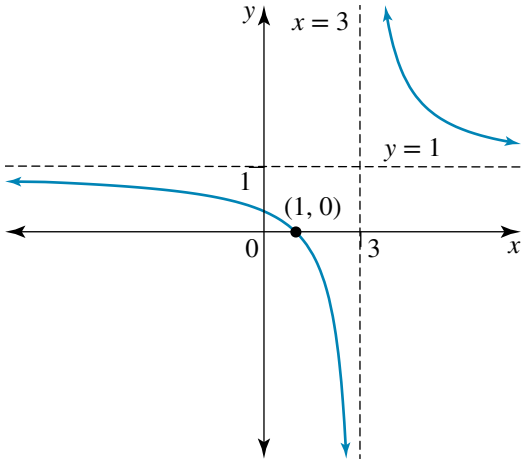
9. Sketch the graph of the following functions, stating the domain and range.

a. $y = \dfrac{1}{x+1} - 3$ **b.** $y = 4 - \dfrac{3}{x-3}$

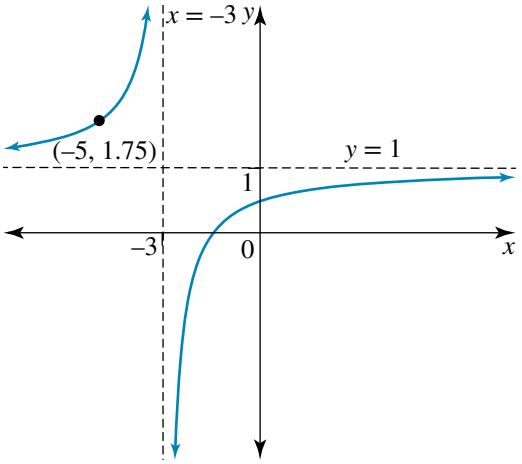
c. $y = -\dfrac{5}{3+x}$ **d.** $y = -\left(1 + \dfrac{5}{2-x}\right)$

10. Deduce the equations of each of these graphs.

a.



b.



11. A hyperbola is undefined when $x = \frac{1}{4}$. As $x \to -\infty$ its graph approaches the line $y = -\frac{1}{2}$ from below. The graph cuts the x-axis where $x = 1$.

a. Determine the equation of the hyperbola.

b. Write the function in mapping notation.

12. Determine the domain and range of the hyperbola with equation $xy - 4y + 1 = 0$.

13. a. If $\frac{11 - 3x}{4 - x} = a - \frac{b}{4 - x}$ calculate the values of a and b.

b. Hence, sketch the graph of $y = \frac{11 - 3x}{4 - x}$.

c. For what values of x is $\frac{11 - 3x}{4 - x} > 0$?

14. Express in the form $y = \frac{a}{bx + c} + d$ and state the equations of the asymptotes for each of the following.

a. $y = \frac{x}{4x + 1}$

b. $(x - 4)(y + 2) = 4$

c. $y = \frac{1 + 2x}{x}$

d. $2xy + 3y + 2 = 0$

15. WE5 The number P of cattle owned by a farmer at a time t years after purchase is modelled by $P = 30 + \frac{100}{2 + t}$.

a. By how many cattle is the herd reduced after the first 2 years?

b. Sketch the graph of the number of cattle over time using the given model and state its domain and range.

c. What is the minimum number the herd of cattle is expected to reach according to this model?



Technology active



16. In an effort to protect a rare species of stick insect, 20 of the species were captured and relocated to a small island where there were few predators. After 2 years the population size grew to 240 stick insects.

A model for the size N of the stick insect population after t years on the island is thought to be defined by the function:

$N: R^+ \cup \{0\} \to R, N(t) = \dfrac{at+b}{t+2}$.

a. Calculate the values of a and b.

b. After what length of time, to the nearest month, did the stick insect population reach 400?

c. Show that $N(t+1) - N(t) = \dfrac{880}{(t+2)(t+3)}$.

d. Hence, or otherwise, find the increase in the stick insect population during the 12^{th} year and compare this with the increase during the 14th year. What is happening to the growth in population?

e. When would the model predict the number of stick insects reaches 500?

f. How large can the stick insect population grow?

17. a. Sketch the graphs of $xy = 1$ and $x^2 - y^2 = 2$ using a graphics calculator and give the equations of their asymptotes.

b. These hyperbolas are the same but they are sketched on different orientations of the axes. Suggest a way to transform one graph into the other.

18. Use technology to sketch $y = \dfrac{x+1}{x+2}$ together with its asymptotes and use the graphing screen to obtain:

a. the number of intersections of $y = x$ with $y = \dfrac{x+1}{x+2}$

b. the values of k for which $y = x + k$ intersects $y = \dfrac{x+1}{x+2}$ once, twice or not at all.

4.3 Inverse proportion

The graph of the hyperbolic relationship $y = \dfrac{k}{x}$, where k is a constant, is also known as an **inverse proportion** graph. While the standard hyperbolic graph maps into two diagonally opposite quadrants, most inverse proportion graphs involve coordinates limited to a single quadrant, usually quadrant 1.

Consider the time taken to travel a fixed distance of 60 km.

The time to travel a fixed distance depends on the speed of travel. For a distance of 60 km, the times taken for some different speeds are shown in the table.

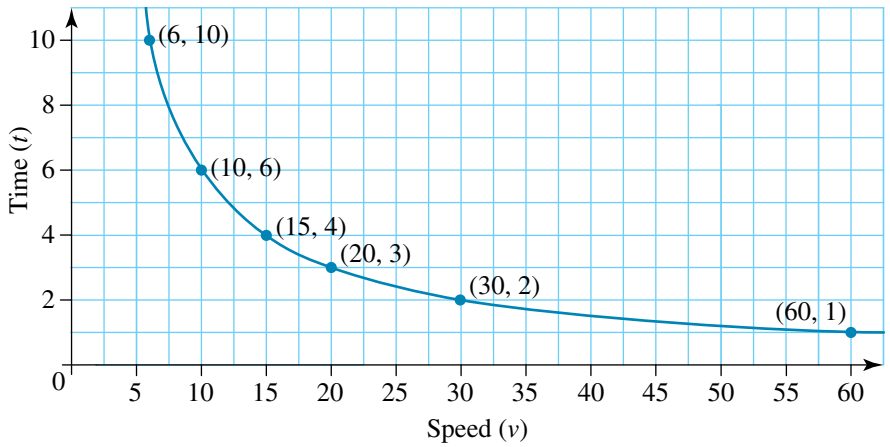
Speed, v (km/h)	10	15	20	30
Time, t (hours)	6	4	3	2

As the speed increases, the time will decrease; as the speed decreases, the time will increase. The time is inversely proportional to the speed, or the time varies inversely as the speed.

From the table:

$$t \times v = 60$$
$$\therefore t = \frac{60}{v}$$

This is the equation of a hyperbola where v is the independent variable and t the dependent variable.



The graph of time versus speed exhibits the asymptotic behaviour typical of a hyperbola. As the speed increases, becoming faster and faster, the time decreases, becoming smaller and smaller. In other words, as $v \to \infty$, $t \to 0$, but t can never reach zero nor can speed reach infinity. Further, as $v \to 0$, $t \to \infty$. Only one branch of the hyperbola is given since neither time nor speed can be negative.

In general, the following rules apply.

- **'y is inversely proportional to x' is written as $y \propto \frac{1}{x}$.**
- **If y is inversely proportional to x, then $y = \frac{k}{x}$ where k is the constant of proportionality.**
- **This relationship can also be expressed as $xy = k$ so if the product of two variables is constant, the variables are in inverse proportion.**

If $y = \frac{k}{x}$, then it could also be said that y is directly proportional to $\frac{1}{x}$; the graph of y against $\frac{1}{x}$ is linear.

Functions of variables may be in inverse proportion. For example, the strength of a radio signal I varies inversely as the square of the distance d from the transmitter, so $I = \frac{k}{d^2}$.

WORKED EXAMPLE 6

Boyle's Law says that if the temperature of a given mass of gas remains constant, its volume V is inversely proportional to the pressure P.
A container of volume 100 cm^3 is filled with a gas under a pressure of 75 cm of mercury.

a. Find the relationship between the volume and pressure.

b. The container is connected by a hose to an empty container of volume 50 cm^3. Find the pressure in the two containers.

THINK	WRITE
a. 1. Write the rule for the inverse proportion relation.	**a.** $V \propto \frac{1}{P}$ $\therefore V = \frac{k}{P}$
2. Use the given data to find k and hence the rule.	Substitute $V = 100,\ P = 75.$ $100 = \frac{k}{75}$ $k = 100 \times 75$ $= 7500$ Hence $V = \frac{7500}{P}$.
b. 1. State the total volume.	**b.** The two containers are connected and can be thought of as one. Therefore, the combined volume is $100 + 50 = 150\,\text{cm}^3$. $V = \frac{7500}{P}$
2. Calculate the pressure for this volume.	When $V = 150$, $150 = \frac{7500}{P}$ $P = \frac{7500}{150}$ $= 50$
3. State the answer.	The gas in the containers is under a pressure of 50 cm of mercury.

studyon

Units 1 & 2 › Area 2 › Sequence 3 › Concept 2

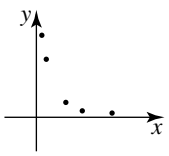
Inverse proportion Summary screen and practice questions

Exercise 4.3 Inverse proportion

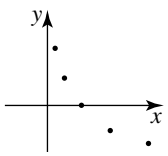
Technology free

1. MC Identify which graph(s) demonstrates y as being inversely proportional to x.

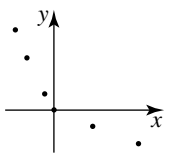
i.



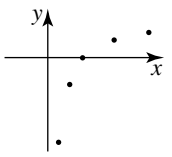
ii.



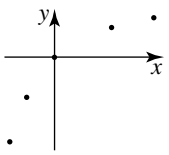
iii.



iv.



v.



A. i only **B.** i, ii and iii **C.** iv and v **D.** i, ii, and iv

2. Determine the conditions under which the general equation of the hyperbola, $y = \frac{a}{x-c} + d$, describes y as being inversely proportional to x.
3. Sort the following equations into two groups according to whether the relationship between x and y is inversely proportional $\left(y = \frac{k}{x}\right)$ or directly proportional $(y = kx)$.

a. $5 = xy$ **b.** $10y = 4x$ **c.** $y^2 = \frac{3}{x^2}$ **d.** $4y = x^{-1}$ **e.** $12 = \frac{y}{x}$

4. Identify which of the data set(s) below demonstrates that y is inversely proportional to x and write an expression for y in terms of x that describes that data set.

a.

x	−3	−2	−1	0	1	2	3
y	−8.1	−2.4	−0.3	0	0.3	2.4	8.1

b.

x	−2	−1	0	1	2	3
y	−24	−6	0	−6	−24	−54

c.

x	0	0.5	1	1.5	2
y	0	1.13	1.6	1.96	2.26

d.

x	1	2	4	5	10
y	5	2.5	1.25	1	0.5

e.

x	−3	−2	−1	0	1	2
y	40.5	12	1.5	0	−1.5	−12

5. The time, t, taken to travel a fixed distance of 180 km is given by $t = \frac{k}{v}$, where v is the speed of travel.
   a. Determine the value of the constant of proportionality, k.
   b. Sketch a graph to show the nature of the relationship between the time and the speed.
   c. Calculate the speed that needs to be maintained if the entire journey needs to be completed in 2.25 hours.
6. The frequency, f Hz, at which a plucked guitar string vibrates can be calculated from the equation $f = \sqrt{\frac{T}{4LM}}$, where T is the tension of the guitar string, L is the length of the string and M is the string's mass.



   a. MC Which of the following pairs of variables vary inversely with each other?
   A. f and L
   B. T and M
   C. L and M
   D. f and $\sqrt{T}$
   b. i. Rearrange the equation so that M is the subject of the equation.
   ii. Could it be said that M is inversely proportional to f? Explain your response.

Technology active

7. Sketch the graphs of $y = \frac{1}{x}$, $y = \frac{3}{x}$, and $y = \frac{1}{2x}$ on the same set of axes.
Use your graphs to make a generalisation as to how the constant of proportionality affects the graph shape.

8. During a science experiment Leanne varies the resistance, R ohms, in an electric circuit and measures the resulting current, I amperes. Her data is shown in the table below.

R	100	120	140	160	180	200
I	0.240	0.200	0.171	0.150	0.133	0.120

a. Graph the data and demonstrate that current is inversely proportional to resistance.
b. Use the graph and the data to determine the value of the constant of proportionality, correct to the nearest whole number.

9. WE6 Provided that air temperature and humidity remain constant, the frequency, f Hz, at which a tuning fork vibrates is inversely proportional to the wavelength, λ m, of the soundwaves it produces. A tuning fork that vibrates at a frequency of 256 Hz produces sound waves with a wavelength of 1.33 metres.
a. Find the relationship between frequency and wavelength, writing the constant of proportionality to the nearest whole number.
b. A tuning fork with a frequency of 400 Hz is now used. What is the wavelength of the sound waves it produces?

10. Ashok is doing an experiment in which small samples of two different metals — aluminium and iron — are heated from 25 °C to 200 °C by placing them on a hotplate. The aluminium and iron samples have the same thickness but are different sizes. He has 6 pieces of aluminium but only 5 pieces of iron. Ashok's hypothesis is that the time, t seconds, taken for the top surface of each metal sample to reach 200 °C is inversely proportional to its surface area, A cm^2. The results of his experiment are shown in the tables below.
Aluminium samples:

A	2	4	6	8	10	12
t	58	29	20	15	12	10

Iron samples:

A	5	7	9	11	13
t	101	72	56	46	39

a. Display Ashok's results on the same set of axes.
b. Assuming that $t \propto \frac{1}{A}$, then the relationship between t and A can be modelled as $t = \frac{k}{A}$, where k is a constant and will have different values for aluminium and iron. Find an approximate value for k for:
i. aluminium
ii. iron.
Ashok repeats his experiment with samples of iron and aluminium that both have surface areas of 25 cm^2 .

c. Predict which metal sample would reach 200 °C first.
d. How much time would pass before the other metal sample also reached 200 °C?

11. For her science assignment, Rachel has to find the relationship between the intensity of the light, I, and the distance between the observer and the source of light, d. From the experiments she obtains the following results.

d	1	1.5	2	2.5	3	3.5	4
I	270	120	68	43	30	22	17

Rachel thinks that her data indicates that intensity of the light is inversely proportional to the distance between the observer and the light source. Her sister, Magda, suggests that the intensity is inversely proportional to the distance squared. Decide which sister is correct and explain the mathematical basis upon which you made your decision.

12. Apple pickers at a farm are paid $30 for each bin of apples that they fill. One average, a novice apple picker will fill 4 bins in an 8-hour day while experienced pickers will fill 10 bins in the same period of time.



a. Use this information to derive an equation in which a picker's hourly wage, W, is inversely proportional to the average time, T hours, that a picker takes to fill a bin.
b. How many bins does a picker fill over an 8-hour day if they earn $70/hour (round your answer to the nearest whole number).
c. The owner of the apple farm realises that the hardest thing about getting the apples picked is getting the pickers to turn up for work. As a result, he decides to offer all of the pickers $50 for turning up in the morning and then an hourly rate depending on their level of experience: novices will earn $10/hour, while experienced pickers will earn $40/hour. The pickers will still work an 8-hour day. Explain — using mathematical methods to support your response — how the change affects novice pickers.

4.4 The circle

The **circle** is an example of a many-to-many **relation**. A circle is not a function. A relation is a set of ordered pairs. All functions are relations but not all relations are functions.

4.4.1 Equation of a circle

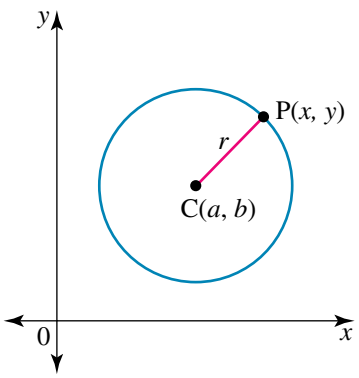
To obtain the equation of a circle, consider a circle of radius r and centre at the point C (a, b).

Let P (x, y) be any point on the circumference. CP, of length r, is the radius of the circle.

Using the formula for the distance between two points:

$$\sqrt{(x-a)^2+(y-b)^2} = \text{CP} = r$$

$$(x-a)^2+(y-b)^2 = r^2$$



The equation of a circle with centre (h, k) and radius r is:

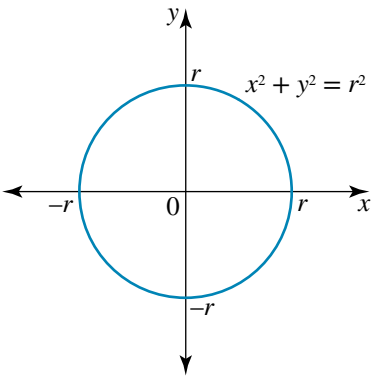
$$(x-a)^2 + (y-b)^2 = r^2.$$

The endpoints of the horizontal diameter have coordinates $(a-r, b)$ and $(a+r, b)$; the endpoints of the vertical diameter are $(a, b-r)$ and $(a, b+r)$. These points, together with the centre point, are usually used to sketch the circle. The intercepts with the coordinate axes are not always calculated.

The domain and range are obtained from the endpoints of the horizontal and vertical diameters.

The circle with the centre (a, b) and radius r has domain $[a-r, a+r]$ and range $[b-r, b+r]$.

If the centre is at $(0, 0)$, then the circle has equation $x^2 + y^2 = r^2$, with domain $[-r, r]$ and range $[-r, r]$.



WORKED EXAMPLE 7

Sketch the graphs of the following circles. State the domain and range of each.

a. $x^2 + (y-3)^2 = 1$ **b.** $(x+3)^2 + (y+2)^2 = 9$

THINK

a. 1. This circle has centre $(0, 3)$ and radius 1.

2. On a set of axes mark the centre and four points; 1 unit (the radius) left and right of the centre, and 1 unit (the radius) above and below the centre.

3. Draw a circle which passes through these four points.

4. State the domain.

5. State the range.

b. 1. This circle has centre $(-3, -2)$ and radius 3.

2. On a set of axes mark the centre and four points; 3 units left and right of the centre, and 3 units above and below the centre.

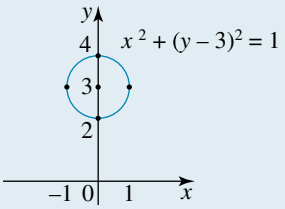
3. Draw a circle which passes through these four points.

4. State the domain.

5. State the range.

WRITE

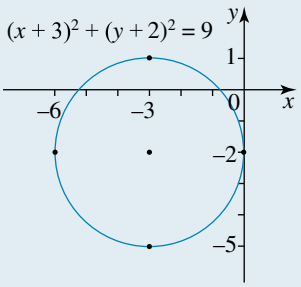
a.



Domain is $[-1, 1]$.

Range is $[2, 4]$.

b.



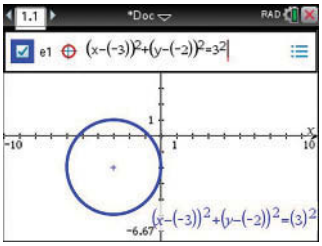
Domain is $[-6, 0]$.

Range is $[-5, 1]$.

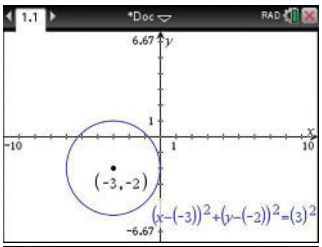
TI | THINK

WRITE

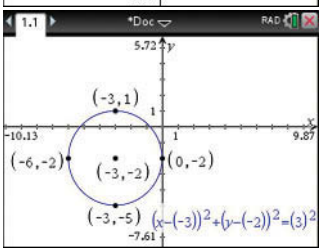
b.1. On a Graphs page, press MENU then select
3: Graph Entry/Edit
3: Equation Templates
3: Circle
1: Center form
$(x-h)^2+(y-k)^2=r^2$
Complete the entry line as
$(x-(-3))^2+(y-(-2))^2=3^2$
then press ENTER.



2. To find the coordinates of the center of the circle, press MENU then select:
6: Analyze Graph
8: Analyze Conics
1: Center
Click on the center of the circle, then press ENTER.



3. To plot maximum, minimum, leftmost and rightmost points on the graph, press MENU then select
5: Trace
1: Graph trace
Type '−3' then press ENTER twice. Use the left/right arrows to move to the opposite side of the circle then type '−3' again and press ENTER twice. Repeat this process to plot the points $(-6,-2)$ and $(0,-2)$.



4. The domain and range can be read from the graph.

The domain is $[-6, 0]$ and the range is $[-5, 1]$.

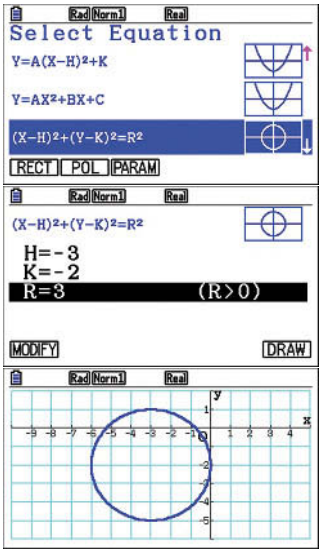
CASIO | THINK

WRITE

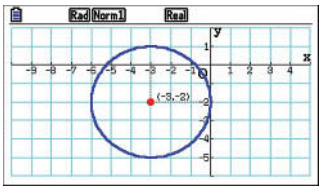
b.1. On a Conic Graphs screen, use the up/down arrows to select the equation for a circle, then press EXE.

Complete the fields as
H = −3
K = −2
R = 3
then press EXE.

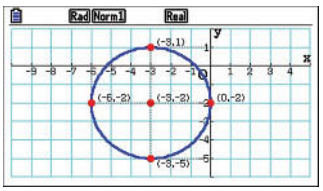
Select DRAW by pressing F6.



2. To find the coordinates of the center of the circle, select G-Solve by pressing F5, then select CENTER by pressing F1. Press EXE.



3. To plot maximum, minimum, leftmost and rightmost points on the graph, select Trace by pressing F1. Use the left/right arrows to move to the point $(-6,-2)$ then press EXE.
Repeat this process to plot the points $(0,-2)$, $(-3,-5)$ and $(-3,1)$.



4. The domain and range can be read from the graph.

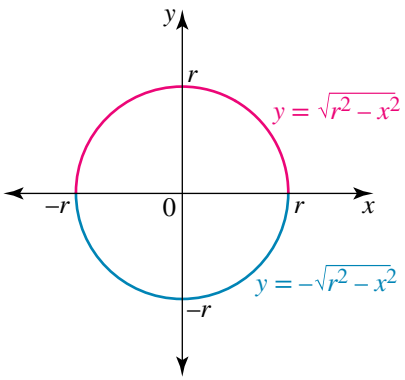
The domain is $[-6, 0]$ and the range is $[-5, 1]$.

4.4.2 Semicircles

The equation of the circle $x^2+y^2=r^2$ can be rearranged to make y the subject.

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

The equation of the circle can be expressed as $y = \pm\sqrt{r^2 - x^2}$. This form of the equation indicates two semicircle functions which together make up the whole circle.

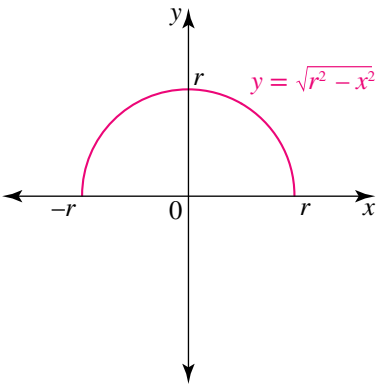


For $y = +\sqrt{r^2 - x^2}$, the y-coordinates must be positive (or zero) so this is the equation of the **semicircle** which lies above the x-axis.

For $y = -\sqrt{r^2 - x^2}$, the y-coordinates must be negative (or zero) so this is the equation of the semicircle which lies below the x-axis.

The semicircle $y = \sqrt{r^2 - x^2}$

The semicircle with equation $y = \sqrt{r^2 - x^2}$ is a function with a many-to-one correspondence. It is the top half of the circle, with centre (0, 0), radius r, domain $[-r, r]$ and range $[0, r]$.



The domain can be deduced algebraically since $\sqrt{r^2 - x^2}$ is only real if $r^2 - x^2 \geq 0$. From this the domain requirement $-r \leq x \leq r$ can be obtained.

For the circle with centre (a, b) and radius r, rearranging its equation $(x - a)^2 + (y - b)^2 = r^2$ gives the equation of the top, or upper, semicircle as $y = \sqrt{r^2 - (x - a)^2} + b$.

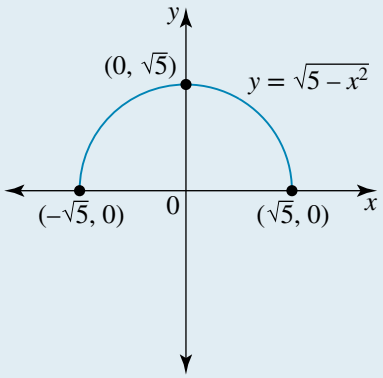
on Resources

Interactivity: Graph plotter: Circles, semicircles and regions (int-2571)

WORKED EXAMPLE 8

a. Sketch the graph of $y = \sqrt{5 - x^2}$ and state the domain and range.
b. For the circle with equation $4x^2 + 4y^2 = 1$, give the equation of its lower semicircle and state its domain and range.

THINK	WRITE
a. 1. State the centre and radius of the circle this semicircle is part of.	**a.** $y = \sqrt{5 - x^2}$ is the equation of a semicircle in the form of $y = \sqrt{r^2 - x^2}$. Centre: (0, 0) Radius: $r^2 = 5 \Rightarrow r = \sqrt{5}$ since r cannot be negative.
2. Sketch the graph.	This is an upper semicircle.
3. Read from the graph its domain and range.	Domain $[-\sqrt{5}, \sqrt{5}]$; range $[0, \sqrt{5}]$



b. 1. Rearrange the equation of the circle to make y the subject and state the equation of the lower semicircle.

b. $4x^2 + 4y^2 = 1$

Rearrange:

$4y^2 = 1 - 4x^2$

$$y^2 = \frac{1 - 4x^2}{4}$$

$$y = \pm\sqrt{\frac{1}{4} - x^2}$$

Therefore the lower semicircle has the equation $y = -\sqrt{\frac{1}{4} - x^2}$.

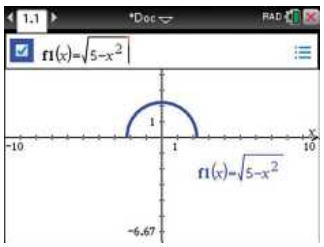
2. State the domain and range.

The domain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This is the lower semicircle, so the range is $\left[-\frac{1}{2}, 0\right]$.

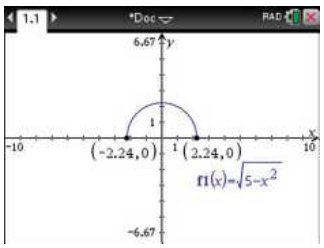
TI | THINK

a.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = \sqrt{5 - x^2}$ then press ENTER.

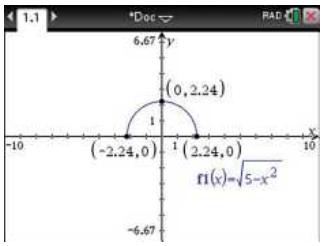
WRITE



2. To find the x-intercepts, press MENU then select:
6: Analyze Graph
1: Zero
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER. Move the cursor to the right of the x-intercept when prompted for upper bound, then press ENTER. Repeat this process to find the other x-intercept.



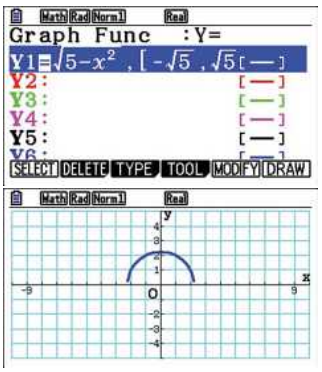
3. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound then, press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



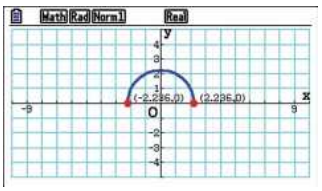
CASIO | THINK

a.1. On a Graph screen, complete the entry line for Y1 as $Y1 = \sqrt{5 - x^2}$, $[-\sqrt{5}, \sqrt{5}]$ then press EXE. Select DRAW by pressing F6.

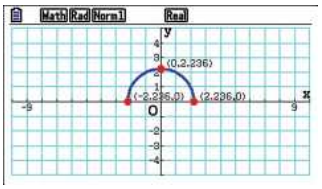
WRITE



2. To find the x-intercepts, select G-Solv by pressing F5, then select ROOT by pressing F1. Press EXE then use the left/right arrows to move across to the other x-intercept and press EXE.



3. To find the maximum, select G-Solv by pressing G5, then select MAX by pressing F2. Press EXE.



4. State the domain and range. Note: The calculator only displays approximate values for points of interest on the screen, however, exact values should be used when stating the domain and range.	The domain is $[-\sqrt{5}, \sqrt{5}]$ and the range is $[0, \sqrt{5}]$.

4. State the domain and range. Note: The calculator only displays approximate values for points of interest on the screen, however, exact values should be used when stating the domain and range.	The domain is $[-\sqrt{5}, \sqrt{5}]$ and the range is $[0, \sqrt{5}]$.

4.4.3 General form of the equation of a circle

The general form of the equation of a circle is the expanded form of $(x-a)^2+(y-b)^2=r^2$. Expanding gives $x^2+y^2-2ax-2by+a^2+b^2-r^2=0$. This shows that three pieces of information are needed to calculate a, b and r in order to determine the equation.

The general form is converted into the standard centre–radius form by completing the square both on the x terms and on the y terms.

WORKED EXAMPLE 9

Find the centre, radius, domain and range of the circle with equation $2x^2+2y^2+12x-4y+3=0$.

THINK	WRITE
1. Express the equation in the form where the coefficients of x^2 and y^2 are both 1.	$2x^2+2y^2+12x-4y+3=0$ Divide both sides by 2. $\therefore x^2+y^2+6x-2y+\frac{3}{2}=0$
2. Group the terms in x together and the terms in y together, and complete the squares.	$x^2+6x+y^2-2y=-\frac{3}{2}$ $(x^2+6x+9)-9+(y^2-2y+1)-1=-\frac{3}{2}$ $(x+3)^2+(y-1)^2=-\frac{3}{2}+9+1$ $(x+3)^2+(y-1)^2=\frac{17}{2}$
3. State the centre and radius.	Centre $(-3, 1)$; radius $\sqrt{\frac{17}{2}}=\frac{\sqrt{34}}{2}$
4. State the domain and range.	Domain $\left[-3-\frac{\sqrt{34}}{2}, -3+\frac{\sqrt{34}}{2}\right]$ Range $\left[1-\frac{\sqrt{34}}{2}, 1+\frac{\sqrt{34}}{2}\right]$

studyon

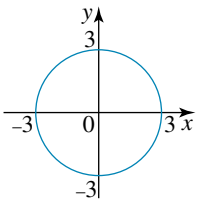
Units 1 & 2 > Area 2 > Sequence 3 > Concept 3

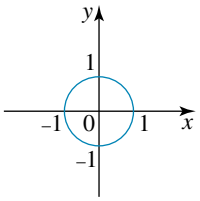
Equation of a circle Summary screen and practice questions

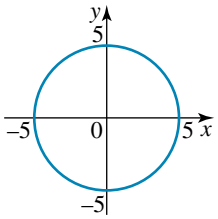
Exercise 4.4 The circle

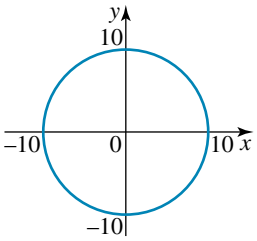
Technology free

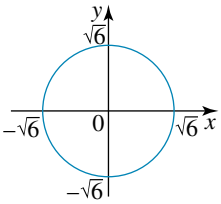
1. State the equation of each of the circles graphed below.

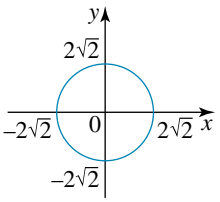
a.


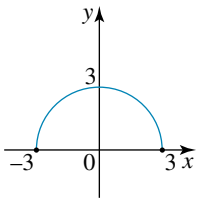
b.


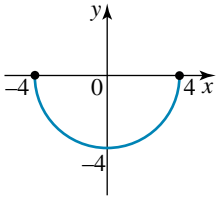
c.


d.


e.


f.


g.


h.


2. State the domain and range of each circle in question **1**.

3. Sketch the graph of each of the following relations.

a. $x^2 + y^2 = 4$

b. $x^2 + y^2 = 16$

c. $x^2 + y^2 = 49$

d. $x^2 + y^2 = 7$

e. $x^2 + y^2 = 12$

f. $x^2 + y^2 = \frac{1}{4}$

4. **WE7** Sketch the graph of the following circles. State the domain and range of each.

a. $x^2 + (y + 2)^2 = 1$

b. $x^2 + (y - 2)^2 = 4$

c. $(x - 4)^2 + y^2 = 9$

d. $(x - 2)^2 + (y + 1)^2 = 16$

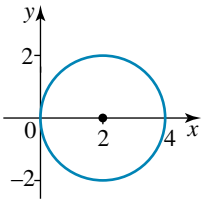
e. $(x + 3)^2 + (y + 2)^2 = 25$

f. $(x - 3)^2 + (y - 2)^2 = 9$

g. $(x + 5)^2 + (y - 4)^2 = 36$

h. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}$

5. **MC** Consider the circle below.



a. The equation of the circle is:

A. $x^2 + (y - 2)^2 = 4$

B. $(x - 2)^2 + y^2 = 16$

C. $(x + 2)^2 + y^2 = 16$

D. $(x - 2)^2 + y^2 = 4$

b. The range of the relation is:

A. R

B. $[-2, 2]$

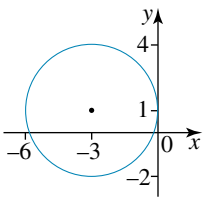
C. $[0, 4]$

D. $[2, 4]$

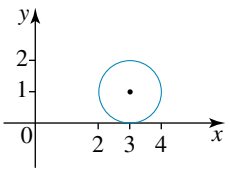
6. MC Consider the equation $(x+3)^2 + (y-1)^2 = 1$.

a. The graph which represents this relation is:

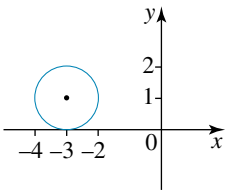
A.



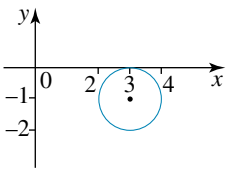
B.



C.



D.



b. The domain of the relation is:

A. $[-3.5, -2.5]$ **B.** $(-4, -2)$ **C.** R **D.** $[-4, -2]$

7. Classify each of the following equations as describing either a function (F) or a non-function (N).

a. $y = \pm\sqrt{81 - x^2}$ **b.** $y = \sqrt{4 - x^2}$

c. $y = -\sqrt{1 - x^2}$ **d.** $y = \sqrt{\frac{1}{9} - x^2}$

e. $y = -\sqrt{\frac{1}{4} - x^2}$ **f.** $y = \sqrt{5 - x^2}$

g. $y = \pm\sqrt{10 - x^2}$ **h.** $x^2 + y^2 = 3, -\sqrt{3} \leq x \leq 0$

8. WE8 **a.** Sketch the graph of $y = \sqrt{7 - x^2}$ and state the domain and range.

b. For the circle with equation $9x^2 + 9y^2 = 1$, give the equation of its upper semicircle and state its domain and range.

9. WE9 Sketch the following circles and state the centre, radius, domain and range of each.

a. $x^2 + (y-1)^2 = 1$

b. $(x+2)^2 + (y+4)^2 = 9$

c. $16x^2 + 16y^2 = 81$

d. $x^2 + y^2 - 6x + 2y + 6 = 0$

e. $16x^2 + 16y^2 - 16x - 16y + 7 = 0$

f. $(2x+6)^2 + (6-2y)^2 = 4$



10. Form the equations of the following circles from the given information.

a. Centre $(-8, 9)$; radius 6

b. Centre $(7, 0)$; radius $2\sqrt{2}$

c. Centre $(1, 6)$ and containing the point $(-5, -4)$

d. Endpoints of a diameter are $\left(-\frac{4}{3}, 2\right)$ and $\left(\frac{4}{3}, 2\right)$.

Technology active

11. A region is bounded by a circle described by the equation $(x-2)^2 + (y+4)^2 = 25$. Evaluate whether the following points lie (i) inside the circle, (ii) on the circle, or (iii) outside the circle.

a. $(2, 1)$ **b.** $(0, 0)$ **c.** $(1, 3)$ **d.** $(4, -3)$ **e.** $(5, 3)$

12. Write a general rule that can be used to determine if a point (m, n) lies inside the region bounded by the equation $(x-h)^2+(y-k)^2=r^2$.

13. **a.** Deduce the two values of a so that the point $(a, 2)$ lies on the circle $x^2+y^2+8x-3y+2=0$.

b. Identify the equation of the semicircle (of the given circle) on which these points lie.

14. The points $(0, 4)$ and $(4, 0)$ lie on the circle $x^2+y^2=16$. Decide whether it is possible to draw two other circles with different radii to this one that also pass through this pair of points. Explain your decision.

15. **a.** Calculate the coordinates of the points of intersection of the line $y=2x$ and the circle $(x-2)^2+(y-2)^2=1$.

b. Calculate the coordinates of the points of intersection of $y=7-x$ with the circle $x^2+y^2=49$. On a diagram, sketch the region $\{(x, y): y \geq 7-x\} \cap \{(x, y): x^2+y^2 \leq 49\}$.

16. Circular ripples are formed when a water drop hits the surface of a pond. If one ripple is represented by the equation $x^2+y^2=4$ and then 3 seconds later by $x^2+y^2=190$, where the length of measurements are in centimetres:



a. Find the radius (in cm) of the ripple in each case.

b. Calculate how fast the ripple is moving outwards. (State your answers to 1 decimal place.)

17. A circle passes through the three points $(1, 0)$, $(0, 2)$ and $(0, 8)$. The general equation of the circle is $x^2+y^2+ax+by+c=0$.

a. Calculate the values of a, b and c.

b. Determine the coordinates of the centre and the length of the radius.

c. Sketch the circle labelling all intercepts with the coordinate axes with their coordinates.

d. Calculate the shortest distance from the origin to the circle, giving your answer correct to 2 decimal places.

e. What is the greatest distance from the origin to the circle? Express the answer correct to 2 decimal places.

18. Consider the circle with equation $x^2+y^2-2x-4y-20=0$.

a. Calculate the exact length of the intercept, or chord, cut off on the x-axis by the circle.

b. Using clearly explained mathematical analysis, calculate the exact distance of the centre of the circle from the chord joining the points $(5, -1)$ and $(4, 6)$.

4.5 The sideways parabola

4.5.1 The relation $y^2 = x$

Here we shall consider a one-to-many relation (correspondence).

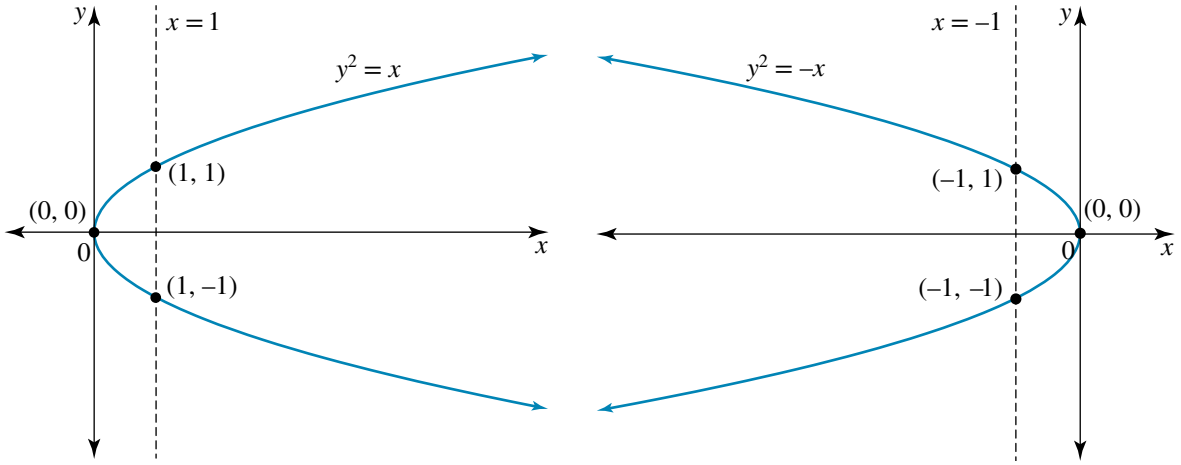
The relation $y^2=x$ cannot be a function since, for example, $x=1$ is paired with both $y=1$ and $y=-1$; the graph of $y^2=x$ therefore fails the vertical line test for a function.

The shape of the graph of $y^2=x$ could be described as a **sideways parabola** opening to the right, like the reflector in a car's headlight.

Key features of the graph of $y^2=x$ are:

- domain $R^+ \cup \{0\}$ with the graph opening to the right
- range R
- turning point, usually called a vertex, at $(0, 0)$
- axis of symmetry is horizontal with equation $y=0$ (the x-axis)
- one-to-many relation.

The graph of $y^2=-x$ will open to the left, with domain $R^- \cup \{0\}$.



Resources

Interactivity: Vertical and horizontal line test (int-2570)

4.5.2 Transformations of the graph of $y^2 = x$

Horizontal and vertical translations are identifiable from the coordinates of the vertex, just as they are for translations of the parabolic function $y = x^2$; the analysis of the curve is very similar to that applied to the parabolic function.

From the equation $(y - d)^2 = a(x - c)$ we can deduce:

- The vertex has coordinates (c, d), due to the horizontal and vertical translations c and d, respectively.
- The axis of symmetry has equation $y = d$.
- If $a > 0$, the graph opens to the right; if $a < 0$, it opens to the left.
- There is always one x-intercept obtained by substituting $y = 0$.
- There may be two, one or no y-intercepts, determined by substituting $x = 0$ and solving the resulting quadratic equation for y.

By considering the sign of a and the position of the vertex, it is possible to deduce whether or not there will be a y-intercept. If there is no y-intercept, this consideration can avoid wasted effort in attempting to solve a quadratic equation for which there are no real solutions.

If the equation of the graph is not given in the vertex form $(y - d)^2 = a(x - c)$, completing the square on the y terms may be necessary to transform the equation into this form.

Resources

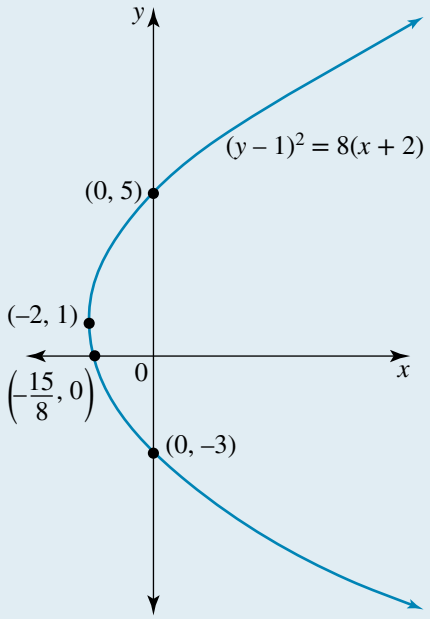
Interactivity: Graph plotter: $y^2 = x$ (int-2574)

WORKED EXAMPLE 10

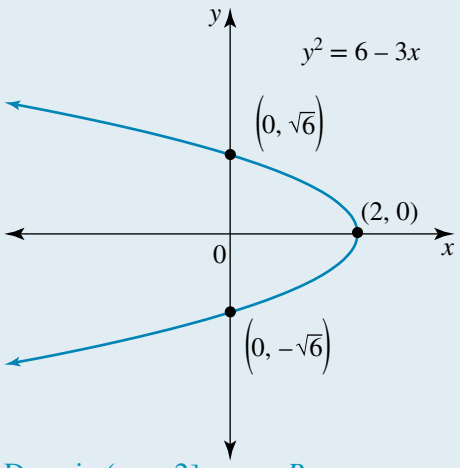
For each of the following relations, state the coordinates of the vertex and sketch the graph stating its domain and range.

a. $(y - 1)^2 = 8(x + 2)$

b. $y^2 = 6 - 3x$

THINK	WRITE
a. 1. State the coordinates of the vertex.	**a.** $(y-d)^2 = a(x-c)$ has vertex (c, d) $(y-1)^2 = 8(x+2)$ has vertex $(-2, 1)$
2. Calculate any intercepts with the axes.	x-intercept: let $y = 0$ $(-1)^2 = 8(x+2)$ $8x = -15$ $\therefore x = -\frac{15}{8}$ x-intercept $\left(-\frac{15}{8}, 0\right)$ y-intercepts: let $x = 0$ $(y-1)^2 = 16$ $y - 1 = \pm 4$ $\therefore y = -3$ or $y = 5$ y-intercepts $(0, -3)$, $(0, 5)$
3. Sketch the graph showing the key features and state the domain and range.	 Domain $[-2, \infty)$ and range R
b. 1. Express the equation in the form $(y-d)^2 = a(x-c)$ and state the vertex.	**b.** $y^2 = 6 - 3x$ $= -3(x-2)$ Vertex is $(2, 0)$.
2. Calculate any intercepts with the axes.	x-intercept is the vertex $(2, 0)$. y-intercepts: in $y^2 = 6 - 3x$, let $x = 0$ $y^2 = 6$ $\therefore y = \pm\sqrt{6}$ y-intercepts $(0, \pm\sqrt{6})$

3. Sketch the graph and state the domain and range.

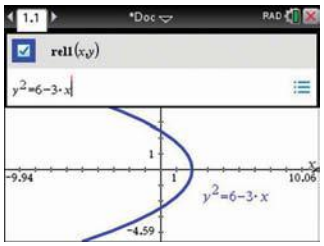


Domain $(-\infty, 2]$; range R

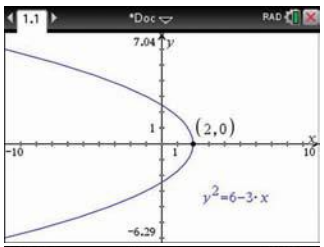
TI | THINK

b.1. On a Graphs page, press MENU then select
3: Graph Entry/Edit
2: Relation
Complete the entry line as $y^2 = 6 - 3x$
then press ENTER.

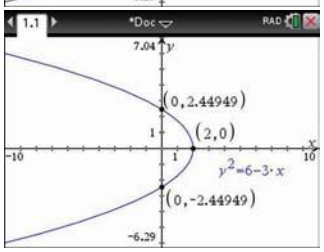
WRITE



2. To label the vertex, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



3. To find the y-intercepts, press MENU then select
5: Trace
1: Graph Trace
Use the left/right arrows to move the cursor to a y-intercept, then press ENTER. Repeat to find the other y-intercept.



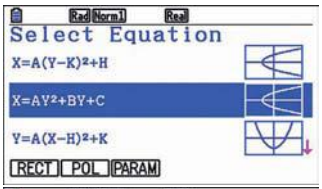
4. State the domain, range, and coordinates of the vertex.

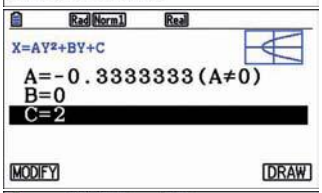
The domain is $(-\infty, 2]$, the range is R and the vertex is located at $(2, 0)$.

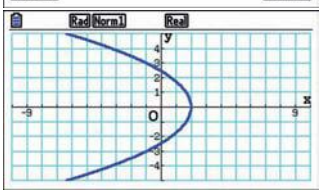
CASIO | THINK

b.1. On a Conic Graphs screen, select the second option for the equation of a sideways parabola, then press EXE.
Rearrange the given equation to make x the subject:
$y^2 = 6 - 3x \Rightarrow x = -\frac{1}{3}y^2 + 2$
Complete the fields as
A $= -1 \div 3$
B $= 0$
C $= 2$ then press EXE.
Select DRAW by pressing F6.

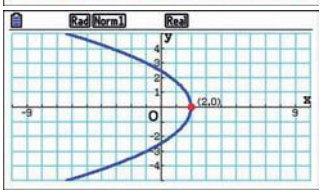
WRITE



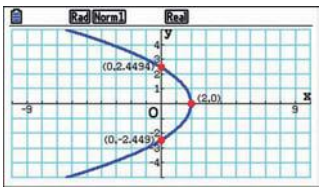




2. To label the vertex, select G-Solv by pressing F5, then select Vertex by pressing F4. Press EXE.



3. To find the y-intercepts, select G-Solv by pressing F5, press F6 to scroll across to more menu options, then select Y-ICEPT by pressing F2. Press EXE, then use the up/down arrows to move to the other y-intercept and press EXE.



4. State the domain, range, and coordinates of the vertex.

The domain is $(-\infty, 2]$, the range is R and the vertex is located at $(2, 0)$.

WORKED EXAMPLE 11

Express the equation $y^2 + 4y - 3x + 7 = 0$ in the form $(y - d)^2 = a(x - c)$, and hence state the coordinates of the vertex and the domain.

THINK

1. Complete the square on the y terms.
2. Read the coordinates of the vertex.
3. The coefficient of x is positive, so the graph opens to the right.

WRITE

$$y^2 + 4y - 3x + 7 = 0$$
$$(y^2 + 4y + 4) - 4 - 3x + 7 = 0$$
$$(y + 2)^2 - 3x + 3 = 0$$
$$(y + 2)^2 = 3x - 3$$
$$(y + 2)^2 = 3(x - 1)$$

The vertex is located at $(1, -2)$.

The domain is $[1, \infty)$.

4.5.3 Determining the rule for the sideways parabola

Since the most common form given for the equation of the sideways parabola is the vertex form $(y - d)^2 = a(x - c)$, once the coordinates of the vertex are known, a second point can be used to obtain the value of a. Other sets of three pieces of information and analysis could also determine the equation, including that the axis of symmetry lies midway between the y-intercepts.

WORKED EXAMPLE 12

a. Determine the equation of the relation with rule $(y - d)^2 = a(x - c)$ and vertex (3, 5) which passes through the point (5, 3).

b. Determine the equation of the sideways parabola which contains the three points (0, 0), (0, −4), (3, 2).

THINK

a. 1. Substitute the coordinates of the vertex into the general form of the equation.

2. Use the given point on the graph to determine the remaining unknown constant.

3. State the equation.

b. 1. Calculate the equation of the axis of symmetry. *Note:* An alternative approach would be to set up a system of 3 simultaneous equations using the coordinates of the 3 given points.

WRITE

a. $(y - d)^2 = a(x - f)$

Vertex (3, 5)

$\Rightarrow (y - 5)^2 = a(x - 3)$

Point (5, 3) is on the curve.

$\Rightarrow (3 - 5)^2 = a(5 - 3)$

$$4 = 2a$$
$$\therefore a = 2$$

The equation is $(y - 5)^2 = 2(x - 3)$.

b. Two of the given points, (0, 0) and (0, −4), lie on the y-axis, so the axis of symmetry lies midway between these two points.

	Equation of axis of symmetry is: $y = \frac{y_1 + y_2}{2}$ $= \frac{0 + (-4)}{2}$ $= -2$ $y = -2$ is the equation of the axis of symmetry.
2. Substitute the equation of the axis of symmetry into the general equation of a sideways parabola.	Let the equation be $(y - d)^2 = a(x - c)$. Axis of symmetry $y = -2$ $\therefore (y + 2)^2 = a(x - c)$
3. Use the third point and one of the y-intercepts to form a system of two simultaneous equations.	Substitute the point (0, 0). $(2)^2 = a(-c)$ $4 = -ac$ Substitute the point (3, 2). $(2 + 2)^2 = a(3 - c)$ $16 = a(3 - c)$ $= 3a - ac$
4. Solve the simultaneous equations to obtain a and c.	$4 = -ac$ [1] $16 = 3a - ac$ [2] Equation [2] – equation [1] $12 = 3a$ $a = 4$ Equation [1] $\Rightarrow c = -1$
5. State the answer.	The equation of the sideways parabola is $(y + 2)^2 = 4(x + 1)$.

studyon

Units 1 & 2 > Area 2 > Sequence 3 > Concept 4

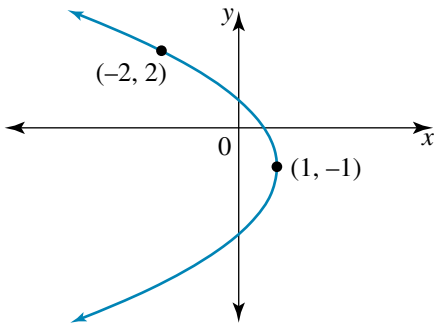
The sideways parabola Summary screen and practice questions

Exercise 4.5 The sideways parabola

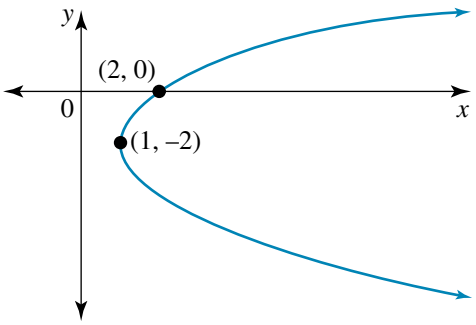
Technology free

1. Recall why the equation of the sideways parabola cannot be described as a function.
2. WE10 For each of the following relations state the coordinates of the vertex and sketch the graph, stating its domain and range.
   a. $(y + 3)^2 = 4(x - 1)$
   b. $(y - 3)^2 = -9x$
3. Express the relation given by $y^2 + 8y - 3x + 20 = 0$ in the form $(y - d)^2 = a(x - c)$ and hence state the coordinates of its vertex and the equation of its axis of symmetry.

4. WE12 **a.** Determine the equation of the relation with rule $(y-d)^2 = a(x-c)$ which passes through the point $(-10, 0)$ and has a vertex at $(4, -7)$.
   b. Determine the equation of the sideways parabola which contains the points $(0, 0)$, $(0, 6)$ and $(9, -3)$.
5. A sideways parabola touches the y-axis at $y = 3$ and cuts the x-axis at $x = 2$. Form the equation of the parabola.
6. Consider the relation $S = \{(x, y) : (y+2)^2 = 9(x-1)\}$.
   Determine the coordinates of its vertex and x-intercept and hence sketch its graph.
7. The relation $(y-a)^2 = b(x-c)$ has a vertex at $(2, 5)$ and cuts the x-axis at $x = -10.5$. Determine the values of a, b and c, and hence state the equation of the relation and its domain and range.
8. On the same diagram, sketch the graphs of $y^2 = x$, $y^2 = 4x$ and $y^2 = \frac{1}{4}x$ and comment on the effect of the change of the coefficient of the x-term.
9. Sketch the following, labelling the coordinates of the vertex and any axis intercepts.
   a. $(y+1)^2 = 3x$
   b. $9y^2 = x + 1$
   c. $(y+2)^2 = 8(x-3)$
   d. $(y-4)^2 = 2x + 1$
10. Sketch the following, stating the coordinates of the vertex and the exact coordinates of any intercept with the axes.
    a. $y^2 = -2x$
    b. $(y+1)^2 = -2(x-4)$
    c. $(6-y)^2 = -8 - 2x$
    d. $x = -(2y-6)^2$
11. WE11 Express the following equations in the form $(y-d)^2 = a(x-c)$ and hence state the coordinates of the vertex and the domain.
    a. $y^2 + 16y - 5x + 74 = 0$
    b. $y^2 - 3y + 13x - 1 = 0$
    c. $(5+2y)^2 = 8 - 4x$
    d. $(5-y)(1+y) + 5(x-1) = 0$
12. **a.** Form the equation of the graph of the parabola relation shown.



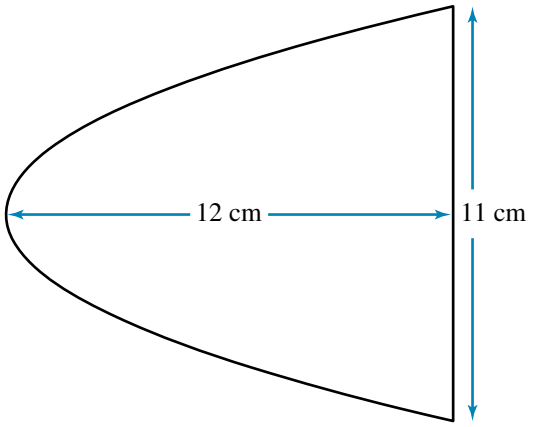
   b. Give a possible equation for the graph shown.



   c. A curve in the shape of a sideways parabola touches the y-axis and passes through the points $(1, 12)$ and $(1, -4)$.
      i. State the equation of its axis of symmetry.
      ii. Determine the equation of the curve.

d. The reflector in a car's headlight has a parabolic shape.





Placing the coordinate axes with the origin at the vertex of the parabola, form the equation of the parabola relative to these axes.

13. Consider the curve with equation $y^2 = -8x$.

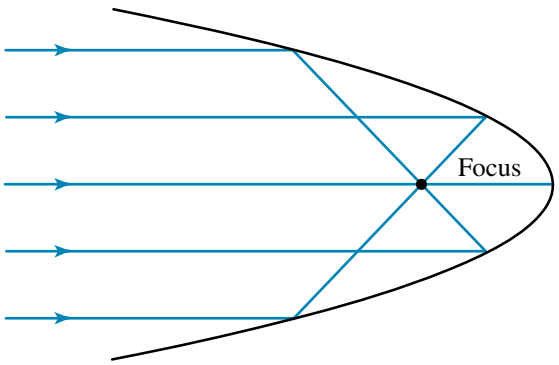
a. State the domain of the curve and show that the point $P\left(-3,\ 2\sqrt{6}\right)$ lies on the curve. Identify which branch of the curve it lies on.

b. Show that both the vertex V and the point $P\left(-3,\ 2\sqrt{6}\right)$ are at positions which are equidistant from the point F$(-2,\ 0)$ and the vertical line D with equation $x = 2$.

c. Q is a point on the other branch of the curve to P, where $x = a$, $a < 0$. Express the coordinates of Q in terms of a and show that Q is also equidistant from the point F$(-2,\ 0)$ and the vertical line D with equation $x = 2$.

d. A property of a parabola is that rays travelling parallel to its axis of symmetry are all reflected through a point called the focus. A radio telescope is designed on this principle so that signals received from outer space will be concentrated at its focus.

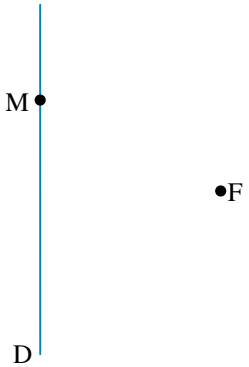




Consider the equation $y^2 = -8x$ as a two-dimensional model of a telescope dish. Its focus is the point F$(-2,\ 0)$. A signal, travelling parallel to the axis of symmetry, strikes the dish at the point $P\left(-3, 2\sqrt{6}\right)$ and is reflected through the focus F, striking the curve at point Q where $x = a$, $a < 0$. Calculate the value of a.

Technology active

14. A parabola can be defined as the path traced out by the set of points which lie in positions that are equidistant from a fixed point and a fixed line. Using Cabri or other geometry software, construct a vertical line and label it D; select a point to the right of the line D and label it F.



Construct the following:

- a point M on the line D
- the perpendicular line to D through M
- the perpendicular bisector of MF
- the intersection point P of these last two lines
- the locus of P as M moves along line D.

a. What shape is the locus path?

b. Erase the locus and create, then measure, the line segments FP and PM. What do you observe about the measurements? Move F and M to test whether your observation continues to hold. What conclusion can you form?

4.6 Review: exam practice

A summary of this chapter is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC If $f(x) = \frac{2}{x} + 1$, then $f(x) + 2$ will have:

A. the horizontal asymptote $y = 2$

B. the horizontal asymptote $y = 1$

C. the horizontal asymptote $y = 3$

D. the vertical asymptote $x = 2$.

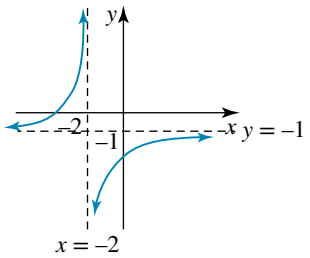
2. MC The equation of the graph shown is likely to be:

A. $y = -\frac{2}{x-2} - 1$

B. $y = 1 - \frac{2}{x+2}$

C. $y = -\frac{2}{x+1} - 2$

D. $y = \frac{-2}{x+2} - 1$



3. The graph of $y = \frac{1}{x}$ was dilated by the factor of 4 in the y direction, reflected in the x-axis and then translated 2 units to the left and 1 unit down.

a. State the equation of the asymptotes.

b. State the domain and range.

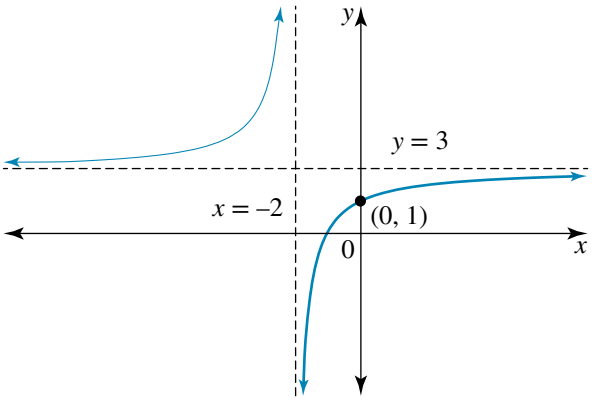
c. State the equation of the new graph.

d. Sketch the graph.

4. Sketch the graph of each of the following, clearly showing the position of the asymptotes and the intercepts with the axes.

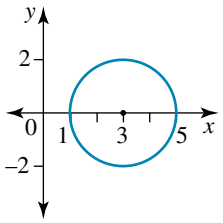
a. $y = \frac{2}{x-2}$ **b.** $y = -\frac{4}{x} - 1$ **c.** $y = \frac{2}{x-4} + 2$

5. Obtain the equation of the hyperbola shown in the diagram.



6. MC The equation of the circle shown is:

A. $(x+3)^2 + y^2 = 4$
B. $(x-3)^2 + y^2 = 2$
C. $(x+3)^2 + y^2 = 2$
D. $(x-3)^2 + y^2 = 4$



The circle with equation $(x+1)^2 + (y-4)^2 = 9$ applies to questions **7** and **8**.

7. MC The domain is:

A. $[-10, 8]$
B. $[-2, 4]$
C. $[-4, 2]$
D. $(-2, 4)$

8. MC The range is:

A. $[-7, -1]$
B. $[-5, 13]$
C. $[1, 7]$
D. $[-3, 3]$

9. MC A circle has its centre at $(4, -2)$ and a radius of $\sqrt{5}$ The equation of the circle is:

A. $(x-4)^2 + (y+2)^2 = 25$
B. $(x-4)^2 + (y+2)^2 = 5$
C. $(x+4)^2 + (y-2)^2 = 5$
D. $(x+4)^2 + (y-2)^2 = 25$

10. a. Sketch the graph of the relation $x^2 + y^2 = 100$.

b. From this relation form two one-to-one functions and state the domain and range of each.

11. Sketch each of the following:

a. $y = -\sqrt{4 - x^2}$
b. $x = (y+2)(y-4)$.

Complex familiar

12. The graph of $(y-c)^2 = a(x-b)$ has a vertex at $(-2, 5)$ and passes through the point $(6, 1)$. Determine the values of a, b and c.

13. a. Specify the centre and radius of the circle $2x^2 + 2y^2 - 12x + 8y + 3 = 0$.

b. At what values for x does the circle cross the line $y = 0$?

14. Determine the radius of the circle that is described by the equation $x^2 + y^2 - 8x + 4y + 11 = 0$.

Note: You may choose to use technology to answer questions **15**–**20**.

15. Determine the equation of the sideways parabola that contains the three points $(1, 5)$, $(3.5, 0)$, $(7, -1)$.

16. An eagle soars from the top of a cliff that is 48.4 metres above the ground and then descends towards unsuspecting prey below.



The eagle's height, h metres above the ground, at time t seconds can be modelled by the equation $h = 50 + \frac{a}{t - 25}$, where $0 \leq t < 25$ and a is a constant.

a. Determine the value of a.

b. Calculate the eagle's height above the ground after:

i. 5 seconds

ii. 20 seconds.

c. After how many seconds will the eagle reach the ground?

Complex unfamiliar

17. A circle described by the equation $(x - 2)^2 + (y - 2)^2 = 4$ is intersected at three points by a sideways parabola such that the parabola and the circle have the same axis of symmetry. Two of the intersection points lie at opposite ends of the circle's diameter while the third point is at the parabola's vertex. Determine the equation of the sideways parabola.

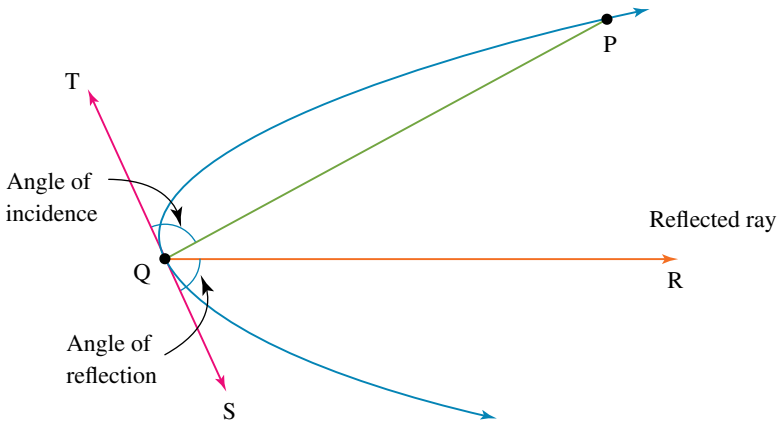
18. A small container with height, h cm, has a longitudinal cross-section in the shape of the curve with equation $h = \frac{2}{x - 2} - 1$, $2 < x \leq 4$, and its reflection in the line $x = 2$.

a. Sketch this container and state the diameter of its circular base.

b. If the diameter of the top of the container is 1 cm, calculate its height.

c. If the container is filled with a powder to a height of 1.5 cm, what is the surface area of the powder?

19. Light rays are reflected off parabolic mirrors in such a way that the angle of incidence the incoming ray of light makes with the tangent to the parabola is equal to the angle of departure made between the departing ray and the tangent. This is illustrated in the diagram shown, where TS is a tangent to the parabola at point Q, the incoming ray of light is PQ, the reflected ray is QR and the angles PQT and SQR are the angles of incidence and reflection respectively.



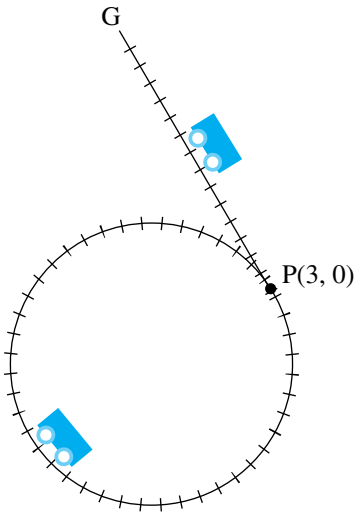
Let the equation of the parabola in the diagram be $y^2 = 4x$.

a. Show that P(9, 6) lies on the parabola.

b. Show that the line with equation $3y + 9x + 1 = 0$ is a tangent to the parabola and give the coordinates of the point Q, its point of contact with the parabola.

c. Calculate the magnitude of the angle PQT, the angle of incidence (angle of arrival) between the light ray PQ and the tangent at Q.

20. A toy train runs around a circular track that can be described by the equation $(x-1)^2 + \left(y+\frac{3}{2}\right)^2 = \frac{25}{4}$. There is also a straight track tangential to the circle at point P, which leads to a train depot at point G. All units are in metres.



a. Form the equation of the straight track PG.

b. A vertical line connects the depot at G with the centre of the circular track. Determine the coordinates of G.

c. One train engine continues to travel round and round the circular track at a speed of π m/s, while a second train engine continues to shunt backwards and forwards between P and G at a speed of 1 m/s. If the small child playing with the trains releases both engines at the same time from point P, how long does it take before they collide at P?

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 4 Inverse proportions and graphs of relations

Exercise 4.2 The hyperbola

1. **a.** Dilation in the y direction by a factor of 2

b. Dilation in the y direction by a factor of 3, reflection in the x-axis

c. Translation 6 units to the right

d. Dilation in the y direction by a factor of 2, translation 4 units to the left

e. Translation 7 units up

f. Dilation in the y direction by a factor of 2, translation 5 units down

g. Translation 4 units to the left, translation 3 units down

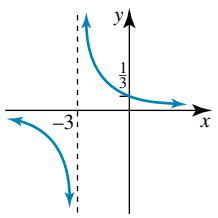
h. Dilation in the y direction by a factor of 2, translation 3 units to the right, translation 6 units up

i. Dilation in the y direction by a factor of 4, reflection in the x-axis, translation 1 unit to the right, translation 4 units down

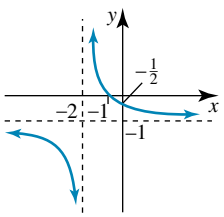
2. **a.** v **b.** iii **c.** i **d.** v, iii
e. v, ii, iii **f.** i, iii **g.** v, i, iv **h.** ii, iv

3. **a.** **i.** $x = 4, y = 0$ **ii.** Domain : $R\backslash\{4\}$ **iii.** Range : $R\backslash\{0\}$
b. **i.** $x = 0, y = 2$ **ii.** Domain : $R\backslash\{0\}$ **iii.** Range : $R\backslash\{2\}$
c. **i.** $x = 3, y = 2$ **ii.** Domain : $R\backslash\{3\}$ **iii.** Range : $R\backslash\{2\}$
d. **i.** $x = -1, y = -1$ **ii.** Domain : $R\backslash\{-1\}$ **iii.** Range : $R\backslash\{-1\}$
e. **i.** $x = m, y = n$ **ii.** Domain : $R\backslash\{m\}$ **iii.** Range : $R\backslash\{n\}$
f. **i.** $x = b, y = a$ **ii.** Domain : $R\backslash\{b\}$ **iii.** Range : $R\backslash\{a\}$

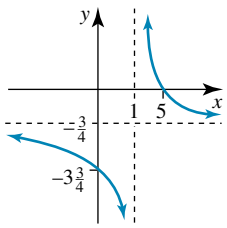
4. **a.**



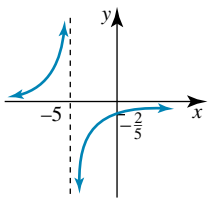
b.



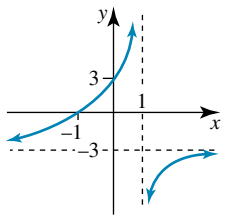
c.



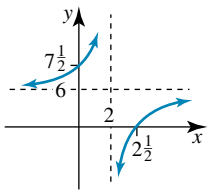
d.



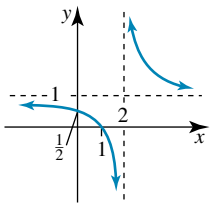
e.



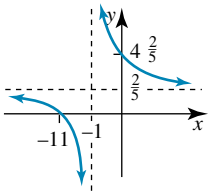
f.



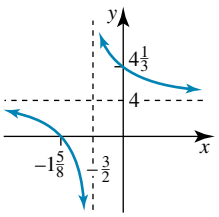
g.



h.

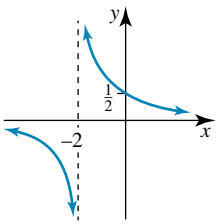


i.

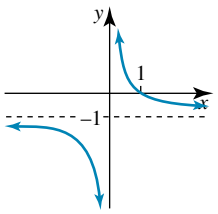


5. D

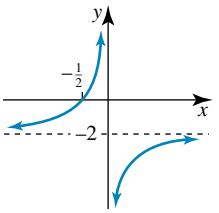
6. **a.**



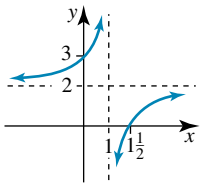
b.



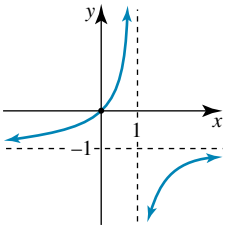
c.



d.



e.



7. **a.** Vertical asymptote $x = -\frac{2}{3}$; horizontal asymptote $y = 2$

b. $y = \frac{-1}{x-4} + \frac{1}{2}$

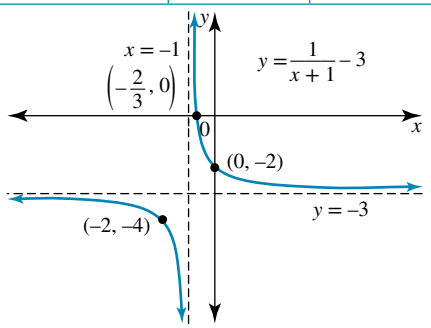
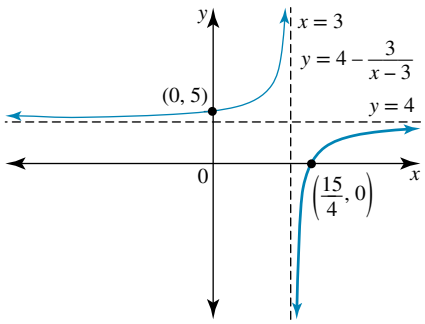
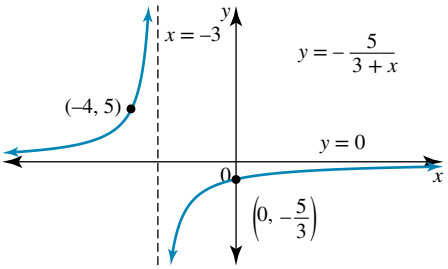
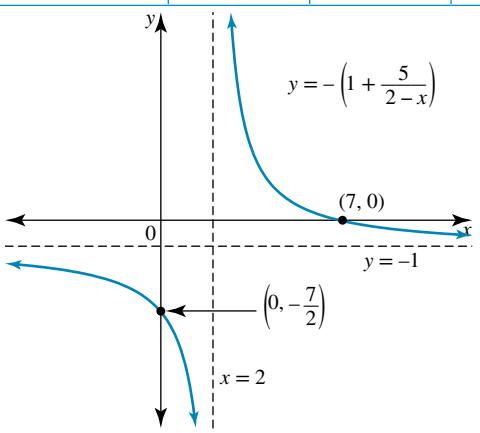
8. **a.** $x = -5, y = 2$

b. $x = 0, y = -3$

c. $x = 0, y = 0$

d. $x = -14, y = -\frac{3}{4}$

9.

	Asymptotes	y-intercept	x-intercept	Domain	Range	Point
a	$x = -1, y = -3$	$(0, -2)$	$\left(-\frac{2}{3}, 0\right)$	$R\backslash\{-1\}$	$R\backslash\{-3\}$	$(-2, -4)$
						
b	$x = 3, y = 4$	$(0, 5)$	$\left(\frac{15}{4}, 0\right)$	$R\backslash\{3\}$	$R\backslash\{4\}$	
						
c	$x = -3, y = 0$	$\left(0, -\frac{5}{3}\right)$	none	$R\backslash\{-3\}$	$R\backslash\{0\}$	$(-4, 5)$
						
d	$x = 2, y = -1$	$\left(0, -\frac{7}{2}\right)$	$(7, 0)$	$R\backslash\{2\}$	$R\backslash\{-1\}$	
						

10. a. $y = \dfrac{2}{x-3} + 1$ **b.** $y = \dfrac{-1.5}{x+3} + 1$

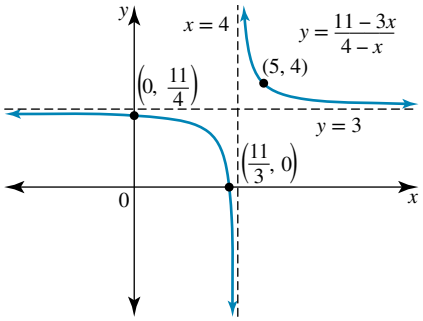
11. a. $y = \dfrac{\frac{3}{8}}{x - \frac{1}{4}} - \dfrac{1}{2} = \dfrac{3}{8x-2} - \dfrac{1}{2}$

b. $f: R \setminus \left\{\dfrac{1}{4}\right\} \to R,\ f(x) = \dfrac{3}{8x-2} - \dfrac{1}{2}$

12. Domain $R\setminus\{4\}$; range $R\setminus\{0\}$

13. a. $a = 3;\ b = 1$

b.



c. $x < \dfrac{11}{3}$ or $x > 4$

14. a. $y = \dfrac{-1}{16x+4} + \dfrac{1}{4};\ x = -\dfrac{1}{4};\ y = \dfrac{1}{4}$

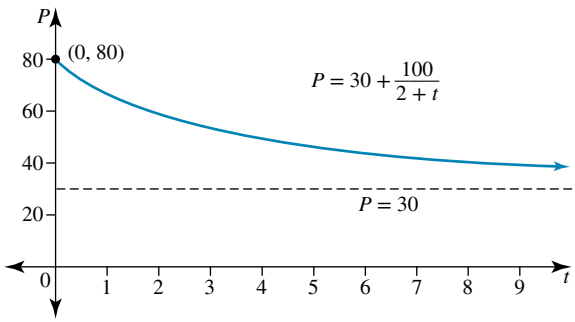
b. $y = \dfrac{4}{x-4} - 2;\ x = 4;\ y = -2$

c. $y = \dfrac{1}{x} + 2;\ x = 0;\ y = 2$

d. $y = \dfrac{-2}{2x+3};\ x = -\dfrac{3}{2};\ y = 0$

15. a. Reduced by 25 cattle

b. Domain $\{t : t \geq 0\}$; range $(30,\ 80)$



c. The number of cattle will never go below 30.

16. a. $a = 460;\ b = 40$

b. 12 years 8 months

c. Sample responses can be found in the worked solutions in the online resources.

d. Increases by approximately 5 insects in 12th year and 3 in 14th year; growth is slowing.

e. Never reaches 500 insects

f. Cannot be larger than 460

17. a. Asymptotes $x = 0$, $y = 0$ and $y = -x$, $y = x$

b. For the graph of $xy = 1$, rotate the axes 45° anticlockwise.

18. a. Two intersections

b. One intersection if $k = 1$ or $k = 5$; two intersections if $k < 1$ or $k > 5$; no intersections if $1 < k < 5$

Exercise 4.3 Inverse proportion

1. A

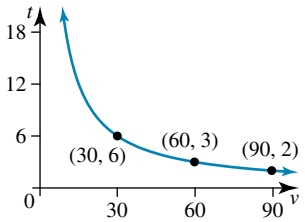
2. When $c = 0$ and $d = 0$

3. Inversely: a, c, d; directly: b, e

4. d; $y = \dfrac{5}{x}$

5. a. The constant is the distance; $k = 180$

b.



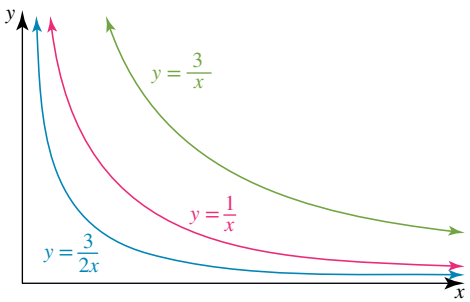
c. 80 km/h

6. a. C

b. i. $M = \dfrac{T}{4Lf^2}$;

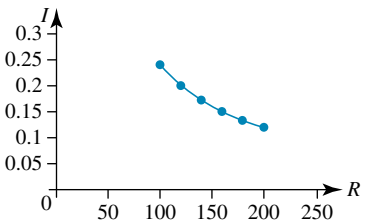
ii. No, M is inversely proportional to f^2.

7.



The constant of proportionality affects the steepness of the graph.

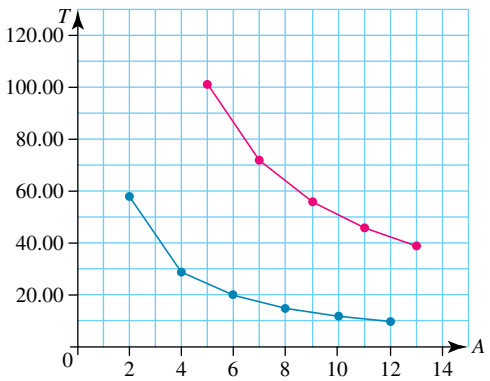
8. a.



b. 24

9. a. $f = \dfrac{340}{\lambda}$ **b.** 0.85 m

10. a.



b. i. $k \approx 120$; **ii.** $k \approx 505$

c. aluminium

d. 15.4 seconds

11. Magda is correct.

12. a. $W = \dfrac{30}{T}$

b. 19 bins

c. Under the old scheme, a novice picker would earn \$120 $(4 \text{ bins/day} \times \$30/\text{bin})$ for an 8-hour day of picking. Under the new scheme, a novice picker will still only fill 4 bins in an 8-hour day but now they are paid \$50 + (\$10/hour × 8 hours/day) = \$130 for an 8-hour work day. Therefore, the novice pickers will be better off under the new scheme.

Exercise 4.4 The circle

1. a. $x^2 + y^2 = 9$ **b.** $x^2 + y^2 = 1$ **c.** $x^2 + y^2 = 25$

d. $x^2 + y^2 = 100$ **e.** $x^2 + y^2 = 6$ **f.** $x^2 + y^2 = 8$

g. $y = \sqrt{9 - x^2}$ **h.** $y = -\sqrt{16 - x^2}$

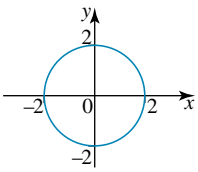
2. a. Both $[-3, 3]$ **b.** Both $[-1, 1]$

c. Both $[-5, 5]$ **d.** Both $[-10, 10]$

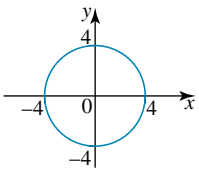
e. Both $[-\sqrt{6}, \sqrt{6}]$ **f.** Both $[-2\sqrt{2}, 2\sqrt{2}]$

g. $[-3, 3]$, $[0, 3]$ **h.** $[-4, 4]$, $[-4, 0]$

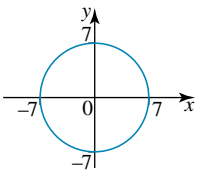
3. a.



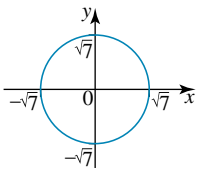
b.



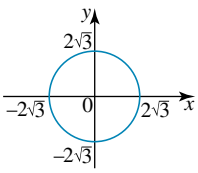
c.



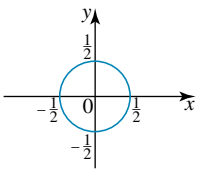
d.



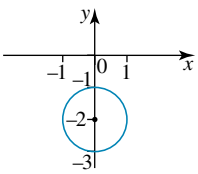
e.



f.

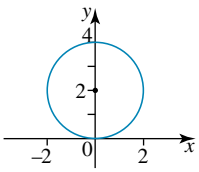


4. a.



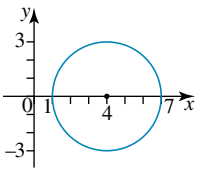
$[-1, 1]$ and $[-3, -1]$

b.



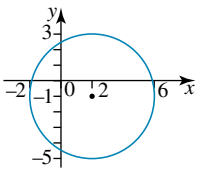
$[-2, 2]$ and $[0, 4]$

c.



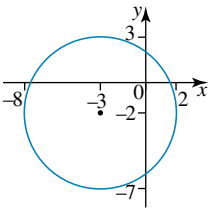
$[1, 7]$ and $[-3, 3]$

d.



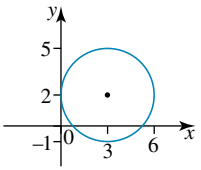
$[-2, 6]$ and $[-5, 3]$

e.



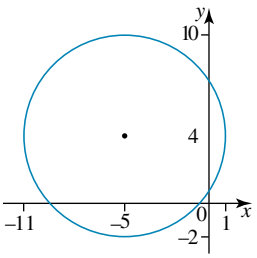
$[-8, 2]$ and $[-7, 3]$

f.



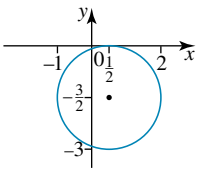
$[0, 6]$ and $[-1, 5]$

g.



$[-11, 1]$ and $[-2, 10]$

h.



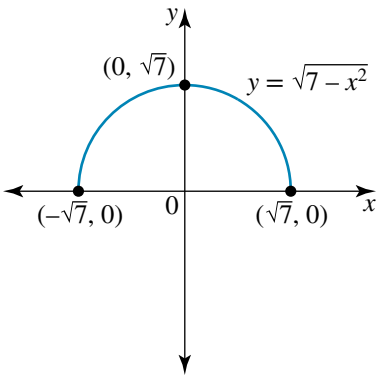
$[-1, 2]$ and $[-3, 0]$

5. a. D **b.** B

6. a. C **b.** D

7. F: b, c, d, e, f
N: a, g, h

8. a.



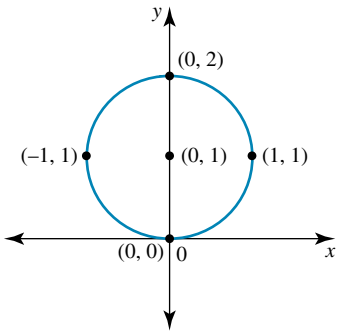
Domain $\left[-\sqrt{7}, \sqrt{7}\right]$; range $\left[0, \sqrt{7}\right]$

b. $y = \sqrt{\dfrac{1}{9} - x^2}$

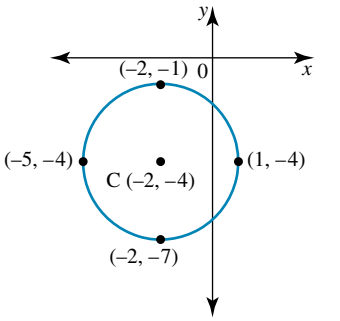
Domain $\left[-\dfrac{1}{3}, \dfrac{1}{3}\right]$; range $\left[0, \dfrac{1}{3}\right]$

9.

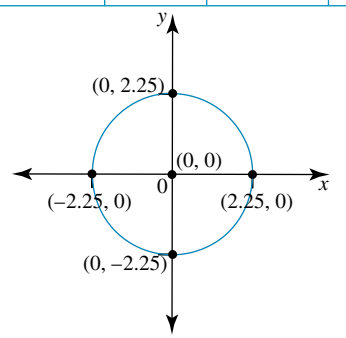
	Centre	Radius	Domain	Range
a	(0, 1)	1	[−1, 1]	[0, 2]



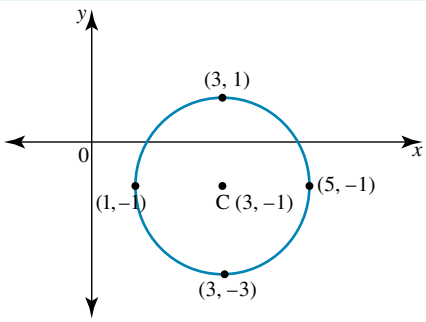
b	(−2, −4)	3	[−5, 1]	[−7, −1]



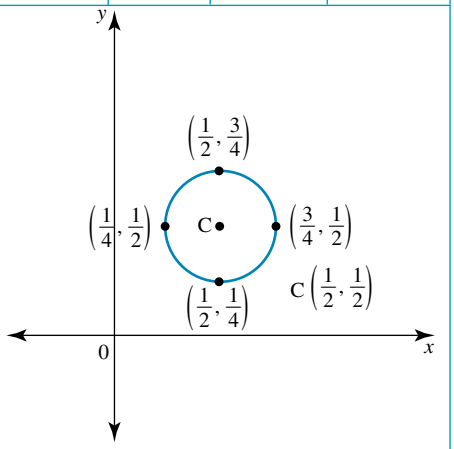
c	(0, 0)	$\frac{9}{4}$	$\left[-\frac{9}{4}, \frac{9}{4}\right]$	$\left[-\frac{9}{4}, \frac{9}{4}\right]$



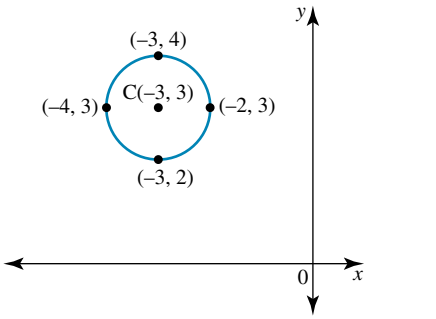
d	(3, −1)	2	[1, 5]	[−3, 1]



	Centre	Radius	Domain	Range
e	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{4}$	$\left[\frac{1}{4}, \frac{3}{4}\right]$	$\left[\frac{1}{4}, \frac{3}{4}\right]$



f	(−3, 3)	1	[−4, −2]	[2, 4]



10. **a.** $(x + 8)^2 + (y - 9)^2 = 36$
b. $(x - 7)^2 + y^2 = 8$
c. $(x - 1)^2 + (y - 6)^2 = 136$
d. $9x^2 + 9(y - 2)^2 = 16$

11. **a.** on **b.** outside **c.** outside **d.** inside **e.** outside

12. (m, n) lies inside the region of the circle if $(m - h)^2 + (n - k)^2 < r^2$

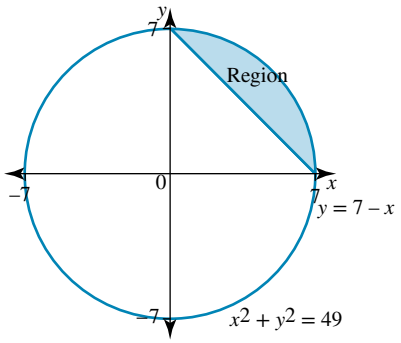
13. **a.** $a = 0$ and $a = -8$

b. $\therefore y = \sqrt{\frac{65}{4} - (x + 4)^2} + \frac{3}{2}$

14. Yes. Sample responses can be found in the worked solutions in the online resources.

15. **a.** $\left(\frac{7}{5}, \frac{14}{5}\right)$ and (1,2).

b. (0, 7) , (7, 0); region is inside circle and above the line (boundaries included)

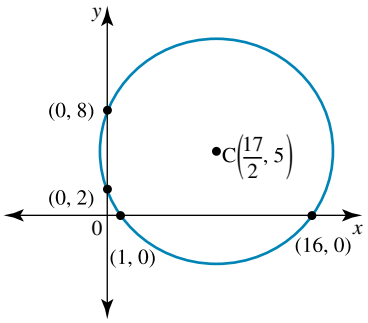


16. a. 2 cm, 13.8 cm **b.** 3.9 cm/s

17. a. $a = -17$, $b = -10$, $c = 16$

b. Centre $\left(\frac{17}{2}, 5\right)$; radius $= \frac{5\sqrt{13}}{2}$

c. x-intercepts $(1, 0)$, $(16, 0)$; y-intercepts $(0, 2)$, $(0, 8)$



d. 0.85 units

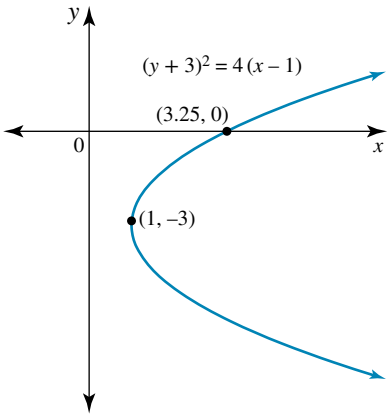
e. 18.88 units

18. a. $2\sqrt{21}$ **b.** $\frac{5\sqrt{2}}{2}$

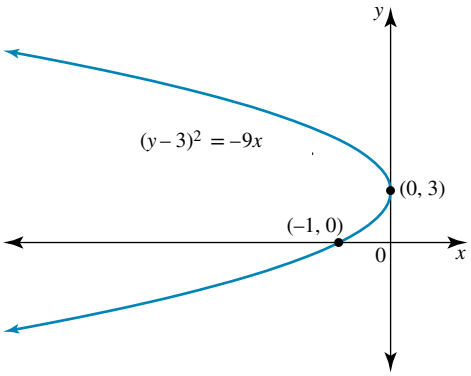
Exercise 4.5 The sideways parabola

1. It is not a function as each value of x maps to more than one value of y.

2. a. Vertex $(1, -3)$



b. Vertex $(0, 3)$



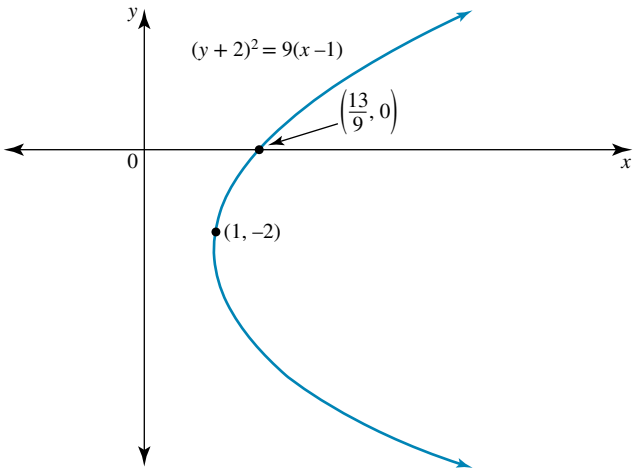
3. $(y + 4)^2 = 3\left(x - \frac{4}{3}\right)$; vertex $\left(\frac{4}{3}, -4\right)$; axis of symmetry $y = -4$

4. a. $(y + 7)^2 = -\frac{7}{2}(x - 4)$

b. $(y - 3)^2 = 3(x + 3)$

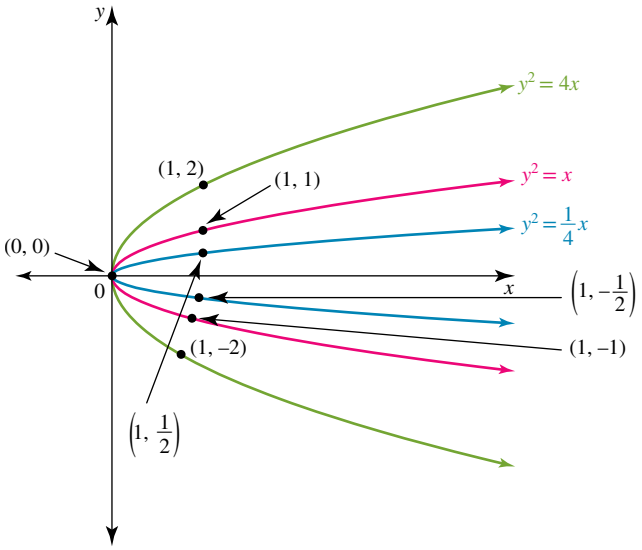
5. $(y - 3)^2 = \frac{9}{2}x$

6. Vertex $(1, -2)$; x-intercept $\left(\frac{13}{9}, 0\right)$



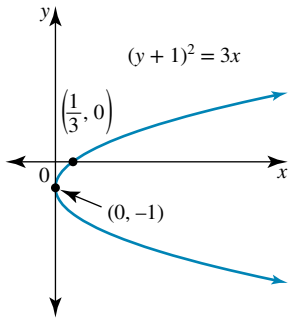
7. $a = 5$, $b = -2$, $c = 2$; $(y - 5)^2 = -2(x - 2)$; domain $(-\infty, 2]$; range R

8. Increasing the coefficient of x makes the graph wider (in the y direction) or more open.

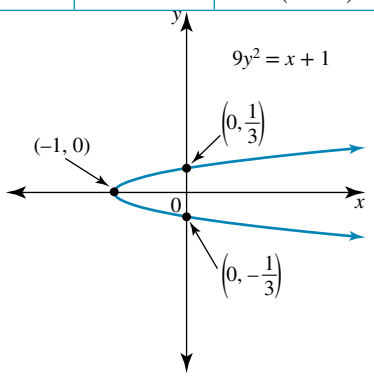


9.

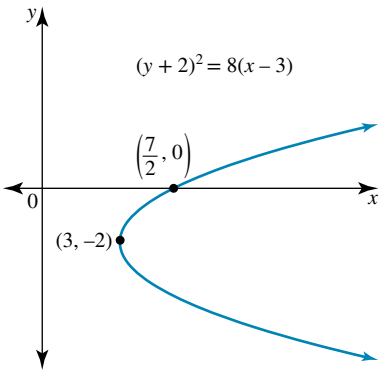
	Vertex	*x*-intercept	*y*-intercepts
a	$(0, -1)$	$\left(\frac{1}{3}, 0\right)$	$(0, -1)$



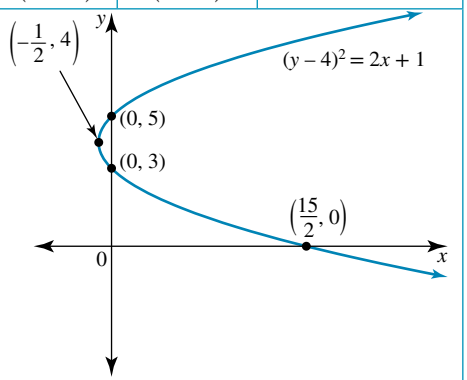
b	$(-1, 0)$	$(-1, 0)$	$\left(0, \pm\frac{1}{3}\right)$



c	$(3, -2)$	$\left(\frac{7}{2}, 0\right)$	none

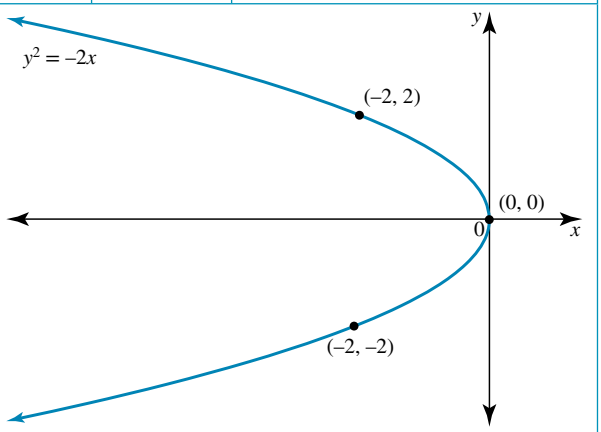


d	$\left(-\frac{1}{2}, 4\right)$	$\left(\frac{15}{2}, 0\right)$	$(0, 3), (0, 5)$

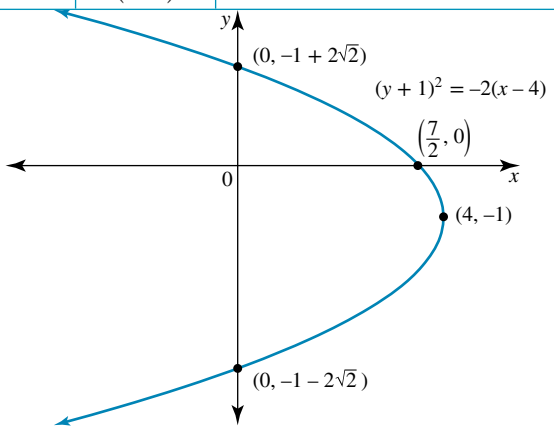


10.

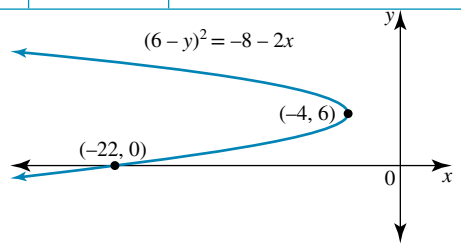
	Vertex	*x*-intercept	*y*-intercepts
a	$(0, 0)$	$(0, 0)$	$(0, 0)$



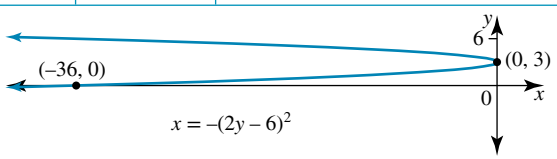
b	$(4, -1)$	$\left(\frac{7}{2}, 0\right)$	$(0, -1 \pm 2\sqrt{2})$



c	$(-4, 6)$	$(-22, 0)$	none



d	$(0, 3)$	$(-36, 0)$	$(0, 3)$



11. a. $(y+8)^2 = 5(x-2)$; vertex $(2, -8)$; domain $[2, \infty)$

b. $\left(y-\frac{3}{2}\right)^2 = -13\left(x-\frac{1}{4}\right)$; vertex $\left(\frac{1}{4}, \frac{3}{2}\right)$; domain $\left(-\infty, -\frac{1}{4}\right]$

c. $\left(y+\frac{5}{2}\right)^2 = -(x-2)$; vertex $\left(2, -\frac{5}{2}\right)$; domain $(-\infty, 2]$

d. $(y-2)^2 = 5\left(x+\frac{4}{5}\right)$; vertex $\left(-\frac{4}{5}, 2\right)$; domain $\left[-\frac{4}{5}, \infty\right)$

12. a. $(y+1)^2 = -3(x-1)$

b. $(y+2)^2 = 4(x-1)$

c. i. $y = 4$ **ii.** $(y-4)^2 = 64x$

d. $y^2 = \frac{121}{48}x$

13. a. Domain $(-\infty, 0]$; P lies on upper branch

b. V is 2 units from both F and line D and P is 5 units from F and line D.

c. $(a, -\sqrt{-8a})$; Q is $2 - a$ units from F and line D

d. $a = -\frac{4}{3}$

14. a. A sideways parabola opening to the right

b. Distances are equal. Any point on the parabola is equidistant from the point F and the line D.

4.6 Review: exam practice

Simple familiar

1. C

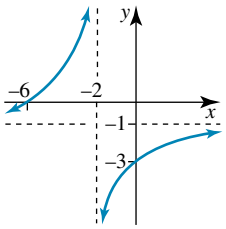
2. D

3. a. $x = -2, y = -1$

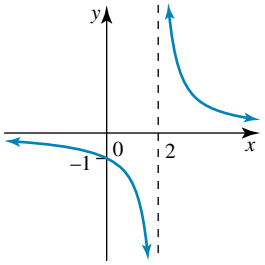
b. Domain: $R\backslash\{-2\}$, range: $R\backslash\{-1\}$

c. $y = \frac{-4}{x+2} - 1$

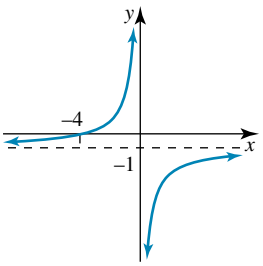
d.



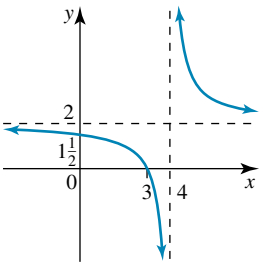
4. a.



b.



c.



5. $y = 3 - \frac{4}{x+2}$

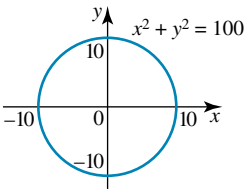
6. D

7. C

8. C

9. B

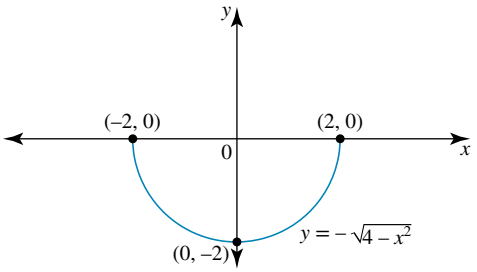
10. a.



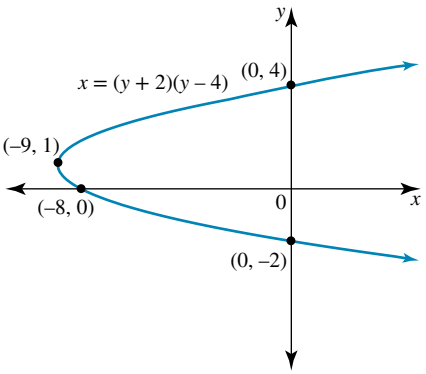
b. $f_1 : [-10, 10] \to R, f(x) = \sqrt{(100 - x^2)}$ with Domain $f = [-10, 10]$, range $f = [0, 10]$ and

$f_2 : [-10, 10] \to R, f(x) = -\sqrt{(100 - x^2)}$ with Domain $f = [-10, 10]$, range $f = [-10, 0]$

11. a.



b.



Complex familiar

12. $a = 2$, $b = -2$, $c = 5$

13. a. $(3, -2)$; $\sqrt{\frac{23}{2}}$

b. $x = 3 + \sqrt{\frac{15}{2}}$ and $x = 3 - \sqrt{\frac{15}{2}}$

14. 3 units

15. $(y - 3)^2 = 2(x + 1)$

16. a. $a = 40$

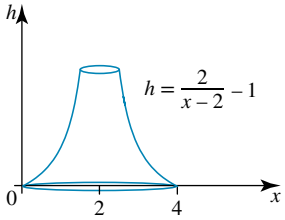
b. i. 48 m **ii.** 42 m

c. 24.2 s

Complex unfamiliar

17. $(y - 2)^2 = 2x$ or $(y - 2)^2 = -2(x - 4)$

18. a. Diameter 4 cm



b. Height 3 cm

c. $r = 0.8$ cm, $A = 0.64\pi$ cm^2

19. a. Let $y = 6$

$36 = 4x$

$x = 9$

Hence, P(9, 6) lies on the sideways parabola.

b. One point of intersection, $Q\left(\frac{1}{9}, -\frac{2}{3}\right)$

c. 71.6°

20. a. $y = -\frac{4}{3}x + 4$

b. $\left(1, \frac{8}{3}\right)$

c. 20 seconds

CHAPTER 5
Powers and polynomials

5.1 Overview

5.1.1 Introduction



A polynomial is an algebraic expression, usually with several terms. Each term can be a variable, a constant or a combination of both. Polynomials are used widely in many areas of mathematics and science. Mathematicians were able to solve second degree polynomial equations (quadratic equations) in 400 BCE. However, it wasn't until the early 16th century that progress was made on solving equations involving higher powers of x. In 1732 Leonhard Euler devised a general method for solving equations of the type $ax^3 + bx^2 + cx + d = 0$ and soon after Lodovico Ferrari found a general solution to fourth degree equations of the type $ax^4 + bx^3 + cx^2 + dx + e = 0$. No general formula to find all the roots of any 5th degree or higher equation exists, but various special solution techniques have been developed over the years. The advent of computers and calculators has provided mathematicians with new approaches to solving higher order polynomials.

Many important developments in mathematics, engineering, science and medicine have occurred through the use of polynomial functions. Polynomial modelling functions play an important part in solving real life problems. For example, they are used to design roller coasters, roads, buildings and other structures, or to describe and predict traffic patterns. Economists use polynomial functions to predict growth patterns, medical researchers use them to describe the growth of bacterial colonies, and meteorologists used them to understand weather patterns.

LEARNING SEQUENCE

5.1 Overview
5.2 Polynomials
5.3 Graphs of cubic polynomials
5.4 The factor and remainder theorems
5.5 Solving cubic equations
5.6 Cubic models and applications
5.7 Graphs of quartic polynomials
5.8 Solving polynomial equations
5.9 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.2 Polynomials

A **polynomial** is an algebraic expression in which the power of the variable is a positive whole number. For example, $3x^2 + 5x - 1$ is a quadratic polynomial in the variable x but $\frac{3}{x^2} + 5x - 1$, i.e. $3x^{-2} + 5x - 1$, is not a polynomial because of the $3x^{-2}$ term. Similarly, $\sqrt{3}x + 5$, is a linear polynomial but $3\sqrt{x} + 5$, that is, $3x^{\frac{1}{2}} + 5$, is not a polynomial because the power of the variable x is not a whole number. Note that the coefficients of x, x^2 etc., can be positive or negative integers, rational or irrational real numbers.

5.2.1 Classification of polynomials

- The **degree of a polynomial** is the highest power of the variable.
  For example, linear polynomials have degree 1, quadratic polynomials have degree 2 and cubic polynomials have degree 3.
- The **leading term** is the term containing the highest power of the variable.
- If the coefficient of the leading term is 1 then the polynomial is said to be monic.
- The **constant term** is the term that does not contain the variable.

A polynomial of degree n has the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $n \in N$ and the coefficients $a_n, a_{n-1}, \ldots a_1, a_0 \in R$. The leading term is $a_n x^n$ and the constant term is a_0.

WORKED EXAMPLE 1

Select the polynomials from the following list of algebraic expressions and for these polynomials, state the degree, the coefficient of the leading term, the constant term and the type of coefficients.

A. $5x^3 + 2x^2 - 3x + 4$ **B.** $5x - x^3 + \frac{x^4}{2}$ **C.** $4x^5 + 2x^2 + 7x^{-3} + 8$

THINK	WRITE
1. Check the powers of the variable x in each algebraic expression.	A and B are polynomials since all the powers of x are positive integers. C is not a polynomial due to the $7x^{-3}$ term.
2. For polynomial A, state the degree, the coefficient of the leading term and the constant term.	Polynomial A: the leading term of $5x^3 + 2x^2 - 3x + 4$ is $5x^3$. Therefore, the degree is 3 and the coefficient of the leading term is 5. The constant term is 4.
3. Classify the coefficients of polynomial A as elements of a subset of R.	The coefficients in polynomial A are integers. Therefore, A is a polynomial over Z.
4. For polynomial B, state the degree, the coefficient of the leading term and the constant term.	Polynomial B: the leading term of $5x - x^3 + \frac{x^4}{2}$ is $\frac{x^4}{2}$. Therefore, the degree is 4 and the coefficient of the leading term is $\frac{1}{2}$. The constant term is 0.
5. Classify the coefficients of polynomial B as elements of a subset of R.	The coefficients in polynomial B are rational numbers. Therefore, B is a polynomial over Q.

5.2.2 Polynomial notation

- The polynomial in variable x is often referred to as $P(x)$.
- The value of the polynomial $P(x)$ when $x = a$ is written as $P(a)$.
- $P(a)$ is evaluated by substituting a in place of x in the $P(x)$ expression.

WORKED EXAMPLE 2

a. If $P(x) = 5x^3 + 2x^2 - 3x + 4$ calculate $P(-1)$.
b. If $P(x) = ax^2 - 2x + 7$ and $P(4) = 31$, obtain the value of a.

THINK	WRITE
a. Substitute -1 in place of x and evaluate.	**a.** $P(x) = 5x^3 + 2x^2 - 3x + 4$ $P(-1) = 5(-1)^3 + 2(-1)^2 - 3(-1) + 4$ $= -5 + 2 + 3 + 4$ $= 4$
b. 1. Find an expression for $P(4)$ by substituting 4 in place of x, and then simplify.	**b.** $P(x) = ax^2 - 2x + 7$ $P(4) = a(4)^2 - 2(4) + 7$ $= 16a - 1$
2. Equate the expression for $P(4)$ with 31.	$P(4) = 31$ $\Rightarrow 16a - 1 = 31$
3. Solve for a.	$16a = 32$ $a = 2$

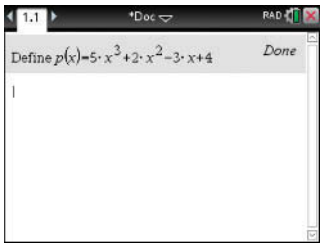
TI | THINK

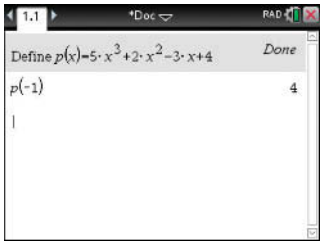
a. 1. On a Calculator page, press MENU then select
1: Actions
1: Define
Complete the entry line as
Define $p(x) = 5x^3 + 2x^2 - 3x + 4$
then press ENTER.

2. Complete the next entry line as $p(-1)$, then press ENTER.

3. The answer appears on the screen.

WRITE





$P(-1) = 4$.

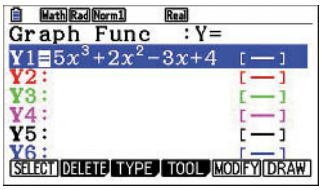
CASIO | THINK

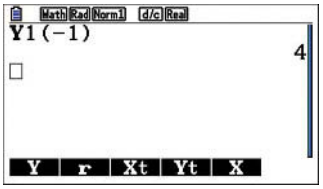
a. 1. On a Graph screen, complete the entry line for $y1$ as $y1 = 5x^3 + 2x^2 - 3x + 4$ then press EXE.

2. On a Run-Matrix screen, press VARS then select GRAPH by pressing F4. Select Y by pressing F1 and complete the entry line as $Y1(-1)$, then press EXE.

3. The answer appears on the screen.

WRITE





$P(-1) = 4$.

5.2.3 Identity of polynomials

If two polynomials are **identically equal** then the coefficients of like terms are equal. **Equating coefficients** means that if $ax^2 + bx + c \equiv 2x^2 + 5x + 7$ then $a = 2$, $b = 5$ and $c = 7$. The identically equal symbol '$\equiv$' means the equality holds for all values of x. For convenience, however, we shall replace this symbol with the equality symbol '=' in working steps.

WORKED EXAMPLE 3

Calculate the values of a, b and c so that $x(x - 7) \equiv a(x - 1)^2 + b(x - 1) + c$.

THINK	WRITE
1. Expand each bracket and express both sides of the equality in expanded polynomial form.	$x(x - 7) \equiv a(x - 1)^2 + b(x - 1) + c$ $\therefore x^2 - 7x = a\left(x^2 - 2x + 1\right) + bx - b + c$ $\therefore x^2 - 7x = ax^2 + (-2a + b)x + (a - b + c)$
2. Equate the coefficients of like terms.	Equate the coefficients. x^2: $1 = a$ [1] x: $-7 = -2a + b$ [2] Constant: $0 = a - b + c$ [3]
3. Solve the system of simultaneous equations.	Since $a = 1$, substitute $a = 1$ into equation [2]. $-7 = -2(1) + b$ $b = -5$ Substitute $a = 1$ and $b = -5$ into equation [3]. $0 = 1 - (-5) + c$ $c = -6$
4. State the answer.	$\therefore a = 1, b = -5, c = -6$

5.2.4 Expansion of cubic and quadratic polynomials from factors

The expansion of linear factors, for example $(x + 1)(x + 2)(x - 7)$ may result in a polynomial.

To expand, each term in one bracket must be multiplied by the terms in the other brackets, then like terms collected to simplify the expression.

WORKED EXAMPLE 4

Expand and simplify:

a. $x(x + 2)(x - 3)$ **b.** $(x - 1)(x + 5)(x + 2)$.

THINK	WRITE
a. 1. Write the expression.	**a.** $x(x + 2)(x - 3)$
2. Expand the last two linear factors.	$= x(x^2 - 3x + 2x - 6)$ $= x(x^2 - x - 6)$
3. Multiply the expression in the grouping symbols by x.	$= x^3 - x^2 - 6x$

b. 1. Write the expression.	**b.** $(x-1)(x+5)(x+2)$
2. Expand the last two linear factors.	$= (x-1)(x^2+2x+5x+10)$
3. Multiply the expression in the second pair of grouping symbols by x and then by -1.	$= (x-1)(x^2+7x+10)$
	$= x^3+7x^2+10x-x^2-7x-10$
4. Collect like terms.	$= x^3+6x^2+3x-10$

5.2.5 Operations on polynomials

The addition, subtraction and multiplication of two or more polynomials results in another polynomial. For example, if $P(x) = x^2$ and $Q(x) = x^3 + x^2 - 1$, then $P(x) + Q(x) = x^3 + 2x^2 - 1$, a polynomial of degree 3; $P(x) - Q(x) = -x^3 + 1$, a polynomial of degree 3; and $P(x)Q(x) = x^5 + x^4 - x^2$, a polynomial of degree 5.

WORKED EXAMPLE 5

Given $P(x) = 3x^3 + 4x^2 + 2x + m$ and $Q(x) = 2x^2 + kx - 5$, find the values of m and k for which $2P(x) - 3Q(x) = 6x^3 + 2x^2 + 25x - 25$.

THINK	WRITE
1. Form a polynomial expression for $2P(x) - 3Q(x)$ by collecting like terms together.	$2P(x) - 3Q(x)$ $= 2(3x^3 + 4x^2 + 2x + m) - 3(2x^2 + kx - 5)$ $= 6x^3 + 2x^2 + (4 - 3k)x + (2m + 15)$
2. Equate the two expressions for $2P(x) - 3Q(x)$.	Hence, $6x^3 + 2x^2 + (4 - 3k)x + (2m + 15)$ $= 6x^3 + 2x^2 + 25x - 25$
3. Calculate the values of m and k.	Equate the coefficients of x. $4 - 3k = 25$ $k = -7$ Equate the constant terms. $2m + 15 = -25$ $m = -20$
4. State the answer.	Therefore, $m = -20, k = -7$.

5.2.6 Division of polynomials

There are several methods for performing the **division of polynomials**. Here, two 'by-hand' methods will be shown.

The inspection method for division

The division of one polynomial by another polynomial of equal or lesser degree can be carried out by expressing the numerator in terms of the denominator.

To divide $(x + 3)$ by $(x - 1)$, or to find $\frac{x+3}{x-1}$, write the numerator $x + 3$ as $(x - 1) + 1 + 3 = (x - 1) + 4$.

$$\frac{x+3}{x-1} = \frac{(x-1)+4}{x-1}$$

This expression can then be split into the sum of **partial fractions** as:

$$\begin{aligned}\frac{x+3}{x-1} &= \frac{(x-1)+4}{x-1} \\ &= \frac{x-1}{x-1} + \frac{4}{x-1} \\ &= 1 + \frac{4}{x-1}\end{aligned}$$

The division is in the form: $\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$

In the language of division, when the dividend $(x+3)$ is divided by the divisor $(x-1)$ it gives a quotient of 1 and a remainder of 4. Note that from the division statement $\frac{x+3}{x-1} = 1 + \frac{4}{x-1}$ we can write $x+3 = 1 \times (x-1) + 4$.

This is similar to the division of integers. For example, 7 divided by 2 gives a quotient of 3 and a remainder of 1.

$$\frac{7}{2} = 3 + \frac{1}{2}$$
$$\therefore 7 = 3 \times 2 + 1$$

This inspection process of division can be extended, with practice, to division involving non-linear polynomials. It could be used to show that $\frac{x^2+4x+1}{x-1} = \frac{x(x-1)+5(x-1)+6}{x-1}$ and therefore $\frac{x^2+4x+1}{x-1} = x+5+\frac{6}{x-1}$. This result can be verified by checking that $x^2+4x+1 = (x+5)(x-1)+6$.

WORKED EXAMPLE 6

a. Calculate the quotient and the remainder when $(x+7)$ is divided by $(x+5)$.

b. Use the inspection method to find $\frac{3x-4}{x+2}$.

THINK

a. 1. Write the division of the two polynomials as a fraction.

2. Write the numerator in terms of the denominator.

3. Split into partial fractions.

4. Simplify.

WRITE

a. $\frac{x+7}{x+5} = \frac{(x+5)-5+7}{x+5}$

$= \frac{(x+5)+2}{x+5}$

$= \frac{(x+5)}{x+5} + \frac{2}{x+5}$

$= 1 + \frac{2}{x+5}$

5. State the answer.

The quotient is 1 and the remainder is 2.

b. 1. Express the numerator in terms of the denominator.

b. The denominator is $(x + 2)$.

Since $3(x + 2) = 3x + 6$, the numerator is

$3x - 4 = 3(x + 2) - 6 - 4$

$\therefore 3x - 4 = 3(x + 2) - 10$

2. Split the given fraction into its partial fractions.

$$\frac{3x-4}{x+2} = \frac{3(x+2)-10}{x+2}$$

$$= \frac{3(x+2)}{(x+2)} - \frac{10}{x+2}$$

3. Simplify and state the answer.

$$= 3 - \frac{10}{x+2}$$

$$\therefore \frac{3x-4}{x+2} = 3 - \frac{10}{x+2}$$

Algorithm for long division of polynomials

The inspection method of division is very efficient, particularly when the division involves only linear polynomials. However, it is also possible to use the long-division **algorithm** to divide polynomials.

Resources

Interactivity: Division of polynomials (int-2564)

The steps in the long-division algorithm are:

1. **Divide the leading term of the divisor into the leading term of the dividend.**
2. **Multiply the divisor by this quotient.**
3. **Subtract the product from the dividend to form a remainder of lower degree.**
4. **Repeat this process until the degree of the remainder is lower than that of the divisor.**

To illustrate this process, consider $\left(x^2 + 4x + 1\right)$ divided by $(x - 1)$. This is written as:

$$x - 1\overline{)x^2 + 4x + 1}$$

Step 1. The leading term of the divisor $(x - 1)$ is x; the leading term of the dividend $\left(x^2 + 4x + 1\right)$ is x^2.

Dividing x into x^2, we get $\frac{x^2}{x} = x$. We write this quotient x on top of the long-division symbol.

$$\begin{array}{r} x \\ x - 1\overline{)x^2 + 4x + 1} \end{array}$$

Step 2. The divisor $(x - 1)$ is multiplied by the quotient x to give $x(x - 1) = x^2 - x$. This product is written underneath the terms of $(x^2 + 4x + 1)$; like terms are placed in the same columns.

$$\begin{array}{r} x \\ x - 1 \overline{)x^2 + 4x + 1} \\ x^2 - x \end{array}$$

Step 3. $x^2 - x$ is subtracted from $(x^2 + 4x + 1)$. This cancels out the x^2 leading term to give $x^2 + 4x + 1 - (x^2 - x) = 5x + 1$.

$$\begin{array}{r} x \\ x - 1 \overline{)x^2 + 4x + 1} \\ \underline{-(x^2 - x)} \\ 5x + 1 \end{array}$$

The division statement, so far, would be $\frac{x^2 + 4x + 1}{x - 1} = x + \frac{5x + 1}{x - 1}$. This is incomplete since the remainder $(5x + 1)$ is not of a smaller degree than the divisor $(x - 1)$. The steps in the algorithm must be repeated with the same divisor $(x - 1)$ but with $(5x + 1)$ as the new dividend.

Continue the process.

Step 4. Divide the leading term of the divisor $(x - 1)$ into the leading term of $(5x + 1)$; this gives $\frac{5x}{x} = 5$. Write this as $+5$ on the top of the long-division symbol.

$$\begin{array}{r} x + 5 \\ x - 1 \overline{)x^2 + 4x + 1} \\ \underline{-(x^2 \ - x)} \\ 5x + 1 \end{array}$$

Step 5. Multiply $(x - 1)$ by 5 and write the result underneath the terms of $(5x + 1)$.

$$\begin{array}{r} x + 5 \\ x - 1 \overline{)x^2 + 4x + 1} \\ \underline{-(x^2 - x)} \\ 5x + 1 \\ 5x - 5 \end{array}$$

Step 6. Subtract $(5x - 5)$ from $(5x + 1)$.

$$\begin{array}{r} x + 5 \leftarrow \text{Quotient} \\ x - 1 \overline{)x^2 + 4x + 1} \\ \underline{-(x^2 - x)} \\ 5x + 1 \\ \underline{-(5x - 5)} \\ 6 \leftarrow \text{Remainder} \end{array}$$

The remainder is of lower degree than the divisor so no further division is possible and we have reached the end of the process.

Thus: $\frac{x^2 + 4x + 1}{x - 1} = x + 5 + \frac{6}{x - 1}$

This method can be chosen instead of the inspection method, or if the inspection method becomes harder to use.

WORKED EXAMPLE 7

a. **Given $P(x) = 4x^3+6x^2 - 5x + 9$, use the long-division method to divide $P(x)$ by $(x + 3)$ and state the quotient and the remainder.**

b. **Use the long-division method to calculate the remainder when $\left(3x^3 + \frac{5}{3}x\right)$ is divided by $(5 + 3x)$.**

THINK	WRITE
a. 1. Set up the long division.	**a.** $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$
2. The first stage of the division is to divide the leading term of the divisor into the leading term of the dividend.	$4x^2$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$
3. The second stage of the division is to multiply the result of the first stage by the divisor. Write this product placing like terms in the same columns.	$4x^2$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ $4x^3 + 12x^2$
4. The third stage of the division is to subtract the result of the second stage from the dividend. This will yield an expression of lower degree than the original dividend.	$4x^2$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ $-(4x^3 + 12x^2)$ $- 6x^2 - 5x + 9$
5. The algorithm needs to be repeated. Divide the leading term of the divisor into the leading term of the newly formed dividend.	$4x^2 - 6x$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ $-(4x^3 + 12x^2)$ $- 6x^2 - 5x + 9$
6. Multiply the result by the divisor and write this product keeping like terms in the same columns.	$4x^2 - 6x$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ $-(4x^3 + 12x^2)$ $- 6x^2 - 5x + 9$ $- 6x^2 - 18x$
7. Subtract to yield an expression of lower degree. *Note:* The degree of the expression obtained is still not less than the degree of the divisor so the algorithm will need to be repeated again.	$4x^2 - 6x$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ $-(4x^3 + 12x^2)$ $- 6x^2 - 5x + 9$ $-(- 6x^2 - 18x)$ $13x + 9$
8. Divide the leading term of the divisor into the dividend obtained in the previous step.	$4x^2 - 6x + 13$ $x + 3\overline{)4x^3 + 6x^2 - 5x + 9}$ $-(4x^3 + 12x^2)$ $- 6x^2 - 5x + 9$ $-(- 6x^2 - 18x)$ $13x + 9$

9. Multiply the result by the divisor and write this product keeping like terms in the same columns.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\
(4x^3 + 12x^2) \\
\hline
-6x^2 - 5x + 9 \\
-(-6x^2 - 18x) \\
\hline
13x + 9 \\
13x + 39
\end{array}
$$

10. Subtract to yield an expression of lower degree.
Note: The term reached is a constant so its degree is less than that of the divisor. The division is complete.

$$
\begin{array}{r}
4x^2 - 6x + 13 \\
x + 3\overline{)4x^3 + 6x^2 - 5x + 9} \\
-(4x^3 + 12x^2) \\
\hline
-6x^2 - 5x + 9 \\
-(-6x^2 - 18x) \\
\hline
13x + 9 \\
-(13x + 39) \\
\hline
-30
\end{array}
$$

11. State the answer.

$$\frac{4x^3 + 6x^2 - 5x + 9}{x + 3} = 4x^2 - 6x + 13 - \frac{30}{x + 3}$$

The quotient is $4x^2 - 6x + 13$ and the remainder is -30.

b. $3x^3 + 0x^2 + \frac{5}{3}x + 0 \div 3x + 5$

1. Set up the division, expressing both the divisor and the dividend in decreasing powers of x. This creates the columns for like terms.

2. Divide the leading term of the divisor into the leading term of the dividend, multiply this result by the divisor and then subtract this product from the dividend.

3. Repeat the three steps of the algorithm using the dividend created by the first application of the algorithm.

$$
\begin{array}{r}
x^2 - \frac{5}{3}x \\
3x + 5\overline{)3x^3 + 0x^2 + \frac{5}{3}x + 0} \\
-(3x^3 + 5x^2) \\
\hline
-5x^2 + \frac{5}{3}x + 0 \\
-\left(-5x^2 - \frac{25}{3}x\right) \\
\hline
10x + 0
\end{array}
$$

4. Repeat the algorithm using the dividend created by the second application of the algorithm.

$$
\begin{array}{r}
x^2 - \frac{5}{3}x + \frac{10}{3} \\
3x + 5 \overline{)\,3x^3 + 0x^2 + \frac{5}{3}x + 0} \\
-(3x^3 + 5x^2) \\
\hline
-5x^2 + \frac{5}{3}x + 0 \\
-\left(-5x^2 - \frac{25}{3}x\right) \\
\hline
10x + 0 \\
-\left(10x + \frac{50}{3}\right) \\
\hline
-\frac{50}{3}
\end{array}
$$

5. State the answer.

$$\frac{3x^3 + \frac{5}{3}x}{3x + 5} = x^2 - \frac{5}{3}x + \frac{10}{3} + \frac{-\frac{50}{3}}{3x + 5}$$
$$= x^2 - \frac{5}{3}x + \frac{10}{3} - \frac{50}{3(3x + 5)}$$

The remainder is $-\frac{50}{3}$.

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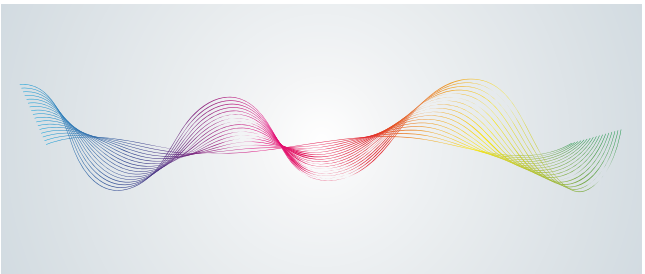
Units 1 & 2 > Area 2 > Sequence 4 > Concept 1

Polynomials Summary screen and practice questions

Exercise 5.2 Polynomials

Technology free

1. WE1 Select the polynomials from the following list of algebraic expressions and state their degree, the coefficient of the leading term, the constant term and the type of coefficients.
   a. $30x + 4x^5 - 2x^3 + 12$
   b. $\frac{3x^2}{5} - \frac{2}{x} + 1$
   c. $5.6 + 4x - 0.2x^2$



2. Of the following expressions which are polynomials? State their degree. For those which are not polynomials, state a reason why not.
   a. $7x^4 + 3x^2 + 5$
   b. $9 - \frac{5}{2}x - 4x^2 + x^3$
   c. $-9x^3 + 7x^2 + 11\sqrt{x} - \sqrt{5}$
   d. $\frac{6}{x^2} + 6x^2 + \frac{x}{2} - \frac{2}{x}$

3. Consider the following list of algebraic expressions.

a. $3x^5 + 7x^4 - \frac{x^3}{6} + x^2 - 8x + 12$ **b.** $9 - 5x^4 + 7x^2 - \sqrt{5}x + x^3$

c. $\sqrt{4x^5} - \sqrt{5}x^3 + \sqrt{3}x - 1$ **d.** $2x^2\left(4x - 9x^2\right)$

e. $\frac{x^6}{10} - \frac{2x^8}{7} + \frac{5}{3x^2} - \frac{7x}{5} + \frac{4}{9}$ **f.** $\left(4x^2 + 3 + 7x^3\right)^2$

a. Select the polynomials from the list and for each of these polynomials state:

i. its degree **ii.** the type of coefficients

iii. the leading term **iv.** the constant term.

b. Give a reason why each of the remaining expressions is not a polynomial.

4. a. If $P(x) = -x^3 + 2x^2 + 5x - 1$, calculate $P(1)$.

b. If $P(x) = 2x^3 - 4x^2 + 3x - 7$, calculate $P(-2)$.

c. For $P(x) = 3x^3 - x^2 + 5$, calculate $P(3)$ and $P(-x)$.

d. For $P(x) = x^3 + 4x^2 - 2x + 5$, calculate $P(-1)$ and $P(2a)$.



5. Given $P(x) = 2x^3 + 3x^2 + x - 6$, evaluate the following.

a. $P(3)$ **b.** $P(-2)$

c. $P(1)$ **d.** $P(0)$

e. $P\left(-\frac{1}{2}\right)$ **f.** $P(0.1)$

6. If $P(x) = x^2 - 7x + 2$, obtain expressions for the following.

a. $P(a) - P(-a)$ **b.** $P(1 + h)$ **c.** $P(x + h) - P(x)$

7. a. WE2 If $P(x) = 7x^3 - 8x^2 - 4x - 1$ calculate $P(2)$.

b. If $P(x) = 2x^2 + kx + 12$ and $P(-3) = 0$, find k.

8. a. If $P(x) = ax^2 + 9x + 2$ and $P(1) = 3$, determine the value of a.

b. Given $P(x) = -5x^2 + bx - 18$, calculate the value of b if $P(3) = 0$.

c. Given $P(x) = -2x^3 + 3x^2 + kx - 10$, calculate the value of k if $P(-1) = -7$.

d. If $P(x) = x^3 - 6x^2 + 9x + m$ and $P(0) = 2P(1)$, determine the value of m.

9. If $P(x) = -2x^3 + 9x + m$ and $P(1) = 2P(-1)$, determine the value of m.

10. a. If $P(x) = 4x^3 + kx^2 - 10x - 4$ and $P(1) = 15$, obtain the value of k.

b. If $Q(x) = ax^2 - 12x + 7$ and $Q(-2) = -5$, obtain the value of a.

c. If $P(x) = x^3 - 6x^2 + nx + 2$ and $P(2) = 3P(-1)$, obtain the value of n.

d. If $Q(x) = -x^2 + bx + c$ and $Q(0) = 5$ and $Q(5) = 0$, obtain the values of b and c.

11. WE3 Calculate the values of a, b and c so that $(2x + 1)(x - 5) \equiv a(x + 1)^2 + b(x + 1) + c$.

12. a. Determine the values of a and b so that $x^2 + 10x + 6 \equiv x(x + a) + b$.

b. Express $8x - 6$ in the form $ax + b(x + 3)$.

c. Express the polynomial $6x^2 + 19x - 20$ in the form $(ax + b)(x + 4)$.

d. Obtain the values of a, b and c so that $x^2 - 8x = a + b(x + 1) + c(x + 1)^2$ for all values of x.

13. Express $(x + 2)^3$ in the form $px^2(x + 1) + qx(x + 2) + r(x + 3) + t$.

14. WE4a Expand and simplify each of the following.

a. $x(x - 9)(x + 2)$ **b.** $-3x(x - 4)(x + 4)$

15. WE4b Expand and simplify each of the following.

a. $(x - 2)(x + 4)(x - 5)$ **b.** $(x + 6)(x - 1)(x + 1)$

c. $(x + 2)(x - 7)^2$ **d.** $(x + 1)(x - 1)(x + 1)$

16. WE5 Given $P(x) = 4x^3 - px^2 + 8$ and $Q(x) = 3x^2 + qx - 7$, determine the values of p and q for which $P(x) + 2Q(x) = 4x^3 + x^2 - 8x - 6$.

17. a. If $P(x) = 2x^2 - 7x - 11$ and $Q(x) = 3x^4 + 2x^2 + 1$, determine each of the following, expressing the terms in descending power of x.

i. $Q(x) - P(x)$ **ii.** $3P(x) + 2Q(x)$ **iii.** $P(x)Q(x)$

b. If $P(x)$ is a polynomial of degree m and $Q(x)$ is a polynomial of degree n where $m > n$, state the degree of:

i. $P(x) + Q(x)$ **ii.** $P(x) - Q(x)$ **iii.** $P(x)\,Q(x)$



18. a. **WE6** Calculate the quotient and the remainder when $(x - 12)$ is divided by $(x + 3)$.

b. Use the inspection method to obtain $\dfrac{4x + 7}{2x + 1}$.

19. a. **WE7** Given $P(x) = 2x^3 - 5x^2 + 8x + 6$, divide $P(x)$ by $(x - 2)$ and state the quotient and the remainder.

b. Use the long-division method to calculate the remainder when $(x^3 + 10)$ is divided by $(1 - 2x)$.

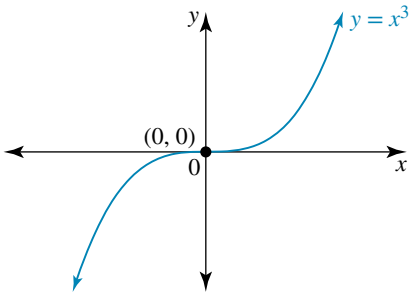
5.3 Graphs of cubic polynomials

The graph of the general cubic polynomial has an equation of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real constants and $a \neq 0$. Since a cubic polynomial may have up to three linear factors, its graph may have up to three x-intercepts. The shape of its graph is affected by the number of x-intercepts.

5.3.1 The graph of $y = x^3$ and transformations

The graph of the simplest cubic polynomial has the equation $y = x^3$.

The 'maxi–min' point at the origin is sometimes referred to as a 'saddle point'. Formally, it is called a **stationary point of inflection** (or inflexion as a variation of spelling). It is a key feature of this cubic graph.



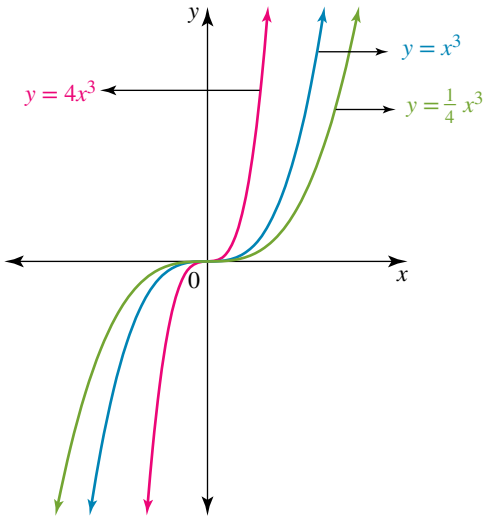
Key features of the graph of $y = x^3$:

- (0,0) is a stationary point of inflection.
- The shape of the graph changes from concave down to concave up at the stationary point of inflection.
- There is only one x-intercept.
- As the values of x increase positively, the behaviour of the graph shows its y-values also increase positively. This means that as $x \to \infty$, $y \to \infty$. This is read as 'as x approaches infinity, y approaches infinity'.
- As the values of x increase negatively, the behaviour of the graph shows its y-values also increase negatively. This means that as $x \to -\infty$, $y \to -\infty$.
- The graph starts from below the x-axis and increases as x increases.

Once the basic shape is known, the graph can be dilated, reflected and translated in much the same way as the parabola $y = x^2$.

Dilation

The graph of $y = 4x^3$ will be narrower than the graph of $y = x^3$ due to the dilation factor of 4 from the x-axis. Similarly, the graph of $y = \frac{1}{4}x^3$ will be wider than the graph of $y = x^3$ due to the dilation factor of $\frac{1}{4}$ from the x-axis.



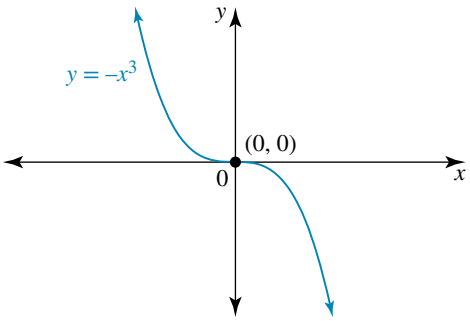
Reflection

The graph of $y = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis.

For the graph of $y = -x^3$ note that:

- as $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to \infty$
- the graph starts from above the x-axis and decreases as x increases
- at (0, 0), the stationary point of inflection, the graph changes from concave up to concave down.

Note: Reflection in the y-axis has the same result, $y = (-x)^3 = -x^3$.

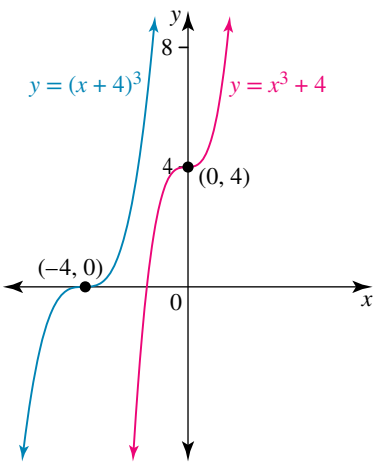


Translation

The graph of $y = x^3 + 4$ is obtained when the graph of $y = x^3$ is translated vertically upwards by 4 units. The stationary point of inflection is at the point (0, 4).

The graph of $y = (x + 4)^3$ is obtained when the graph of $y = x^3$ is translated horizontally 4 units to the left. The stationary point of inflection is at the point $(-4, 0)$.

The transformations from the basic parabola $y = x^2$ are recognisable from the equation $y = a(x - b)^2 + c$, and the equation of the graph of $y = x^3$ can be transformed to a similar form.



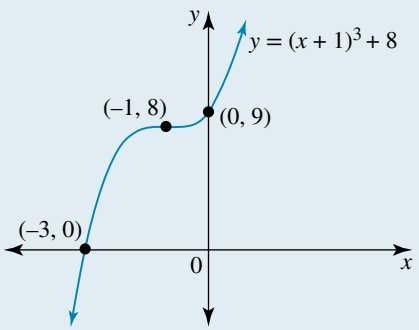
The key features of the graph of $y = a(x - b)^3 + c$ are as follows.

- **Stationary point of inflection at (b, c)**
- **Change of concavity at the stationary point of inflection**
- **If $a > 0$, the graph starts below the x-axis and increases, like $y = x^3$.**
- **If $a < 0$, the graph starts above the x-axis and decreases, like $y = -x^3$.**
- **The one x-intercept is found by solving $a(x - b)^3 + c = 0$.**
- **The y-intercept is found by substituting $x = 0$.**

WORKED EXAMPLE 8

Sketch:

a. $y = (x + 1)^3 + 8$ **b.** $y = 6 - \frac{1}{2}(x - 2)^3$.

THINK	WRITE
a. 1. State the point of inflection.	**a.** $y = (x + 1)^3 + 8$ Point of inflection is $(-1, 8)$.
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = (1)^3 + 8$ $= 9$ $\Rightarrow (0,9)$
3. Calculate the x-intercept.	x-intercept: let $y = 0$ $(x + 1)^3 + 8 = 0$ $(x + 1)^3 = -8$ Take the cube root of both sides: $x + 1 = \sqrt[3]{-8}$ $x + 1 = -2$ $x = -3$ $\Rightarrow (-3, 0)$
4. Sketch the graph. Label the key points and ensure the graph changes concavity at the point of inflection.	The coefficient of x^3 is positive so the graph starts below the x-axis and increases. 
b. 1. Rearrange the equation to the $y = a(x - b)^3 + c$ form and state the point of inflection.	**b.** $y = 6 - \frac{1}{2}(x - 2)^3$ $= -\frac{1}{2}(x - 2)^3 + 6$ Point of inflection: $(2, 6)$

2. Calculate the y-intercept.

y-intercept: let $x = 0$

$$y = -\frac{1}{2}(-2)^3 + 6$$
$$= 10$$
$$\Rightarrow (0,10)$$

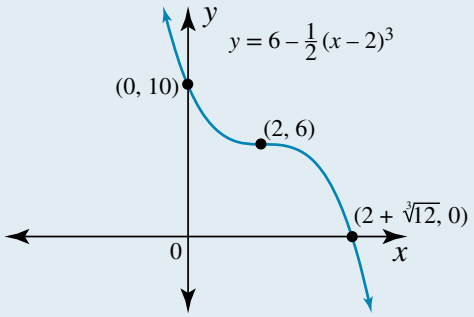
3. Calculate the x-intercept.
Note: A decimal approximation helps locate the point.

x-intercept: let $y = 0$

$$-\frac{1}{2}(x-2)^3 + 6 = 0$$
$$\frac{1}{2}(x-2)^3 = 6$$
$$(x-2)^3 = 12$$
$$x - 2 = \sqrt[3]{12}$$
$$x = 2 + \sqrt[3]{12}$$
$$\Rightarrow \left(2 + \sqrt[3]{12}, 0\right) \approx (4.3,0)$$

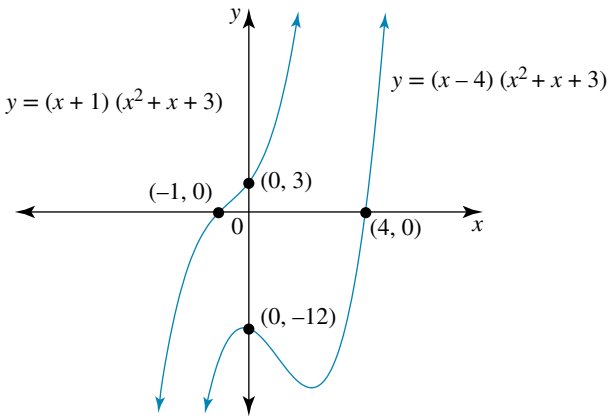
4. Sketch the graph showing all key features.

$a < 0$ so the graph starts above the x-axis and decreases.



Cubic graphs with one x-intercept but no stationary point of inflection

There are cubic graphs which have one x-intercept but no stationary point of inflection. The equations of such cubic graphs cannot be expressed in the form $y = a(x-b)^3 + c$. Their equations can be expressed as the product of a linear factor and a quadratic factor which is **irreducible**, meaning the quadratic has no real factors.



Technology is often required to sketch such graphs. Two examples, $y = (x+1)\left(x^2+x+3\right)$ and $y = (x-4)\left(x^2+x+3\right)$, are shown in the diagram. Each has a linear factor and the discriminant of the quadratic factor $x^2 + x + 3$ is negative; this means it cannot be further factorised over R.

Both graphs maintain the **long-term behaviour** exhibited by all cubic with a positive leading-term coefficient; that is, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

Every cubic polynomial must have at least one linear factor in order to maintain this long-term behaviour.

5.3.2 Cubic graphs with three *x*-intercepts

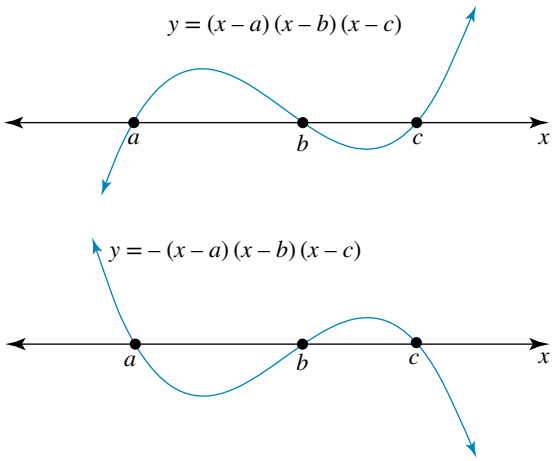
For the graph of a cubic polynomial to have three *x*-intercepts, the polynomial must have three distinct linear factors. This is the case when the cubic polynomial expressed as the product of a linear factor and a quadratic factor is such that the quadratic factor has two distinct linear factors.

This means that the graph of a monic cubic with an equation of the form $y = (x - a)(x - b)(x - c)$ where $a, b, c \in R$ and $a < b < c$ will have the shape of the graph shown.

If the graph is reflected in the *x*-axis, its equation is of the form $y = -(x-a)(x-b)(x-c)$ and the shape of its graph satisfies the long-term behaviour that as $x \to \pm\infty$, $y \to \mp\infty$.

It is important to note the graph is not a quadratic so the maximum and minimum turning points do not lie halfway between the *x*-intercepts. In a later chapter we will learn how to locate these points without using technology.

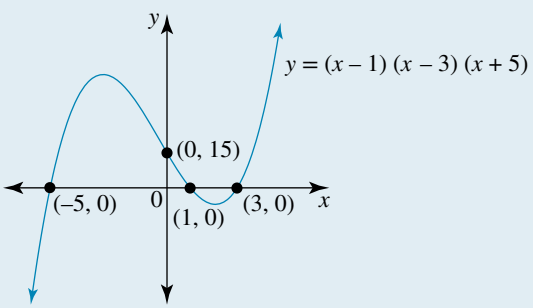
To sketch the graph, it is usually sufficient to identify the *x*- and *y*-intercepts and to ensure the shape of the graph satisfies the long-term behaviour requirement determined by the sign of the leading term.



WORKED EXAMPLE 9

Sketch the following without attempting to locate turning points.

a. $y = (x - 1)(x - 3)(x + 5)$ **b.** $y = (x+1)(2x - 5)(6 - x)$

THINK	WRITE
a. 1. Calculate the *x*-intercepts.	**a.** $y = (x - 1)(x - 3)(x + 5)$ *x*-intercepts: let $y = 0$ $(x - 1)(x - 3)(x + 5) = 0$ $x = 1,\ x = 3,\ x = -5$ $\Rightarrow (-5, 0), (1, 0), (3, 0)$ are the *x*-intercepts.
2. Calculate the *y*-intercept.	*y*-intercept: let $x = 0$ $y = (-1)(-3)(5)$ $= 15$ $\Rightarrow (0, 15)$ is the *y*-intercept.
3. Determine the shape of the graph.	Multiplying together the terms in *x* from each bracket gives x^3, so its coefficient is positive. The shape is of a positive cubic.
4. Sketch the graph.	

b. 1. Calculate the x-intercepts.

b. $y = (x + 1)(2x - 5)(6 - x)$

x-intercepts: let $y = 0$

$(x + 1)(2x - 5)(6 - x) = 0$

$x + 1 = 0, 2x - 5 = 0, 6 - x = 0$

$x = -1,\ x = 2.5,\ x = 6$

$\Rightarrow (-1, 0),\ (2.5,\ 0),\ (6, 0)$ are the x-intercepts.

2. Calculate the y-intercept.

y-intercept: let $x = 0$

$y = (1)(-5)(6)$

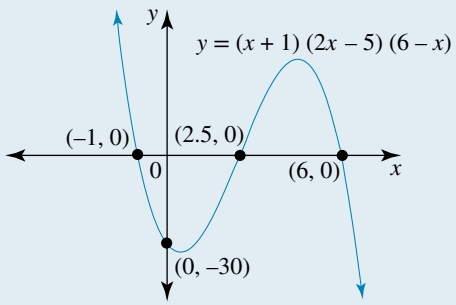
$= -30$

$\Rightarrow (0, -30)$ is the y-intercept.

3. Determine the shape of the graph.

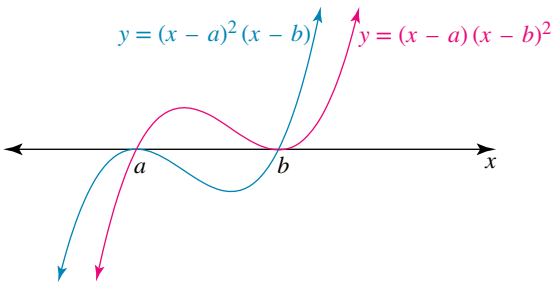
Multiplying the terms in x from each bracket gives $(x) \times (2x) \times (-x) = -2x^3$ so the shape is of a negative cubic.

4. Sketch the graph.



5.3.3 Cubic graphs with two x-intercepts

If a cubic has two x-intercepts, one at $x = a$ and one at $x = b$, then in order to satisfy the long-term behaviour required of any cubic, the graph either touches the x-axis at $x = a$ and turns, or it touches the x-axis at $x = b$ and turns. One of the x-intercepts must be a turning point.



Thinking of the cubic polynomial as the product of a linear and a quadratic factor, for its graph to have two instead of three x-intercepts, the quadratic factor must have two identical factors. Either the factors of the cubic are $(x - a)(x - a)(x - b) = (x - a)^2(x - b)$ or the factors are $(x - a)(x - b)(x - b) = (x - a)(x - b)^2$. The repeated factor identifies the x-intercept which is the turning point. The repeated factor is said to be of multiplicity 2 and the single factor of multiplicity 1.

The graph of a cubic polynomial with equation of the form $y = (x - a)^2 (x - b)$ has a turning point on the x-axis at $(a, 0)$ and a second x-intercept at $(b, 0)$. The graph is said to *touch* the x-axis at $x = a$ and *cut* it at $x = b$.

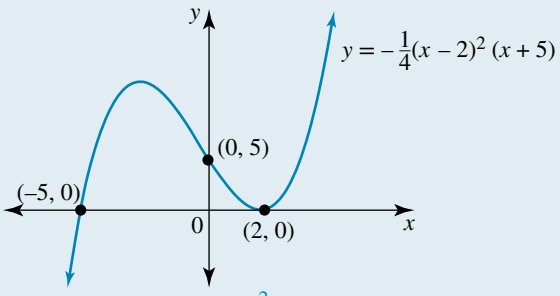
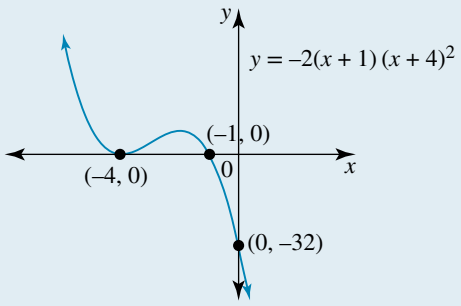
Although the turning point on the x-axis must be identified when sketching the graph, there will be a second turning point that cannot yet be located without technology.

Note that a cubic graph whose equation has a repeated factor of multiplicity 3, such as $y = (x - b)^3$, would have only one x-intercept as this is a special case of $y = a(x - b)^3 + c$ with $c = 0$. The graph would cut the x-axis at its stationary point of inflection $(b, 0)$.

WORKED EXAMPLE 10

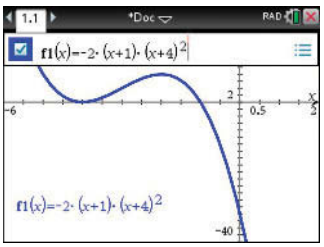
Sketch the graphs of:

a. $y = \frac{1}{4}(x-2)^2(x+5)$ **b.** $y = -2(x+1)(x+4)^2$

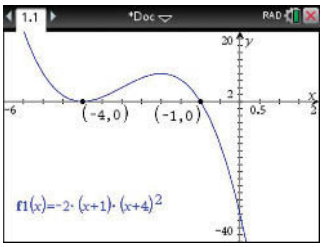
THINK	WRITE
a. 1. Calculate the x-intercepts and interpret the multiplicity of each factor.	**a.** $y = \frac{1}{4}(x-2)^2(x+5)$ x-intercepts: let $y = 0$ $\frac{1}{4}(x-2)^2(x+5) = 0$ $\therefore x = 2$ (touch), $x = -5$ (cut) x-intercept at $(-5, 0)$ and turning-point x-intercept at $(2,0)$
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = \frac{1}{4}(-2)^2(5)$ $= 5$ $\Rightarrow (0, 5)$
3. Sketch the graph.	
b. 1. Calculate the x-intercepts and interpret the multiplicity of each factor.	**b.** $y = -2(x+1)(x+4)^2$ x-intercepts: let $y = 0$ $-2(x+1)(x+4)^2 = 0$ $(x+1)(x+4)^2 = 0$ $\therefore x = -1$ (cut), $x = -4$ (touch) x-intercept at $(-1, 0)$ and turning-point x-intercept at $(-4, 0)$
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = -2(1)(4)^2$ $= -32$ y-intercept at $(0, -32)$
3. Sketch the graph.	

TI | THINK **WRITE**

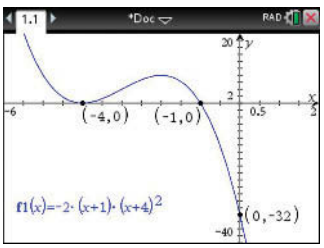
b. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = -2(x+1)(x+4)^2$ then press ENTER.



2. To find the x-intercepts, press MENU then select
6: Analyze Graph
1: Zero
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER. Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.
Repeat this step to find the other x-intercept.

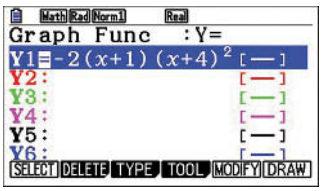


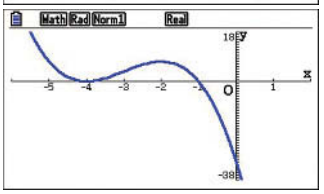
3. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



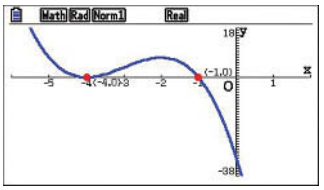
CASIO | THINK **WRITE**

b. 1. On a Graph screen, complete the entry line for y1 as $y1 = -2(x+1)(x+4)^2$ then press EXE. Select DRAW by pressing F6.

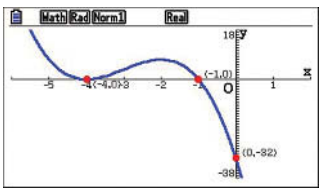




2. To find the x-intercepts, select G-Solv by pressing SHIFT then F5, then select ROOT by pressing F1. Press EXE. Use the left/right arrows to move across to the next x-intercept, then press EXE.



3. To find the y-intercept, select G-Solv by pressing SHIFT then F5, then select Y-ICEPT by pressing F4. Press EXE.



5.3.4 Cubic graphs in the general form $y = ax^3 + bx^2 + cx + d$

If the cubic polynomial with equation $y = ax^3 + bx^2 + cx + d$ can be factorised, then the shape of its graph and its key features can be determined. Standard factorisation techniques such as grouping terms together may be sufficient, or the factor theorem (to be covered in 5.4 section) may be required in order to obtain the factors.

The sign of a, the coefficient of x^3, determines the long-term behaviour the graph exhibits. For $a > 0$ as $x \to \pm\infty$, $y \to \pm\infty$; for $a < 0$ as $x \to \pm\infty$, $y \to \mp\infty$.

The value of d determines the y-intercept.

The linear or quadratic factors determine the x-intercepts and the multiplicity of each factor will determine how the graph intersects the x-axis.

Every cubic graph must have at least one x-intercept and hence the polynomial must have at least one linear factor. Considering a cubic as the product of a linear and a quadratic factor, it is the quadratic factor which determines whether there is more than one x-intercept.

Graphs which have only one x-intercept may be of the form $y = a(x-b)^3 + c$ where the stationary point of inflection is a major feature. Recognition of this equation from its expanded form would require the expansion of a perfect cube to be recognised, since $a\left(x^3 - 3x^2b + 3xb^2 - b^3\right) + c = a(x-b)^3 + c$. However, as previously noted, not all graphs with only one x-intercept have a stationary point of inflection.

WORKED EXAMPLE 11

Sketch the graph of $y = x^3 - 3x - 2$, without attempting to obtain any turning points that do not lie on the coordinate axes.

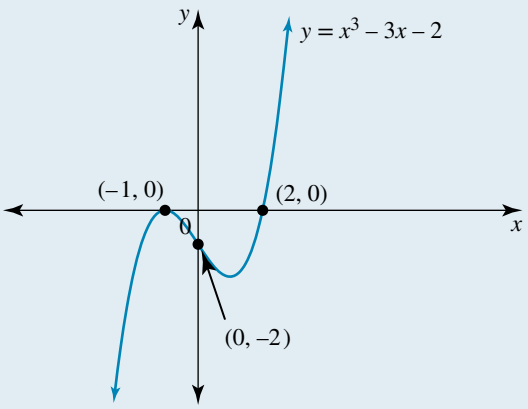
THINK

1. Obtain the y-intercept first since it is simpler to obtain from the expanded form.
2. Factorisation will be needed in order to obtain the x-intercepts.
3. It can be shown by using technology, or the factor theorem to be covered later, that $x^3 - 3x - 2 = (x + 1)^2(x - 2)$.
4. What is the nature of these x-intercepts?
5. Sketch the graph.

WRITE

$y = x^3 - 3x - 2$
y-intercept: $(0, -2)$
x-intercepts: let $y = 0$
$x^3 - 3x - 2 = 0$
$\Rightarrow (x + 1)^2(x - 2) = 0$
$\therefore x = -1, 2$

$y = P(x) = (x + 1)^2 (x - 2)$
$x = -1$ (touch) and $x = 2$ (cut)
Turning point at $(-1, 0)$



5.3.5 Determining the equation of cubic graph

The equation $y = ax^3 + bx^2 + cx + d$ contains four unknown coefficients that need to be specified, so four pieces of information are required to determine the equation of a cubic graph, unless the equation is written in turning point form when 3 pieces of information are required.

As a guide:

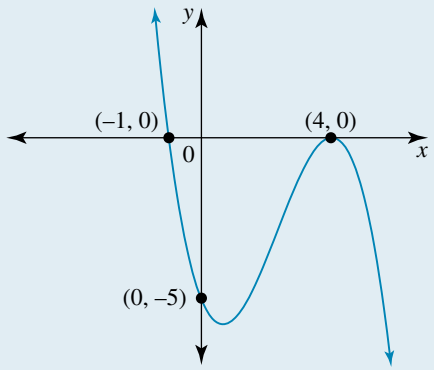
- **If there is a stationary point of inflection given, use the $y = a(x - b)^3 + c$ form.**
- **If the x-intercepts are given, use the $y = a(x - x_1)(x - x_2)(x - x_3)$ form, or the repeated factor form $y = a(x - x_1)^2(x - x_2)$ if there is a turning point at one of the x-intercepts.**
- **If four points on the graph are given, use the $y = ax^3 + bx^2 + cx + d$ form.**

WORKED EXAMPLE 12

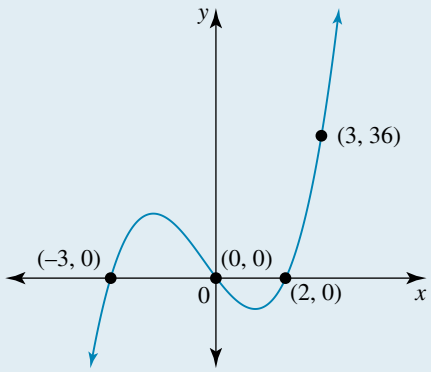
Determine the equation for each of the following graphs.

a. The graph of a cubic polynomial which has a stationary point of inflection at the point (−7, 4) and an *x*-intercept at (1, 0).

b.



c.



THINK

a. 1. Consider the given information and choose the form of the equation to be used.

2. Calculate the value of a.
Note: The coordinates of the given points show the y-values decrease as the x-values increase, so a negative value for a is expected.

3. Write the equation of the graph.

b. 1. Consider the given information and choose the form of the equation to be used.

2. Calculate the value of a.

3. Write the equation of the graph.

WRITE

a. Stationary point of inflection is given.
Let $y = a(x - b)^3 + c$.
Point of inflection is $(-7, 4)$.
$\therefore y = a(x + 7)^3 + 4$.

Substitute the given x-intercept point $(1, 0)$.

$$0 = a(8)^3 + 4$$

$$(8)^3 a = -4$$

$$a = \frac{-4}{8 \times 64}$$

$$a = -\frac{1}{128}$$

The equation is $y = -\frac{1}{128}(x + 7)^3 + 4$.

b. Two x-intercepts are given.
One shows a turning point at $x = 4$ and the other a cut at $x = -1$.
Let the equation be $y = a(x + 1)(x - 4)^2$.
Substitute the given y-intercept point $(0, -5)$.

$$-5 = a(1)(-4)^2$$

$$-5 = a(16)$$

$$a = -\frac{5}{16}$$

The equation is $y = -\frac{5}{16}(x + 1)(x - 4)^2$.

c. 1. Consider the given information and choose the form of the equation to be used.

2. Calculate the value of a.

3. Write the equation of the graph.

c. Three x-intercepts are given.
Let the equation be
$y = a(x + 3)(x - 0)(x - 2)$
$\therefore y = ax(x + 3)(x - 2)$
Substitute the given point $(3, 36)$.
$36 = a\,(3)\,(6)\,(1)$
$36 = 18a$
$a = 2$
The equation is $y = 2x\,(x + 3)\,(x - 2)$.

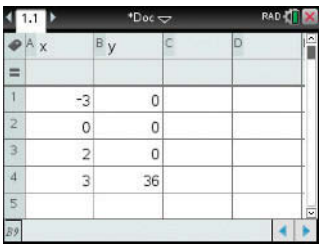
TI | THINK

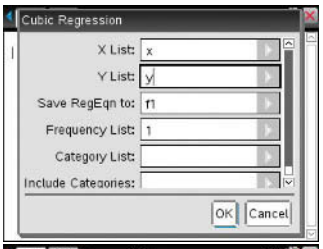
c. 1. On a Lists & Spreadsheet page, label the first column x and the second column y. Enter the x-coordinates of the given points in the first column and the corresponding y-coordinates in the second column.

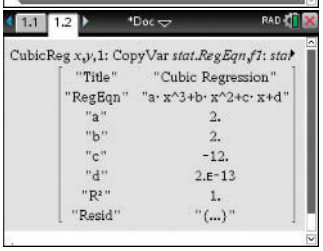
2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
7: Cubic Regression …
Complete the fields as
X List: x
Y List: y
then select OK.

3. The answer appears on the screen.

WRITE







$y = 2x^3 + 2x^2 - 12x$

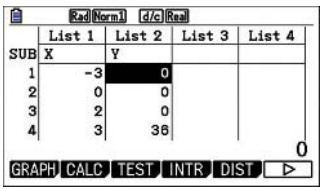
CASIO | THINK

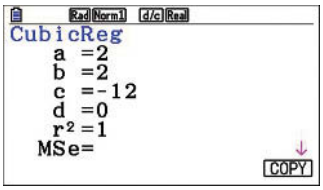
c. 1. On a Statistics screen,relabel List 1 as x and List 2 as y. Enter the x-coordinates of the given points in the first column and the corresponding y-coordinates in the second column.

2. Select CALC by pressing F2, select REG by pressing F3, then select X^3 by pressing F4.

3. The answer appears on the screen.

WRITE





$y = 2x^3 + 2x^2 - 12x$

on Resources

Interactivity: Graph plotter: Cubic polynomials (int-2566)

studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 2

Cubic polynomials Summary screen and practice questions

Exercise 5.3 Graphs of cubic polynomials

Technology free

1. State the coordinates of the point of inflection for each of the following.
   a. $y = (x-7)^3$ b. $y = x^3 - 7$ c. $y = -7x^3$
   d. $y = 2 - (x-2)^3$ e. $y = \frac{1}{6}(x+5)^3 - 8$ f. $y = -\frac{1}{2}(2x-1)^3 + 5$
2. **WE8** Sketch the graph of the polynomials.
   a. $y = (x-1)^3 - 8$
   b. $y = 1 - \frac{1}{36}(x+6)^3$



3. a. Sketch the graph $y = -x^3 + 1$. Include all important features.
   b. Sketch the graph $y = 2(3x-2)^3$. Include all important features.
   c. Sketch the graph $y = 2(x+3)^3 - 16$. Include all important features.
   d. Sketch the graph $y = (3-x)^3 + 1$. Include all important features.
4. **WE9** Sketch the following, without attempting to locate turning points.
   a. $y = (x+1)(x+6)(x-4)$ b. $y = (x-4)(2x+1)(6-x)$
5. Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.
   a. $y = (x-2)(x+1)(x+4)$ b. $y = -0.5x(x+8)(x-5)$
   c. $y = (x+3)(x-1)(4-x)$ d. $y = \frac{1}{4}(2-x)(6-x)(4+x)$
   e. $y = 0.1(2x-7)(x-10)(4x+1)$ f. $y = 2\left(\frac{x}{2} - 1\right)\left(\frac{3x}{4} + 2\right)\left(x - \frac{5}{8}\right)$

Technology active

6. **WE10** Sketch the graphs of these polynomials.
   a. $y = \frac{1}{9}(x-3)^2(x+6)$ b. $y = -2(x-1)(x+2)^2$
7. Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.
   a. $y = -(x+4)^2(x-2)$ b. $y = 2(x+3)(x-3)^2$ c. $y = (x+3)^2(4-x)$
   d. $y = \frac{1}{4}(2-x)^2(x-12)$ e. $y = 3x(2x+3)^2$ f. $y = -0.25x^2(2-5x)$
8. Sketch the graphs of the following, showing any intercepts with the coordinate axes and any stationary point of inflection.
   a. $y = (x+3)^3$ b. $y = (x+3)^2(2x-1)$
   c. $y = (x+3)(2x-1)(5-x)$ d. $2(y-1) = (1-2x)^3$
   e. $4y = x(4x-1)^2$ f. $y = -\frac{1}{2}(2-3x)(3x+2)(3x-2)$
9. Factorise, if possible, and sketch the graphs of the cubic polynomials with equations given by the following.
   a. $y = 9x^2 - 2x^3$ b. $y = 9x^3 - 4x$ c. $y = 9x^2 - 3x^3 + x - 3$
   d. $y = 9x(x^2 + 4x + 3)$ e. $y = 9x^3 + 27x^2 + 27x + 9$ f. $y = -9x^3 - 9x^2 + 9x + 9$

10. **WE11** Sketch the graph of $y = x^3 - 3x^2 - 10x + 24$ without attempting to obtain any turning points that do not lie on the coordinate axes.



11. **a.** Determine the x- and y-intercepts of the cubic graph $y = -x^3 - 3x^2 + 16x + 48$.
Factorise the cubic equation to express it as a product of linear factors by grouping and hence, sketch the graph.

b. The partially factorised form of $2x^3 + x^2 - 13x + 6$ is $(x - 2)(2x^2 + 5x - 3)$. Complete the factorisation and sketch the graph of $y = 2x^3 + x^2 - 13x + 6$, showing all intercepts with the coordinate axes.

c. Determine the x- and y-intercepts of the cubic graph $y = x^3 + 5x^2 - x - 5$. (To determine the linear factors use the grouping technique).
Hence, sketch the graph.

d. Partially factorised, $-x^3 - 5x^2 - 3x + 9$ is expressed as $(x - 1)(-x^2 - 6x - 9)$. Complete the factorisation and sketch the graph of $y = -x^3 - 5x^2 - 3x + 9$ showing all intercepts with the coordinate axes.

12. Factorise as a product of linear factors and sketch, without attempting to locate any turning points that do not lie on the coordinate axes.

a. $y = 2x^3 - 3x^2 - 17x - 12$
Factorised $y = (x + 1)(2x + 3)(x - 4)$

b. $y = 6 - 55x + 57x^2 - 8x^3$
Factorised partially $y = (x - 1)(-8x^2 + 49x - 6)$

c. $y = 6x^3 - 13x^2 - 59x - 18$
Factorised partially $(x + 2)(6x^2 - 25x - 9)$

d. $y = -\frac{1}{2}x^3 + 14x - 24$
Factorised partially $-2y = (x - 2)(x^2 + 2x - 24)$

13. **a.** Sketch the graph of $y = -x^3 + 3x^2 + 10x - 30$ without attempting to obtain any turning points that do not lie on the coordinate axes.

b. By expressing $y = x^3 + 3x^2 + 3x + 2$ in the form $a(x - b)^3 + c$ determine coordinates of the stationary point of inflection of the graph and sketch the graph.

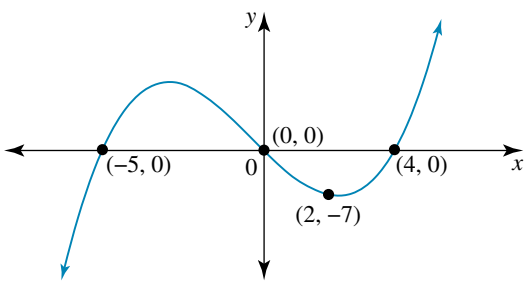
14. **a.** Express $-\frac{1}{2}x^3 + 6x^2 - 24x + 38$ in the form $a\,(x - b)^3 + c$.

b. Hence sketch the graph of $y = -\frac{1}{2}x^3 + 6x^2 - 24x + 38$.

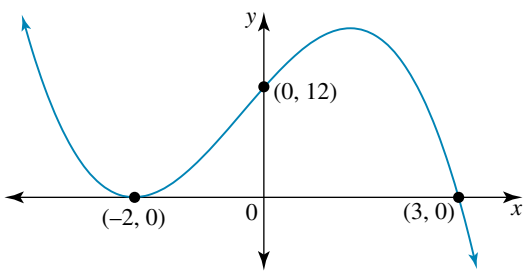
15. **WE12** Determine the equation of each of the following graphs.

a. The graph of a cubic polynomial which has a stationary point of inflection at the point $(3, -7)$ and an x-intercept at $(10, 0)$.

b.



c.



16. Use simultaneous equations to determine the equation of the cubic graph containing the points $(0, 3)$, $(1, 4)$, $(-1, 8)$, $(-2, 7)$.

17. **a.** The graph of a cubic polynomial of the form $y = a(x - b)^3 + c$ has a stationary point of inflection at $(3, 9)$ and passes through the origin. Form the equation of the graph.

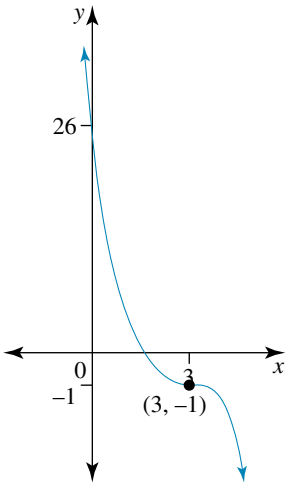
b. The graph of the form $y = a(x - b)^3 + c$ has a stationary point of inflection at $(-2, 2)$ and a y-intercept at $(0, 10)$. Determine the equation.

c. The graph of the form $y = a(x - b)^3 + c$ has stationary point of inflection at $(0, 4)$ and passes through the x-axis at $\left(\sqrt[3]{2}, 0\right)$. Determine the equation.

d. The graph of $y = x^3$ is translated 5 units to the left and 4 units upwards. State its equation after these translations take place.

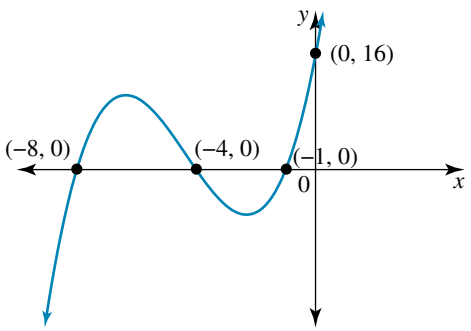
e. After the graph $y = x^3$ has been reflected about the x-axis, translated 2 units to the right and translated downwards 1 unit, what would its equation become?

f. The graph shown has a stationary point of inflection at $(3, -1)$. Determine its equation.

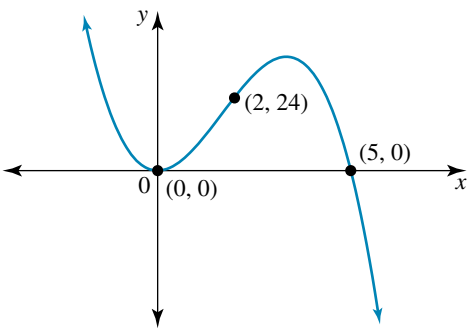


18. Determine the equation for each of the following graphs of cubic polynomials.

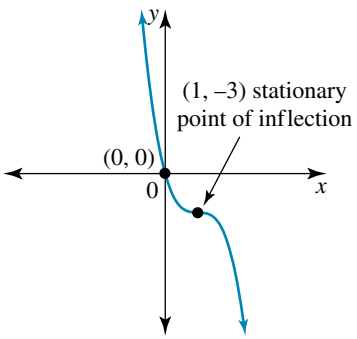
a.



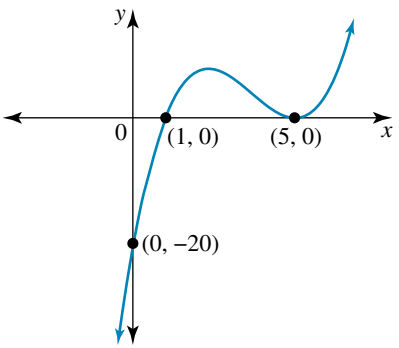
b.



c.



d.



19. **a.** Using technology sketch, locating turning points, the graph of $y = x^3 + 4x^2 - 44x - 96$.

b. Show that the turning points are not placed symmetrically in the interval between the adjoining x-intercepts.

20. Using technology sketch the following, locating intercepts with the coordinate axes and any turning points. Express values to 1 decimal place where appropriate.

a. $y = 10x^3 - 20x^2 - 10x - 19$

b. $y = -x^3 + 5x^2 - 11x + 7$

c. $y = 9x^3 - 70x^2 + 25x + 500$

5.4 The factor and remainder theorems

The remainder obtained when dividing $P(x)$ by the linear divisor $(x-a)$ is of interest because if the remainder is zero, then the divisor must be a linear factor of the polynomial. To pursue this interest we need to be able to calculate the remainder quickly without the need to do a lengthy division.

5.4.1 The remainder theorem

The actual division, as we know, will result in a quotient and a remainder. This is expressed in the division statement $\frac{P(x)}{x-a} = \text{quotient} + \frac{\text{remainder}}{x-a}$.

Since $(x-a)$ is linear, the remainder will be some constant term independent of x.
From the division statement it follows that:

$$P(x) = (x-a) \times \text{quotient} + \text{remainder}$$

If we let $x = a$, this makes $(x-a)$ equal to zero and the statement becomes:

$$P(a) = 0 \times \text{quotient} + \text{remainder}$$

Therefore:

$$P(a) = \text{remainder}$$

This result is known as the **remainder theorem**.

> **If a polynomial $P(x)$ is divided by $(x-a)$ then the remainder is $P(a)$.**

Note that:

- If $P(x)$ is divided by $(x+a)$ then the remainder would be $P(-a)$ since replacing x by $-a$ would make the $(x+a)$ term equal zero.
- If $P(x)$ is divided by $(ax+b)$ then the remainder would be $P\left(-\frac{b}{a}\right)$ since replacing x by $-\frac{b}{a}$ would make the $(ax+b)$ term equal zero.

WORKED EXAMPLE 13

Find the remainder when $P(x) = x^3 - 3x^2 - 2x + 9$ is divided by:
a. $x - 2$ **b. $2x + 1$.**

THINK	WRITE
a. 1. What value of x will make the divisor zero?	**a.** $(x-2) = 0 \Rightarrow x = 2$
2. Write an expression for the remainder.	$P(x) = x^3 - 3x^2 - 2x + 9$ Remainder is $P(2)$.
3. Evaluate to obtain the remainder.	$P(2) = (2)^3 - 3(2)^2 - 2(2) + 9$ $= 1$ The remainder is 1.

b. 1. Find the value of x which makes the divisor zero.

2. Write an expression for the remainder and evaluate it.

b. $(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$

Remainder is $P\left(-\frac{1}{2}\right)$.

$$P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + 9$$

$$= -\frac{1}{8} - \frac{3}{4} + 1 + 9$$

$$= 9\frac{1}{8}$$

Remainder is $9\frac{1}{8}$.

5.4.2 The factor theorem

We know 4 is a factor of 12 because it divides 12 exactly, leaving no remainder. Similarly, if the division of a polynomial $P(x)$ by $(x - a)$ leaves no remainder, then the divisor $(x - a)$ must be a factor of the polynomial $P(x)$.

$$P(x) = (x - a) \times \text{quotient} + \text{remainder}$$

If the remainder is zero, then $P(x) = (x - a) \times$ quotient.
Therefore $(x - a)$ is a factor of $P(x)$.
This is known as the **factor theorem**.

If $P(x)$ is a polynomial and $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

Conversely, if $(x - a)$ is a factor of a polynomial $P(x)$ then $P(a) = 0$.
a is a **zero** of the polynomial.

It also follows from the remainder theorem that if $P\left(-\frac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of $P(x)$ and $-\frac{b}{a}$ is a zero of the polynomial.

WORKED EXAMPLE 14

a. Show that $(x + 3)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.
b. Determine the polynomial $P(x) = ax^3 + bx + 2$ which leaves a remainder of -9 when divided by $(x - 1)$ and is exactly divisible by $(x + 2)$.

THINK

a. 1. State how the remainder can be calculated when $Q(x)$ is divided by the given linear expression.

WRITE

a. $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$
When is divided by $(x + 3)$, the remainder equals $Q(-3)$.

2. Evaluate the remainder.

$$Q(-3) = 4(-3)^4 + 4(-3)^3 - 25(-3)^2 - (-3) + 6$$
$$= 324 - 108 - 225 + 3 + 6$$
$$= 0$$

3. It is important to explain in the answer why the given linear expression is a factor.

Since $Q(-3) = 0$, $(x + 3)$ is a factor of $Q(x)$.

b. 1. Express the given information in terms of the remainders.

b. $P(x) = ax^3 + bx + 2$

Dividing by $(x - 1)$ leaves a remainder of -9.

$\Rightarrow P(1) = -9$

Dividing by $(x + 2)$ leaves a remainder of 0.

$\Rightarrow P(-2) = 0$

2. Set up a pair of simultaneous equations in a and b.

$$P(1) = a + b + 2$$
$$a + b + 2 = -9$$
$$\therefore a + b = -11 \quad [1]$$
$$P(-2) = -8a - 2b + 2$$
$$-8a - 2b + 2 = 0$$
$$\therefore 4a + b = 1 \quad [2]$$

3. Solve the simultaneous equations.

$$a + b = -11 \quad [1]$$
$$4a + b = 1 \quad [2]$$

Equation [2] – equation [1]:

$$3a = 12$$
$$a = 4$$

Substitute $a = 4$ into equation [1].

$$4 + b = -11$$
$$b = -15$$

4. Write the answer.

$\therefore P(x) = 4x^3 - 15x + 2$

5.4.3 Factorising polynomials

When factorising a cubic or higher-degree polynomial, the first step should be to check if any of the standard methods for factorising can be used. In particular, look for a common factor, then look to see if a grouping technique can produce either a common linear factor or a difference of two squares. If the standard techniques do not work then the remainder and factor theorems can be used to factorise, since the zeros of a polynomial enable linear factors to be formed.

Cubic polynomials may have up to three zeros and therefore up to three linear factors. For example, a cubic polynomial $P(x)$ for which it is known that $P(1) = 0$, $P(2) = 0$, and $P(-4) = 0$, has 3 zeros: $x = 1$, $x = 2$ and $x = -4$. From these, its three linear factors $(x - 1)$, $(x - 2)$ and $(x + 4)$ are formed.

Integer zeros of a polynomial may be found through a trial-and-error process where factors of the polynomial's constant term are tested systematically. For the polynomial $P(x) = x^3 + x^2 - 10x + 8$, the constant term is 8 so the possibilities to test are $1, -1, 2, -2, 4, -4, 8$ and -8. This is a special case of what is known as the **rational root theorem**.

The rational solutions to the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$, where the coefficients are integers and a_n and a_0 are non-zero, will have solutions $x = \frac{p}{q}$ (in simplest form), where p is a factor of a_0 and q is a factor of a_n.

In practice, not all of the zeros need to be, nor necessarily can be, found through trial and error. For a cubic polynomial it is sufficient to find one zero by trial and error and form its corresponding linear factor using the factor theorem. Dividing this linear factor into the cubic polynomial gives a quadratic quotient and zero remainder, so the quadratic quotient is also a factor. The standard techniques for factorising quadratics can then be applied.

For the division step, long division could be used; however, it is more efficient to use a division method based on equating coefficients. With practice, this can usually be done by inspection. To illustrate, $P(x) = x^3 + x^2 - 10x + 8$ has a zero of $x = 1$ since $P(1) = 0$. Therefore $(x - 1)$ is a linear factor and $P(x) = (x - 1)\left(ax^2 + bx + c\right)$.

Note that the x^3 term of $(x - 1)\left(ax^2 + bx + c\right)$ can only be formed by the product of the x term in the first bracket with the x^2 term in the second bracket; likewise, the constant term of $(x - 1)\left(ax^2 + bx + c\right)$ can only be formed by the product of the constant terms in the first and second brackets.

The coefficients of the quadratic factor are found by equating coefficients of like terms in $x^3 + x^2 - 10x + 8 = (x - 1)\left(ax^2 + bx + c\right)$.

For x^3: $1 = a$

For constants: $8 = -1 \times c \Rightarrow c = -8$

This gives $x^3 + x^2 - 10x + 8 = (x - 1)\left(x^2 + bx - 8\right)$ which can usually be written down immediately.

For the right-hand expression $(x - 1)\left(x^2 + bx - 8\right)$, the coefficient of x^2 is formed after a little more thought. An x^2 term can be formed by the product of the x term in the first bracket with the x term in the second bracket and also by the product of the constant term in the first bracket with the x^2 term in the second bracket.

$$x^3 + x^2 - 10x + 8 = (x - 1)\,(x^2 + bx - 8)$$

$$1 = b - 1$$

Equating coefficients of x^2: $\therefore b = 2$

If preferred, the coefficients of x could be equated or used to check the validity of the answer.

It follows that:

$$\begin{aligned} P(x) &= (x - 1)\left(x^2 + 2x - 8\right) \\ &= (x - 1)(x - 2)(x + 4) \end{aligned}$$

WORKED EXAMPLE 15

a. Factorise $P(x) = x^3 - 2x^2 - 5x + 6$.

b. Given that $(x + 1)$ and $(5 - 2x)$ are factors of $P(x) = -4x^3 + 4x^2 + 13x + 5$ completely factorise $P(x)$.

THINK	WRITE
a. 1. The polynomial does not factorise by a grouping technique so a zero needs to be found. The factors of the constant term are potential zeros.	**a.** $P(x) = x^3 - 2x^2 - 5x + 6$ The factors of 6 are ± 1, ± 2, ± 3 and ± 6.
2. Use the remainder theorem to test systematically until a zero is obtained. Then use the factor theorem to state the corresponding linear factor.	$P(1) = 1 - 2 - 5 + 6$ $= 0$ $\therefore (x - 1)$ is a factor.

3. Express the polynomial in terms of a product of the linear factor and a general quadratic factor.

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(ax^2 + bx + c)$

4. State the values of a and c.

For the coefficient of x^3 to be 1, $a = 1$.
For the constant term to be 6, $c = -6$.
$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$

5. Calculate the value of b.

Equating the coefficients of x^2 gives:

$x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$

$-2 = b - 1$
$b = -1$
$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

6. Factorise the quadratic factor so the polynomial is fully factorised into its linear factors.

Hence:
$P(x) = x^3 - 2x^2 - 5x + 6$
$= (x - 1)(x^2 - x - 6)$
$= (x - 1)(x - 3)(x + 2)$

b. 1. Multiply the two given linear factors to form the quadratic factor.

b. $P(x) = -4x^3 + 4x^2 + 13x + 5$
Since $(x + 1)$ and $(5 - 2x)$ are factors, then $(x + 1)(5 - 2x) = -2x^2 + 3x + 5$ is a quadratic factor.

2. Express the polynomial as a product of the quadratic factor and a general linear factor.

The remaining factor is linear.
$\therefore P(x) = (x + 1)(5 - 2x)(ax + b)$
$= (-2x^2 + 3x + 5)(ax + b)$

3. Find a and b.

$-4x^3 + 4x^2 + 13x + 5 = (-2x^2 + 3x + 5)(ax + b)$
Equating coefficients of x^3 gives:
$-4 = -2a$
$\therefore a = 2$
Equating constants gives:
$5 = 5b$
$\therefore b = 1$

4. State the answer.

$-4x^3 + 4x^2 + 13x + 5 = (-2x^2 + 3x + 5)(2x + 1)$
$= (x + 1)(5 - 2x)(2x + 1)$
$\therefore P(x) = (x + 1)(5 - 2x)(2x + 1)$

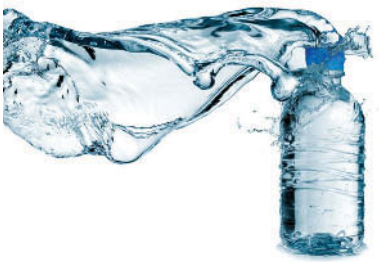
on Resources

Interactivity: The remainder and factor theorems (int-2565)

Exercise 5.4 The factor and remainder theorems

Technology free

1. **WE13** Determine the value of the remainder when $P(x) = x^3 + 4x^2 - 3x + 5$ is divided by:
   a. $x + 2$ b. $2x - 1$.
2. a. Without actual division, calculate the remainder when $3x^2 + 8x - 5$ is divided by $(x - 1)$.
   b. Without dividing, calculate the remainder when $-x^3 + 7x^2 + 2x - 12$ is divided by $(x + 1)$.
   c. When $ax^2 - 4x - 9$ is divided by $(x - 3)$, the remainder is 15. Calculate the value of a.
   d. When $x^3 + x^2 + kx + 5$ is divided by $(x + 2)$, the remainder is -5. Calculate the value of k.
3. **MC** Select the correct statement for the remainder when $P(x)$ is divided by $(2x + 9)$.
   A. The remainder is $P(9)$.
   B. The remainder is $P(-9)$.
   C. The remainder is $P\left(-\frac{9}{2}\right)$.
   D. The remainder is $P\left(\frac{9}{2}\right)$.
4. Calculate the remainder without actual division when:
   a. $x^3 - 4x^2 - 5x + 3$ is divided by $(x - 1)$
   b. $6x^3 + 7x^2 + x + 2$ is divided by $(x + 1)$
   c. $-2x^3 + 2x^2 - x - 1$ is divided by $(x - 4)$
   d. $x^3 + x^2 + x - 10$ is divided by $(2x + 1)$
   e. $27x^3 - 9x^2 - 9x + 2$ is divided by $(3x - 2)$
   f. $4x^4 - 5x^3 + 2x^2 - 7x + 8$ is divided by $(x - 2)$.



5. If $x^3 - kx^2 + 4x + 8$ leaves a remainder of 29 when it is divided by $(x - 3)$, determine the value of k.
6. a. **WE14** Show that $(x - 2)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.
   b. Determine the polynomial $P(x) = 3x^3 + ax^2 + bx - 2$ which leaves a remainder of -22 when divided by $(x + 1)$ and is exactly divisible by $(x - 1)$.
7. a. When $P(x) = x^3 - 2x^2 + ax + 7$ is divided by $(x + 2)$, the remainder is 11. Determine the value of a.
   b. If $P(x) = 4 - x^2 + 5x^3 - bx^4$ is exactly divisible by $(x - 1)$, determine the value of b.
   c. If $2x^3 + cx^2 + 5x + 8$ has a remainder of 6 when divided by $(2x - 1)$, determine the value of c.
   d. Given that each of $x^3 + 3x^2 - 4x + d$ and $x^4 - 9x^2 - 7$ have the same remainder when divided by $(x + 3)$, find the value of d.
8. Given $(2x + a)$ is a factor of $12x^2 - 4x + a$, obtain the value(s) of a.
9. a. Calculate the values of a and b for which $Q(x) = ax^3 + 4x^2 + bx + 1$ leaves a remainder of 39 when divided by $(x - 2)$, given $(x + 1)$ is a factor of $Q(x)$.
   b. Dividing $P(x) = \frac{1}{3}x^3 + mx^2 + nx + 2$ by either $(x - 3)$ or $(x + 3)$ results in the same remainder. If that remainder is three times the remainder left when $P(x)$ is divided by $(x - 1)$, determine the values of m and n.
10. a. **WE15** Factorise $P(x) = x^3 + 3x^2 - 13x - 15$.
    b. Given that $(x + 1)$ and $(3x + 2)$ are factors of $P(x) = 12x^3 + 41x^2 + 43x + 14$, completely factorise $P(x)$.

11. **a.** Show that $x + 4$ is a factor of $3x^3 + 11x^2 - 6x - 8$.

b. Show that $x - 5$ is not a factor of $x^3 + 6x^2 + x - 30$.

c. Show that $2x - 1$ is a factor of $6x^3 + 7x^2 - 9x + 2$.

d. Show that $x - 1$ is not a factor of $2x^3 + 13x^2 + 5x - 6$.

e. Given $x + 3$ is a factor of $x^3 - 13x + a$, determine the value of a.

f. Given $2x - 5$ is a factor of $4x^3 + kx^2 - 9x + 10$, determine the value of k.

12. **a.** Given $(x - 4)$ is a factor of $P(x) = x^3 - x^2 - 10x - 8$, fully factorise $P(x)$.

b. Given $(x + 12)$ is a factor of $P(x) = 3x^3 + 40x^2 + 49x + 12$, fully factorise $P(x)$.

c. Given $(5x + 1)$ is a factor of $P(x) = 20x^3 + 44x^2 + 23x + 3$, fully factorise $P(x)$.

d. Given $(4x - 3)$ is a factor of $P(x) = -16x^3 + 12x^2 + 100x - 75$, fully factorise $P(x)$.

e. Given $(8x - 11)$ and $(x - 3)$ are factors of $P(x) = -8x^3 + 59x^2 - 138x + 99$, fully factorise $P(x)$.

f. Given $(3x - 5)$ is a factor of $P(x) = 9x^3 - 75x^2 + 175x - 125$, fully factorise $P(x)$.



13. Fully factorise the following.

a. $x^3 + 5x^2 + 2x - 8$

b. $x^3 + 10x^2 + 31x + 30$

c. $2x^3 - 13x^2 + 13x + 10$

d. $-18x^3 + 9x^2 + 23x - 4$

e. $x^3 - 7x + 6$

f. $x^3 + x^2 - 49x - 49$



Technology active

14. Given the zeros of the polynomial $P(x) = 12x^3 + 8x^2 - 3x - 2$ are not integers, use the rational root theorem to calculate one zero and hence find the three linear factors of the polynomial.

15. **a.** A monic polynomial of degree 3 in x has zeros of 5, 9 and -2. Express this polynomial in:

i. factorised form **ii.** expanded form.

b. A polynomial of degree 3 has a leading term with coefficient -2 and zeros of -4, -1 and $\frac{1}{2}$. Express this polynomial in:

i. factorised form **ii.** expanded form.

16. **a.** The polynomial $24x^3 + 34x^2 + x - 5$ has three zeros, none of which are integers. Calculate the three zeros and express the polynomial as the product of its three linear factors.

b. The polynomial $P(x) = 8x^3 + mx^2 + 13x + 5$ has a zero of $\frac{5}{2}$.

i. State a linear factor of the polynomial.

ii. Fully factorise the polynomial.

iii. Calculate the value of m.

c. **i.** Factorise the polynomials $P(x) = x^3 - 12x^2 + 48x - 64$ and $Q(x) = x^3 - 64$.

ii. Hence, show that $\frac{P(x)}{Q(x)} = 1 - \frac{12x}{x^2 + 4x + 16}$.

d. A cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ has integer coefficients and $P(0) = 9$. Two of its linear factors are $\left(x - \sqrt{3}\right)$ and $\left(x + \sqrt{3}\right)$. Calculate the third linear factor and obtain the values of b, c and d.

17. Specify the remainder when $\left(9 + 19x - 2x^2 - 7x^3\right)$ is divided by $\left(x - \sqrt{2} + 1\right)$.

5.5 Solving cubic equations

5.5.1 Polynomial equations

If a cubic or any polynomial is expressed in factorised form, then the polynomial equation can be solved using the Null Factor Law.

$$(x-a)(x-b)(x-c)=0$$
$$\therefore (x-a)=0,\ (x-b)=0, (x-c)=0$$
$$\therefore x=a, x=b \text{ or } x=c$$

$x=a$, $x=b$ and $x=c$ are called the roots or the solutions to the equation $P(x)=0$.
The factor theorem may be required to express the polynomial in factorised form.
If the equation cannot be expressed in factorised form technology can be used to solve the equation.

5.5.2 Solving cubic equations using the Null Factor Law

If the equation is not expressed in standard cubic polynomial form, $ax^3+bx^2+cx+d=0$, it must first be rearranged to $P(x)=0$ before the Null Factor Law can applied.

WORKED EXAMPLE 16

Solve:
a. $x^3=9x$ **b.** $-2x^3+4x^2+70x=0$ **c.** $2x^3-11x^2+18x-9=0$.

THINK

a. 1. Write the equation.
2. Rearrange so all terms are on the left.
3. Take out a common factor of x.
4. Factorise the expression in the grouping symbols using the difference of square rule.
5. Use the Null Factor Law to solve.

b. 1. Write the equation.
2. Take out a common factor of $-2x$.
3. Factorise the expression in the grouping symbols.
4. Use the Null Factors law to solve.

c. 1. Name the polynomial.
2. Use the factor theorem to find a factor (search for a value a such that $P(a)=0$). Consider factors of the constant term (that is, factors of 9 such as 1, 3). The simplest value to try is 1.

WRITE

a. $x^3=9x$
$x^3-9x=0$
$x(x^2-9)=0$
$x(x+3)(x-3)=0$
$x=0,\ x+3=0 \text{ or } x-3=0$
$x=0,\ x=-3 \text{ or } x=3$

b. $-2x^3+4x^2+70x=0$
$-2x(x^2-2x-35)=0$
$2x(x-7)(x+5)=0$
$-2x=0,\ x-7=0 \text{ or } x+5=0$
$x=0,\ x=7 \text{ or } x=-5$

c. Let $P(x)=2x^3-11x^2+18x-9$.
$P(1)=2-11+18-9$
$=0$

3. Use long or short division to find another factor of $P(x)$.

$$
\begin{array}{r}
2x^2 - 9x + 9 \\
x - 1 \overline{)2x^3 - 11x^2 + 18x - 9} \\
\underline{2x^3 - 2x^2} \\
-9x^2 + 18x \\
\underline{-9x^2 + 9x} \\
9x - 9 \\
\underline{9x - 9} \\
0
\end{array}
$$

4. Factorise the quadratic factor.

$$P(x) = (x - 1)(2x^2 - 9x + 9)$$

$$P(x) = (x - 1)(2x - 3)(x - 3)$$

5. Consider the factorised equation to solve

$$\text{For } (x - 1)\ (2x - 3)\ (x - 3) = 0$$

6. Use the Null Factor Law to solve.

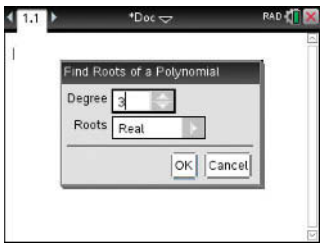
$$x - 1 = 0,\ 2x - 3 = 0 \text{ or } x - 3 = 0$$

$$x = 1,\ x = \frac{3}{2} \text{ or } x = 3$$

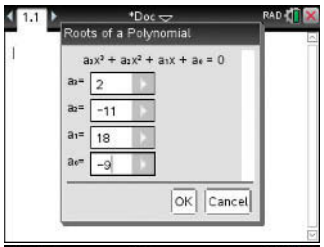
TI | THINK

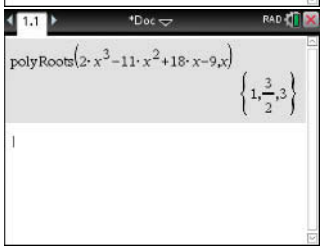
WRITE

c. 1. On a Calculator page, press MENU then select
3: Algebra
3 : Polynomial Tools
1: Find Roots of Polynomial …
Complete the fields as
Degree: 3
Roots: Real
then select OK.



2. Complete the fields for the coefficients as
$a_3 = 2$
$a_2 = -11$
$a_1 = -18$
$a_0 = -9$
then select OK.





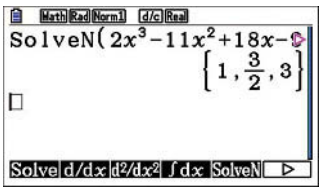
3. The answer appears on the screen.

$$x = 1, x = \frac{3}{2} \text{ or } x = 3$$

CASIO | THINK

WRITE

c. 1. On a Run-Matrix screen, press OPTN then select CALC by pressing F4.
Select Solve N by pressing F5,
then complete the entry line as
Solve N $(2x^3 - 11x^2 + 18x - 9 = 0, x)$
and press EXE.



2. The answer appears on the screen.

$$x = 1, x = \frac{3}{2} \text{ or } x = 3$$

Solve the equation $3x^3 + 4x^2 = 17x + 6$

The equation $3x^3 + 4x^2 = 17x + 6$ can be solved using the factor theorem and the Null Factor Law as shown in Worked example 17. Alternatively, since the equation consists of a cubic polynomial and a linear polynomial it could be solved by graphing each polynomial and finding where the graphs intersect. This method will be demonstrated in section 5.4.

WORKED EXAMPLE 17

Solve the equation $3x^3 + 4x^2 = 17x + 6$

THINK	WRITE
	$3x^3 + 4x^2 = 17x + 6$
1. Rearrange the equation so one side is zero.	$3x^3 + 4x^2 - 17x - 6 = 0$
2. Since the polynomial does not factorise by grouping techniques, use the remainder theorem to find a zero and the factor theorem to form the corresponding linear factor. *Note:* It is simpler to test for integer zeros first.	Let $P(x) = 3x^3 + 4x^2 - 17x - 6$. Test factors of the constant term: $P(1) \neq 0$ $P(-1) \neq 0$ $P(2) = 3(2)^3 + 4(2)^2 - 17(2) - 6$ $= 24 + 16 - 34 - 6$ $= 0$ Therefore $(x - 2)$ is a factor.
3. Express the polynomial as a product of the linear factor and a general quadratic factor.	$3x^3 + 4x^2 - 17x - 6 = (x - 2)(ax^2 + bx + c)$
4. Find and substitute the values of a and c.	$\therefore 3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + bx + 3)$
5. Calculate b.	Equate the coefficients of x^2: $4 = b - 6$ $b = 10$
6. Completely factorise the polynomial.	$3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + 10x + 3)$ $= (x - 2)(3x + 1)(x + 3)$
7. Solve the equation.	The equation $3x^3 + 4x^2 - 17x - 6 = 0$ becomes: $(x - 2)(3x + 1)(x + 3) = 0$ $x - 2 = 0,\ 3x + 1 = 0,\ x + 3 = 0$ $x = 2,\ x = -\frac{1}{3},\ x = -3$

5.5.3 Intersections of cubic graphs with linear and quadratic graphs

If $P(x)$ is a cubic polynomial and $Q(x)$ is either a linear or a quadratic polynomial, then the intersection of the graphs of $y = P(x)$ and $y = Q(x)$ occurs when $P(x) = Q(x)$. Hence the x-coordinates of the points of intersection are the roots of the equation $P(x) - Q(x) = 0$. This is a cubic equation since $P(x) - Q(x)$ is a polynomial of degree 3.

WORKED EXAMPLE 18

Sketch the graphs of $y = x(x - 1)(x + 1)$ and $y = x$ and calculate the coordinates of the points of intersection.

THINK	WRITE
1. Sketch the graphs.	$y = x(x - 1)(x + 1)$ This is a positive cubic.

x-intercepts: let $y = 0$

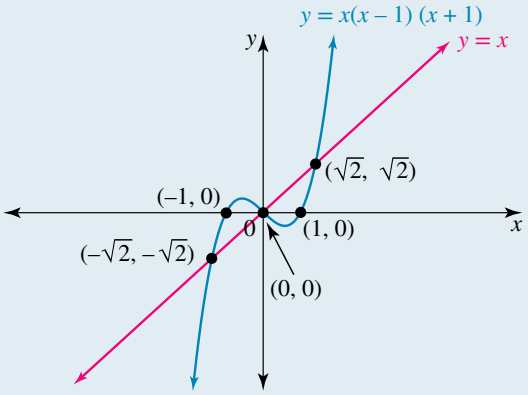
$x(x-1)(x+1) = 0$

$x = 0,\ x = \pm 1$

$(-1,\ 0)$, $(0,\ 0)$, $(1,\ 0)$ are the three x-intercepts.

y-intercept is $(0, 0)$.

Line: $y = x$ passes through $(0,\ 0)$ and $(1,\ 1)$.



2. Calculate the coordinates of the points of intersections.

At intersection:

$x(x-1)(x+1) = x$

$x\left(x^2 - 1\right) - x = 0$

$x^3 - 2x = 0$

$x\left(x^2 - 2\right) = 0$

$x = 0,\ x^2 = 2$

$x = 0,\ x = \pm\sqrt{2}$

Substituting these x-values in the equation of the line $y = x$, the points of intersection are $(0,\ 0)$, $\left(\sqrt{2},\ \sqrt{2}\right)$, $\left(-\sqrt{2},\ -\sqrt{2}\right)$.

on Resources

Interactivity: Graph plotter: 1, 2, 3 intercepts (int-2567)

studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 4

Solving cubic equations and inequations Summary screen and practice questions

Exercise 5.5 Solving cubic equations

Technology free

1. Solve the following.
   a. $2x^3 - 50x = 0$ b. $-4x^3 + 8x = 0$ c. $x^3 - 5x^2 + 6x = 0$ d. $x^3 + 6x = 4x^2$
2. Use the factor theorem to solve the following.
   a. $x^3 - 3x^2 - 6x + 8 = 0$ b. $-4x^3 + 16x^2 - 9x - 9 = 0$
   c. $-2x^3 - 9x^2 - 7x + 6 = 0$ d. $2x^3 + 4x^2 - 2x - 4 = 0$
3. MC A solution of $x^3 - 9x^2 + 15x + 25 = 0$ is $x = 5$. How many other (distinct) solution are there?
   A. 0 B. 1 C. 2 D. 3
4. Solve $P(x) = 0$.
   a. $P(x) = x^3 + 4x^2 - 3x - 18$ b. $P(x) = 3x^3 - 13x^2 - 32x + 12$
   c. $P(x) = -x^3 + 12x - 16$ d. $P(x) = 8x^3 - 4x^2 - 32x - 20$
5. WE16 Solve for x, $2x^4 + 3x^3 - 8x^2 - 12x = 0$.
6. WE17 Solve the equation $6x^3 + 13x^2 = 2 - x$.
7. Solve the following equations for x.
   a. $(x + 4)(x - 3)(x + 5) = 0$ b. $2(x - 7)(3x + 5)(x - 9) = 0$
   c. $x^3 - 13x^2 + 34x + 48 = 0$ d. $2x^3 + 7x^2 = 9$
   e. $3x^2(3x + 1) = 4(2x + 1)$ f. $8x^4 + 158x^3 - 46x^2 - 120x = 0$
8. a. Show that $(x - 2)$ is a factor of $P(x) = x^3 + 6x^2 - 7x - 18$ and hence fully factorise $P(x)$ over R.
   b. Show that $(3x - 1)$ is the only real linear factor of $3x^3 + 5x^2 + 10x - 4$.
   c. Show that $(2x^2 - 11x + 5)$ is a factor of $2x^3 - 21x^2 + 60x - 25$ and hence calculate the roots of the equation $2x^3 - 21x^2 + 60x - 25 = 0$.
9. a. If $(x^2 - 4)$ divides $P(x) = 5x^3 + kx^2 - 20x - 36$ exactly, fully factorise $P(x)$ and hence obtain the value of k.
   b. If $x = a$ is a solution to the equation $ax^2 - 5ax + 4(2a - 1) = 0$, find possible values for a.
   c. The polynomials $P(x) = x^3 + ax^2 + bx - 3$ and $Q(x) = x^3 + bx^2 + 3ax - 9$ have a common factor of $(x + a)$. Calculate a and b and fully factorise each polynomial.
   d. $(x + a)^2$ is a repeated linear factor of the polynomial $P(x) = x^3 + px^2 + 15x + a^2$. Show there are two possible polynomials satisfying this information and, for each , calculate the values of x which give the roots of the equation $x^3 + px^2 + 15x + a^2 = 0$.
10. WE18 Sketch the graphs of $y = (x + 2)(x - 1)^2$ and $y = -3x$ and calculate the coordinates of the points of intersection.
11. a. Give the equation of the graph which has the same shape as $y = -2x^3$ and a point of inflection at $(-6, -7)$.
    b. Calculate the y-intercept of the graph which is created by translating the graph of $y = x^3$ two units to the right and four units down.
    c. A cubic graph has a stationary point of inflection at $(-5, 2)$ and a y-intercept of $(0, -23)$. Calculate its exact x-intercept.
    d. A curve has the equation $y = ax^3 + b$ and contains the points $(1, 3)$ and $(-2, 39)$. Calculate the coordinates of its stationary point of inflection.
12. Find the coordinates of the points of intersection of the following.
    a. $y = 2x^3$ and $y = x^2$ b. $y = 2x^3$ and $y = x - 1$
    c. Illustrate the answers to parts a and b with a graph.

Technology active

13. a. The number of solutions to the equation $x^3 + 2x - 5 = 0$ can be found by determining the number of intersections of the graphs of $y = x^3$ and a straight line. What is the equation of this line and how many solutions does $x^3 + 2x - 5 = 0$ have?

b. Use a graph of a cubic and a linear polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.

c. Use a graph of a cubic and a quadratic polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.

d. Solve the equation $x^3 + 3x^2 - 4x = 0$.

14. The graph with equation $y = (x + a)^3 + b$ passes through the three points $(0, 0)$, $(1, 7)$, $(2, 26)$.

a. Use this information to determine the values of a and b.

b. Find the points of intersection of the graph with the line $y = x$.

c. Sketch both graphs in part **b** on the same axes.

15. a. Show the line $y = 3x + 2$ is a tangent to the curve $y = x^3$ at the point $(-1, -1)$.

b. What are the coordinates of the point where the line cuts the curve?

c. Sketch the curve and its tangent on the same axes.

d. Investigate for what values of m will the line $y = mx + 2$ have one, two or three intersections with the curve $y = x^3$.



16. A graph of a cubic polynomial with equation $y = x^3 + ax^2 + bx + 9$ has a turning point at $(3, 0)$.

a. State the factor of the equation with greatest multiplicity.

b. Determine the other x-intercept.

c. Calculate the values of a and b.

5.6 Cubic models and applications

Practical situations which use cubic polynomials as models are likely to require a restriction of the possible values the variable may take. This is called a **domain** restriction. The domain is the set of possible values of the variable that the polynomial may take. We shall look more closely at domains in other chapters.

The polynomial model should be expressed in terms of one variable.

Applications of cubic models where a maximum or minimum value of the model is sought will require identification of turning point coordinates. In a later chapter we will see how this is done. For now, obtaining turning points may require the use of graphing technology.

WORKED EXAMPLE 19

A rectangular storage container is designed to have an open top and a square base.

The base has side length x cm and the height of the container is h cm. The sum of its dimensions (the sum of the length, width and height) is 48 cm.

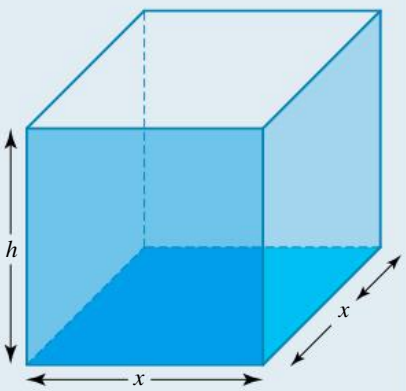
a. Express h in terms of x.

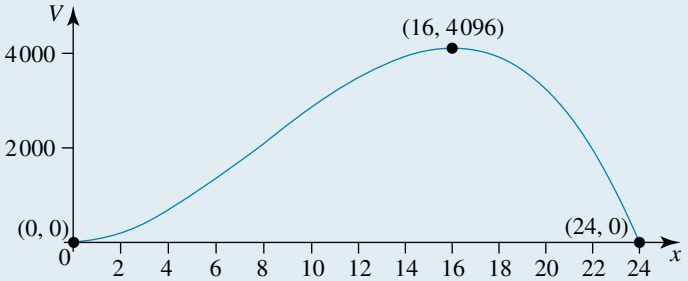
b. Show that the volume V cm^3 of the container is given by $V = 48x^2 - 2x^3$.

c. State any restrictions on the values x can take.

d. Sketch the graph of V against x for appropriate values of x, given its maximum turning point has coordinates (16, 4096).

e. Calculate the dimensions of the container with the greatest possible volume.

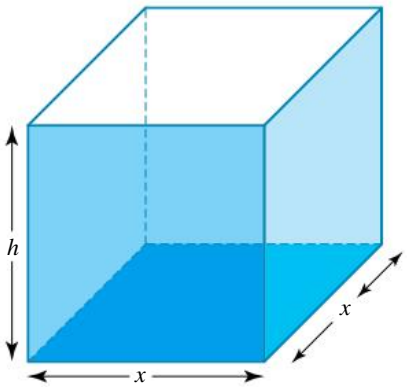


THINK	WRITE
a. Write the given information as an equation connecting the two variables.	**a.** Sum of dimensions is 48 cm. $x + x + h = 48$ $h = 48 - 2x$
b. Use the result from part **a** to express the volume in terms of one variable and prove the required statement.	**b.** The formula for volume of a cuboid is $V = lwh$ $\therefore V = x^2h$ Substitute $h = 48 - 2x$. $V = x^2(48 - 2x)$ $\therefore V = 48x^2 - 2x^3$, as required
c. State the restrictions. *Note*: It could be argued that the restriction is $0 < x < 24$ because when $x = 0$ or $x = 48$ there is no storage container, but we are adopting the closed convention.	**c.** Length cannot be negative, so $x \geq 0$. Height cannot be negative, so $h \geq 0$. $48 - 2x \geq 0$ $-2x \geq -48$ $\therefore x \leq 24$ Hence the restriction is $0 \leq x \leq 24$.
d. Draw the cubic graph but only show the section of the graph for which the restriction applies. Label the axes with the appropriate symbols and label the given turning point.	**d.** $V = 48x^2 - 2x$ $= 2x^2(24 - x)$ x-intercepts: let $V = 0$ $x^2 = 0$ or $24 - x = 0$ $\therefore x = 0$ (touch), $x = 24$ (cut) $(0, 0), (24, 0)$ are the x-intercepts. This is a negative cubic. Maximum turning point $(16, 4096)$ Draw the section for which $0 \leq x \leq 24$. 
e. 1. Calculate the required dimensions. *Note:* The maximum turning point (x, V) gives the maximum value of V and the value of x when this maximum occurs.	**e.** The maximum turning point is (16, 4096). This means the greatest volume is 4096 cm^3. It occurs when $x = 16$. $\therefore h = 48 - 2(16) \Rightarrow h = 16$ Dimensions: length = 16 cm, width = 16 cm, height = 16 cm
2. State the answer.	The container has the greatest volume when it is a cube of edge 16 cm.

Exercise 5.6 Cubic models and applications

Technology active

1. **WE19** A rectangular storage container is designed to have an open top and a square base.



The base has side length x metres and the height of the container is h metres. The total length of its 12 edges is 6 metres.

a. Express h in terms of x.

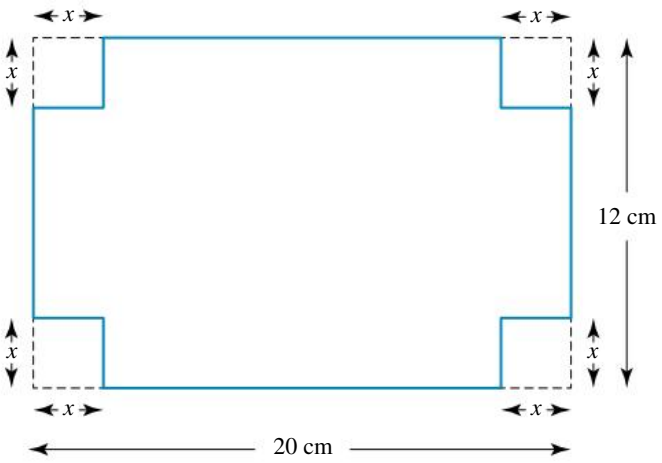
b. Show that the volume $V\ m^3$ of the container is given by $V = 1.5x^2 - 2x^3$.

c. State any restrictions on the values x can take.

d. Sketch the graph of V against x for appropriate values of x, given its maximum turning point has coordinates $(0.5, 0.125)$.

e. Calculate the dimensions of the container with the greatest possible volume.

2. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length x cm out of the four corners and folding the flaps up.





The box has length l cm, width w cm and volume $V\,\text{cm}^3$.

a. Express l, w and h in terms of x and hence express V in terms of x.

b. State any restrictions on the values of x.

c. Sketch the graph of V against x for appropriate values of x, given the unrestricted graph would have turning points at $x = 2.43$ and $x = 8.24$.

d. Calculate the length and width of the box with maximum volume and give this maximum volume to the nearest whole number.

3. The cost C dollars for an artist to produce x sculptures by contract is given by $C = x^3 + 100x + 2000$. Each sculpture is sold for $500 and as the artist only makes the sculptures by order, every sculpture produced will be paid for. However, too few sales will result in a loss to the artist.



a. Show the artist makes a loss if only 5 sculptures are produced and a profit if 6 sculptures are produced.

b. Show that the profit, P dollars, from the sale of x sculptures is given by $P = -x^3 + 400x - 2000$.

c. What will happen to the profit if a large number of sculptures are produced? Why does this effect occur?

d. Calculate the profit (or loss) from the sale of:

i. 16 sculptures

ii. 17 sculptures.

e. Use the above information to sketch the graph of the profit P for $0 \leq x \leq 20$. Place its intersection with the x-axis between two consecutive integers but don't attempt to obtain its actual x-intercepts.

f. In order to guarantee a profit is made, how many sculptures should the artist produce?

4. The number of bacteria in a slow-growing culture at time t hours after 9 am is given by $N = 54 + 23t + t^3$.

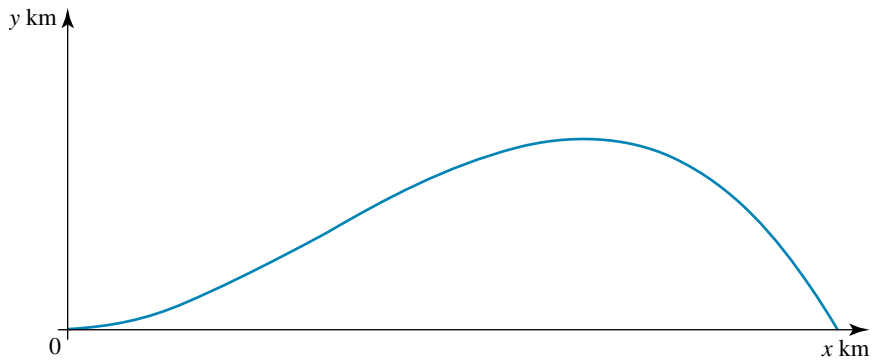
a. What is the initial number of bacteria at 9 am?

b. How long does it take for the initial number of bacteria to double?

c. How many bacteria are there by 1 pm?

d. Once the number of bacteria reaches 750, the experiment is stopped. At what time of the day does this happen?

5. Engineers are planning to build an underground tunnel through a city to ease traffic congestion. The cross-section of their plan is bounded by the curve shown.

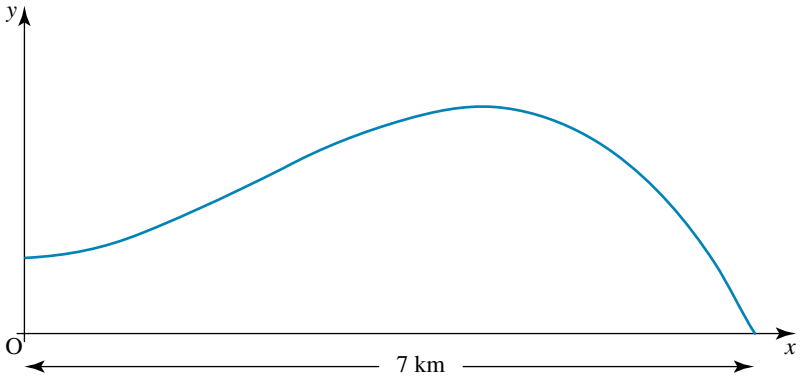


The equation of the bounding curve is $y = ax^2(x - b)$ and all measurements are in kilometres.

It is planned that the greatest breadth of the bounding curve will be 6 km and the greatest height will be 1 km above this level at a point 4 km from the origin.

a. Determine the equation of the bounding curve.

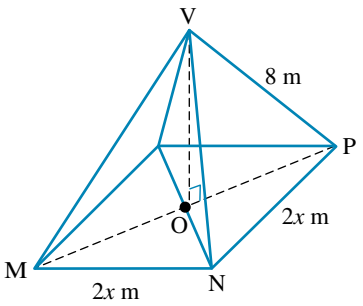
b. If the greatest breadth of the curve was extended to 7 km, what would be the greatest height of the curve above this new lowest level?



6. Find the smallest positive integer and the largest negative integer for which the difference between the square of 5 more than this number and the cube of 1 more than the number exceeds 22.

7. A tent used by a group of bushwalkers is in the shape of a square-based right pyramid with a slant height of 8 metres.

For the figure shown, let OV, the height of the tent, be h metres and the edge of the square base be $2x$ metres.



a. Use Pythagoras' theorem to express the length of the diagonal of the square base of the tent in terms of x.

b. Use Pythagoras' theorem to show $2x^2 = 64 - h^2$.

c. The volume V of a pyramid is found using the formula $V = \frac{1}{3}Ah$ where A is the area of the base of the pyramid. Use this formula to show that the volume of space contained within the bushwalkers' tent is given by $V = \frac{1}{3}\left(128h - 2h^3\right)$.

d. **i.** If the height of the tent is 3 metres, what is the volume?

ii. What values for the height does this mathematical model allow?

e. Sketch the graph of $V = \frac{1}{3}\left(128h - 2h^3\right)$ for appropriate values of h and estimate the height for which the volume is greatest.

f. The greatest volume is found to occur when the height is half the length of the base. Use this information to calculate the height which gives the greatest volume and compare this value with your estimate from your graph in part **e**.

8. A cylindrical storage container is designed so that it is open at the top and has a surface area of 400π cm^2. Its height is h cm and its radius is r cm.



a. Show that $h = \frac{400 - r^2}{2r}$.

b. Show that the volume V cm^3 the container can hold is given by $V = 200\pi r - \frac{1}{2}\pi r^3$.

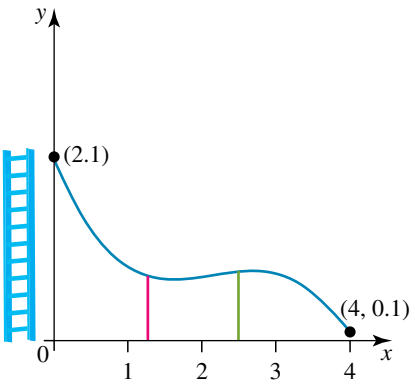
c. State any restrictions on the values r can take.

d. Sketch the graph of V against r for appropriate values of r.

e. Find the radius and height of the container if the volume is 396π cm^3.

f. State the maximum possible volume to the nearest cm^3 if the maximum turning point on the graph of $y = 200\pi x - \frac{1}{2}\pi x^3$ has an x-coordinate of $\frac{20}{\sqrt{3}}$.

9. A new playground slide for children is to be constructed at a local park. At the foot of the slide the children climb a vertical ladder to reach the start of the slide. The slide must start at a height of 2.1 metres above the ground and end at a point 0.1 metres above the ground and 4 metres horizontally from its foot. A model for the slide is $h = ax^3 + bx^2 + cx + d$ where h metres is the height of the slide above ground level at a horizontal distance of x metres from its foot. The foot is at the origin.



The ladder supports the slide at one end and the slide also requires two vertical struts as support. One strut of length 1 metre is placed at a point 1.25 metres horizontally from the foot of the slide and the other is placed at a point 1.5 metres horizontally from the end of the slide and is of length 1.1 metres.

a. Give the coordinates of 4 points which lie on the cubic graph of the slide.

b. State the value of d in the equation of the slide.

c. Form a system of 3 simultaneous equations, the solutions to which give the coefficients a, b, c in the equation of the slide.

d. The equation of the slide can be shown to be $y = -0.164x^3 + x^2 - 1.872x + 2.1$. Use this equation to calculate the length of a third strut thought necessary at $x = 3.5$. Give your answer to 2 decimal places.

10. Since 1988, the world record times for the men's 100-m sprint can be roughly approximated by the cubic model $T(t) = -0.00005(t - 6)^3 + 9.85$ where T is the time in seconds and t is the number of years since 1988.

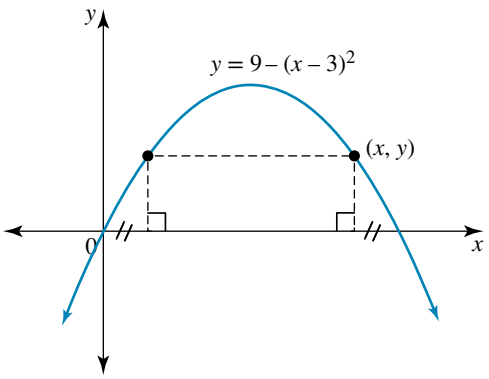


a. In 1991 the world record was 9.86 seconds and in 2008 the record was 9.72 seconds. Compare these times with those predicted by the cubic model.

b. Sketch the graph of T versus t from 1988 to 2008.

c. What does the model predict for 2016? Is the model likely to be a good predictor beyond 2016?

11. A rectangle is inscribed under the parabola $y = 9 - (x - 3)^2$ so that two of its corners lie on the parabola and the other two lie on the x-axis at equal distances from the intercepts the parabola makes with the x-axis.



a. Calculate the x-intercepts of the parabola.

b. Express the length and width of the rectangle in terms of x.

c. Hence show that the area of the rectangle is given by $A = -2x^3 + 18x^2 - 36x$.

d. For what values of x is this a valid model of the area?

e. Calculate the value(s) of x for which $A = 16$.

12. A pathway through the countryside passes through 5 scenic points. Relative to a fixed origin, these points have coordinates $A(-3, 0)$, $B\left(-\sqrt{3}, -12\sqrt{3}\right)$, $C\left(\sqrt{3}, 12\sqrt{3}\right)$, $D(3, 0)$; the fifth scenic point is the origin, $O(0, 0)$. The two-dimensional shape of the path is a cubic polynomial.



a. State the maximum number of turning points and x-intercepts that a cubic graph can have.

b. Determine the equation of the pathway through the 5 scenic points.

c. Sketch the path, given that points B and C are turning points of the cubic polynomial graph.

d. It is proposed that another pathway be created to link B and C by a direct route. Show that if a straight-line path connecting B and C is created, it will pass through O and give the equation of this line.

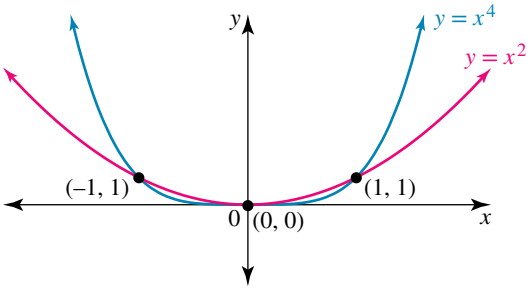
e. An alternative plan is to link B and C by a cubic path which has a stationary point of inflection at O. Determine the equation of this path.

5.7 Graphs of quartic polynomials

A quartic polynomial is a polynomial of degree 4 and is of the form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$ and $a,\ b,\ c,\ d,\ e \in R$.

5.7.1 Graphs of quartic polynomials of the form $y = a(x - b)^4 + c$

The simplest quartic polynomial graph has the equation $y = x^4$. As both negative and positive numbers raised to an even power, in this case 4, will be positive, the long-term behaviour of the graph of $y = x^4$ must be that as $x \to -\infty$ or as $x \to \infty$, then $y \to \infty$.



The graph of $y = x^4$ is similar to that of the parabola $y = x^2$. Both graphs are concave up with a minimum turning point at $(0, 0)$ and both contain the points $(-1, 1)$ and $(1, 1)$. However, for the intervals where $x < -1$ and $x > 1$, the graph of $y = x^4$ lies above the parabola. This is because $x^4 > x^2$ for these intervals. Likewise, the graph of $y = x^4$ lies below that of the parabola for the intervals $-1 < x < 0$ and $0 < x < 1$, since $x^4 < x^2$ for these intervals.

Despite these differences, the two graphs are of sufficient similarity to enable us to obtain the key features of graphs of quartic polynomials of the form $y = a(x - b)^4 + c$ in much the same manner as for quadratics of the form $y = a(x - b)^2 + c$.

Under a dilation of a units, a horizontal translation of b units and a vertical translation of c units, the graph of $y = x^4$ is transformed to that of $y = a(x - b)^4 + c$.

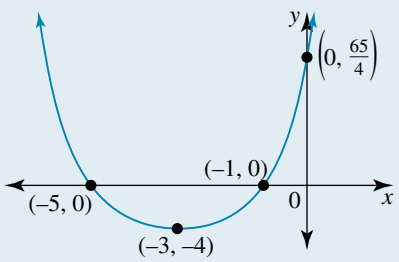
The graph of $y = a(x - b)^4 + c$ has the following features.

- **A turning point with coordinates (b, c)**
- **If $a > 0$, the turning point is a minimum and if $a < 0$ it is a maximum.**
- **Axis of symmetry with equation $x = b$**
- **Zero, one or two x-intercepts. These are obtained as the solution to the equation $a(x - b)^4 + c = 0$.**

WORKED EXAMPLE 20

Sketch the graphs of:

a. $y = \frac{1}{4}(x+3)^4 - 4$ **b.** $y = -(3x-1)^4 - 7$

THINK	WRITE
a. 1. State the coordinates and type of turning point.	**a.** $y = \frac{1}{4}(x+3)^4 - 4$ Turning point is $(-3, -4)$. As $a = \frac{1}{4}$, $a > 0$, so the turning point is a minimum.
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = \frac{1}{4}(3)^4 - 4$ $= \frac{81}{4} - \frac{16}{4}$ $= \frac{65}{4}$ y-intercept: $\left(0, \frac{65}{4}\right)$
3. Determine whether there will be any x-intercepts.	As the y-coordinate of the minimum turning point is negative, the concave up graph must pass through the x-axis.
4. Calculate the x-intercepts. *Note:* $\pm$ is needed in taking the fourth root of each side.	x-intercepts: let $y = 0$ $\frac{1}{4}(x+3)^4 - 4 = 0$ $\frac{1}{4}(x+3)^4 = 4$ $\therefore (x+3)^4 = 16$ Take the fourth root of both sides. $(x+3) = \pm\sqrt[4]{16}$ $x + 3 = \pm 2$ $\therefore x = -5$ or $x = -1$ x-intercepts: $(-5, 0)$ and $(-1, 0)$
5. Sketch the graph.	

b. 1. Express the equation in the form $y = a(x-b)^4 + c$.

b. $y = -(3x-1)^4 - 7$

$= -\left(3\left(x - \frac{1}{3}\right)\right)^4 - 7$

$= -81\left(x - \frac{1}{3}\right)^4 - 7$

2. State the coordinates of the turning point and its type.

The graph has a maximum turning point at $\left(\frac{1}{3}, -7\right)$.

3. Calculate the y-intercept.

y-intercept: let $x = 0$ in the original form

$y = -(3x-1)^4 - 7$

$= -(-1)^4 - 7$

$= -(1) - 7$

$= -8$

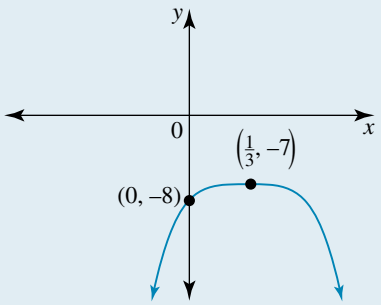
y-intercept: $(0, -8)$

4. Determine whether there will be any x-intercepts.

As the y-coordinate of the maximum turning point is negative, the concave down graph will not pass through the x-axis.

5. Sketch the graph.

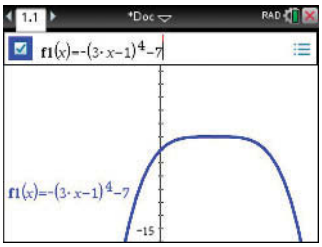
The graph is symmetric about its axis of symmetry, $x = \frac{1}{3}$.



TI | THINK

b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = -(3x-1)^4 - 7$, then press ENTER.

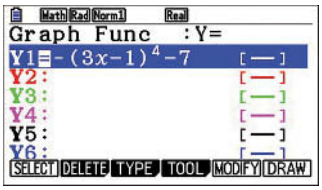
WRITE

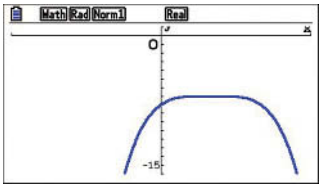


CASIO | THINK

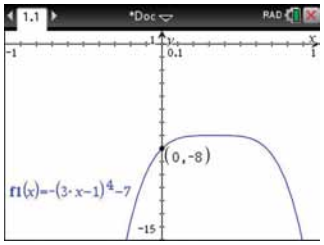
b.1. On a Graph screen, complete the entry line for $y1$ as $y1 = -(3x-1)^4 - 7$, then press EXE. Select DRAW by pressing F6.

WRITE

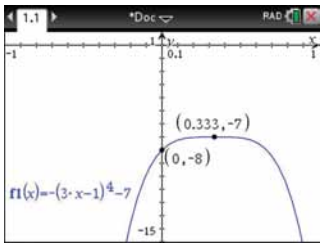




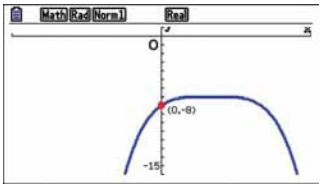
2. To find the *y*-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



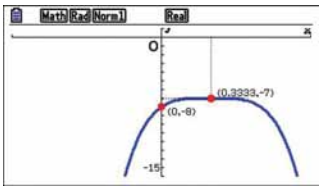
3. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



2. To find the *y*-intercept, select G-Solv by pressing SHIFT then F5, then select Y-ICEPT by pressing F4. Press EXE.



3. To find the maximum, select G-Solv by pressing SHIFT then F5, then select MAX by pressing F2. Press EXE.



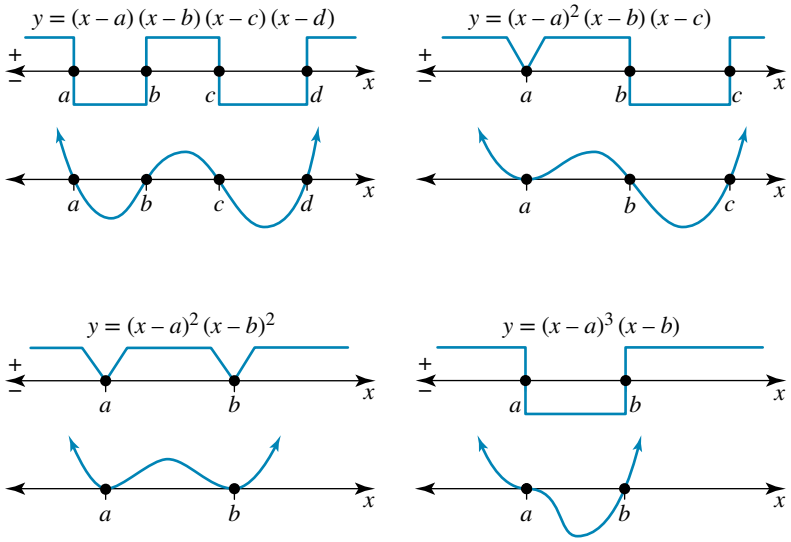
5.7.2 Quartic polynomials which can be expressed as the product of linear factors

Not all quartic polynomials have linear factors. However, the graphs of those which can be expressed as the product of linear factors can be readily sketched by analysing these factors.

A quartic polynomial may have up to 4 linear factors since it is of fourth degree.
The possible combinations of these linear factors are:

- **four distinct linear factors** $y = (x - a)(x - b)(x - c)(x - d)$
- **one repeated linear factor** $y = (x - a)^2(x - b)(x - c)$
- **two repeated linear factors** $y = (x - a)^2(x - b)^2$
- **one factor of multiplicity 3** $y = (x - a)^3(x - b)$
- **one factor of multiplicity 4** $y = (x - a)^4$.
  This case in which the graph has a minimum turning point at $(a, 0)$ **has already been considered.**

Given the long-term behaviour of a quartic polynomial whereby $y \to \infty$ as $x \to \pm\infty$ for a positive coefficient of the term in x^4, the sign diagrams and accompanying shape of the graphs must be of the form shown in the diagrams.



For a negative coefficient of x^4, $y \to -\infty$ as $x \to \pm\infty$, so the sign diagrams and graphs are inverted.

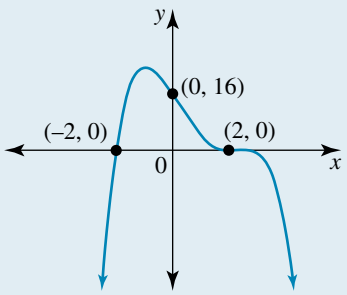
The single factor identifies an *x*-intercept where the graph cuts the axis; a factor of multiplicity 2 identifies an *x*-intercept which is a turning point; and the factor of multiplicity 3 identifies an *x*-intercept which is a stationary point of inflection.

WORKED EXAMPLE 21

Sketch the graph of $y = (x + 2)(2 - x)^3$.

THINK	WRITE
1. Calculate the *x*-intercepts.	$y = (x + 2)(2 - x)^3$ *x*-intercepts: let $y = 0$ $(x + 2)(2 - x)^3 = 0$ $\therefore x + 2 = 0$ or $(2 - x)^3 = 0$ $\therefore x = -2$ or $x = 2$ *x*-intercepts: $(-2, 0)$ and $(2, 0)$
2. Interpret the nature of the graph at each *x*-intercept.	Due to the multiplicity of each factor, at $x = -2$ the graph cuts the *x*-axis and at $x = 2$ it saddle-cuts the *x*-axis. The point $(2, 0)$ is a stationary point of inflection.
3. Calculate the *y*-intercept.	*y*-intercept: let $x = 0$ $y = (2)(2)^3$ $= 16$ *y*-intercept: $(0, 16)$
4. Determine the sign of the coefficient of the leading term and identify the long-term behaviour of the graph.	Leading term is $(x)(-x)^3 = -x^4$. The coefficient of the leading term is negative, so as $x \to \pm\infty$ then $y \to -\infty$. This means the sketch of the graph must start and finish below the *x*-axis.

5. Sketch the graph.



Resources

Interactivity: Graph plotter: Polynomials of higher degree (int-2569)

studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 6

Quartic polynomials Summary screen and practice questions

Exercise 5.7 Graphs of quartic polynomials

Technology active

1. a. On the same set of axes, sketch the graphs of $y = x^4$, $y = 2x^4$ and $y = \frac{1}{2}x^4$. Label the points for which $x = -1$, 0 and 1.

b. On the same set of axes, sketch the graphs of $y = x^4$, $y = -x^4$, $y = -2x^4$ and $y = (-2x)^4$. Label the points for which $x = -1$, 0 and 1.

c. On the same set of axes, sketch the graphs of $y = x^4$, $y = -(x + 1)^4$ and $y = (1 - x)^4$. Label the points for which $x = -1$, 0 and 1.

d. On the same set of axes, sketch the graphs of $y = x^4$, $y = x^4 + 2$ and $y = -x^4 - 1$. Label the points for which $x = -1$, 0 and 1.

2. WE20 Sketch the following graphs.

a. $y = (x - 2)^4 - 1$ **b.** $y = -(2x + 1)^4$

3. a. State the coordinates and nature of the turning point of the graph of $y = \frac{1}{8}(x + 2)^4 - 2$ and sketch the graph.

b. After the graph of $y = x^4$ has been reflected about the x-axis, translated 1 unit to the right and translated downwards 1 unit, state:

i. the coordinates and nature of its turning point

ii. its equation and sketch its graph.

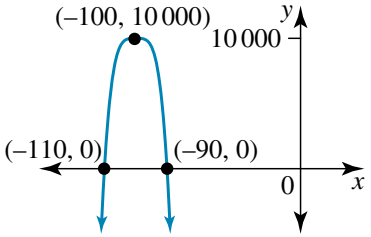
c. The equation of the graph of a quartic polynomial is of the form $y = a(x - b)^4 + c$. Determine the equation given there is a y-intercept at $(0, 64)$ and a turning point at $(4, 0)$.

4. Sketch the following graphs, identifying the coordinates of the turning point and any point of intersection with the coordinate axes.

a. $y = (x-1)^4 - 16$ **b.** $y = \frac{1}{9}(x+3)^4 + 12$ **c.** $y = 250 - 0.4(x+5)^4$

d. $y = -\left(6(x-2)^4 + 11\right)$ **e.** $y = \frac{1}{8}(5x-3)^4 - 2$ **f.** $y = 1 - \left(\frac{2-7x}{3}\right)^4$

5. Determine a possible equation for each of the following.



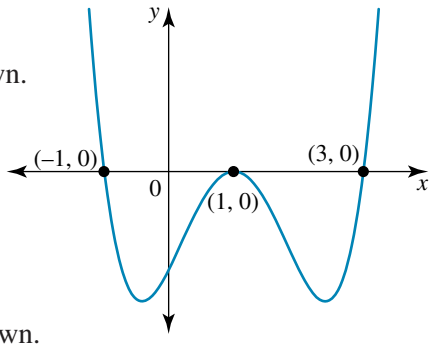
a. A quartic graph with the same shape as $y = \frac{2}{3}x^4$ but whose turning point has the coordinates $(-9, -10)$.

b. The curve with the equation $y = a(x+b)^4 + c$ which has a minimum turning point at $(-3, -8)$ and passes through the point $(-4, -2)$.

c. A curve has the equation $y = (ax+b)^4$ where $a > 0$ and $b < 0$. The points $(0, 16)$ and $(2, 256)$ lie on the graph.

d. The graph shown has the equation $y = a(x-b)^4 + c$.

6. a. Sketch the graph of $y = -(x+2)(x-3)(x-4)(x+5)$ showing all intercepts with the coordinate axes.



b. The graph of a quartic polynomial with three x-intercepts is shown.

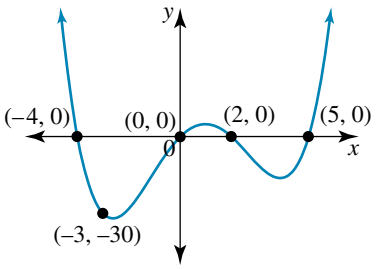
i. For each of the three x-intercepts, state the corresponding factor in the equation of the graph.

ii. Write the form of the equation.

iii. The graph cuts the y-axis at $(0, -6)$. Determine the equation of the graph.

7. WE21 Sketch the graph of $y = (x+2)^2(2-x)^2$.

8. Give a suitable equation for the graph of the quartic polynomial shown.



9. Sketch the following quartic polynomials without attempting to locate any turning points that do not lie on the coordinate axes.

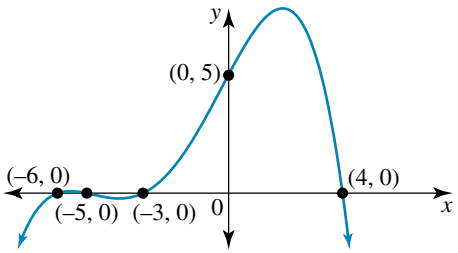
a. $y = (x+8)(x+3)(x-4)(x-10)$ **b.** $y = -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10)$

c. $y = -2(x+7)(x-1)^2(2x-5)$ **d.** $y = \frac{2}{3}x^2(4x-15)^2$

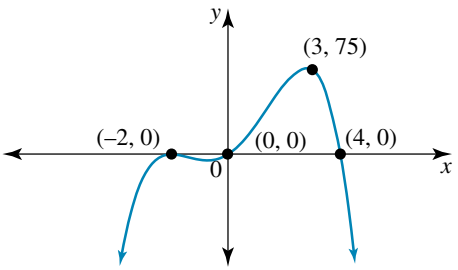
e. $y = 3(1+x)^3(4-x)$ **f.** $y = (3x+10)(3x-10)^3$

10. For each of the following quartic graphs, form a possible equation.

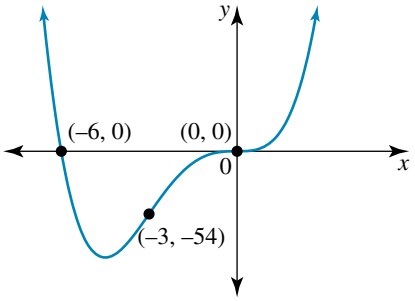
a.



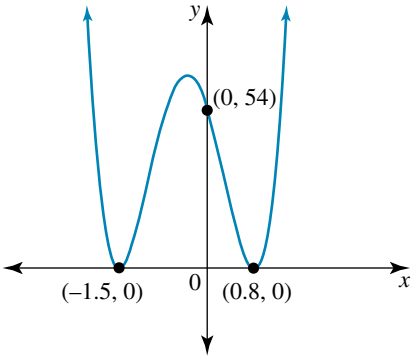
b.



c.



d.



11. a. A graph with the equation $y = a(x - b)^4 + c$ has a maximum turning point at $(-2, 4)$ and cuts the y-axis at $y = 0$. Determine its equation.

b. Sketch the graph of the equation, showing all intercepts and turning points.

12. The graph of $y = a(x + b)^4 + c$ passes through the points $(-2, 3)$ and $(4, 3)$.

a. State the equation of its axis of symmetry.

b. Given the greatest y-value the graph reaches is 10, state the coordinates of the turning point of the graph.

c. Determine the equation of the graph.

d. Calculate the coordinates of the point of intersection with the y-axis.

e. Calculate the value(s) of any intercepts the graph makes with the x-axis.

f. Sketch the graph.

13. Use technology to sketch the graph of $y = x^4 - x^3 - 12x^2 - 4x + 4$, locating turning points and intersections with the coordinate axes. Express coordinates to 2 decimal places where appropriate.

5.8 Solving polynomial equations

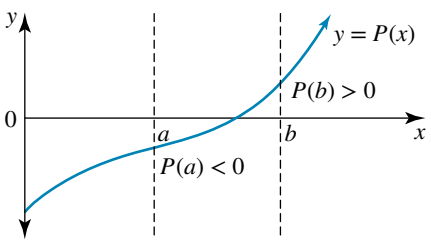
For any polynomial $P(x)$ the values of the x-axial intercepts of the graph of $y = P(x)$ are the roots of the polynomial equation $P(x) = 0$. These roots can always be obtained if the polynomial is linear or quadratic, or if the polynomial can be expressed as a product of linear factors. However, there are many polynomial equations that cannot be solved by an algebraic method. In such cases, if an approximate value of a root can be estimated, then this value can be improved upon by a method of **iteration**. An iterative procedure is one which is repeated by using the values obtained from one stage to calculate the value of the next stage and so on.

Existence of roots

For a polynomial $P(x)$, if $P(a)$ and $P(b)$ are of opposite signs, then there is at least one value of $x \in (a, b)$ where $P(x) = 0$.

For example, in the diagram shown, $P(a) < 0$ and $P(b) > 0$. The graph cuts the x-axis at a point for which $a < x < b$.

This means that the equation $P(x) = 0$ has a root which lies in the interval (a, b). This gives an estimate of the root. Often the values of a and b are integers and these may be found through trial and error. Alternatively, if the polynomial graph has been sketched, it may be possible to obtain their values from the graph. Ideally, the values of a and b are not too far apart in order to avoid, if possible, there being more than one x-intercept of the graph, or one root of the polynomial equation, that lies between them.



5.8.1 The method of bisection

Either of the values of a and b for which $P(a)$ and $P(b)$ are of opposite sign provides an estimate for one of the roots of the equation $P(x) = 0$. The **method of bisection** is a procedure for improving this estimate by halving the interval in which the root is known to lie.

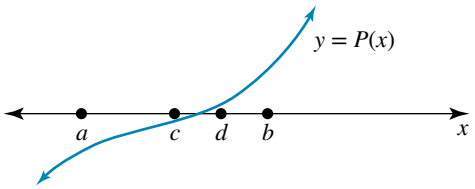
Let c be the midpoint of the interval $[a, b]$ so $c = \frac{1}{2}(a + b)$.

The value $x = c$ becomes an estimate of the root.

By testing the sign of $P(c)$ it can be determined whether the root lies in $(a, c]$ or in $[c, b)$.

In the diagram shown, $P(a) < 0$ and $P(c) < 0$ so the root does not lie between a and c. It lies between c and b since $P(c) < 0$ and $P(b) > 0$.

The midpoint d of the interval $[c, b]$ can then be calculated. The value of d may be an acceptable approximation to the root. If not, the accuracy of the approximation can be further improved by testing which of $[c, d]$ and $[d, b]$ contains the root and then halving that interval and so on. The use of some form of technology helps considerably with the calculations as it can take many iterations to achieve an estimate that has a high degree of accuracy.

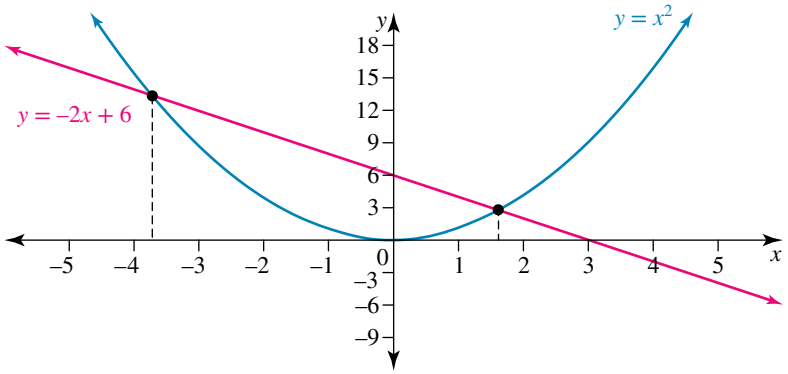


Any other roots of the polynomial equation may be estimated by the same method once an interval in which each root lies has been established.

5.8.2 Using the intersections of two graphs to estimate solutions to equations

Consider the quadratic equation $x^2 + 2x - 6 = 0$. Although it can be solved algebraically to give $x = \pm\sqrt{7} - 1$, we shall use it to illustrate another non-algebraic method for solving equations as shown in 5.5.3. If the equation is rearranged to $x^2 = -2x + 6$, then any solutions to the equation are the x-coordinates of any points of intersection of the parabola $y = x^2$ and the straight line $y = -2x + 6$.

Both of these polynomial graphs are relatively simple graphs to draw. The line can be drawn accurately using its intercepts with the coordinate axes, and the parabola can be drawn with reasonable accuracy by plotting some points that lie on it. The diagram of their graphs shows there are two points of intersection and hence that the equation $x^2 + 2x - 6 = 0$ has two roots.



Estimates of the roots can be read from the graph. One root is approximately $x = -3.6$ and the other is approximately $x = 1.6$. (This agrees with the actual solutions which, to 3 decimal places, are $x = -3.646$ and $x = 1.646$).

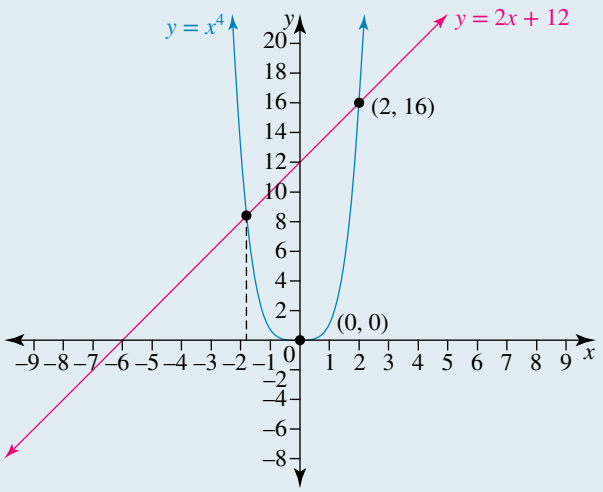
Alternatively, we can confidently say that one root lies in the interval $[-4, -3]$ and the other in the interval $[1, 2]$ and by applying the method of bisection the roots could be obtained to a greater accuracy than that of the estimates that were read from the graph.

To use the graphical method to solve the polynomial equation $H(x) = 0$:

- **Rearrange the equation into the form $P(x) = Q(x)$ where each of the polynomials $P(x)$ and $Q(x)$ have graphs that can be drawn quite simply and accurately.**
- **Sketch the graphs of $y = P(x)$ and $y = Q(x)$ with care.**
- **The number of intersections determines the number of solutions to the equation $H(x) = 0$.**
- **The x-coordinates of the points of intersection are the solutions to the equation.**
- **Estimate these x-coordinates by reading off the graph.**
- **Alternatively, an interval in which the x-coordinates lie can be determined from the graph and the method of bisection applied to improve the approximation.**

WORKED EXAMPLE 22

Use a graphical method to estimate any solutions to the equation $x^4 - 2x - 12 = 0$.

THINK	WRITE
1. Rearrange the equation so that it is expressed in terms of two familiar polynomials.	$x^4 - 2x - 12 = 0$ $x^4 = 2x + 12$
2. State the equations of the two polynomial graphs to be drawn.	The solutions to the equation are the x-coordinates of the points of intersection of the graphs of $y = x^4$ and $y = 2x + 12$.
3. Determine any information which will assist you to sketch each graph with some accuracy.	The straight line $y = 2x + 12$ has a y-intercept at $(0, 12)$ and an x-intercept at $(-6, 0)$. The quartic graph $y = x^4$ has a minimum turning point at $(0, 0)$ and contains the points $(\pm 1, 1)$ and $(\pm 2, 16)$.
4. Carefully sketch each graph on the same set of axes.	
5. State the number of solutions to the original equation given.	As there are two points of intersection, the equation $x^4 - 2x - 12 = 0$ has two solutions.

6. Use the graph to obtain the solutions.

From the graph it is clear that one point of intersection is at (2, 16), so $x = 2$ is an exact solution of the equation.
An estimate of the x-coordinate of the other point of intersection is approximately -1.7, so $x = -1.7$ is an approximate solution to the equation.

5.8.3 Estimating coordinates of turning points

If the linear factors of a polynomial are known, sketching the graph of the polynomial is a relatively easy task to undertake. Turning points, other than those which lie on the x-axis, have largely been ignored, or, at best, allowed to occur where our pen and 'empathy' for the polynomial have placed them. Promises of rectifying this later when calculus is studied have been made. While this remains the case, we will address this unfinished aspect of our graph-sketching by considering a numerical method of systematic trial and error to locate the approximate position of a turning point.

For any polynomial with zeros at $x = a$ and $x = b$, its graph will have at least one turning point between $x = a$ and $x = b$.

To illustrate, consider the graph of $y = (x + 2)(x - 1)(x - 4)$. The factors show there are three x-intercepts: one at $x = -2$, a second one at $x = 1$ and a third at $x = 4$.

There would be a turning point between $x = -2$ and $x = 1$, and a second turning point between $x = 1$ and $x = 4$. The first turning point must be a maximum and the second one must be a minimum in order to satisfy the long-term behaviour requirements of a positive cubic polynomial.

The interval in which the x-coordinate of the maximum turning point lies can be narrowed using a table of values.

x	-2	-1.5	-1	-0.5	0	0.5	1
y	0	6.875	10	10.125	8	4.375	0

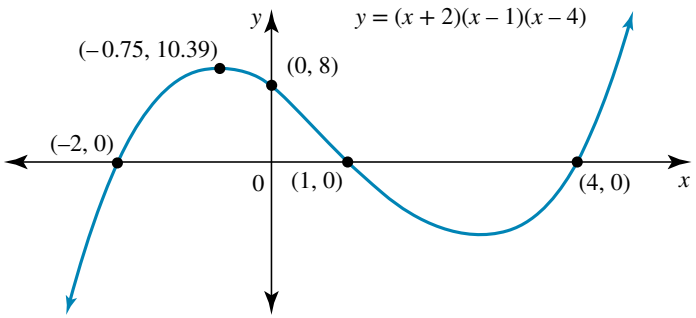
As a first approximation, the maximum turning point lies near the point $(-0.5, 10.125)$. Zooming in further around $x = -0.5$ gives greater accuracy.

x	-0.75	-0.5	-0.25
y	10.390625	10.125	9.2989

An improved estimate is that the maximum turning point lies near the point $(-0.75, 10.39)$. The process could continue by zooming in around $x = -0.75$ if greater accuracy is desired.

An approximate position of the minimum turning point could be estimated by the same numerical method of systematic trial and error.

The shape of this positive cubic with a y-intercept at $(0, 8)$ could then be sketched.



An alternative approach

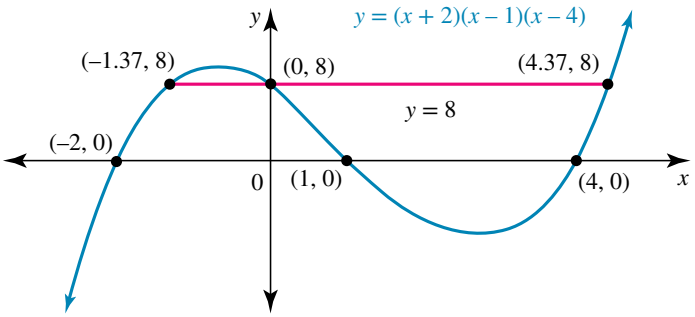
For any polynomial $P(x)$, if $P(a) = P(b)$ then its graph will have at least one turning point between $x = a$ and $x = b$. This means for the cubic polynomial shown in the previous diagram, the maximum turning point must lie between the x-values for which $y = 8$ (the y-intercept value).

Substitute $y = 8$ into $y = (x + 2)(x - 1)(x - 4)$:

$$\begin{aligned} x^3 - 3x^2 - 6x + 8 &= 8 \\ x^3 - 3x^2 - 6x &= 0 \\ x\left(x^2 - 3x - 6\right) &= 0 \\ x = 0 \text{ or } x^2 - 3x - 6 &= 0 \end{aligned}$$

As the cubic graph must have a maximum turning point, the quadratic equation must have a solution. Solving it would give the negative solution as $x = -1.37$. Rather than test values between $x = -2$ and $x = 1$ as we have previously, the starting interval for testing values could be narrowed to between $x = -1.37$ and $x = 0$.

The positive solution $x = 4.37$ indicates that the minimum turning point lies between $x = 0$ and $x = 4.37$. In this case the interval between the two positive x-intercepts provides a narrower and therefore better interval to zoom into.



WORKED EXAMPLE 23

a. State an interval in which the x-coordinate of the minimum turning point on the graph of $y = x(x - 2)(x + 3)$ must lie.

b. Use a numerical method to zoom in on this interval and hence estimate the position of the minimum turning point of the graph, with the x-coordinate correct to 1 decimal place.

THINK

a. 1. State the values of the x-intercepts in increasing order.

2. Determine the pair of x-intercepts between which the required turning point lies.

b. 1. Construct a table of values which zooms in on the interval containing the required turning point.

2. State an estimate of the position of the turning point.

WRITE

a. $y = x(x - 2)(x + 3)$
The x-intercepts occur when $x = 0, x = 2, -3$.
In increasing order they are $x = -3$, $x = 0$, $x = 2$.
The graph is a positive cubic so the first turning point is a maximum and the second is a minimum.
The minimum turning point must lie between $x = 0$ and $x = 2$.

b. Values of the polynomial calculated over the interval $[0, 2]$ are tabulated.

x	0	0.5	1	1.5	2
y	0	−2.625	−4	−3.375	0

The turning point is near $(1, -4)$.

3. Zoom in further to obtain a second estimate.

Zooming in around $x = 1$ gives the table of values:

x	0.9	1	1.1	1.2
y	−3.861	−4	−4.059	−4.032

4. State the approximate position.

The minimum turning point is approximately $(1.1, -4.059)$.

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Units 1 & 2 › Area 2 › Sequence 4 › Concept 7

Solving polynomial equations Summary screen and practice questions

Exercise 5.8 Solving polynomial equations

Technology active

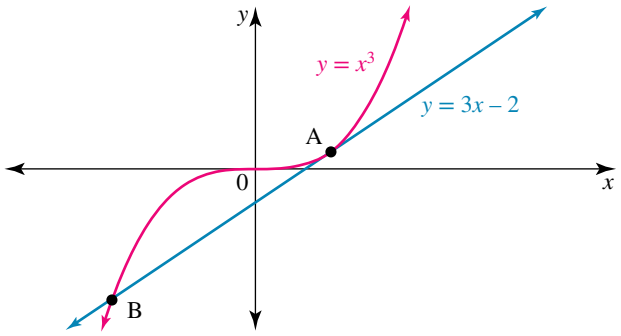
1. MC For a polynomial equation $P(x) = 0$, it is known that a solution to this equation lies in the interval $x \in (b, c)$. Which one of the following statements supports this conclusion?
   A. $P(b) > 0, P(c) > 0$ **B.** $P(b) < 0, P(c) < 0$ **C.** $P(b) > 0, P(c) < 0$
2. The equation $x^3 + 7x - 14 = 0$ is known to have exactly one solution.
   a. Show that this solution does not lie between $x = -2$ and $x = -1$.
   b. Show that the solution does lie between $x = 1$ and $x = 2$.
   c. Use the midpoint of the interval $[1, 2]$ to deduce a narrower interval $[a, b]$ in which the root of the equation lies.
3. Let $P(x) = 3x^2 - 3x - 1$. The equation $P(x) = 0$ has two solutions, one negative and one positive.
   a. Evaluate $P(-2)$ and $P(0)$ and hence explain why the negative solution to the equation lies in the interval $[-2, 0]$.
   b. Carry out the method of bisection twice to narrow the interval in which the negative solution lies.
   c. Using your answer from part b, state an estimate of the negative solution to the equation.
4. For each of the following polynomials, show that there is a zero of each in the interval $[a, b]$.
   a. $P(x) = x^2 - 12x + 1, a = 10, b = 12$
   b. $P(x) = -2x^3 + 8x + 3, a = -2, b = -1$
   c. $P(x) = x^4 + 9x^3 - 2x + 1, a = -2, b = 1$
   d. $P(x) = x^5 - 4x^3 + 2, a = 0, b = 1$
5. The following polynomial equations are formed using the polynomials in question 4. Use the method of bisection to obtain two narrower intervals in which the root lies and hence give an estimate of the root which lies in the interval $[a, b]$.
   a. $x^2 - 12x + 1 = 0, a = 10, b = 12$
   b. $-2x^3 + 8x + 3 = 0, a = -2, b = -1$
   c. $x^4 + 9x^3 - 2x + 1 = 0, a = -2, b = 1$
   d. $x^5 - 4x^3 + 2 = 0, a = 0, b = 1$



6. Consider the cubic polynomial $P(x) = x^3 + 3x^2 - 7x - 4$.
   a. Show the equation $x^3 + 3x^2 - 7x - 4 = 0$ has a root which lies between $x = 1$ and $x = 2$.
   b. State a first estimate of the root.
   c. Carry out two iterations of the method of bisection by hand to obtain two further estimates of this root.
   d. Continue the iteration using technology until the error in using this estimate as the root of the equation is less than 0.05.
7. The quadratic equation $5x^2 - 26x + 24 = 0$ has a root in the interval for which $1 \leq x \leq 2$.
   a. Use the method of bisection to obtain this root correct to 1 decimal place.
   b. What is the other root of this equation?
   c. Comment on the efficiency of the method of bisection.
8. Consider the polynomial defined by the rule $y = x^4 - 3$.
   a. Complete the table of values for the polynomial.

x	-2	-1	0	1	2
y					

   b. Hence, state an interval in which $\sqrt[4]{3}$ lies.
   c. Use the method of bisection to show that $\sqrt[4]{3} = 1.32$ to 2 decimal places.
9. The graph of $y = x^4 - 2x - 12$ has two x-intercepts.
   a. Construct a table of values for this polynomial rule for $x = -3, -2, -1, 0, 1, 2, 3$.
   b. Hence state an exact solution to the equation $x^4 - 2x - 12 = 0$.
   c. State an interval within which the other root of the equation lies and use the method of bisection to obtain an estimate of this root correct to 1 decimal place.
10. Consider the polynomial equation $P(x) = x^3 + 5x - 2 = 0$.
    a. Determine an interval $[a, b]$, $a, b \in Z$ in which there is a root of this equation.
    b. Use the method of bisection to obtain this root with an error less than 0.1.
    c. State the equations of two graphs, the intersections of which would give the number of solutions to the equation.
    d. Sketch the two graphs and hence state the number of solutions to the equation $P(x) = x^3 + 5x - 2 = 0$. Does the diagram support the answer obtained in part **b**?
11. **WE23** Use a graphical method to estimate any solutions to the equation $x^4 + 3x - 4 = 0$.
12. Use a graphical method to estimate any solutions to the equation $x^3 - 6x + 4 = 0$.
13. The diagram shows that the line $y = 3x - 2$ is a tangent to the curve $y = x^3$ at a point A and that the line intersects the curve again at a point B.



   a. Form the polynomial equation $P(x) = 0$ for which the x-coordinates of the points A and B are solutions.
   b. Describe the number and multiplicity of the linear factors of the polynomial specified in part **a**.

c. Use an algebraic method to calculate the exact roots of the polynomial equation specified in part **a** and hence state the coordinates of the points A and B.

d. Using the graph, state how many solutions there are to the equation $x^3 - 3x + 1 = 0$.

14. a. **WE24** State an interval in which the x-coordinate of the maximum turning point on the graph of $y = -x(x + 2)(x - 3)$ must lie.

b. Use a numerical method to zoom in on this interval and hence estimate the position of the maximum turning point of the graph with the x-coordinate correct to 1 decimal place.

15. Use a numerical systematic trial and error process to estimate the position of the following turning points. Express the x-coordinate correct to 1 decimal place.

a. The maximum turning point of $y = (x + 4)(x - 2)(x - 6)$

b. The minimum turning point of $y = x(2x + 5)(2x + 1)$

c. The maximum and minimum turning points of $y = x^2 - x^4$

16. Consider the cubic polynomial $y = 2x^3 - x^2 - 15x + 9$.

a. State the y-intercept.

b. What other points on the graph have the same y-coordinate as the y-intercept?

c. Between which two x-values does the maximum turning point lie?

d. Use a numerical method to zoom in on this interval and hence estimate the position of the maximum turning point of the graph, with the x-coordinate correct to 1 decimal place.

17. For the following polynomials, $P(0) = d$. Solve the equations $P(x) = d$ and hence state intervals in which the turning points of the graphs of $y = P(x)$ lie.

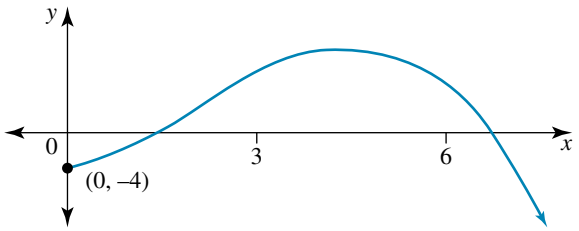
a. $P(x) = x^3 - 3x^2 - 4x + 9$

b. $P(x) = x^3 - 12x + 18$

c. $P(x) = -2x^3 + 10x^2 - 8x + 1$

d. $P(x) = x^3 + x^2 + 7$

18. The weekly profit y, in tens of dollars, from the sale of $10x$ containers of whey protein sold by a health food business is given by $y = -x^3 + 7x^2 - 3x - 4$, $x \geq 0$.



The graph of the profit is shown in the diagram.

a. Show that the business first started to make a profit when the number of containers sold was between 10 and 20.

b. Use the method of bisection to construct two further intervals for the value of x required for the business to first start making a profit.

c. Use the graph to state an interval in which the greatest profit lies.

d. Use a numerical systematic trial and error process to estimate the number of containers that need to be sold for greatest profit, and state the greatest profit to the nearest dollar.

e. As the containers are large, storage costs can lower profits. State an estimate from the graph of the number of containers beyond which no profit is made and improve upon this value with a method of your choice.



19. A rectangular sheet of cardboard measures 18 cm by 14 cm. Four equal squares of side length x cm are cut from the corners and the sides are folded to form an open rectangular box in which to place some clothing.



a. Express the volume of the box in terms of x.

b. State an interval within which lies the value of x for which the volume is greatest.

c. Use technology to systematically test values in order to determine, to 3 decimal places, the side length of the square needed for the volume of the box to be greatest.

5.9 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Factorise $P(x) = x^3 + 5x^2 + 3x - 9$ into linear factors.

2. Given $(x - 2)$ and $(x + 1)$ are factors of $P(x) = 6x^4 - 17x^3 - 11x^2 + 32x + 20$, determine all the linear factors of this polynomial.

3. The polynomial $P(x) = x^3 - ax^2 + bx - 3$ leaves a remainder of 2 when it is divided by $(x - 1)$ and a remainder of -4 when it is divided by $(x + 1)$. Calculate the values of a and b.



4. Sketch the following graphs.

a. $y = 8 - (x + 3)^3$

b. $y = -2(4 - x)^2(5 + x)$

c. $y = (8x - 3)^3$

d. $y = 2x^3 - x$

5. Solve the following.

a. $(x + 4)(x + 1)^2(x - 3) = 0$

b. $(x - 5)^3(3x + 7) = 0$

6. Sketch the graph of $y = -x^3 + 6x^2 - 11x + 6$ and hence state the values of x for which y is positive.

7. MC Select the correct statement about the graph of $y = \frac{1}{2}(x + 6)^4 - 3$.

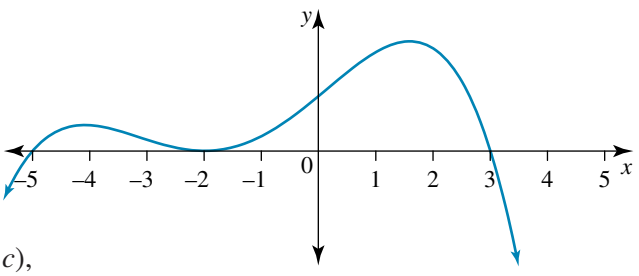
A. There is a maximum turning point at $(6, -3)$.

B. There is a minimum turning point at $(6, -3)$.

C. There is a stationary point of inflection at $(-6, -3)$.

D. There is a minimum turning point at $(-6, -3)$.

8. MC A possible equation for the quartic graph shown could be:



A. $y = (x + 5)(x + 2)^2(x - 3)$

B. $y = -(x - 5)(x - 2)^2(x + 3)$

C. $y = (x + 5)(x + 2)^2(3 - x)$

D. $y = -(x + 4)^2(x - 2)^2$

9. MC If $x^3 - 2x^2 - 3x + 10 \equiv (x + 2)(ax^2 + bx + c)$, then the values of a, b and c are, respectively:

A. $1, -2, 5$ **B.** $1, 0, 5$ **C.** $-2, -3, 10$ **D.** $1, -4, 5$

10. MC If $P(x) = 3 + kx - 5x^2 + 2x^3$ and $P(-1) = 8$, then k is equal to:

A. 0 **B.** 4 **C.** -12 **D.** 12

11. MC The curve defined by $y = a(x + b)^4 + c$ has a turning point at $(0, -7)$ and passes through the point $(-1, -10)$. The sum $a + b + c$ is equal to:

A. -4 **B.** -6 **C.** -7 **D.** -10

12. MC The equation $6x^3 - 7x + 5 = 0$ has only one solution. This solution lies in the interval for which:

A. $-3 \leq x \leq -2$ **B.** $-2 \leq x \leq -1$ **C.** $-1 \leq x \leq 0$ **D.** $0 \leq x \leq 1$

Complex familiar

13. Divide $(2x^3 - 3x^2 + x - 1)$ by $(x + 2)$ and state the quotient and the remainder.

14. Consider the cubic polynomial $P(x) = 8x^3 - 34x^2 + 33x - 9$.

a. Show that $(x - 3)$ is a factor of $P(x)$.

b. Hence, completely factorise $P(x)$.

c. The graph of the polynomial $y = P(x) = 8x^3 - 34x^2 + 33x - 9$ has turning points at $(0.62,\ 0.3)$ and $(2.2, -15.8)$. Sketch the graph labelling all key points with their coordinates.

d. Calculate $\{x: P(x) = -9\}$.

e. For what values of k will the line $y = k$ intersect the graph of $y = P(x)$ in:

i. 3 places

ii. 2 places

iii. 1 place?

15. The revenue $(\$)$ from the sale of x thousand items is given by $R(x) = 6(2x^2 + 10x + 3)$ and the manufacturing cost ($) of x thousand items is $C(x) = x(6x^2 - x + 1)$.

a. State the degree of $R(x)$ and of $C(x)$.

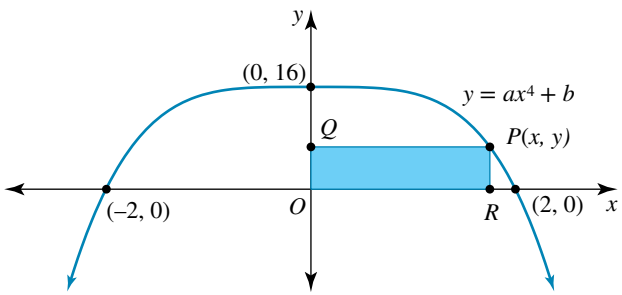
b. Calculate the revenue and the cost if 1000 items are sold and explain whether a profit is made.

c. Show that the profit ($) from the sale of x thousand items is given by $P(x) = -6x^3 + 13x^2 + 59x + 18$.

d. Given the graph of $y = -6x^3 + 13x^2 + 59x + 18$ cuts the x-axis at $x = -2$, sketch the graph of $y = P(x)$ for appropriate values of x.

e. If a loss occurs when the number of items manufactured is d, state the smallest value of d.

16. The cross-section of a mountain range with equation $y = ax^4 + b$ is shown in the diagram.



A small tunnel is to be built through the mountain. Its cross-section is the shaded rectangle PQOR and the point P (x, y) lies on the curve $y = ax^4 + b$.

a. Calculate the values of a and b and state the equation of the cross-section curve.

b. Express the area of the rectangular cross-section of the tunnel in terms of x.

c. The area of the cross-section of the tunnel is 15 square metres. Show that either $x = 1$ or $x^4 + x^3 + x^2 + x - 15 = 0$.

d. **i.** Determine two natural numbers between which there is a root of the equation $x^4 + x^3 + x^2 + x - 15 = 0$.

ii. Calculate an estimate, β, of this root using three iterations of the method of bisection.

e. For which of the two values $x = 1$ or $x = \beta$ is the height of the tunnel the smaller?

Complex unfamiliar



17. Relative to a reference point O, two towns A and B are located at the points $(1,\ 20)$ and $(5,\ 12)$, respectively. A freeway passing through A and B can be considered to be a straight line.

a. Determine the equation of the line modelling the freeway. Prior to the freeway being built, the road between A and B followed a scenic route modelled by the equation $y = a(2x - 1)(x - 6)(x + b)$ for $0 \leq x \leq 8$.

b. Using the fact this road goes through towns A and B, show that $a = \frac{2}{3}$ and $b = -7$.

c. What are the coordinates of the endpoints where the scenic route starts and finishes?

d. On the same diagram, sketch the scenic route and the freeway. Any endpoints and intercepts with the axes should be given and the positions of the points A and B should be marked on your graph.

e. The freeway meets the scenic route at three places. Calculate the coordinates of these three points.

f. Which of the three points found in part **e** is closest to the reference point O?



18. Consider the graph of $y = P(x)$ where $P(x) = x^3 + 3x^2 + 2x + 5$.

a. Solve the equation $P(x) = 5$ and hence state the intervals between which the turning points of the graph would lie.

b. Use a systematic numerical method to estimate the coordinates of the turning points to 2 decimal places.

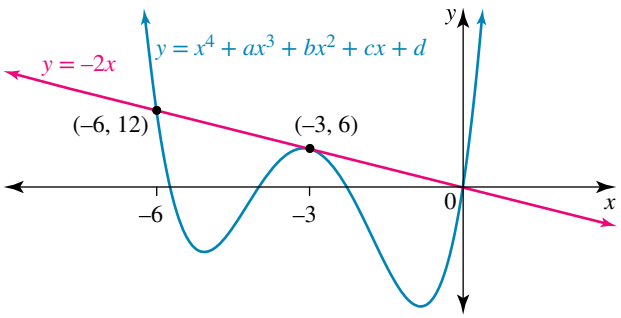
c. Explain why the graph can have only one x-intercept.

d. Locate an interval in which the x-intercept lies. Use the method of bisection to generate 3 narrower intervals in which the x-intercept lies.

e. State an estimate of the x-intercept to 1 decimal place.

f. Use the information gathered to sketch the graph.

19. A quartic polynomial is defined by the rule $y = x^4 + ax^3 + bx^2 + cx + d$ where $a, b, c, d \in R$. The line $y = -2x$ is a tangent to the graph of this polynomial, touching it at the point $(-3, 6)$. The line also cuts the graph at the origin and at the point $(-6, 12)$, as shown in the diagram.



a. State the value of d.

b. Form the equation $P(x) = 0$ for which the x-coordinates of the points of intersection of the two graphs $y = x^4 + ax^3 + bx^2 + cx + d$ and $y = -2x$ are the solutions.

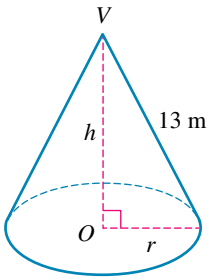
c. Use the information shown in the diagram to write down the factors of the equation $P(x) = 0$ and hence calculate the values of a, b and c.

d. **i.** State the rule for the quartic polynomial $y = x^4 + ax^3 + bx^2 + cx + d$ shown in the diagram and show that its graph has an x-intercept at $x = -4$.

ii. Calculate the exact values of its other x-intercepts.

20. The slant height of a right conical tent has a length of 13 metres.

For the figure shown, O is the centre of the circular base of radius r metres. OV, the height of the tent, is h metres.



a. Calculate the height of the cone if the radius of the base is $\frac{13\sqrt{6}}{3}$ metres.

b. Express the volume V in terms of h, given that the formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

c. State any restrictions on the values h can take and sketch the graph of V against h for these restrictions.

d. Express the volume as multiples of π for $h = 7$, $h = 8$, $h = 9$ and hence obtain the integer a so that the greatest volume occurs when $a < h < a + 1$.

e. i. Using the midpoint of the interval $[a, a + 1]$ as an estimate for h, calculate r.

ii. Use the estimates for h and r to calculate an approximate value for the maximum volume, to the nearest whole number.

f. i. The greatest volume is found to occur when $r = \sqrt{2}\,h$. Use this information to calculate the height and radius which give the greatest volume.

ii. Specify the greatest volume to the nearest whole number and compare this value with the approximate value obtained in part **e**.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 5 Powers and polynomials

Exercise 5.2 Polynomials

1. A: Degree 5; leading coefficient 4; constant term 12; coefficients $\in$ Z
   C: Degree 2; leading coefficient -0.2; constant term 5.6; coefficients $\in$ Q
2. a. polynomial of degree 4
   b. polynomial of degree 3
   c. not a polynomial due to the $\sqrt{x}$ term
   d. not a polynomial due to the $\frac{6}{x^2}$ term and the $\frac{2}{x}$ term
3. a. A, B, D, F are polynomials.

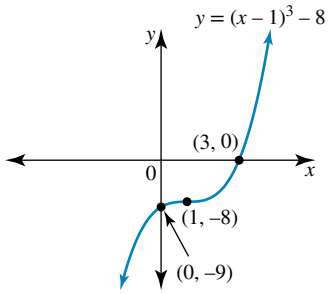
	Degree	Type of coefficient	Leading term	Constant term
A	5	Q	$3x^5$	12
B	4	R	$-5x^4$	9
D	4	Z	$-18x^4$	0
F	6	N	$49x^6$	9

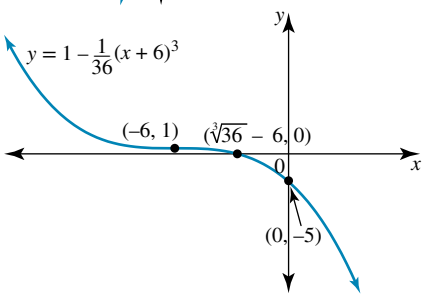
   b. C is not a polynomial due to $\sqrt{4x^5} = 2x^{\frac{5}{2}}$ term.
   E is not a polynomial due to $\frac{5}{3x^2} = \frac{5}{3}x^{-2}$ term.
4. a. $P(1) = 5$
   b. $P(-2) = -45$
   c. $P(3) = 77$, $P(-x) = -3x^3 - x^2 + 5$
   d. $P(-1) = 10$, $P(2a) = 8a^3 + 16a^2 - 4a + 5$.
5. a. 78 b. -12
   c. 0 d. -6
   e. -6 f. -5.868
6. a. $-14a$ b. $h^2 - 5h - 4$
   c. $2xh + h^2 - 7h$
7. a. 15 b. 10
8. a. $a = -8$ b. $b = 21$
   c. $k = 2$ d. $m = -8$
9. 21
10. a. $k = 25$ b. $a = -9$
    c. $n = -\frac{1}{5}$ d. $b = 4$; $c = 5$
11. $a = 2$; $b = -13$; $c = 6$
12. a. $a = 10, b = 6$
    b. $8x - 6 = 10x - 2(x + 3)$
    c. $6x^2 + 19x - 20 = (6x - 5)(x + 4)$
    d. $a = 9, b = -10, c = 1$
13. $(x + 2)^3 = x^2(x + 1) + 5x(x + 2) + 2(x + 3) + 2$
14. a. $x^3 - 7x^2 - 18x$
    b. $48x - 3x^3$
15. a. $x^3 - 3x^2 - 18x + 40$
    b. $x^3 + 6x^2 - x - 6$
    c. $x^3 - 12x^2 + 21x + 98$
    d. $x^3 + x^2 - x - 1$
16. $p = 5$; $q = -4$
17. a. i. $3x^4 + 7x + 12$
    ii. $6x^4 + 10x^2 - 21x - 31$
    iii. $6x^6 - 21x^5 - 29x^4 - 14x^3 - 20x^2 - 7x - 11$
    b. i. m ii. m iii. $m + n$
18. a. $\frac{x - 12}{x + 3} = 1 - \frac{15}{x + 3}$; quotient is 1; remainder is -15.
    b. $\frac{4x + 7}{2x + 1} = 2 + \frac{5}{2x + 1}$
19. a. $\frac{2x^3 - 5x^2 + 8x + 6}{x - 2} = 2x^2 - x + 6 + \frac{18}{x - 2}$; quotient is $2x^2 - x + 6$; remainder is 18.
    b. $\frac{x^3 + 10}{1 - 2x} = -\frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{8} + \frac{81}{8(1 - 2x)}$; remainder is $\frac{81}{8}$.

Exercise 5.3 Graphs of cubic polynomials

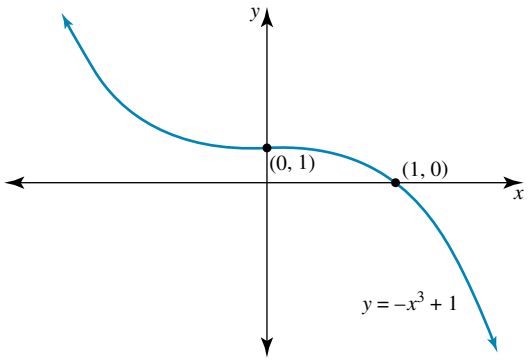
1. a. $(7, 0)$ b. $(0, -7)$
   c. $(0, 0)$ d. $(2, 2)$
   e. $(-5, -8)$ f. $\left(\frac{1}{2}, 5\right)$
2.

	Inflection point	y-intercept	x-intercept
a.	$(1, -8)$	$(0, -9)$	$(3, 0)$
b.	$(-6, 1)$	$(0, -5)$	$(-2.7, 0)$ approx.

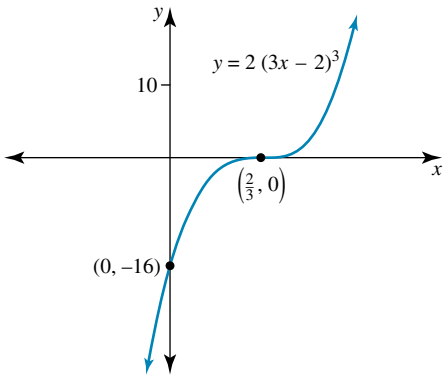
   a.


   b.


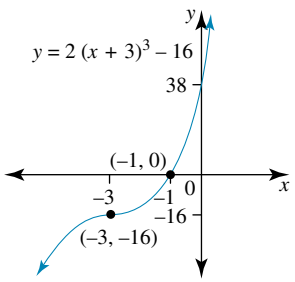
3. a.



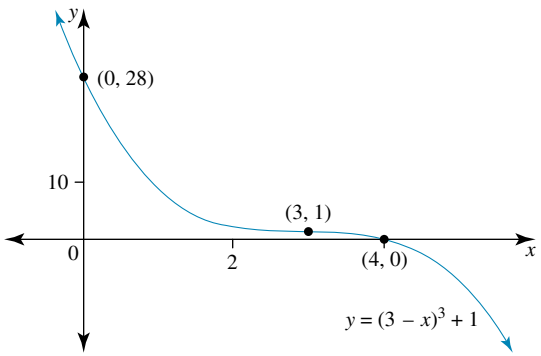
b.



c.



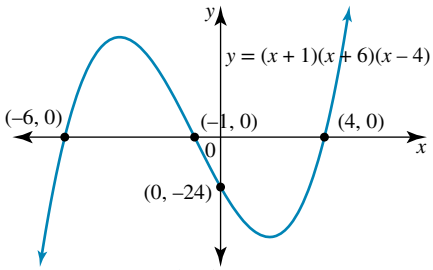
d.



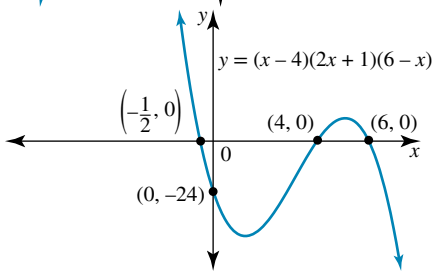
4.

	y-intercept	*x*-intercepts
a.	(0, −24)	(−6, 0), (−1, 0), (4, 0)
b.	(0, −24)	$\left(-\frac{1}{2}, 0\right)$, (6, 0), (6, 0)

a.



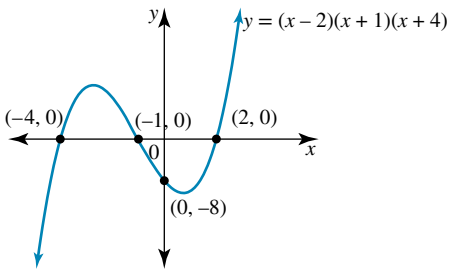
b.



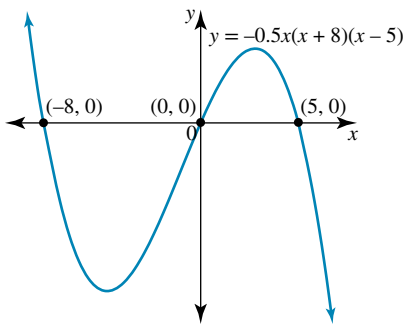
5.

	y-intercept	*x*-intercepts
a.	(0, −8)	(−4, 0), (−1, 0), (2, 0)
b.	(0, 0)	(−8, 0), (0, 0), (5, 0)
c.	(0, −12)	(−3, 0), (1, 0), (4, 0)
d.	(0, 12)	(−4, 0), (2, 0), (6, 0)
e.	(0, 7)	$\left(-\frac{1}{4}, 0\right)$, $\left(\frac{7}{2}, 0\right)$, (10, 0)
f.	$\left(0, \frac{5}{2}\right)$	$\left(-\frac{8}{3}, 0\right)$, $\left(\frac{5}{8}, 0\right)$, (2, 0)

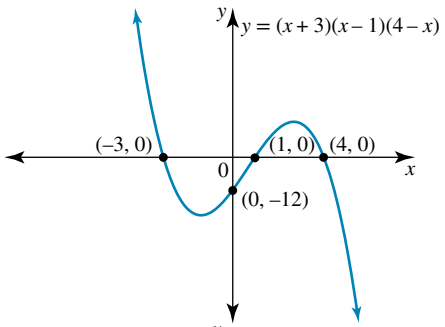
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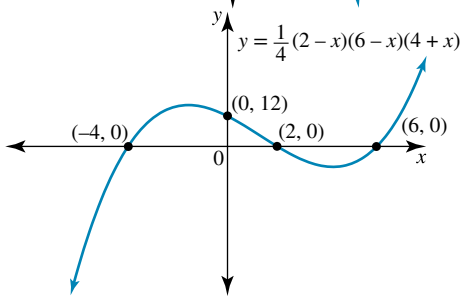
b.



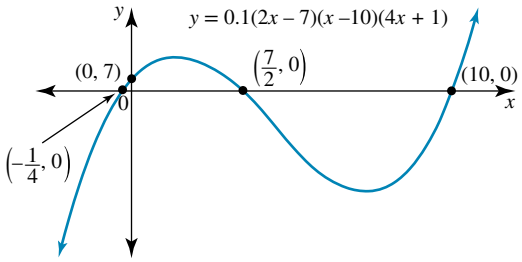
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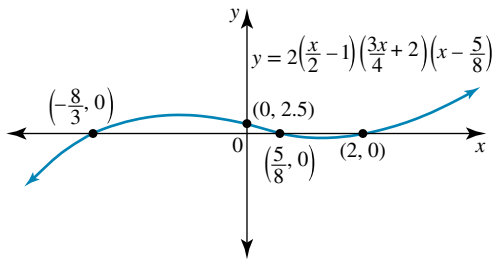
d.



e.



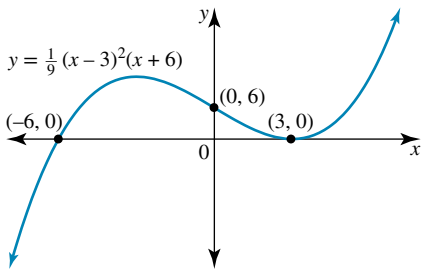
f.



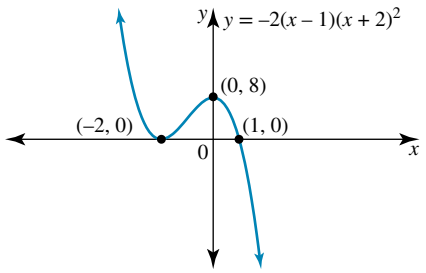
6.

	y-intercept	*x*-intercept
a.	(0, 6)	(−6, 0) and (3, 0) which is a turning point
b.	(0, 8)	(−2, 0) is a turning point and (1, 0)

a.



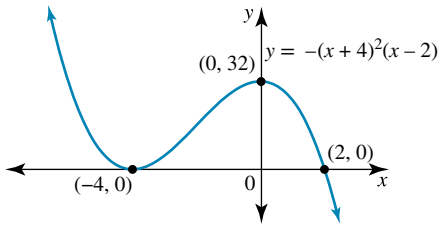
b.



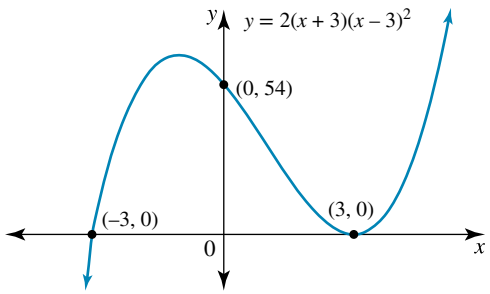
7.

	y-intercept	*x*-intercepts
a.	(0, 32)	(−4, 0) is a turning point; (2, 0) is a cut
b.	(0, 54)	(3, 0) is a turning point; (−3, 0) is a cut
c.	(0, 36)	(−3, 0) is a turning point; (4, 0) is a cut
d.	(0, −12)	(2, 0) is a turning point; (12, 0) is a cut
e.	(0, 0)	$\left(-\frac{3}{2}, 0\right)$ is a turning point; (0, 0) is a cut
f.	(0, 0)	(0, 0) is a turning point; (0.4, 0) is a cut

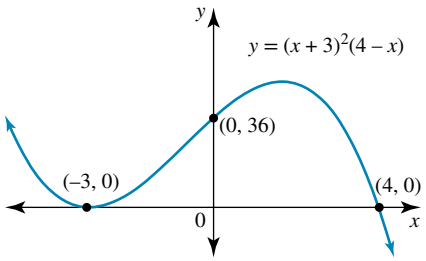
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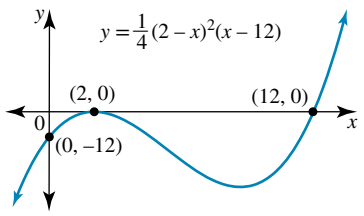
b.



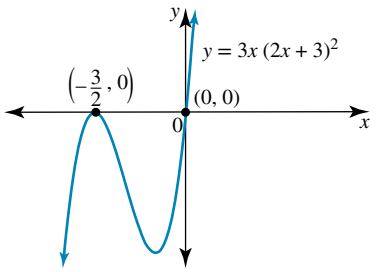
c.



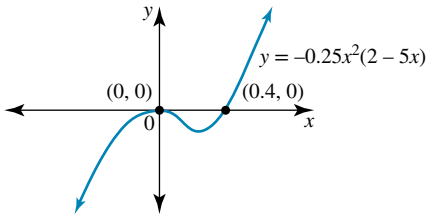
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e.

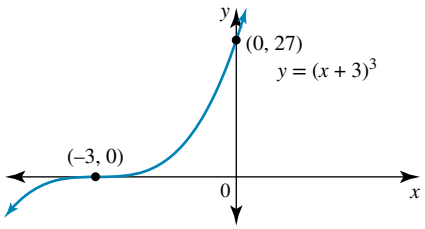


f.

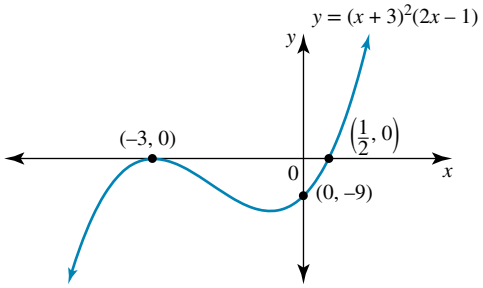


8. See table at the bottom of the page.*

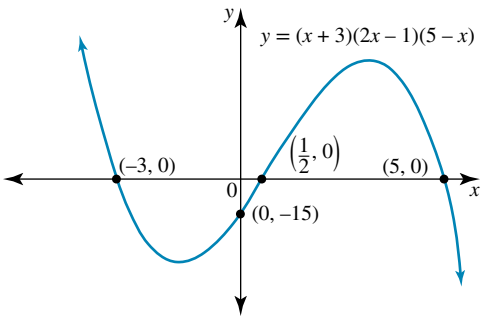
a.



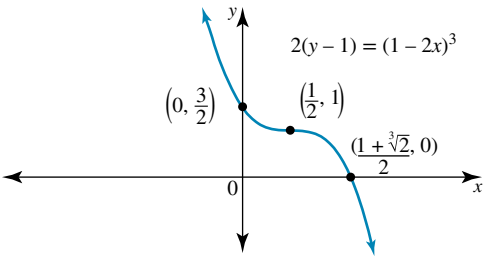
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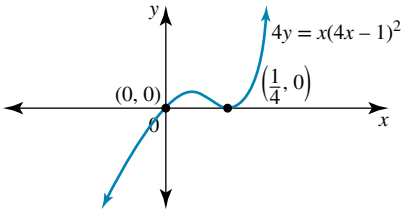
c.



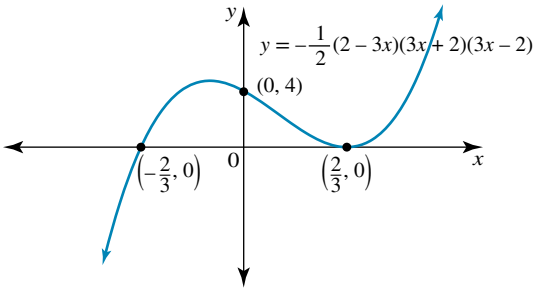
d.



e.



f.

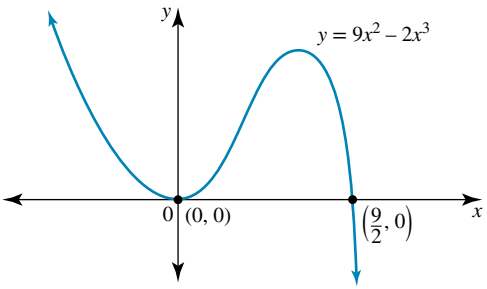


*8.

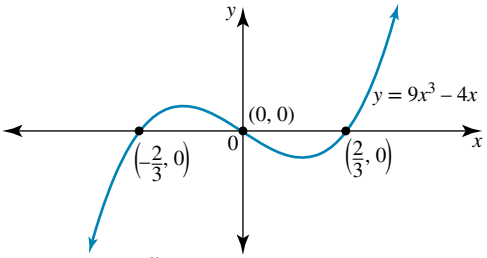
	Stationary point of inflection	*y*-intercept	*x*-intercepts
a.	$(-3, 0)$	$(0, 27)$	$(-3, 0)$
b.	none	$(0, -9)$	$(-3, 0)$ is a turning point; $\left(\frac{1}{2}, 0\right)$ is a cut
c.	none	$(0, -15)$	$(-3, 0)$, $\left(\frac{1}{2}, 0\right)$, $(5, 0)$
d.	$\left(\frac{1}{2}, 1\right)$	$\left(0, \frac{3}{2}\right)$	(1.1, 0) approx.
e.	none	$(0, 0)$	$\left(\frac{1}{4}, 0\right)$ is a turning point; $(0, 0)$ is a cut
f.	none	$(0, 4)$	$\left(\frac{2}{3}, 0\right)$ is a turning point; $\left(-\frac{2}{3}, 0\right)$ is a cut

9. See table at the bottom of the page.*

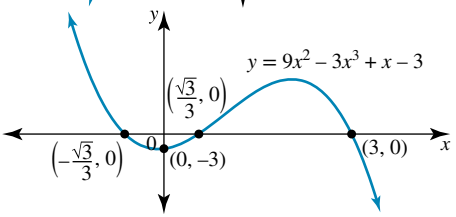
a.



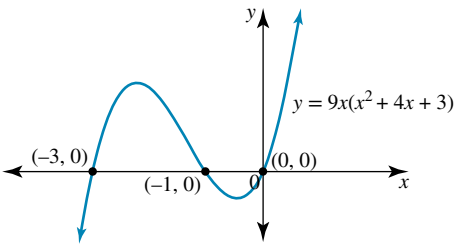
b.



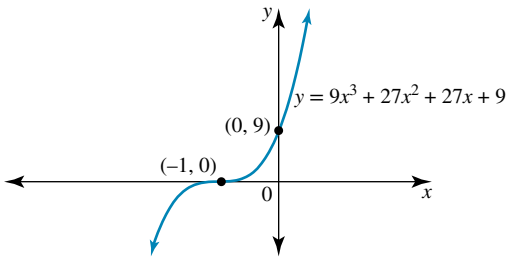
c.



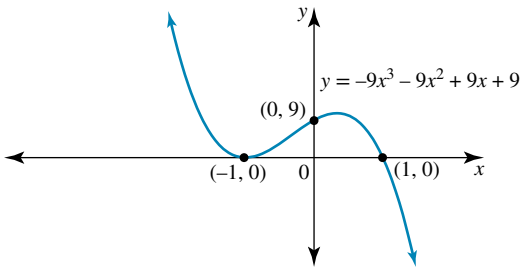
d.



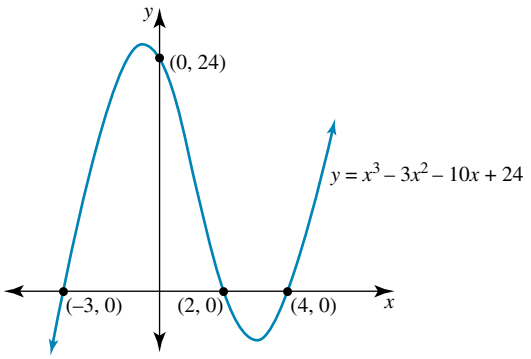
e.



f.



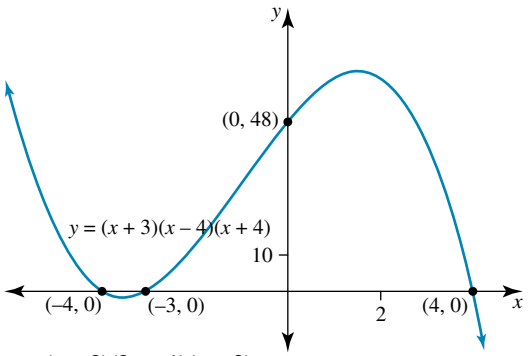
10.



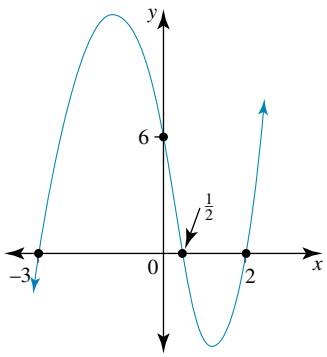
*9.

	Factorised form	Stationary point of inflection	*y*-intercept	*x*-intercepts
a.	$y = x^2(9 - 2x)$	none	(0, 0)	(0, 0) is a turning point; $\left(\frac{9}{2}, 0\right)$ is a cut
b.	$y = x(3x - 2)(3x + 2)$	none	(0, 0)	$\left(-\frac{2}{3}, 0\right)$, (0, 0) $\left(\frac{2}{3}, 0\right)$
c.	$(x - 3)\left(1 - \sqrt{3x}\right)\left(1 + \sqrt{3x}\right)$	none	(0, –3)	$\left(-\frac{\sqrt{3}}{3}, 0\right), \left(\frac{\sqrt{3}}{3}, 0\right), (3, 0)$
d.	$y = 9x(x + 1)(x + 3)$	none	(0, 0)	(–3, 0), (–1, 0), (0, 0)
e.	$y = 9(x + 1)^3$	(–1, 0)	(0, 9)	(–1, 0)
f.	$y = -9(x + 1)^2(x - 1)$	None	(0, 9)	(–1, 0) is a turning point; (1, 0) is a cut

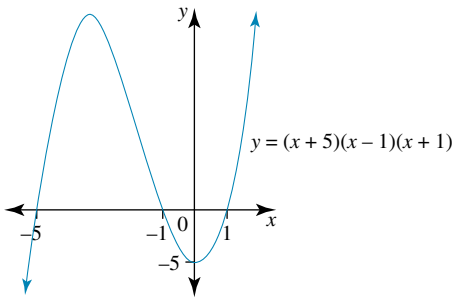
11. a.



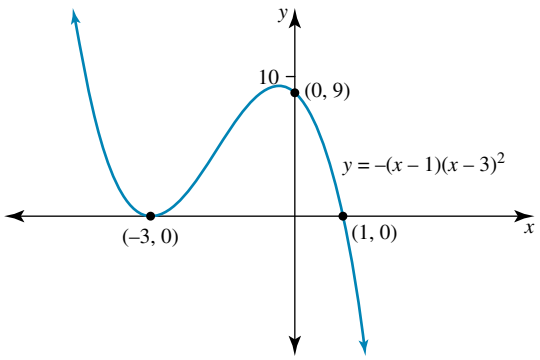
b. $y = (x - 2)(2x - 1)(x + 3)$



c.

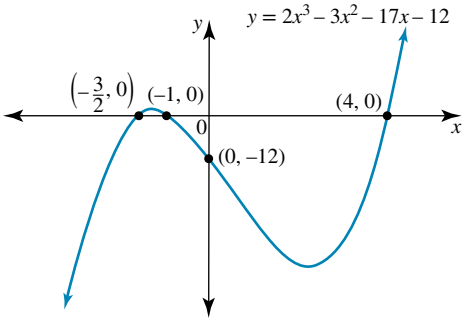


d.

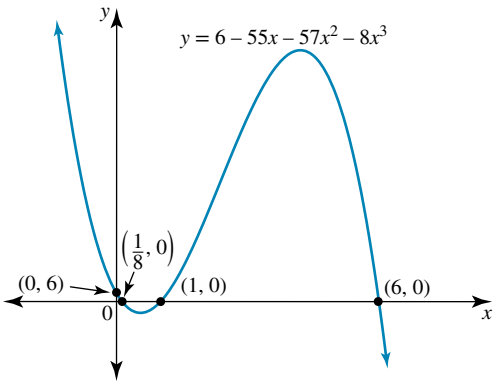


12. See table at the bottom of the page.*

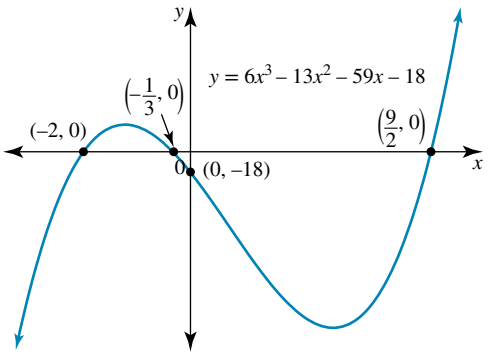
a.



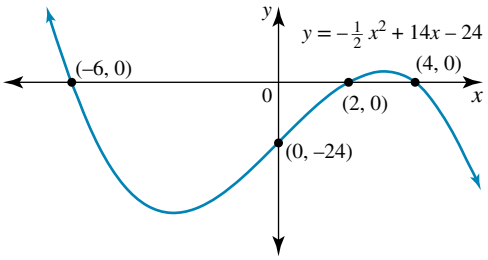
b.



c.



d.

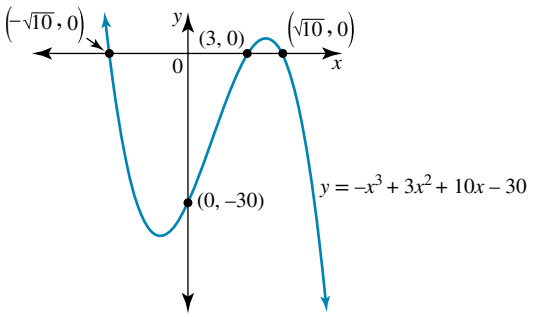


*** 12.**

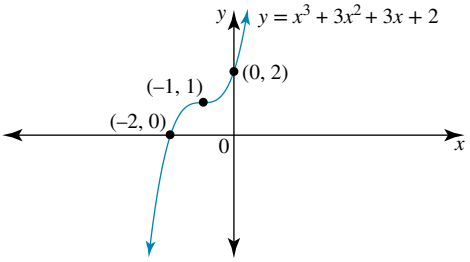
	Factorised form	y-intercept	x-intercepts
a.	$y = (x + 1)(2x + 3)(x - 4)$	$(0, -12)$	$\left(-\frac{3}{2}, 0\right), (-1, 0), (4, 0)$
b.	$y = -(x - 1)(8x - 1)(x - 6)$	$(0, 6)$	$\left(\frac{1}{8}, 0\right), (1, 0), (6, 0)$
c.	$y = (x + 2)(3x + 1)(2x - 9)$	$(0, -18)$	$(-2, 0), \left(-\frac{1}{3}, 0\right), \left(\frac{9}{2}, 0\right)$
d.	$y = -\frac{1}{2}(x - 2)(x + 6)(x - 4)$	$(0, -24)$	$(-6, 0), (2, 0), (4, 0)$

13. a. $-x^3 + 3x^2 + 10x - 30 = -(x-3)\left(x-\sqrt{10}\right)\left(x+\sqrt{10}\right)$

y-intercept	*x*-intercepts
$(0, -30)$	$\left(-\sqrt{10}, 0\right), (3, 0), \left(\sqrt{10}, 0\right)$

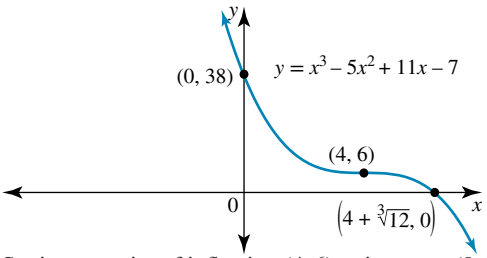


b. Stationary point of inflection $(-1, 1)$; y-intercept $(0, 2)$; x intercept $(-2, 0)$



14. a. $-\frac{1}{2}(x-4)^3 + 6$

b.



Stationary point of inflection $(4, 6)$; y-intercept $(0, 38)$; x-intercept approximately $(6.3, 0)$

15. a. $y = \frac{1}{49}(x-3)^3 - 7$

b. $y = 0.25x(x+5)(x-4)$

c. $y = -(x+2)^2(x-3)$

16. $y = 2x^3 + 3x^2 - 4x + 3$

17. a. $y = \frac{1}{3}(x-3)^3 + 9$

b. $y = (x+2)^3 + 2$

c. $y = -2x^3 + 4$

d. $y = (x+5)^3 + 4$

e. $y = -(x-2)^3 - 1$

f. $y = -(x-3)^3 - 1$

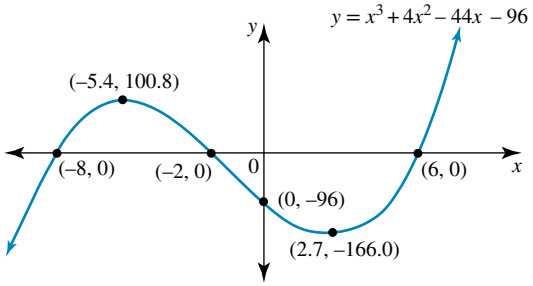
18. a. $y = \frac{1}{2}(x+8)(x+4)(x+1)$

b. $y = -2x^2(x-5)$

c. $y = -3(x-1)^3 - 3$

d. $y = \frac{4}{5}(x-1)(x-5)^2$

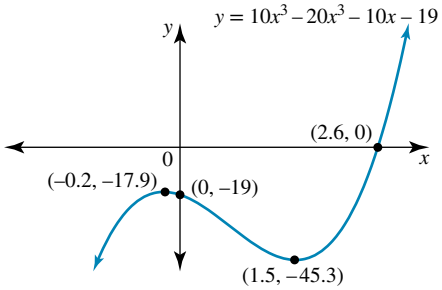
19. a. Maximum turning point $(-5.4, 100.8)$; minimum turning point $(2.7, -166.0)$; y-intercept $(0, -96)$; x-intercepts $(-8, 0)$, $(-2, 0)$, $(6, 0)$



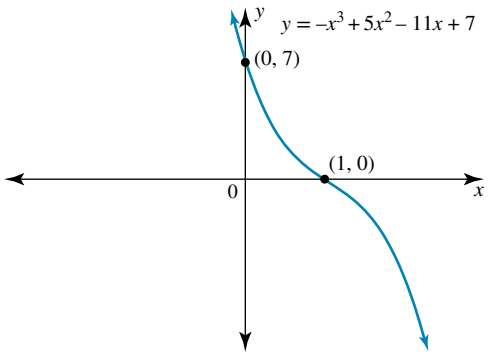
b. See the worked solutions in your eBookPLUS for the proof.

20. See table at the bottom of the page.*

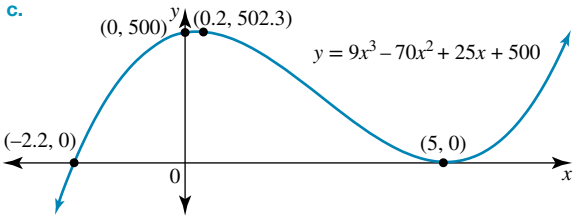
a.



b.



c.



* **20.**

	Maximum turning point	Minimum turning point	*y*-intercept	*x*-intercepts
a.	$(-0.2, -17.9)$	$(1.5, -45.3)$	$(0, -19)$	$(2.6, 0)$
b.	None	None	$(0, 7)$	$(1, 0)$
c.	$(0.2, 502.3)$	$(5, 0)$	$(0, 500)$	$(-2.2, 0), (5, 0)$

Exercise 5.4 The factor and remainder theorems

1. a. The remainder is 19. **b.** The remainder is $\frac{37}{8}$.

2. a. 6 **b.** -6
c. $a = 4$ **d.** $k = 3$

3. C

4. a. -5 **b.** 2 **c.** -101
d. $-10\frac{3}{8}$ **e.** 0 **f.** 26

5. $k = 2$

6. a. Proof required to show $Q(2) = 0$
b. $a = -9$; $b = 8$; $P(x) = 3x^3 - 9x^2 + 8x - 2$

7. a. -10 **b.** 8 **c.** -19 **d.** -19

8. $a = 0$; $a = -1$

9. a. $a = 2$; $b = 3$
b. $m = -\frac{2}{3}$; $n = -3$

10. a. $P(-1) = 0 \Rightarrow (x + 1)$ is a factor; $P(x) = (x + 1)(x + 5)(x - 3)$
b. $P(x) = (x + 1)(3x + 2)(4x + 7)$

11. a. Show $P(-4) = 0$
b. Show $P(5) \neq 0$
c. Show $P\left(\frac{1}{2}\right) = 0$
d. $P(1) = 14$ so $P(1) \neq 0$.
e. $a = -12$
f. $k = -8$

12. a. $(x - 4)(x + 1)(x + 2)$
b. $(x + 12)(3x + 1)(x + 1)$
c. $(5x + 1)(2x + 3)(2x + 1)$
d. $(4x - 3)(5 - 2x)(5 + 2x)$
e. $-(x - 3)^2(8x - 11)$
f. $(3x - 5)^2(x - 5)$

13. a. $(x - 1)(x + 2)(x + 4)$
b. $(x + 2)(x + 3)(x + 5)$
c. $(x - 2)(2x + 1)(x - 5)$
d. $(x + 1)(3x - 4)(1 - 6x)$
e. $(x - 1)(x + 3)(x - 2)$
f. $(x + 1)(x - 7)(x + 7)$

14. Linear factors are $(2x - 1)$, $(2x + 1)$ and $(3x + 2)$.

15. a. i. $(x - 5)(x - 9)(x + 2)$ **ii.** $x^3 - 12x^2 + 17x + 90$
b. i. $(x + 4)(x + 1)(1 - 2x)$ **ii.** $-2x^3 - 9x^2 - 3x + 4$

16. a. $(2x + 1)(3x - 1)(4x + 5)$
b. i. $(2x - 5)$
ii. $(2x - 5)(4x + 1)(x - 1)$
iii. $m = -26$
c. i. $P(x) = (x - 4)^3$; $Q(x) = (x - 4)\left(x^2 + 4x + 16\right)$
ii. Proof required. See the worked solutions in your eBookPLUS.
d. Third factor is $(x - 3)$; $b = c = -3$; $d = 9$

17. $-12\sqrt{2} + 33$

Exercise 5.5 Solving cubic equations

1. a. $-5, 0, 5$ **b.** $-\sqrt{2}, 0, \sqrt{2}$
c. $0, 2, 3$ **d.** 0

2. a. $-2, 1, 4$ **b.** $-\frac{1}{2}, \frac{3}{2}, 3$
c. $-3, -2, \frac{1}{2}$ **d.** $-2, -1, 1$

3. B

4. a. $-3, 2$ **b.** $-2, \frac{1}{3}, 6$
c. $-4, 2$ **d.** $-1, \frac{5}{2}$

5. $x = 0$, $x = -\frac{3}{2}$, $x = 2$, $x = -2$

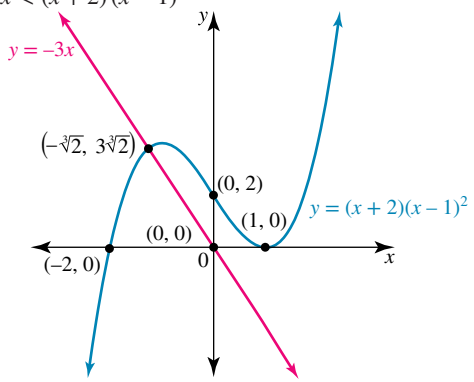
6. $x = -2, \frac{1}{3}, -\frac{1}{2}$

7. a. $-4, 3, -5$ **b.** $7, -\frac{5}{3}, 9$
c. $-1, 6, 8$ **d.** $1, -\frac{3}{2}, -3$
e. $1, -\frac{2}{3}$ **f.** $0, 1, -\frac{3}{4}, -20$

8. a. Proof required; $(x - 2)\left(x + 4 - \sqrt{7}\right)\left(x + 4 + \sqrt{7}\right)$. See the worked solutions in your eBookPLUS.
b. Proof required. See the worked solutions in your eBookPLUS.
c. $(2x - 1)(x - 5)^2$; equation has roots $x = \frac{1}{2}$, $x = 5$. See the worked solutions in your eBookPLUS.

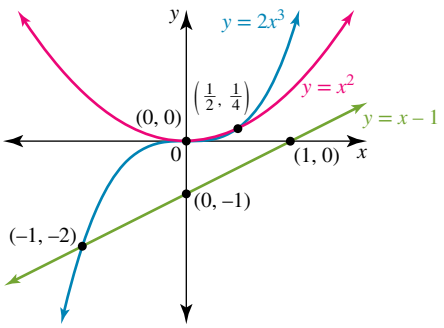
9. a. $(x - 2)(x + 2)(5x + 9)$; $k = 9$
b. $a = 1$; $a = 2$
c. $a = -3$; $b = 1$; $P(x) = (x - 3)\left(x^2 + 1\right)$, $Q(x) = (x - 3)(x + 3)(x + 1)$
d. $x^3 + 7x^2 + 15x + 9 : x = -3,\ x = -1$; $x^3 - 9x^2 + 15x + 25 : x = 5,\ x = -1$

10. Point of intersection is $\left(-\sqrt[3]{2}, 3\sqrt[3]{2}\right)$. When $x > -\sqrt[3]{2}$, $-3x < (x + 2)(x - 1)^2$



11. a. $y = -2(x + 6)^3 - 7$ **b.** $(0, -12)$
c. $\left(\sqrt[3]{10} - 5, 0\right)$ **d.** $(0, 7)$

12. a. $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{4}\right)$
b. $(-1, -2)$
c.



13. a. $y = -2x + 5$; 1 solution

b. $y = x^3 + 3x^2, y = 4x$; 3 solutions

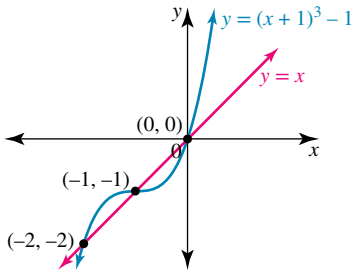
c. There are 3 solutions. One method is to use $y = x^3$, $y = -3x^2 + 4x$.

d. $x = -4,\ 0,\ 1$

14. a. $a = 1;\ b = -1$

b. $(-2, -2)$, $(-1, -1)$, $(0, 0)$

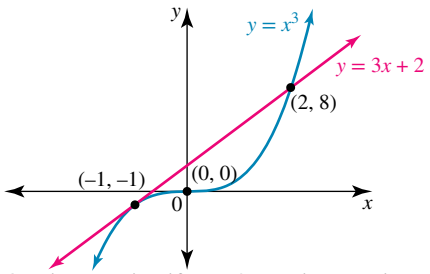
c.



15. a. See the worked solutions in your eBookPLUS for a sample proof.

b. $(2, 8)$

c.



d. One intersection if $m < 3$; two intersections if $m = 3$; three intersections if $m > 3$

16. a. $(x - 3)^2$

b. $(-1, 0)$

c. $a = -5,\ b = 3$

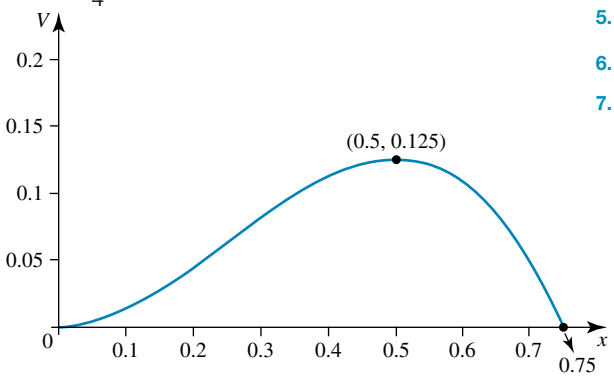
Exercise 5.6 Cubic models and applications

1. a. $h = \dfrac{3 - 4x}{2}$

b. See the worked solutions in your eBookPLUS for a sample proof.

c. $0 \leq x \leq \dfrac{3}{4}$

d.



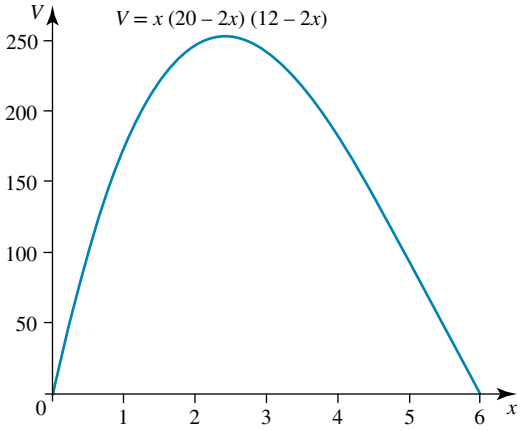
x-intercepts at $x = 0$ (touch), $x = 0.75$ (cut); shape of a negative cubic

e. Cube of edge 0.5 m

2. a. $l = 20 - 2x;\ w = 12 - 2x;\ V = (20 - 2x)(12 - 2x)x$

b. $0 \leq x \leq 6$

c.



x-intercepts at $x = 10, x = 6, x = 0$ but since $0 \leq x \leq 6$, the graph won't reach $x = 10$; shape is of a positive cubic.

d. Length 15.14 cm; width 7.14 cm; height 2.43 cm; greatest volume 263 cm^3

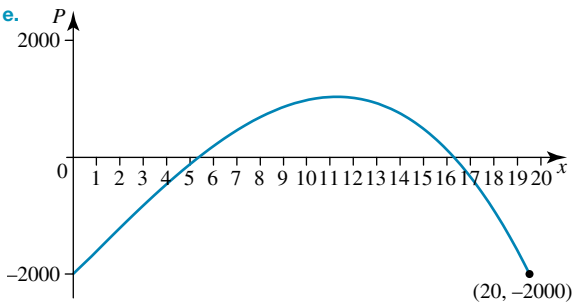
3. a. Making 5 sculptures gives a loss of \$125.
Making 6 sculptures gives a profit of \$184.

b. See the worked solutions in your eBookPLUS for a sample proof.

c. Too many and the costs outweigh the revenue from the sales. A negative cubic tends to $-\infty$ as x becomes very large.

d. **i.** Profit \$304 **ii.** Loss \$113

e.



x-intercepts lie between 5 and 6 and between 16 and 17.

f. Between 6 and 16 inclusive

4. a. 54 **b.** 2 hours **c.** 210 **d.** 5 pm

5. a. $y = -\dfrac{1}{32}x^2(x - 6)$ **b.** $-\dfrac{49}{32}$ km

6. $-4,\ 1$

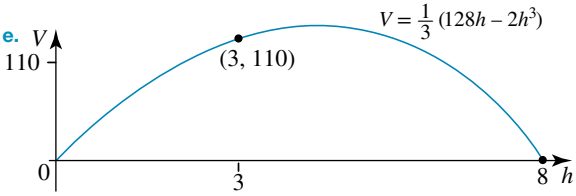
7. a. $2\sqrt{2}x$

b. See the worked solutions in your eBookPLUS for sample proofs.

c. Making 5 sculptures gives a loss of \$125.
Making 6 sculptures gives a profit of \$184.

d. **i.** 110 m^3 **ii.** Mathematically $0 \leq h \leq 8$

e.



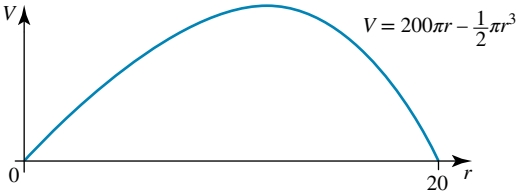
Max volume when h is between 4 and 5 (estimates will vary).

f. Height $\dfrac{8}{\sqrt{3}} \approx 4.6$ m

8. a, b. See the worked solutions in your eBookPLUS for sample proofs.

c. $0 \leq r \leq 20$

d.



e. Radius 2 cm, height 99 cm or radius 18.9 cm, height 1.1 cm

f. 4837 cm^3

9. a. $(0, 2.1)$, $(1.25, 1)$, $(2.5, 1.1)$, $(4, 0.1)$

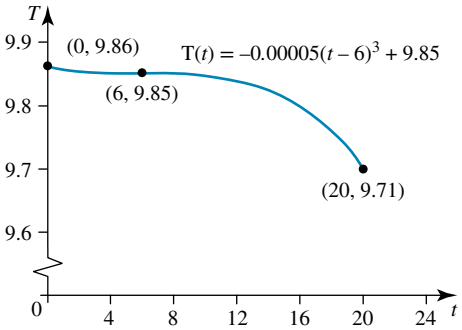
b. $d = 2.1$

c. $125a + 100b + 80c = -70.4$
$125a + 50b + 20c = -8$
$64a + 16b + 4c = -2$

d. 0.77 m

10. a. $T(3) = 9.85, T(20) = 9.71$

b.



c. $T(28) = 9.32$; unlikely, but not totally impossible. Model is probably not a good predictor beyond 2016.

11. a. $x = 0, x = 6$

b. Length $2x - 6$; width $6x - x^2$

c. See the worked solutions in your eBookPLUS for a sample proof.

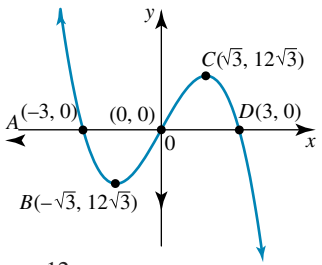
d. $3 \leq x \leq 6$

e. $x = 4, x = \dfrac{5 + \sqrt{33}}{2}$

12. a. 3 x-intercepts; 2 turning points

b. $y = -2x\left(x^2 - 9\right)$

c.

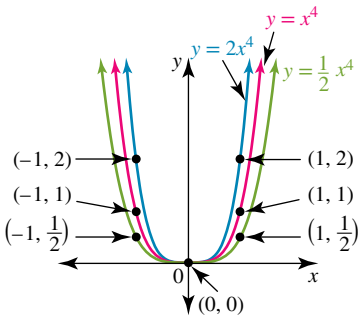


d. $y = 12x$

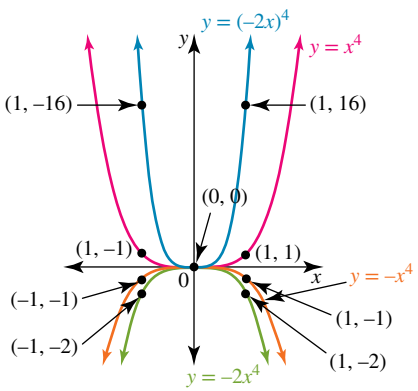
e. $y = 4x^3$

Exercise 5.7 Graphs of quartic polynomials

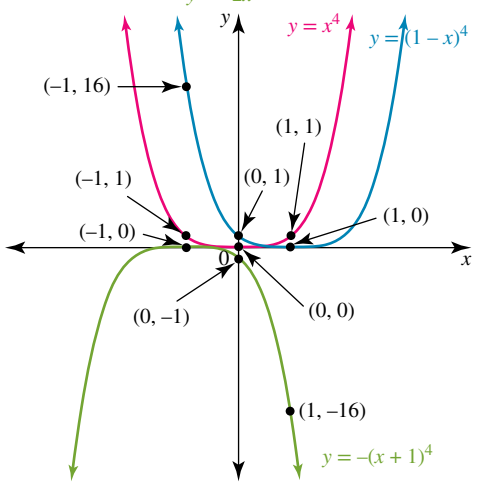
1. a.



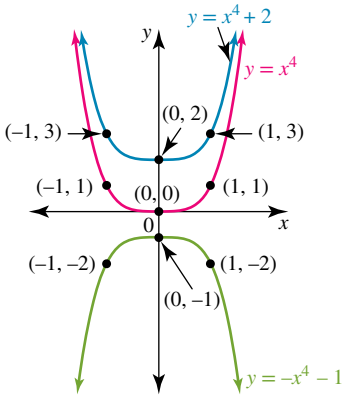
b.



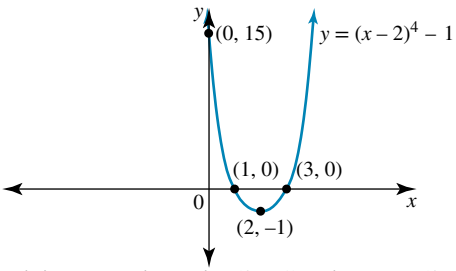
c.



d.

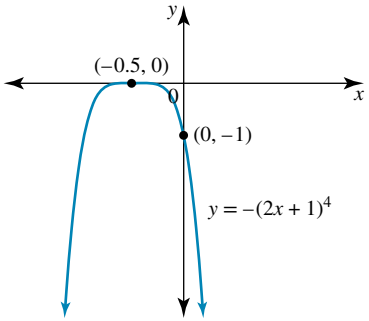


2. a.



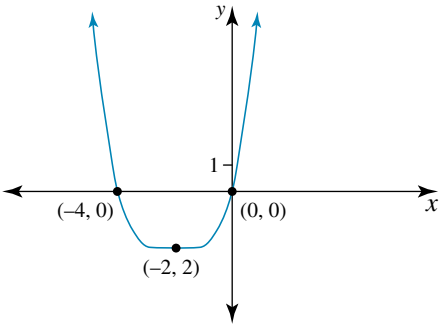
Minimum turning point $(2, -1)$; y-intercept $(0, 15)$; x-intercepts $(1, 0)$, $(3, 0)$

b.



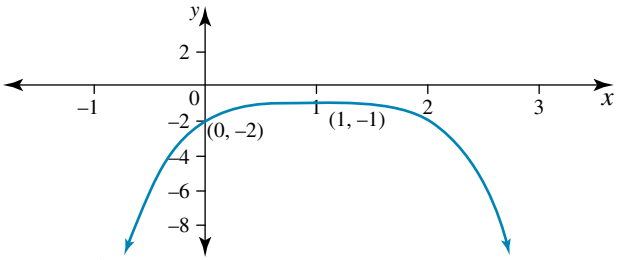
x-intercept and maximum turning point $\left(-\frac{1}{2}, 0\right)$;
y-intercept $(0, -1)$

3. a. turning point is $(-2, -2)$.



b. i. maximum turning point $(1, -1)$

ii. $y = -(x - 1)^4 - 1$

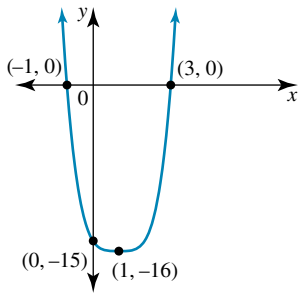


c. $y = \frac{1}{4}(x - 4)^4$

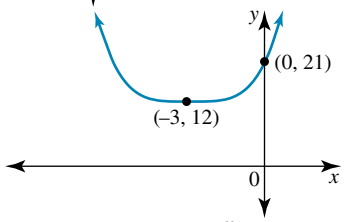
4.

	Turning point	y-intercept	x-intercepts
a.	$(1, -16)$ minimum	$(0, -15)$	$(-1, 0)$, $(3, 0)$
b.	$(-3, 12)$ minimum	$(0, 21)$	none
c.	$(-5, 250)$ maximum	$(0, 0)$	$(-10, 0)$, $(0, 0)$
d.	$(2, -11)$ maximum	$(0, -107)$	none
e.	$\left(\frac{3}{5}, -2\right)$ minimum	$\left(0, \frac{65}{8}\right)$	$\left(\frac{1}{5}, 0\right)$, $(1, 0)$
f.	$\left(\frac{2}{7}, 1\right)$ maximum	$\left(0, \frac{65}{81}\right)$	$\left(-\frac{1}{7}, 0\right)$, $\left(\frac{5}{7}, 0\right)$

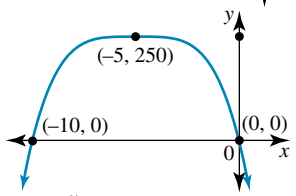
a.



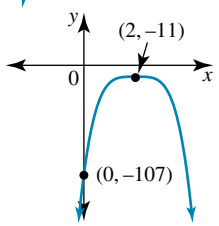
b.



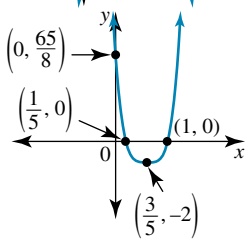
c.



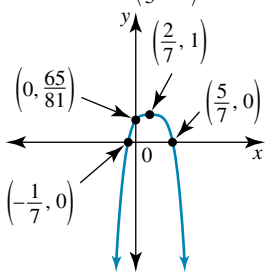
d.



e.



f.



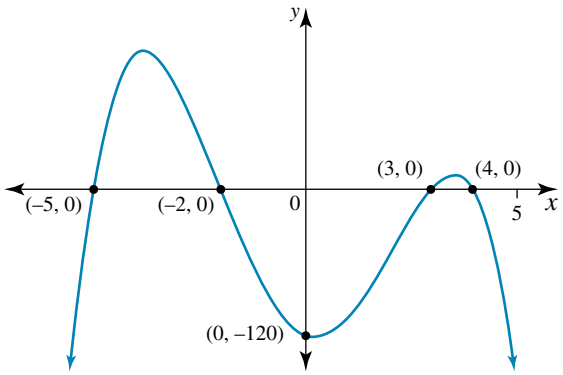
5. a. $y = \frac{2}{3}(x+9)^4 - 10$

b. $y = 6(x+3)^4 - 8$

c. $y = (3x-2)^4$

d. $y = -(x+100)^4 + 10\,000$

6. a.

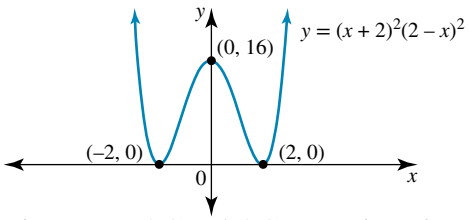


b. i. cut at $x = -1 \Rightarrow (x+1)$ is a factors, touch at $x = 1 \Rightarrow (x-1)^2$ is a factor, cut at $x = 3 \Rightarrow (x-3)$ is a factor.

ii. $y = a(x+1)(x-1)^2(x-3)$

iii. $y = 2(x+1)(x-1)^2(x-3)$

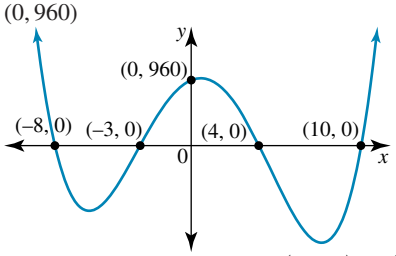
7.



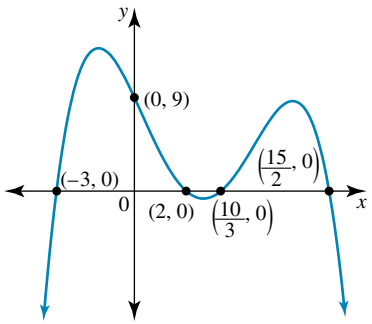
x-intercepts $(-2, 0)$ and $(2, 0)$ are turning points; y-intercept $(0, 16)$

8. $y = \frac{1}{4}x(x+4)(x-2)(x-5)$

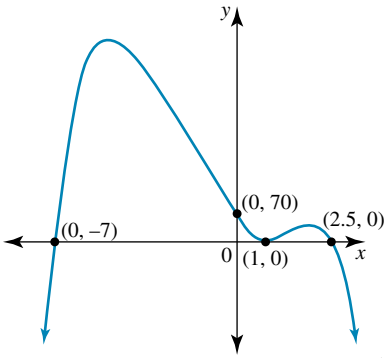
9. a. x-intercepts $(-8,\ 0)$, $(-3,\ 0)$, $(4,\ 0)$, $(10,\ 0)$; y-intercept $(0, 960)$



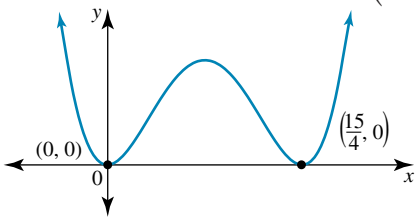
b. x-intercepts $(-3, 0)$, $(2, 0)$, $\left(\frac{15}{2}, 0\right)$, $\left(\frac{10}{3}, 0\right)$; y-intercept $(0, 9)$



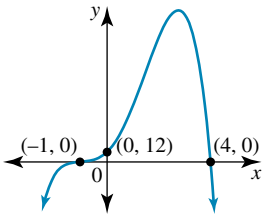
c. x-intercepts $(-7, 0)$, $(2.5, 0)$; turning point $(1, 0)$; y-intercept $(0, 70)$



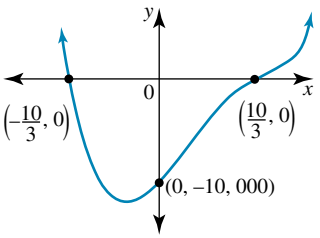
d. x-intercepts and turning points $(0, 0)$, $\left(\frac{15}{4}, 0\right)$



e. x-intercept and stationary point of inflection $(-1, 0)$; x-intercept $(4, 0)$; y-intercept $(0, 12)$



f. x-intercept and stationary point of inflection $\left(\frac{10}{3}, 0\right)$; x-intercept $\left(-\frac{10}{3}, 0\right)$; y – intercept $(0,\ -10\,000)$



10. a. $y = -\frac{1}{72}(x+6)(x+5)(x+3)(x-4)$

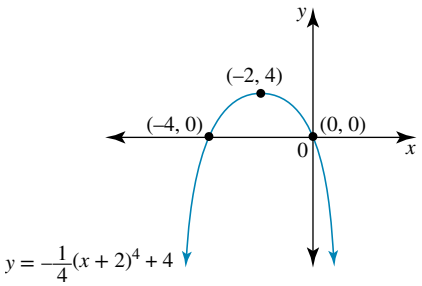
b. $y = -x(x-4)(x+2)^2$

c. $y = \frac{2}{3}x^3(x+6)$

d. $y = \frac{3}{8}(2x+3)^2(5x-4)^2$

11. a. $y = -\frac{1}{4}(x+2)^4 + 4$

b.



12. a. $x = 1$

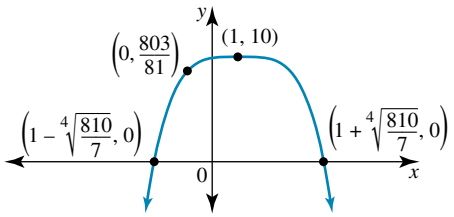
b. $(1, 10)$

c. $y = -\frac{7}{81}(x-1)^4 + 10$

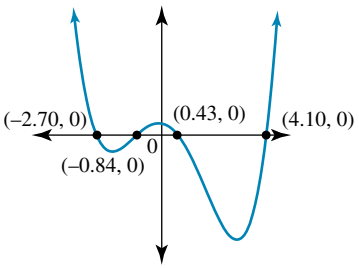
d. $\left(0, \frac{803}{81}\right)$

e. $\left(1 - \sqrt[4]{\frac{810}{7}}, 0\right), \left(1 + \sqrt[4]{\frac{810}{7}}, 0\right)$

f.



13.



x-intercepts are $(-2.70,\ 0), (-0.84,\ 0), (0.43,\ 0), (4.10,\ 0)$; minimum turning points $(-2,\ -12)$, $(2.92,\ -62.19)$; maximum turning point $(-0.17, 4.34)$

Exercise 5.8 Solving polynomial equations

1. C

2. a. both $P(-2) < 0$ and $P(-1) < 0$

b. $P(1) < 0$ and $P(2) > 0$

c. $\left[\frac{3}{2}, 2\right]$

3. a. $P(-2) = 17 > 0$ and $P(0) = -1 < 0$.

b. $[-1, 0], [-0.5, 0]$

c. $x = -0.25$

4. a. $P(10) = -19,\ P(12) = 1$

b. $P(-2) = 3,\ P(-1) = -3$

c. $P(-2) = -51,\ P(1) = 9$

d. $P(0) = 2,\ P(1) = -1$

5. a. $[11, 12], [11.5, 12]$; $x = 11.75$

b. $[-2,\ -1.5], [-2,\ -1.75]$; $x = -1.875$

c. $[-2,\ -0.5], [-1.25,\ -0.5]$; $x = -0.875$

d. $[0.5, 1], [0.75, 1]$; $x = 0.875$

6. a. $P(1) < 0, P(2) > 0$ **b.** $x = 1.5$

c. $x = 1.5$; $x = 1.75$ **d.** $x = 1.875$

7. a. $x = 1.2$ **b.** $x = 4$

c. Method of bisection very slowly converges towards the solution.

8. a.

x	-2	-1	0	1	2
y	13	-2	-3	-2	13

b. $x \in [1,\ 2]$

c. See the worked solutions in your eBookPLUS for a sample proof.

9. a.

x	-3	-2	-1	0	1	2	3
y	75	8	-9	-12	-13	0	63

b. $x = 2$

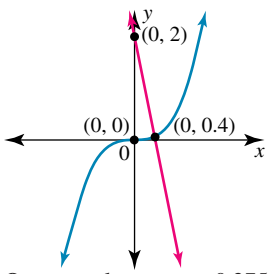
c. $[-2,\ -1]$; $x = -1.7$

10. a. $[0, 1]$

b. $x = 0.375$

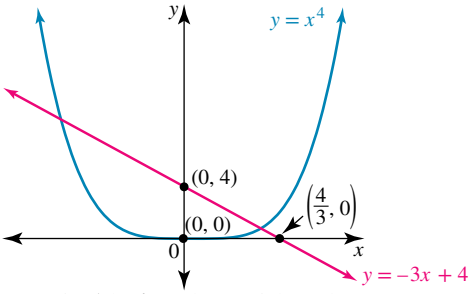
c. $y = x^3$ and $y = -5x + 2$ (or other)

d.



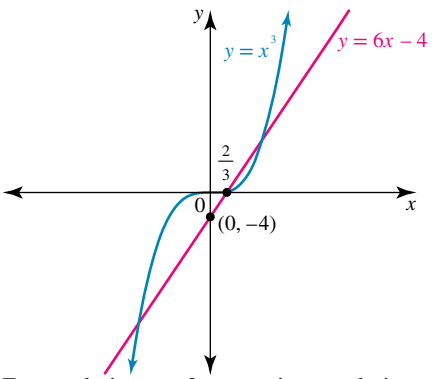
One root close to $x = 0.375$

11.



$x = -1.75$ (estimate); $x = 1$ (exact)

12.



Exact solution $x = 2$; approximate solutions $x = -2.7$ and $x = 0.7$

13. a. $x^3 - 3x + 2 = 0$

b. Two factors, one of multiplicity 2, one of multiplicity 1

c. $x = -2, x = 1$, A $(1, 1)$, B $(-2, -8)$

d. Three solutions

14. a. $x \in [0,\ 3]$ **b.** $(1.8, 8.208)$

15. a. $(-1.6, 65.664)$

b. $(-0.2, -0.552)$

c. Maximum turning points approximately $(-0.7, 0.2499)$ and $(0.7, 0.2499)$; minimum turning point exactly $(0, 0)$

16. a. $(0, 9)$

b. $\left(-\frac{5}{2}, 9\right)$ and $(3, 9)$

c. Between $x = -\frac{5}{2}$ and $x = 0$

d. $(-1.4, 22.552)$

17. a. $x \in [-1, 0]$ and $x \in [0, 4]$

b. $x \in \left[-2\sqrt{3}, 0\right]$ and $x \in \left[0, 2\sqrt{3}\right]$

c. $x \in [0, 1]$ and $x \in [1, 4]$

d. $x \in [-1, 0]$ and at the point $(0, 7)$

18. a. $y = 0$ for $x \in [1, 2]$

b. $x \in [1, 1.5]$; $x \in [1, 1.25]$

c. $x \in [3, 6]$

d. 44 containers; \$331

e. 65 or more containers

19. a. $V = x(18 - 2x)(14 - 2x)$

b. Between $x = 0$ and $x = 7$

c. 2.605 cm

5.9 Review: exam practice

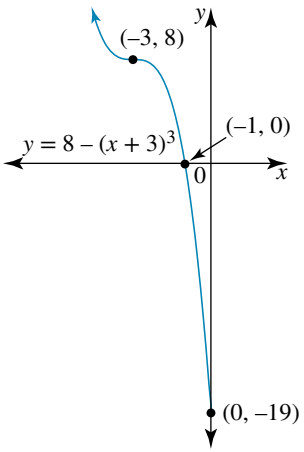
Simple familiar

1. $(x - 1)(x + 3)^2$

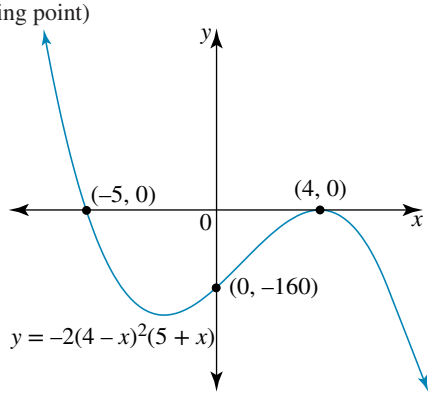
2. $(x - 2)(x + 1)(3x + 2)(2x - 5)$

3. $a = -2, b = 2$

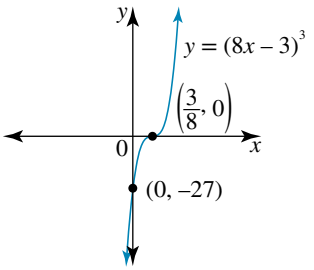
4. a. Stationary point of inflection $(-3, 8)$; y-intercept $(0, -19)$; x-intercept $(-1, 0)$



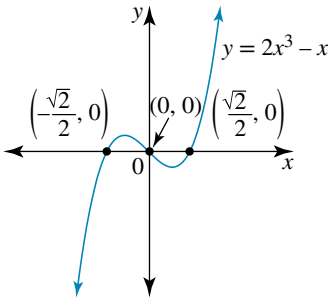
b. y-intercept $(0, -160)$; x-intercepts $(-5, 0)$, $(4, 0)$ (turning point)



c. Stationary point of inflection $\left(\frac{3}{8}, 0\right)$; y-intercept $(0, -27)$



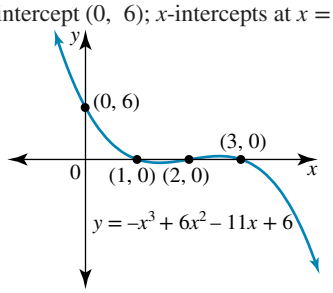
d. y-intercept $(0, 0)$; x-intercepts $\left(-\frac{\sqrt{2}}{2}, 0\right), (0, 0), \left(\frac{\sqrt{2}}{2}, 0\right)$



5. a. $x = -4, x = -1, x = 3$

b. $x = 5, x = -\frac{7}{3}$

6. y-intercept $(0, 6)$; x-intercepts at $x = 1$, $x = 2$, $x = 3$



$x < 1$ and $2 < x < 3$

7. D

8. C

9. D

10. C

11. D

12. B

13. Quotient $2x^2 - 7x + 15$; remainder -31

14. a. $P(3) = 8(3)^3 - 34(3)^2 + 33(3) - 9$

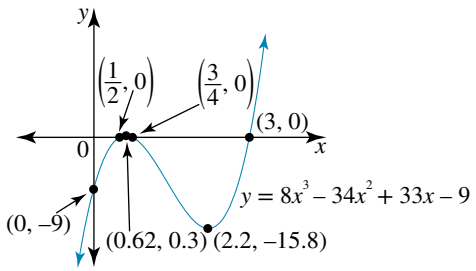
$= 216 - 306 + 99 - 9$

$= 0$

Since $P(3) = 0$, then $(x - 3)$ is a factor.

b. $(x - 3)(4x - 3)(2x - 1)$

c.



d. $\left\{0, \frac{3}{2}, \frac{11}{4}\right\}$

e. i. $-15.8 < k < 0.3$

ii. $k = 0.3,\ k = -15.8$

iii. $k < -15.8$ or $k > 0.3$

15. a. R has degree 2; C has degree 3

b. Revenue \$90; cost \$6; profit \$84

c. The profit is revenue R − cost C.

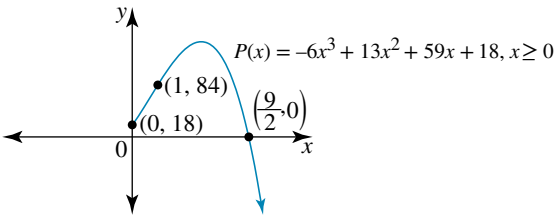
$\therefore P(x) = R(x) - C(x)$

$= 6(2x^2 + 10x + 3) - x(6x^2 - x + 1)$

$= 12x^2 + 60x + 18 - 6x^3 + x^2 - x$

$\therefore P(x) = -6x^3 + 13x^2 + 59x + 18$

d. Restriction $x \geq 0$; x-intercept $(4.5, 0)$



e. $d = 4501$

16. a. $a = -1,\ b = 16,\ y = -x^4 + 16$

b. $A = -x^5 + 16x$

c. Let $A = 15$

$-x^5 + 16x = 15$

$x^5 - 16x + 15 = 0$

Let $x = 1$

LHS $= 1 - 16 + 15$

$= 0$

$=$ RHS

Therefore $x = 1$ is a root of the equation and $(x - 1)$ is a factor of $x^5 - 16x + 15$.

$x^5 - 16x + 15 = (x - 1)(x^4 + ax^3 + bx^2 + cx - 15)$

Equate coefficients of x^4: $0 = a - 1$

$\therefore a = 1$

Equate coefficients of x^3: $0 = -1 + b$

$\therefore b = 1$

Equate coefficients of x^2: $0 = -1 + c$

$\therefore c = 1$

Hence $x^5 - 16x + 15 = (x - 1)(x^4 + x^3 + x^2 + x - 15)$

When $x^5 - 16x + 15 = 0$, either $x = 1$ or

$x^4 + x^3 + x^2 + x - 15 = 0$.

d. i. Between $x = 1$ and $x = 2$

ii. $\beta = 1.625$

e. Smaller height if width is β metres

17. a. $y = -2x + 22$

b. $y = a(2x - 1)(x - 6)(x + b),\ 0 \leq x \leq 8$

Substitute point A $(1, 20)$.

$20 = a(2 - 1)(1 - 6)(1 + b)$

$20 = -5a(1 + b)$

$a(1 + b) = -4$....(1)

Substitute point B $(5, 12)$.

$12 = a(10 - 1)(5 - 6)(5 + b)$

$12 = -9a(5 + b)$

$3a(5 + b) = -4$....(2)

Divide equation (2) by equation (1).

$\frac{3a(5 + b)}{a(1 + b)} = \frac{-4}{-4}$

$\frac{3(5 + b)}{1 + b} = 1,\ a \neq 0$

$15 + 3b = 1 + b$

$2b = -14$

$b = -7$

Substitute $b = -7$ into equation (1).

$a(1 - 7) = -4$

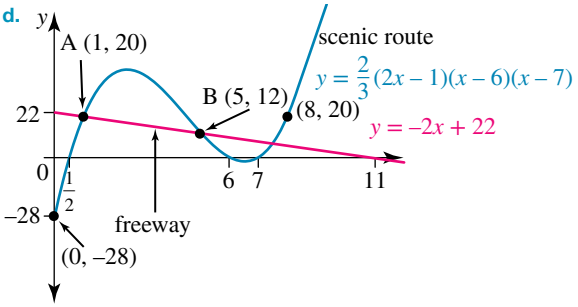
$-6a = -4$

$a = \frac{4}{6}$

$= \frac{2}{3}$

c. Endpoints: $(0, -28), (8, 20)$

d.



e. $(1, 20), (5, 12)$ and $\left(\frac{15}{2}, 7\right)$

f. $\left(\frac{15}{2}, 7\right)$

18. a. $x = -2, x = -1, x = 0$; maximum turning point $x \in [-2, -1]$; minimum turning point $x \in [-1, 0]$

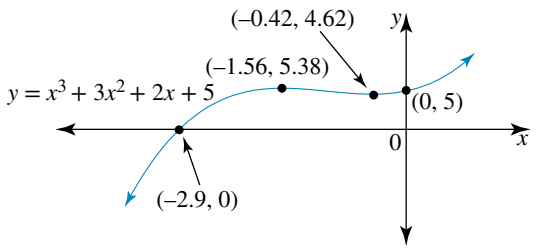
b. $(-1.58, 5.38)$ and $(-0.42, 4.62)$

c. Both turning points lie above the x-axis.

d. $[-3, -2]$, $[-3, -2.5]$, $[-3, -2.75]$, $[-3, -2.875]$

e. $(-2.9, 0)$ (answers may vary)

f.



19. a. $d = 0$

b. $x^4 + ax^3 + bx^2 + (c+2)x = 0$

c. $a = 12, b = 45, c = 52$

d. i. The rule for the quartic polynomial $y = x^4 + ax^3 + bx^2 + cx + d$ shown in the diagram is $y = x^4 + 12x^3 + 45x^2 + 52x$.

Let $x = -4$

$y = (-4)^4 + 12(-4)^3 + 45(-4)^2 + 52(-4)$

$= 256 - 768 + 720 - 208$

$= 976 - 976$

$= 0$

There is an x-intercept at $x = -4$.

ii. $(x+4)$ is a factor and so is x.

$y = x^4 + 12x^3 + 45x^2 + 52x$

$= x(x^3 + 12x^2 + 45x + 52)$

$= x((x+4)(x^2 + nx + 13))$

$= x((x+4)(x^2 + 8x + 13))$

$= x(x+4)(x^2 + 18x + 13)$

Let $y = 0$

$\therefore x = 0, x = -4$ or\, $x^2 + 8x + 13 = 0$

Solving the quadratic equation by completing the square gives:

$(x^2 + 8x + 16) - 16 + 13 = 0$

$(x+4)^2 = 3$

$x + 4 = \pm\sqrt{3}$

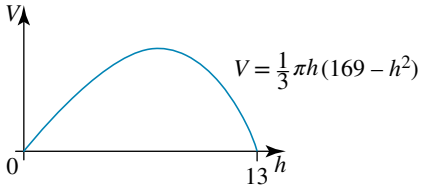
$x = -4 \pm \sqrt{3}$

The x-intercepts apart from $(-4, 0)$ are $(0, 0)$, $\left(-4 - \sqrt{3}, 0\right)$ and $\left(-4 + \sqrt{3}, 0\right)$.

20. a. $\frac{13\sqrt{3}}{3}$ metres

b. $V = \frac{1}{3}\pi h(169 - h^2)$

c. $0 \leq h \leq 13$



d. $V(7) = 280\pi = V(8)$, $V(9) = 264\pi$, $a = 7$

e. i. $h = 7.5$, $r = 10.62$

ii. $886\,\text{m}^3$

f. i. $h = \frac{13}{\sqrt{3}}$, $r = \frac{13\sqrt{6}}{3}$

ii. $886\,\text{m}^3$

TOPIC 2 Functions and graphs

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



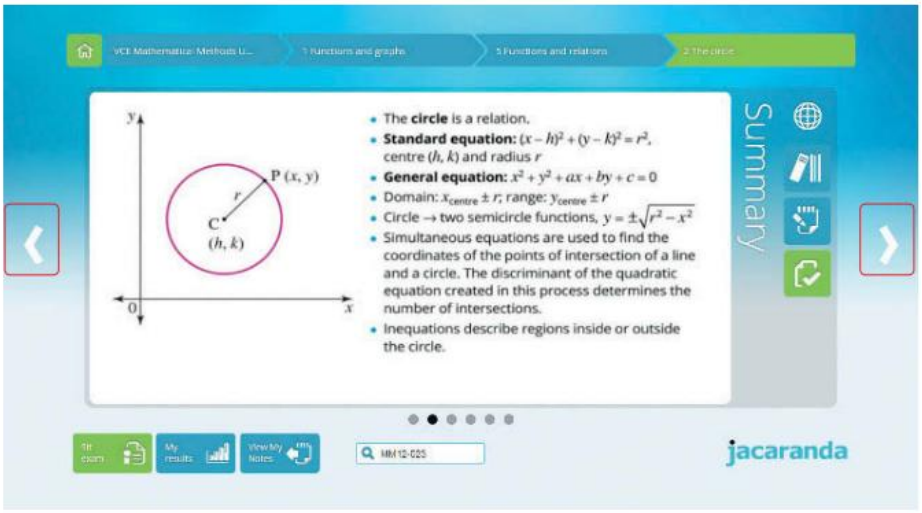
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 1

Mathematical Methods: Problem solving and modelling task

Unit

Unit 1 Algebra, statistics and functions

Topic

Topic 2: Functions and graphs

Conditions

Duration	Mode	Individual/group
4 Weeks	Written report, up to 10 pages (maximum 2000 words) excluding appendix	Individual

Resources permitted
The use of technology is required, for example: • computer • internet • spreadsheet program • calculator • other software/technology

Milestones		
Week 4		
Week 5		
Week 6		
Week 7 (assessment submission)		

Criterion*	Marks allocated	Result
Formulate • Assessment objectives 1, 2, 5	4	
Solve • Assessment objectives 1, 6	7	
Evaluate and verify • Assessment objectives 4, 5	5	
Communicate • Assessment objectives 3	4	
Total	20	

Scaffolding
Please refer to the flow chart on the following page describing an appropriate approach to problem solving and modelling.

*Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1*, Brisbane, 2018.
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Context



All structures, regardless of their function, are made up of combinations of only a few basic shapes: beams, slabs, trusses, arches and domes. The characteristics of these design features can all be expressed mathematically. Architects, builders and engineers can create buildings ranging from the most utilitarian garage to majestic cathedrals, and all of them can be described in terms of mathematical equations.

Bridges made up of arches, such as the William Jolley Bridge and the Goodwill Bridge, can be described mathematically in terms of polynomials, with the arch's keystone located at the curve's turning point.

Consider the Story Bridge shown in the image. The uppermost line of girders (the horizontal support beams) between its two highest points forms a curve. Can this curve be described by a polynomial?



Task

Develop a function that models the shape of the curved section between the two highest points of a bridge of your choosing. Your choice of bridge must be able to be described as a polynomial. Produce a report which explains how you developed and refined your model.

In presenting the development of your model in the report, you must consider:

- at least two polynomial functions and
- a piece-wise (hybrid) function.

To complete this task, you must:

- Present your findings as a report based on the approach to problem-solving and mathematical modelling outlined in the Mathematical Methods syllabus and on the flow chart on the following page to develop your response.
- Respond with a range of understanding and skills, such as using mathematical language, appropriate calculations, tables of data, graphs and diagrams.
- Provide a unique response that highlights the real-life application of mathematics.
- Respond using a written report format that can be read and interpreted independently of the instrument task sheet.
- Use both analytic procedures and technology.

Stimulus

Further reading:

- Passy's World of Mathematics, 'Sydney Harbour Bridge Mathematics', **www.passyworldofmathematics.com/sydney-harbour-bridge-mathematics/**
- Donna Levandoski Wade, 'The Math in the Design and Building of Bridges', **www.teachersinstitute.yale.edu/curriculum/units/2006/4/06.04.06.x.html**
- State Library of Queensland, 'The Story Bridge', **www.slq.qld.gov.au/showcase/hornibrook/the-hornibrook-story-bridge**

Formulate

Determine what the question is asking you to solve, then design a plan to solve it. Your plan needs to translate into a mathematically purposeful representation. Using your mathematical experience and knowledge, determine the appropriate mathematical and/statistical principles, concepts, techniques and technology that are required to solve the problem. Then list the mathematical techniques and technology you will use to develop the response.

Consider how you will determine the data values that define the shape of the bridge, and what methods you will use to generate the model. You must identify and document appropriate assumptions, variables and other observations based on the logic of a proposed solution and/or model.

In mathematical modelling, formulating a model involves the process of mathematisation — moving from the real world to the mathematical world.

Solve

Draw on your knowledge of mathematical and statistical procedures, concepts and techniques to solve the mathematical problem to be addressed through your model. There are a number of possible approaches, but include synthesising and refining the polynomial model, generating and testing the feasibility of the sum of even-powered terms polynomial, and standard mathematical techniques.

This process involves critically examining your initial observations and assumptions and making necessary changes to conclude whether the problem can be solved with a robust solution.

Solutions can be found using algebraic, graphic and technological methods.

Is it solved?

Evaluate and verify

Examine the reasonableness of the solution you have generated. This involves justifying the solution in relation to the initial problem. To justify a solution you must examine the strengths and weaknesses of the solution/model. Weaknesses or limitations of the model may be reduced by further refinement of the original model. Mathematical modelling of the bridge of your choice involves providing a model which matches as closely as possible the real-world bridge it has been designed to address.

Use both a residual analysis and the correlation coefficient to interpret the result of the mathematics compared with the original task.

Is the solution verified?

Communicate

The solution and model developed to model the bridge of your choice must clearly and fully communicated so that it may be evaluated and used by others. Your response should be communicated using mathematical, statistical and everyday language. Discuss your solution/model with reference to key results, strengths and weaknesses and recommendations for future modelling.

CHAPTER 6
Counting and probability

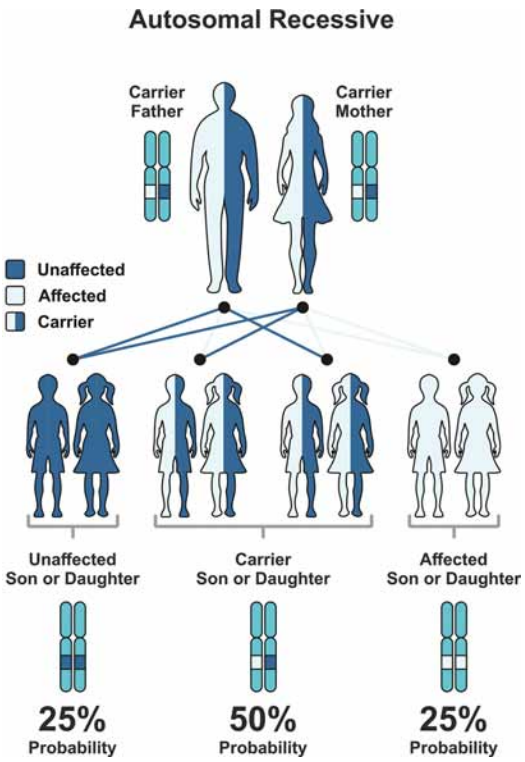
6.1 Overview

6.1.1 Introduction

It is absolutely certain that the sun will rise tomorrow. You will never see a pink elephant. You are more likely to be bitten by a toddler than by a shark. From the earliest times, humans have understood that some events were more likely to happen than others. The ancient Romans invented the insurance policy on this basis and gamblers since the dawn of time have made and lost money according to their sense of how likely it was they would get the right card, roll the right die or pick the right horse.

While popularity of the study of probability for hundreds of years was chiefly due to its usefulness in games of chance, it was eventually recognised that probability could be used to model the future behaviour of many phenomena by analysing how they had behaved in the past. Today, probability shapes many of our society's most important decisions. The decisions of farmers are based on the probability of rainfall, drought and the spread of disease through crops and herds. Firefighters rely on probability to predict the way in which a fire will spread through a building or through bushland. Geneticists use probability to determine how likely it is for a gene to be passed from generation to generation within a population. From being simply a way for a gambler to get an inside edge, probability has become one of the most important mathematical tools of the modern age.

Probability can be used to estimate the chances of specific genes being passed down through families



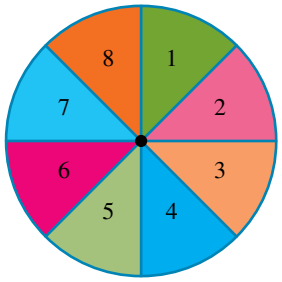
LEARNING SEQUENCE

6.1 Overview
6.2 Fundamentals of probability
6.3 Relative frequency
6.4 Conditional probability
6.5 Independence
6.6 Permutations and combinations
6.7 Pascal's triangle and binomial expansions
6.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

6.2 Fundamentals of probability

6.2.1 Notation and fundamentals: outcomes, sample spaces and events



Consider the experiment or trial of spinning a wheel which is divided into eight equal sectors, with each sector marked with one of the numbers 1 to 8. If the wheel is unbiased, each of these numbers is equally likely to occur.

The **outcome** of each trial is one of the eight numbers.

The **sample space**, ξ, is the set of all possible outcomes: $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

An **event** is a particular set of outcomes which is a subset of the sample space. For example, if M is the event of obtaining a number which is a multiple of 3, then $M = \{3, 6\}$. This set contains two outcomes. This is written in set notation as $n(M) = 2$.

The **probability** of an event is the long-term proportion, or relative frequency, of its occurrence.

For any event A, the probability of its occurrence is $\mathrm{P}(A) = \dfrac{n(A)}{n(\xi)}$.

Hence, for the event M:

$$\begin{aligned}\mathrm{P}(M) &= \frac{n(M)}{n(\xi)} \\ &= \frac{2}{8} \\ &= \frac{1}{4}\end{aligned}$$

This value does not mean that a multiple of 3 is obtained once in every four spins of the wheel. However, it does mean that after a very large number of spins of the wheel, the proportion of times that a multiple of 3 would be obtained approaches $\frac{1}{4}$. The closeness of this proportion to $\frac{1}{4}$ would improve in the long term as the number of spins is further increased.

For any event A, $0 \leq \mathrm{P}(A) \leq 1$.

- If $\mathrm{P}(A) = 0$ then it is not possible for A to occur. For example, the chance that the spinner lands on a negative number would be zero.
- If $\mathrm{P}(A) = 1$ then the event A is certain to occur. For example, it is 100% certain that the number the spinner lands on will be smaller than 9.

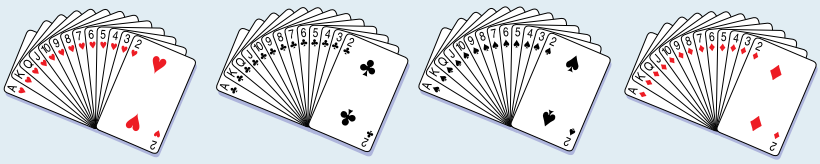
The probability of each outcome $\mathrm{P}(1) = \mathrm{P}(2) = \mathrm{P}(3) = \ldots = \mathrm{P}(8) = \frac{1}{8}$ for this spinning wheel. As each outcome is equally likely to occur, the outcomes are **equiprobable**. In other situations, some outcomes may be more likely than others.

For any sample space, $\mathrm{P}(\xi) = \dfrac{n(\xi)}{n(\xi)} = 1$ and the sum of the probabilities of each of the outcomes in any sample space must total 1.

WORKED EXAMPLE 1

A card is chosen from a standard deck. List the following outcomes in order from least likely to most likely.

Selecting a picture card **Selecting an ace**
Selecting a diamond **Selecting a black card**



THINK	WRITE
	The order of events in ascending order of likelihood:
There are 4 aces in the deck.	• selecting an ace
There are 12 picture cards in the deck.	• selecting a picture card
There are 13 diamonds in the deck.	• selecting a diamond
There are 26 black cards in the deck.	• selecting a black card.

In the above worked example, we have been able to calculate which event is more likely by counting the number of ways an event may occur. This is not always possible. In some cases we need to use general knowledge to describe the chance of an event occurring.

Consider the following probability problems.

Problem A

'The letters of the alphabet are written on cards and one card is selected at random. Which letter has the greater chance of being chosen, E or Q?'

Each letter has an equal chance of being chosen because there is one chance that E will be chosen and one chance that Q will be chosen.

Problem B

'Stacey sticks a pin into a page of a book and then writes down the letter nearest to the pin. Which letter has the greater chance of being chosen, E or Q?'

Problem B is more difficult to answer because each letter does not occur with equal frequency. However, we know from our experience with the English language that Q will occur much less often than most other letters. We can therefore say that E will occur more often than Q.

This is an example of using knowledge of the world to make predictions about which event is more likely to occur. In this way, we make predictions about everyday things such as the weather and which football team will win on the weekend.

WORKED EXAMPLE 2

Weather records show that it has rained on Christmas Day 12 times in the last 80 years. Describe the chance of it raining on Christmas Day this year.

THINK	WRITE
It has rained only 12 times on the last 80 Christmas Days. This is much less than half of all Christmas Days.	It is unlikely that it will rain on Christmas Day this year.

Complementary events

For the spinner example, the event that the number is not a multiple of 3 is the **complement** of the event M. The complementary event is written as M' or as $\overline{M}$.

$$\begin{aligned} \mathrm{P}(M') &= 1 - \mathrm{P}(M) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

For any complementary events, $\mathrm{P}(A) + \mathrm{P}(A') = 1$ and therefore $\mathrm{P}(A') = 1 - \mathrm{P}(A)$.

WORKED EXAMPLE 3

A spinning wheel is divided into eight sectors, each of which is marked with one of the numbers 1 to 8. This wheel is biased so that P (8) = 0.3, while the other numbers are equiprobable.
a. Calculate the probability of obtaining the number 4.
b. If A is the event the number obtained is even, calculate P(A) and P(A').

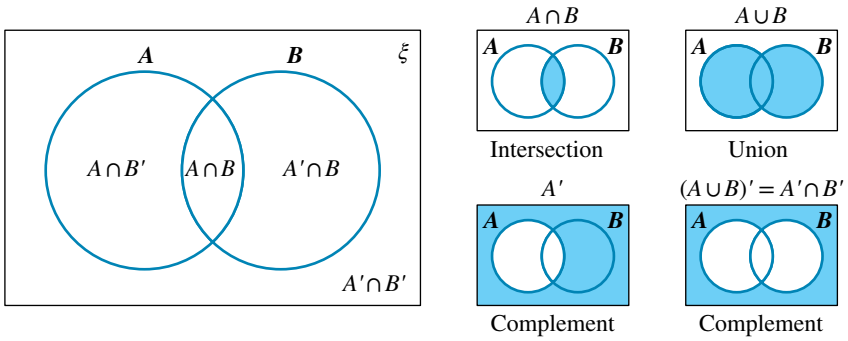
THINK	WRITE
a. 1. State the complement of obtaining the number 8 and the probability of this.	**a.** The sample space contains the numbers 1 to 8 so the complement of obtaining 8 is obtaining one of the numbers 1 to 7. As P (8) = 0.3 then the probability of not obtaining 8 is 1 − 0.3 = 0.7.
2. Calculate the required probability.	Since each of the numbers 1 to 7 are equiprobable, the probability of each number is $\frac{0.7}{7} = 0.1$. Hence, P (4) = 0.1.
b. 1. Identify the elements of the event.	**b.** $A = \{2,4,6,8\}$
2. Calculate the probability of the event.	$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(2 \text{ or } 4 \text{ or } 6 \text{ or } 8) \\ &= \mathrm{P}(2) + \mathrm{P}(4) + \mathrm{P}(6) + \mathrm{P}(8) \\ &= 0.1 + 0.1 + 0.1 + 0.3 \\ &= 0.6\end{aligned}$

3. State the complementary probability.

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - 0.6 \\ &= 0.4 \end{aligned}$$

6.2.2 Venn diagrams

A **Venn diagram** can be useful for displaying the union and intersection of sets. Such a diagram may be helpful in displaying compound events in probability, as illustrated for the sets or events A and B.



The information shown in the Venn diagram may be the actual outcomes for each event, or it may only show a number which represents the number of outcomes for each event. Alternatively, the Venn diagram may show the probability of each event. The total probability is 1; that is, $P(\xi) = 1$.

The addition formula

The number of elements contained in set A is denoted by $n(A)$.

The Venn diagram illustrates that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Hence, dividing by the number of elements in the sample space gives:

$$\frac{n(A \cup B)}{n(\xi)} = \frac{n(A)}{n(\xi)} + \frac{n(B)}{n(\xi)} - \frac{n(A \cap B)}{n(\xi)}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

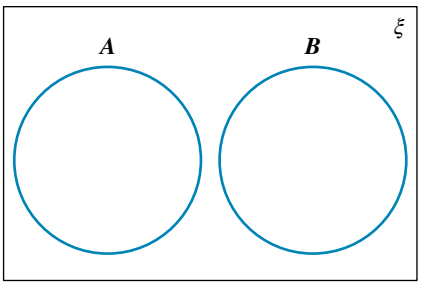
The result is known as the addition formula.

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

If the events A and B are **mutually exclusive** then they cannot occur simultaneously. For mutually exclusive events, $n(A \cap B) = 0$ and therefore $P(A \cap B) = 0$.

The addition formula for mutually exclusive events becomes:

$$P(A \cup B) = P(A) + P(B)$$



Venn diagram for mutually exclusive events

WORKED EXAMPLE 4

From a survey of a group of 50 people it was found that in the past month 30 members of the group had made a donation to a local charity, 25 had donated to an international charity and 20 had made donations to both local and international charities.

Let L be the set of people donating to a local charity and let I be the set of people donating to an international charity.

a. Draw a Venn diagram to illustrate the results of this survey.

b. One person from the group is selected at random. Using appropriate notation, calculate the probability that this person donated to a local charity but not an international one.

c. What is the probability that the person in part a did not make a donation to either type of charity?

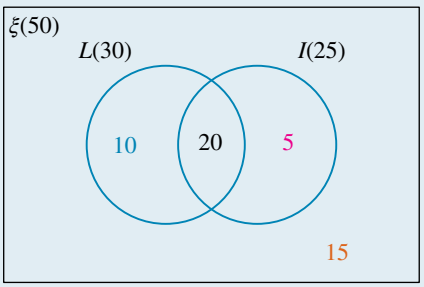
d. Calculate the probability that the person in part a donated to at least one of the two types of charity.

THINK

a. Show the given information on a Venn diagram and complete the remaining sections using arithmetic.

WRITE

a. Given: $n(\xi) = 50$, $n(L) = 30$, $n(I) = 25$ and $n(L \cap I) = 20$



b. 1. State the required probability using set notation.

b. $P(L \cap I') = \dfrac{n(L \cap I')}{n(\xi)}$

2. Identify the value of the numerator from the Venn diagram and calculate the probability.

$$P(L \cap I') = \frac{10}{50} = \frac{1}{5}$$

3. Express the answer in context.

The probability that the randomly chosen person donated to a local charity but not an international one is 0.2.

c. 1. State the required probability using set notation.

c. $P(L' \cap I') = \dfrac{n(L' \cap I')}{n(\xi)}$

2. Identify the value of the numerator from the Venn diagram and calculate the probability.

$$P(L' \cap I') = \frac{15}{50} = \frac{3}{10}$$

3. Express the answer in context.

The probability that the randomly chosen person did not donate is 0.3.

d. 1. State the required probability using set notation.

d. $P(L \cup I) = \dfrac{n(L \cup I)}{n(\xi)}$

2. Identify the value of the numerator from the Venn diagram and calculate the probability.

$$P(L \cup I) = \frac{10 + 20 + 5}{50}$$
$$= \frac{35}{50}$$
$$= \frac{7}{10}$$

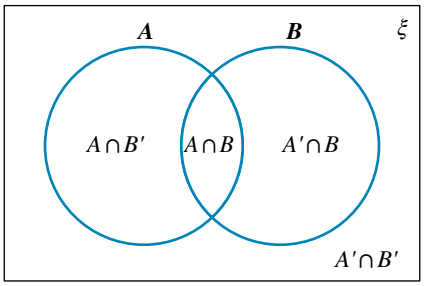
3. Express the answer in context.

The probability that the randomly chosen person donated to at least one type of charity is 0.7.

6.2.3 Probability tables

For situations involving two events, a probability table can provide an alternative to a Venn diagram. Consider the Venn diagram shown.

A **probability table** presents any known probabilities of the four compound events $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$ in rows and columns.



	B	B'	
A	$P(A \cap B)$	$P(A \cap B')$	$P(A)$
A'	$P(A' \cap B)$	$P(A' \cap B')$	$P(A')$
	$P(B)$	$P(B')$	$P(\xi) = 1$

This allows the table to be completed using arithmetic calculations since, for example, $P(A) = P(A \cap B) + P(A \cap B')$ and $P(B) = P(A \cap B) + P(A' \cap B)$.

The probabilities of complementary events can be calculated using the formula $P(A') = 1 - P(A)$.

To obtain $P(A \cup B)$, the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ can be used.

Probability tables are also known as **Karnaugh maps**.

WORKED EXAMPLE 5

$P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.2$

a. Construct a probability table for the events A and B.

b. Calculate $P(A' \cup B)$.

THINK

a. 1. Enter the given information in a probability table.

WRITE

a Given: $P(A) = 0.4$, $P(B) = 0.7$, $P(A \cap B) = 0.2$ and also $P(\xi) = 1$

	B	B'	
A	0.2		0.4
A'			
	0.7		1

2. Add in the complementary probabilities.

$P(A') = 1 - 0.4 = 0.6$ and $P(B') = 1 - 0.7 = 0.3$

	B	B'	
A	0.2		0.4
A'			0.6
	0.7	0.3	1

3. Complete the remaining sections using arithmetic.

For the first row, $0.2 + 0.2 = 0.4$
For the first column, $0.2 + 0.5 = 0.7$

	B	B'	
A	0.2	0.2	0.4
A'	0.5	0.1	0.6
	0.7	0.3	1

b. 1. State the addition formula.

2. Use the values in the probability table to carry out the calculation.

b $P(A' \cup B) = P(A') + P(B) - P(A' \cap B)$

From the probability table, $P(A' \cap B) = 0.5$

$\therefore P(A' \cup B) = 0.6 + 0.7 - 0.5$

$= 0.8$

studyon

Units 1 & 2 | Area 3 | Sequence 1 | Concept 1

Probability review Summary screen and practice questions

6.2.4 Tree diagrams

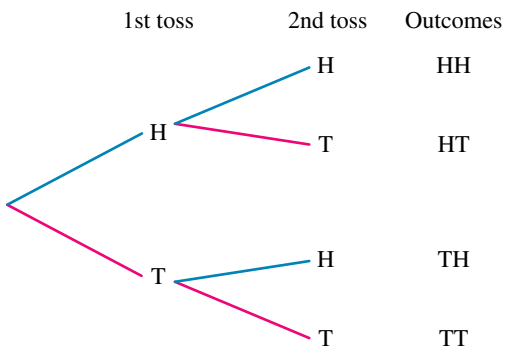
Tree diagrams are useful displays of two or three stage events.

Simple tree diagram

The outcome of each toss of a coin is either Heads (H) or Tails (T). For two tosses, the outcomes are illustrated by the tree diagram shown.

The sample space consists of the four equally likely outcomes HH, HT, TH and TT. This means the probability of obtaining two Heads (HH) in two tosses of a coin would be $\frac{1}{4}$.

The tree diagram could be extended to illustrate repeated tosses of the coin.



WORKED EXAMPLE 6

Two coins are tossed.

a. Find all the possible outcomes, namely 'Head-Head', 'Head-Tail' … and so on.

b. Find the probability of:

i. no (0) Heads turning up

ii. exactly 1 Head turning up

iii. at least one Head turning up

iv. exactly 2 Heads turning up.

THINK

a. 1. (a) It is reasonable to assume that the outcome of the first coin toss is independent of the second toss.

(b) Display the possible outcomes for the first event, namely the first coin toss.

Note: The two lines are called *branches*, one for each possible event (H or T) in the event space {H, T}.

(c) Label one branch H for Heads, the other one T for Tails.

2. Show the probabilities for each branch. In this case they are both 0.5.

3. Repeat for the second coin toss.

Notes

1. If a Head turns up on the 1st toss, either a Head or a Tail turns up on the second toss.

2. Observe the pattern of the H branch being above the T branch. Try to follow this pattern.

4. Apply the multiplication rule for independent events for each pathway. The first pathway is 'HH'. There are 4 pathways in all. Note that the sum of all 4 final probabilities = 1 because when two coins are tossed one of the 4 outcomes must occur. The probability of a certainty occurring is 1.

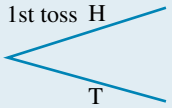
b. Use the tree diagram to answer the question.

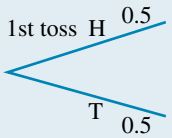
Note: There is only one way of turning up 2 Heads. There is only one way of turning up 2 Tails. There are two ways of turning up 1 Head and 1 Tail.

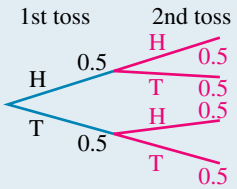
P(1 Head) = P(HT) + P(TH).

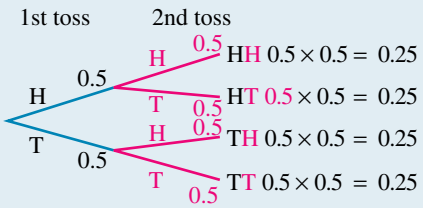
WRITE

a.









b. i. P(TT) = 0.25

ii. P(1 Head) = P(HT) + P(TH)
= 0.5

iii. P(at least 1 Head)
= P(HH) + P(HT) + P(TH)
= 0.25 + 0.25 + 0.25
= 0.75

iv. P(HH) = 0.25

Exercise 6.2 Fundamentals of probability

Technology free

1. For each of the events below, describe the chance of it occurring as impossible, unlikely, even chance (fifty-fifty), probable or certain.
   a. Rolling a die and getting a negative number
   b. Rolling a die and getting a positive number
   c. Rolling a die and getting an even number
   d. Selecting a card from a standard deck and getting a red card
   e. Selecting a card from a standard deck and getting a spot (numbered) card
   f. Selecting a card from a standard deck and getting an ace
   g. Reaching into a moneybox and selecting a 30c piece
   h. Selecting a blue marble from a bag containing 3 red, 3 green and 6 blue marbles
2. WE1 A standard six-sided die is thrown and the number rolled is noted. List the following events in order from least likely to most likely.
   a. Rolling an even number
   b. Rolling a number less than 3
   c. Rolling a 6
   d. Rolling a number greater than 2
3. MC The ski season opens on the first weekend of June. At a particular ski resort there has been sufficient snow for skiing on that weekend on 32 of the last 40 years. Which of the following statements is true?
   A. Sufficient snow on the opening day of the ski season is impossible.
   B. It is unlikely there will be sufficient snow at the opening of the ski season this year.
   C. There is a fifty-fifty chance there will be sufficient snow at the opening of the ski season this year.
   D. It is probable there will be sufficient snow at the opening of the ski season this year.
4. WE2 On a production line, light globes are tested to see how long they will last. After testing 1000 light globes it is found that 960 will burn for more than 1500 hours. Wendy purchases a light globe. Describe the chance of the light globe burning for more than 1500 hours.
5. Of 12 000 new cars sold last year, 1500 had a major mechanical problem during the first year. Edwin purchases a new car. Describe the chance of Edwin having a major mechanical problem in the first year.
6. A bag contains 12 counters: 7 are orange, 4 are red and 1 is yellow. One counter is selected at random from the bag. Calculate the probability that the counter chosen is:
   a. yellow b. red c. orange.
7. WE3 A spinning wheel is divided into eight sectors, each of which is marked with one of the numbers 1 to 8. This wheel is biased so that $\mathrm{P}(6) = \frac{9}{16}$ while the other numbers are equiprobable.
   a. Calculate the probability of obtaining the number 1.
   b. If A is the event that a prime number is obtained, calculate $\mathrm{P}(A)$ and $\mathrm{P}(A')$.
8. A bag contains 20 balls of which 9 are green and 6 are red. One ball is selected at random.
   a. What is the probability that this ball is:
      i. either green or red
      ii. not red
      iii. neither green nor red?
   b. How many additional red balls must be added to the original bag so that the probability that the chosen ball is red is 0.5?

9. **WE4** From a group of 42 students it was found that 30 students studied Mathematical Methods and 15 studied Geography. Ten of the Geography students did not study Mathematical Methods.
Let M be the set of students studying Mathematical Methods and let G be the set of students studying Geography.
   a. Draw a Venn diagram to illustrate this situation.
   One student from the group is selected at random.
   b. Using appropriate notation, calculate the probability that this student studies Mathematical Methods but not Geography.
   c. What is the probability that this student studies neither Mathematical Methods nor Geography?
   d. Calculate the probability that this student studies only one of Mathematical Methods or Geography.

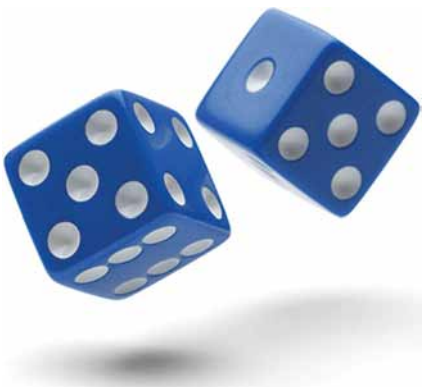
10. From a set of 18 cards numbered $1, 2, 3, \ldots, 18$, one card is drawn at random.
Let A be the event of obtaining a multiple of 3, B be the event of obtaining a multiple of 4 and let C be the event of obtaining a multiple of 5.
   a. List the elements of each event and then illustrate the three events as sets on a Venn diagram.
   b. Which events are mutually exclusive?
   c. State the value of $\mathrm{P}(A)$.
   d. Calculate the following.
      i. $\mathrm{P}(A \cup B)$ ii. $\mathrm{P}(A \cap B')$ iii. $\mathrm{P}((A \cup B \cup C)')$

11. **WE5** Given $\mathrm{P}(A) = 0.65$, $\mathrm{P}(B) = 0.5$ and $\mathrm{P}(A' \cap B') = 0.2$:
   a. construct a probability table for the events A and B
   b. calculate $\mathrm{P}(B' \cup A)$.

12. For two events A and B it is known that $\mathrm{P}(A \cup B) = 0.75$, $\mathrm{P}(A') = 0.42$ and $\mathrm{P}(B) = 0.55$.
   a. Form a probability table for these two events.
   b. State $\mathrm{P}(A' \cap B')$.
   c. Show that $\mathrm{P}(A \cup B)' = \mathrm{P}(A' \cap B')$.
   d. Show that $\mathrm{P}(A \cap B) = 1 - \mathrm{P}(A' \cup B')$.
   e. Draw a Venn diagram for the events A and B.

13. **WE6** A coin is tossed three times.
   a. Draw a simple tree diagram to show the possible outcomes.
   b. What is the probability of obtaining at least one Head?
   c. Calculate the probability of obtaining either exactly two Heads or two Tails.

14. Two unbiased dice are rolled and the larger of the two numbers is noted.
If the two dice show the same number, then the sum of the two numbers is recorded. Use a table to show all the possible outcomes. Hence calculate the probability that the result is:
   a. 5
   b. 10
   c. a number greater than 5
   d. 7
   e. either a two-digit number or a number greater than 6
   f. not 9.



15. A coin is tossed three times. Show the sample space on a tree diagram and hence calculate the probability of getting:
   a. 2 Heads and 1 Tail
   b. either 3 Heads or 3 Tails
   c. a Head on the first toss of the coin
   d. at least 1 Head
   e. no more than 1 Tail.

16. The 3:38 train to the city is late on average 1 day out of 3. Draw a probability tree to show the outcomes on three consecutive days. Hence calculate the probability that the train is:



a. late on 1 day
b. late on at least 2 days
c. on time on the last day
d. on time on all 3 days.

Technology active

17. Two hundred people applied to do their driving test in October. The results are shown below.

Gender	Passed	Failed
Male	73	26
Female	81	20

a. Calculate the probability that a person selected at random has failed the test.
b. What is the probability that a person selected at random is a female who passed the test?

18. A sample of 100 first-year university science students were asked if they study physics or chemistry. It was found that 63 study physics, 57 study chemistry and 4 study neither.



A student is then selected at random. What is the probability that the student studies:
a. either physics or chemistry but not both
b. both physics and chemistry?
In total, there are 1200 first-year university science students.
c. Estimate the number of students who are likely to study both physics and chemistry.
Two students are chosen at random from the total number of students. Find the probability that:
d. both students study physics and chemistry
e. each student studies just one of the two subjects
f. one of the two students studies neither physics nor chemistry.

19. In a table tennis competition, each team must play every other team twice.
a. How many games must be played if there are 5 teams in the competition?
b. How many games must be played if there are n teams in the competition?
A regional competition consists of 16 teams, labelled A, B, C, …, N, O, P.
c. How many games must each team play?
d. What is the total number of games played?

20. **a.** When you roll a die, what is the theoretical probability of rolling a 1?
b. Roll a die 120 times and record each result in the table below.

Number	Occurrences	Percentage of throws
1		
2		
3		
4		
5		
6		

c. How close are the experimental results to the theoretical results that were expected? Would you expect the difference between the experimental results and the theoretical results to increase or decrease if the die were rolled 1200 times? Explain your answer.

6.3 Relative frequency

6.3.1 Relative frequency

You are planning to go skiing on the first weekend in July. The trip is costing you a lot of money and you don't want your money wasted on a weekend without snow. So what is the chance of it snowing on that weekend? We can use past records only to estimate that chance.

If we know that it has snowed on the first weekend of July for 54 of the last 60 years, we could say that the chance of snow this year is very high. To measure that chance, we calculate the relative frequency of snow on that weekend. We do this by dividing the number of times it has snowed by the number of years we have examined. In this case, we can say the relative frequency of snow on the first weekend in July is $54 \div 60 = 0.9$.

The relative frequency is usually expressed as a decimal or percentage and is calculated using the formula:

$$\text{relative frequency} = \frac{\text{number of times an event has occurred}}{\text{number of trials}}$$

In this formula, a *trial* is the number of times the probability experiment has been conducted.

The formula for relative frequency is similar to that for probability.

The term *relative frequency* refers to actual data obtained, but the term *probability* generally refers to theoretical data unless experimental probability is specifically stated.

WORKED EXAMPLE 7

The weather has been fine on Christmas Day in Brisbane for 32 of the past 40 Christmas Days. Calculate the relative frequency of fine weather on Christmas Day.

THINK	WRITE
1. Write the formula.	$\text{Relative frequency} = \frac{\text{number of times an event has occurred}}{\text{number of trials}}$
2. Substitute the number of fine Christmas Days (32) and the number of trials (40).	$\text{Relative frequency} = \frac{32}{40}$
3. Calculate the relative frequency as a decimal.	$= 0.8$

The relative frequency is used to assess the quality of products. This is done by finding the relative frequency of defective products.

WORKED EXAMPLE 8

A tyre company tests its tyres and finds that 144 out of a batch of 150 tyres will withstand 20 000 km of normal wear. Find the relative frequency of tyres that will last 20 000 km. Give the answer as a percentage.

THINK	WRITE
1. Write the formula.	$\text{Relative frequency} = \frac{\text{number of times an event has occurred}}{\text{number of trials}}$

2. Substitute 144 (the number of times the event occurred) and 150 (number of trials).	$\text{Relative frequency} = \frac{144}{150}$
3. Calculate the relative frequency.	$= 0.96$
4. Convert the relative frequency to a percentage.	$= 96\%$

Relative frequencies can be used to solve many practical problems.

WORKED EXAMPLE 9

A batch of 200 light globes was tested. The batch is considered unsatisfactory if more than 15% of globes burn for less than 1000 hours. The results of the test are in the table below.

Number of hours	Number of globes
less than 500	4
500–750	12
750–1000	15
1000–1250	102
1250–1500	32
more than 1500	35

Determine if the batch is unsatisfactory.

THINK	WRITE
1. Count the number of light globes that burn for less than 1000 hours.	31 light globes burn for less than 1000 hours.
2. Write the formula.	$\text{Relative frequency} = \frac{\text{number of times an event has occurred}}{\text{number of trials}}$
3. Substitute 31 (number of times the event occurs) and 200 (number of trials).	$\text{Relative frequency} = \frac{31}{200}$
4. Calculate the relative frequency.	$= 0.155$
5. Convert the relative frequency to a percentage.	$= 15.5\%$
6. Make a conclusion about the quality of the batch of light globes.	More than 15% of the light globes burn for less than 1000 hours and so the batch is unsatisfactory.

Exercise 6.3 Relative frequency

Technology free

1. MC A study of cricket players found that of 150 players, 36 batted left handed. What is the relative frequency of left-handed batsmen?
   - **A.** 0.24
   - **B.** 0.36
   - **C.** 0.54
   - **D.** 0.64
2. MC Five surveys were conducted and the following results were obtained. Which result has the highest relative frequency?
   - **A.** Of 1500 P-plate drivers, 75 had been involved in an accident.
   - **B.** Of 1200 patients examined by a doctor, 48 had to be hospitalised.
   - **C.** Of 20 000 people at a football match, 950 were attending their first match.
   - **D.** Of 300 drivers breath tested, 170 were found to be over the legal limit.
3. WE7 At the opening of the ski season, there has been sufficient snow for skiing for 37 out of the past 50 years. Calculate the relative frequency of sufficient snow at the beginning of the ski season.
4. A biased coin has been tossed 100 times with the result of 79 Heads. Calculate the relative frequency of the coin landing Heads.
5. Of eight Maths tests done by a class during a year, Peter has topped the class three times. Calculate the relative frequency of Peter topping the class.
6. Farmer Jones has planted a wheat crop. For the wheat crop to be successful, Farmer Jones needs 500 mm of rain to fall over the spring months. Past weather records show that this has occurred on 27 of the past 60 years. Find the relative frequency of:
   - **a.** sufficient rainfall
   - **b.** insufficient rainfall.
7. WE8 Of 300 cars coming off an assembly line, 12 are found to have defective brakes. Calculate the relative frequency of a car having defective brakes. Give the answer as a percentage.
8. A survey of 25 000 new car buyers found that 750 had a major mechanical problem in the first year of operation. Calculate the relative frequency of:
   - **a.** having mechanical problems in the first year
   - **b.** not having mechanical problems in the first year.
9. In an electronics factory, 15 out of every 1400 graphics cards is faulty. Calculate the relative frequency of getting a fully functional graphics card. Express your answer as a percentage.
10. During an election campaign 2000 people were asked for their voting preferences. One thousand and fifty said that they would vote for the government, 875 said they would vote for the opposition and the remainder were undecided. What is the relative frequency of:
    - **a.** government voters
    - **b.** opposition voters
    - **c.** undecided voters?



11. Research over the past 25 years shows that each November there is an average of two wet days on Sunnybank Island. Travelaround Tours offer one-day tours to Sunnybank Island at a cost of $150 each, with a money back guarantee against rain.
    a. What is the relative frequency of wet November days as a percentage?
    b. If Travelaround Tours take 1200 bookings for tours in November, how many refunds would they expect to give?
12. An average of 200 robberies takes place each year in the town of Amiak. There are 10 000 homes in this town.
    a. What is the relative frequency of robberies per home in Amiak?
    b. Each robbery results in an average insurance claim of $20 000. What would be the minimum premium per home the insurance company would need to charge to cover these claims?
13. WE9 A car maker recorded the first time that its cars came in for mechanical repairs. The results are in the table below.

Time taken	Number of cars
0–<3 months	5
3–<6 months	12
6–<12 months	37
1–<2 years	49
2–<3 years	62
3 years or more	35



    a. The assembly line will need to be upgraded if the relative frequency of cars needing mechanical repair in the first year is greater than 25%. Determine if this will be necessary.
    b. Determine, as a percentage, the relative frequency of:
        i. a car needing mechanical repair in the first 3 months
        ii. a car needing mechanical repair in the first 2 years
        iii. a car not needing mechanical repair in the first 3 years.

Technology active

14. A manufacturer of shock absorbers measures the distance that its shock absorbers can travel before they must be replaced. The results are in the table below.

Number of kilometres	Number of shock absorbers
0–<20 000	1
20 000–<40 000	2
40 000–<60 000	46
60 000–<80 000	61
80 000–<100 000	90

What is the maximum distance the manufacturer will guarantee so that the relative frequency of the shock absorbers lasting is greater than 0.985?

15. A mini-lottery game may be simulated as follows. Each game consists of choosing two numbers from the whole numbers 1 to 6. The cost to play one game is $1. A prize of $10 is paid for both numbers correct. No other prizes are awarded.

    Use technology to simulate a game by generating 2 random numbers between 1 and 6. Do this 40 times (that is, 'play' 40 games of lotto) and enter your results in an appropriately constructed frequency table.

    a. A particular player, Ethan, always chooses the numbers 1 and 2. How many times would Ethan win in your simulation?

    b. How much would Ethan win or lose overall based on the simulation?

16. Choose one of the topics below (or another of your choice) and calculate the relative frequency of the event. Most of the information needed can be found from books or the internet.

    a. Examine weather records and find out the relative frequency of rain on New Year's Eve in Brisbane.

    b. Choose your favourite sporting team. Find the relative frequency of them winning over the past three seasons.

    c. Find the relative frequency of the stock market rising for three consecutive days.

    d. Check the NRL or AFL competitions and find the relative frequencies of win, loss and draw for each team.



6.4 Conditional probability

6.4.1 Introduction

Some given information may reduce the number of elements in a sample space. For example, in two tosses of a coin, if it is known that at least one Head is obtained then this reduces the sample space from {HH, HT, TH, TT} to {HH, HT, TH}. This affects the probability of obtaining two Heads.

The probability of obtaining two Heads given at least one Head has occurred is called **conditional probability**. It is written as $\mathrm{P}(A|B)$ where A is the event of obtaining two Heads and B is the event of at least one Head. The event B is the conditional event known to have occurred. Since event B has occurred, the sample space has been reduced to three elements. This means $\mathrm{P}(A|B) = \frac{1}{3}$.

When no information is given about what has occurred, the sample space contains four elements and the probability of obtaining two Heads is $\mathrm{P}(A) = \frac{1}{4}$.

On Resources

Interactivity: Conditional probability and independence (int-6292)

WORKED EXAMPLE 10

The table shows the results of a survey of 100 people aged between 16 and 29 about their preferred choice of food when eating at a café.

	Vegetarian (V)	Non-vegetarian (V')	
Male (M)	18	38	56
Female (F)	25	19	44
	43	57	100

One person is selected at random from those surveyed. Identify the event and use the table to calculate the following.

a. $P(M \cap V)$ **b.** $P(M|V)$ **c.** $P(V'|F)$ **d.** $P(V)$

THINK

a. 1. Describe the event $M \cap V$.

2. Calculate the probability.

b. 1. Describe the conditional probability.

2. State the number of elements in the reduced sample space.

3. Calculate the required probability.

c. 1. Identify the event.

2. State the number of elements in the reduced sample space.

3. Calculate the probability.

WRITE

a. The event $M \cap V$ is the event the selected person is both male and preferred vegetarian.

$$P(M \cap V) = \frac{n(M \cap V)}{n(\xi)}$$
$$= \frac{18}{100}$$
$$\therefore P(M \cap V) = 0.18$$

b. The event $M|V$ is the event of a person being male given that the person prefers vegetarian.

Since the person is known to prefer vegetarian, the sample space is reduced to $n(V) = 43$ people.

Of the 43 vegetarians, 18 are male.

$$\therefore P(M|V) = \frac{n(M \cap V)}{n(V)}$$
$$= \frac{18}{43}$$

c. The event $V'|F$ is the event of a person preferring non-vegetarian given the person is female.

Since the person is known to be female, the sample space is reduced to $n(F) = 44$ people.

Of the 44 females, 19 prefer non-vegetarian.

$$\therefore P(V'|F) = \frac{n(V' \cap F)}{n(F)}$$
$$= \frac{19}{44}$$

d. 1. State the event.

2. Calculate the required probability.
Note: This is not a conditional probability.

d. The event V is the event the selected person prefers vegetarian.

$$\mathrm{P}(V) = \frac{n(V)}{n(\xi)}$$
$$= \frac{43}{100}$$
$$\therefore \mathrm{P}(V) = 0.43$$

6.4.2 Formula for conditional probability

Consider the events A and B:

$\mathrm{P}(A) = \dfrac{n(A)}{n(\xi)}$, $\mathrm{P}(B) = \dfrac{n(B)}{n(\xi)}$ and $\mathrm{P}(A \cap B) = \dfrac{n(A \cap B)}{n(\xi)}$

For the conditional probability, $\mathrm{P}(A|B)$, the sample space is reduced to $n(B)$.

$$\mathrm{P}(A|B) = \frac{n(A \cap B)}{n(B)}$$
$$= n(A \cap B) \div n(B)$$
$$= \frac{n(A \cap B)}{n(\xi)} \div \frac{n(B)}{n(\xi)}$$
$$= \mathrm{P}(A \cap B) \div \mathrm{P}(B)$$
$$= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Hence, the conditional probability formula is:

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$$

This formula illustrates that if the events A and B are mutually exclusive so that $\mathrm{P}(A \cap B) = 0$, then $\mathrm{P}(A|B) = 0$. That is, if B occurs then it is impossible for A to occur.

However, if B is a subset of A so that $\mathrm{P}(A \cap B) = \mathrm{P}(B)$, then $\mathrm{P}(A|B) = 1$. That is, if B occurs, then it is certain that A will occur.

6.4.3 Multiplication of probabilities

Consider the conditional probability formula for $\mathrm{P}(B|A)$:

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$$

Since $B \cap A$ is the same as $A \cap B$, then $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$.

Rearranging, the formula for multiplication of probabilities is formed.

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \times \mathbf{P}(B|A)$$

For example, the probability of obtaining first an aqua (A) and then a black (B) ball when selecting two balls without replacement from a bag containing 16 balls, 6 of which are aqua and 10 of which are black, would be $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B|A) = \frac{6}{16} \times \frac{10}{15}$.

The multiplication formula can be extended. For example, the probability of obtaining 3 black balls when selecting three balls without replacement from the bag containing 16 balls, 10 of which are black, would be $\frac{10}{16} \times \frac{9}{15} \times \frac{8}{14}$.

WORKED EXAMPLE 11

a. If $\mathrm{P}(A) = 0.6$, $\mathrm{P}(A|B) = 0.6125$ and $\mathrm{P}(B') = 0.2$, calculate $\mathrm{P}(A \cap B)$ and $\mathrm{P}(B|A)$.

b. Three girls each select one ribbon at random, one after the other, from a bag containing 8 green ribbons and 10 red ribbons. What is the probability that the first girl selects a green ribbon and both the other girls select a red ribbon?

THINK	WRITE
a. 1. State the conditional probability formula for $\mathrm{P}(A \mid B)$.	**a** $\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
2. Obtain the value of $\mathrm{P}(B)$.	For complementary events: $\mathrm{P}(B) = 1 - \mathrm{P}(B')$ $= 1 - 0.2$ $= 0.8$
3. Use the formula to calculate $\mathrm{P}(A \cap B)$.	$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ $0.6125 = \frac{\mathrm{P}(A \cap B)}{0.8}$ $\mathrm{P}(A \cap B) = 0.6125 \times 0.8$ $= 0.49$
4. State the conditional probability formula for $\mathrm{P}(B|A)$.	$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$
5. Calculate the required probability.	$\mathrm{P}(B \cap A) = \mathrm{P}(A \cap B)$ $\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$ $= \frac{0.49}{0.6}$ $= \frac{49}{60}$
b. 1. Define the events.	**b** Let G be the event a green ribbon is chosen and R be the event a red ribbon is chosen.

2. Describe the sample space.	There are 18 ribbons in the bag forming the elements of the sample space. Of these 18 ribbons, 8 are green and 10 are red.
3. State the probability the first ribbon selected is green.	$\mathrm{P}(G) = \frac{8}{18}$
4. Calculate the conditional probability the second ribbon is red by reducing the number of elements in the sample space.	Once a green ribbon has been chosen, there are 7 green and 10 red ribbons remaining, giving a total of 17 ribbons in the bag. $\therefore \mathrm{P}(R\|G) = \frac{10}{17}$
5. Calculate the conditional probability the third ribbon is red by reducing the number of elements in the sample space.	Once a green and a red ribbon have been chosen, there are 7 green and 9 red ribbons remaining, giving a total of 16 ribbons in the bag. $\therefore \mathrm{P}(R\|G \cap R) = \frac{9}{16}$
6. Calculate the required probability.	$\mathrm{P}(G \cap R \cap R) = \frac{8}{18} \times \frac{10}{17} \times \frac{9}{16}$ $= \frac{5}{34}$

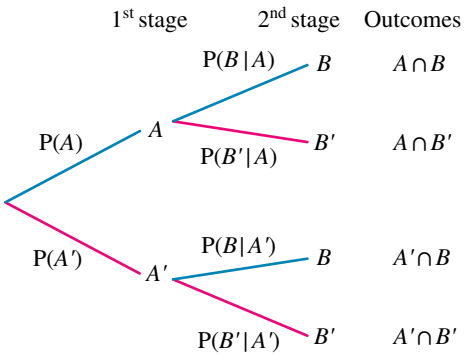
Note: $\mathrm{P}(G \cap R \cap R) = \mathrm{P}(G) \times \mathrm{P}(R|G) \times \mathrm{P}(R|G \cap R)$

6.4.4 Probability tree diagrams

The sample space of a two-stage trial where the outcomes of the second stage are dependent on the outcomes of the first stage can be illustrated with a probability tree diagram.

Each branch is labelled with its probability; conditional probabilities are required for the second-stage branches. Calculations are performed according to the addition and multiplication laws of probability.

The formula for multiplication of probabilities is applied by multiplying the probabilities that lie along the respective branches to calculate the probability of an outcome. For example, to obtain the probability of A and B occurring, we need to multiply the probabilities along the branches A to B since $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B|A)$.



The addition formula for mutually exclusive events is applied by adding the results from separate outcome branches to calculate the union of any of the four outcomes. For example, to obtain the probability that A occurs, add together the results from the two branches where A occurs. This gives $\mathrm{P}(A) = \mathrm{P}(A)\mathrm{P}(B|A) + \mathrm{P}(A)\mathrm{P}(B'|A)$.

- **Multiply along the branch**
- **Add the results from each complete branch**

WORKED EXAMPLE 12

A box of chocolates contains 6 soft-centre and 4 hard-centre chocolates. A chocolate is selected at random and once eaten, a second chocolate is chosen.
Let S_i be the event a soft-centre chocolate is chosen on the i^{th} selection and H_i be the event that a hard-centre chocolate is chosen on the i^{th} selection, $i = 1, 2$.
a. Deduce the value of $P(H_2|S_1)$.
b. Construct a probability tree diagram to illustrate the possible outcomes.
c. What is the probability that the first chocolate has a hard centre and the second a soft centre?
d. Calculate the probability that either both chocolates have soft centres or both have hard centres.

THINK

a. 1. Identify the meaning of $P(H_2 | S_1)$.

2. State the required probability.

b. Construct the two-stage probability tree diagram.

c. Identify the appropriate branch and multiply along it to obtain the required probability.
Note: The multiplication law for probability is
$P(H_1 \cap S_2) = P(H_1) \times P(S_2|H_1)$.

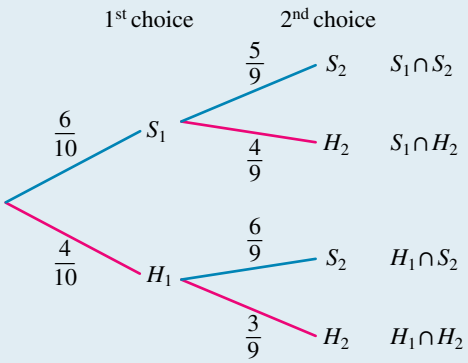
d. 1. Identify the required outcome.

2. Calculate the probabilities along each relevant branch.

WRITE

a. $P(H_2|S_1)$ is the probability that the second chocolate has a hard centre given that the first has a soft centre.
If a soft centre has been chosen first, there remain in the box 5 soft- and 4 hard-centre chocolates.
$\therefore P(H_2 | S_1) = \frac{4}{9}$

b.



c. The required outcome is $H_1 \cap S_2$.

$$P(H_1 \cap S_2) = \frac{4}{10} \times \frac{6}{9}$$
$$= \frac{2}{5} \times \frac{2}{3}$$
$$= \frac{4}{15}$$

The probability is $\frac{4}{15}$.

d. The probability that both chocolates have the same type of centre is $P((S_1 \cap S_2) \cup (H_1 \cap H_2))$.

$$P(S_1 \cap S_2) = \frac{6}{10} \times \frac{5}{9}$$
$$P(H_1 \cap H_2) = \frac{4}{10} \times \frac{3}{9}$$

3. Use the addition law for mutually exclusive events by adding the probabilities from the separate branches.

The probability that both chocolates have the same type of centre is:

$$\frac{6}{10} \times \frac{5}{9} + \frac{4}{10} \times \frac{3}{9} = \frac{30}{90} + \frac{12}{90}$$
$$= \frac{42}{90}$$
$$= \frac{7}{15}$$

The probability both centres are the same type is $\frac{7}{15}$.

studyon

Units 1 & 2 > Area 3 > Sequence 1 > Concept 2

Conditional probability Summary screen and practice questions

Exercise 6.4 Conditional probability

Technology free

1. MC If A is the event that a person is less than 30 years old and B is the event that a person likes broccoli, which of the following expressions represents the probability of a person being over 30 years of age given that they like broccoli?
   A. $P(A' \cap B)$ **B.** $P(B'|A)$ **C.** $P(A'|B)$ **D.** $P(B|A')$
2. WE10 The table shows the results of a survey of 100 people aged between 16 and 29 who were asked whether they rode a bike and what drink they preferred.
   One person is selected at random from those surveyed. Identify the event and use the table to calculate the following.
   a. $P(B' \cap C')$ **b.** $P(B'|C')$ **c.** $P(C|B)$ **d.** $P(B)$

	Drink containing caffeine (C)	**Caffeine-free drink (C')**	
Bike rider (B)	28	16	44
Non–bike rider (B')	36	20	56
	64	36	100

3. Two six-sided dice are rolled. Calculate the probability that:
   a. the sum of 8 is obtained
   b. a sum of 8 is obtained given the numbers are not the same
   c. the sum of 8 is obtained but the numbers are not the same
   d. the numbers are not the same given the sum of 8 is obtained.

4. **WE11** **a.** If $P(A') = 0.6$, $P(B|A) = 0.3$ and $P(B) = 0.5$, calculate $P(A \cap B)$ and $P(A|B)$.
   b. Three girls each select one ribbon at random, one after the other, from a bag containing 8 green ribbons and 4 red ribbons. What is the probability that all three girls select a green ribbon?
5. If $P(A) = 0.61$, $P(B) = 0.56$ and $P(A \cup B) = 0.81$, calculate the following.
   a. $P(A|B)$ **b.** $P(A|A \cup B)$ **c.** $P(A|A' \cup B)$
6. **WE12** A box of jubes contains 5 green jubes and 7 red jubes. One jube is selected at random and once eaten, a second jube is chosen.
   Let G_i be the event a green jube is chosen on the i^{th} selection and R_i the event that a red jube is chosen on the i^{th} selection, $i = 1, 2$.
   a. Deduce the value of $P(G_2|R_1)$.
   b. Construct a probability tree diagram to illustrate the possible outcomes.
   c. What is the probability that the first jube is green and the second is red?
   d. Calculate the probability that either both jubes are green or both are red.
7. To get to school Rodney catches a bus and then walks the remaining distance.
   If the bus is on time, Rodney has a 98% chance of arriving at school on time. However, if the bus is late, Rodney's chance of arriving at school on time is only 56%. On average the bus is on time 90% of the time.
   a. Draw a probability tree diagram to describe the given information, defining the symbols used.
   b. Calculate the probability that Rodney will arrive at school on time.
8. Two unbiased dice are rolled and the sum of the topmost numbers is noted. Given that the sum is less than 6, find the probability that the sum is an even number.
9. Two unbiased dice are rolled. Find the probability that the sum is greater than 8, given that a 5 appears on the first die.
10. Given $P(A) = 0.7, P(B) = 0.3$ and $P(A \cup B) = 0.8$, find the following.
    a. $P(A \cap B)$ **b.** $P(A|B)$ **c.** $P(B|A)$ **d.** $P(A|B')$
11. Given $P(A) = 0.6, P(B) = 0.5$ and $P(A \cup B) = 0.8$, find the following.
    a. $P(A \cap B)$ **b.** $P(A|B)$ **c.** $P(B|A)$ **d.** $P(A|B')$
12. Given $P(A) = 0.6$, $P(B) = 0.7$ and $P(A \cap B) = 0.4$, find the following.
    a. $P(A \cup B)$ **b.** $P(A|B)$ **c.** $P\left(B|A'\right)$ **d.** $P(A'|B')$
13. Two cards are drawn randomly from a standard pack of 52 cards. Find the probability that:
    a. both cards are diamonds
    b. at least 1 card is a diamond
    c. both cards are diamonds, given that at least one card is a diamond
    d. both cards are diamonds, given that the first card drawn is a diamond.



14. A bag contains 5 red marbles and 7 green marbles. Two marbles are drawn from the bag, one at a time, without replacement. Find the probability that:
    a. both marbles are green
    b. at least 1 marble is green
    c. both marbles are green given that at least 1 is green
    d. the first marble drawn is green given that the marbles are of different colours.
15. Sarah and Kate sit a Biology exam. The probability that Sarah passes the exam is 0.9 and the probability that Kate passes the exam is 0.8. Find the probability that:
    a. both Sarah and Kate pass the exam
    b. at least one of the two girls passes the exam
    c. only 1 girl passes the exam, given that Sarah passes.

16. In a survey designed to check the number of male and female smokers in a population, it was found that there were 32 male smokers, 41 female smokers, 224 female non-smokers and 203 male non-smokers. A person is selected at random from this group of people. Find the probability that the person selected is:

a. a non-smoker

b. male

c. female, given that the person is a non-smoker.

Technology active

17. In a sample of 1000 people, it is found that:

- 82 people are overweight and suffer from hypertension
- 185 are overweight but do not suffer from hypertension
- 175 are not overweight but suffer from hypertension
- 558 are not overweight and do not suffer from hypertension.

A person is selected at random from the sample. Find the probability that the person:

a. is overweight

b. suffers from hypertension

c. suffers from hypertension given that the person is overweight

d. is overweight given that the person does not suffer from hypertension.

18. A group of 400 people were tested for allergic reactions to two new medications. The results are shown in the table below.

	Allergic reaction	**No allergic reaction**
Medication A	25	143
Medication B	47	185

If a person is selected at random from the group, calculate the following.

a. The probability that the person suffers an allergic reaction

b. The probability that the person was administered medication A

c. Given there was an allergic reaction, the probability that medication B was administered

d. Given the person was administered medication A, the probability that the person did not have an allergic reaction

19. When Incy Wincy Spider climbs up the waterspout, the chances of his falling are affected by whether or not it is raining. When it is raining, the probability that he will fall is 0.84. When it is not raining, the probability that he will fall is 0.02. On average it rains 1 day in 5 around Incy Wincy Spider's spout.
Draw a probability tree to show the sample space and hence find the probability that:

a. Incy Wincy will fall and it is raining

b. Incy Wincy falls, given that it is raining

c. it is raining given that Incy Wincy makes it to the top of the spout.



20. In tenpin bowling, a game is made up of 10 frames. Each frame represents one turn for the bowler. In each turn, a bowler is allowed up to 2 rolls of the ball to knock down the 10 pins. If the bowler knocks down all 10 pins with the first ball, this is called a strike. If it takes 2 rolls of the ball to knock the 10 pins, this is called a spare. Otherwise it is called an open frame.
On average, Richard hits a strike 85% of the time. If he needs a second roll of the ball then, on average, he will knock down the remaining pins 97% of the time. While training for the club championship, Richard plays a game.
Draw a tree diagram to represent the outcomes of the first 2 frames of his game.
Hence find the probability that:

a. both are strikes

b. all 10 pins are knocked down in both frames

c. the first frame is a strike, given that the second is a strike.



6.5 Independence

6.5.1 Introduction

If a coin is tossed twice, the chance of obtaining a Head on the coin on its second toss is unaffected by the result of the first toss. The probability of a Head on the second toss given a Head is obtained on the first toss is still $\frac{1}{2}$.

Events which have no effect on each other are called **independent events**. For such events, $\mathrm{P}(A|B) = \mathrm{P}(A)$. The given information does not affect the chance of event A occurring.

Events which do affect each other are dependent events. For dependent events $\mathrm{P}(A|B) \neq \mathrm{P}(A)$ and the conditional probability formula is used to evaluate $\mathrm{P}(A|B)$.

6.5.2 Test for mathematical independence

While it may be obvious that the chance of obtaining a Head on a coin on its second toss is unaffected by the result of the first toss, in more complex situations it can be difficult to intuitively judge whether events are independent or dependent. For such situations there is a test for mathematical independence that will determine the matter.

The multiplication formula states $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B|A)$.

If the events A and B are independent then $\mathrm{P}(B|A) = \mathrm{P}(B)$.

Hence for independent events:

$$\boldsymbol{P(A \cap B) = P(A) \times P(B)}$$

This result is used to test whether events are mathematically independent or not.

WORKED EXAMPLE 13

Consider the trial of tossing a coin twice.
Let A be the event of at least one Tail, B be the event of either two Heads or two Tails, and C be the event that the first toss is a Head.

a. List the sample space and the set of outcomes in each of A, B and C.
b. Test whether A and B are independent.
c. Test whether B and C are independent.
d. Use the addition formula to calculate $\mathrm{P}(B \cup C)$.

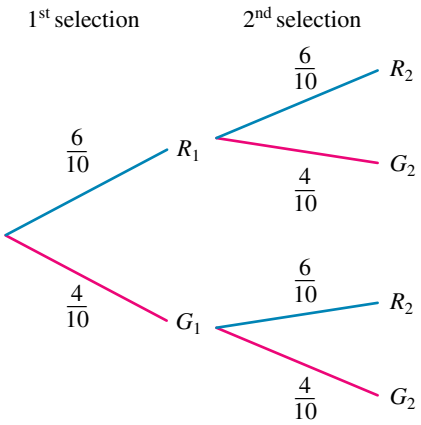
THINK	WRITE
a. 1. List the elements of the sample space.	**a.** The sample space is the set of equiprobable outcomes $\{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$.
2. List the elements of A, B and C.	$A = \{\mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$ $B = \{\mathrm{HH}, \mathrm{TT}\}$ $C = \{\mathrm{HH}, \mathrm{HT}\}$
b. 1. State the test for independence.	**b.** A and B are independent if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$.
2. Calculate the probabilities needed for the test for independence to be applied.	$\mathrm{P}(A) = \frac{3}{4}$ and $\mathrm{P}(B) = \frac{2}{4}$ Since $A \cap B = \{\mathrm{TT}\}$, $\mathrm{P}(A \cap B) = \frac{1}{4}$.
3. Determine whether the events are independent.	Substitute values into the formula $\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$. $\mathrm{LHS} = \frac{1}{4}$ $\mathrm{RHS} = \frac{3}{4} \times \frac{2}{4}$ $= \frac{3}{8}$ Since LHS $\neq$ RHS, the events A and B are not independent.
c. 1. State the test for independence.	**c.** B and C are independent if $\mathrm{P}(B \cap C) = \mathrm{P}(B)\,\mathrm{P}(C)$.
2. Calculate the probabilities needed for the test for independence to be applied.	$\mathrm{P}(B) = \frac{2}{4}$ and $\mathrm{P}(C) = \frac{2}{4}$ Since $\mathrm{P}(B \cap C) = \{\mathrm{HH}\}$, $\mathrm{P}(B \cap C) = \frac{1}{4}$.
3. Determine whether the events are independent.	Substitute values in $\mathrm{P}(B \cap C) = \mathrm{P}(B)\,\mathrm{P}(C)$. $\mathrm{LHS} = \frac{1}{4}$ $\mathrm{RHS} = \frac{2}{4} \times \frac{2}{4}$ $= \frac{1}{4}$ Since LHS = RHS, the events B and C are independent.
d. 1. State the addition formula for $\mathrm{P}(B \cup C)$.	**d.** $\mathrm{P}(B \cup C) = \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(B \cap C)$

2. Replace $P(B \cap C)$.	Since B and C are independent, $P(B \cap C) = P(B)P(C)$. $\therefore \quad P(B \cup C) = P(B) + P(C) - P(B) \times P(C)$
3. Complete the calculation.	$P(B \cup C) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$ $= \frac{3}{4}$

6.5.3 Independent trials

Consider choosing a ball from a bag containing 6 red and 4 green balls, noting its colour, returning the ball to the bag and then choosing a second ball. These trials are independent as the chance of obtaining a red or green ball is unaltered for each draw. This is an example of **sampling with replacement**.

The probability tree diagram is as shown.



The second stage branch outcomes are not dependent on the results of the first stage.

The probability that both balls are red is $P(R_1R_2) = \frac{6}{10} \times \frac{6}{10}$.

As the events are independent:

$$\begin{aligned} P(R_1 \cap R_2) &= P(R_1) \times P(R_2|R_1) \\ &= P(R_1) \times P(R_2) \end{aligned}$$

However, if **sampling without replacement**, then $P(R_1 \cap R_2) = \frac{6}{10} \times \frac{5}{9}$ since the events are not independent.

Sequences of independent events

If the events A, B, C, … are independent, then $P(A \cap B \cap C \cap \ldots) = P(A) \times P(B) \times P(C) \times \ldots$

WORKED EXAMPLE 14

Three overweight people, Ari, Barry and Chris, commence a diet. The chances that each person sticks to the diet are 0.6, 0.8 and 0.7, respectively, independent of each other. What is the probability that:

a. all three people stick to the diet

b. only Chris sticks to the diet

c. at least one of the three people sticks to the diet?

THINK	WRITE
a. 1. Define the independent events.	**a.** Let A be the event that Ari sticks to the diet, B be the event that Barry sticks to the diet and C be the event that Chris sticks to the diet. $P(A) = 0.6, P(B) = 0.8$ and $P(C) = 0.7$

2. State an expression for the required probability.
3. Calculate the required probability.

The probability that all three people stick to the diet is $P(A \cap B \cap C)$.

Since the events are independent,

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \times P(B) \times P(C) \\ &= 0.6 \times 0.8 \times 0.7 \\ &= 0.336 \end{aligned}$$

The probability all three stick to the diet is 0.336.

b. 1. State an expression for the required probability.

2. Calculate the probability.

Note: For complementary events, $P(A') = 1 - P(A)$.

b. If only Chris sticks to the diet then neither Ari nor Barry do. The probability is $P(A' \cap B' \cap C)$.

$$\begin{aligned} P(A' \cap B' \cap C) &= P(A') \times P(B') \times P(C) \\ &= (1 - 0.6) \times (1 - 0.8) \times 0.7 \\ &= 0.4 \times 0.2 \times 0.7 \\ &= 0.056 \end{aligned}$$

The probability only Chris sticks to the diet is 0.056.

c. 1. Express the required event in terms of its complementary event.

2. Calculate the required probability.

c. The event that at least one of the three people sticks to the diet is the complement of the event that no one sticks to the diet.

$$\begin{aligned} &P(\text{at least one sticks to the diet}) \\ &= 1 - P(\text{no one sticks to the diet}) \\ &= 1 - P(A' \cap B' \cap C') \\ &= 1 - 0.4 \times 0.2 \times 0.3 \\ &= 1 - 0.024 \\ &= 0.976 \end{aligned}$$

The probability that at least one person sticks to the diet is 0.976.

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Units 1 & 2 > Area 3 > Sequence 1 > Concept 3

Independence Summary screen and practice questions

Exercise 6.5 Independence

Technology free

1. Identify which of the following are independent events.
   - **a.** Event 1: Rolling a 5 on a die; Event 2: getting a head on a tossed coin
   - **b.** Event 1: Removing a green ball from a bag containing green, yellow and red balls without replacement; Event 2: Removing a red ball from the same bag
   - **c.** Event 1: A cyclone strikes Brisbane; Event 2: Brisbane airport is closed down
   - **d.** Event 1: Drawing a black card from a deck of cards with replacement; Event 2: Drawing a picture card from the deck of cards
2. State the condition under which two events A and B are defined as being independent.

3. **WE13** Consider the experiment of tossing a coin twice.
   Let A be the event the first toss is a Tail, B the event of one Head and one Tail and C be the event of no more than one Tail.
   a. List the sample space and the set of outcomes in each of A, B and C.
   b. Test whether A and B are independent.
   c. Test whether B and C are independent.
   d. Use the addition formula to calculate $\mathrm{P}(B \cup A)$.
4. Two unbiased six-sided dice are rolled. Let A be the event the same number is obtained on each die and B be the event the sum of the numbers on each die exceeds 8.
   a. Are events A and B mutually exclusive? Justify your answer.
   b. Are events A and B independent? Justify your answer.
   c. If C is the event the sum of the two numbers equals 8, determine whether B and C are:
      i. mutually exclusive
      ii. independent.
5. **WE14** Three underweight people, Ava, Bambi and Chi, commence a carbohydrate diet. The chances that each person sticks to the diet are 0.4, 0.9 and 0.6 respectively, independent of each other. What is the probability that:
   a. all three people stick to the diet
   b. only Ava and Chi stick to the diet
   c. at least one of the three people does not stick to the diet?
6. A box of toy blocks contains 10 red blocks and 5 yellow blocks. A child draws out two blocks at random.
   a. Draw the tree diagram if the sampling is with replacement and calculate the probability that one block of each colour is obtained.
   b. Draw the tree diagram if the sampling is without replacement and calculate the probability that one block of each colour is obtained.
   c. If the child was to draw out three blocks, rather than two, calculate the probability of obtaining three blocks of the same colour if the sampling is with replacement.
7. Two events A and B are such that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.8$ and $\mathrm{P}(A \cup B) = 0.94$. Determine whether A and B are independent.
8. Events A and B are such that $\mathrm{P}(A) = 0.75$, $\mathrm{P}(B) = 0.64$ and $\mathrm{P}(A \cup B) = 0.91$. Determine whether events A and B are independent.
9. A family owns two cars, A and B. Car A is used 65% of the time, car B is used 74% of the time and at least one of the cars is used 97% of the time. Determine whether the two cars are used independently.
10. The probability that a male is colourblind is 0.05 while the probability that a female is colourblind is 0.0025.
    If there is an equal number of males and females in a population, find the probability that a person selected at random from the population is:
    a. female and colourblind
    b. colourblind given that the person is female
    c. male given that the person is colourblind.
    If two people are chosen at random, find the probability that:
    d. both are colourblind males
    e. one is colourblind given a male and a female are chosen.

11. A survey of 200 people was carried out to determine the number of traffic violations committed by different age groups. The results are shown in the table below.

	Number of violations		
Age group	**0**	**1**	**2**
Under 25	8	30	7
25 – 45	47	15	2
45 – 65	45	18	3
65+	20	5	0



If one person is selected at random from the group, find the probability that:

a. the person belongs to the under-25 age group

b. the person has had at least one traffic violation

c. given that the person has had at least 1 traffic violation, he or she has had only 1 violation

d. the person is 38 and has had no traffic violations

e. the person is under 25, given that he/she has had 2 traffic violations.

12. In an attempt to determine the efficacy of a test used to detect a particular disease, 100 subjects, of which 27 had the disease, were tested. A positive result means the test detected the disease and a negative result means the test did not detect the disease. Only 23 of the 30 people who tested positive actually had the disease. Draw up a two-way table to show this information and hence find the probability that a subject selected at random:

a. does not have the disease

b. tested positive but did not have the disease

c. had the disease given that the subject tested positive

d. did not have the disease, given that the subject tested negative

13. Events A and B are independent. If $\mathrm{P}(B) = \frac{2}{3}$ and $\mathrm{P}(A|B) = \frac{4}{5}$, find:

a. $\mathrm{P}(A)$ **b.** $\mathrm{P}(B|A)$ **c.** $\mathrm{P}(A \cap B)$ **d.** $\mathrm{P}(A \cup B)$.

14. Events A and B are such that $\mathrm{P}(B) = \frac{3}{5}$, $\mathrm{P}(A|B) = \frac{1}{3}$ and $\mathrm{P}(A \cup B) = \frac{23}{30}$.

a. Find $\mathrm{P}(A \cap B)$.

b. Find $\mathrm{P}(A)$.

c. Determine whether events A and B are independent.

15. Events A and B are independent such that $\mathrm{P}(A \cup B) = 0.8$ and $\mathrm{P}(A|B') = 0.6$. Find $\mathrm{P}(B)$.

16. 600 people were surveyed about whether they watched movies on their TV or on their devices. They are classified according to age, with the following results.

	Age 15 to 30	**Age 30 to 70**	
TV	95	175	270
Device	195	135	330
	290	310	600



Based on these findings, is the medium used to watch movies independent of a person's age?

Technology active

17. A popular fast-food restaurant has studied the customer service provided by a sample of 100 of its employees across Australia. They wanted to know if employees who were with the company longer received more positive feedback from customers than newer employees.

The results of their study are shown below.

	Positive customer feedback (P)	**Negative customer feedback (P′)**	
Employed 2 years or more *(T)*	34	16	50
Employed fewer than 2 years *(T′)*	24	26	50
	58	42	100

Are the events P and T independent? Justify your answer.

18. Roll two fair dice and record the number uppermost on each. Let A be the event of rolling a 6 on one die, B be the event of rolling a 3 on the other, and C be the event of the product of the numbers on the dice being at least 20.

a. Are the events A and B independent? Explain.

b. Are the events B and C independent? Explain.

c. Are the events A and C independent? Explain.



6.6 Permutations and combinations

When calculating the probability of an event A from the fundamental rule $\mathrm{P}(A) = \dfrac{n(A)}{n(\xi)}$, the number of elements in both A and the sample space need to be able to be counted. Here we shall consider two counting techniques, one where order is important and one where order is not important. Respectively, these are called **arrangements** and **selections** or, alternatively, **permutations** and **combinations**.

6.6.1 Arrangements or permutations

The arrangement AB is a different arrangement to BA.

If two of three people designated by A, B and C are to be placed in a line, the possible arrangements are AB, BA, AC, CA, BC, CB. There are 6 possible arrangements or permutations.

Rather than list the possible arrangements, the number of possibilities could be calculated as follows using a **box table**.

3	2

↑

There are 3 people who can occupy the left position.

Once that position is filled this leaves 2 people who can occupy the remaining position.

Multiplying these figures together gives the total number of $3 \times 2 = 6$ arrangements. This is an illustration of the **multiplication principle**.

Multiplication principle

If there are m ways of doing the first procedure and for each one of these there are n ways of doing the second procedure, then there are $m \times n$ ways of doing the first **and** the second procedures. This can be extended.

Suppose either two or three of four people A, B, C, D are to be arranged in a line. The possible arrangements can be calculated as follows:

Arrange two of the four people:

4	3

This gives $4 \times 3 = 12$ possible arrangements using the multiplication principle.

Arrange three of the four people:

4	3	2

This gives $4 \times 3 \times 2 = 24$ possible arrangements using the multiplication principle.

The total number of arrangements of either two or three from the four people is $12+24 = 36$. This illustrates the **addition principle**.

Addition principle for mutually exclusive events

For mutually exclusive procedures, if there are m ways of doing one procedure and n ways of doing another procedure, then there are $m + n$ ways of doing one **or** the other procedure.

AND $\times$ (Multiplication principle)

OR $+$ (Addition principle)

WORKED EXAMPLE 15

Consider the set of five digits {2, 6, 7, 8, 9}.
Assume no repetition of digits in any one number can occur.
a. How many three-digit numbers can be formed from this set?
b. How many numbers with at least four digits can be formed?
c. How many five-digit odd numbers can be formed?
d. A five-digit number is chosen at random. What is the probability it will be an odd number?

THINK	WRITE
a. 1. Draw a box table with three divisions.	**a.** There are five choices for the first digit, leaving four choices for the second digit and then three choices for the third digit.
	5 | 4 | 3
2. Calculate the answer.	Using the multiplication principle, there are $5 \times 4 \times 3 = 60$ possible three-digit numbers that could be formed.
b. 1. Interpret the event described.	**b.** At least four digits means either four-digit or five-digit numbers are to be counted.

2. Draw the appropriate box tables.

For four-digit numbers:

5	4	3	2

For five-digit numbers:

5	4	3	2	1

3. Calculate the answer.

There are $5 \times 4 \times 3 \times 2 = 120$ four-digit numbers and $5 \times 4 \times 3 \times 2 \times 1 = 120$ five-digit numbers. Using the addition principle there are $120 + 120 = 240$ possible four- or five-digit numbers.

c. 1. Draw the box table showing the requirement imposed on the number.

c. For the number to be odd its last digit must be odd, so the number must end in either 7 or 9. This means there are two choices for the last digit.

				2

2. Complete the box table.

Once the last digit has been formed, there are four choices for the first digit then three choices for the second digit, two choices for the third digit and one choice for the fourth digit.

4	3	2	1	2

3. Calculate the answer.

Using the multiplication principle, there are $4 \times 3 \times 2 \times 1 \times 2 = 48$ odd five-digit numbers possible.

d. 1. Define the sample space and state $n(\xi)$.

d The sample space is the set of five-digit numbers. From part **b**, $n(\xi) = 120$.

2. State the number of elements in the required event.

Let A be the event the five-digit number is odd. From part **c**, $n(A) = 48$.

3. Calculate the probability.
Note: The last digit must be odd. Of the five possible last digits, two are odd. Hence the probability the number is odd is $\frac{2}{5}$.

$$\begin{aligned} \mathrm{P}(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{48}{120} \\ &= \frac{2}{5} \end{aligned}$$

The probability the five-digit number is odd is $\frac{2}{5}$.

Factorial notation

The number of ways that four people can be arranged in a row can be calculated using the box table shown.

4	3	2	1

Using the multiplication principle, the total number of arrangements is $4 \times 3 \times 2 \times 1 = 24$. This can be expressed using factorial notation as 4!.

> **In general, the number of ways of arranging n objects in a row is $n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$.**

6.6.2 Arrangements in a circle

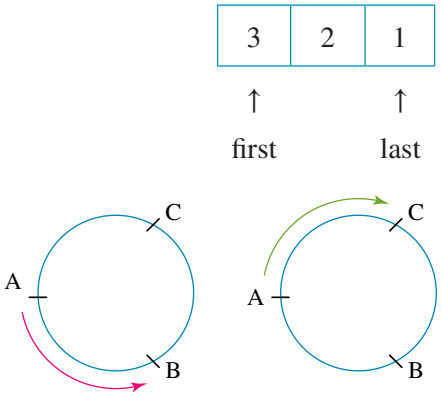
Consider arranging the letters A, B and C.

If the arrangements are in a row then there are $3! = 6$ arrangements.

In a row arrangement there is a first and a last position.

In a circular arrangement there is no first or last position; order is only created clockwise or anticlockwise from one letter once this letter is placed. This means ABC and BCA are the same circular arrangement, as the letters have the same anticlockwise order relative to A.

There are only two distinct circular arrangements of three letters: ABC or ACB as shown.



> **$n!$ is the number of ways of arranging n objects in a row.**
> **$(n-1)!$ is the number of ways of arranging n objects in a circle.**

Three objects A, B and C arranged in a row in 3! ways; three objects arranged in a circle in $(3-1)! = 2!$ ways.

WORKED EXAMPLE 16

A group of 7 students queue in a straight line at a canteen to buy a drink.

a. In how many ways can the queue be formed?

b. The students carry their drinks to a circular table. In how many different seating arrangements can the students sit around the table?

c. This group of students have been shortlisted for the Mathematics, History and Art prizes. What is the probability that one person in the group receives all three prizes?

THINK	WRITE
a. 1. Use factorial notation to describe the number of arrangements.	**a.** Seven people can arrange in a straight line in 7! ways.
2. Calculate the answer.	Since $7! = 7 \times 6 \times 5!$ and $5! = 120$, $7! = 7 \times 6 \times 120$ $= 7 \times 720$ $= 5040$
Note: Technology could be used to evaluate the factorial.	There are 5040 ways in which the students can form the queue.
b. 1. State the rule for circular arrangements.	**b.** For circular arrangements, 7 people can be arranged in $(7-1)! = 6!$ ways.
2. State the answer.	Since $6! = 720$, there are 720 different arrangements in which the 7 students may be seated.

c. 1. Form the number of elements in the sample space.

c. There are three prizes. Each prize can be awarded to any one of the 7 students.

7	7	7

The total number of ways the prizes can be awarded is $7 \times 7 \times 7$. $\therefore n(\xi) = 7 \times 7 \times 7$

2. Form the number of elements in the event under consideration.

Let A be the event that the same student receives all three prizes. There is only one arrangement in which one particular student wins all three prizes. However, since it can be any of the seven students, we must multiply this one arrangement by seven, so there are seven arrangements where one person receives all three prizes.

$\therefore n(A) = 7$

3. Calculate the required probability.

$$P(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{7}{7 \times 7 \times 7}$$
$$= \frac{1}{49}$$

The probability that one student receives all three prizes is $\frac{1}{49}$.

TI | THINK

a. 1. On a Calculator page, complete the entry line as
7!
then press ENTER.
Note: The factorial symbol (!) can be found by pressing the ° button.

2. The answer appears on the screen.

b. 1. On a Calculator page. complete the entry line as
6!
then press ENTER.
Note: The factorial symbol (!) can be found by pressing the ° button.

2. The answer appears on the screen.

WRITE



5040



720

CASIO | THINK

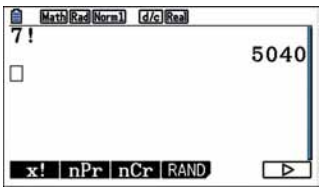
a. 1. On a Run-Matrix screen, complete the entry line as
7!
then press EXE.
Note: The factorial symbol (!) can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting PROB by pressing F3.
Select $x!$ by pressing F1.

2. The answer appears on the screen.

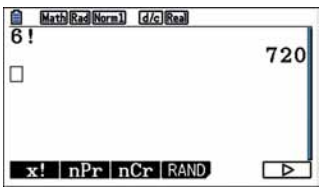
b 1. On a Run-Matrix screen, complete the entry line as
6!
then press EXE.
Note: The factorial symbol (!) can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting PROB by pressing F3. Select $x!$ by pressing F1.

2. The answer appears on the screen.

WRITE



5040



720

6.6.3 Arrangements with objects grouped together

Where a group of objects are to be together, treat them as one unit in order to calculate the number of arrangements. Having done this, then allow for the number of internal rearrangements within the objects grouped together, and apply the multiplication principle.

For example, consider arranging the letters A, B, C and D with the restriction that ABC must be together.

Treating ABC as one unit would mean there are two objects to arrange: D and the unit (ABC).

Two objects arrange in 2! ways.

For each of these arrangements, the unit (ABC) can be internally arranged in 3! ways.

The multiplication principle then gives the total number of possible arrangements which satisfy the restriction is $2! \times 3!$ or $2 \times 6 = 12$ arrangements.

6.6.4 Arrangements where some objects may be identical

The word SUM has three distinct letters. These letters can be arranged in 3! ways. The six arrangements can be listed as 3 pairs SUM and MUS, USM and UMS, SMU and MSU formed when the S and the M are interchanged.

Now consider the word MUM. Although this word also has three letters, two of the letters are identical. Interchanging the two M's will not create a new arrangement. SUM and MUS where the S and the M are interchanged are different, but MUM and MUM are the same.

There are only three arrangements: MUM, UMM and MMU.

The number of arrangements is $\frac{3!}{2!} = 3$.

This can be considered as cancelling out the rearrangements of the 2 identical letters from the total number of arrangements of 3 letters.

The number of arrangements of n objects, p of which are of one type, q of which are of another type, is $\frac{n!}{p!\,q!\,\ldots}$.

WORKED EXAMPLE 17

Consider the two words 'PARALLEL' and 'LINES'.

a. How many arrangements of the letters of the word LINES have the vowels grouped together?

b. How many arrangements of the letters of the word LINES have the vowels separated?

c. How many arrangements of the letters of the word PARALLEL are possible?

d. What is the probability that in a randomly chosen arrangement of the word PARALLEL, the letters A are together?

THINK	WRITE
a. 1. Group the required letters together.	a. There are two vowels in the word LINES. Treat these letters, I and E, as one unit.
2. Arrange the unit of letters together with the remaining letters.	Now there are four groups to arrange: (IE), L, N, S. These arrange in 4! ways.

3. Use the multiplication principle to allow for any internal rearrangements.

The unit (IE) can internally rearrange in 2! ways.
Hence, the total number of arrangements is:
$4! \times 2!$
$= 24 \times 2$
$= 48$

b. 1. State the method of approach to the problem.

b. The number of arrangements with the vowels separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.

2. State the total number of arrangements.

The five letters of the word LINES can be arranged in $5! = 120$ ways.

3. Calculate the answer.

From part **a**, there are 48 arrangements with the two vowels together.
Therefore, there are $120 - 48 = 72$ arrangements in which the two vowels are separated.

c. 1. Count the letters, stating any identical letters.

c. The word PARALLEL contains 8 letters of which there are 2 A's and 3 L's.

2. Using the rule $\frac{n!}{p!\ q!\ \ldots}$ state the number of distinct arrangements.

There are $\frac{8!}{2! \times 3!}$ arrangements of the word PARALLEL.

3. Calculate the answer.

$$\frac{8!}{2! \times 3!} = \frac{8 \times 7 \times 6 \times 5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}}$$
$$= 3360$$

There are 3360 arrangements.

d. 1. State the number of elements in the sample space.

d. There are 3360 total arrangements of the word PARALLEL, so $n(\xi) = 3360$ or $\frac{8!}{2! \times 3!}$.

2. Group the required letters together.

For the letters A to be together, treat these two letters as one unit. This creates seven groups (AA), P, R, L, L, E, L of which three are identical L's.

3. Calculate the number of elements in the event.

The seven groups arrange in $\frac{7!}{3!}$ ways. As the unit (AA) contains two identical letters, there are no distinct internal rearrangements of this unit that need to be taken into account.
Hence $\frac{7!}{3!}$ is the number of elements in the event.

4. Calculate the required probability.
Note: It helps to use factorial notation in the calculations.

The probability that the A's are together

$$= \frac{\text{number of arrangements with the As together}}{\text{total number of arrangements}}$$

$$= \frac{7!}{3!} \div \frac{8!}{2! \times 3!}$$

$$= \frac{7!}{3!} \times \frac{2! \times 3!}{8 \times 7!}$$

$$= \frac{2}{8}$$

$$= \frac{1}{4}$$

Formula for permutations

The number of arrangements of ***n*** objects taken r at a time is shown in the box table.

n	$n-1$	$n-2$	… … … … … … … … … … … … …	$n-r+1$
↑	↑	↑		↑
1^{st}	2^{nd}	3^{rd}		r^{th} object

The number of arrangements equals $n(n-1)(n-2)\ldots(n-r+1)$.
This can be expressed using factorial notation as:

$$n(n-1)(n-2)\ldots(n-r+1) = n(n-1)(n-2)\ldots(n-r+1) \times \frac{(n-r)(n-r-1)\times\ldots 2\times 1}{(n-r)(n-r-1)\times\ldots 2\times 1}$$

$$= \frac{n!}{(n-r)!}$$

The formula for the number of permutations or arrangements of n objects taken r at a time is ${}^{n}\mathrm{P}_{r} = \frac{n!}{(n-r)!}$.
Although we have preferred to use a box table, it is possible to count arrangements using this formula.

6.6.5 Combinations or selections

Now we shall consider the counting technique for situations where order is unimportant. This is the situation where the selection AB is the same as the selection BA. For example the entry Alan and Bev is no different to the entry Bev and Alan as a pair of mixed doubles players in a tennis match: they are the same entry.

The number of combinations of ***r*** objects from a total group of ***n*** distinct objects is calculated by counting the number of arrangements of the objects ***r*** at a time and then dividing that by the number of ways each group of these ***r*** objects can rearrange between themselves. This is done in order to cancel out counting these as different selections.

The number of combinations is therefore $\frac{{}^{n}\mathrm{P}}{r!} = \frac{n!}{(n-r)!\,r!}$.

The symbol for the number of ways of choosing r objects from a total of n objects is ${}^{n}\mathrm{C}_{r}$ or $\binom{n}{r}$.

The number of combinations of r objects from a total of n objects is ${}^nC_r = \frac{n!}{r!\,(n-r)!}$, $0 \leq r \leq n$ where r and n are non-negative integers.

Drawing on that knowledge, we have:

- ${}^nC_0 = 1 = {}^nC_n$, there being only one way to choose none or all of the n objects.
- ${}^nC_1 = n$, there being n ways to choose one object from a group of n objects.
- ${}^nC_r = {}^nC_{n-r}$ since choosing r objects must leave behind a group of $(n - r)$ objects and vice versa.

Calculations

The formula is always used for calculations in selection problems. Most calculators have a ${}^nC_r$ key to assist with the evaluation when the figures become large.

Both the multiplication and addition principles apply and are used in the same way as for arrangements.

The calculation of probabilities from the rule $P(A) = \frac{n(A)}{n(\xi)}$ requires that the same counting technique used for the numerator is also used for the denominator. We have seen for arrangements that it can assist calculations to express numerator and denominator in terms of factorials and then simplify. Similarly for selections, express the numerator and denominator in terms of the appropriate combinatoric coefficients and then carry out the calculations.

on Resources

Interactivity: Counting techniques (int-6293)

WORKED EXAMPLE 18

A committee of 5 students is to be chosen from 7 boys and 4 girls.

a. How many committees can be formed?

b. How many of the committees contain exactly 2 boys and 3 girls?

c. How many committees have at least 3 girls?

d. What is the probability of the oldest and youngest students both being on the committee?

THINK

a. 1. As there is no restriction, choose the committee from the total number of students.

2. Use the formula ${}^nC_r = \frac{n!}{r! \times (n-r)!}$ to calculate the answer.

WRITE

a. There are 11 students in total from whom 5 students are to be chosen. This can be done in ${}^{11}C_5$ ways.

$${}^{11}C_5 = \frac{11!}{5! \times (11-5)!}$$

$$= \frac{11!}{5! \times 6!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5! \times 6!}$$

$$= \frac{11 \times \cancel{10}^{2} \times \cancel{9} \times \cancel{8}^{3} \times 7}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 462$$

There are 462 possible committees.

b. 1. Select the committee to satisfy the given restriction.

b. The 2 boys can be chosen from the 7 boys available in ${}^7C_2$ ways.
The 3 girls can be chosen from the 4 girls available in ${}^4C_3$ ways.

2. Use the multiplication principle to form the total number of committees.
Note: The upper numbers on the combinatoric coefficients sum to the total available, $7 + 4 = 11$, while the lower numbers sum to the number that must be on the committee, $2 + 3 = 5$.

The total number of committees that contain two boys and three girls is ${}^7C_2 \times {}^4C_3$.

3. Calculate the answer.

$$\begin{aligned} {}^7C_2 \times {}^4C_3 &= \frac{7!}{2! \times 5!} \times 4 \\ &= \frac{7 \times 6}{2!} \times 4 \\ &= 21 \times 4 \\ &= 84 \end{aligned}$$

There are 84 committees possible with the given restriction.

c. 1. List the possible committees which satisfy the given restriction.

c. As there are 4 girls available, at least 3 girls means either 3 or 4 girls.
The committees of 5 students which satisfy this restriction have either 3 girls and 2 boys, or they have 4 girls and 1 boy.

2. Write the number of committees in terms of combinatoric coefficients.

3 girls and 2 boys are chosen in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy are chosen in ${}^4C_4 \times {}^7C_1$ ways.

3. Use the addition principle to state the total number of committees.

The number of committees with at least three girls is ${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$.

4. Calculate the answer.

$$\begin{aligned} {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 &= 84 + 1 \times 7 \\ &= 91 \end{aligned}$$

There are 91 committees with at least 3 girls.

d. 1 State the number in the sample space.

d. The total number of committees of 5 students is ${}^{11}C_5 = 462$ from part **a**.

2. Form the number of ways the given event can occur.

Each committee must have 5 students. If the oldest and youngest students are placed on the committee, then 3 more students need to be selected from the remaining 9 students to form the committee of 5. This can be done in ${}^9C_3$ ways.

3. State the probability in terms of combinatoric coefficients.

Let A be the event the oldest and the youngest students are on the committee.

$$\begin{aligned} \mathrm{P}(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{{}^9C_3}{{}^{11}C_5} \end{aligned}$$

4. Calculate the answer.

$$\begin{aligned} P(A) &= \frac{9!}{3! \times 6!} \div \frac{11!}{5! \times 6!} \\ &= \frac{9!}{3! \times 6!} \times \frac{5! \times 6!}{11!} \\ &= \frac{1}{3!} \times \frac{5!}{11 \times 10} \\ &= \frac{5 \times 4}{110} \\ &= \frac{2}{11} \end{aligned}$$

The probability of the committe containing the youngest and the oldest students is $\frac{2}{11}$.

TI | THINK

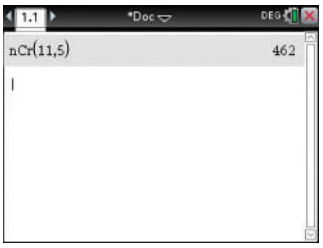
a. 1. On a Calculator page, press MENU then select
5: Probability
3: Combinations
Complete the entry line as
nCr(11,5)
then press ENTER.

2. The answer appears on the screen.

b. 1. On a Calculator page, complete the entry line as
nCr (7,2) × nCr (4,3)
then press ENTER.
Note: nCr can be found by pressing MENU then selecting
5: Probability
3: Combinations.

2. The answer appears on the screen.

WRITE



462



84

CASIO | THINK

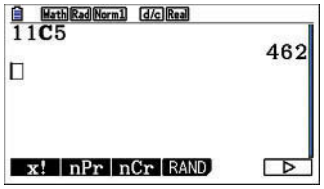
a. 1. On a Run-Matrix screen, type '11', press OPTN then press F6 to scroll across to more menu options.
Select PROB by pressing F3 then select nCr by pressing F3. Type '5' then press EXE.

2. The answer appears on the screen.

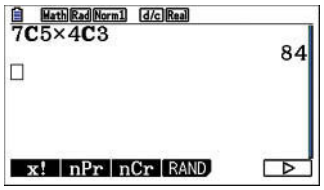
b. 1. On the Run-Matrix screen, complete the entry line as
7nCr2 × 4nCr3
then press EXE.
Note: nCr can be found by pressing OPTN then pressing F6 to scroll across to more menu options.
Select PROB by pressing F3 then select nCr by pressing F3.

2. The answer appears on the screen.

WRITE



462



84

studyon

Units 1 & 2 | Area 3 | Sequence 1 | Concept 4

Permutations and combinations Summary screen and practice questions

Exercise 6.6 Permutations and combinations

Technology active

1. **MC** Identify which one of the following permutations cannot be calculated.
   A. ${}^{1000}P_{100}$ **B.** ${}^{4}P_{8}$ **C.** ${}^{1}P_{0}$ **D.** ${}^{8}P_{8}$
2. **a.** State the difference between permutations and combinations.
   b. Demonstrate that ${}^{5}P_{2} = 2{}^{5}C_{2}$.
3. Two letters are to be chosen from A, B and C. List the number of ways this may done if:
   a. the order of the letters in each pair is not important
   b. the order of the letters in each pair is important.
4. **MC** Which of the following best describes the comparison between the numbers 100! and 94!?
   A. 100! is 6 more than 94!
   B. 100! is 6 times bigger than 94!
   C. 100! is ${}^{100}P_{6}$ times bigger than 94!
   D. 100! is 100! times bigger than 94!
5. **WE15** Consider the set of five digits {3, 5, 6, 7, 9}.
   Assume no repetition of digits in any one number can occur.
   a. How many four-digit numbers can be formed from this set?
   b. How many numbers with at least three digits can be formed?
   c. How many five-digit even numbers can be formed?
   d. One of the five-digit numbers is chosen at random. What is the probability it will be an even number?
6. A car's number plate consists of two letters of the English alphabet followed by three of the digits 0 to 9, followed by one single letter. Repetition of letters and digits is allowed.
   a. How many such number plates are possible?
   b. How many of the number plates use the letter X exactly once?
   c. What is the probability the first two letters are identical and all three numbers are the same, but the single letter differs from the other two?
7. **WE16** A group of six students queue in a straight line to take out books from the library.
   a. In how many ways can the queue be formed?
   b. The students carry their books to a circular table. In how many different seating arrangements can the students sit around the table?
   c. This group of students have been shortlisted for the Mathematics, History and Art prizes. What is the probability that one person in the group receives all three prizes?
8. A group of four boys seat themselves in a circle.
   a. How many different seating arrangements of the boys are possible?
   b. Four girls join the boys and seat themselves around the circle so that each girl is sitting between two boys. In how many ways can the girls be seated?
9. **WE17** Consider the words SIMULTANEOUS and EQUATIONS.
   a. How many arrangements of the letters of the word EQUATIONS have the letters Q and U grouped together?
   b. How many arrangements of the letters of the word EQUATIONS have the letters Q and U separated?
   c. How many arrangements of the letters of the word SIMULTANEOUS are possible?
   d. What is the probability that in a randomly chosen arrangement of the word SIMULTANEOUS, both the letters U are together?
10. In how many ways can the thirteen letters PARALLEL LINES be arranged:
    a. in a row
    b. in a circle
    c. in a row with the vowels together?

11. **WE18** A committee of 5 students is to be chosen from 6 boys and 8 girls.

- **a.** How many committees can be formed?
- **b.** How many of the committees contain exactly 2 boys and 3 girls?
- **c.** How many committees have at least 4 boys?
- **d.** What is the probability of neither the oldest nor the youngest student being on the committee?

12. A cricket team of eleven players is to be selected from a list of 3 wicketkeepers, 6 bowlers and 8 batsmen. What is the probability the team chosen consists of one wicketkeeper, four bowlers and six batsmen?

13. **a.** Baby Amelie has 10 different bibs and 12 different body suits. How many different combinations of bib and body suit can she wear?



- **b.** On her way to work each morning, Christine has the option of taking the motorway or the highway. She then must travel through some suburban streets to get to work. She has the option of three different routes through the suburban streets. If Christine wishes to take a different route to work each day, on how many days will she be able to take a different route before she must use a route already travelled?
- **c.** When selecting his new car, Abdul has the option of a manual or an automatic. He is also offered a choice of 5 exterior colours, leather or vinyl seats, 3 interior colours and the options of individual seat heating and self-parking. How many different combinations of new car can Abdul choose from?



- **d.** Sarah has a choice of 3 hats, 2 pairs of sunglasses, 7 T-shirts and 5 pairs of shorts. When going out in the sun, she chooses one of each of these items to wear. How many different combinations are possible?
- **e.** In order to start a particular game, each player must roll an unbiased die, then select a card from a standard pack of 52. How many different starting combinations are possible?
- **f.** On a recent bushwalking trip, a group of friends had a choice of travelling by car, bus or train to the Blue Mountains. They decided to walk to one of the waterfalls, then a mountain-top scenic view. How many different trips were possible if there are 6 different waterfalls and 12 mountain-top scenic views?

14. **a.** Registration plates on a vehicle consist of 2 letters, followed by 2 digits, followed by another letter. How many different number plates are possible if repetitions are allowed?

- **b.** How many five-letter words can be formed using the letters B, C, D, E, G, I and M if repetitions are allowed?
- **c.** A die is rolled three times. How many possible outcomes are there?
- **d.** How many three-digit numbers can be formed using the digits 2, 3, 4, 5, 6, 7 if repetitions are allowed?
- **e.** Three friends on holidays decide to stay at a hotel which has 4 rooms available. In how many ways can the rooms be allocated if there are no restrictions and each person has their own room?

15. Eight people, consisting of 4 boys and 4 girls, are to be arranged in a row. Find the number of ways this can be done if:
    a. there are no restrictions
    b. the boys and girls are to alternate
    c. the end seats must be occupied by a girl
    d. the brother and sister in the group must not sit together
    e. the girls must sit together.
16. How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 given that no repetitions are allowed, the number cannot start with 0, and the following condition holds:
    a. there are no other restrictions
    b. the number must be even
    c. the number must be less than 400
    d. the number is made up of odd digits only?
17. In how many ways can 6 men and 3 women be arranged at a circular table if:
    a. there are no restrictions
    b. the men can only be seated in pairs?
18. a. A set of 12 mugs that are identical except for the colour are to be placed on a shelf. In how many ways can this be done if 4 of the mugs are blue, 3 are orange and 5 are green?
    b. In how many ways can the 12 mugs from part **a** be arranged in 2 rows of 6 if the green ones must be in the front row?



19. How many words can be formed from the letters of the word BANANAS given the following conditions?
    a. All the letters are used.
    b. A four-letter word is to be used including at least one A.
    c. A four-letter word using all different letters is to be used.
20. a. In how many ways can 7 men be selected from a group of 15 men?
    b. How many five-card hands can be dealt from a standard pack of 52 cards?
    c. How many five-card hands that contain all 4 aces, can be dealt from a standard pack of 52 cards?
    d. In how many ways can 3 prime numbers be selected from the set containing the first 10 prime numbers?
21. A panel of 8 is to be selected from a group of 8 men and 10 women. Find how many panels can be formed if:
    a. there are no restrictions
    b. there are 5 men and 3 women on the panel
    c. there are at least 6 men on the panel
    d. two particular men cannot both be included
    e. a particular man and woman must both be included.
22. Consider the universal set $S = \{2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 15, 17\}$.
    a. How many subsets of S are there?
    b. Determine the number of subsets whose elements are all even numbers.
    c. What is the probability that a subset selected at random will contain only even numbers?
    d. Find the probability that a subset selected at random will contain at least 3 elements.
    e. Find the probability that a subset selected at random will contain exactly 3 numbers, all of which are prime.



23. A representative sport committee consisting of 7 members is to be formed from 21 tennis players, 17 squash players and 18 badminton players. Find the probability that the committee will contain:

- **a.** 7 squash players
- **b.** at least 5 tennis players
- **c.** at least one representative from each sport
- **d.** exactly 3 badminton players, given that it contains at least 1 badminton player.

24. Four three-digit numbers are selected at random from all the possible three-digit numbers. Giving your answer as a decimal, find the probability that:

- **a.** the digits are all primes
- **b.** the number has just a single repeated digit
- **c.** the digits are perfect squares
- **d.** there are no repeated digits
- **e.** the number lies between 300 and 400 inclusive, given that the number is greater than 200.

6.7 Pascal's triangle and binomial expansions

6.7.1 Pascal's triangle



Although known to Chinese mathematicians many centuries earlier, the following pattern is named after the seventeenth century French mathematician Blaise Pascal. **Pascal's triangle** contains many fascinating patterns. Each row from row 1 onwards begins and ends with '1'. Each other number along a row is formed by adding the two terms to its left and right from the preceding row.

Row 0					1				
Row 1				1		1			
Row 2			1		2		1		
Row 3		1		3		3		1	
Row 4	1		4		6		4		1

The numbers in each row are called **binomial coefficients**.

The numbers 1, 2, 1 in row 2 are the coefficients of the terms in the expansion of $(a + b)^2$.

$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$

The numbers 1, 3, 3, 1 in row 3 are the coefficients of the terms in the expansion of $(a + b)^3$.

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Each row of Pascal's triangle contains the coefficients in the expansion of a power of $(a + b)$. Such expansions are called binomial expansions because of the two terms a and b in the brackets.

Row n contains the coefficients in the binomial expansion $(a + b)^n$.

To expand $(a + b)^4$ we would use the binomial coefficients, 1, 4, 6, 4, 1, from row 4 to obtain:

$$\begin{aligned}(a + b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

Notice that the powers of a decrease by 1 as the powers of b increase by 1, with the sum of the powers of a and b always totalling 4 for each term in the expansion of $(a + b)^4$.

For the expansion of $(a - b)^4$ the signs would alternate:

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

By extending Pascal's triangle, higher powers of such binomial expressions can be expanded.

WORKED EXAMPLE 19

Form the rule for the expansion of $(a - b)^5$ and hence expand $(2x - 1)^5$.
Note: **It is appropriate to use technology to perform expansions such as this.**

THINK	WRITE
1. Choose the row in Pascal's triangle which contains the required binomial coefficients.	For $(a - b)^5$, the power of the binomial is 5. Therefore the binomial coefficients are in row 5. The binomial coefficients are: 1, 5, 10, 10, 5, 1
2. Write down the required binomial expansion.	Alternate the signs: $(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$
3. State the values to substitute in place of a and b.	To expand $(2x - 1)^5$, replace a by $2x$ and b by 1.
4. Write down the expansion.	$(2x - 1)^5$ $= (2x)^5 - 5(2x)^4(1) + 10(2x)^3(1)^2 - 10(2x)^2(1)^3 + 5(2x)(1)^4 - (1)^5$
5. Evaluate the coefficients and state the answer.	$= 32x^5 - 5 \times 16x^4 + 10 \times 8x^3 - 10 \times 4x^2 + 10x - 1$ $= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$ $\therefore\ (2x - 1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

Resources

Interactivity: Pascal's triangle (int-2554)

6.7.2 Formula for binomial coefficients

Each of the terms in the rows of Pascal's triangle can be expressed using factorial notation. For example, row 3 contains the coefficients 1, 3, 3, 1.

These can be written as $\frac{3!}{0! \times 3!}, \frac{3!}{1! \times 2!}, \frac{3!}{2! \times 1!}, \frac{3!}{3! \times 0!}$. (Remember that 0! was defined to equal 1.)

The coefficients in row 5 (1, 5, 10, 10, 5, 1) can be written as:

$$\frac{5!}{0! \times 5!}, \frac{5!}{1! \times 4!}, \frac{5!}{2! \times 3!}, \frac{5!}{3! \times 2!}, \frac{5!}{4! \times 1!}, \frac{5!}{5! \times 0!}$$

The third term of row 4 would equal $\frac{4!}{2! \times 2!}$ and so on.

The $(r + 1)$th term of row n would equal $\frac{n!}{r! \times (n - r)!}$. This is normally written using the notations ${}^nC_r$ or $\left(\frac{n}{r}\right)$.

These expressions for the binomial coefficients are referred to as **combinatoric coefficients**. They occur frequently in other branches of mathematics including probability theory. Blaise Pascal is regarded as the 'father of probability' and it could be argued he is best remembered for his work in this field.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^nC_r \text{ where } r \leq n \text{ and } r, n \text{ are non negative whole numbers.}$$

6.7.3 Pascal's triangle with combinatoric coefficients

Pascal's triangle can now be expressed using this notation:

Row 0				$\binom{0}{0}$					
Row 1			$\binom{1}{0}$		$\binom{1}{1}$				
Row 2		$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$			
Row 3	$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$		
Row 4	$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$

Binomial expansions can be expressed using this notation for each of the binomial coefficients.

The expansion $(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$.

Note the following patterns:

- $\binom{n}{0} = 1 = \binom{n}{n}$ (the start and end of each row of Pascal's triangle)
- $\binom{n}{1} = n = \binom{n}{n-1}$ (the second from the start and the second from the end of each row) and $\binom{n}{r} = \binom{n}{n-1}$

While most calculators have a ${}^nC_r$ key to assist with the evaluation of the coefficients, the formula for $\binom{n}{r}$ or ${}^nC_r$ should be known. Some values of $\binom{n}{r}$ can be committed to memory.

WORKED EXAMPLE 20

Evaluate $\binom{8}{3}$.

THINK

1. Apply the formula.

WRITE

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Let $n = 8$ and $r = 3$.

$$\binom{8}{3} = \frac{8!}{3!\,(8-3)!}$$
$$= \frac{8!}{3!\,5!}$$

2. Write the largest factorial in terms of the next largest factorial and simplify.

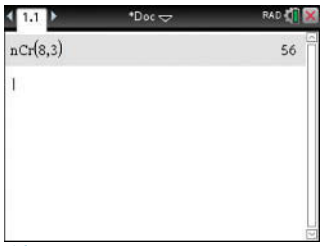
3. Calculate the answer.

$$= \frac{8 \times 7 \times 6 \times \cancel{5!}}{3!\,\cancel{5!}}$$
$$= \frac{8 \times 7 \times 6}{3!}$$
$$= \frac{8 \times 7 \times \cancel{6}}{\cancel{3 \times 2} \times 1}$$
$$= 8 \times 7$$
$$= 56$$

TI | THINK

1. On a Calculator page, press MENU then select 5: Probability 3: Combinations. Complete the entry line as nCr (8, 3) then press ENTER.
2. The answer appears on the screen.

WRITE



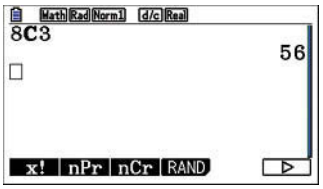
56

CASIO | THINK

a.1. On a Run-Matrix screen, type '8', press OPTN then press F6 to scroll across to more menu options. Select PROB by pressing F3 then select nCr by pressing F3. Type '3' then press EXE.

2. The answer appears on the screen.

WRITE



56

Hence, generalising, in the binomial expansion of $(a+b)^n$, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r}$, where $\binom{n}{r}$ is the number of ordered selections of n objects, in which r are alike of one type and $n-r$ are alike of another type.

We have:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

This can be written using sigma notation as:

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

We now have a simple formula for calculating binomial coefficients.

Important features to note in the expansion are:

1. The powers of a decrease as the powers of b increase.
2. The sum of the powers of a and b for each term in the expansion is equal to n.
3. As we would expect, $\binom{n}{0} = \binom{n}{n} = 1$ since there is only 1 way of selecting none, or of selecting all n objects.
4. If we look at the special case of:

   $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$ then $\binom{n}{r}$, or ${}^nC_r$, is the coefficient of x^r.

WORKED EXAMPLE 21

Write down the expansion of the following.

a. $(p+q)^4$ **b.** $(x-y)^3$ **c.** $(2m+5)^3$

THINK	WRITE
a. 1. Determine the values of n, a and b in the expansion of $(a+b)^n$.	**a.** $n=4;\ a=p;\ b=q$
2. Use the formula to write the expansion and simplify.	$(p+q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^3q + \binom{4}{2}p^2q^2 + \binom{4}{3}pq^3 + \binom{4}{4}q^4$ $= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + p^4$
b. 1. Determine the values of n, a and b in the expansion of $(a+b)^n$.	**b.** $n=3;\ a=x;\ b=-y$
2. Use the formula to write the expansion and simplify.	$(x-y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2(-y) + \binom{3}{2}x(-y)^2 + \binom{3}{3}(-y)^3$ $= x^3 - 3x^2y + 3xy^2 - y^3$
c. 1. Determine the values of n, a and b in the expansion of $(a+b)^n$.	**c.** $n=3;\ a=2m;\ b=5$
2. Use the formula to write the expansion and simplify.	$(2m+5)^3 = \binom{3}{0}(2m)^3 + \binom{3}{1}(2m)^2(5) + \binom{3}{2}(2m)(5)^2 + \binom{3}{3}(5)^3$ $= 8m^3 + 60m^2 + 150m + 125$

WORKED EXAMPLE 22

a. Write down the notation for the eighth coefficient in the 13th row of Pascal's triangle.

b. Calculate the value of $\binom{7}{5}$.

c. Verify that $\binom{16}{5} = \binom{16}{11}$.

THINK	WRITE
a. 1. Write down the values of n and r. Remember, numbering starts at 0.	**a.** $n = 13$ and $r = 7$
2. Substitute into the notation.	8th coefficient in the 13th row is ${}^{13}C_7$.

b. 1. Use the formula.

b. $\binom{7}{5} = \frac{7!}{(7-5)!\,5!}$

$= \frac{7 \times 6}{2!}$

$= 21$

2. Verify using technology.

${}^{7}C_{5} = 21$

c. 1. Use the formula to express one side.

c. $\binom{16}{11} = \frac{16!}{(16-11)!\,11!}$

$= \frac{16!}{5!\,11!}$

$= \frac{16!}{11!\,5!}$

$= \binom{16}{5}$

2. Verify your answer by evaluating each side.

$\binom{16}{5} = 4368$

$\binom{16}{11} = 4368$

6.7.4 Extending the binomial expansion to probability

Consider the following example.

A coin is biased in such a way that the probability of tossing a Head is 0.6. The coin is tossed three times. The outcomes and their probabilities are shown below.

$$P(TTT) = (0.4)^3 \text{ or } P(0H) = (0.4)^3$$

$$\left.\begin{aligned} P(HTT) &= (0.6)(0.4)^2 \\ P(THT) &= (0.6)(0.4)^2 \\ P(TTH) &= (0.6)(0.4)^2 \end{aligned}\right\} \rightarrow P(1H) = \binom{3}{1}(0.6)(0.4)^2 \text{ or } P(1H) = 3(0.6)(0.4)^2$$

$$\left.\begin{aligned} P(HHT) &= (0.6)^2(0.4) \\ P(HTH) &= (0.6)^2(0.4) \\ P(THH) &= (0.6)^2(0.4) \end{aligned}\right\} \rightarrow P(2H) = \binom{3}{2}(0.6)^2(0.4) \text{ or } P(2H) = 3(0.6)^2(0.4)$$

$$P(HHH) = 0.6^3 \rightarrow P(3H) = 0.6^3$$

Since this represents the sample space, the sum of these probabilities is 1.

So we have $P(0H) + P(1H) + P(2H) + P(3H) = 1$

$$\binom{3}{0}(0.4)^3 + \binom{3}{1}(0.4)^2(0.6) + \binom{3}{2}(0.4)(0.6)^2 + \binom{3}{3}0.6^3 = 1$$

$$\rightarrow 1 = (0.4)^3 + 3(0.4)^2(0.6) + 3(0.4)(0.6)^2 + 0.6^3$$

Note that $\binom{3}{2}$, which is the number of ways in which we can get 2 Heads from 3 tosses, is also the number of ordered selections of 3 objects where 2 are alike of one type (Heads) and 1 of another (Tails), as was noted previously.

Now compare the following expansion:

$$(q+p)^3 = q^3 + 3p^2q + 3pq^2 + p^3$$

We see that if we let $p = 0.6$ (the probability of getting a Head on our biased die) and $q = 0.4$ (the probability of not getting a Head), we have identical expressions. Hence:

$$\begin{aligned}(0.4+0.6)^3 &= (0.4)^3 + 3(0.4)^2(0.6) + 3(0.4)(0.6)^2 + 0.6^3 \\ &= \binom{3}{0}(0.4)^3 + \binom{3}{1}(0.4)^2(0.6) + \binom{3}{2}(0.4)(0.6)^2 + \binom{3}{3}0.6^3\end{aligned}$$

So the coefficients in the binomial expansion are equal to the number of ordered selections of 3 objects of just 2 types.

If we were to toss the coin n times, with p being the probability of a Head and q being the probability of not Head (Tail), we have the following.

The probability of 0 Heads (n Tails) in n tosses is $\binom{n}{0}q^n$

The probability of 1 Head ($n-1$ Tails) in n tosses is $\binom{n}{1}q^{n-1}p$

The probability of 2 Heads ($n-2$ Tails) in n tosses is $\binom{n}{2}q^{n-2}p^2$

These probabilities are known as **binomial probabilities**.

In general, the probability of getting r favourable and $n-r$ non-favourable outcomes, in n repetitions of an experiment, is $\binom{n}{r}q^{n-r}p^r$, where $\binom{n}{r}$ is the number of ways of getting r favourable outcomes.

So again equating the sum of all the probabilities in the sample space to the binomial expansion of $(q+p)^n$, we have:

$$(q+p)^n = \binom{n}{0}q^n + \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 + \ldots + \binom{n}{r}q^{n-r}p^r + \ldots \binom{n}{n-1}qp^{n-1} + \binom{n}{n}p^n$$

WORKED EXAMPLE 23

A fair die is rolled 5 times. Find the probability of obtaining:

a. exactly four 2s

b. all results greater than 3.

THINK	WRITE
a. 1. Define and assign values to variables. The number of 2s obtained is exactly four.	$n = 5$ Let r = the number of 2s obtained = 4 p = probability of getting a 2 = $\frac{1}{6}$ $q = 1 - p = \frac{5}{6}$

2. Substitute the values into the rule.

$$\text{P (exactly four 2s)} = {}^5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$$

3. Evaluate.

$$= 5 \times \frac{1}{1296} \times \frac{5}{6}$$

$$= \frac{25}{7776}$$

b. 1. Define and assign values to variables. We need five occasions on which the roll is greater than 3.

$n = 5$

$r = 5$

$p =$ probability of getting a number greater than 3

$p = \frac{1}{2}$

$q = 1 - p = \frac{1}{2}$

2. Substitute the values into the rule.

$$\text{P (all results higher than 3)} = {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0$$

3. Evaluate.

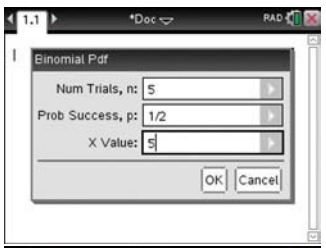
$$= 1 \times \frac{1}{32} \times 1$$

$$= \frac{1}{32}$$

TI | THINK

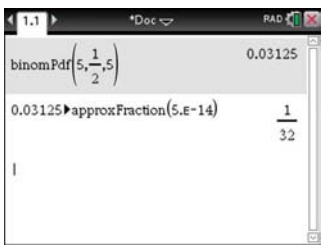
b.1. On a Calculator page, press MENU then select
5: Probability
5: Distributions
A: Binomial Pdf …
Complete the fields as:
Num Trials, n: 5
Prob Success, p:1/2
X Value: 5
Then select OK.

WRITE





2. To convert the decimal answer to a fraction, use the up arrow to highlight the previous answer then press ENTER to copy and paste it on the next entry line. Press MENU then select
2: Number
2: Approximate to Fraction.



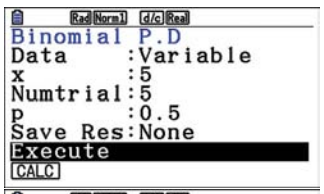
3. The answer appears on the screen.

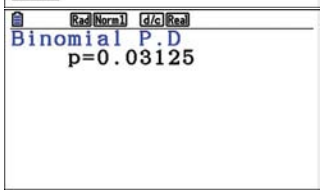
$\frac{1}{32}$

CASIO | THINK

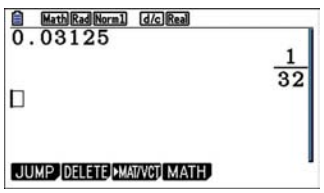
b.1. On a Statistics screen, select DIST by pressing F5, then select BINOMIAL by pressing F5. Select Bpd by pressing F1, then select Var by pressing F2. Complete the fields as:
Data: Variable
x: 5
Numtrial: 5
p: 0.5
Save Res: None
Select Execute then press EXE.

WRITE





2. On a Run-Matrix screen, complete the entry line as 0.03125
then press EXE.
Press the (−) button to convert the decimal to a fraction.



3. The answer appears on the screen.

$\frac{1}{32}$

Exercise 6.7 Pascal's triangle and binomial expansions

Technology free

1. MC Rachel sits a multiple choice test containing 20 questions, each with four possible answers. If she guesses every answer, identify the expression representing the probability of Rachel getting 11 questions correct.

   A. ${}^{20}C_{11}\left(\frac{1}{4}\right)^{20}\left(\frac{3}{4}\right)^{9}$ **B.** ${}^{20}C_{11}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$

   C. ${}^{20}C_{10}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$ **D.** ${}^{20}C_{11}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{11}$

2. WE19 Form the rule for the expansion of $(a-b)^6$ and hence expand $(2x-1)^6$.
3. Expand $(3x+2y)^4$.
4. WE20 Evaluate $\binom{7}{4}$.
5. Find an algebraic expression for $\binom{n}{2}$ and use this to evaluate $\binom{21}{2}$.
6. WE21 Write down the expansion of each of the following.

   a. $(a+b)^4$ **b.** $(2+x)^4$ **c.** $(t-2)^3$

7. Write down the expansion of each of the following.

   a. $(m+3b)^2$ **b.** $(2d-x)^4$ **c.** $\left(h+\frac{2}{h}\right)^3$

8. WE22 **a.** Write down the notation for the 4th coefficient in the 7th row of Pascal's triangle.

   b. Evaluate $\binom{9}{6}$.

   c. Show that $\binom{18}{12}=\binom{18}{6}$.

   d. Show that $\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$.

9. **a.** Verify that $\binom{22}{8}=\binom{22}{14}$.

   b. Express $\binom{19}{15}+\binom{19}{14}$ in simplified ${}^{n}C_{r}$ form.

10. Expand and simplify each of the following.

    a. $(x+y)^3$ **b.** $(a+2)^4$ **c.** $(m-3)^4$ **d.** $(2-x)^5$

11. Expand and simplify each of the following.

    a. $\left(1-\frac{2}{x}\right)^3$ **b.** $\left(1+\frac{p}{q}\right)^4$ **c.** $\left(3-\frac{m}{2}\right)^4$ **d.** $\left(2x-\frac{1}{x}\right)^3$

12. For each of the following find the term specified in the expansion.
    a. The 3rd term in $(2w - 3)^5$
    b. The 5th term in $\left(3 - \frac{1}{b}\right)^7$
    c. The constant term in $\left(y - \frac{3}{y}\right)^4$
13. **WE23** A fair coin is tossed four times. Find the probability of obtaining:
    a. Heads on every toss
    b. two Heads and two Tails.
14. Seven per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 50. What is the probability of five items being defective in a box?
15. A weighted coin is biased such that a Tail comes up 60% of the time. The coin is tossed 5 times. What is the probability of getting four Tails?
16. On a certain evening, during a ratings period, two television stations put their best shows on against each other. The ratings showed that 39% of people watched Channel 6 while only 30% of people watched Channel 8. The rest watched other channels. A random sample of 10 people were surveyed the next day. Find the probability that:
    a. exactly six watched Channel 6
    b. exactly four watched Channel 8.

Technology active

17. Simplify $1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6$.
18. The ratio of the coefficients of the 4th and 5th terms in the expansion of $(ax + 2y)^5$ is 3 : 1. Find the value of a.
19. The first 3 terms in the expansion of $(1 + kx)^n$ are 1, $2x$ and $\frac{3}{2}x^2$. Find the values of k and n.
20. a. Three consecutive terms of Pascal's triangle are in the ratio 13:8:4. Find the three terms.
    b. Find the values of a and b given $\left(3 + \sqrt{2}\right)^9 = a + b\sqrt{2}$.
    c. In the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, the coefficient of x^3 is 70. Find the value of n.

6.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. The numbers 1 to 5 are written on the front of 5 cards that are turned face down. Michelle then chooses one card at random. She wants to choose a number greater than 2. List the sample space and all favourable outcomes.
2. From every 100 televisions on a production line, two are found to be defective. If a television is chosen at random, find the relative frequency of defective televisions.
3. Given $P(A) = 0.4$, $P(A \cup B) = 0.58$ and $P(B|A) = 0.3$:
    a. calculate $P(B)$
    b. determine whether A and B are independent.
4. **MC** Let the universal set ζ be the set of integers from 1 to 50 inclusive. Three subsets, A, B and C, are defined as:
    A. is the set of positive multiples of 3 less than 50
    B. is the set of positive prime numbers less than 50
    C. is the set of positive multiples of 10 less than or equal to 50.
    Which of the following statements is correct?
    A. $B \subseteq C$
    B. $A \cap B = \varnothing$
    C. A and B are independent.
    D. B and C are mutually exclusive.

5. One bag contains 2 green and 6 blue counters. A second identical bag contains 4 green counters. A bag is selected at random and a counter is drawn.
   a. Draw a probability tree diagram for the possible outcomes.
   b. What is the probability that the counter is green?
6. There are 14 teams competing in a rugby championship. In how many different ways can the first, second and third positions be filled?
7. MC Consider the binomial expansion of $(x + 2)^6$. Which of the following is true?
   A. The first term is 64.
   B. The coefficient of the third term is 60.
   C. There are six terms in the expansion.
   D. The fourth term is $160x^4$.
8. How many committees consisting of 3 men and 4 women are possible if there are 7 men and 6 women available for selection to the committee?
9. At the carnival, the mystery house offers a choice of different adventures to those who dare to enter. Adventurers can enter the house through any one of three doors. Behind each door there are four mystery envelopes containing maps leading to different adventures. How many choices of adventures are possible?
10. It is found that 150 of every thousand 17-year-old drivers will be involved in an accident within one year of having their driver's licence.
    a. What is the relative frequency of a 17-year-old driver having an accident?
    b. If the average cost to an insurance company of each accident is $5000, what would be the minimum premium that an insurance company should charge a 17-year-old driver?



11. How many different nine-letter arrangements can be formed from the letters of the word EMERGENCY?
12. The number of children per family for the 40 students in a class was recorded in the table below.

Number of children	Frequency
1	7
2	12
3	13
4	5
5	2
6	1

What is the probability that a family selected has at least three children?

Complex familiar

13. A class of 60 students is surveyed and it is found that 35 of the students study Japanese, 45 study General Mathematics and 10 study neither Japanese nor General Mathematics.
    a. Draw a Venn diagram to represent this information. Ensure that numbers are calculated for all regions of the diagram.
    b. What is the probability that a student selected at random studies both Japanese and General Mathematics?
    c. Calculate the probability that:
       i. a student studies Japanese given that they study General Mathematics.
       ii. a Japanese student studies General Mathematics.
14. A three-digit number is formed using the digits 2, 4 and 7.
    a. Explain why it is more likely that an even number will be formed than an odd number.
    b. Which is more likely to be formed: a number less than 400 or a number greater than 400?

15. A new lie detector machine is being tested to determine its degree of accuracy. One hundred men, 20 of whom were known liars, were tested on the machine. Some of the results are shown below.

	Liars (L)	Honest people (H)	Totals
Correctly tested (C)	18		85
Incorrectly tested (I)			
Totals			100

a. Complete the table.
Find the probability that a person selected at random:
b. was incorrectly tested
c. was honest and correctly tested
d. was correctly tested, given that he was honest.

16. Forty per cent of all Australians have poor diets. A random survey of 20 Australians is taken. Find the probability that:
a. fewer than 3 have poor diets
b. at least 18 do not have poor diets.

Complex unfamiliar

17. John lives in Mackay. Last year he flew to New York to visit his sister. He was about to collect his luggage at the JFK terminal when he bumped into an old friend from school. He thought, 'This is incredible. I probably only know about 100 people and I meet one on the other side of the world. The probability of this must be about 100 out of 7 000 000 000.' Assuming there are about 7 billion people in the world, comment on the likelihood of John's accidental meeting.



18. **a.** Expand $\left(x^2 - \frac{3}{x}\right)^4$ in decreasing powers of x.
b. Find the term independent of y in the expansion of $\left(\frac{3}{y^2} - y\right)^{12}$.

19. Jo, who owns a corner grocery, imports tins of chickpeas and lentils. When unpacking the tins, Jo finds that one box contains 10 tins that have lost their labels. The tins are identical but after looking through his invoices, Jo has calculated that 7 of the tins contain chickpeas and 3 contain lentils. He decides to take them home since he is unable to sell them without a label. He wants to use the chickpeas to make some hummus, so he opens the tins at random until he opens a tin of chickpeas.
a. What is the minimum number of tins Jo must open to ensure he opens a tin of chickpeas?
b. Find the probability that the chickpeas are in the third tin he opens.



In fact, Jo found the chickpeas in the second tin. After making the hummus, Jo decides he will open one tin each day and use whatever it contains.

c. What is the probability that he opens at least one tin of each over the next 3 nights?

20. Twenty-two items are to be sold at a small charity auction. The amount, in dollars, paid for each article is to be decided by the throw of a dart at a dartboard. The rules are as follows.



The price of any item is the number scored multiplied by $100.

- If more than one person wants to bid for an item, the person with the highest score wins.
- If a buyer misses the board the buyer can throw the dart again until he/she hits the board, unless there are others who want to buy the same item, in which case the buyer does not get a second chance.
- Anyone hitting the inner bull will automatically win and will be sold the item for the modest price of $5000.
- Anyone hitting the outer bull will automatically win over any score other than an inner bull, and will be sold the item for $2500.

Aino and Bryan attend the auction. The associated probabilities are as follows:

For Aino:

The probability of hitting any one of the numbers from 1 to 20 is 0.045; the probability of hitting the inner bull is 0.005; and the probability of hitting the outer bull is 0.006.

For Bryan:

The probability of hitting any one of the numbers from 1 to 20 is 0.046; the probability of hitting the inner bull is 0.004; and the probability of hitting the outer bull is 0.003.

a. Find the probability that Aino misses the board.

Aino is the only buyer interested in a particular item. What is the probability that:

b. she needs two attempts to buy her item?

c. she buys the item for $900 with her first throw of the dart?

d. she only needs the one throw but her item costs more than $1900?

Aino then decides to buy a second item in which Bryan is also interested. They each throw a dart.

e. What is the probability that Bryan wins with his first dart?

f. Given that Bryan has already scored a 15 on his first throw, what is the probability that Aino will beat Bryan's score on her first throw?

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 6 Counting and probability

Exercise 6.2 Fundamentals of probability

1. **a.** Impossible **b.** Certain
c. Even chance **d.** Even chance
e. Probable **f.** Unlikely
g. Impossible **h.** Even chance

2. Rolling a 6; Rolling a number less than 3; Rolling an even number; Rolling a number greater than 2

3. D

4. It is highly probable that Wendy's bulb will last longer than 1500 hours.

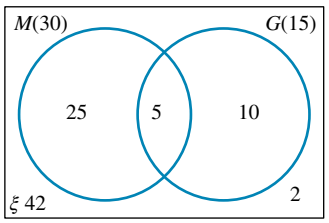
5. It is highly unlikely that Edwin will have major mechanical problems in the first year of purchase.

6. **a.** $\frac{1}{12}$ **b.** $\frac{4}{12}$ **c.** $\frac{7}{12}$

7. **a.** $\frac{1}{16}$ **b.** $\frac{3}{4}$

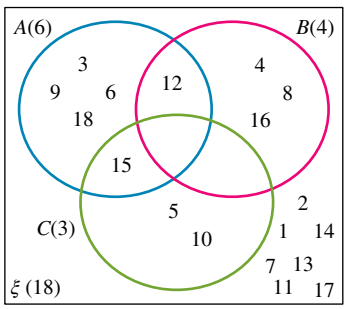
8. **a.** **i.** $\frac{3}{4}$ **ii.** $\frac{7}{10}$; **iii.** $\frac{1}{4}$
b. An additional 8 red balls must be added

9. **a.**



b. $\frac{25}{42}$ **c.** $\frac{1}{21}$ **d.** $\frac{5}{6}$

10. **a.**



b. B and C are mutually exclusive
c. $\frac{1}{3}$
d. **i.** $\frac{4}{9}$ **ii.** $\frac{5}{18}$ **iii.** $\frac{7}{18}$

11. **a.**

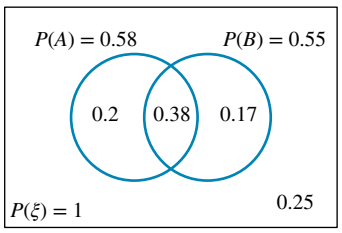
	B	B'	
A	0.35	0.3	0.65
A'	0.15	0.2	0.35
	0.5	0.5	1

b. 0.85

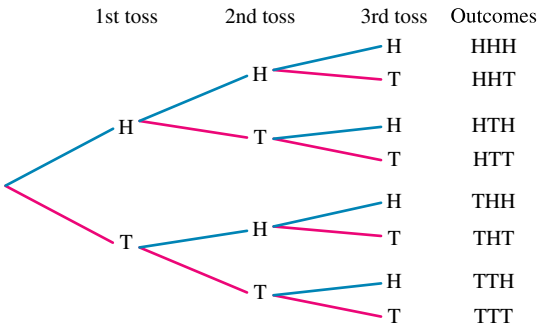
12. **a.**

	B	B'	
A	0.38	0.2	0.58
A'	0.17	0.25	0.42
	0.55	0.45	1

b. 0.25
c. See the worked solutions in your eBookPLUS for the proof.
d. See the worked solutions in your eBookPLUS for the proof.
e.



13. **a.**



b. $\frac{7}{8}$
c. $\frac{3}{4}$

14. **a.** $\frac{2}{9}$ **b.** $\frac{1}{36}$ **c.** $\frac{7}{18}$ **d.** 0 **e.** $\frac{1}{12}$
f. 1

15. **a.** $\frac{3}{8}$ **b.** $\frac{1}{4}$ **c.** $\frac{1}{2}$ **d.** $\frac{7}{8}$ **e.** $\frac{1}{2}$

16. **a.** $\frac{4}{9}$ **b.** $\frac{7}{27}$ **c.** $\frac{2}{3}$ **d.** $\frac{8}{27}$

17. **a.** $\frac{23}{100}$ **b.** $\frac{81}{200}$

18. **a.** $\frac{18}{25}$ **b.** $\frac{6}{25}$
c. 288 students **d.** 0.0576
e. 0.2304 **f.** 0.0784

19. **a.** 20 games **b.** $n(n-1)$
c. 30 games **d.** 240 games

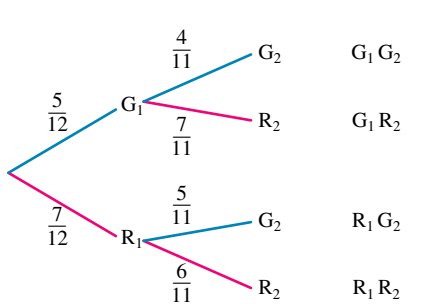
20. Student investigation

Exercise 6.3 Relative frequency

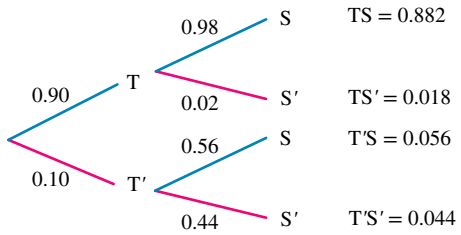
1. A
2. D
3. 0.74
4. 0.79
5. 0.375
6. **a.** 0.45 **b.** 0.55
7. 4%
8. **a.** 0.03 **b.** 0.97
9. 98.9%
10. **a.** 0.525 **b.** 0.4375 **c.** 0.0375
11. **a.** 6.67% **b.** 80
12. **a.** 0.02 **b.** $400
13. **a.** The assembly line will need upgrading.
    b. **i.** 2.5%; **ii.** 51.5%; **iii.** 17.5%
14. Maximum distance is 40 000 km
15. Student investigation
16. Student research task

Exercise 6.4 Conditional probability

1. C
2. **a.** $\frac{1}{5}$ **b.** $\frac{5}{9}$ **c.** $\frac{7}{11}$ **d.** $\frac{11}{25}$
3. **a.** $\frac{5}{36}$ **b.** $\frac{2}{15}$ **c.** $\frac{1}{9}$ **d.** $\frac{4}{5}$
4. **a.** $\frac{6}{25}$ **b.** $\frac{14}{55}$
5. **a.** $\frac{9}{14}$ **b.** $\frac{61}{81}$ **c.** $\frac{12}{25}$
6. **a.** $\frac{5}{11}$
   b.



   c. $\frac{35}{132}$ **d.** $\frac{31}{66}$
7. **a.**



   b. 0.938
8. $\frac{2}{5}$
9. $\frac{1}{2}$
10. **a.** 0.2 **b.** $\frac{2}{3}$ **c.** $\frac{2}{7}$ **d.** $\frac{5}{7}$
11. **a.** 0.3 **b.** $\frac{3}{5}$ **c.** $\frac{1}{2}$ **d.** $\frac{3}{5}$
12. **a.** 0.9 **b.** $\frac{4}{7}$ **c.** $\frac{3}{4}$ **d.** $\frac{1}{3}$
13. **a.** $\frac{1}{17}$ **b.** $\frac{15}{34}$ **c.** $\frac{2}{15}$ **d.** $\frac{4}{17}$
14. **a.** $\frac{7}{22}$ **b.** $\frac{28}{33}$ **c.** $\frac{3}{8}$ **d.** $\frac{1}{2}$
15. **a.** 0.72 **b.** 0.98 **c.** 0.2
16. **a.** $\frac{427}{500}$ **b.** $\frac{47}{100}$ **c.** $\frac{32}{61}$
17. **a.** $\frac{267}{1000}$ **b.** $\frac{257}{1000}$ **c.** $\frac{82}{267}$ **d.** $\frac{185}{743}$
18. **a.** $\frac{9}{50}$ **b.** $\frac{21}{50}$ **c.** $\frac{47}{232}$ **d.** $\frac{143}{168}$
19. **a.** 0.168 **b.** 0.84 **c.** 0.0392
20. **a.** 0.7225 **b.** 0 991 020 25 **c.** 0 85

Exercise 6.5 Independence

1. **a.** Independent **b.** Dependent
    c. Dependent **d.** Independent
2. $P(B|A) = P(B), P(A \cap B) = P(A) \times P(B)$
3. **a.** $\xi = \{HH, HT, TH, TT\}$
    $A = \{TH, TT\}$
    $B = \{HT, TH\}$
    $C = \{HH, HT, TH\}$
    b. A and B are independent.
    c. B and C are not independent.
    d. $\frac{3}{4}$
4. **a.** Not mutually exclusive
    b. Not independent
    c. **i.** B and C are mutually exclusive.
    ii. A and B are not independent.
5. **a.** 0.216 **b.** 0.024 **c.** 0.784
6. **a.** $\frac{4}{9}$ **b.** $\frac{10}{21}$ **c.** $\frac{1}{3}$
7. They are independent.
8. They are independent.
9. The cars are not used independently.
10. **a.** 0.001 25 **b.** 0.0025 **c.** 0.9524
    d. 0.000 625 **e.** 0.052 25
11. **a.** $\frac{9}{40}$ **b.** $\frac{2}{5}$ **c.** $\frac{17}{20}$ **d.** $\frac{47}{200}$ **e.** $\frac{7}{12}$
12. **a.** 0.73 **b.** $\frac{7}{100}$ **c.** $\frac{23}{30}$ **d.** $\frac{33}{35}$
13. **a.** $\frac{4}{5}$ **b.** $\frac{2}{3}$ **c.** $\frac{8}{15}$ **d.** $\frac{14}{15}$
14. **a.** $\frac{1}{5}$ **b.** $\frac{11}{30}$
    c. The events are not independent.
15. 0.5

16. Medium and age are not independent.
17. P and T are not independent.
18. a. A and B are not independent.
    b. B and C are not independent.
    c. A and C are not independent.

Exercise 6.6 Permutations and combinations

1. B
2. a. Sample responses can be found in the worked solutions in the online resources.
   b. Sample responses can be found in the worked solutions in the online resources.
3. a. {AB, AC, BC}
   b. {AB, AC, BA, BC, CA, CB}
4. C
5. a. 120 b. 300 c. 24 d. $\frac{1}{5}$
6. a. 17 576 000
   b. 1 875 000
   c. $\frac{1}{2600}$
7. a. 720 b. 120 c. $\frac{1}{36}$
8. a. 6 b. 24
9. a. 80 640 b. 282 240
   c. 119 750 400 d. $\frac{1}{6}$
10. a. 64 864 800 b. 4 989 600 c. 43 545 600
11. a. 2002 b. 840 c. 126 d. $\frac{36}{91}$
12. $\frac{45}{442}$
13. a. 120 b. 6 days c. 240
    d. 210 e. 312 f. 216
14. a. 1 757 600 b. 16 807 c. 216
    d. 216 e. 24
15. a. 40 320 b. 1152 c. 8640
    d. 30 240 e. 2880
16. a. 648 b. 328 c. 216 d. 60
17. a. 40 320 b. 3600
18. a. 27 720 b. 210
19. a. 420 b. 408 c. 24
20. a. 6435 b. 2 598 960
    c. 48 d. 120
21. a. 43 758 b. 6720 c. 1341
    d. 35 750 e. 8008
22. a. 4096 b. 31 c. $\frac{31}{4096}$
    d. $\frac{4017}{4096}$ e. $\frac{35}{4096}$
23. a. $\frac{1}{11\,925}$ b. $\frac{19}{312}$
    c. $\frac{210\,721}{248\,040} = 0.85$ d. $\frac{24\,877}{73\,099\,048} = 0.27$
24. a. 0.0006 b. 0.0052 c. 2.69×10^{-7}
    d. 0.268 e. 0.000 242 2

Exercise 6.7 Pascal's triangle and binomial expansions

1. B
2. $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$
3. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$
4. 35
5. 210
6. a. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
   b. $16 + 32x + 24x^2 + 8x^3 + x^4$
   c. $t^3 - 6t^2 + 12t - 8$
7. a. $m^2 + 6bm + 9b^2$
   b. $16d^4 - 32d^3x + 24d^2x^2 - 8dx^3 + x^4$
   c. $h^3 + 6h + \frac{12}{h} + \frac{8}{h^3}$
8. a. ${}^7C_3$
   b. 84
   c. Sample responses can be found in the worked solutions in the online resources.
   d. Sample responses can be found in the worked solutions in the online resources.
9. a. Sample responses can be found in the worked solutions in the online resources.
   b. ${}^{19}C_{15} + {}^{19}C_{14}$
10. a. $x^3 + 3x^2y + 3xy^2 + y^3$
    b. $a^4 + 8a^3 + 24a^2 + 32a + 16$
    c. $m^4 - 12m^3 + 54m^2 - 108m + 81$
    d. $32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$
11. a. $1 - \frac{6}{x} + \frac{12}{x^2} - \frac{8}{x^3}$
    b. $1 + \frac{4p}{q} + \frac{6p^2}{q^2} + \frac{4p^3}{q^3} + \frac{p^4}{q^4}$
    c. $81 - 54m + \frac{27}{2}m^2 - \frac{3}{2}m^3 + \frac{1}{16}m^4$
    d. $8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}$
12. a. $720w^3$ b. $\frac{945}{b^4}$ c. 54
13. a. 0.0625 b. 0.375
14. 0.135 93
15. 0.2592
16. a. 0.1023 b. 0.2001
17. $(1 - m)^6$
18. $a = 3$ (assuming a is non-zero)
19. $n = 4$ and $k = \frac{1}{2}$
20. a. 125 970, 77 520, 38 760
    b. $a = 318\,195$; $b = 224\,953$
    c. $n = 16$

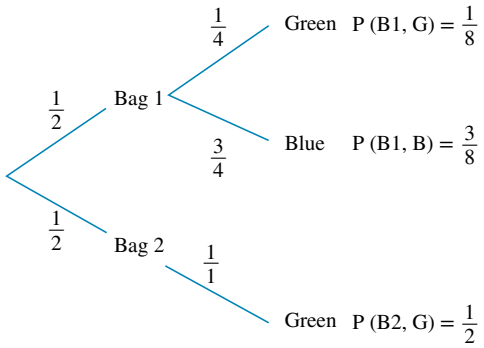
6.8 Review: exam practice

Simple familiar

1. S = {1, 2, 3, 4, 5} ; Favourable outcomes = {3, 4, 5}
2. 0.02
3. a. 0.3
   b. A and B are independent.

4. D

5. a.



b. $\frac{5}{8}$

6. 2184

7. B

8. 525

9. 12

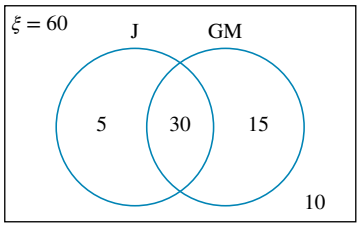
10. a. 0.15 **b.** \$750

11. 60 480

12. $\frac{21}{40}$

Complex familiar

13. a.



b. 0.5

c. i. 0.67 **ii.** 0.86

14. a. Sample responses can be found in the worked solutions in the online resources.

b. Numbers greater than 400 are more likely to be formed.

15. a.

	Liars (L)	Honest people (H)	Totals
Correctly tested (C)	18	67	85
Incorrectly tested (I)	2	13	15
Totals	20	80	100

b. $\frac{3}{20}$

c. $\frac{67}{100}$

d. $\frac{67}{80}$

16. a. 0.003 61 **b.** 0.003 61

Complex unfamiliar

17. Sample responses can be found in the worked solutions in the online resources.

18. a. $x^8 - 12x^5 + 54x^2 - 108x^{-1} + 81x^{-4}$

b. 40 095

19. a. 4 **b.** $\frac{7}{120}$ **c.** $\frac{9}{14}$

20. a. 0.089 **b.** 0.081 079 **c.** 0.045

d. 0.056 **e.** 0.482 165 **f.** 0.236

TOPIC 3 Counting and probability

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



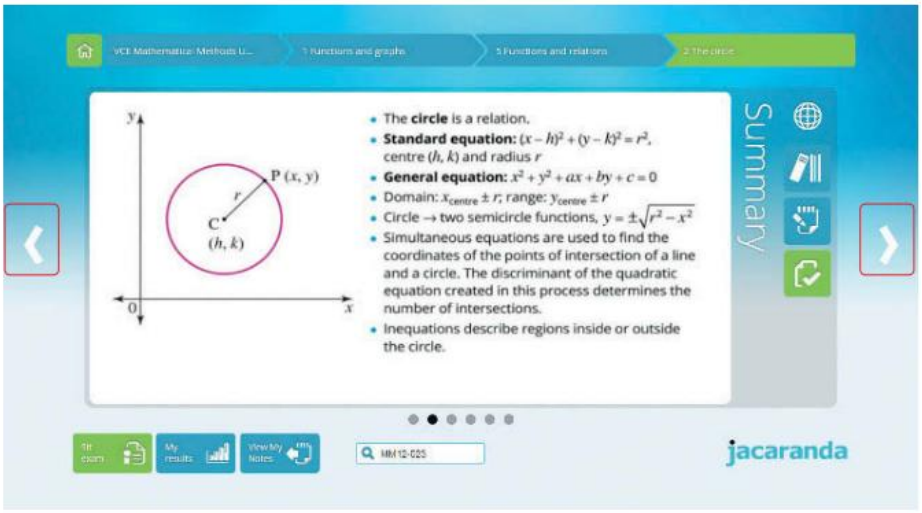
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 7
Indices

7.1 Overview

7.1.1 Introduction

Exponential (or indicial) notation provides a convenient means of writing multiples of the same variable in a succinct form. The French mathematician René Descartes — best known to us for his philosophy 'I think, therefore I am' — was the first to describe indices as we use them today. In his 1637 text *Discours de la méthode pour bien conduire sa raison et chercher* (roughly translated as 'Discourse on the methodology of logic and reasoning'), he described the notation that he used to write multiples of the same variable:

> **'… in order to multiply a by itself we write aa or a^2, and a^3 in order to multiply it once more by a, and thus to infinity'**



This notation is at its most useful when we are using numbers that involve very large or very small powers of ten. Hence, we write the speed of light as 3×10^8 metres per second rather than 300 000 000 metres per second and the mass of a carbon atom as 1.997×10^{-23} grams instead of 0.000 000 000 000 000 000 000 001 997 grams. As these extraordinary numbers are most commonly used in fields such as chemistry or physics, this form of notation is referred to as scientific notation (we shall examine this in more detail during the course of this chapter).

Without exponential notation it would take us a very long time to write numbers such as the googol (equal to 10^{100}), let alone the googolplex, which is $10^{10^{100}}$. In fact, astrophysicist Carl Sagan once noted that to write the decimal representation of the googolplex would take a piece of paper bigger than the boundaries of the known universe!

LEARNING SEQUENCE

7.1 Overview
7.2 Index laws
7.3 Negative and rational indices
7.4 Indicial equations and scientific notation
7.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

7.2 Index laws

7.2.1 Introduction

Recall that a number, a, which is multiplied by itself n times can be represented in **index notation**.

$$\underbrace{a \times a \times a \times \ldots \times a}_{n \text{ lots of } a} = a^n$$

Index (or power or exponents) → n

Base → a

where a is the base number and n is the **index** (or power or exponent).

a^n is read as 'a to the power of n' or 'a to the n'.

When the number 8 is expressed as a power of 2, it is written as $8 = 2^3$. In this form, the base is 2 and the power (also known as the index or **exponent**) is 3. The form of 8 expressed as 2^3 is known both as its index form with index 3 and base 2, and its **exponential form** with exponent 3 and base 2. The words 'index' and 'exponent' are interchangeable.

For any positive number n where $n = a^x$, the statement $n = a^x$ is called an index or exponential statement. For our study:

- the base is a where $a \in R^+ \setminus \{1\}$
- the exponent, or index, is x where $x \in R$
- the number n is positive, so $a^x \in R^+$.

Index laws control the simplification of expressions which have the same base.

7.2.2 Review of the index laws

Multiplication

When multiplying two numbers in index form with the same *base, add* the indices. $a^m \times a^n = a^{m+n}$

For example, $2^3 \times 2^4 = 2 \times 2 \times 2 \quad \times \quad 2 \times 2 \times 2 \times 2 = 2^7$

Division

When dividing two numbers in index form with the same *base, subtract* the indices. $a^m \div a^n = a^{m-n}$

For example, $2^6 \div 2^2 = \dfrac{2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2} = 2^4$

Raising to a power

To raise an indicial expression to a power, *multiply* the indices. $(a^m)^n = a^{m \times n} = a^{mn}$

For example, $\left(2^4\right)^3 = 2^4 \times 2^4 \times 2^4 = 2^{4+4+4} = 2^{(4 \times 3)} = 2^{12}$

Raising to the power of zero

Any number raised to the power of zero is equal to *one*. $a^0 = 1, \; a \neq 0$

For example $2^3 \div 2^3 = 2^{3-3} = 2^0$ [1]

or $2^3 \div 2^3 = (2 \times 2 \times 2) \div (2 \times 2 \times 2)$

$= 8 \div 8$

$= 1$

So $2^3 \div 2^3 = 1$ [2]

Using [1] and [2] we have $2^0 = 1$.

7.2.3 Products and quotients

Note the following examples of products or quotients.

$$(ab)^n = a^n b^n$$

Example: $(2 \times 3)^4 = (2 \times 3) \times (2 \times 3) \times (2 \times 3) \times (2 \times 3)$
$= 2 \times 3 \times 2 \times 3 \times 2 \times 3 \times 2 \times 3$
$= 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3$
$= 2^4 \times 3^4$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Example: $\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right)$
$= \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3}$
$= \frac{2^4}{3^4}$

WORKED EXAMPLE 1

Simplify each of the following.

a. $2x^3y^2 \times 4x^2y$ **b.** $(2x^2y^3)^2 \times xy^4$ **c.** $(3a)^5 b^6 \div 9a^4b^3$ **d.** $\frac{8p^6m^2 \times (3p)^3m^5}{6p^4m}$

THINK

a. 1. Collect 'plain' numbers (2 and 4) and terms with the same base.

2. Simplify by multiplying plain numbers and adding powers with the same base.

b. 1. Remove the bracket by multiplying the powers. (The power of the 2 inside the bracket is 1.)

2. Convert 2^2 to a plain number (4) first and collect terms with the same base.

3. Simplify by adding powers with the same base.

c. 1. Write the quotient as a fraction.

2. Remove the bracket by multiplying the powers.

3. Simplify by first cancelling plain numbers.

WRITE

a. $2x^3y^2 \times 4x^2y$
$= 2 \times 4 \times x^3 \times x^2 \times y^2 \times y$
$= 8x^5y^3$

b. $(2x^2y^3)^2 \times xy^4$
$= 2^2 \times x^4 \times y^6 \times xy^4$
$= 4 \times x^4 \times x \times y^6 \times y^4$
$= 4x^5y^{10}$

c. $(3a)^5 b^6 \div 9a^4b^3 = \frac{(3a)^5b^6}{9a^4b^3}$
$= \frac{243a^5b^6}{9a^4b^3}$
$= \frac{27a^5b^6}{a^4b^3}$

4. Complete simplification by subtracting powers with the same base.
(*Note:* $a^1 = a$)

$= 27ab^3$

d. 1. Expand the brackets by raising each term to the power of 3.

2. Convert 3^3 to 27 and collect 'like' pronumerals.

3. Simplify by first reducing the plain numbers, then simplify the pronumerals by adding the indices for multiplication and subtracting the indices for division.

4. Simplify the indices of each base.

d. $\dfrac{8p^6m^2 \times (3p)^3m^5}{6p^4m} = \dfrac{8p^6m^2 \times 3^3p^3m^5}{6p^4m}$

$= \dfrac{8 \times 27 \times p^6 \times p^3 \times m^2 \times m^5}{6p^4m}$

$= 36p^{6+3-4}m^{2+5-1}$

$= 36p^5m^6$

WORKED EXAMPLE 2

Simplify $\dfrac{6a^4b^3}{16a^7b^6} \div \left(\dfrac{3a^2b}{2a^3b^2}\right)^3$.

THINK

1. Write the expression.
2. Change the division sign to multiplication and replace the second term with its reciprocal (turn the second term upside down).
3. Remove the brackets by multiplying the powers.
4. Collect plain numbers and terms with the same base.
5. Cancel plain numbers and apply index laws.
6. Simplify.

WRITE

$\dfrac{6a^4b^3}{16a^7b^6} \div \left(\dfrac{3a^2b}{2a^3b^2}\right)^3$

$= \dfrac{6a^4b^3}{16a^7b^6} \times \left(\dfrac{2a^3b^2}{3a^2b}\right)^3$

$= \dfrac{6a^4b^3}{16a^7b^6} \times \dfrac{2^3a^9b^6}{3^3a^6b^3}$

$= \dfrac{6 \times 8a^{4+9-7-6}b^{3+6-6-3}}{16 \times 27}$

$= \dfrac{a^0b^0}{9}$

$= \dfrac{1}{9}$

Expressions involving just numbers and numerical indices can be simplified using index laws and then evaluated.

WORKED EXAMPLE 3

Write in simplest index notation and evaluate.

a. $2^3 \times 16^2$ **b.** $\dfrac{9^5 \times 3^4}{27^3}$

THINK	WRITE
a. 1. Rewrite the bases in terms of their prime factors.	**a.** $2^3 \times 16^2 = 2^3 \times (2 \times 2 \times 2 \times 2)^2$
2. Simplify the brackets using index notation.	$= 2^3 \times (2^4)^2$
3. Remove the brackets by multiplying the powers.	$= 2^3 \times 2^8$
4. Simplify by adding the powers.	$= 2^{11}$
5. Evaluate as a basic number.	$= 2048$
b. 1. Rewrite the bases in terms of their prime factors.	**b.** $\frac{9^5 \times 3^4}{27^3} = \frac{(3 \times 3)^5 \times 3^4}{(3 \times 3 \times 3)^3}$
2. Simplify the brackets using index notation.	$= \frac{(3^2)^5 \times 3^4}{(3^3)^3}$
3. Remove the brackets by multiplying the powers.	$= \frac{3^{10} \times 3^4}{3^9}$
4. Write in simplest index form.	$= 3^5$
5. Evaluate as a basic number.	$= 243$

Complex expressions involving terms with different bases have to be simplified by replacing each base with its prime factors.

WORKED EXAMPLE 4

Simplify $\frac{3^{4n} \times 18^{n+1}}{6^{3n-2}}$.

THINK	WRITE
1. Rewrite the bases in terms of their prime factors.	$\frac{3^{4n} \times 18^{n+1}}{6^{3n-2}} = \frac{3^{4n} \times (3 \times 3 \times 2)^{n+1}}{(2 \times 3)^{3n-2}}$
2. Simplify the brackets using index notation.	$= \frac{3^{4n} \times (3^2 \times 2^1)^{n+1}}{(2 \times 3)^{3n-2}}$
3. Remove the brackets by multiplying powers.	$= \frac{3^{4n} \times 3^{2n+2} \times 2^{n+1}}{2^{3n-2} \times 3^{3n-2}}$
4. Collect terms with the same base by adding the powers in the products and subtracting the powers in the quotients.	$= 3^{4n+2n+2-(3n-2)} \times 2^{n+1-(3n-2)}$
5. Simplify.	$= 3^{6n+2-3n+2} \times 2^{n+1-3n+2}$
	$= 3^{3n+4} \times 2^{-2n+3}$
	$= 3^{3n+4} \times 2^{3-2n}$

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Units 1 & 2 > Area 4 > Sequence 1 > Concept 1

Index laws Summary screen and practice questions

Exercise 7.2 Index laws

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1. **WE1a** Simplify each of the following.
   a. $x^2 \times x^5 \times x^3$ b. $m^3 \times m^2 p \times p^4$ c. $4y^3 \times 2y \times y^7$
2. **WE1b** Simplify each of the following.
   a. $5^2 \times 5^7 \times \left(5^3\right)^3$ b. $(xy)^3 \times x^4 y^5$ c. $\left(2x^4\right)^2 \times \left(4x^2\right)^5$
   d. $3m^2 p^5 \times \left(mp^2\right)^3 \times 2m^4 p^6$ e. $5x^2 y^3 \times \left(5xy^2\right)^4 \times \left(5x^2 y\right)^2$
3. **WE1c** Simplify each of the following.
   a. $a^7 b^8 \div a^2 b^5$ b. $2a^{12} b^9 \div (2a)^3 b^4$ c. $\left(3x^5\right) y^{11} \div 6x^2 y^2$
   d. $p^{13} q^{10} \div \left(pq^4\right)^2$ e. $\left(4mn^4\right)^2 \div 14n^3$
4. Simplify each of the following.
   a. $\dfrac{a^3 b^4}{ab^2}$ b. $25r^{15} s^{10} t^4 \div r^5 \left(s^5\right)^2 (5t)^3$
   c. $\dfrac{15a^6 b^7}{3a^3 b^4}$ d. $\dfrac{24x^4 y^7}{20x^2 y^3}$
5. Simplify each of the following.
   a. $\dfrac{6p^8 m^4 \times 2p^7 m^6}{9p^5 m^2}$ b. $\dfrac{(3x)^2 y^2 \times 5x^6 y^3}{10x^7 y}$ c. $\dfrac{14u^{11} v^9 \times \left(3u^2\right)^3 v}{21u^6 v^5}$
   d. $\dfrac{\left(5e^3\right)^2 f^4 \times 8e^4 f^3}{20ef^5}$ e. $\dfrac{6w^2 t^7 \times 9w^4 t^{12}}{(3w)^5 t^{13}}$
6. Simplify each of the following.
   a. $\dfrac{(2x)^4 y \times \left(3x^7 y\right)^2}{18x^5 (2y)^3}$ b. $\dfrac{\left(-3x^3 y^2\right)^3}{2x^3 y^6} \times \dfrac{6x^7 y^5}{\left(x^2 y\right)^2}$ c. $\dfrac{(-3mp)^2 \times 4m^4 p}{12 (mp)^2}$
   d. $\dfrac{m^3 p^4 \times \left(mp^3\right)^2}{\left(-mp^2\right)^4}$ e. $\dfrac{4\left(u^7 v^6\right)^3}{\left(-2u^3 v^2\right)^2 \times u^4 \left(3v^5\right)^2}$
7. **WE2** Simplify each of the following.
   a. $\dfrac{15a^8 b^3}{9a^4 b^5} \div \left(\dfrac{2a^3 b}{3ab^2}\right)^2$ b. $\dfrac{5k^{12} d}{\left(2k^3\right)^2} \div \dfrac{6kd^4}{25\left(k^2 d^3\right)^3}$
   c. $\dfrac{4g^4 \left(2p^{11}\right)^2}{g^3 p^7} \div \dfrac{8g^4 p}{(2gp)^3}$ d. $\left(\dfrac{3jn^2}{n^5}\right)^3 \div \dfrac{\left(4j^2 n\right)^2}{n^{13} (2j)^4}$
   e. $\dfrac{x^4 y^7}{x^3 y^2} \div \dfrac{x^3 y^2}{x^5 y}$ f. $\dfrac{6x^3 y^8}{\left(x^2 y^3\right)^3} \div \dfrac{\left(2xy^3\right)^2}{8x^5 y^7}$
8. Simplify each of the following.
   a. $\dfrac{3p^3 m^4}{p^1 m^2}$ b. $\dfrac{6x^6 y^5}{x^5 y^3} \times \dfrac{x^4}{(2y)^2}$ c. $\dfrac{3ab^3}{-ab} \div \left(\dfrac{a^2 b}{a^5}\right)^2$
9. Simplify each of the following.
   a. $\dfrac{x^{n+1} \times y^5 \times z^{4-n}}{x^{n-2} \times y^{4-n} \times z^{3-n}}$ b. $\dfrac{\left(x^n y^{m+3}\right)^2}{x^{n+2} y^{3-m}} \times \dfrac{x^2 y}{x^{n-5} \times y^{5-3m}}$
10. **WE3** Write in simplest index notation.
    a. $2^4 \times 4^2 \times 8$
    b. $3^7 \times 9^2 \times 27^3 \times 81$
    c. $5^3 \times 15^2 \times 3^2$

11. Write in simplest index notation.

a. $20^5 \times 8^4 \times 125$ **b.** $\dfrac{3^4 \times 27^2}{6^4 \times 3^5}$ **c.** $\dfrac{8 \times 5^2}{2^3 \times 10}$

12. Write in simplest index notation and evaluate.

a. $\dfrac{(625)^4}{(5^3)^5}$ **b.** $\dfrac{(25)^4}{(125)^3}$ **c.** $\dfrac{4^{11} \div 8^2}{16^3}$ **d.** $\dfrac{27^2 \times 81}{9^3 \times 3^5}$

13. WE4 Simplify each of the following.

a. $\dfrac{2^n \times 9^{2n+1}}{6^{n-2}}$ **b.** $\dfrac{25^{3n} \times 5^{n-3}}{5^{4n+3}}$ **c.** $\dfrac{12^{x-2} \times 4^x}{6^{x-2}}$

d. $\dfrac{12^{n-3} \times 27^{1-n}}{9^{2n} \times 8^{n-1} \times 16^n}$ **e.** $\dfrac{4^n \times 7^{n-3} \times 49^{3n+1}}{14^{n+2}}$

14. Simplify each of the following.

a. $\dfrac{35^2 \times 5^5 \times 7^6}{25^4 \times 49^3}$ **b.** $\dfrac{3^{5n-4} \times 16^n \times 9^3}{4^{n+1} \times 18^{1-n} \times 6^{3-2n}}$

c. $*\dfrac{3^n + 3^{n+1}}{3^n + 3^{n-1}}$ **d.** $*\dfrac{5^n - 5^{n+1}}{5^{n+1} + 5^n}$

**Hint*: Factorise the numerator and denominator first.

15. Simplify the following.

$$\frac{36^{2n} \times 6^{n+3}}{216^{n-2}}$$

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16. Write in simplest index notation and evaluate.

a. $\dfrac{4^5}{2^7}$ **b.** $9^4 \times 3^5 \div 27$ **c.** $\dfrac{(16^2)^3}{(2^5)^4}$ **d.** $\dfrac{27^2}{(3^2)^3}$

7.3 Negative and rational indices

7.3.1 Negative indices

Wherever possible, negative index numbers should be expressed with positive index numbers using the simple rule below.

When an index number is moved from the numerator to denominator or vice versa, the sign of the power changes.

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

This is easily verified as follows.

$$\begin{aligned}\frac{1}{a^n} &= \frac{a^0}{a^n} \text{ since } a^0 = 1\\ &= a^{0-n} \text{ using division rule for indices}\\ &= a^{-n} \text{ simplifying the index}\end{aligned}$$

In other words, $\dfrac{a^{-n}}{1} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = \dfrac{a^n}{1}$

Note: Change the level, change the sign.

WORKED EXAMPLE 5

Express each of the following with positive index numbers.

a. $\left(\frac{5}{8}\right)^{-4}$ **b.** $\frac{x^4y^{-2} \times (x^2y)^{-5}}{x^{-3}y^3}$

THINK

a. 1. Remove the brackets by raising the denominator and numerator to the power of −4.

2. Interchange the numerator and denominator, changing the signs of the powers.

3. Simplify by expressing as a fraction to the power of 4.

b. 1. Remove the brackets by multiplying powers.

2. Collect terms with the same base by adding the powers on the numerator and subtracting the powers on the denominator.

3. Rewrite the answer with positive powers.

WRITE

a. $\left(\frac{5}{8}\right)^{-4} = \frac{5^{-4}}{8^{-4}}$

$= \frac{8^4}{5^4}$

$= \left(\frac{8}{5}\right)^4$

b. $\frac{x^4y^{-2} \times (x^2y)^{-5}}{x^{-3}y^3} = \frac{x^4y^{-2} \times x^{-10}y^{-5}}{x^{-3}y^3}$

$= \frac{x^{-6}y^{-7}}{x^{-3}y^3}$

$= x^{-6-(-3)}y^{-7-3}$

$= x^{-3}y^{-10}$

$= \frac{1}{x^3y^{10}}$

7.3.2 Fractional indices

Up to now, we have only looked at cases in which the indices have all been integers. However, an index can be any number.

Since $\left(a^{\frac{1}{2}}\right)^2 = a$ and $\left(\sqrt{a}\right)^2 = a$, then $a^{\frac{1}{2}} = \sqrt{a}$.

Thus, surds such as $\sqrt{3}$ can be written in index form as $3^{\frac{1}{2}}$.

The symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, $\ldots \sqrt[n]{\ }$ are radical signs. Any radical can be converted to and from a fractional index using the index law:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

A combination of index laws allows $a^{\frac{m}{n}}$ to be expressed as $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m$. It can also be expressed as $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \text{ or } a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

WORKED EXAMPLE 6

Evaluate each of the following without technology.

a. $16^{\frac{3}{2}}$ **b.** $\left(\frac{9}{25}\right)^{-\frac{3}{2}}$

THINK

a. 1. Rewrite the base number in terms of its prime factors.

2. Remove the brackets by multiplying the powers.

3. Evaluate as a basic number.

b. 1. Rewrite the base numbers of the fraction in terms of their prime factors.

2. Remove the brackets by multiplying the powers.

3. Rewrite with positive powers by interchanging the numerator and denominator.

4. Evaluate the numerator and denominator as basic numbers.

WRITE

a. $16^{\frac{3}{2}} = \left(2^4\right)^{\frac{3}{2}}$

$= 2^6$

$= 64$

b. $\left(\frac{9}{25}\right)^{-\frac{3}{2}} = \left(\frac{3^2}{5^2}\right)^{-\frac{3}{2}}$

$= \frac{3^{-3}}{5^{-3}}$

$= \frac{5^3}{3^3}$

$= \frac{125}{27}$

WORKED EXAMPLE 7

Simplify the following, expressing your answer with positive indices.

a. $\sqrt[3]{64} \times \sqrt[4]{512}$ **b.** $\sqrt[3]{x^2y^6} \div \sqrt{x^3y^5}$

THINK

a. 1. Write the expression.

2. Write using fractional indices.

3. Write 64 and 512 in index form.

4. Multiply the powers.

5. Simplify the powers.

WRITE

a. $\sqrt[3]{64} \times \sqrt[4]{512}$

$= 64^{\frac{1}{3}} \times 512^{\frac{1}{4}}$

$= \left(2^6\right)^{\frac{1}{3}} \times \left(2^9\right)^{\frac{1}{4}}$

$= 2^2 \times 2^{\frac{9}{4}}$

$= 2^{\frac{17}{4}}$

b. 1. Write the expression.

2. Express the roots in index notation.

3. Remove the brackets by multiplying the powers.

b. $\sqrt[3]{x^2y^6} \div \sqrt{x^3y^5}$

$= \left(x^2y^6\right)^{\frac{1}{3}} \div \left(x^3y^5\right)^{\frac{1}{2}}$

$= x^{\frac{2}{3}}y^2 \div x^{\frac{3}{2}}y^{\frac{5}{2}}$

4. Collect terms with the same base by subtracting the powers. $= x^{\frac{2}{3}-\frac{3}{2}} y^{2-\frac{5}{2}}$

5. Simplify the powers. $= x^{-\frac{5}{6}} y^{-\frac{1}{2}}$

6. Rewrite with positive powers. $= \dfrac{1}{x^{\frac{5}{6}} y^{\frac{1}{2}}}$

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Units 1 & 2 > Area 4 > Sequence 1 > Concept 2

Indicial equations Summary screen and practice questions

Exercise 7.3 Negative and rational indices

Technology free

1. MC

a. The exact value of 6^{-2} is:

A. -12 **B.** -36 **C.** $-\dfrac{1}{6}$ **D.** $\dfrac{1}{36}$

b. The exact value of $\left(\dfrac{27}{8}\right)^{-\frac{1}{3}}$ is:

A. $-\dfrac{2}{3}$ **B.** 6 **C.** $\dfrac{2}{3}$ **D.** $-\dfrac{3}{2}$

c. $\sqrt[3]{25} \times \sqrt{125}$ simplifies to:

A. $25^{\frac{5}{6}}$ **B.** $5^{\frac{7}{6}}$ **C.** $5^{\frac{3}{2}}$ **D.** $5^{\frac{13}{6}}$

2. WE5a Express each of the following with positive index numbers.

a. 6^{-3} **b.** 5^{-4} **c.** $\left(\dfrac{3}{5}\right)^{-2}$ **d.** $\left(\dfrac{7}{4}\right)^{-5}$

3. Express each of the following with positive index numbers.

a. $\left(\dfrac{1}{9}\right)^{-2}$ **b.** $\left(64^{-2}\right)^{3}$ **c.** $(-3)^{-1}$ **d.** $\left(\dfrac{3^4}{2^3}\right)^{-4}$

4. WE5b Simplify each of the following, expressing your answer with positive index numbers.

a. $\dfrac{(-2)^3 \times 2^{-4}}{2^{-3}}$ **b.** $\dfrac{\left(x^{-2}\right)^3 \times \left(y^4\right)^{-2}}{x^{-5} \times \left(y^{-2}\right)^3}$ **c.** $\dfrac{(-m)^2 \times m^{-3}}{\left(p^{-2}\right)^{-1} \times p^{-4}}$

5. Simplify each of the following, expressing your answer with positive index numbers.

a. $\dfrac{x^5}{x^{-3}} \div \dfrac{\left(x^4\right)^{-2}}{\left(x^2\right)^{-3}}$ **b.** $\dfrac{\left(3^{-2}\right)^2 \times \left(2^{-5}\right)^{-1}}{\left(2^4\right)^{-2} \times \left(3^4\right)^{-3}}$ **c.** $\dfrac{x^3 y^{-2} \times \left(xy^2\right)^{-3}}{\left(2x^3\right)^2 \times \left(y^{-3}\right)^2}$

6. **WE6** Evaluate the following without using technology.

a. $9^{\frac{1}{2}}$ **b.** $27^{\frac{1}{3}}$ **c.** $625^{\frac{1}{4}}$ **d.** $256^{\frac{1}{8}}$

e. $8^{\frac{2}{3}}$ **f.** $81^{\frac{3}{4}}$ **g.** $125^{\frac{4}{3}}$ **h.** $\left(\frac{8}{125}\right)^{\frac{1}{3}}$

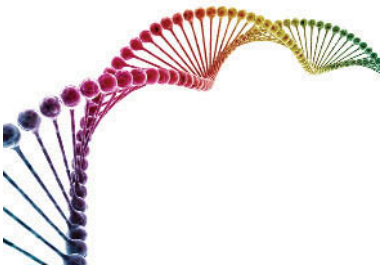
7. Evaluate the following without using technology.

a. $\left(\frac{16}{81}\right)^{\frac{1}{4}}$ **b.** $\left(\frac{25}{16}\right)^{\frac{3}{2}}$

c. $\left(\frac{27}{64}\right)^{\frac{2}{3}}$ **d.** $32^{-\frac{2}{5}}$

e. $81^{-\frac{3}{4}}$ **f.** $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

g. $\left(\frac{16}{121}\right)^{-\frac{1}{2}}$ **h.** $\left(\frac{125}{216}\right)^{-\frac{1}{3}}$



8. **WE7** Simplify each of the following, expressing your answer with positive indices.

a. $\sqrt{9} \times \sqrt[3]{81}$ **b.** $x^{\frac{2}{3}} \times x^{\frac{1}{6}}$

c. $x^{-\frac{3}{4}} \times x^{\frac{9}{8}}$ **d.** $x^{\frac{5}{2}} \div \left(x^{\frac{1}{3}}\right)^4$

e. $\sqrt[3]{(xy^3)} \div \sqrt{(x^2y)}$ **f.** $\sqrt[5]{32} \times \sqrt[4]{8}$

9. Simplify each of the following, expressing your answer with positive indices.

a. $2^{\frac{5}{4}} \times 4^{-\frac{1}{2}} \times 8^{-\frac{2}{3}}$ **b.** $27^{-\frac{1}{4}} \times 9^{\frac{2}{3}} \times 3^{-\frac{5}{4}}$ **c.** $\dfrac{18^{\frac{1}{2}}}{9^{\frac{4}{3}} \times 4^{\frac{3}{4}}}$

d. $\left(\sqrt[4]{x^3}\right)^{\frac{2}{3}} \times \left(\sqrt[3]{x^4}\right)^{\frac{3}{8}}$ **e.** $\dfrac{(64m^6)^{\frac{4}{3}}}{4m^{-2}}$ **f.** $\dfrac{\sqrt{x^3}}{\sqrt{x}}$

10. Simplify each of the following, expressing your answer with positive indices.

a. $\dfrac{1}{\sqrt{x^{-4}}}$ **b.** $\dfrac{(x+1)^2}{\sqrt{x+1}}$ **c.** $\sqrt{x} - \dfrac{1}{\sqrt{x}}$

d. $\sqrt{x+2} + \dfrac{x}{\sqrt{x+2}}$ **e.** $(y-4)\sqrt{y-4}$ **f.** $(p+3)(p+3)^{-\frac{2}{5}}$

11. **a.** Express the following in index (exponent) form.

i. $\sqrt{a^3b^4}$ **ii.** $\sqrt{\dfrac{a^5}{b^{-4}}} \times \sqrt[3]{a^2b}$

b. Express the following in surd form.

i. $a^{\frac{1}{2}} \div b^{\frac{3}{2}}$ **ii.** $2^{\frac{5}{2}}$ **iii.** $3^{-\frac{2}{5}}$

12. Evaluate the following without using technology.

a. $4^{\frac{3}{2}}$ **b.** $3^{-1} + 5^0 - 2^2 \times 9^{-\frac{1}{2}}$

c. $2^3 \times \left(\frac{4}{9}\right)^{-\frac{1}{2}} \div \left(6 \times \left(3^{-2}\right)^2\right)$ **d.** $\dfrac{15 \times 5^{\frac{3}{2}}}{125^{\frac{1}{2}} - 20^{\frac{1}{2}}}$

13. Simplify and express the answer with positive indices.

a. $\dfrac{3\left(x^2y^{-2}\right)^3}{\left(3x^4y^2\right)^{-1}}$

b. $\dfrac{2a^{\frac{2}{3}}b^{-3}}{3a^{\frac{1}{3}}b^{-1}} \times \dfrac{3^2 \times 2 \times (ab)^2}{\left(-8a^2\right)^2 b^2}$

c. $\dfrac{\left(2mn^{-2}\right)^{-2}}{m^{-1}n} \div \dfrac{10n^4m^{-1}}{3\left(m^2n\right)^{\frac{3}{2}}}$

d. $\dfrac{4m^2n^{-2} \times -2\left(m^2n^{\frac{3}{2}}\right)^2}{\left(-3m^3n^{-2}\right)^2}$

e. $\dfrac{m^{-1} - n^{-1}}{m^2 - n^2}$

f. $\sqrt{4x-1} - 2x(4x-1)^{-\frac{1}{2}}$

14. Express $\dfrac{2^{1-n} \times 8^{1+2n}}{16^{1-n}}$ as a power of 2.

7.4 Indicial equations and scientific notation

7.4.1 Indicial equations

An **indicial equation** has the unknown variable as an exponent. In this section we shall consider indicial equations which have rational solutions.

7.4.2 Method of equating indices

If index laws can be used to express both sides of an equation as single powers of the same base, then this allows indices to be equated. For example, if an equation can be simplified to the form $2^{3x} = 2^4$, then for the equality to hold, $3x = 4$. Solving this linear equation gives the solution to the indicial equation as $x = \dfrac{4}{3}$.

In the case of an inequation, a similar method is used. If $2^{3x} < 2^4$, for example, then $3x < 4$. Solving this linear inequation gives the solution to the indicial inequation as $x < \dfrac{4}{3}$.

To solve for the exponent x in equations of the form $a^x = n$:

- **Express both sides as powers of the same base.**
- **Equate the indices and solve the equation formed to obtain the solution to the indicial equation.**

Inequations are solved in a similar manner. However, you need to ensure the base a is greater than 1 prior to expressing the indices with the corresponding order sign between them.

For example, you first need to write $\left(\dfrac{1}{2}\right)^x < \left(\dfrac{1}{2}\right)^3$ as $2^{-x} < 2^{-3}$ and then solve the linear inequation $-x < -3$ to obtain the solution $x > 3$.

WORKED EXAMPLE 8

Find the value of x in each of the following equations.

a. $3^x = 81$ **b.** $4^{x-1} = 256$

THINK	WRITE
a. 1. Write the equation.	**a.** $3^x = 81$
2. Express both sides to the same base.	$3^x = 3^4$
3. Equate the powers.	Therefore, $x = 4$,

b. 1.	Write the equation.	**b.** $4^{x-1} = 256$
2.	Express both sides to the same base.	$4^{x-1} = 4^4$
3.	Equate the powers.	Therefore, $x - 1 = 4$,
4.	Solve the linear equation for x by adding one to both sides.	$x = 5$

WORKED EXAMPLE 9

Solve $5^{3x} \times 25^{4-2x} = \frac{1}{125}$ for x.

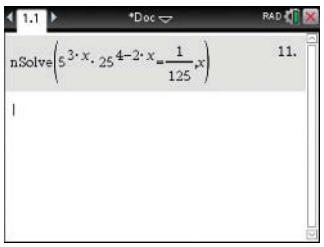
THINK	WRITE
1. Use the index laws to express the left-hand side of the equation as a power of a single base.	$5^{3x} \times 25^{4-2x} = \frac{1}{125}$ $5^{3x} \times 5^{2(4-2x)} = \frac{1}{125}$ $5^{3x+8-4x} = \frac{1}{125}$ $5^{8-x} = \frac{1}{125}$
2. Express the right-hand side as a power of the same base.	$5^{8-x} = \frac{1}{5^3}$ $5^{8-x} = 5^{-3}$
3. Equate indices and calculate the required value of x.	Equating indices, $8 - x = -3$ $x = 11$

TI | THINK

1. On a Calculator page, press MENU then select
3: Algebra
1: Numerical Solve
Complete the entry line as nSolve $\left(5^{3x} \times 25^{4-2x} = \frac{1}{125}, x\right)$ then press ENTER.

2. The answer appears on the screen.

WRITE



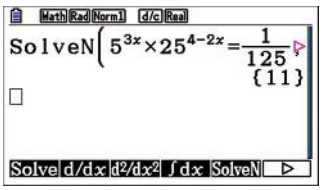
$x = 11$

CASIO | THINK

1. On the Run-Matrix screen, press OPTN then select CALC by pressing F4. Select SolveN by pressing F5 then complete the entry line as SolveN $\left(5^{3x} \times 25^{4-2x} = \frac{1}{125}, x\right)$ and press EXE.

2. The answer appears on the screen.

WRITE



$x = 11$

7.4.3 Indicial equations which reduce to quadratic form

The technique of substitution to form a quadratic equation may be applicable to indicial equations.

To solve equations of the form $p \times a^{2x} + q \times a^x + r = 0$:

- Note that $a^{2x} = (a^x)^2$.
- Reduce the indicial equation to quadratic form by using a substitution for a^x.
- Solve the quadratic and then substitute back for a^x.
- Since a^x must always be positive, solutions for x can only be obtained for $a^x > 0$; reject any negative or zero values for a^x.

WORKED EXAMPLE 10

Solve $3^{2x} - 6 \times 3^x - 27 = 0$ for x.

THINK	WRITE
1. Use a substitution technique to reduce the indicial equation to quadratic form. *Note:* The subtraction signs prevent the use of index laws to express the left-hand side as a power of a single base.	$3^{2x} - 6 \times 3^x - 27 = 0$ Let $a = 3^x$ $\therefore a^2 - 6a - 27 = 0$
2. Solve the quadratic equation.	$(a - 9)(a + 3) = 0$ $a = 9, a = -3$
3. Substitute back and solve for x.	Replace a by 3^x. $\therefore 3^x = 9$ or $3^x = -3$ (reject negative value) $3^x = 9$ $\therefore 3^x = 3^2$ $\therefore x = 2$

7.4.4 Scientific notation (standard form)

Index notation provides a convenient way to express numbers which are either very large or very small. Writing a number as $a \times 10^b$ (the product of a number a where $1 \leq a < 10$ and a power of 10) is known as writing the number in **scientific notation** (or **standard form**). The age of the earth since the Big Bang is estimated to be 4.54×10^9 years, while the mass of a carbon atom is approximately 1.994×10^{-23} grams. These numbers are written in scientific notation.

To convert scientific notation back to a basic numeral:

- **Move the decimal point b places to the right if the power of 10 has a positive index, in order to obtain the large number $a \times 10^b$ represents**
  Or
- **Move the decimal point b places to the left if the power of 10 has a negative index, in order to obtain the small number $a \times 10^{-b}$ represents. This is because multiplying by 10^{-b} is equivalent to dividing by 10^b.**

7.4.5 Significant figures

When a number is expressed in scientific notation as either $a \times 10^b$ or $a \times 10^{-b}$, the number of digits in a determines the number of **significant figures** in the basic numeral. The age of the Earth is 4.54×10^9 years in scientific notation or 4 540 000 000 years to three significant figures. To one significant figure, the age would be 5 000 000 000 or 5×10^9 years.

WORKED EXAMPLE 11

a. Express each of the following numerals in scientific notation and state the number of significant figures each numeral contains.
i. 3 266 400 **ii. 0.009 876 03**

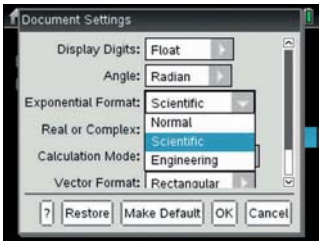
b. Express the following as basic numerals.
i. 4.54×10^9 **ii. 1.037×10^{-5}**

THINK	WRITE
a. i. 1. Write the given number as a value between 1 and 10 multiplied by a power of 10, discarding any trailing zeroes. *Note:* The number is large so the power of 10 should be positive.	**a. i.** In scientific notation, $3\,266\,400 = 3.2\,664 \times 10^6$
2. Count the number of digits in the number a in the scientific notation form $a \times 10^b$ and state the number of significant figures.	There are 5 significant figures in the number 3 266 400.
ii. 1. Write the given number as a value between 1 and 10 multiplied by a power of 10. *Note:* The number is small so the power of 10 should be negative.	**ii.** $0.009\,876\,03 = 9.87\,603 \times 10^{-3}$
2. Count the number of digits in the number a in the scientific notation form and state the number of significant figures.	0.009 876 03 has 6 significant figures.
b. i. 1. Perform the multiplication. *Note:* The power of 10 is positive, so a large number should be obtained.	**b. i.** 4.54×10^9 Move the decimal point 9 places to the right. $\therefore 4.54 \times 10^9 = 4\,540\,000\,000$
ii. 2. Perform the multiplication. *Note:* The power of 10 is negative, so a small number should be obtained.	**ii.** 1.037×10^{-5} Move the decimal point 5 places to the left. $\therefore 1.037 \times 10^{-5} = 0.00\,001\,037$

TI | THINK

a.i1. On the Home screen, select
5: Settings
2: Document settings
then change the Exponential Format to Scientific and select OK.

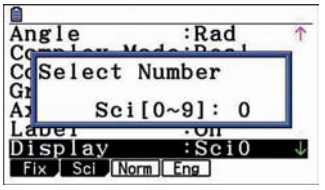
WRITE



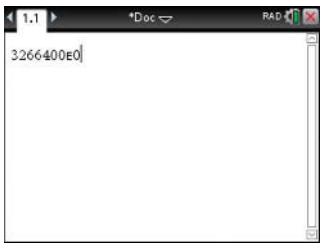
CASIO | THINK

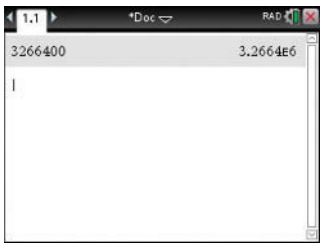
a.i1. On a Run-Matrix screen, press SHIFT then MENU. Change the Display mode to Scientific, then type '0' and press ENTER.
Press EXIT.

WRITE



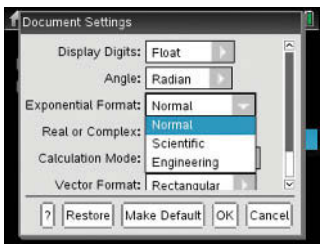
2. On a Calculator page, complete the entry line as 3 266 400E0, then press ENTER.
Note: The EE button is located above the π button.



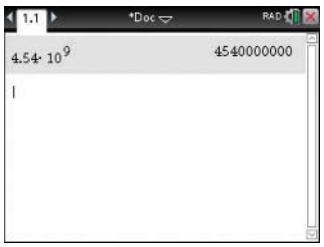


3. The answer appears on the screen.
$3\,266\,400 = 3.2664 \times 10^6$

b.i1. Change the Exponential Format back to Normal.

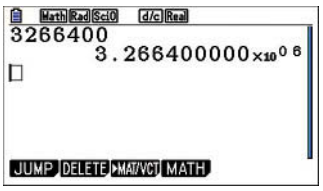


2. On a Calculator page, complete the entry line as 4.54×10^9 then press ENTER.



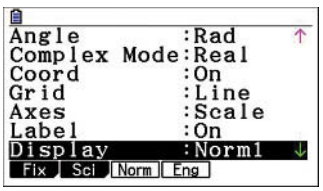
3. The answer appears on the screen.
$4.54 \times 10^9 = 4540000000$

2. Complete the entry line as 3266400, then press EXE.

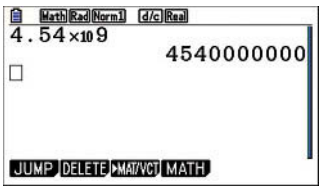


3. The answer appears on the screen.
$3\,266\,400 = 3.2664 \times 10^6$

b.i1. On the Run-Matrix screen, change the Display mode back to Normal.



2. Complete they entry line as 4.54×10^9, then press EXE.



3. The answer appears on the screen.
$4.54 \times 10^9 = 4540000000$

studyon

Units 1 & 2 > Area 4 > Sequence 1 > Concept 3

Scientific notation Summary screen and practice questions

Exercise 7.4 Indicial equations and scientific notation

Technology free

1. **WE8a** Solve for x in each of the following equations.

a. $2^x = 32$ **b.** $5^x = 625$ **c.** $3^x = 243$

d. $10^{-x} = \frac{1}{100}$ **e.** $4^{-x} = 16$ **f.** $6^x = \frac{1}{216}$

g. $3^{-x} = \frac{1}{81}$ **h.** $2^{-x} = 1$ **i.** $8^x = 2^6$

2. **WE8b** Solve for n in each of the following equations.

a. $2^{3n+1} = 64$ **b.** $5^{2n+3} = 25$ **c.** $3^{2-n} = 27$

d. $16^{n+3} = 2^3$ **e.** $49^{5-3n} = \frac{1}{7}$ **f.** $36^{4n-3} = 216$

3. **WE9** Solve for x in each of the following.

a. $4^{2x} = 8^{x-1}$ **b.** $27^{4-x} = 9^{2x+1}$ **c.** $16^{3x+1} = 128^{x-2}$ **d.** $25^{2x-3} = \frac{1}{125}$

e. $32^{5-x} = 4^{3x+2}$ **f.** $64^{2-3x} = 16^{x+1}$ **g.** $9^{3x+5} = \frac{1}{243}$ **h.** $16^{4-3x} = \frac{1}{8^{x+3}}$

4. Solve $\frac{2^{5x-3} \times 8^{9-2x}}{4^x} = 1$ for x.

5. Solve the following inequations.

a. $2 \times 5^x + 5^x < 75$ **b.** $\left(\frac{1}{9}\right)^{2x-3} > \left(\frac{1}{9}\right)^{7-x}$

6. Solve for x in each of the following equations.

a. $2^x \times 8^{3x-1} = 64$ **b.** $5^{2x} \times 125^{3-x} = 25$ **c.** $3^{4x} \times 27^{x+3} = 81$

d. $16^{x+4} \times 2^{3+2x} = 4^{5x}$ **e.** $3125 \times 25^{2x+1} = 5^{3x+4}$ **f.** $\frac{81^{2-x}}{27^{x+3}} = 9^{2x}$

7. **WE10** Solve for x in each of the following.

a. $3^{2x} - 4(3^x) + 3 = 0$ **b.** $2^{2x} - 6(2^x) + 8 = 0$ **c.** $3\left(2^{2x}\right) - 36(2^x) + 96 = 0$

d. $2\left(5^{2x}\right) - 12(5^x) + 10 = 0$ **e.** $3\left(4^{2x}\right) = 15(4^x) - 12$ **f.** $25^x - 30(5^x) + 125 = 0$

8. **a.** Express $\frac{32 \times 4^{3x}}{16^x}$ as a power of 2.

b. Express $\frac{3^{1+n} \times 81^{n-2}}{243^n}$ as a power of 3.

c. Express $0.001 \times \sqrt[3]{10} \times 100^{\frac{5}{2}} \times (0.1)^{-\frac{2}{3}}$ as a power of 10.

d. Express $\frac{5^{n+1} - 5^n}{4}$ as a power of 5.

9. Solve for x in each of the following.

a. $2^{2x} \times 8^{2-x} \times 16^{-\frac{3x}{2}} = \frac{2}{4^x}$ **b.** $25^{3x-3} \leq 125^{4+x}$

c. $9^x \div 27^{1-x} = \sqrt{3}$ **d.** $\left(\frac{2}{3}\right)^{3-2x} > \left(\frac{27}{8}\right)^{-\frac{1}{3}} \times \frac{1}{\sqrt{2\frac{1}{4}}}$

10. Use a suitable substitution to solve the following equations.

a. $3^{2x} - 10 \times 3^x + 9 = 0$ **b.** $24 \times 2^{2x} + 61 \times 2^x = 2^3$

c. $25^x + 5^{2+x} - 150 = 0$ **d.** $(2^x + 2^{-x})^2 = 4$

e. $10^x - 10^{2-x} = 99$ **f.** $2^{3x} + 3 \times 2^{2x-1} - 2^x = 0$

Technology active

11. **WE11** Express each of the following numbers in scientific notation, and state the number of significant figures each number contains.

a. **i.** 1 409 000 **ii.** 0.0 001 306

b. Express the following as basic numerals.

i. 3.04×10^5

ii. 5.803×10^{-2}

12. **a.** Express in scientific notation:

i. $-0.0\,000\,000\,506$

ii. the diameter of the Earth, given its radius is 6370 km

iii. $3.2 \times 10^4 \times 5 \times 10^{-2}$

iv. the distance between Roland Garros and Kooyong tennis stadiums of 16878.7 km.



b. Express the following as a basic numeral.

i. $6.3 \times 10^{-4} + 6.3 \times 10^4$ **ii.** $\left(1.44 \times 10^6\right)^{\frac{1}{2}}$

13. Express the following to 2 significant figures.

a. 60 589 people attended a football match.

b. The probability of winning a competition is 1.994×10^{-2}.

c. The solution to an equation is $x = -0.00634$.

d. The distance flown per year by the Royal Flying Doctor Service is 26 597 696 km.

14. **a.** Simplify $\dfrac{x^2y^{-2}}{2x^{\frac{1}{3}}\sqrt{y^5}}$.

b. Solve the following equations.

i. $5^x \times 25^{2x} = \dfrac{1}{5}$ to obtain x exactly

ii. $5^x \times 25^{2x} = 0.25$ to obtain x to 4 significant figures

7.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC When simplified, $\dfrac{\left(2xy^3\right)^2}{7x^3} \times \dfrac{3x^5y^2}{4y}$ is equal to:

A. $\dfrac{x^4y^7}{7}$ **B.** $\dfrac{3x^4y^7}{7}$ **C.** $\dfrac{3y^7}{x^2}$ **D.** $\dfrac{3x^4}{y^6}$

2. MC $\dfrac{5m^4p^2}{2m^3p} \div \dfrac{\left(5m^2p^6\right)^3}{3m^7p}$ may be simplified to:

A. $\dfrac{m^2}{47p^{16}}$ **B.** $\dfrac{3m^{10}}{2p^{32}}$ **C.** $\dfrac{3m^2}{50p^{16}}$ **D.** $\dfrac{m^{15}}{p^{29}}$

3. MC The value of $5^{-2}\left(\dfrac{64}{125}\right)^{-\frac{1}{3}}$ is:

A. $\dfrac{1}{20}$ **B.** 5 **C.** $\dfrac{4}{5}$ **D.** $\dfrac{5}{4}$

4. MC If $25^{2-x} = 125$, then x is equal to:

A. 1 **B.** $\dfrac{1}{2}$ **C.** -1 **D.** 2

5. MC In scientific notation, $\left(3.2 \times 10^{-2}\right) \times \left(5 \times 10^5\right)$ would equal:

A. 2.56×10^8 **B.** 16×10^{-10} **C.** 1.6×10^{-9} **D.** 1.6×10^4

6. Simplify $\left(9a^3b^{-4}\right)^{\frac{1}{2}} \times 2\left(a^{\frac{1}{2}}b^{-2}\right)^{-2}$.

7. Evaluate $27^{-\frac{2}{3}} + \left(\dfrac{49}{81}\right)^{\frac{1}{2}}$.

8. Simplify the following expression with positive indices.

$\left(16x^{-6}y^{10}\right)^{\frac{1}{2}} \div \sqrt[3]{\left(27x^3y^9\right)}$

9. Solve the following equations.

   a. $2x^5 = 100$ b. $8^{x+1} \times 2^{2x} = 4^{3x-1}$

10. Express the following in surd form.

    a. $a^{\frac{1}{2}} \div b^{\frac{3}{2}}$ b. $2^{\frac{5}{2}}$ c. $3^{-\frac{2}{5}}$

11. Calculate $\left(4 \times 10^6\right)^2 \times \left(5 \times 10^{-3}\right)$ without using technology.

12. Evaluate the following without using technology.

    a. $4^{\frac{3}{2}}$ b. $3^{-1} + 5^0 - 2^2 \times 9^{-\frac{1}{2}}$

Complex familiar

13. Simplify $\dfrac{20p^5}{m^3q^{-2}} \div \dfrac{5\left(p^2q^{-3}\right)^2}{-4m^{-1}}$.

14. Solve $2^x - 48 \times 2^{-x} = 13$ for x.

15. Solve for x.

    a. $4^{5x} + 4^{5x} = \dfrac{8}{2^{4x-5}}$ b. $5^{\frac{2x}{3}} \times 5^{\frac{3x}{2}} = 25^{x+4}$

16. Simplify $\dfrac{a - a^{-1}}{a + 1}$, expressing your answer with positive indices.

Complex unfamiliar

17. If $x = 3^{\frac{1}{3}} + 3^{-\frac{1}{3}}$, show that $x^3 - 3x = \dfrac{10}{3}$.

18. a. Solve the pair of simultaneous equations for x and y.

$$5^{2x-y} = \frac{1}{125}$$
$$10^{2y-6x} = 0.01$$

    b. Solve the pair of simultaneous equations for a and k.

$$a \times 2^{k-1} = 40$$
$$a \times 2^{2k-2} = 10$$



19. If $\left(\dfrac{2x^2}{3a}\right)^{n-1} \div \left(\dfrac{3x}{a}\right)^{n+1} = \left(\dfrac{x}{4}\right)^3$, determine the values of the constants a and n.

20. Given that $x^2 = 2^{32}$ and $x^x = 2^y$, determine the value of y. Assume that $x > 0$.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 7 Indices

Exercise 7.2 Index laws

1. **a.** x^{10} **b.** m^5p^5 **c.** $8y^{11}$

2. **a.** 5^{18} **b.** x^7y^8
c. $2^{12}x^{18}$ **d.** $6m^9p^{17}$
e. $5^7x^{10}y^{13}$

3. **a.** a^5b^3 **b.** $\frac{a^9b^5}{4}$ **c.** $\frac{x^3y^9}{2}$
d. $p^{11}q^2$ **e.** $\frac{8m^2n^5}{7}$

4. **a.** a^2b^2 **b.** $\frac{r^{10}t}{5}$
c. $5a^3b^3$ **d.** $\frac{6x^2y^4}{5}$

5. **a.** $\frac{4p^{10}m^8}{3}$ **b.** $\frac{9xy^4}{2}$ **c.** $18u^{11}v^5$
d. $10e^9f^2$ **e.** $\frac{2wt^6}{9}$

6. **a.** x^{13} **b.** $-81x^9y^3$ **c.** $3m^4p$
d. mp^2 **e.** $\frac{u^{11}v^4}{9}$

7. **a.** $\frac{15}{4}$ **b.** $\frac{125k^{11}d^6}{24}$ **c.** $16p^{17}$
d. $27j^3n^2$ **e.** x^3y^4 **f.** 12

8. **a.** $3p^2m^2$ **b.** $\frac{3x^5}{2}$ **c.** $-3a^6$

9. **a.** $x^3y^{1+n}z$ **b.** x^5y^{6m-1}

10. **a.** 2^{11} **b.** 3^{24} **c.** $5^5 \times 3^4$

11. **a.** $2^{22} \times 5^8$ **b.** $\frac{3}{2^4}$ **c.** $\frac{5}{2}$

12. **a.** 5 **b.** $\frac{1}{5}$
c. 16 **d.** $\frac{1}{3}$

13. **a.** $2^2 \times 3^{3n+4}$ **b.** 5^{3n-6}
c. 2^{3x-2} **d.** $2^{-5n-3} \times 3^{-6n}$
e. $2^{n-2} \times 7^{6n-3}$

14. **a.** $\frac{49}{5}$ **b.** $3^{9n-3} \times 2^{5n-6}$
c. 3 **d.** $-\frac{2}{3}$

15. 6^{2n+9}

16. **a.** 8 **b.** 59 049
c. 16 **d.** 1

Exercise 7.3 Negative and rational indices

1. **a.** D **b.** C **c.** D

2. **a.** $\frac{1}{6^3}$ **b.** $\frac{1}{5^4}$
c. $\frac{5^2}{3^2}$ **d.** $\frac{4^5}{7^5}$

3. **a.** 9^2 **b.** $\frac{1}{64^6}$
c. $\frac{1}{-3}$ **d.** $\frac{2^{12}}{3^{16}}$

4. **a.** -2^2 **b.** $\frac{1}{xy^2}$ **c.** $\frac{p^2}{m}$

5. **a.** x^{10} **b.** $3^8 \times 2^{13}$ **c.** $\frac{1}{4x^6y^2}$

6. **a.** 3 **b.** 3 **c.** 5
d. 2 **e.** 4 **f.** 27
g. 625 **h.** $\frac{2}{5}$

7. **a.** $\frac{2}{3}$ **b.** $\frac{125}{64}$ **c.** $\frac{9}{16}$
d. $\frac{1}{4}$ **e.** $\frac{1}{27}$ **f.** $\frac{9}{4}$
g. $\frac{11}{4}$ **h.** $\frac{6}{5}$

8. **a.** $3^{\frac{7}{3}}$ **b.** $x^{\frac{5}{6}}$ **c.** $x^{\frac{3}{8}}$
d. $x^{\frac{7}{6}}$ **e.** $\frac{y^{\frac{1}{2}}}{x^{\frac{2}{3}}}$ **f.** $2^{\frac{7}{4}}$

9. **a.** $\frac{1}{2^{\frac{7}{4}}}$ **b.** $\frac{1}{3^{\frac{2}{3}}}$ **c.** $\frac{1}{2 \times 3^{\frac{5}{3}}}$
d. x **e.** $64m^{10}$ **f.** x

10. **a.** x^2 **b.** $(x+1)^{1\frac{1}{2}}$ **c.** $\frac{x-1}{x^{\frac{1}{2}}}$
d. $\frac{2x+2}{(x+2)^{\frac{1}{2}}}$ **e.** $(y-4)^{\frac{3}{2}}$ **f.** $(p+3)^{\frac{3}{5}}$

11. **a.** **i.** $a^{\frac{3}{2}}b^2$ **ii.** $a^{\frac{19}{6}}b^{\frac{7}{3}}$
b. **i.** $\sqrt{\frac{a}{b^3}}$ **ii.** $\sqrt{32}$ **iii.** $\sqrt[5]{\frac{1}{9}}$

12. **a.** 8 **b.** 0
c. 162 **d.** 25

13. **a.** $\frac{9x^{10}}{y^4}$ **b.** $\frac{3}{16a^{\frac{5}{3}}b^2}$ **c.** $\frac{3m^3n^{\frac{1}{2}}}{40}$
d. $\frac{-8n^5}{9}$ **e.** $\frac{-1}{mn(m+n)}$ **f.** $\frac{2x-1}{(4x-1)^{\frac{1}{2}}}$

14. 2^{9n}

Exercise 7.4 Indicial equations and scientific notation

1. **a.** 5 **b.** 4 **c.** 5 **d.** 2 **e.** -2
f. -3 **g.** 4 **h.** 0 **i.** 2

2. **a.** $\frac{5}{3}$ **b.** $-\frac{1}{2}$ **c.** -1
d. $-\frac{9}{4}$ **e.** $\frac{11}{6}$ **f.** $\frac{9}{8}$

3. **a.** -3 **b.** $\frac{10}{7}$ **c.** $-\frac{18}{5}$
d. $\frac{3}{4}$ **e.** $\frac{21}{11}$ **f.** $\frac{4}{11}$
g. $-\frac{5}{2}$ **h.** $\frac{25}{9}$

4. 8

5. **a.** $x < 2$ **b.** $x < \frac{10}{3}$

6. **a.** $\frac{9}{10}$ **b.** 7 **c.** $\frac{-5}{7}$
d. $\frac{19}{4}$ **e.** –3 **f.** $\frac{-1}{11}$

7. **a.** $x = 0$ or $x = 1$ **b.** $x = 2$ or $x = 1$
c. $x = 3$ or $x = 2$ **d.** $x = 0$ or $x = 1$
e. $x = 0$ or $x = 1$ **f.** $x = 2$ or $x = 1$

8. **a.** 2^{5+2x} **b.** 3^{-7}
c. 10^3 **d.** 5^n

9. **a.** $x = 1$ **b.** $x \leq 6$
c. $\frac{7}{10}$ **d.** $x > \frac{1}{2}$

10. **a.** $x = 0$ or $x = 2$ **b.** $x = -3$
c. $x = 1$ **d.** $x = 0$
e. $x = 2$ **f.** $x = -1$

11. **a.** **i.** 1.409×10^6; 4 significant figures
ii. 1.306×10^{-4}; 4 significant figures
b. **i.** 304 000 **ii.** 0.058 03

12. **a.** **i.** -5.06×10^{-8} **ii.** 1.274×10^4 km
iii. 1.6×10^3 **iv.** 1.68787×10^4 km
b. **i.** 63 000.000 63 **ii.** 1200

13. **a.** 61 000 **b.** 0.020
c. -0.0063 **d.** 27 000 000 km

14. **a.** $\frac{x^{\frac{5}{3}}}{2y^{\frac{9}{2}}}$
b. **i.** $-\frac{1}{5}$ **ii.** -0.1723

7.5 Review: exam practice

1. B

2. C

3. A

4. B

5. D

6. $6a^{\frac{1}{2}}b^2$

7. $\frac{8}{9}$

8. $\frac{4y^2}{3x^4}$

9. **a.** $50^{\frac{1}{5}}$ **b.** 5

10. **a.** $\sqrt{\frac{a}{b^3}}$ **b.** $\sqrt{32}$
c. $\sqrt[5]{\frac{1}{9}}$

11. 8×10^{10}

12. **a.** 8 **b.** 0

13. $\frac{-16pq^8}{m^4}$

14. $x = 4$

15. **a.** $\frac{1}{2}$ **b.** 48

16. $\frac{a-1}{a}$

17. For fully worked solutions, access your online eBookPLUS.

18. **a.** $x = 4, y = 11$ **b.** $a = 160, k = -1$

19. $a = \pm\left(3^6 \times 2^{-\frac{11}{2}}\right), n = 6$

20. $y = 2^{20}$

TOPIC 4 Exponential functions 1

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



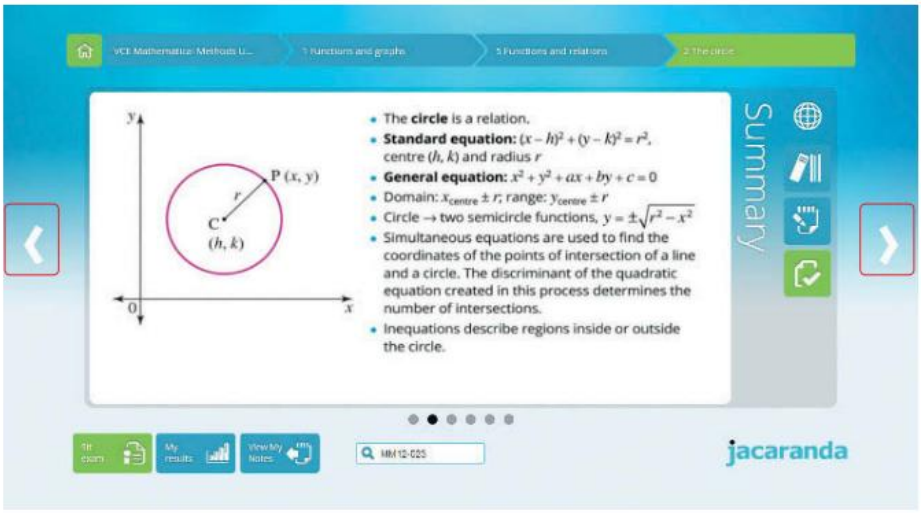
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 8
Geometric sequences

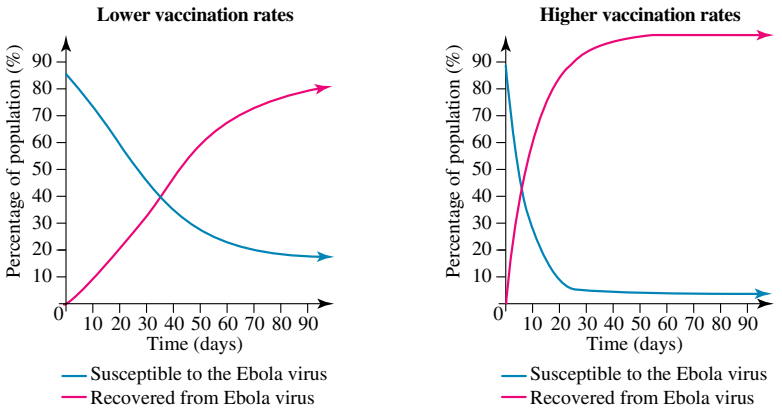
8.1 Overview

8.1.1 Introduction

Geometric sequences and exponential growth or exponential decay can be observed in population and bacterial growth. A simple biology experiment involves spreading a sample on a petri dish and observing the growth of bacterial cultures. The rate of initial growth is typically observed to be exponential in nature, until it slows down due to running out of food and/or space, or by interacting with conflicting colonies. Following a stationary phase, the death of a bacterial colony is typically observed to follow exponential decay.

In a similar vein, mathematical models were used during the West African Ebola virus epidemic, which occurred between 2013 and 2016. In trial vaccination programs, models attempted to estimate the percentages of the population susceptible to, and recovered from, the Ebola virus, depending upon the speed with which the vaccinations were administered. The results of the vaccine program demonstrated exponential growth and exponential decay can be observed in the number of people susceptible to the Ebola virus and the number of people recovered from the Ebola virus respectively. The results show the significance of administering early vaccinations and gives us an idea of how a population will react to and recover from such an epidemic.

Exponential growth and exponential decay can be observed in the number of people susceptible to the Ebola virus and the number of people recovered from the Ebola virus respectively. These graphs show the significance of administering early vaccinations and gives us an idea of how a population will react to and recover from such an epidemic.



LEARNING SEQUENCE

8.1 Overview
8.2 Recursive definition and the general term of geometric sequences
8.3 The sum of a geometric sequence
8.4 Geometric sequences in context
8.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Recursive definition and the general term of geometric sequences

8.2.1 Common ratio of geometric sequences

A **geometric sequence** is one in which the first term is multiplied by a number, known as the **common ratio**, to create the second term, which is then multiplied by the common ratio to create the third term, and so on. This creates a sequence of numbers whose consecutive terms increase or decrease in the same ratio.

For example, 1, 3, 9, 27, 81is a geometric sequence because each term is obtained by multiplying the previous term by 3. Alternatively, 1, 3, 6, 10, 15 is not a geometric sequence because each consecutive term is not increasing by the same ratio. We use the same notation to represent geometric sequences as for arithmetic sequences (see Chapter 1).

The terms of a geometric sequence are written as $t_1, t_2, t_3, \ldots, t_n$. The common ratio is denoted r.

WORKED EXAMPLE 1

Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of t_1 and r.

a. **20, 40, 80, 160, 320, …**

b. **8, 4, 2, 1, $\frac{1}{2}$, ...**

c. **3, −9, 27, −81, …**

d. **2, 4, 6, 8, 10, …**

THINK	WRITE
a. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.	**a.** $\frac{t_2}{t_1} = \frac{40}{20}$ $= 2$ $\frac{t_3}{t_2} = \frac{80}{40}$ $= 2$ $\frac{t_4}{t_3} = \frac{160}{80}$ $= 2$ $\frac{t_5}{t_4} = \frac{320}{160}$ $= 2$
2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is t_1 and the common difference is r.	The ratios between consecutive terms are all 2, so this is a geometric sequence. $t_1 = 20, r = 2$

b. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

b. $\frac{t_2}{t_1} = \frac{4}{8}$

$= \frac{1}{2}$

$\frac{t_3}{t_2} = \frac{2}{4}$

$= \frac{1}{2}$

$\frac{t_4}{t_3} = \frac{1}{2}$

$\frac{t_5}{t_4} = \frac{\left(\frac{1}{2}\right)}{1}$

$= \frac{1}{2}$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is t_1 and the common difference is r.

The ratios between consecutive terms are all $\frac{1}{2}$ so this is a geometric sequence.

$t_1 = 8, r = \frac{1}{2}$

c. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

c. $\frac{t_2}{t_1} = \frac{-9}{3}$

$= -3$

$\frac{t_3}{t_2} = \frac{27}{-9}$

$= -3$

$\frac{t_4}{t_3} = \frac{-81}{27}$

$= -3$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is t_1 and the common difference is r.

The ratios between consecutive terms are all -3, so this is a geometric sequence.

$t_1 = 3, r = -3$

d. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

d. $\frac{t_2}{t_1} = \frac{4}{2}$

$= 2$

$\frac{t_3}{t_2} = \frac{6}{4}$

$= \frac{3}{2}$

$\frac{t_4}{t_3} = \frac{8}{6}$

$= \frac{4}{3}$

$$\frac{t_5}{t_4} = \frac{10}{8}$$
$$= \frac{5}{4}$$

2. If the ratios between consecutive terms are constant, then the sequence is geometric.

All of the ratios between consecutive terms are different, so this is not a geometric sequence.

8.2.2 The recursive definition of a geometric sequence

Since each consecutive term in a geometric sequence increases by the same ratio, we can establish a recursive function of a geometric sequence as follows:

$$t_{n+1} = rt_n$$

where r is the common ratio. This is also a first-order recurrence relation because each term depends on the previous one.

WORKED EXAMPLE 2

Determine the recursive functions that represent the following geometric sequences.
a. 4, 16, 64, 256, 1024 **b. 81, −27, 9, −3, 1**

THINK

a. 1. Determine the value of r.

WRITE

$$r = \frac{t_2}{t_1}$$
$$= \frac{16}{4}$$
$$= 4$$

2. Substitute the value of r into the recursive formula for geometric sequences.

$$t_{n+1} = rt_n$$
$$t_{n+1} = 4t_n$$

b. 1. Determine the value of r.

$$r = \frac{t_2}{t_1}$$
$$= \frac{-27}{81}$$
$$= -\frac{1}{3}$$

2. Substitute the value of r into the recursive formula for geometric sequences.

$$t_{n+1} = rt_n$$
$$t_{n+1} = -\frac{1}{3}t_n$$

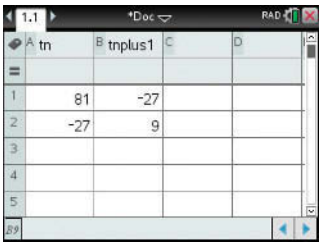
TI | THINK

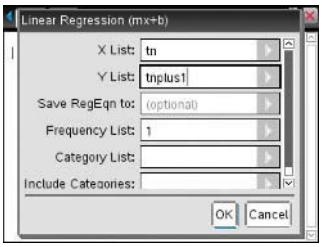
b. 1. On a Lists & Spreadsheet page, label the first column *tn* (t_n) and the second column *tnplus1* (t_{n+1}). Enter the first and second terms of the given sequence in the first column, then enter the second and third terms in the second column.

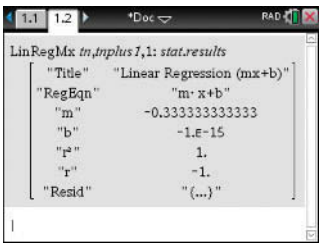
2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
3: Linear Regression ($mx + b$) …
Complete the fields as
X List: tn
Y List: tnplus1
then select OK.

3. The answer appears on the screen.

WRITE







The answer is given in the form $y = mx + b$, where $y = t_n$, $m = -\frac{1}{3}$, $x = t_{n+1}$ and $b = 0$.

$t_{n+1} = -\frac{1}{3}t_n$.

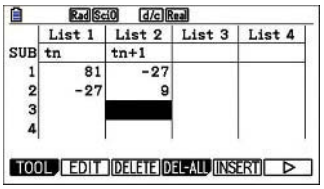
CASIO | THINK

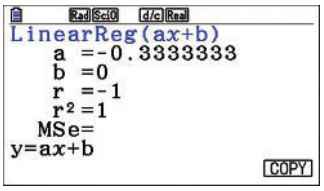
b. 1. On a Statistics screen, label List 1 as *tn* (t_n) and List 2 as *tn* + 1 (t_{n+1}). Enter the first and second terms of the given sequence in the first column, then enter the second and third terms in the second column.

2. Select CALC by pressing F2, then select REG by pressing F3. Select X by pressing F1, then select $ax + b$ by pressing F1.

3. The answer appears on the screen.

WRITE





The answer is given in the form $y = ax + b$, where $y = t_n$, $a = -\frac{1}{3}$, $x = t_{n+1}$ and $b = 0$.

$t_{n+1} = -\frac{1}{3}t_n$.

8.2.3 The general term of the geometric sequence

We can use the recursive definition of a geometric sequence to come up with a formula for the general term of a geometric sequence. Let us start by writing down the recurrence relation for values up to $n = 3$.

$$n = 1 \quad t_2 = rt_1$$
$$n = 2 \quad t_3 = rt_2$$
$$n = 3 \quad t_4 = rt_3$$

Substituting the first equation into the second, and then the result of that into the third, gives the following progression.

$$t_2 = rt_1$$

$$\begin{aligned} t_3 &= rt_2 \\ \text{Substitute } t_2 &= rt_1 \\ \Rightarrow t_3 &= r(rt_1) \\ &= r^2t_1 \end{aligned}$$

$$
\begin{aligned}
t_4 &= rt_3 \\
\text{Substitute } t_3 &= r^2 t_1 \\
\Rightarrow t_4 &= r\left(r^2 t_1\right) \\
&= r^3 t_1
\end{aligned}
$$

Once we've written t_2, t_3, t_4 in terms of t_1, a pattern emerges.

$$
\begin{aligned}
t_2 &= rt_1 \\
t_3 &= r^2 t_1 \\
t_4 &= r^3 t_1
\end{aligned}
$$

We can see that for each t_n on the left-hand side, the exponent of r is equal to $n - 1$. This allows us to establish the following formula for the general term of a geometric sequence.

$$t_n = t_1 r^{n-1}$$

This formula can be used to determine any term in a geometric sequence provided we know the values of t_1 and r.

on Resources

Interactivity: Terms of a geometric sequence (int-6260)

WORKED EXAMPLE 3

Determine the equations that represent the following geometric sequences.

a. 7, 28, 112, 448, 1792, …

b. 8, −4, 2, −1, $\frac{1}{2}$, …

THINK	WRITE
a. 1. Determine the values of t_1 and r.	**a.** $t_1 = 7$ $r = \frac{t_2}{t_1}$ $= \frac{28}{7}$ $= 4$
2. Substitute the values for t_1 and r into the formula for geometric sequences.	$t_n = t_1 r^{n-1}$ $= 7 \times 4^{n-1}$

b. 1. Determine the values of t_1 and r.

b. $t_1 = 8$

$$r = \frac{t_2}{t_1} = \frac{-4}{8} = -\frac{1}{2}$$

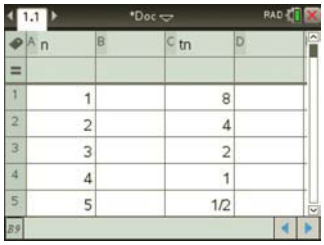
2. Substitute the values for t_1 and r into the formula for geometric sequences.

$$t_n = t_1 r^{n-1} = 8 \times \left(-\frac{1}{2}\right)^{n-1}$$

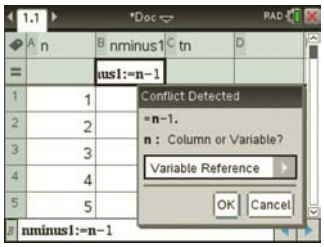
TI | THINK

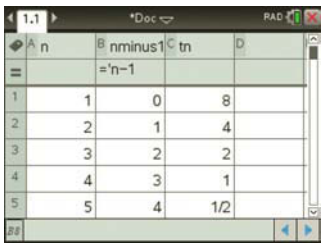
WRITE

b. 1. On a Lists & Spreadsheet page, label the first column n and the third column tn (t_n). Enter the numbers 1 to 5 in the first column, and the magnitudes (ignore the sign) of terms of the given sequence in the third column.

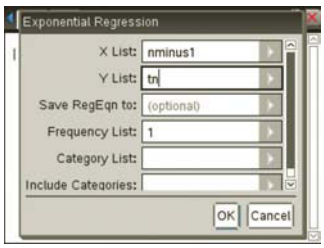


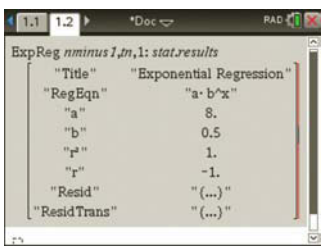
2. Label the second column $nminus1$ ($n - 1$), then select the function entry cell below the label $nminus1$ and complete the entry line as $= n - 1$, then press ENTER. Select the Variable reference for n when prompted, then select OK.





3. On a Calculator page, press MENU then select
6: Statistics1: Stat Calculations
A: Exponential Regression …
Complete the fields as
X List: nminus1
Y List: tn
then select OK.

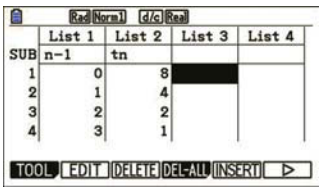




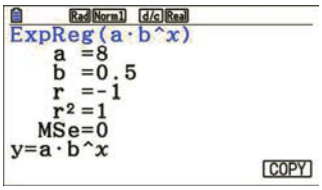
CASIO | THINK

WRITE

b. 1. On a Statistics screen, label List 1 as $n - 1$ and List 2 as tn (t_n). Enter the numbers 0 to 4 in the first column, and the magnitudes (ignore the sign) of terms of the given sequence in the second column.



2. Select CALC by pressing F2, then select REG by pressing F3. Press F6 to move to across to more menu options, select EXP by pressing F2, then select ab^x by pressing F2.



3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 8$, $b = -\frac{1}{2}$ (the common ratio needs to be negative as the signs of the terms in the sequence alternate), and $x = n - 1$.

$$t_n = 8 \times \left(-\frac{1}{2}\right)^{n-1}.$$

4. The answer appears on the screen.	The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 8$, $b = -\frac{1}{2}$ (the common ratio needs to be negative as the signs of the terms in the sequence alternate), and $x = n - 1$. $t_n = 8 \times \left(-\frac{1}{2}\right)^{n-1}$.

WORKED EXAMPLE 4

a. Find the 20th term of the geometric sequence with $t_1 = 5$ and $r = 2$.
b. A geometric sequence has a first term of 3 and a 20th term of 1 572 864. Find the common ratio between consecutive terms of the sequence.
c. Find the first term of a geometric series with a common ratio of 2.5 and a 5th term of 117.1875.

THINK	WRITE
a. 1. Identify the known values in the question.	**a.** $t_1 = 5$ $r = 2$ $n = 20$
2. Substitute these values into the geometric sequence formula and solve to find the missing value.	$t_n = t_1 r^{n-1}$ $t_{20} = 5 \times 2^{20-1}$ $= 5 \times 2^{19}$ $= 2\,621\,440$
3. Write the answer.	The 20th term of the sequence is 2 621 440.
b. 1. Identify the known values in the question.	**b.** $t_{20} = 1\,572\,864$ $t_1 = 3$ $n = 20$
2. Substitute these values into the formula to calculate the common ratio and solve to find the missing value.	$r = \left(\frac{t_n}{t_1}\right)^{\frac{1}{n-1}}$ $= \left(\left(\frac{1\,572\,864}{3}\right)\right)^{\frac{1}{20-1}}$ $= 524\,288^{\frac{1}{19}}$ $= 2$
3. Write the answer.	The common ratio between consecutive terms of the sequence is 2.
c. 1. Identify the known values in the question.	**c.** $t_5 = 117.1875$ $r = 2.5$ $n = 5$

2. Substitute these values into the formula to calculate the first term and solve to find the missing value.

$$t_1 = \frac{t_n}{r^{n-1}}$$
$$= \frac{117.1875}{2.5^{5-1}}$$
$$= \frac{117.1875}{2.5^4}$$
$$= \frac{117.1875}{39.0625}$$
$$= 3$$

3. Write the answer.

The first term of the sequence is 3.

TI | THINK

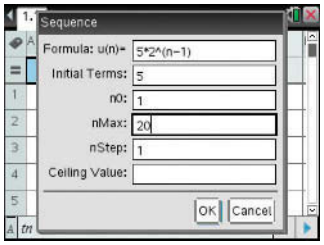
a. 1. Define the rule for the sequence from the given information.

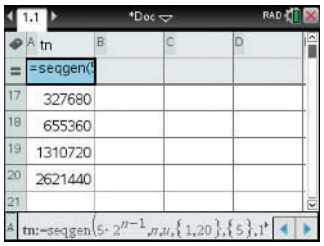
2. On a Lists & Spreadsheet page, label the first column $tn\,(t_n)$. Select the function cell below *tn*, and press MENU then select
3: Data
1: Generate Sequence
Complete the fields as
Formula:
$u\,(n) = 5 \times 2\text{^}(n-1)$
Initial Terms: 5
n0: 1
nMax: 20
nStep: 1
then select OK.

3. The answer appears on the screen.

WRITE

$t_n = t_1 r^{n-1}$
$= 5 \times 2^{n-1}$





The answer can be found in cell A20:
$t_{20} = 2621440$.

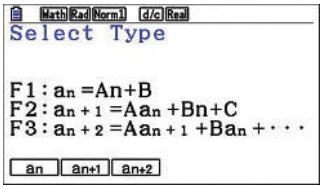
CASIO | THINK

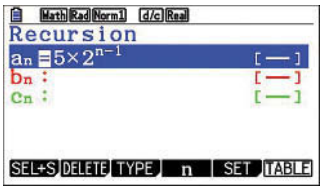
a. 1. Define the rule for the sequence from the given information.

2. On a Recursion screen, select TYPE by pressing F3, then select the first option by pressing F1. Complete the entry line for a_n as $a_n = 5 \times 2^{n-1}$ then press EXE.
Note: For *n* press F1.

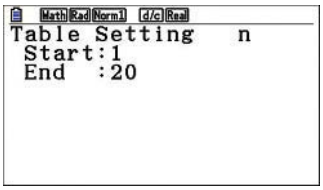
WRITE

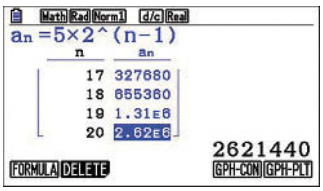
$t_n = t_1 r^{n-1}$
$= 5 \times 2^{n-1}$





3. Select SET by pressing F5, then compete the fields as
Start: 1
End: 20
and press ENTER.
Select TABLE by pressing F6.

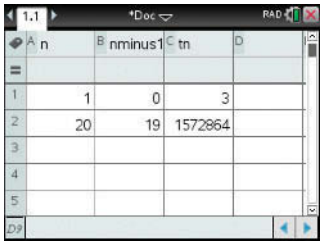




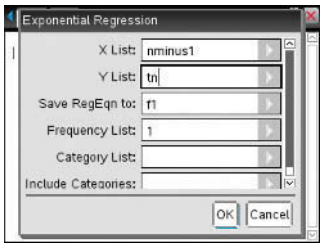
4. The answer appears on the screen.

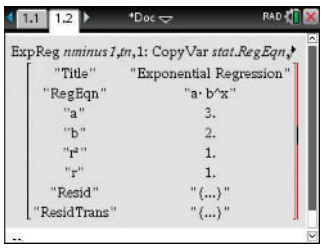
The answer can be found in the last row:
$t_{20} = 2621440$.

b. 1. On a Lists & Spreadsheet page, label the first column as n, the second column as *nminus1* $(n-1)$ and the third column as *tn* (t_n). Enter the numbers 1 and 20 in the first column, the numbers 0 and 19 in the second column, and the numbers 3 and 1572864 in the third column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
A: Exponential Regression …
Complete the fields as
X List: nminus1
Y List: tn
then select OK.

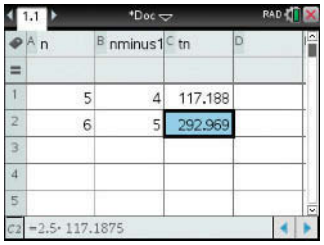




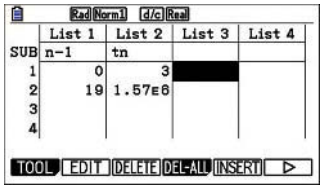
3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2$ (the common ratio) , and $x = n - 1$. Hence the common ratio between consecutive terms is 2.

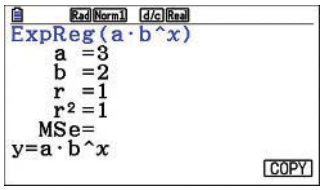
c. 1. On a Lists & Spreadsheet page, label the first column as n, the second column as *nminus1* $(n-1)$ and the third column as *tn* (t_n). Enter the numbers 5 and 6 in the first column, the numbers 4 and 5 in the second column, and the numbers 117.1875 and 2.5×117.1875 in the third column.



b. 1. On a Statistics screen, label the first column as $n - 1$ and the second column as *tn* (t_n). Enter the numbers 0 and 19 in the first column, and the numbers 3 and 1572864 in the second column.



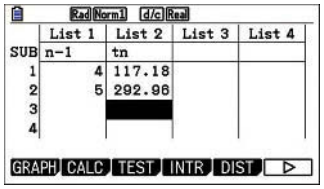
2. Select CALC by pressing F2, then select REG by pressing F3. Press F6 to move to across to more menu options, select EXP by pressing F2, then select ab^x by pressing F2.



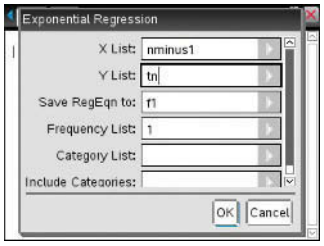
3. The answer appears on the screen.

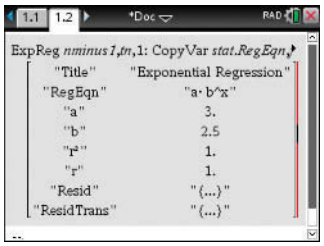
The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2$ (the common ratio) , and $x = n - 1$. Hence the common ratio between consecutive terms is 2.

c. 1. On a Statistics screen, label the first column as $n - 1$ and the second column as *tn* (t_n). Enter the numbers 4 and 5 in the first column, and the numbers 117.1875 and 2.5×117.1875 in the third column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
A: Exponential Regression …
Complete the fields as
X List: nminus1
Y List: tn
then select OK.

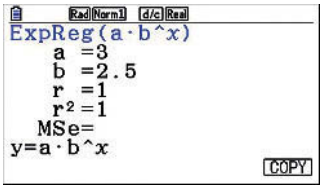




3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2.5$ (the common ratio), and $x = n - 1$. Hence, the first term of the sequence is 3.

2. Select CALC by pressing F2, then select REG by pressing F3. Press F6 to move to across to more menu options, select EXP by pressing F2, then select ab^x by pressing F2.



3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2.5$ (the common ratio), and $x = n - 1$. Hence, the first term of the sequence is 3.

studyon

Units 1 & 2 | Area 5 | Sequence 1 | Concept 1

The recursive definition and general term of a geometric sequence Summary screen and practice questions

Exercise 8.2 Recursive definition and the general term of geometric sequences

Technology free

1. **WE1** Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of t_1 and r.
   a. 3, 6, 12, 24, 48, … **b.** $\frac{1}{2}, \frac{5}{4}, \frac{25}{8}, \frac{125}{16}, \ldots$ **c.** 9, 6, 3, 0, −3, … **d.** $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, \ldots$
2. Find the missing values in the following geometric sequences.
   a. 1, 6, c, 216, 1296 **b.** 3, g, h, −24, 48 **c.** p, q, s, 300, 1500 **d.** −6, 24, u, w
3. **WE2** Determine the recursive functions that represent the following geometric sequences.
   a. −2, −6, −18, −54, … **b.** 512, −128, 32, −8, 2, …
   c. 0.12, 0.6, 3, 15, … **d.** $\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \ldots$
4. Determine the first four terms of the geometric sequences represented by the following recursive functions.
   a. $t_{n+1} = -2t_n$, $t_1 = 0.13$ **b.** $t_{n+1} = \frac{1}{3}t_n$, $t_1 = 2$
5. What is the value of x in the following geometric sequences?
   a. x, 14, 28, … **b.** $2x$, $4x$, $8 + 6x$, …
   c. $x + 1$, $3x + 3$, $10x + 5$, … **d.** $2 - x$, $3x - 6$, $9x$, …

6. **WE3** Determine the equations that represent the following geometric sequences.
   a. $-1, -5, -25, -125, -625, \ldots$
   b. $7, -3.5, 1.75, -0.875, 0.4375, \ldots$
   c. $\frac{5}{6}, \frac{5}{9}, \frac{10}{27}, \frac{20}{81}, \frac{40}{243}, \ldots$

Technology active

7. Determine the first 5 terms of the following geometric sequences.
   a. $t_n = (-2)^{n-1} \times \frac{1}{3}$
   b. $t_n = \left(\frac{3}{5}\right)^{n-1} \times \frac{5}{3}$
   c. $t_n = \left(-\frac{2}{5}\right)^{n-1} \times \frac{3}{8}$
   d. $t_n = \left(-\frac{2}{7}\right)^{n-1} \times \frac{4}{9}$

8. a. **WE4** Determine the 15th term of the geometric sequence with $t_1 = 4$ and $r = 3$.
   b. A geometric sequence has a first term of 2 and a 12th term of 97 656 250. Find the common ratio between consecutive terms of the sequence.
   c. Determine the first term of a geometric series with a common ratio of $-\frac{1}{2}$ and a 6th term of 13.125.

9. a. Determine the 11th term of the geometric sequence with a first value of 1.2 and a common ratio of 4.
   b. A geometric sequence has a first term of -1.5 and a 10th term of 768. Find the common ratio between consecutive terms of the sequence.
   c. Determine the first term of a geometric series with a common ratio of 0.4 and a 6th term of 6.5536.

10. a. Determine the first four terms of the geometric sequence where the 6th term is 243 and the 8th term is 2187.
    b. Determine the first four terms of the geometric sequence where the 3rd term is 331 and the 5th term is 8275.

11. A geometric sequence has a 1st term of 200 and a 6th term of 2.048. Identify the values of the 2nd, 3rd, 4th and 5th terms.



12. The 1st term of a geometric sequence is 13 and the 3rd term of the same sequence is 117.
    a. Explain why there are two possible values for the common ratio of the sequence.
    b. Calculate both possible values of the 6th term of the sequence.

13. Identify the missing values in the following geometric sequences. There may be more than one solution for each missing value.
    a. $1, a, 4, b, 16$
    b. $-2, e, f, -54$
    c. $\frac{1}{4}, j, k, -128$
    d. $\frac{1}{8}, s, t, u, 2$



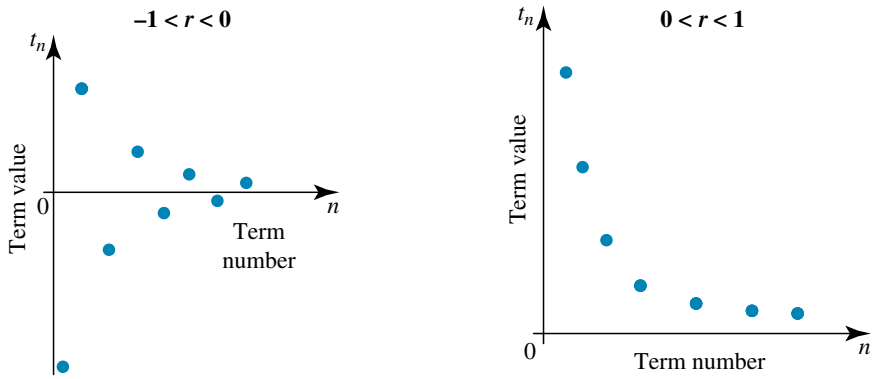
14. Write down a sequence that can be considered both an arithmetic and a geometric sequence. For such sequences, what are the restrictions (if any) on the first term t_1, the common ratio r and the common difference d?

8.3 The sum of a geometric sequence

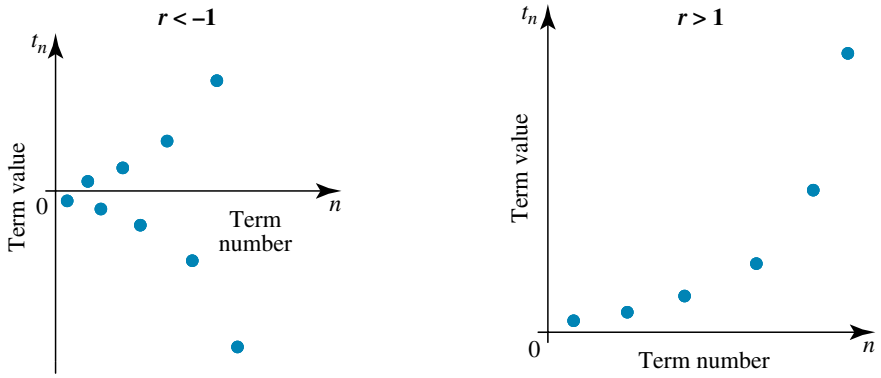
8.3.1 Limiting behaviour as $n \to \infty$

The common ratio r is key to understanding what happens when we progress further down a geometric sequence.

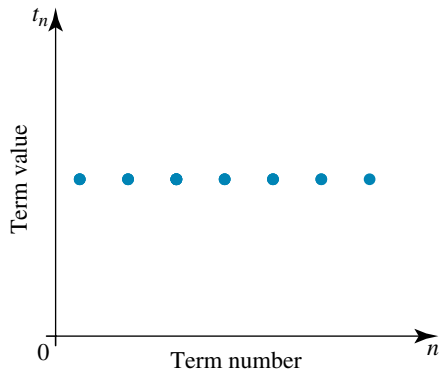
If $-1 < r < 1$, then the terms of a geometric series become smaller and approach zero.



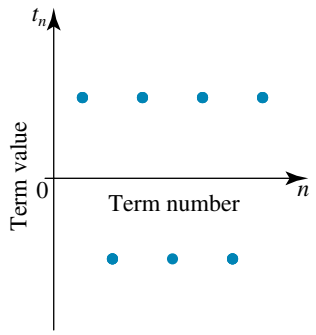
If $r < -1$ or $r > 1$, then the terms of a geometric series move from the starting point at an exponential rate.



If $r = 1$, then every term of the geometric series is the same.



If $r = -1$, then the terms oscillate between two values.



on Resources

Interactivity: Geometric sequences (int-6259)

8.3.2 The sum of the first *n* terms of a geometric sequence

When the terms of a geometric sequence are added, a **geometric series** is formed.

So 3, 6, 12, 24, 48, ... is a geometric sequence whereas $3 + 6 + 12 + 24 + 48 + \ldots$ is a geometric series.

The sum of the first n terms of a geometric sequence is given by S_n and it is calculated using the equation:

$$S_n = t_1 \frac{r^n - 1}{r - 1}$$

where t_1 is the first term of the geometric sequence and r is the common ratio.

To understand how this formula is derived, let us start by writing the summation in terms of $t_1, t_2, \ldots, t_n$.

$$S_n = t_1 + t_2 + \ldots + t_{n-1} + t_n$$

Using the formula for a general term of a geometric sequence, we can write every term on the right-hand side in terms of t_1.

$$S_n = t_1 + rt_1 + \cdots + r^{n-2}t_1 + r^{n-1}t_1 \qquad [1]$$

Multiplying both sides by r we get the following.

$$rS_n = rt_1 + r^2t_1 + \cdots + r^{n-2}t_1 + r^{n-1}t_1 + r^nt_1 \; [2]$$

Subtracting [2] from [1] gives us the following.

$$rS_n - S_n = r^nt_1 - t_1$$
$$\Rightarrow S_n\,(r-1) = t_1\,(r^n - 1)$$

And, finally, rearranging this equation to make S_n the subject:

$$S_n = t_1 \frac{(r^n - 1)}{r - 1}$$

This formula is useful if $r < -1$ or $r > 1$, for example, if r is 2, 10, 3.3, -4, -1.2.
By calculating $S_n - rS_n$ instead of $rS_n - S_n$, we obtain an alternative form of the formula. That is:

$$S_n - rS_n = t_1 - t_1 r^n$$
$$S_n (1 - r) = t_1 (1 - r^n)$$
$$S_n = t_1 \frac{(1 - r^n)}{1 - r}$$

This formula is useful if r is in between -1 and 1 (shown as $-1 < r < 1$).

WORKED EXAMPLE 5

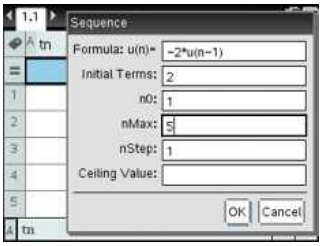
a. Calculate the sum of the first five terms of the geometric sequence with $t_1 = 2$ and $r = -2$.
b. The sum of the first four terms of a geometric sequence is 10 and the common ratio is $r = 3$. What is the value of the first term?

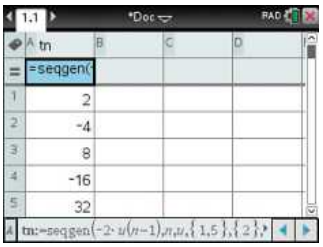
THINK	WRITE
a. 1. State the known values.	$t_1 = 2, r = -2, n = 5$
2. Substitute the values into the equation and solve for the answer.	$S_n = t_1 \frac{(r^n - 1)}{r - 1}$
	$S_5 = 2 \frac{((-2)^5 - 1)}{-2 - 1}$
	$= 2 \times \frac{-33}{-3}$
	$= 2 \times 11$
	$S_5 = 22$
b. 1. State the known values.	$n = 4, r = 3, S_4 = 10$
2. Substitute the known values into the equation and solve for the answer.	$S_n = t_1 \frac{(r^n - 1)}{r - 1}$
	$10 = t_1 \frac{(3^4 - 1)}{3 - 1}$
	$10 = t_1 \frac{80}{2}$
	$10 = t_1 \times 40$
	$t_1 = \frac{10}{40}$
	$t_1 = \frac{1}{4}$

TI | THINK

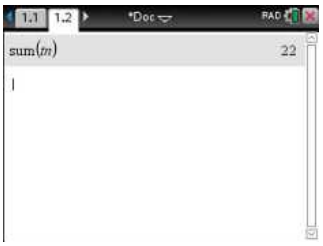
a. 1. On a Lists & Spreadsheet page, label the first column *tn* (t_n). Select the function cell below *tn*, and press MENU then select
3: Data
1: Generate Sequence
Complete the fields as
Formula:
$u(n) = -2 \times u(n-1)$
Initial Terms: 2
n0: 1
nMax: 5
nStep: 1
then select OK.

WRITE





2. On a Calculator page, press MENU then select
6: Statistics
3: List Math
5: Sum of Elements.
Press VAR then select *tn* from the list and press ENTER.



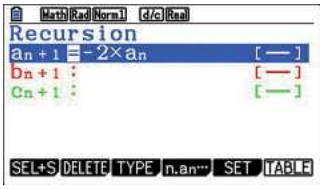
3. The answer appears on the screen.

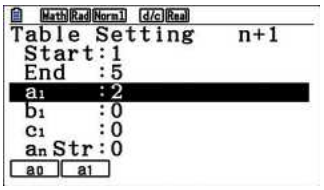
$S_5 = 22$

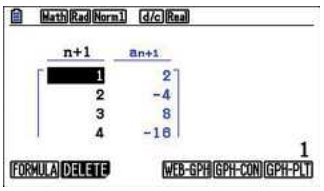
CASIO | THINK

a. 1. On a Recursion screen, select TYPE by pressing F3, then select the second option by pressing F2. Complete the entry line for a_n as $a_{n+1} = -2 \times a_n$ then press EXE.
Note: For *n* press F1.
Select SET by pressing F5 and complete the fields as
Start: 1
End: $5a_1$:2
then press EXE. Select TABLE by pressing F6.

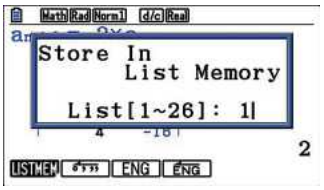
WRITE







2. Press OPTN then select LISTMEM by pressing F1. Type '1' then press EXE. On the Run-Matrix screen, press OPTN then select LIST by pressing F1. Press F6 twice to scroll to more menu options, then select Sum by pressing F1. Press F6 to scroll to more menu options and select List by pressing F1. Type '1' then press EXE.



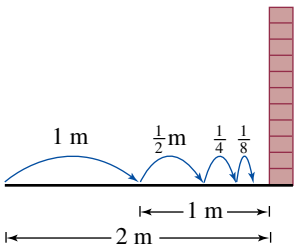


3. The answer appears on the screen.

$S_5 = 22$

8.3.3 The sum of an infinite geometric sequence

If you are 2 metres away from a wall and you move 1 metre (or half-way) towards the wall and then move $\frac{1}{2}$ metre (or half-way again) towards the wall and continue to do this, will you reach the wall? When will you reach the wall?



Consider the following geometric sequence:

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ This is an infinite geometric sequence since it continues on with an infinite number of terms.

Each term in the sequence is less than the previous term by a factor of $\frac{1}{2}$, that is, $r = 0.5$.

If we were to add n terms of this sequence together, we would have:

$$\begin{aligned} s_n &= \frac{1 \times (1 - 0.5^n)}{1 - 0.5} \\ &= \frac{1 - 0.5^n}{0.5} \\ &= \frac{1}{0.5} - \frac{0.5^n}{0.5} \\ &= 2 - 0.5^{n-1} \end{aligned}$$

Consider 0.5^{n-1} in the above equation. As n becomes very large, the term 0.5^{n-1} becomes very small. Try this with your calculator.

Let $n = 5$, $0.5^{n-1} = 0.5^4 = 0.0625$; therefore, $S_5 = 2 - 0.0625 = 1.9375$

Let $n = 10$, $0.5^{n-1} = 0.5^9 = 0.001\,95$; therefore, $S_{10} = 2 - 0.001\,95 = 1.998\,05$

Let $n = 20$, $0.5^{n-1} = 0.5^{19} = 0.000\,001\,9$; therefore, $S_{20} = 2 - 0.000\,001\,9 = 1.999\,998\,1$

We can see that as n becomes larger, 0.5^{n-1}becomes smaller. If n were to approach infinity (note that you can never reach infinity, you can only approach it), then the value of 0.5^{n-1} would approach zero. So, $S_n = 2 - 0.5^{n-1}$ would become $S_\infty = 2$.

It is possible to generalise this in order to find the sum of an infinite geometric sequence. We use the symbol S_∞which is referred to as the sum to infinity of a geometric sequence.

We start by writing out the summation in terms of t_1:

$$\begin{aligned} S_\infty &= t_1 + t_2 + t_3 + \ldots \\ \Rightarrow S_\infty &= t_1 + rt_1 + r^2 t_1 + \ldots \end{aligned}$$

Factorising r on the right-hand side gives us:

$$S_\infty = r\left(r^{-1}t_1 + t_1 + rt_1 + r^2 t_1 + \ldots\right)$$

We can see that:

$$\begin{aligned} S_\infty &= r\left(r^{-1}t_1 + S_\infty\right) \\ \Rightarrow S_\infty &= t_1 + rS_\infty \end{aligned}$$

Rearranging to make S_∞ the subject:

$$\begin{aligned} S_\infty - rS_\infty &= t_1 \\ S_\infty(1 - r) &= t_1 \end{aligned}$$

Finally:

$$S_\infty = \frac{t_1}{1 - r}$$

As $n \to \infty$, the sum a geometric sequence in terms of n is given by S_∞ and is calculated using the following equation.

$$\boldsymbol{S_\infty = \frac{t_1}{1 - r}}$$

where t_1 is the first term of the geometric sequence and r is the common ratio.

Remember that this equation only provides a non-infinite solution when $-1 < r < 1$.

WORKED EXAMPLE 6

a. Calculate the sum of the infinite geometric sequence with $t_1 = 3$ and $r = \frac{1}{3}$.

b. Calculate the sum of the infinite geometric sequence 8, 4, 2, 1, …

THINK	WRITE
a. 1. Substitute the values into the equation.	$S_\infty = \frac{t_1}{1-r}$ $= \frac{3}{1-\frac{1}{3}}$
2. Multiply the numerator and denominator of the fraction by the denominator of r.	$= \frac{3}{1-\frac{1}{3}} \times \frac{3}{3}$ $= \frac{9}{3-1}$ $= \frac{9}{2}$
b. 1. Determine r and t_1 from the provided sequence.	$t_1 = 8$ $r = \frac{t_2}{t_1} = \frac{4}{8} = \frac{1}{2}$
2. Substitute r and t_1 into the equation.	$S_\infty = \frac{t_1}{1-r}$ $= \frac{8}{1-\frac{1}{2}}$ $= \frac{8}{1-\frac{1}{2}} \times \frac{2}{2}$ $= \frac{16}{2-1}$ $= 16$

studyon

Units 1 & 2 > Area 5 > Sequence 1 > Concept 2

The sum of a geometric sequence Summary screen and practice questions

Exercise 8.3 The sum of a geometric sequence

Technology free

1. **WE5** Calculate the sum of the first five terms of the geometric sequence with $t_1 = -4$ and $r = -2$.
2. The sum of the first four terms of a geometric sequence is 10 and the common ratio is $r = 3$. What is the value of the first term?
3. Calculate the sum of the following infinite geometric sequences:
   a. $56, -8, \frac{8}{7}, -\frac{8}{49}, \ldots$
   b. $-12, -2, -\frac{1}{3}, -\frac{1}{18}, \ldots$

4. Calculate the sum of the infinite geometric sequence with $t_1 = -3$ and $r = \frac{5}{9}$.
5. Calculate the sum of the following infinite geometric sequences.
   a. $4, \frac{5}{2}, \frac{25}{16}, \frac{125}{128}, \ldots$
   b. $21, -9, \frac{27}{7}, -\frac{81}{49}, \ldots$
6. A geometric sequence has $t_1 = -10$ and $S_2 = -5$. What is the value of the common ratio?
7. A geometric sequence has $t_1 = 3$ and $S_3 = 9$. What are the two possible values of the common ratio?
8. A geometric sequence has $t_1 = 5$ and $S_3 = \frac{35}{4}$. Assuming that r is positive, what are the third and fourth terms of the geometric sequence?
9. A geometric sequence has $t_1 = 4$, $r = -2$ and the sum of the first n terms is $S_n = 172$. What is the value of n?

Technology active

10. On the first day Jenny hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.
    a. Write the geometric sequence for the first five days of the above real-life situation.
    b. Find the value of r.
    c. How many people are told of the rumour on the 12th day?



11. Decay of radioactive material is modelled as a geometric sequence where $r = \frac{1}{2}$. If there are initially 20 million radioactive atoms, write the first 7 terms of the sequence.
12. A failing machine produces 35 metres of spouting in the first hour, 21 metres in the second hour and 12.6 in the third hour. If this pattern continues and 280 metres of spouting is required, how far short of the quota will the machine fall?
13. a. Find the sum of the first 12 terms of the geometric sequence 2, 6, 18, 54, 162, ...
    b. Find the sum of the first 7 terms of the geometric sequence 5, 35, 245, 1715, 12 005, ...
    c. Find the sum of the first 15 terms of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, ...
    d. Find the sum of the first 11 terms of the geometric sequence 3.1, 9.3, 27.9, 83.7, 251.1, ...
14. The second term of a geometric sequence is -20 and the fifth is -1280. What is the sum of the first 12 terms of the sequence?
15. **WE6** Calculate the sum of the infinite geometric sequence with $t_1 = 20$ and $r = \frac{1}{10}$.
16. A nail penetrates 20 mm with the first hit of a hammer, 12 mm with the 2nd hit and 7.2 mm with the 3rd. If this pattern continues, will the 50 mm long nail ever be completely hammered in?



17. An infinite geometric sequence is such that $S_\infty = 120$. What is the range of possible values of t_1?
18. What happens to the sum of an infinite geometric series when $r = -1$?
    Let us start by exploring the infinite sum $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$, which can be represented by a geometric sequence with $t_1 = 1$ and $r = -1$.
    a. Evaluate the summation by applying brackets to adjacent pairs of $+1$ and -1. How many different finite results can you obtain? What conclusions can you draw from this?
    b. Let us assume the infinite sum has an arbitrary finite value, for example, $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$. What do we know about $1 - S$? How can we use this result to find a value for S and what conclusions can we draw from this?

8.4 Geometric sequences in context

8.4.1 Growth and decay in the real world

Growth and decay of discrete variables is constantly found in real-life situations. Some examples are increasing or decreasing populations and an increase or decrease in financial investments. Some of these geometric models are presented here.

WORKED EXAMPLE 7

A computer system decreases in value each year by 15% of the previous year's value. Write an expression for the value of the computer, which shall be referred to as V_n, at the beginning of n years. Its initial purchase price is given as $V_1 = \$12\,000$.

THINK	WRITE
1. This is a geometric sequence since there is a 15% decrease on the previous year's value. Find t_1 and r. *Note:* Since this is a decreasing value, r is a value less than 1.	$t_1 = 12\,000$ $r = 1 - 15\%$ $= 1 - 0.15$ $= 0.85$
2. We want an expression for the value at the beginning of n years. Use $t_n = t_1 r^{n-1}$ which gives the value of the nth term. Use V_n instead of t_n.	$V_n = 12\,000 \times (0.85)^{n-1}$
3. Write your answer.	The value of the computer is given by the expression $V_n = 12\,000\,(0.85)^{n-1}$.

8.4.2 Compound interest

We can invest money over an extended period of time — typically several years — and let it grow exponentially on its own by taking advantage of **compound interest**. The amount of money invested grows by a ratio per given period, meaning that we can model this growth using a geometric sequence.

Compound interest is calculated by using the formula $A = P(1 + i)^n$, where:

- A is the total amount of the investment
- P is the principal
- i is the percentage rate of interest per compounding period divided by 100
- n is the number of compounding periods.

We can see similarities between this compound interest formula and the formula for the general term of a geometric sequence $t_n = r^{n-1}t_1$, where:

- A is equivalent to t_n
- $(1 + i)$ is equivalent to the common ratio
- P is equivalent to t_1.

WORKED EXAMPLE 8

Alexis puts $2000 into an investment account that earns compound interest at a rate of 0.5% per month.

a. Set up an equation that represents Alexis's situation as a geometric sequence, where t_n is the amount in Alexis' account after n months.

b. Use your equation from part a to determine the amount in Alexis's account at the end of each of the first 6 months.

c. Calculate the amount in Alexis's account at the end of 15 months.



THINK	WRITE
a. 1. Determine the amounts in the account after each of the first two months.	**a.** Let $n = 1$: $A = (1 + i)^n \times P$ $= 2000\left(1 + \frac{0.5}{100}\right)^1$ $= 2000 \times 1.005$ $= 2010$ Let $n = 2$: $A = (1 + i)^n \times P$ $= 2000\left(1 + \frac{0.5}{100}\right)^2$ $= 2000 \times 1.005^2$ $= 2020.05$
2. Calculate the common ratio between consecutive terms.	$r = \frac{t_2}{t_1}$ $= \frac{2020.05}{2010}$ $= 1.005$
3. State the known values in the geometric sequence equation.	$t_1 = 2010, r = 1.005$
4. Substitute these values into the geometric sequence equation.	$t_n = 2010 \times 1.005^{n-1}$
b. 1. Use the equation from part a to find the values of t_3, t_4, t_5 and t_6. Round all values correct to 2 decimal places.	**b.** $t_3 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{3-1}$ $= 2010 \times 1.005^2$ $= 2030.150...$ ≈ 2030.15 $t_4 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{4-1}$ $= 2010 \times 1.005^3$ $= 2040.301...$ ≈ 2040.30

$t_5 = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{5-1}$
$= 2010 \times 1.005^4$
$= 2050.502...$
≈ 2050.50
$t_6 = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{6-1}$
$= 2010 \times 1.005^5$
$= 2060.755...$
≈ 2060.76

2. Write the answer.

The amounts in Alexis' account at the end of each of the first 6 months are \$2010, \$2020.05, \$2030.15, \$2040.30, \$2050.50 and \$2060.76.

c. 1. Use the equation from part **a** to find the values of t_{15}, rounding your answer correct to 2 decimal place.

c. $t_{15} = 2010 \times 1.005^{n-1}$
$= 2010 \times 1.005^{15-1}$
$= 2010 \times 1.005^{14}$
$= 2155.365...$
≈ 2155.37

2. Write the answer.

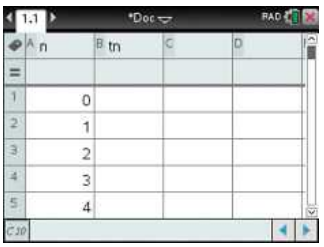
After 15 months Alexis has \$2155.37 in her account.

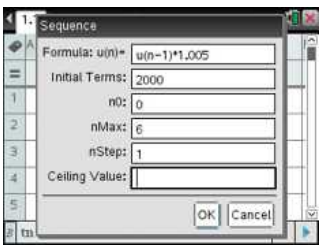
TI | THINK

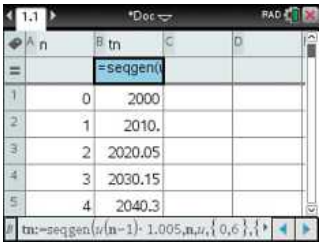
b. 1. On a Lists & Spreadsheet page, label the first column n and the second column *tn* (t_n). Enter the numbers 0 to 6 in the first column.

2. Select the function cell below *tn*, and press MENU then select
3: Data
1: Generate Sequence
Complete the fields as
Formula:
$u(n) = u(n-1) \times 1.005$
Initial Terms: 2000
n0: 0
nMax: 6
nStep: 1
then select OK.

WRITE





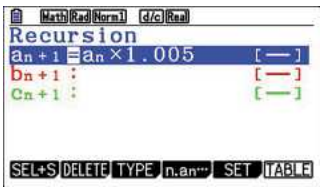


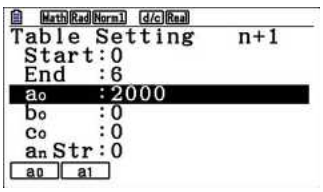
CASIO | THINK

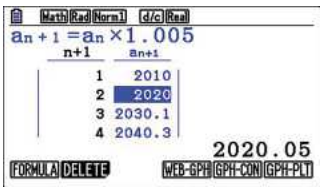
b. 1. On a Recursion screen, select TYPE by pressing F3, then select the second option by pressing F2.
Complete the entry line for a_n as
$a_{n+1} = a_n \times 1.005$
then press EXE.
Note: For *n* press F1.

2. Select SET by pressing F5 and complete the fields as
Start: 0
End: 6
a_0: 2000
then press EXE. Select TABLE by pressing F6.

WRITE







3 3. The answer appears on the screen.	The amounts in Alexis' account at the end of each of the first 6 months are $2010, $2020.05, $2030.15, $2040.30, $2050.50 and $2060.76.
3. The answer appears on the screen.	The amounts in Alexis' account at the end of each of the first 6 months are $2010, $2020.05, $2030.15, $2040.30, $2050.50 and $2060.76. *Note:* The values in the table have been rounded to 1 decimal place. Highlight each value in turn to view more decimal places.

8.4.3 Reducing balance depreciation

An individual or company owning expensive assets will be interested in their depreciation values as they can be claimed as expenses on their tax return. An item can be depreciated by a percentage of its previous future value over a specified period — typically a year. Since the sequence of future values decays in an exponential fashion, we can model this scenario using a geometric sequence.

WORKED EXAMPLE 9

A solar hot water system purchased for $1250 is depreciated by the reducing balance method at a rate of 8% p.a.

a. Set up an equation to determine the value of the Solar hot water system after *n* years of use.

b. Use your equation from part a to determine the future value of the solar hot water system after 6 years of use (correct to the nearest cent).



THINK

a. 1. Calculate the common ratio by identifying the value of the item in any given year as a percentage of the value in the previous year.
Convert the percentage to a ratio by dividing by 100.

2. Calculate the value of the solar hot water system after 1 year of use.

3. Substitute the values of t_1 and r into the geometric sequence equation.

b. 1. Substitute $n = 6$ into the equation determined in part a. Give your answer correct to the nearest cent.

2. Write the answer.

WRITE

a. $100\% - 8\% = 92\%$
Each year the value of the item is 92% of the previous value.

$$92\% = \frac{92}{100}$$
$$= 0.92$$
$$r = 0.92$$
$$t_1 = 1250 \times 0.92$$
$$= 1150$$
$$t_n = 1150 \times 0.92^{n-1}$$

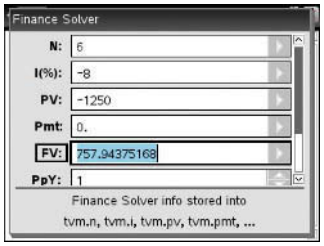
b. $t_n = 1150 \times 0.92^{n-1}$
$t_6 = 1150 \times 0.92^{6-1}$
$= 1150 \times 0.92^5$
$= 757.943...$
$= 757.94$
After 6 years the book value of the solar hot water system is $757.94.

TI | THINK

b. 1. On a Calculator page, press MENU then select
8: Finance
1: Finance Solver …
Complete the fields as
N: 6
I(%): −8
PV: −1250
Pmt: 0
FV:
PpY: 1
CpY: 1
PmtAt: END.
Place the cursor in the field for FV and press ENTER.

2. The answer appears on the screen.

WRITE



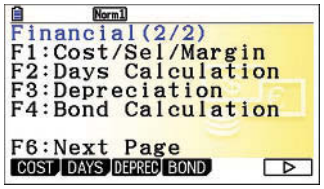
After 6 years the book value of the solar hot water system is $757.94.

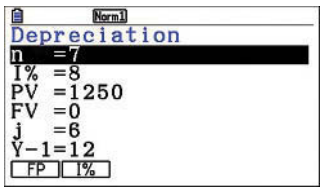
CASIO | THINK

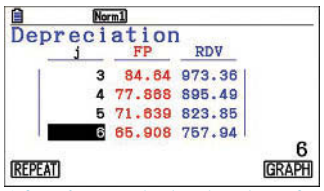
b. 1. On a Financial screen, press F6 to scroll to more menu options, then select Depreciation by pressing F3.
Complete the fields as
$n = 7$
I% = 8
PV = 1250
FV = 0
j = 6
Y − 1 = 12
Select FP by pressing F2, the select FP again by pressing F1. Select TABLE by pressing F6.

2. The answer appears on the screen.

WRITE







After 6 years the book value of the solar hot water system is $757.94.

studyon

Units 1 & 2 > Area 5 > Sequence 1 > Concept 3

Modelling with a geometric sequence Summary screen and practice questions

Exercise 8.4 Geometric sequences in context

Technology active

1. **WE7** The population of a town is decreasing by 10% each year. Find an expression for the population of the town, which will be referred to as P_n. The population in the first year, P_1, was 10 000.
2. The population of the newly established town of Alansford in its first year was 6000. It is predicted that the town's population will increase by 10% each year. If this were to be the case, find:
   a. the population of the town in its 10th year
   b. in which year the population of Alansford would reach 25 000.
3. **WE8** Hussein puts $2500 into an investment that earns compound interest at a rate of 0.3% per month.
   a. Set up an equation that represents Hussein's situation as a geometric sequence, where t_n is the amount in Hussein's account after n months.
   b. Use your equation from part **a** to determine the amount in Hussein's account after each of the first 6 months.
   c. Calculate the amount in Hussein's account at the end of 15 months.
4. Tim sets up an equation to model the amount of his money in a compound interest investment account after n months. His equation is $t_n = 4515.75 \times 1.0035^{n-1}$, where t_n is the amount in his account after n months.
   a. How much did Tim invest in the account?
   b. What is the annual interest rate of the investment?



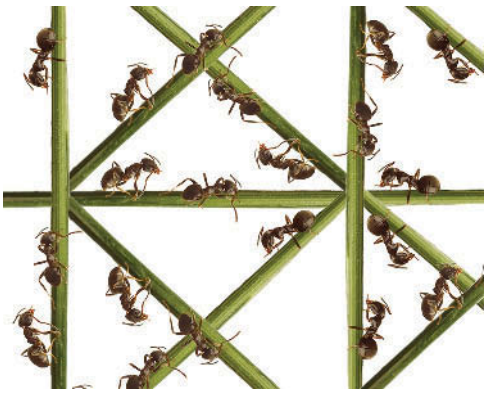
5. **WE9** A refrigerator purchased for $1470 is depreciated by the reducing balance method at a rate of 7% p.a.
   a. Set up an equation to determine the value of the refrigerator after n years of use.
   b. Use your equation from part **a** to determine the future value of the refrigerator after 8 years of use.
6. Ivy buys a new oven and decides to depreciate the value of the oven by the reducing balance method. Ivy's equation for the value of the oven after n years is $t_n = 1665 \times 0.925^{n-1}$.
   a. How much did the oven cost?
   b. What is the annual rate of depreciation for the oven?



7. The promoters of 'Fleago' flea powder assert that continued application of the powder will reduce the number of fleas on a dog by 15% each week. At the end of week 1, Fido the dog has 200 fleas left on him and his owner continues to apply the powder.
   a. How many fleas would Fido be expected to have on him at the end of the 4th week?
   b. How many weeks would Fido have to wait before the number of fleas on him had dropped to less than 50?



8. A company exported $300 000 worth of manufactured goods in its first year of production. According to the business plan of the company, this amount should increase each year by 7.5%.
   a. How much would the company be expected to export in its 5th year?
   b. What is the total amount exported by the company in its first 7 years of operation?
9. The number of ants in a colony doubles every week. If there are 2944 ants in the colony at the end of 8 weeks, how many ants were in the colony at the end of the first week?



10. Jonas starts a new job with a salary of $55 000 per year and the promise of a 3% pay rise for each subsequent year in the job.
    a. Write an equation to determine Jonas' salary in his nth year in the job.
    b. How much will Jonas earn in his 5th year in the job?
11. Julio's parents invest $5000 into a college fund on his 5th birthday. The fund pays a compound interest rate of 5.5% p.a. How much will the fund be worth when Julio turns 18?

12. A meteoroid is burning up as it passes through the Earth's atmosphere. For every 5 km it travels, the mass of the meteoroid decreases by 5%. At the start of its descent into the Earth's atmosphere, at 100 km above ground level, the mass of the meteoroid is 675 g.



a. Formulate an equation to determine the mass of the meteoroid after each 5-km increment of its descent.

b. What is the mass of the meteoroid when it hits the Earth, correct to 2 decimal places?

13. The number of pieces of stone used to build a pyramid decreases in a ratio of $\frac{1}{3}$ for each layer of the pyramid. The pyramid has 9 layers. The top (9th) layer of the pyramid needed only 2 stones.

a. How many stones were needed for the base layer of the pyramid?

b. Write an equation to express how many stones were needed for the nth layer of the pyramid.

c. How many stones were needed for the entire pyramid?

14. The populations of Melbourne and Sydney are projected to grow steadily over the next 20 years. A government agency predicts that the population of Melbourne will grow at a steady rate of 2.6% per year and the population of Sydney will grow at a steady rate of 1.7% per year.



a. If the current population of Melbourne is 4.35 million, formulate an equation to estimate the population of Melbourne after n years.

b. If the current population of Sydney is 4.65 million, formulate an equation to estimate the population of Sydney after n years.

c. Using technology of your choice, determine how long it will take for the population of Melbourne to exceed the population of Sydney. Give your answer correct to the nearest year.

8.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC There is a geometric sequence for which t_1 is a positive number and $r = -2$. It is true to say that:

A. only one term of the sequence is a positive number

B. the 3rd term will be a negative number

C. the 3rd term will be less than the 2nd term

D. the 5th term will be greater than the 6th term

2. MC The sum of the first 10 terms of the geometric sequence 2.25, 4.5, 9, 18, 36, ... is closest to:

A. 1149.75 **B.** 2301.75 **C.** 5318.81 **D.** 6648.51

3. MC Which of the following is a geometric sequence?

A. $2, -2, 2, -2, 2, \ldots$ **B.** $2, 4, 6, 8, 10, \ldots$

C. $1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$ **D.** $4, -4, 2, -2, 1, \ldots$

4. MC The 19th term of the geometric sequence 3.25, 6.5, 13, 26, 52, ... is:

A. 425 984 **B.** 851 968 **C.** 1 703 936 **D.** 41 978 243

5. MC The 3rd term of a geometric sequence is 19.35 and the 6th is 522.45. The 12th term of the sequence is:
   A. 16 539.15 **B.** 417 629.75 **C.** 126 955.35 **D.** 380 866.05
6. Calculate the value of the term specified for the given geometric sequences.
   a. Find the 18th term of the geometric sequence 8, 16, 32, 64, 128, ...
   b. Find the 9th term of the geometric sequence $-2, -8, -32, -128, -512, \ldots$
7. Determine the first five terms of the following arithmetic sequences.
   a. $t_n = -2 \times 3^{n-1}$
   b. $t_n = 4 \times \left(\frac{1}{3}\right)^{n-1}$
   c. $t_n = \frac{1}{4} \times \left(-\frac{3}{2}\right)^{n-1}$
   d. $t_n = \frac{1}{7} \times \left(\frac{2}{5}\right)^{n-1}$
8. A total of $7000 is invested in an account, which earns compound interest of 6% per annum compounding monthly. After 3 years, how much is in the account?
9. Calculate the sum to infinity of the following geometric sequences.
   a. 50, 25, 12.5, 6.25, 3.125, ...
   b. 20, 16, 12.8, 10.24, 9.192, ...
10. Write down the first 3 terms of the geometric sequence for which:
    a. $r = 0.6$ and $S_\infty = 25$
    b. $r = -0.2$ and $S_\infty = 3\frac{1}{3}$
    c. $r = 0.6$ and $S_\infty = -60$.
11. The batteries in a toy soldier are running down. The toy soldier marches 50 cm in the first minute, 30 cm in the second minute, 18 cm in the next and so on. By how much does the toy soldier fall short of marching 1.5 m?



12. What is the first term of the geometric sequence for which $r = -0.5$ and $S_\infty = 5\frac{1}{3}$?

Complex familiar

13. Blood donations at a suburban location increased by 40 each year. If there were 520 donations in the first year:
    a. How many donations were made in the 15th year?
    b. What was the total number of donations made over those 15 years?
14. Anya invests $25 000 in an account earning compound interest of 10% per annum compounding quarterly.
    a. Calculate the amount in the account after 3 years.
    b. Calculate how long it would take to have $40 965.41 in her account.
15. An irrigation system sprays 25 mm of water over a crop in its 1st month, 20 mm in the 2nd month and 16 mm in the 3rd month. If the crop requires 100 mm of water during its lifetime, how far short or how far over is the irrigation system in supplying the correct amount?



16. Helena receives $15 627.12 after closing an investment account that earned compound interest of 9% per annum compounding every 6 months. If Helena originally deposited $12 000 in the account, for how long was it in the account?

Complex unfamiliar

17. Rosanna decided to test the ball rebound height of a basketball. She dropped the basketball from a height of 500 cm and noted that each successive rebound was two-fifths of the previous height.

a. Set up a recurrence relation to model the bounce height of the ball.

b. Use your relation to estimate the heights of the first 5 rebounds, correct to 2 decimal places.

c. Sketch the graph of the first 5 bounces against the rebound height.



18. As part of an experiment, bacteria are grown in a laboratory. On day 7 of the experiment the bacteria count is 10 935. The number of bacteria has tripled each day.

a. Calculate the number of bacteria expected on the 9th day.

b. Determine the original number of bacteria.

19. Determine the third term of a geometric progression in which $t_3 + t_6 - t_5 = 12$ and $t_4 + t_7 - t_6 = 24$.

20. A particular geometric sequence is defined as $t_{n+1} - t_n = 4^n$, where $t_1 = 1$. Determine the value of t_8.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 8 Geometric sequences

Exercise 8.2 Recursive definition and the general term of geometric sequences

1. **a.** $t_1 = 3, r = 2$ **b.** $t_1 = \frac{1}{2}, r = \frac{5}{2}$
   c. Not a geometric sequence **d.** $t_1 = \frac{1}{2}, r = \frac{2}{5}$
2. **a.** $c = 36$
   b. $g = -6, h = 12$
   c. $p = 2.4, q = 12, s = 60$
   d. $u = -96, w = 384$
3. **a.** $t_{n+1} = 3t_n$ **b.** $t_{n+1} = -\frac{1}{4}t_n$
   c. $t_{n+1} = 5t_n$ **d.** $t_{n+1} = \frac{1}{2}t_n$
4. **a.** $0.13, -0.26, 0.52, -1.04.$ **b.** $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}.$
5. **a.** $x = 7$ **b.** $x = 4$
   c. $x = 4$ **d.** $x = 1$
6. **a.** $t_n = -1 \times 5^{n-1}$
   b. $t_n = 7 \times (-0.5)^{n-1}$
   c. $t_n = \frac{5}{6} \times \left(\frac{2}{3}\right)^{n-1}$
7. **a.** $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \frac{16}{3}$
   b. $\frac{5}{3}, 1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}$
   c. $\frac{3}{8}, -\frac{3}{20}, \frac{3}{50}, -\frac{3}{125}, \frac{6}{625}$
   d. $\frac{4}{9}, -\frac{8}{63}, \frac{16}{441}, -\frac{32}{3087}, \frac{64}{21\,609}$
8. **a.** $t_{15} = 19\,131\,876$
   b. $r = 5$
   c. $t_1 = -420$
9. **a.** $t_{11} = 12\,582\,91.1$
   b. $r = -2$
   c. $t_1 = 640$
10. **a.** 1, 3, 9, 27 or $-1, 3, -9, 27$
    b. 13.24, 66.2, 331, 1655 or 13.24, $-66.2, 331, -1655$
11. 80, 32, 12.8, 5.12
12. **a.** $r = \pm 3$; There are 2 possible values for the common ratio because the square root of 9 could be positive or negative 3.
    b. $t_6 = \pm 3159$
13. **a.** $a = \pm 2, b = \pm 8$ **b.** $e = -6, f = -18$
    c. $j = -2, k = 16$ **d.** $t = \frac{1}{2}, s = \pm\frac{1}{4}, u = \pm 1$
14. Sample responses can be found in the worked solutions in the online resources.

Exercise 8.3 The sum of a geometric sequence

1. $S_5 = -22$
2. $t_1 = \frac{1}{4}$
3. **a.** $S_\infty = 49$ **b.** $S_\infty = -\frac{72}{7}$
4. $S_\infty = -\frac{27}{4}$
5. **a.** $S_\infty = \frac{32}{3}$ **b.** $S_\infty = \frac{147}{10}$
6. $r = -\frac{1}{2}$
7. $r = -2$ or $r = 1$
8. $t_3 = \frac{5}{4}, t_4 = \frac{5}{8}$
9. $n = 7$
10. **a.** 1, 2, 4, 8, 16
    b. $r = 2$
    c. 2048 people are told the rumour on the 12th day.
11. 20 million, 10 million, 5 million, 2.5 million, 1.25 million, 625 000, 312 500
12. 192.5 m
13. **a.** $S_{12} = 531\,440$ **b.** $S_7 = 686\,285$
    c. $S_{15} = 36\,043.7$ **d.** $S_{11} = 274\,576.3$
14. $S_{12} = -27962025$
15. $S_\infty = \frac{200}{9}$
16. Yes, $S_\infty = 50$ mm
17. $0 < t_1 < 240$
18. Sample responses can be found in the worked solutions in the online resources.

Exercise 8.4 Geometric sequences in context

1. $P_n = 10000 \times (0.9)^{n-1}$
2. **a.** 14 147 **b.** 16th year
3. **a.** $t_n = 2507.5 \times 1.003^{n-1}$
   b. \$2507.50, \$2515.02, \$2522.57, \$2530.14, \$2537.73, \$2545.34
   c. \$2614.89
4. **a.** \$4500 **b.** 4.2%
5. **a.** $t_n = 1367.1 \times 0.93^{n-1}$ **b.** \$822.59
6. **a.** \$1800 **b.** 7.5%
7. **a.** 123 **b.** 10 weeks
8. **a.** \$400 640.74 **b.** $S_7 = \$2\,636\,196.56$
9. 23
10. **a.** $t_n = 55\,000 \times 1.03^{n-1}$ **b.** \$61 902.98

11. $10 028.87

12. a. $t_n = 641.25 \times 0.95^{n-1}$ **b.** 241.98 g

13. a. 13 122

b. $t_n = 13\,122 \times \left(\frac{1}{3}\right)^{n-1}$

c. 19 682

14. a. $t_n = 4\,463\,100 \times 1.026^{n-1}$

b. $t_n = 4\,729\,050 \times 1.017^{n-1}$

c. 8 years

8.5 Review: exam practice

1. D

2. B

3. A

4. B

5. D

6. a. 1 048 576 **b.** −131 072

7. a. −2, −6, −18, −54, −162 **b.** $4, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \frac{4}{81}$

c. $\frac{1}{4}, -\frac{3}{8}, \frac{9}{16}, -\frac{27}{32}, \frac{81}{64}$ **d.** $\frac{1}{7}, \frac{2}{35}, \frac{4}{175}, \frac{8}{875}, \frac{16}{4375}$

8. $8376.76

9. a. $S_\infty = 100$ **b.** $S_\infty = 100$

10. a. 10, 6, 3.6

b. 4, −0.8, 0.16

c. −24, −14.4, −8.64

11. 25 cm

12. $t_1 = 8$

13. a. 1080 **b.** 12 000

14. a. $33 622.22 **b.** 5 years

15. 25 mm too much

16. 3 years

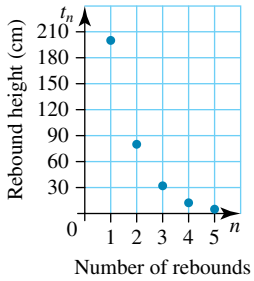
17. a. $t_{n+1} = \frac{2}{5}t_n,\ t_1 = 200$

b. 200 cm, 80 cm, 32 cm, 12.80 cm, 5.12 cm

c.

Bounce number	1	2	3	4	5
Rebound height (cm)	200	80	32	12.8	5.12

The points to be plotted are (1, 200), (2, 80), (3, 32), (4, 12.8) and (5, 5.12)



18. a. 98 415 bacteria **b.** 15 bacteria

19. $\frac{12}{5} = 2\frac{2}{5}$

20. $t_8 = 21845$

TOPIC 5 Arithmetic and geometric sequences and series 2

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



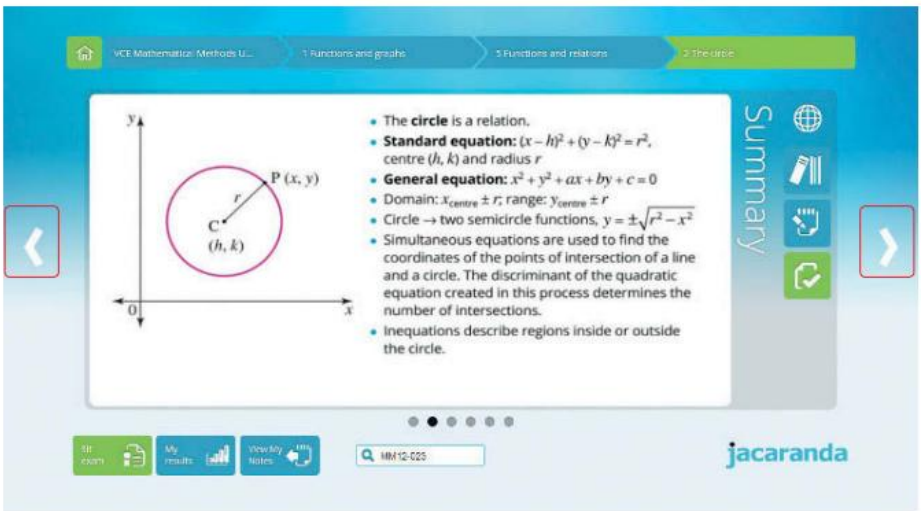
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 2

Mathematical Methods: Unit 1 examination

Unit

Unit 1: Algebra, statistics and functions

Topic

Topic 1: Arithmetic and geometric sequences and series 1
Topic 2: Functions and graphs
Topic 3: Counting and probability
Topic 4: Exponential functions 1
Topic 5: Arithmetic and geometric sequences and series 2

Conditions

Technique	Response type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 60 minutes Paper 2: 60 minutes	5 minutes

Resources	Instructions
• QCAA formula sheet • Notes not permitted. • Paper 1: Calculator not permitted • Paper 2: Non-CAS graphics calculator permitted	• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	46	54	

*Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1,* Brisbane, 2018.

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.
A detailed breakdown of the examination marks summary can be found in the PDF version of this assessment instrument in your eBookPLUS.

Paper 1: Technology free

Part A: Simple familiar — total marks: 27

Question 1 (6 marks)

Consider the linear function $f:[-2, 5) \in R, f(x) = 3x - 4$.

a. State the domain and determine the range of f.

b. Mathematically justify and communicate the transformations that $f(x)$ will need to undergo to map onto the function $g(x) = 3(x + 2) - 2$.

Question 2 (3 marks)

Calculate the fifth term of a geometric sequence defined by $t_{n+1} = -3t_n$, where $t_1 = \frac{1}{3}$?

Question 3 (5 marks)

Determine the sum of the first ten terms of the following sequence.

$$1, 4, 7, 10, \ldots$$

Question 4 (5 marks)

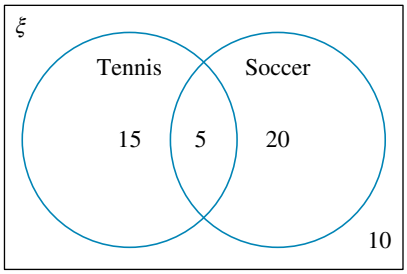
Consider the equation $y = 3x^2 - 15x + 18$.
Determine:

a. where the graph of the equation crosses the x-axis

b. the coordinates of the vertex.

Question 5 (3 marks)

Fifty students at a sports camp were asked what sport they represented their school in. The results are shown in the Venn diagram.



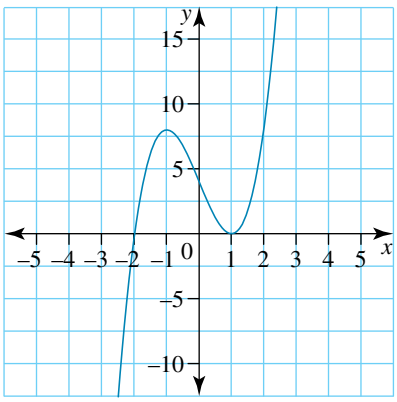
a. Calculate the probability that a student chosen at random:
   i. represents the school in soccer
   ii. represents the school in tennis but not in soccer.

b. Calculate the probability that a student represents the school in tennis, given that they represent the school in soccer.

Question 6 (5 marks)

a. Identify the family of polynomials to which the graph shown below belongs.

b. Determine the equation of the graph.



Part B: Complex familiar — total marks: 9

Question 7 (4 marks)

Simplify and express the answer with positive indices:

$$\frac{s^2\left(t^{-2}\right)^3}{\left(2s^{\frac{1}{2}}t^3\right)^2} \div \frac{t^5\left(2s^{-\frac{1}{3}}\right)^4}{2^2 s^{-4} t^5}$$

Question 8 (5 marks)

Consider the equation of the circle $x^2 + y^2 - 4x + 2y - 4 = 0$.

a. Determine the centre and the radius of the circle.

b. Hence, calculate the area of the circle (in terms of π).

Part C: Complex unfamiliar — total marks: 10

Question 9 (4 marks)

In a game, a 6-sided die is thrown.
If a 3 or less is thrown on the 6-sided die, a 4-sided die is thrown and the results are added together to give a score.
If a 4 or greater is thrown on the 6-sided die, a 4-sided die is thrown and the difference between the results will be the score.
Determine the score with the highest probability of occurring.
Justify your response by using a table, tree diagram or some other graphical method.

Question 10 (6 marks)

Determine the points where the line $y = 2x - 4$ intersects the hyperbola $y = \frac{16}{x}$?

Paper 2: Technology active

Part A: Simple familiar — total marks: 33

Question 1 (3 marks)

Events A and B are such that $P(B) = 0.6$, $P(A|B) = 0.3$ and $P(A \cup B) = 0.8$.

a. Find $P(A \cap B)$.

b. Find $P(A)$.

c. Determine whether events A and B are independent.

Question 2 (4 marks)

Determine the 5th term in $(2x + 3y)^7$.

Question 3 (7 marks)

Using technology, sketch the graph of the polynomial $y = 0.6(x - 2)^2(x + 1)(x + 2)$.
From inspection of this graph, deduce for what value(s) of d the equation $0 = 0.6(x - 2)^2(x + 1)(x + 2) + d$ will have exactly 2 solutions.

Question 4 (5 marks)

Consider the infinite geometric sequence –270, 162, –97.2, 58.32, –34.992 …
Determine:

a. the common ratio

b. the twelfth term

c. the sum of the infinite sequence.

Question 5 (8 marks)

The table shown represents the results of a survey investigating the smoking habits of workers in a particular company.

	Smoker	Never smoked	Ex-smoker	Totals
Office staff	32	203	14	249
Factory floor staff	41	224	18	283
IT staff	2	5	1	8
Totals	75	232	33	540

a. Calculate the probability that a person chosen at random is a member of the IT staff.

b. Calculate the probability of an office worker being a non-smoker.

c. Calculate the probability of a smoker or ex-smoker being a member of the factory floor staff.

(NOTE: Give answers correct to 4 decimal places.)

Question 6 (6 marks)

Elke inherits $\$50\,000$ and invests it for 4 years in an account which pays compound interest of 6.5% per annum compounding every 6 months.

a. Calculate how much will be in Elke's account at the end of 4 years.

b. Determine the yearly interest rate the bank would have to offer under the same compounding terms for Elke's account to reach $\$70\,000$ in the same period of time.

Part B: Complex familiar — total marks: 11

Question 7 (6 marks)

Three consecutive terms in a geometric sequence are

$$\ldots\ x-8,\ x+4,\ 3x-12,\ldots$$

Given that the sum of the first 6 terms in the series equals 378, determine the value of the first term in the series.

Question 8 (5 marks)

A sideways parabola with equation $y^2 = (x - 2)$ has two points of intersection with the line $x = 4$.
A second sideways parabola is formed by dilating the first by a factor of 3 from the origin and translating it by d units horizontally to the right. This second parabola intersects the line $x = 4$ at the same two points as the first.
Determine the vertex of the transformed sideways parabola.

Part C: Complex unfamiliar — total marks: 10

Question 9 (10 marks)

A cuboid container with a base length twice its width is to be made with $48\ \text{m}^2$ of metal. Let x represent the container's width and h represent the container's height.

a. Describe how the volume of the container changes as x changes.
b. Determine the dimensions of the container with the largest volume
c. Calculate the maximum volume.

Use mathematical reasoning to justify your response.

CHAPTER 9
Exponential and logarithmic functions

9.1 Overview

9.1.1 Introduction

There is a very old story, that has been told in one form or another in many different cultures, of an emperor who wished to grant a reward to a humble peasant for a good deed. The peasant looked at the emperor's chess board and asked that he be given coins such that he received one coin for the first square of the chessboard, two coins for the second, four coins for the third, eight coins for the fourth and so on. The emperor quickly agreed, thinking that he would end up giving the peasant far less than he was originally planning to give him, not realising that the modest amounts would rapidly increase — so much so, that there were not enough coins in the empire to fulfil the total amount that the peasant was really asking for (the final square on the chessboard would correspond to nearly 2×10^{19} coins!). The growth of the coins in this story is just one example of exponential growth.



Many processes in nature can be modelled by exponential growth. Bacteria, for example, reproduce by binary fission whereby one organism splits to become two. These two organisms then also split to form four and so on, reproducing rapidly to form colonies of millions under the right conditions. Conversely, there are many examples of elements decreasing rapidly.



LEARNING SEQUENCE

9.1 Overview
9.2 Exponential functions
9.3 Logarithmic functions
9.4 Modelling with exponential functions
9.5 Solving equations with indices
9.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

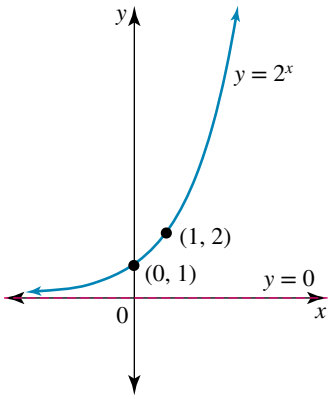
9.2 Exponential functions

Exponential functions are functions of the form $f: R \to R, f(x) = a^x, a \in R^+ \setminus \{1\}$. They provide mathematical models of exponential growth and exponential decay situations such as population increase and radioactive decay respectively.

9.2.1 The graph of $y = a^x$ where $a > 1$

Before sketching such a graph, consider the table of values for the function with rule $y = 2^x$.

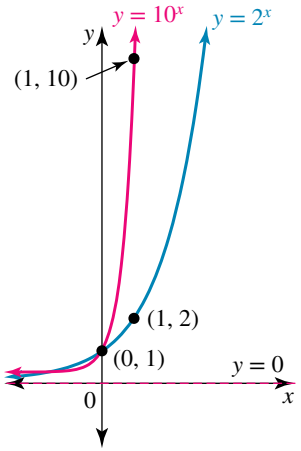
x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



From the table it is evident that $2^x > 0$ for all values of x, and that as $x \to -\infty$, $2^x \to 0$. This means that the graph will have a horizontal asymptote with equation $y = 0$.

It is also evident that as $x \to \infty$, $2^x \to \infty$ with the values increasing rapidly.

Since these observations are true for any function $y = a^x$ where $a > 1$, the graph of $y = 2^x$ will be typical of the basic graph of any exponential with base larger than 1.



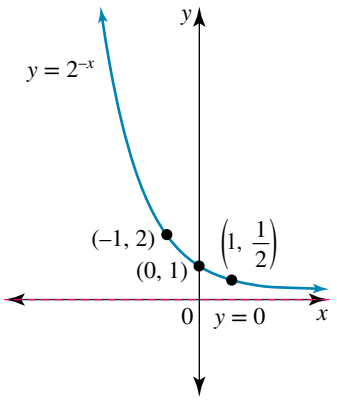
Key features of the graph of $y = 2^x$ and any such function $y = a^x$ where $a > 1$:

- horizontal asymptote with equation $y = 0$
- y-intercept is $(0, 1)$
- shape is of 'exponential growth'
- domain R
- range R^+
- one-to-one relation/function.

For $y = 2^x$, the graph contains the point $(1, 2)$; for the graph of $y = a^x, a > 1$, the graph contains the point $(1, a)$, showing that as the base increases, the graph becomes steeper more quickly for values $x > 0$. This is illustrated by the graphs of $y = 2^x$ and $y = 10^x$, with the larger base giving the steeper graph.

9.2.2 The graph of $y = a^x$ where $0 < a < 1$

An example of a function whose rule is in the form $y = a^x$ where $0 < a < 1$ is $y = \left(\frac{1}{2}\right)^x$. Since $\left(\frac{1}{2}\right)^x = 2^{-x}$, the rule for the graph of this exponential function $y = \left(\frac{1}{2}\right)^x$ where the base lies between 0 and 1 is identical to the rule $y = 2^{-x}$ where the base is greater than 1.



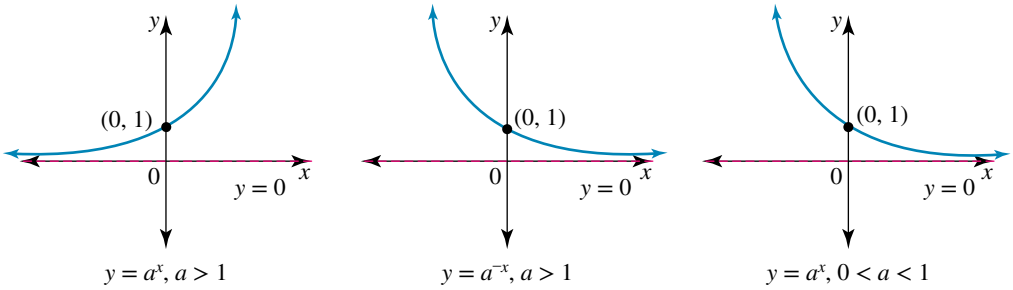
The graph of $y = 2^{-x}$ shown is typical of the graph of $y = a^{-x}$ where $a > 1$ and of the graph of $y = a^x$ where $0 < a < 1$.

Note: The substitution of $-x$ for x always indicates a reflectinon in the y-axis.

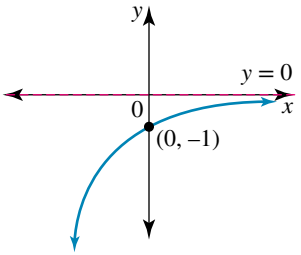
Key features of the graph of $y = 2^{-x}$ and any such function with rule expressed as either $y = a^x$ where $0 < a < 1$ or as $y = a^{-x}$ where $a > 1$:

- horizontal asymptote with equation $y = 0$
- y-intercept is $(0, 1)$
- shape is of 'exponential decay'
- domain R
- range R^+
- one-to-one relation/function
- reflection of $y = 2^x$ in the y-axis.

The basic shape of an exponential function is either one of 'growth' or 'decay'.



As with other functions, the graph of $y = -a^x$ will be inverted (reflected in the x-axis).



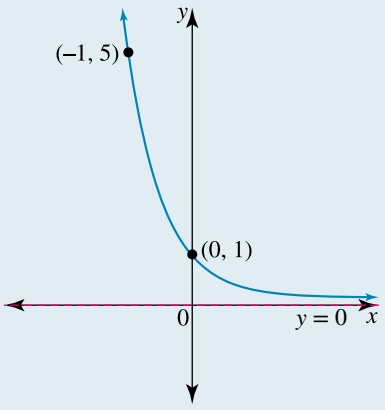
on Resources

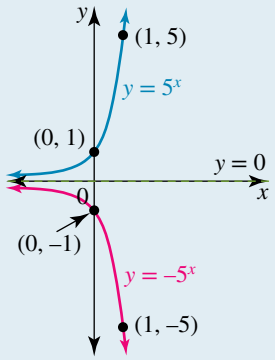
Interactivity: Exponential functions (int-5959)

WORKED EXAMPLE 1

a. **On the same set of axes, sketch the graphs of $y = 5^x$ and $y = -5^x$, stating their ranges.**

b. **Give a possible equation for the graph shown.**



THINK	WRITE
a. 1. Identify the asymptote of the first function.	**a.** $y = 5^x$ The asymptote is the line with equation $y = 0$.
2. Find the y-intercept.	y-intercept: when $x = 0$, $y = 1 \Rightarrow (0,1)$
3. Calculate the coordinates of a second point.	Let $x = 1$. $y = 5^1$ $= 5$ $\Rightarrow (1, 5)$
4. Use the relationship between the two functions to deduce the key features of the second function.	$y = -5^x$ This is the reflection of $y = 5^x$ in the x-axis. The graph of $y = -5^x$ has the same asymptote as that of $y = 5^x$. Equation of its asymptote is $y = 0$. Its y-intercept is $(0, -1)$. Point $(1, -5)$ lies on the graph.
5. Sketch and label each graph.	
6. State the range of each graph.	The range of $y = 5^x$ is R^+ and the range of $y = -5^x$ is R^-.
b. 1. Use the shape of the graph to suggest a possible form for the rule.	**b.** The graph has a 'decay' shape. Let the equation be $y = a^{-x}$.
2. Use a given point on the graph to calculate a.	The point $(-1, 5) \Rightarrow 5 = a^1$ $\therefore \quad a = 5$
3. State the equation of the graph.	The equation of the graph could be $y = 5^{-x}$. The equation could also be expressed as $y = \left(\frac{1}{5}\right)^x$ or $y = 0.2^x$.

9.2.3 Translations of exponential graphs

Once the basic exponential growth or exponential decay shapes are known, the graphs of exponential functions can be transformed in similar ways to graphs of any other functions previously studied.

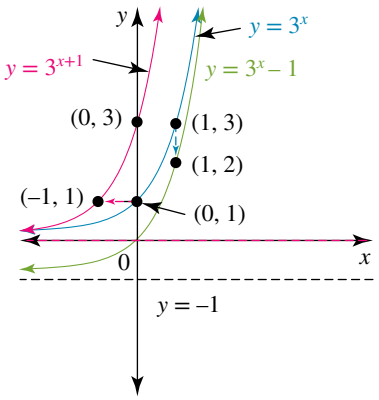
Vertical translations

Under a vertical translation of d units, the graph of $y = a^x$ will become $y = a^x + d$, and the position of the asymptote will be altered to $y = d$. If $d < 0$, the graph will have x-axis intercepts which are found by solving the exponential equation $a^x + d = 0$.

Horizontal translations

Under a horizontal translation of c units, the graph of $y = a^x$ will become $y = a^{x+c}$. The asymptote is unaffected and the intercept on the y-axis will no longer occur at $y = 1$. An additional point to the y-intercept that can be helpful to locate is the one where $x = -c$, since $a^{x\ \ c}$ will equal 1 when $x = -c$.

A horizontal translation and a vertical translation of the graph of $y = 3^x$ are illustrated in the diagram by the graphs of $y = 3^{x+1}$ and $y = 3^x - 1$ respectively. Under the horizontal translation of 1 unit to the left, the point $(0, 1) \rightarrow (-1, 1)$; under the vertical translation of 1 unit down, the point $(1, 3) \rightarrow (1, 2)$.

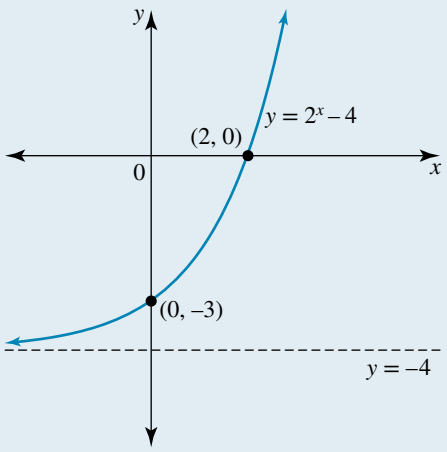


WORKED EXAMPLE 2

Sketch the graphs of each of the following and state the range of each.

a. $y = 2^x - 4$ **b.** $y = 10^{-(x+1)}$

THINK	WRITE
a. 1. State the equation of the asymptote.	**a.** $y = 2^x - 4$ The vertical translation 4 units down affects the asymptote. The asymptote has the equation $y = -4$.
2. Calculate the y-intercept.	y-intercept: let $x = 0$, $y = 1 - 4$ $= -3$ y-intercept is $(0, -3)$.
3. Calculate the x-intercept.	x-intercept: let $y = 0$, $2^x - 4 = 0$ $\therefore \quad 2^x = 4$ $\therefore \quad 2^x = 2^2$ $\therefore \quad x = 2$ x-intercept is $(2, 0)$.
4. Sketch the graph and state the range.	A 'growth' shape is expected since the coefficient of x is positive.



Range is $(-4, \infty)$.

b. $y = 10^{-(x+1)}$

1. Identify the key features from the given equation.

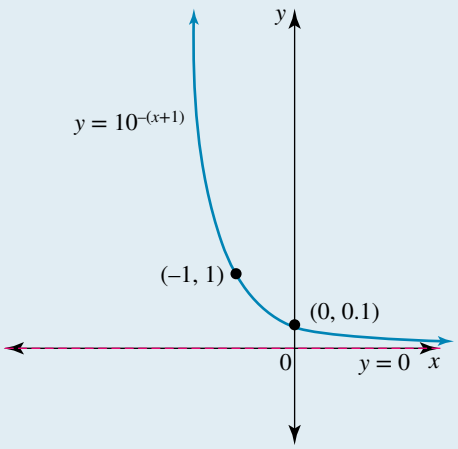
Reflection in the y-axis, horizontal translation 1 unit to the left.
The asymptote will not be affected.
Asymptote: $y = 0$
There is no x-intercept.
y-intercept: let $x = 0$,
$y = 10^{-1}$
$= \frac{1}{10}$
y-intercept is $(0, 0.1)$.

2. Calculate the coordinates of a second point on the graph.

Let $x = -1$
$y = 10^{0}$
$= 1$
The point $(-1, 1)$ lies on the graph.

3. Sketch the graph and state the range.

A 'decay' shape is expected since the coefficient of x is negative.



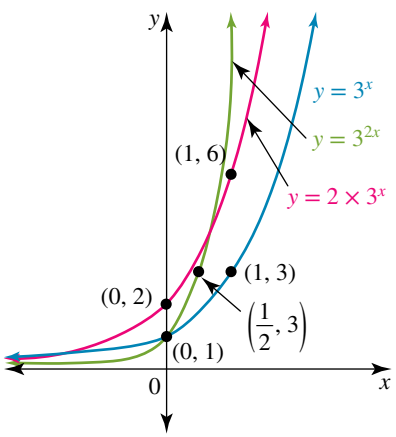
Range is R^+.

Dilations

Exponential functions of the form $y = b \times a^x$ have been dilated by a factor b $(b > 0)$ from the x-axis. This affects the y-intercept, but the asymptote remains at $y = 0$.

Exponential functions of the form $y = a^{kx}$ have been dilated by a factor $\frac{1}{k}$ $(k > 0)$ from the y-axis. This affects the steepness of the graph but does not affect either the y-intercept or the asymptote.

A dilation from the x-axis of factor 2 and a dilation from the y-axis of factor $\frac{1}{2}$ of the graph of $y = 3^x$ are illustrated in the diagram by the graphs of $y = 2 \times 3^x$ and $y = 3^{2x}$ respectively. Under the dilation from the x-axis of factor 2, the point $(1, 3) \rightarrow (1, 6)$; under the dilation from the y-axis of factor $\frac{1}{2}$, the point $(1, 3) \rightarrow \left(\frac{1}{2}, 3\right)$.



Combinations of transformations

Exponential functions with equations of the form $y = b \times a^{k(x+c)} + d$ are derived from the basic graph of $y = a^x$ by applying a combination of transformations. The key features to identify in order to sketch the graphs of such exponential functions are:

- the asymptote
- the y-intercept
- the x-intercept, if there is one.

Another point that can be obtained simply could provide assurance about the shape. Always aim to show at least two points on the graph.

WORKED EXAMPLE 3

Sketch the graphs of each of the following and state the range of each.

a. $y = 10 \times 5^{2x-1}$ **b.** $y = 1 - 4 \times 2^{-x}$

THINK

a. 1. Identify the key features using the given equation.

WRITE

a. $y = 10 \times 5^{2x-1}$

asymptote: $y = 0$

no x-intercept

y-intercept: let $x = 0$

$y = 10 \times 5^{-1}$

$= 10 \times \frac{1}{5}$

$= 2$

y-intercept is $(0, 2)$.

2. Calculate the coordinates of a second point.

Since the horizontal translation is $\frac{1}{2}$ to the right, let $x = \frac{1}{2}$.

$y = 10 \times 5^{2 \times \frac{1}{2} - 1}$

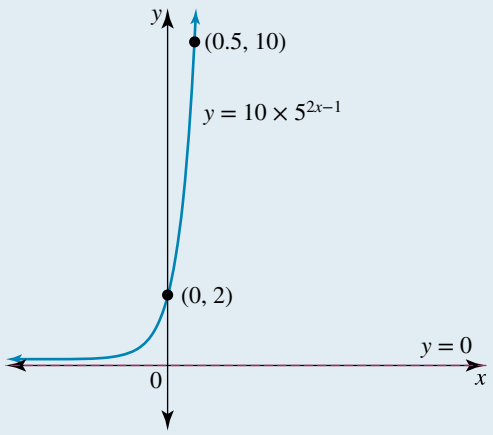
$= 10 \times 5^0$

$= 10 \times 1$

$= 10$

Point $\left(\frac{1}{2}, 10\right)$ lies on the graph.

3. Sketch the graph and state the range.



Range is R^+.

b. 1. Write the equation in the form $y = b \times a^{k(x+c)} + d$ and state the asymptote.

2. Calculate the y-intercept.

3. Calculate the x-intercept.
Note: As the point $(0, -3)$ lies below the asymptote and the graph must approach the asymptote, there will be an x-intercept.

4. Sketch the graph and state the range.

b. $y = 1 - 4 \times 2^{-x}$

$\therefore \quad y = -4 \times 2^{-x} + 1$

Asymptote: $y = 1$

y-intercept: let $x = 0$

$y = -4 \times 2^0 + 1$

$= -4 \times 1 + 1$

$= -3$

y-intercept is $(0, -3)$.

x-intercept: let $y = 0$

$0 = 1 - 4 \times 2^{-x}$

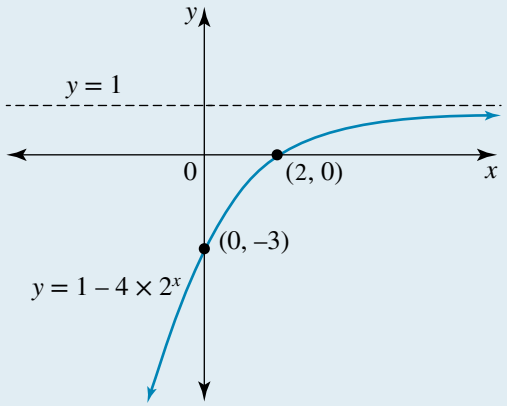
$2^{-x} = \frac{1}{4}$

$\frac{1}{2^x} = \frac{1}{4}$

$2^x = 4$

$x = 2$

The x-intercept is approximately $(2, 0)$.

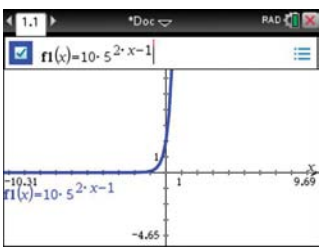


Range is $(-\infty, 1)$.

TI | THINK

a. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 10 \times 5^{2x-1}$ then press ENTER.

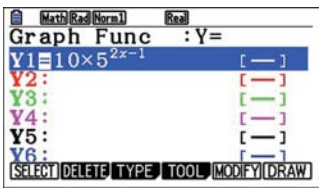
WRITE

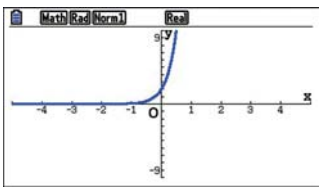


CASIO | THINK

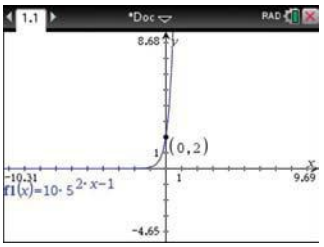
a. 1. On a Graph screen, complete the entry line for Y1 as $Y1 = 10 \times 5^{2x-1}$ then press EXE. Select DRAW by pressing F6.

WRITE





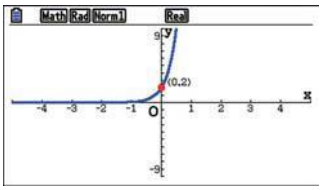
2. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



3. The range can be read from the graph.

The range is R^+.

2. To find the y-intercept, select G-Solv by pressing F5, then select Y-ICEPT by pressing F4. Press EXE.



3. The range can be read from the graph.

The range is R^+.

studyon

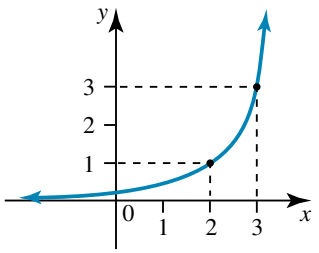
Units 1 & 2 | Area 6 | Sequence 1 | Concepts 1 & 2

Graphs of exponential functions Summary screen and practice questions
Transformation of exponential graphs Summary screen and practice questions

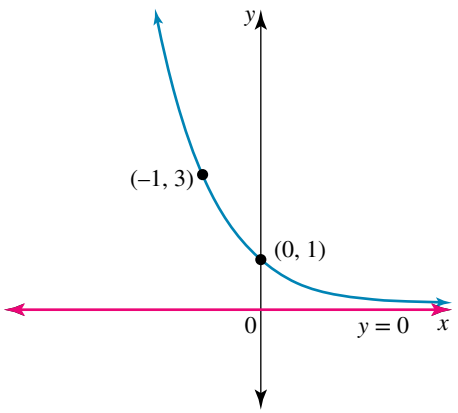
Exercise 9.2 Exponential functions

Technology free

1. MC The rule for the graph at right is:
   - **A.** $y = 3^{x-2}$
   - **B.** $y = 3^x$
   - **C.** $y = 2^{x-3}$
   - **D.** $y = 3^{x+2}$



2. **a.** WE1 On the same set of axes, sketch the graphs of $y = 3^x$ and $y = -3^x$, stating their ranges.
   b. Give a possible equation for the graph shown.



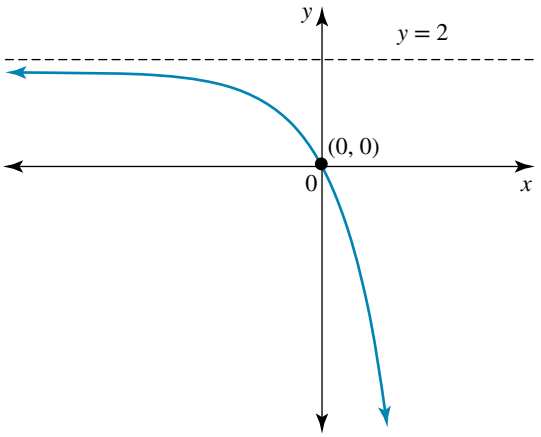
3. Sketch the graphs of $y = (1.5)^x$ and $y = \left(\frac{2}{3}\right)^x$ on the same set of axes.

4. WE2 Sketch the graphs of each of the following and state the range of each.
   a. $y = 4^x - 2$ **b.** $y = 3^{-(x+2)}$

5. Sketch the graph of $y = 4^{x-2} + 1$ and state its range.

6. WE3 Sketch the graphs of each of the following and state the range of each.
   a. $y = \frac{1}{2} \times 10^{1-2x}$ **b.** $y = 1 - 9 \times 3^{-x}$

7. The graph shown has the equation $y = a \times 3^x + b$. Determine the values of a and b.



8. a. i. Sketch, on the same set of axes, the graphs of $y = 4^x$, $y = 6^x$ and $y = 8^x$.

ii. Describe the effect produced by increasing the base.

b. i. Sketch, on the same set of axes, the graphs of $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$.

ii. Express each rule in a different form.

9. a. i. Sketch, on the same set of axes, the graphs of $y = 5^{-x}$, $y = 7^{-x}$ and $y = 9^{-x}$.

ii. Describe the effect produced by increasing the base.

b. i. Sketch, on the same set of axes, the graphs of $y = (0.8)^x$, $y = (1.25)^x$ and $y = (0.8)^{-x}$.

ii. Describe the relationships between the three graphs.

10. Sketch each of the following graphs, showing the asymptote and labelling any intersections with the coordinate axes with their exact coordinates.

a. $y = 5^{-x} + 1$ **b.** $y = 1 - 4^x$ **c.** $y = 3^x - 27$ **d.** $y = 6.25 - (2.5)^{-x}$

11. Sketch the graphs of each of the following.

a. $y = 2^{x-2}$ **b.** $y = -3^{x+2}$ **c.** $y = 4^{x-0.5}$ **d.** $y = 7^{1-x}$

12. Sketch the graphs of each of the following.

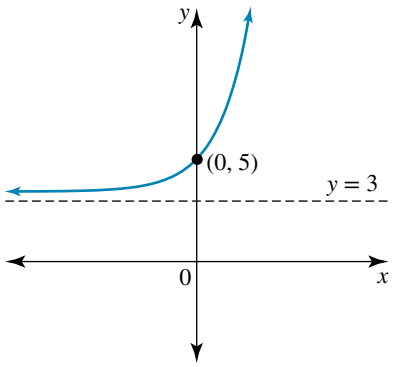
a. $y = 3 \times 2^x$ **b.** $y = 2^{\frac{3x}{4}}$ **c.** $y = -3 \times 2^{-3x}$ **d.** $y = 1.5 \times 10^{-\frac{x}{2}}$

13. a. Sketch the graphs of $y = 3^{2x}$ and $y = 9^x$ and explain the result.

b. i. Use index laws to obtain another form of the rule for $y = 2 \times 4^{0.5x}$.

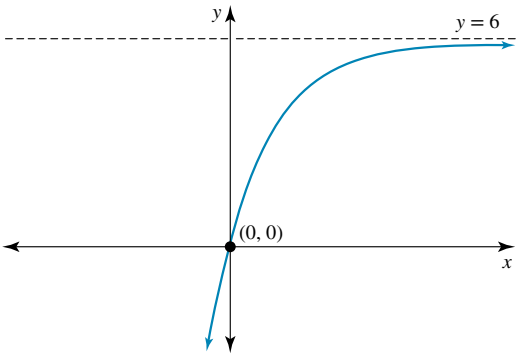
ii. Hence or otherwise, sketch the graph of $y = 2 \times 4^{0.5x}$.

14. a. Determine a possible rule for the given graph in the form $y = a \times 10^x + b$.



b. The graph of an exponential function of the form $y = a \times 3^{kx}$ contains the points (1, 36) and (0, 4). Determine its rule and state the equation of its asymptote.

c. For the graph shown, determine a possible rule in the form $y = a - 2 \times 3^{b-x}$.



d. Express the equation given in part **c** in another form not involving a horizontal translation.

Technology active

15. Use a graphical means to determine the number of intersections between:

a. $y = 2^x$ and $y = -x$, specifying an interval in which the x-coordinate of any point of intersection lies

b. $y = 2^x$ and $y = x^2$

c. $y = e^x$ and $y = 2^x$

d. $y = 2^{-x} + 1$ and $y = \sin(x)$

e. $y = 3 \times 2^x$ and $y = 6^x$, determining the coordinates of any points of intersection algebraically

f. $y = 2^{2x-1}$ and $y = \frac{1}{2} \times 16^{\frac{x}{2}}$, giving the coordinates of any points of intersection.

16. Obtain the coordinates of the points of intersection of $y = 2^x$ and $y = x^2$.

17. Sketch the graphs of $y_1 = 33 - 2(11)^x$ and $y_2 = 33 - 2(11)^{x+1}$ and compare their asymptotes, x- and y-intercepts and the value of their x-coordinates when $y = 10$. What transformation maps y_1 to y_2?

18. In 1772, Johann Bode discovered a curious relationship between pure numbers and the distance of planets from the Sun.

His law consisted of a simple formula relating the number of the planet to its distance from the Sun.

The actual distances of the planets from the Sun are given in the table below. By graphing the distance against 2 raised to the power of the planet number, discover the relationship that Bode found.

(*Hint*: Use either a spreadsheet or technology to graph the data and then find the regression line.)



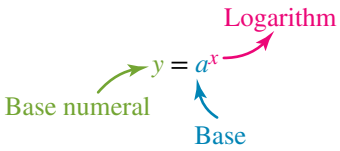
Planet number	Planet	Distance in AU (1AU = distance from the Earth to the Sun)
0	Mercury	0.39
1	Venus	0.72
2	Earth	1
3	Mars	1.52
4	Ceres (dwarf planet)	2.77
5	Jupiter	5.2
6	Saturn	9.54
7	Uranus	19.18
8	Pluto (dwarf planet)	39.4

(*Note*: The discovery of Neptune and large bodies in the Kuiper Belt such as Eris discredited Bode's Law in the eyes of many astronomers. If Neptune's orbit actually falls between Uranus and Pluto at a distance of 30.1 AU, what do you notice about its relationship with Pluto and Bode's Law?)

9.3 Logarithmic functions

9.3.1 Defining logarithms

The index, power or exponent (x) in the indicial equation $y = a^x$ is also known as a logarithm.



This means that $y = a^x$ can be written in an alternative form $\log_a y = x$, which is read as 'the logarithm of y to the base a is equal to x'.

For example, $3^2 = 9$ can be written as $\log_3 9 = 2$.

$10^5 = 100\,000$ can be written as $\log_{10} 100\,000 = 5$.

In general, for $a > 0$ and $a \neq 1$: $a^x = y$ is equivalent to $x = \log_a y$.

Using the indicial equivalent, it is possible to find the exact value of some logarithms.

WORKED EXAMPLE 4

Evaluate the following without technology.

a. $\log_6 216$ **b.** $\log_2\left(\frac{1}{8}\right)$

THINK	WRITE
a. 1. Let x equal the quantity we wish to find.	**a.** Let $x = \log_6 216$
2. Express the logarithmic equation as an indicial equation.	$6^x = 216$
3. Express both sides of the equation to the same base.	$6^x = 6^3$
4. Equate the powers.	$x = 3$
b. 1. Write the logarithm as a logarithmic equation.	**b.** Let $x = \log_2\left(\frac{1}{8}\right)$
2. Express the logarithmic equation as an indicial equation.	$2^x = \frac{1}{8}$ $= \left(\frac{1}{2}\right)^3$ $= \left(2^{-1}\right)^3$
3. Express both sides of the equation to the same base.	$2^x = 2^{-3}$
4. Equate the powers.	$x = -3$

9.3.2 Logarithm laws

The index laws can be used to establish corresponding rules for calculations involving logarithms. These rules are summarised in the following table

Name	Rule	Restrictions
Logarithm of a product	$\log_a(mn) = \log_a m + \log_a n$	$m, n > 0$ $a > 0, a \neq 1$
Logarithm of a quotient	$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$	$m, n > 0$ $a > 0$ and $a \neq 1$
Logarithm of a power	$\log_a m^n = n \log_a m$	$m > 0$ $a > 0$ and $a \neq 1$
Logarithm of the base	$\log_a a = 1$	$a > 0$ and $a \neq 1$
Logarithm of one	$\log_a 1 = 0$	$a > 0$ and $a \neq 1$

It is important to remember that rules 1–3 above work only if the base, a, is the same for each of them. To change bases, the following rule can be applied.

$$\log_a b = \frac{\log_c b}{\log_c a}$$

WORKED EXAMPLE 5

Simplify, and evaluate where possible, each of the following without technology.

a. $\log_{10}5 + \log_{10}4$ **b.** $\log_2 12 + \log_2 8 - \log_2 3$

THINK

a. 1. Apply the 'logarithm of product' rule.

2. Simplify.

b. 1. Multiply the base numerals of the logs being added since their bases are the same.

2. Apply the 'logarithm of a quotient' law.

3. Simplify, noting that 32 is a power of 2.

4. Evaluate using the 'logarithm of a power' and 'logarithm of the base' laws.

WRITE

a. $\log_{10} 5 + \log_{10} 4$
$= \log_{10}(5 \times 4)$
$= \log_{10} 20$

b. $\log_2 12 + \log_2 8 - \log_2 3$
$= \log_2(12 \times 8) - \log_2 3$
$= \log_2(96 \div 3)$
$= \log_2 32$
$= \log_2 2^5$
$= 5 \log_2 2$
$= 5$

WORKED EXAMPLE 6

Simplify $3\log_2 5 - 2\log_2 10$.

THINK

1. Express both terms as logarithms of index numbers.
2. Simplify each logarithm.
3. Apply the 'logarithm of a quotient' law.
4. Simplify.

WRITE

$$\begin{aligned} 3\log_2 5 - 2\log_2 10 &= \log_2 5^3 - \log_2 10^2 \\ &= \log_2 125 - \log_2 100 \\ &= \log_2 (125 \div 100) \\ &= \log_2 \left(\frac{5}{4}\right) \text{ or } \log_2 1.25 \end{aligned}$$

WORKED EXAMPLE 7

Simplify each of the following.

a. $\dfrac{\log_8 49}{\log_8 343}$ **b.** $2\log_{10} x + 1$

THINK

a. 1. Express each base numeral as powers to the same base, 7.

2. Apply the 'logarithm of a power' law.

3. Simply by cancelling out the common factor of $\log_8 7$.

b. 1. Express $2\log_{10} x$ as $\log_{10} x^2$ and 1 as a logarithm to base 10 also.

2. Simplify using the 'logarithm of a product' law.

WRITE

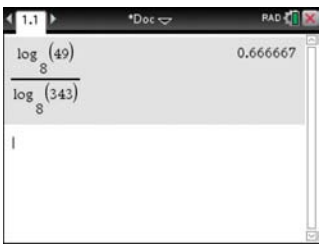
a.
$$\begin{aligned} \frac{\log_8 49}{\log_8 343} &= \frac{\log_8 7^2}{\log_8 7^3} \\ &= \frac{2\log_8 7}{3\log_8 7} \\ &= \frac{2}{3} \end{aligned}$$

b.
$$\begin{aligned} 2\log_{10} x + 1 &= \log_{10} x^2 + \log_{10} 10 \\ &= \log_{10} 10x^2 \end{aligned}$$

TI | THINK

a. 1. On a Calculator page, complete the entry line as $\dfrac{\log_8(49)}{\log_9(343)}$ then press ENTER.

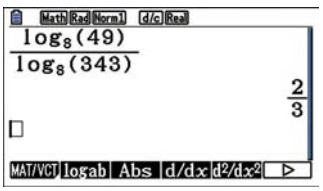
WRITE



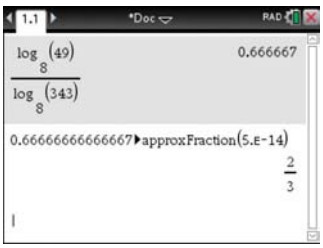
CASIO | THINK

a. 1. On a Run-Matrix screen, complete the entry line as $\dfrac{\log_8(49)}{\log_9(343)}$ then press EXE.
Press the (−) button to convert the decimal to a fraction.
Note: The $\log_a b$ template can be found by pressing F4 to select MATH, then selecting $\log_a b$ by pressing F2.

WRITE



2. Press the up arrow to highlight the answer then press ENTER to copy and paste it on the next entry line. Press MENU then select
2: Number
2: Approximate to Fraction
then press ENTER.



3. The answer appears on the screen.

$$\frac{\log_8(49)}{\log_9(343)} = \frac{2}{3}$$

2. The answer appears on the screen.

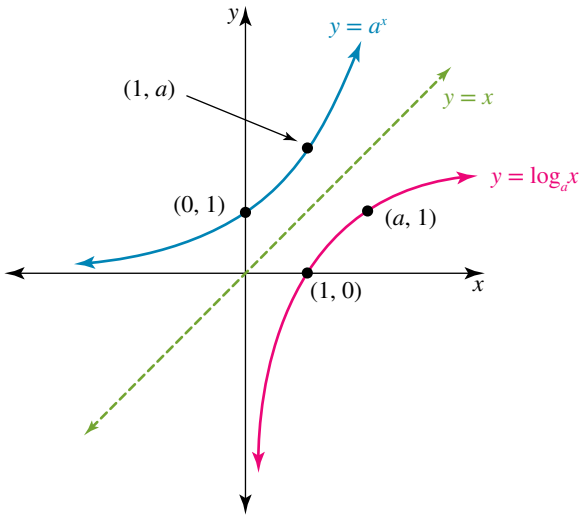
$$\frac{\log_8(49)}{\log_9(343)} = \frac{2}{3}$$

9.3.3 The graph of $y = \log_a(x)$ for $a > 1$

The shape of the basic logarithmic graph with rule $y = \log_a(x)$, $a > 1$ is shown as the reflection in the line $y = x$ of the exponential graph with rule $y = a^x$, $a > 1$.

The graph of $y = \log_a(x)$ is the *inverse function* of $y = a^x$.

The key features of the graph of $y = \log_a(x)$ can be deduced from those of the exponential graph.



$y = a^x$	$y = \log_a(x)$
horizontal asymptote with equation $y = 0$	vertical asymptote with equation $x = 0$
y-intercept $(0, 1)$	x-intercept $(1, 0)$
point $(1, a)$ lies on the graph	point $(a, 1)$ lies on the graph
range R^+	domain R^+
domain R	range R
one-to-one relation	one-to-one relation/function

Note that logarithmic growth is much slower than exponential growth and also note that, unlike a^x which is always positive, $\log_a(x)\begin{cases} > 0, & \text{if } x > 1 \\ = 0, & \text{if } x = 1 \\ < 0, & \text{if } 0 < x < 1 \end{cases}$

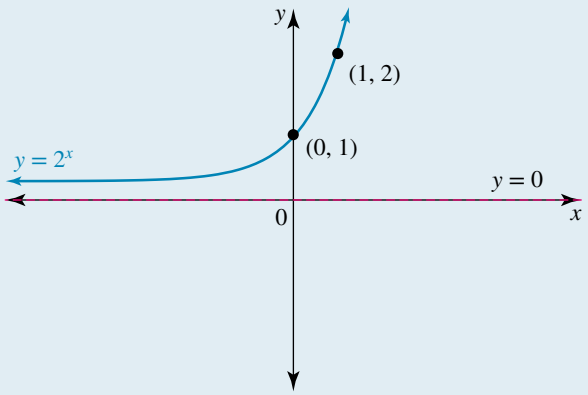
The logarithmic function is formally written as $f: R^+ \to R$, $f(x) = \log_a(x)$.

WORKED EXAMPLE 8

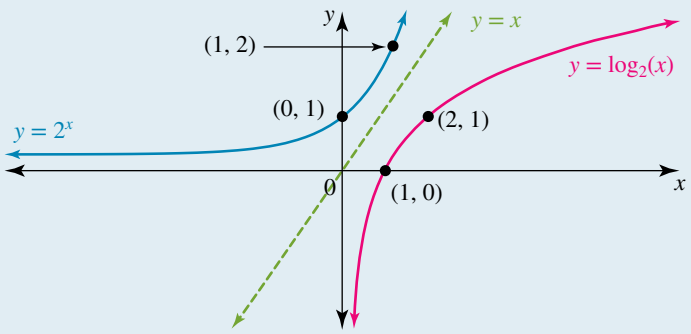
a. Form the exponential rule for the inverse of $y = \log_2(x)$ and hence deduce the graph of $y = \log_2(x)$ from the graph of the exponential.

b. Given the points (1, 2), (2, 4) and (3, 8) lie on the exponential graph in part a, explain how these points can be used to illustrate the logarithm law $\log_2(m) + \log_2(n) = \log_2(mn)$.

THINK	WRITE
a. 1. Form the rule for the inverse by interchanging coordinates and then make y the subject of the rule.	**a.** $y = \log_2(x)$ Inverse: $x = \log_2(y)$ $\therefore y = 2^x$
2. Sketch the exponential function.	$y = 2^x$ Asymptote: $y = 0$ y-intercept: $(0, 1)$ second point: let $x = 1$, $\therefore y = 2$ Point $(1, 2)$ is on the graph.



3. Reflect the exponential graph in the line $y = x$ to form the required graph.	$y = \log_2(x)$ has: asymptote: $x = 0$ x-intercept: $(1, 0)$ second point: $(2, 1)$



b. 1. State the coordinates of the corresponding points on the logarithm graph.

2. State the x- and y-values for each of the points on the logarithmic graph.

3. Use the relationship between the y-coordinates to illustrate the logarithm law.

b. Given the points $(1, 2)$, $(2, 4)$ and $(3, 8)$ lie on the exponential graph, the points $(2, 1)$, $(4, 2)$ and $(8, 3)$ lie on the graph of $y = \log_2(x)$.

$y = \log_2(x)$
point $(2, 1)$: when $x = 2$, $y = 1$
point $(4, 2)$: when $x = 4, y = 2$
point $(8, 3)$: when $x = 8, y = 3$

The sum of the y-coordinates of the points on $y = \log_2(x)$ when $x = 2$ and $x = 4$ equals the y-coordinate of the point on $y = \log_2(x)$ when $x = 8$, as $1 + 2 = 3$.

$\log_2(2) + \log_2(4) = \log_2(8)$
$\log_2(2) + \log_2(4) = \log_2(2 \times 4)$
This illustrates the logarithm law
$\log_2(m) + \log_2(n) = \log_2(mn)$
with $m = 2$ and $n = 4$.

9.3.4 Extension: Transformations of logarithmic graphs

Knowledge of the transformations of graphs enables the graph of any logarithmic function to be obtained from the basic graph of $y = \log_a(x)$. This provides an alternative to sketching the graph as the inverse of that of an exponential function. Further, given the logarithmic graph, the exponential graph could be obtained as the inverse of the logarithmic graph.

The logarithmic graph under a combination of transformations will be studied in Units 3 and 4. In this section we shall consider the effect a single transformation has on the key features of the graph of $y = \log_a(x)$.

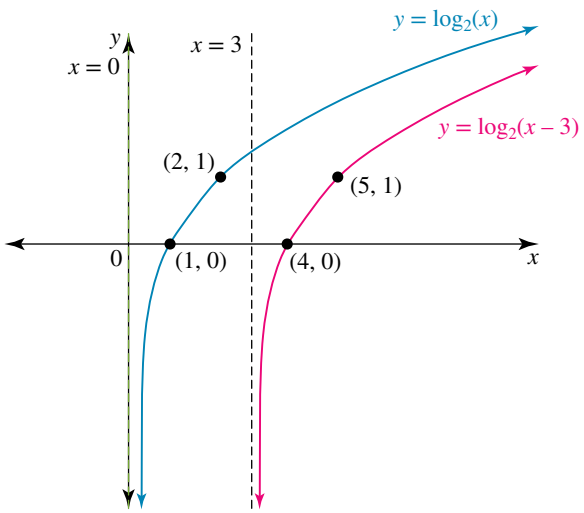
Dilations

Dilations from either coordinate axis are recognisable from the equation of the logarithmic function: for example, $y = 2\log_a(x)$ and $y = \log_a\left(\frac{x}{2}\right)$ would give the images when $y = \log_a(x)$ undergoes a dilation of factor 2 from the x-axis and from the y-axis, respectively. The asymptote at $x = 0$ would be unaffected by either dilation. The position of the x- intercept is affected by the dilation from the y-axis as $(1, 0) \to (2, 0)$. The dilation from the x- axis does not affect the x-intercept.

Horizontal translations

The vertical asymptote will always be affected by a horizontal translation and this affects the domain of the logarithmic function. Under a horizontal translation of c units to the right or left, the vertical asymptote at $x = 0$ must move c units to the right or left respectively. Hence, horizontally translating the graph of $y = \log_a(x)$ by c units to obtain the graph of $y = \log_a(x - c)$ produces the following changes to the key features:

- equation of asymptote: $x = 0 \to x = c$
- domain: $\{x : x > 0\} \to \{x : x > c\}$
- x-intercept: $(1, 0) \to (1 + c, 0)$.



These changes are illustrated in the graph of $y = \log_2(x)$ and its image, $y = \log_2(x-3)$, after a horizontal translation of 3 units to the right.

The diagram shows that the domain of $y = \log_2(x-3)$ is $(3, \infty)$. Its range is unaffected by the horizontal translation and remains R.

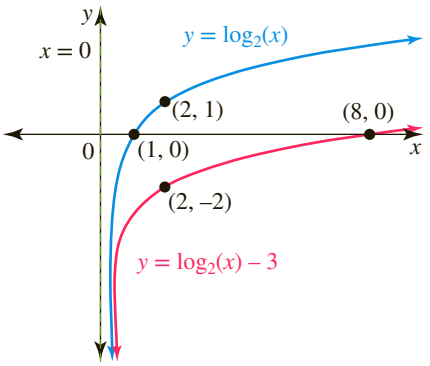
It is important to realise that the domain and the asymptote position can be calculated algebraically, since we only take logarithms of positive numbers. For example, the domain of $y = \log_2(x-3)$ can be calculated by solving the inequation $x - 3 > 0 \Rightarrow x > 3$. This means that the domain is $(3, \infty)$ as the diagram shows. The equation of the asymptote of $y = \log_2(x-3)$ can be calculated from the equation $x - 3 = 0 \Rightarrow x = 3$.

The function defined by $y = \log_a(nx + c)$ would have a vertical asymptote when $nx + c = 0$ and its domain can be calculated by solving $nx + c > 0$.

Vertical translations

When vertically translating the graph of $y = \log_a(x)$ by d units to obtain the graph of $y = \log_a(x) + d$, neither the domain nor the position of the asymptote alters from that of $y = \log_a(x)$. The translated graph will have an x-intercept which can be obtained by solving the equation $\log_a(x) + d = 0$.

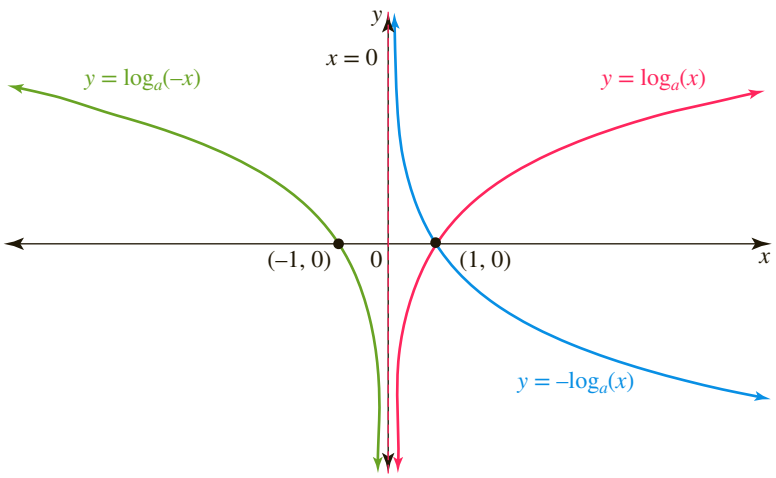
The graph of $y = \log_2(x) - 3$ is a vertical translation down by 3 units of the graph of $y = \log_2(x)$. Solving $\log_2(x) - 3 = 0$ gives $x = 2^3$ so the graph cuts the x-axis at $x = 8$, as illustrated.



Reflections

The graph of $y = -\log_a(x)$ is obtained by inverting the graph of $y = \log_a(x)$; that is, by reflecting it in the x-axis.

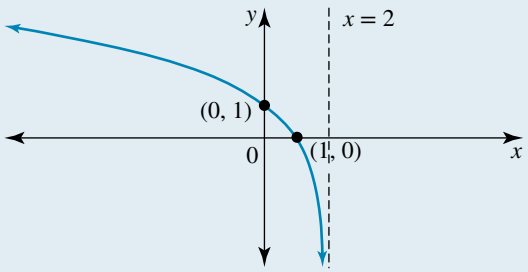
The graph of $y = \log_a(-x)$ is obtained by reflecting the graph of $y = \log_a(x)$ in the y-axis. For $\log_a(-x)$ to be defined, $-x > 0$ so the graph has domain $\{x : x < 0\}$.

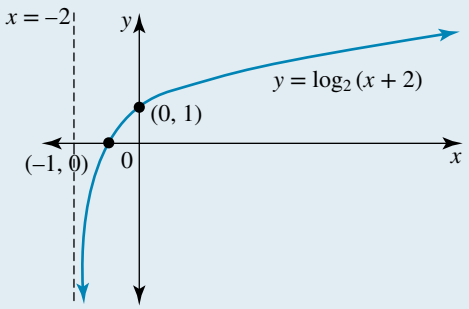


The relative positions of the graphs of $y = \log_a(x)$, $y = -\log_a(x)$ and $y = \log_a(-x)$ are illustrated in the diagram. The vertical asymptote at $x = 0$ is unaffected by either reflection.

WORKED EXAMPLE 9

a. Sketch the graph of $y = \log_2(x + 2)$ and state its domain.
b. Sketch the graph of $y = \log_{10}(x) + 1$ and state its domain.
c. The graph of the function for which $f(x) = \log_2(b - x)$ is shown below.
i. Determine the value of b.
ii. State the domain and range of, and form the rule for, the inverse function.
iii. Sketch the graph of $y = f^{-1}(x)$.



THINK	WRITE
a. 1. Identify the transformation involved.	**a.** $y = \log_2(x + 2)$ Horizontal translation 2 units to the left
2. Use the transformation to state the equation of the asymptote and the domain.	The vertical line $x = 0 - 2 \rightarrow$ the vertical line $x = -2$ under the horizontal translation. The domain is $\{x : x > -2\}$.
3. Calculate any intercepts with the coordinate axes. *Note*: The domain indicates there will be an intercept with the y-axis as well as the x-axis.	y-intercept: when $x = 0$, $y = \log_2(2)$ $= 1$ y-intercept $(0, 1)$ x-intercept: when $y = 0$, $\log_2(x + 2) = 0$ $x + 2 = 2^0$ $x + 2 = 1$ $x = -1$ x-intercept $(-1, 0)$ Check: the point $(1, 0) \rightarrow (-1, 0)$ under the horizontal translation.
4. Sketch the graph.	
b. 1. Identify the transformation involved.	**b.** $y = \log_{10}(x) + 1$ Vertical translation of 1 unit upwards

2. State the equation of the asymptote and the domain.

The vertical transformation does not affect either the position of the asymptote or the domain.
Hence, the equation of the asymptote is $x = 0$.
The domain is R^+.

3. Obtain any intercept with the coordinate axes.

Since the domain is R^+ there is no y-intercept.
x-intercept: when $y = 0$,

$$\log_{10}(x) + 1 = 0$$
$$\log_{10}(x) = -1$$
$$x = 10^{-1}$$
$$= \frac{1}{10} \text{ or } 0.1$$

x-intercept is $(0.1, 0)$.

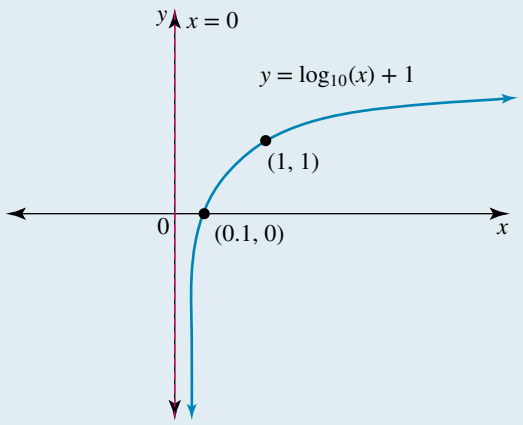
4. Calculate the coordinates of a second point on the graph.

Point: let $x = 1$.

$$y = \log_{10}(1) + 1$$
$$= 0 + 1$$
$$= 1$$

The point $(1, 1)$ lies on the graph.
Check: the point $(1, 0) \rightarrow (1, 1)$ under the vertical translation.

5. Sketch the graph.



c. i. State the equation of the asymptote shown in the graph and use this to calculate the value of b.
Note: The function rule can be rearranged to show the horizontal translation and a reflection in the y-axis.

$$f(x) = \log_2(b - x)$$
$$= \log_2(-(x - b))$$

The horizontal translation determines the position of the asymptote.

c.i $f(x) = \log_2(b - x)$
From the diagram, the asymptote of the graph is $x = 2$.
From the function rule, the asymptote occurs when:

$$b - x = 0$$
$$x = b$$

Hence, $b = 2$.

ii. Give the domain and range of the inverse function.

ii. The given function has domain $(-\infty, 2)$ and range R. Therefore the inverse function has domain R and range $(-\infty, 2)$.

Form the rule for the inverse function by interchanging x- and y-coordinates and rearranging the equation obtained.

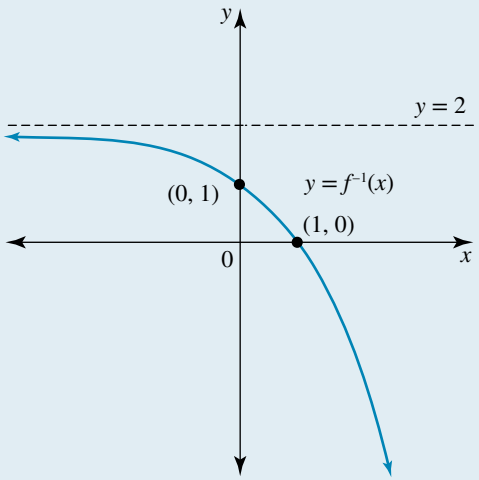
function $f: y = \log_2(2 - x)$
inverse $f^{-1}: x = \log_2(2 - y)$
$2^x = 2 - y$
$\therefore \quad y = 2 - 2^x$
$\therefore \quad f^{-1}(x) = 2 - 2^x$

iii. Use the features of the logarithm graph to deduce the features of the exponential graph.

iii. The key features of f give those for f^{-1}.

$y = f(x)$	$y = f^{-1}(x)$
asymptote: $x = 2$	asymptote $y = 2$
x-intercept $(1, 0)$	y-intercept $(0, 1)$
y-intercept $(0, 1)$	x-intercept $(1, 0)$

Sketch the graph of $y = f^{-1}(x)$.



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Units 1 & 2 | Area 6 | Sequence 1 | Concepts 3, 4 & 5

Defining logarithms Summary screen and practice questions
The basic logarithmic graph Summary screen and practice questions
Transformations of logarithmic graphs Summary screen and practice questions

Exercise 9.3 Logarithmic functions

Technology free

1. Express the following indicial equations in logarithmic form.

a. $2^3 = 8$
b. $3^5 = 243$
c. $5^0 = 1$
d. $0.01 = 10^{-2}$
e. $b^n = a$
f. $2^{-4} = \frac{1}{16}$

2. Express the following logarithmic equations in indicial form.

a. $\log_4 16 = 2$
b. $\log_{10} 1\,000\,000 = 6$
c. $\log_2 \frac{1}{2} = -1$
d. $\log_3 27 = 3$
e. $\log_5 625 = 4$
f. $\log_2 128 = 7$
g. $\log_3 \frac{1}{9} = -2$
h. $\log_b a = x$

3. **WE4** Evaluate each of the following without technology.

a. $\log_2 16$ **b.** $\log_3 81$

c. $\log_5 125$ **d.** $\log_2 \frac{1}{4}$

e. $\log_{10} 1000$ **f.** $\log_{10} (0.000\,01)$

g. $\log_2 0.25$ **h.** $\log_3 \frac{1}{243}$

i. $\log_2 32$ **j.** $\log_2 \frac{1}{64}$

k. $\log_3(-3)$ **l.** $\log_n n^5$

4. **WE5** Simplify, and evaluate where possible, each of the following without technology.

a. $\log_2 8 + \log_2 10$ **b.** $\log_3 7 + \log_3 15$

c. $\log_{10} 20 + \log_{10} 5$ **d.** $\log_6 8 + \log_6 7$

e. $\log_2 20 - \log_2 5$ **f.** $\log_3 36 - \log_3 12$

g. $\log_5 100 - \log_5 8$ **h.** $\log_2 \frac{1}{3} + \log_2 9$

i. $\log_4 25 + \log_4 \frac{1}{5}$ **j.** $\log_{10} 5 - \log_{10} 20$

k. $\log_3 \frac{4}{5} - \log_3 \frac{1}{5}$ **l.** $\log_2 9 + \log_2 4 - \log_2 12$

m. $\log_3 8 - \log_3 2 + \log_2 5$ **n.** $\log_4 24 - \log_4 2 - \log_4 6$

5. **WE6** Simplify each of the following.

a. $3 \log_{10} 5 + \log_{10} 2$ **b.** $2 \log_2 8 + 3 \log_2 3$

c. $2 \log_3 2 + 3 \log_3 1$ **d.** $\log_5 12 - 2 \log_5 2$

e. $4 \log_{10} 2 - 2 \log_{10} 8$ **f.** $\log_3 4^2 - 3 \log_3 2$

g. $\frac{1}{3} \log_2 27 - \frac{1}{2} \log_2 36$ **h.** $\log_2(x - 4) + 3 \log_2 x$

i. $\frac{1}{2} \log_3 16 + 2 \log_3 4$ **j.** $2 \log_{10}(x + 3) - \log_{10}(x - 2)$

6. **WE7a** Simplify the following.

a. $\frac{\log_3 25}{\log_3 125}$ **b.** $\frac{\log_2 81}{\log_2 9}$

c. $\frac{\log_4 36}{\log_4 6}$ **d.** $\frac{2 \log_{10} 8}{\log_{10} 16}$

e. $\frac{3 \log_5 27}{2 \log_5 9}$ **f.** $\frac{4 \log_3 32}{5 \log_3 4}$

g. $\frac{\log_3 x^6}{\log_3 x^2}$ **h.** $\frac{\log_{10} x^3}{\log_{10} \sqrt{x}}$

i. $\frac{\log_5 x^{\frac{3}{2}}}{\log_5 \sqrt{x}}$ **j.** $\frac{2 \log_2 (x + 1)^3}{\log_2 (x + 1)}$

7. WE7b Express each of the following in simplest form:

a. $\log_3 27 + 1$ **b.** $\log_4 16 + 3$ **c.** $3\log_5 2 - 2$

d. $2 + 3\log_{10} x$ **e.** $2\log_2 5 - 3$ **f.** $4\log_3 2 - 2\log_3 6 + 2$

g. $2\log_6 6 - \log_6 4$ **h.** $\frac{1}{2} + 3\log_{10} x^2$

8. a. WE8 Form the exponential rule for the inverse of $y = \log_{10}(x)$ and hence deduce the graph of $y = \log_{10}(x)$ from the graph of the exponential.

b. Given the points (1, 10), (2, 100) and (3, 1000) lie on the exponential graph in part **a**, explain how these points can be used to illustrate the logarithm law $\log_{10}(m) - \log_{10}(n) = \log_{10}\left(\frac{m}{n}\right)$.

9. a. Simplify $\log_6\left(2^{2x} \times 9^x\right)$ using the inverse relationship between exponentials and logarithms.

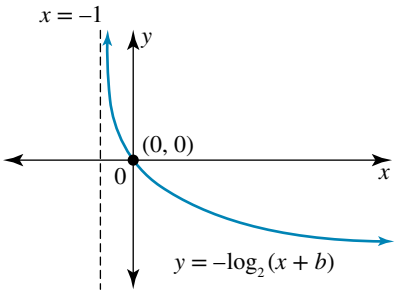
b. Evaluate $2^{-3\log_2(10)}$.

10. Simplify $5^{x\log_5(2) - \log_5(3)}$.

11. a. WE9 Sketch the graph of $y = \log_{10}(x - 1)$ and state its domain.

b. Sketch the graph of $y = \log_5(x) - 1$ and state its domain.

c. The graph of the function for which $f(x) = -\log_2(x + b)$ is shown.



i. Determine the value of b.

ii. State the domain and range of, and form the rule for, the inverse function.

iii. Sketch the graph of $y = f^{-1}(x)$.

12. Consider the function defined by $y = 2 \times (1.5)^{2-x}$.

a. For what value of x does $y = 2$?

b. For what value of y does $x = 0$?

c. Sketch the graph of $y = 2 \times (1.5)^{2-x}$ showing the key features.

d. On the same set of axes sketch the graph of the inverse function.

e. Form the rule for the inverse.

f. Hence state the solution to the equation $2 \times (1.5)^{2-x} = 2 - \log_{1.5}\left(\frac{x}{2}\right)$.

13. a. Evaluate the following.

i. $3^{\log_3(8)}$ **ii.** $10^{\log_{10}(2) + \log_{10}(3)}$

iii. $5^{-\log_5(2)}$ **iv.** $6^{\frac{1}{2}\log_6(25)}$

b. Simplify the following.

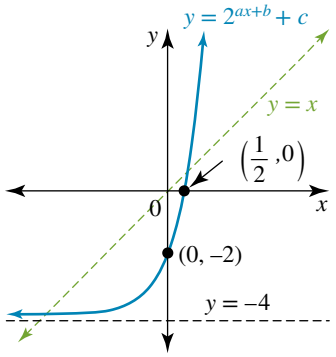
i. $3^{\log_3(x)}$ **ii.** $2^{3\log_2(x)}$

iii. $\log_2(2^x) + \log_3(9^x)$ **iv.** $\log_6\left(\frac{6^{x+1} - 6^x}{5}\right)$

14. Hick's Law arose from research into the time taken for a person to make a decision when faced with a number of possible choices. For n equally probable options, the law is expressed as $t = b\log_2(n + 1)$ where t is the time taken to choose an option, b is a positive constant and $n \geq 2$. Draw a sketch of the time against the number of choices and show that doubling the number of options does not double the time to make the choice between them.



15. The diagram shows the graph of the exponential function $y = 2^{ax+b} + c$. The graph intersects the line $y = x$ twice and cuts the x-axis at $\left(\frac{1}{2}, 0\right)$ and the y-axis at $(0, -2)$.

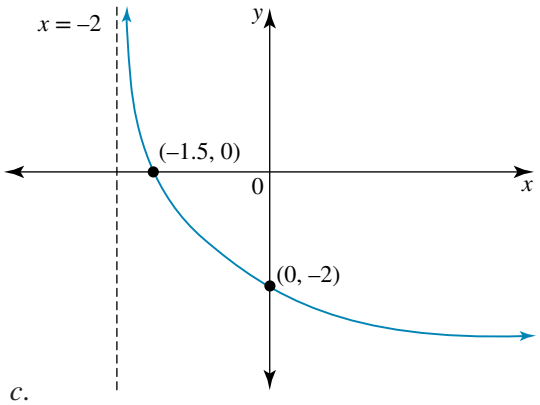


- **a.** Form the rule for the exponential function.
- **b.** Form the rule for the inverse function.
- **c.** For the inverse, state the equation of its asymptote and the coordinates of the points where its graph would cut the x- and y-axes.
- **d.** Copy the diagram and sketch the graph of the inverse on the same diagram. How many points of intersection of the inverse and the exponential graphs are there?
- **e.** The point $\left(\log_2(3), k\right)$ lies on the exponential graph. Calculate the exact value of k.
- **f.** Using the equation for the inverse function, verify that the point $\left(k, \log_2(3)\right)$ lies on the inverse graph.

Technology active

16. **a.** The graph of the function with equation $y = a\log_7(bx)$ contains the points (2,0) and (14, 14). Determine its equation.

- **b.** The graph of the function with equation $y = a\log_3(x) + b$ contains the points $\left(\frac{1}{3}, 8\right)$ and (1,4).
  - **i.** Determine its equation.
  - **ii.** Obtain the coordinates of the point where the graph of the inverse function would cut the y-axis.
- **c.** **i.** For the graph illustrated in the diagram, determine a possible equation in the form $y = a\log_2(x - b) + c$.



  - **ii.** Use the diagram to sketch the graph of the inverse and form the rule for the inverse.
- **d.** Consider the functions f and g for which $f(x) = \log_3(4x + 9)$ and $g(x) = \log_4(2 - 0.1x)$.
  - **i.** Determine the maximal domain of each function.
  - **ii.** State the equations of the asymptotes of the graphs of $y = f(x)$ and $y = g(x)$.
  - **iii.** Calculate the coordinates of the points of intersection of each of the graphs with the coordinate axes.
  - **iv.** Sketch the graphs of $y = f(x)$ and $y = g(x)$ on separate diagrams.

17. Obtain the coordinates of any points of intersection of the graph of $y = 2 \times 3^{\frac{2-x}{2}}$ with its inverse. Express the values to 4 significant figures, where appropriate.

18. Consider the two functions with rules $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$.

- **a.** **i.** Should the graphs of $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$ be the same graphs? Use technology to sketch the graphs of $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$ to verify your answer.
  - **ii.** Give any values of x for which the graphs have the same value and justify algebraically.
- **b.** Sketch the graph of $y = 2\log_3(x)$, stating its domain, range and type of correspondence.
- **c.** Sketch the graph of $y = \log_3\left(x^2\right)$, stating its domain, range and type of correspondence.
- **d.** The graphs in parts b and c are not identical. Explain why this does not contradict the logarithm law $\log_a(m^p) = p\log_a(m)$.

9.4 Modelling with exponential functions

The importance of exponential functions lies in the frequency with which they occur in models of phenomena involving growth and decay situations, in chemical and physical laws of nature and in higher-level mathematical analysis.

9.4.1 Exponential growth and decay models

For time t, the exponential function defined by $y = b \times a^{nt}$ where $a > 1$ represents exponential growth over time if $n > 0$ and exponential decay over time if $n < 0$. The domain of this function would be restricted according to the way the independent time variable t is defined. The rule $y = b \times a^{nt}$ may also be written as $y = b.a^{nt}$.

In some mathematical models such as population growth, the initial population may be represented by a symbol such as N_0. For an exponential decay model, the time it takes for 50% of the initial amount of the substance to decay is called its half-life.

WORKED EXAMPLE 10

The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 2.7^{-kt}$ where Q kg is the amount of the substance present at time t years and Q_0 and k are positive constants.

a. Show that the constant Q_0 represents the initial amount of the substance.

b. If the half-life of the radioactive substance is 100 years, calculate k to one significant figure.

c. If initially there was 25 kg of the radioactive substance, how many kilograms would decay in 10 years? Use the value of k from part b in the calculations.

THINK	WRITE
a. 1. Calculate the initial amount.	**a.** $Q(t) = Q_0 \times 2.7^{-kt}$ The initial amount is the value of Q when $t = 0$. Let $t = 0$: $Q(0) = Q_0 \times 2.7^0$ $= Q_0$ Therefore, Q_0 represents the initial amount of the substance.
b. 1. Form an equation in k from the given information. *Note:* It does not matter that the value of Q_0 is unknown since the Q_0 terms cancel.	**b.** The half-life is the time it takes for 50% of the initial amount of the substance to decay. Since the half-life is 100 years, when $t = 100$, $Q(100) = 50\%$ of Q_0 $Q(100) = 0.50Q_0$ ..... (1) From the equation, $Q(t) = Q_0 \times 2.7^{-kt}$. When $t = 100$, $Q(100) = Q_0 \times 2.7^{-k(100)}$ $\therefore \quad Q(100) = Q_0 \times 2.7^{-100k}$ .... (2) Equate equations (1) and (2): $0.50Q_0 = Q_0 \times 2.7^{-100k}$ Cancel Q_0 from each side: $0.50 = 2.7^{-100k}$

2. Solve the exponential equation to obtain k to the required accuracy.
See page 411 for the rule to convert to a different base.

Convert to the equivalent logarithm form.
$-100k = \log_{2.7}(0.5)$

$$k = -\frac{1}{100}\log_{2.7}(0.5)$$
$$= -\frac{1}{100} \times \frac{\log_{10}(0.5)}{\log_{10}(2.7)}$$
$$\approx 0.007$$

c. 1. Use the values of the constants to state the actual rule for the exponential decay model.

c. $Q_0 = 25,\ k = 0.007$
$\therefore\ Q(t) = 25 \times 2.7^{-0.007t}$

2. Calculate the amount of the substance present at the time given.

When $t = 10$,
$Q(10) = 25 \times 2.7^{-0.07}$
≈ 23.32

3. Calculate the amount that has decayed. *Note:* Using a greater accuracy for the value of k would give a slightly different answer for the amount decayed.

Since $25 - 23.32 = 1.68$, in 10 years approximately 1.68 kg will have decayed.

TI | THINK

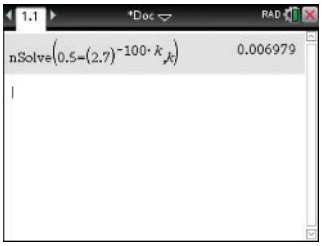
b. 1. Set up an equation to be solved.

2. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as nSolve($0.5 = 2.7^{-100k}, k$) then press ENTER.

3. The answer appears on the screen.

WRITE

When $t = 100$, $Q = 0.5Q_0$.
Substituting these into the equation $Q(t) = Q_0 \times 2.7^{-kt}$ gives
$0.5Q_0 = Q_0 \times 2.7^{-100k}$.
Dividing both sides by Q_0 gives
$0.5 = 2.7^{-100k}$.



$k = 0.007$ (1 sig. figure)

CASIO | THINK

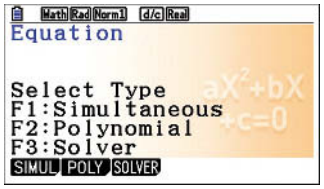
b. 1. Set up an equation to be solved.

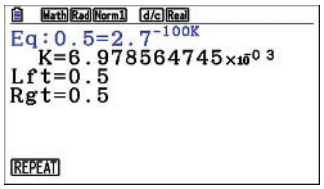
On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as $0.5 = 2.7^{-100k}$ then press EXE.
Select SOLVE by pressing F6.

3. The answer appears on the screen.

WRITE

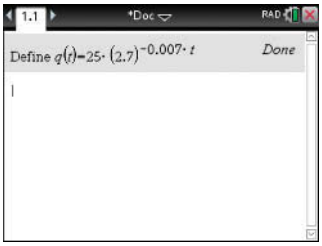
When $t = 100$, $Q = 0.5Q_0$.
Substituting these into the equation $Q(t) = Q_0 \times 2.7^{-kt}$ gives $0.5Q_0 = Q_0 \times 2.7^{-100k}$.
Dividing both sides by Q_0 gives
$0.5 = 2.7^{-100k}$.



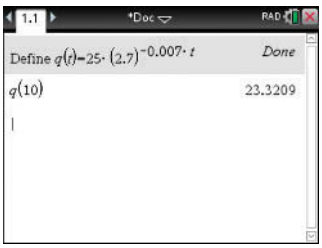


$k = 0.007$ (1 sig. figure)

c. 1. On a Calculator page, press MENU then select
1: Actions
1: Define
then complete the entry line as
Define $q(t) = 25 \times 2.7^{-0.007t}$
then press ENTER.

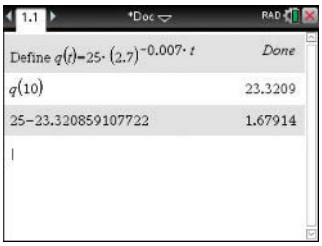


2. Complete the next entry line as $q(10)$
then press ENTER.
Interpret the result.



When $t = 10$, $Q \approx 23.32$.

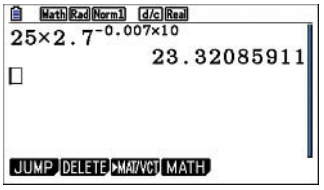
3. Complete the next entry line as
25-Ans
then press ENTER.



4. The answer appears on the screen.

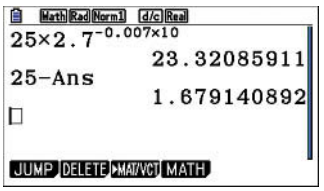
Approximately 1.68 kg will have decayed in 10 years.

c. 1. On a Run-Matrix screen, complete the next entry line as $25 \times 2.7^{-0.007 \times 10}$ then press EXE.
Interpret the result.



When $t = 10$, $Q \approx 23.32$.

2. Complete the next entry line as
25-Ans
then press EXE.



3. The answer appears on the screen.

Approximately 1.68 kg will have decayed in 10 years.

9.4.2 Analysing data

One method for detecting if data has an exponential relationship can be carried out using logarithms. If the data is suspected of following an exponential rule such as $y = A \times 10^{kx}$, then the graph of $\log(y)$ against x should be linear. The reasoning for this is as follows.

$$y = A \times 10^{kx}$$

$$\therefore \frac{y}{A} = 10^{kx}$$

$$\therefore \log\left(\frac{y}{A}\right) = kx$$

$$\therefore \log(y) - \log(A) = kx$$

$$\therefore \log(y) = kx + \log(A)$$

This equation can be written in the form $Y = kx + c$ where $Y = \log(y)$ and $c = \log(A)$.

The graph of Y versus x is a straight line with gradient k and vertical axis Y-intercept $(0, \log(A))$.

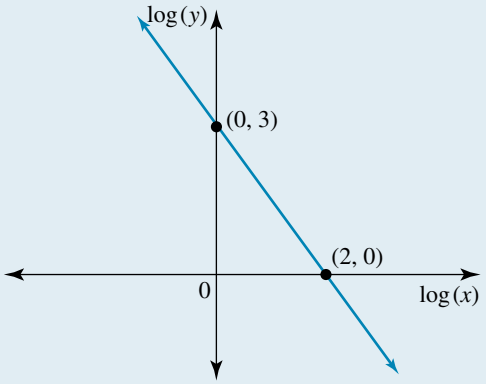
Such an analysis is called a **semi-log plot**. While experimental data is unlikely to give a perfect fit, the equation would describe the line of best fit for the data.

Logarithms can also be effective in determining a power law that connects variables. If the law connecting the variables is of the form $y = x^p$ then $\log(y) = p\log(x)$.

Plotting $\log(y)$ values against $\log(x)$ values will give a straight line of gradient p if the data does follow such a law. Such an analysis is called a **log-log plot**.

WORKED EXAMPLE 11

For a set of data $\{(x, y)\}$, plotting $\log(y)$ versus $\log(x)$ gave the straight line shown in the diagram. Form the equation of the graph and hence determine the rule connecting y and x.



THINK	WRITE
1. State the gradient and the coordinates of the intercept with the vertical axis.	Gradient $= \frac{\text{rise}}{\text{run}}$ $= \frac{-3}{2}$ Intercept with vertical axis: $(0,\ 3)$
2. Form the equation of the line.	Let $Y = \log(y)$ and $X = \log(x)$. The equation of the line is $Y = mX + c$ where $m = -\frac{3}{2}, c = 3$. Therefore the equation of the line is $Y = -\frac{3}{2}X + 3$.
3. Express the equation in terms of the variables marked on the axes of the given graph.	The vertical axis is $\log(y)$ and the horizontal axis is $\log(x)$, so the equation of the graph is $\log(y) = -\frac{3}{2}\log(x) + 3$. $\therefore \log(y) = -1.5\log(x) + 3$
4. Collect the terms involving logarithms together and simplify to create a logarithm statement.	$\log(y) + 1.5\log(x) = 3$ $\log(y) + \log\left(x^{1.5}\right) = 3$ $\log\left(yx^{1.5}\right) = 3$
5. Express the equation with y as the subject. *Note:* Remember the base of the logarithm is 10.	$\therefore \log_{10}\left(yx^{1.5}\right) = 3$ $yx^{1.5} = 10^3$ $y = 1000x^{-1.5}$

studyon

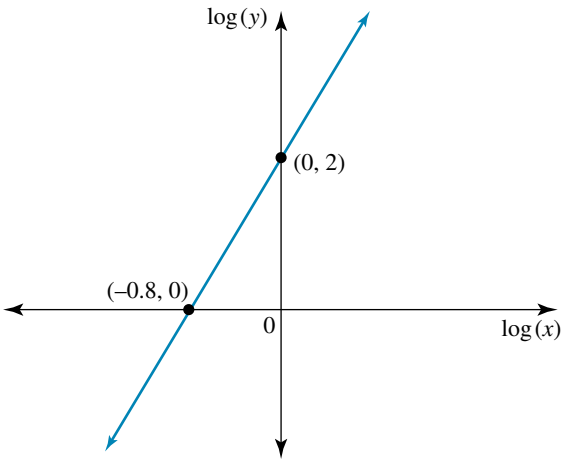
Units 1 & 2 > Area 6 > Sequence 1 > Concept 6

Modelling with exponential functions Summary screen and practice questions

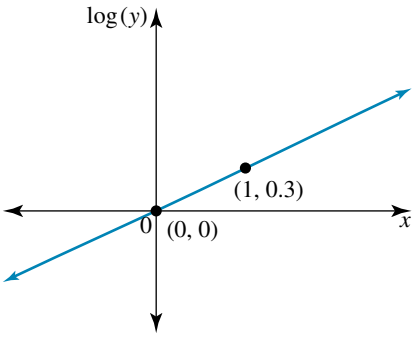
Exercise 9.4 Modelling with exponential functions

Technology active

1. **WE10** The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 1.7^{-kt}$ where Q is the amount of the substance present at time t years and Q_0 and k are positive constants.
   a. Show that the constant Q_0 represents the initial amount of the substance.
   b. If the half-life of the radioactive substance is 300 years, calculate k to one significant figure.
   c. If initially there was 250 kg of the radioactive substance, how many kilograms would decay in 10 years? Use the value of k from part **b** in the calculations.
2. The manager of a small business is concerned about the amount of time she spends dealing with the growing number of emails she receives. The manager starts keeping records and finds the average number of emails received per day can be modelled by $D = 42 \times 2^{\frac{t}{16}}$ where D is the average number of emails received per day t weeks from the start of the records.
   a. How many daily emails on average was the manager receiving when she commenced her records?
   b. After how many weeks does the model predict that the average number of emails received per day will double?
3. **WE11** For a set of data $\{(x, y)\}$, plotting $\log(y)$ versus $\log(x)$ gave the straight line shown in the diagram.
   Form the equation of the graph and hence determine the rule connecting y and x.



4. For a set of data $\{(x, y)\}$, the semi-log plot of $\log(y)$ versus x gave the straight line shown in the diagram.



   Form the equation of the graph and hence determine an exponential rule connecting y and x.
5. The value V of a new car depreciates so that its value after t years is given by $V = V_0 \times 2^{-kt}$.
   a. If 50% of the purchase value is lost in 5 years, calculate k.
   b. How long does it take for the car to lose 75% of its purchase value?

6. The number of *Drosophila* (fruit flies), N, in a colony after t days of observation is modelled by $N = 30 \times 2^{0.072t}$. Give whole-number answers to the following.



- **a.** How many *Drosophila* were present when the colony was initially observed?
- **b.** How many of the insects were present after 5 days?
- **c.** How many days does it take the population number to double from its initial value?
- **d.** Sketch a graph of N versus t to show how the population changes.
- **e.** After how many days will the population first exceed 100?

7. The value of an investment which earns compound interest can be calculated from the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ where P is the initial investment, r the interest rate per annum (yearly), n the number of times per year the interest is compounded and t the number of years of the investment.
An investor deposits \$2000 in an account where interest is compounded monthly.

- **a.** If the interest rate is 3% per annum:
  - **i.** Show that the formula giving the value of the investment is $A = 2000\,(1.0025)^{12t}$.
  - **ii.** Calculate how much the investment is worth after a 6-month period.
  - **iii.** What time period would be needed for the value of the investment to reach \$2500?
- **b.** The investor would like the \$2500 to grow to \$2500 in a shorter time period. What would the interest rate, still compounded monthly, need to be for the goal to be achieved in 4 years?

8. A cup of coffee is left to cool on a kitchen table inside a Brisbane home. The temperature of the coffee T(°C) after t minutes is thought to be given by $T = 85 \times 3^{-0.008t}$.



- **a.** By how many degrees does the coffee cool in 10 minutes?
- **b.** How long does it take for the coffee to cool to 65 °C?
- **c.** Sketch a graph of the temperature of the coffee for $t \in [0, 40]$.
- **d.** By considering the temperature the model predicts the coffee will eventually cool to, explain why the model is not realistic in the long term.

9. The contents of a meat pie immediately after being heated in a microwave have a temperature of 95 °C. The pie is removed from the microwave and left to cool. A model for the temperature of the pie as it cools is given by $T = a \times 3^{-0.13t} + 25$ where T is the temperature after t minutes of cooling.



- **a.** Calculate the value of a.
- **b.** What is the temperature of the contents of the pie after being left to cool for 2 minutes?
- **c.** Determine how long, to the nearest minute, it will take for the contents of the meat pie to cool to 65 °C.
- **d.** Sketch the graph showing the temperature over time and state the temperature to which this model predicts the contents of the pie will eventually cool if left unattended.

10. The barometric pressure P, measured in kilopascals, at height h above sea level, measured in kilometres, is given by $P = P_o \times 10^{-kh}$ where P_o and k are positive constants. The pressure at the top of Mount Everest is approximately one third that of the pressure at sea level.

a. Given the height of Mount Everest is approximately 8848 metres, calculate the value of k to 2 significant figures.
Use the value obtained for k for the remainder of this question.

b. Mount Kilimanjaro has a height of approximately 5895 metres. If the atmospheric pressure at its summit is approximately 48.68 kilopascals, calculate the value of P_o to 3 decimal places.

c. Use the model to estimate the atmospheric pressure to 2 decimal places at the summit of Mont Blanc, 4810 metres, and of Mount Kosciuszko, 2228 metres in height.

d. Draw a graph of the atmospheric pressure against height showing the readings for the four mountains from the above information.



11. The common Indian mynah bird was introduced into Australia in order to control insects affecting market gardens in Melbourne. It is now considered to be Australia's most important pest problem. In 1976, the species was introduced to an urban area in New South Wales. By 1991 the area averaged 15 birds per square kilometre and by 1994 the density reached an average of 75 birds per square kilometre.



A model for the increasing density of the mynah bird population is thought to be $D = D_0 \times 10^{kt}$ where D is the average density of the bird per square kilometre t years after 1976 and D_0 and k are constants.

a. Use the given information to set up a pair of simultaneous equations in D and t.

b. Solve these equations to show that $k = \frac{1}{3}\log(5)$ and $D_0 = 3 \times 5^{-4}$ and hence that $k \approx 0.233$ and $D_0 \approx 0.005$.

c. A project was introduced in 1996 to curb the growth in numbers of these birds. What does the model predict was the average density of the mynah bird population at the time the project was introduced in the year 1996? Use $k \approx 0.233$ and $D_0 \approx 0.005$ and round the answer to the nearest whole number.

d. Sometime after the project is successfully implemented, a different model for the average density of the bird population becomes applicable. This model is given by $D = 30 \times 10^{-\frac{t}{3}} + b$. Four years later, the average density is reduced to 40 birds per square kilometre. How much can the average density expect to be reduced?

12. Carbon dating enables estimates of the age of fossils of once living organisms to be ascertained by comparing the amount of the radioactive isotope carbon-14 remaining in the fossil with the normal amount present in the living entity, which can be assumed to remain constant during the organism's life. It is known that carbon-14 decays with a half-life of approximately 5730 years according to an exponential model of the form $C = C_o \times \left(\frac{1}{2}\right)^{kt}$, where C is the amount of the isotope remaining in the fossil t years after death and C_o is the normal amount of the isotope that would have been present when the organism was alive.



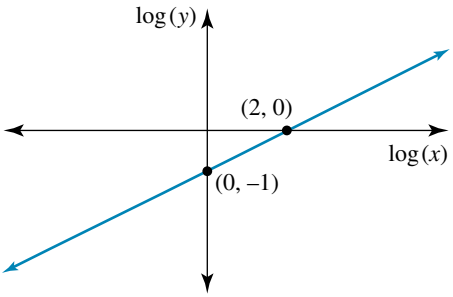
a. Calculate the exact value of the positive constant k.

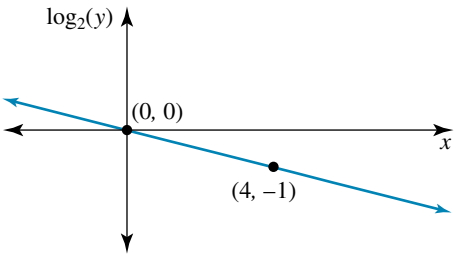
b. The bones of an animal are unearthed during digging explorations by a mining company. The bones are found to contain 83% of the normal amount of the isotope carbon-14. Estimate how old the bones are.

13. a. Obtain the equation of the given linear graphs and hence determine the relationship between y and x.

i. The linear graph of $\log_{10}(y)$ against $\log_{10}(x)$ is shown.

ii. The linear graph of $\log_2(y)$ against x is shown.





b. The acidity of a solution is due to the presence of hydrogen ions. The concentration of these ions is measured by the pH scale calculated as $\text{pH} = -\log\left(\left[\text{H}^+\right]\right)$ where $\left[\text{H}^+\right]$ is the concentration of hydrogen ions.

i. The concentration of hydrogen ions in bleach is 10^{-13} per mole and in pure water the concentration is 10^{-7} per mole. What are the pH readings for bleach and for pure water?

ii. Lemon juice has a pH reading of 2 and milk has a pH reading of 6. Use scientific notation to express the concentration of hydrogen ions in each of lemon juice and milk and then write these concentrations as numerals.



iii. Solutions with pH smaller than 7 are acidic and those with pH greater than 7 are alkaline. Pure water is neutral. How much more acidic is lemon juice than milk?

iv. For each one unit of change in pH, explain the effect on the concentration of hydrogen ions and acidity of a solution.

14. The data shown in the table gives the population of Australia, in millions, in years since 1960.

	1975	**1990**	**2013**
x (years since 1960)	15	30	53
y (population in millions)	13.9	17.1	22.9
$\log(y)$			

a. Complete the third row of the table by evaluating the $\log(y)$ values to 2 decimal places.
b. Plot $\log(y)$ against x and construct a straight line to fit the points.
c. Show that the equation of the line is approximately $Y = 0.006x + 1.05$ where $Y = \log(y)$.
d. Use the equation of the line to show that the exponential rule between y and x is approximately $y = 11.22 \times 10^{0.006x}$.
e. After how many years did the population double the 1960 population?
f. It is said that the population of Australia is likely to exceed 28 million by the year 2030. Does this model support this claim?

15. Experimental data yielded the following table of values.

x	1	1.5	2	2.5	3	3.5	4
y	5.519	6.483	7.615	8.994	10.506	12.341	14.496

a. Enter the data into a calculator and obtain the rule connecting the data using:
i. exponential regression
ii. logarithmic regression.
b. Graph the data on the calculator to confirm which rule better fits the data.

16. Following a fall from his bike, Stephan is feeling some shock but not, initially, a great deal of pain. However, his doctor gives him an injection for relief from the pain that he will start to feel once the shock of the accident wears off. The amount of pain Stephan feels over the next 10 minutes is modelled by the function $P(t) = (200t + 16) \times 2.7^{-t}$, where P is the measure of pain on a scale from 0 to 100 that Stephan feels t minutes after receiving the injection.
a. Give the measure of pain Stephan is feeling:
i. at the time the injection is administered
ii. 15 seconds later when his shock is wearing off but the injection has not reached its full effect.
b. Use technology to draw the graph showing Stephan's pain level over the 10-minute interval and hence give, to 2 decimal places:
i. the maximum measure of pain he feels
ii. the number of seconds it takes for the injection to start lowering his pain level
iii. his pain levels after 5 minutes and after 10 minutes have elapsed.
c. Over the 10-minute interval, when was the effectiveness of the injection greatest?
d. At the end of the 10 minutes, Stephan receives a second injection modelled by $P(t) = (100(t - 10) + a) \times 2.7^{-(t-10)}$, $10 \leq t \leq 20$.
i. Determine the value of a.
ii. Sketch the pain measure over the time interval $t \in [0, 20]$ and label the maximum points with their coordinates.

9.5 Solving equations with indices

9.5.1 Logarithms as operators

Just as both sides of an equation may be raised to a power and the equality still holds, taking logarithms of both sides of an equation maintains the equality.

If $m = n$, then it is true that $\log_a(m) = \log_a(n)$ and vice versa, provided the same base is used for the logarithms of each side.

This application of logarithms can provide an important tool when solving indicial equations.

Consider again the equation $2^x = 5$ where the solution was given as $x = \log_2(5)$.

Take base 10 logarithms of both sides of this equation.

$$2^x = 5$$
$$\log_{10}(2^x) = \log_{10}(5)$$

Using one of the logarithm laws, this becomes $x\log_{10}(2) = \log_{10}(5)$ from which the solution to the indicial equation is obtained as $x = \dfrac{\log_{10}(5)}{\log_{10}(2)}$. This form of the solution can be evaluated on a scientific calculator and is the prime reason for choosing base 10 logarithms in solving the indicial equation.

It also demonstrates that $\log_2(5) = \dfrac{\log_{10}(5)}{\log_{10}(2)}$, which is a particular example of another logarithm law called the *change of base law*.

Change of base law for calculator use

The equation $a^x = p$ for which $x = \log_a(p)$ could be solved in a similar way to $2^x = 5$, giving the solution as $x = \dfrac{\log_{10}(p)}{\log_{10}(a)}$. Thus $\log_a(p) = \dfrac{\log_{10}(p)}{\log_{10}(a)}$. This form enables decimal approximations to logarithms to be calculated on scientific calculators.

The change of base law is the more general statement allowing base a logarithms to be expressed in terms of any other base b as $\log_a(p) = \dfrac{\log_b(p)}{\log_b(a)}$.

Convention

There is a convention that if the base of a logarithm is not stated, this implies it is base 10. As it is on a calculator, $\log(n)$ represents $\log_{10}(n)$. When working with base 10 logarithms it can be convenient to adopt this convention.

WORKED EXAMPLE 12

a. State the exact solution to $5^x = 8$ and calculate its value to 3 decimal places.
b. Calculate the exact value and the value to 3 decimal places of the solution to the equation $2^{1-x} = 6^x$.

THINK	WRITE
a. 1. Convert to the equivalent form and state the exact solution.	**a.** $5^x = 8$ $\therefore \quad x = \log_5(8)$ The exact solution is $x = \log_5(8)$.

2. Use the change of base law to express the answer in terms of base 10 logarithms.

Since

$$\log_a(p) = \frac{\log_{10}(p)}{\log_{10}(a)}$$

then

$$\log_5(8) = \frac{\log_{10}(8)}{\log_{10}(5)}$$

$$\therefore x = \frac{\log_{10}(8)}{\log_{10}(5)}$$

3. Calculate the approximate value.

$\therefore \quad x \approx 1.292$ to 3 decimal places.

b. 1. Take base 10 logarithms of both sides.
Note: The convention is not to write the base 10.

b. $2^{1-x} = 6^x$

Take logarithms to base 10 of both sides:

$\log\left(2^{1-x}\right) = \log(6^x)$

$(1-x)\log(2) = x\log(6)$

2. Apply the logarithm law so that x terms are no longer exponents.

3. Solve the linear equation in x.
Note: This is no different to solving any other linear equation of the form $a - bx = cx$ except the constants a, b, c are expressed as logarithms.

Expand:

$\log(2) - x\log(2) = x\log(6)$

Collect x terms together:

$\log(2) = x\log(6) + x\log(2)$

$= x(\log(6) + \log(2))$

$$x = \frac{\log(2)}{\log(6) + \log(2)}$$

This is the exact solution.

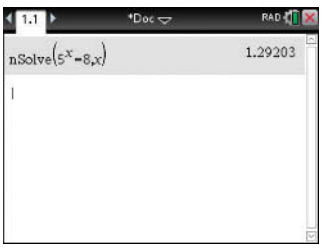
4. Calculate the approximate value.
Note: Remember to place brackets around the denominator for the division.

$x \approx 0.279$ to 3 decimal places.

TI | THINK

a. 1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as
nSolve($5^x = 8, x$)
then press ENTER.

WRITE



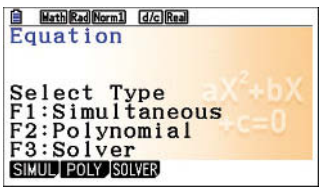
2. The answer appears on the screen.

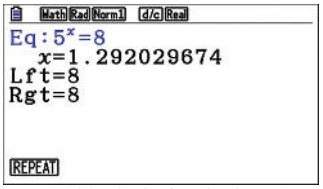
$x = 1.292$ (3 decimal places)

CASIO | THINK

a. 1. On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as
$5^x = 8$
then press EXE.
Select SOLVE by pressing F6.

WRITE

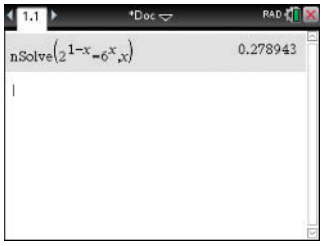




2. The answer appears on the screen.

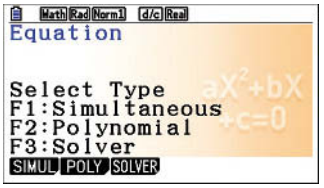
$x = 1.292$ (3 decimal places)

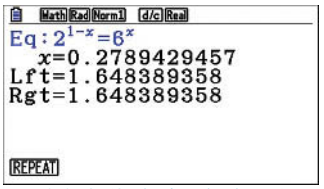
b. 1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as
nSolve($2^{1-x} = 6^x, x$)
then press ENTER.



2. The answer appears on the screen. $x = 0.279$ (3 decimal places)

b. 1. On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as
$2^{1-x} = 6^x$
then press EXE.
Select SOLVE by pressing F6.





2. The answer appears on the screen. $x = 0.279$ (3 decimal places)

9.5.2 Equations containing logarithms

While the emphasis in this chapter is on exponential (indicial) relations for which some knowledge of logarithms is essential, it is important to know that logarithms contribute substantially to Mathematics. As such, some equations involving logarithms are included, allowing further consolidation of the laws which logarithms must satisfy.

Remembering the requirement that x must be positive for $\log_a(x)$ to be real, it is advisable to check any solution to an equation involving logarithms. Any value of x which when substituted back into the original equation creates a '$\log_a$ (negative number)' term must be rejected as a solution. Otherwise, normal algebraic approaches together with logarithm laws are the techniques for solving such equations.

WORKED EXAMPLE 13

Find x if $\log_3 9 = x - 2$.

THINK	WRITE
1. Write the equation.	$\log_3 9 = x - 2$
2. Simplify the logarithm using the 'logarithm of a power' law and the fact that $\log_3 3 = 1$.	$\log_3 3^2 = x - 2$ $2\log_3 3 = x - 2$ $2 = x - 2$
3. Solve for x by adding 2 to both sides.	$x = 4$

WORKED EXAMPLE 14

Solve the equation $\log_6(x) + \log_6(x - 1) = 1$ for x.

THINK	WRITE
1. Apply the logarithm law which reduces the equation to one logarithm term.	$\log_6(x) + \log_6(x - 1) = 1$ $\therefore \log_6(x(x - 1)) = 1$ $\therefore \log_6(x^2 - x) = 1$

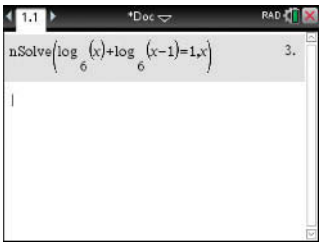
2. Convert the logarithm form to its equivalent form. *Note:* An alternative method is to write $\log_6\left(x^2 - x\right) = \log_6(6)$ from which $x^2 - x = 6$ is obtained.	Converting from logarithm form to index form gives: $x^2 - x = 6^1$ $\therefore x^2 - x = 6$
3. Solve the quadratic equation.	$x^2 - x - 6 = 0$ $\therefore (x - 3)(x + 2) = 0$ $\therefore x = 3,\ x = -2$
4. Check the validity of both solutions in the original equation.	Check in $\log_6(x) + \log_6(x - 1) = 1$ If $x = 3$, LHS $= \log_6(3) + \log_6(2)$ $= \log_6(6)$ $= 1$ $=$ RHS If $x = -2$, LHS $= \log_6(-2) + \log_6(-3)$ which is not admissible. Therefore reject $x = -2$.
5. State the answer.	The solution is $x = 3$.

TI | THINK

1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as nSolve($\log_6(x) + \log_6(x - 1) = 1, x$)
then press ENTER.

2. The answer appears on the screen.

WRITE



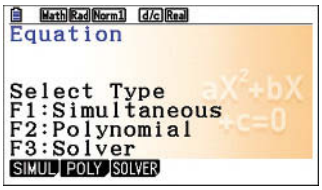
$x = 3$

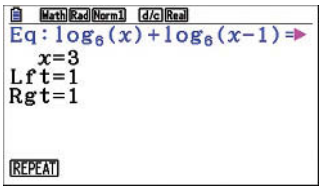
CASIO | THINK

1. On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as $\log_6(x) + \log_6(x - 1) = 1$
then press EXE.
Select SOLVE by pressing F6.
Note: The $\log_a$ b template can be found by pressing OPTN, selecting CALC by pressing F2, then selecting $\log_a b$ by pressing F4.

2. The answer appears on the screen.

WRITE





$x = 3$

studyon

Units 1 & 2 > Area 6 > Sequence 1 > Concept 7

Solving equations with indices Summary screen and practice questions

Exercise 9.5 Solving equations with indices

Technology free

1. **WE13** Find x in each of the following.
   a. $\log_2 4 = x$
   b. $\log_9 1 = x$
   c. $\log_3 27 = x$
   d. $\log_4 256 = x$
   e. $\log_{10} \frac{1}{10} = x$
   f. $\log_3 \frac{1}{9} = x$
   g. $2\log_2 8 = x$
   h. $\log_3 81 = 2x$
   i. $\log_{10} 1000 = 2x - 1$
   j. $2\log_2 32 = 3x + 1$
2. If $\log_2(3) - \log_2(2) = \log_2(x) + \log_2(5)$, solve for x.
3. **WE14** Solve the equation $\log_3(x) + \log_3(2x + 1) = 1$ for x.
4. Solve the equation $\log_6(x) - \log_6(x - 1) = 2$ for x.
5. a. Express the following as a logarithm statement with the index as the subject.
   i. $2^2 = 32$
   ii. $4^{\frac{3}{2}} = 8$
   iii. $10^{-3} = 0.001$
   b. Express the following as an index statement.
   i. $\log_2(16) = 4$
   ii. $\log_9(3) = \frac{1}{2}$
   iii. $\log_{10}(0.1) = -1$
6. Given $\log_a(3) = p$ and $\log_a(5) = q$, express the following in terms of p and q.
   a. $\log_a(15)$
   b. $\log_a(125)$
   c. $\log_a(45)$
   d. $\log_a(0.6)$
   e. $\log_a\left(\frac{25}{81}\right)$
   f. $\log_a\left(\sqrt{5}\right) \times \log_a\left(\sqrt{27}\right)$
7. Express y in terms of x.
   a. $\log_{10}(y) = \log_{10}(x) + 2$
   b. $\log_2\left(x^2\sqrt{y}\right) = x$
   c. $2\log_2\left(\frac{y}{2}\right) = 6x - 2$
   d. $x = 10^{y-2}$
   e. $\log_{10}\left(10^{3xy}\right) = 3$
   f. $10^{3\log_{10}(y)} = xy$
8. Solve the following equations, giving exact solutions.
   a. $2^{2x} - 14 \times 2^x + 45 = 0$
   b. $5^{-x} - 5^x = 4$
   c. $9^{2x} - 3^{1+2x} + 2 = 0$
   d. $\log_a\left(x^3\right) + \log_a\left(x^2\right) - 4\log_a(2) = \log_a(x)$
   e. $\left(\log_2(x)\right)^2 - \log_2\left(x^2\right) = 8$
   f. $\frac{\log_{10}\left(x^3\right)}{\log_{10}(x+1)} = \log_{10}(x)$

Technology active

9. a. **WE12** State the exact solution to $7^x = 15$ and calculate its value to 3 decimal places.
   b. Calculate the exact value and the value to 3 decimal places of the solution to the equation $3^{2x+5} = 4^x$.
10. Solve the following equations correct to 3 decimal places.
    a. $2^x = 11$
    b. $2^x = 0.6$
    c. $3^x = 20$
    d. $3^x = 1.7$
    e. $5^x = 8$
    f. $0.7^x = 3$
    g. $10^{x-1} = 18$
    h. $3^{x+2} = 12$
    i. $2^{2x+1} = 5$
    j. $4^{3x+1} = 24$
    k. $10^{-2x} = 7$
    l. $8^{2-x} = 0.75$
11. Rewrite each of the following in the equivalent index or logarithm form and hence calculate the value of x.
    a. $x = \log_2\left(\frac{1}{8}\right)$
    b. $\log_{25}(x) = -0.5$
    c. $10^{(2x)} = 4$. Express the answer to 2 decimal places.
    d. $3 = e^{-x}$. Express the answer to 2 decimal places.
    e. $\log_x(125) = 3$
    f. $\log_x(25) = -2$

12. a. Express $\log_2(10)$ in terms of $\log_{10}(2)$.

b. State the exact solution and then give the approximate solution to 4 significant figures for each of the following indicial equations.

i. $11^x = 18$ **ii.** $5^{-x} = 8$ **iii.** $7^{2x} = 3$

c. Obtain the approximate solution to 4 significant figures for each of the following inequations.

i. $3^x \leq 10$ **ii.** $5^{-x} > 0.4$

d. Solve the following equations.

i. $2^{\log_5(x)} = 8$ **ii.** $2^{\log_2(x)} = 7$

13. Solve the indicial equations to obtain the value of x to 2 decimal places.

a. $7^{1-2x} = 4$ **b.** $10^{-x} = 5^{x-1}$

c. $5^{2x-9} = 3^{7-x}$ **d.** $10^{3x+5} = 6^{2-3x}$

e. $0.25^{4x} = 0.8^{2-0.5x}$ **f.** $4^{x+1} \times 3^{1-x} = 5^x$

14. a. Give the solution to $12^x = 50$ to 4 significant figures.

b. Give the exact solution to the equation $\log(5x) + \log(x + 5) = 1$.

15. a. Evaluate $\log_{10}(5) + \log_5(10)$ using technology on 'exact' mode and explain the answer obtained.

b. Evaluate $\log_y(x) \times \log_x(y)$ and explain how the result is obtained.

16. The logarithmic slide rule is a compact device for rapidly performing calculations with limited accuracy. The invention of logarithms in 1614 by John Napier made it possible to multiply and divide numbers by the more simple operations of addition and subtraction.



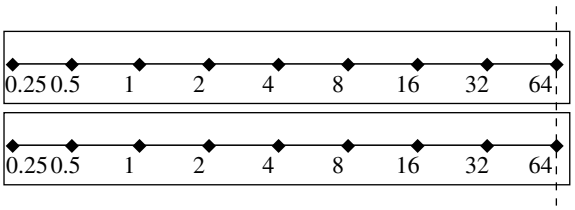
In this investigation we will construct a primitive slide rule.

Number	Power of 2
0.25	−2
0.5	−1
1	0
2	1
4	2
8	3
16	4
32	5
64	6

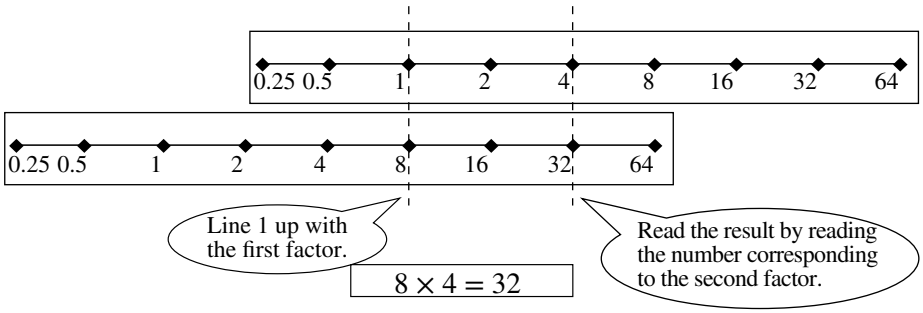
a. Take two strips of card about 3 cm by 20 cm. Mark both cards as shown, using the numbers form the table above.

You will notice that the scale used is a logarithmic scale using 2 as a base. That is, the distance from 1 to 8 is 3 units ($\log_2(8) = 3$).

Also, the distance from 1 to 0.25 is $-2(\log_2(0.25) = -2)$.



To multiply two numbers we need only to add the powers so that $8 \times 4 \rightarrow 2^3 \times 2^2 \rightarrow 2^5 \rightarrow 32$. Thus, multiplying 8 by 4 is equivalent to adding 2 and 3. The operation of multiplication is converted to addition. Your slide rule can be used to perform this addition.



This slide rule is quite primitive and in its present form you would not use it to multiply 5 to 10. However, this principle provided the basis for scientific calculations before the advent of the electronic calculator in the 1960s and 1970s.

b. Use your slide rule to calculate 0.25×32. Check your answer using technology.

c. Use your slide rule to calculate $32 \div 4$. (Remember, division corresponds to a subtraction of exponents.) Check your answer using technology.

d. Construct a base 10 slide rule.

9.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

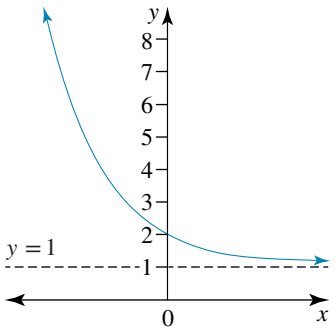
Simple familiar

1. MC The statement $3^5 = 243$ expressed in logarithm form would be:

A. $\log_3(5) = 243$ **B.** $\log_5(3) = 243$

C. $\log_5(243) = 3$ **D.** $\log_3(243) = 5$

2. MC A possible equation for the given graph could be:

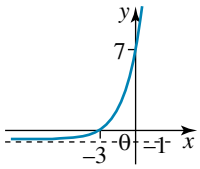


A. $y = 2 \times 3^{-x}$ **B.** $y = 2 \times 3^{x+1}$

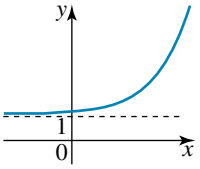
C. $y = -3^x + 1$ **D.** $y = 3^{-x} + 1$

3. MC Identify which of the following graphs best represents the function $y = 2^{x+3} - 1$.

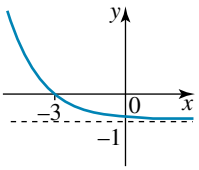
A.



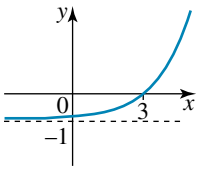
B.



C.



D.



4. MC When expressed in log form, $5^x = 250$ becomes:

A. $\log_x 5 = 250$ **B.** $\log_5 x = 250$ **C.** $\log_5 250 = x$ **D.** $\log_x 250 = 5$

5. MC The value of $\log_7 49 + 3\log_2 8 - 4$ is:

A. 3 **B.** 7 **C.** 0 **D.** 69

6. MC If $25^{2-x} = 125$, then x is equal to:

A. 1 **B.** $\frac{1}{2}$ **C.** -1 **D.** 2

7. Solve the following equations.

a. $2x^5 = 486$ **b.** $8^{x+1} \times 2^{2x} = 4^{3x-1}$

8. Evaluate the following.

a. $\log_6(9) - \log_6\left(\frac{1}{4}\right)$ **b.** $2\log_a(4) + 0.5\log_a(16) - 6\log_a(2)$ **c.** $\frac{\log_a(27)}{\log_a(3)}$

9. Sketch the graphs of the following, stating the equation of the asymptote, the coordinates of any points of intersection with the axes, and the domain and range.

a. $y = 2^{3x} - 1$ **b.** $y = -2 \times 3^{(x-1)}$ **c.** $y = 5 - 5^{-x}$

10. a. Describe the transformations which map $y = \log_3(x) \rightarrow y = -\log_3(x+3)$ and hence state the equation of the asymptote of $y = -\log_3(x+3)$.

b. Give the rule for the inverse of $y = -\log_3(x+3)$, stating its domain and range, and sketch the graphs of $y = -\log_3(x+3)$ and its inverse on the same set of axes.

c. Describe the transformations which map $y = 7^x \rightarrow y = 5 \times 7^{1-x}$ and state the equation of the asymptote of $y = 5 \times 7^{1-x}$.

d. Given $f: R, f(x) = 5 \times 7^{1-x}$, form the inverse function f^{-1}.

You may choose to use technology to answer questions 11 and 12.



11. A number of deer, N, are introduced to a reserve and its population can be predicted by the model $N = 120\left(1.1^t\right)$, where t is the number of years since introduction.

a. Find the initial number of deer in the reserve.

b. Find the number of deer after:

i. 2 years **ii.** 4 years **iii.** 6 years.



12. Prior to a mice plague which lasts 6 months, the population of mice in a country region is estimated to be 10 000. The mice population doubles every month during the plague. If P represents the mice population and t is the number of months after the plague starts:

a. express P as a function of t

b. find the population after

i. 3 months **ii.** 6 months

c. calculate how long it takes the population to reach 100 000 during the plague.

Complex familiar

13. Solve the equations for x.

a. $3^{1-7x} = 81^{x-2} \times 9^{2x}$

b. $2^{2x} - 6 \times 2^{x} - 16 = 0$

c. $\log_5(x+2) + \log_5(x-2) = 1$

d. $2\log_{10}(x) - \log_{10}(101x - 10) = -1$

You may choose to use technology to answer questions 14 and 15.

14. The number of bacteria (N) in a culture is given by the exponential function $N = 12\,000\left(2^{0.125t}\right)$, where t is the number of days.

a. Find the initial number of bacteria in the culture.

b. Find the time taken for the bacteria to reach 32 000.

When the bacteria reach a certain number, they are treated with an anti-bacterial serum. The serum destroys bacteria according to the exponential function $D = N_0 \times 3^{-0.789t}$, where D is the number of bacteria remaining after time t and N_0 is the number of bacteria present at the time the serum is added. The culture is considered cured when the number of bacteria drops below 1000.

c. If the bacteria are treated with the serum when their numbers reach 32 000, find the number of days it takes for the culture to be classed as cured.

d. How much longer would it take the culture to be cured if the serum is applied after 6 weeks?

15. The number of lions, L, in a wildlife park is given by $L = 20\left(10^{0.1t}\right)$, where t is the number of years since counting started. At the same time the number of cheetahs, C, is given by $C = 25\left(10^{0.05t}\right)$.

After how many months are the populations equal and what is this population?



16. Jan's new neighbours are very noisy. It seems to Jan that the neighbours practise playing their electric guitars most evenings until quite late at night. The measure of loudness in decibels (dB) is given by $L = 10\log_{10}\left(I \times 10^{12}\right)$ where L is the number of decibels and I is the intensity of the sound measured in watts per square metre. Given the sound level produced by a guitar is 70 decibels, answer the following questions.

a. Calculate, in standard form, the intensity of the sound each guitar produces.

b. When both guitars are played together, what is the decibel reading?



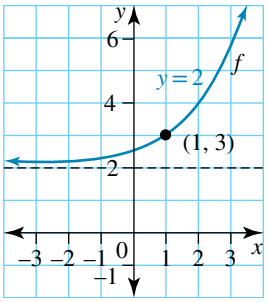
Complex unfamiliar

17. $n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1$

a. Show that $\log(n!) = \log(2) + \log(3) + \ldots + \log(n)$ for $n \in N$.

b. Hence, evaluate $\log(10!) - \log(9!)$.

18. The graph of the function defined by $f: R \to R$, $f(x) = 2^{x+b} + c$ is shown. The graph has an asymptote at $y = 2$ and contains the point $(1, 3)$.



a. Determine the values of b and c.

b. Obtain the coordinates of:

i. point A on $y = f(x)$ for which $f(x) = 6$

ii. the point B on $y = f^{-1}(x)$ for which $f^{-1}(x) = 6$

iii. the point P on $y = f^{-1}(x)$ for which $f^{-1}(x) = 1$.

c. Calculate the area of the triangle ABP.

You may choose to use technology to answer questions 19 and 20.

19. For any integer $x > 1$, it was established in the late nineteenth century that the number of prime numbers less than or equal to x approaches the ratio $\frac{x}{\log_e(x)}$ as x becomes large.

Let the function $p(x) = \frac{x}{\log_e(x)}$ be an estimate of the number of prime numbers less than or equal to x.

Obtain $p(10)$ and $p(30)$ and compare the estimates with the actual number of primes in each case.

20. Polly fills a kettle with water, planning to make a pot of tea. She switches the kettle on and the water heats to boiling point of 100 °C at 10 am, when the kettle automatically switches off. However, Polly is distracted by reading her email and forgets she has put the kettle on. The water in the kettle begins to cool in such a way that the temperature, $T°$, can be modelled by $T = a \times \left(\frac{16}{5}\right)^{-kt}$ where t is the number of minutes since 10 am and a and k are constants.



a. Obtain the value of the constant a.

b. If the temperature of the water in the kettle was 75 °C at 10.12 am, obtain the value of k correct to 2 decimal places.

c. By the time Polly remembers she had put the kettle on, it is 10.30 am. Using the value for k obtained in part **b**, calculate, to the nearest degree, the temperature of the water at this time.

d. At 10.30 am Polly switches the kettle back on and the water is reheated. If the temperature of the water t minutes after 10.30 am is described by $T = 50 \times 2^{\frac{t}{9}}$, at what exact time will the water re-reach its boiling point?

studyon

Units 1 & 2 Sit chapter test

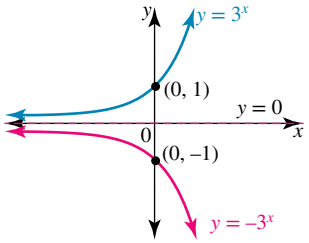
Answers

Chapter 9 Exponential and logarithmic functions

Exercise 9.2 Exponential functions

1. A

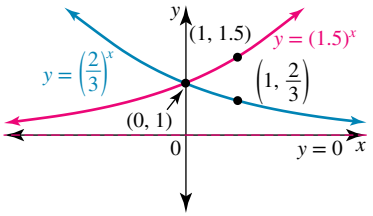
2. a.



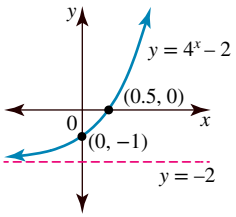
For $y = 3^x$, range is R^+ and for $y = -3^x$, the range is R^-. Asymptote is $y = 0$ for both graphs.

b. $y = 3^{-x}$

3.

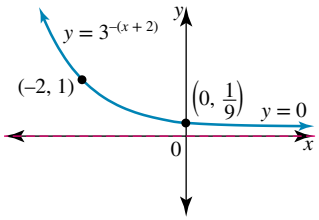


4. a.



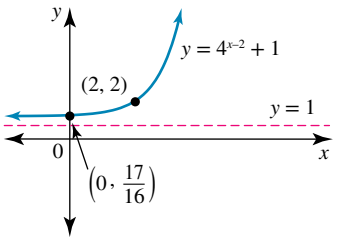
Range is $(-2, \infty)$.

b.



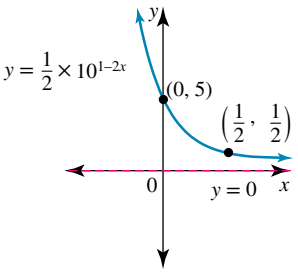
Range is R^+

5.



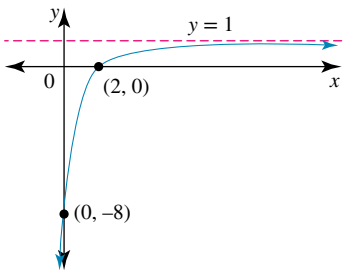
Range is $(1, \infty)$.

6. a.



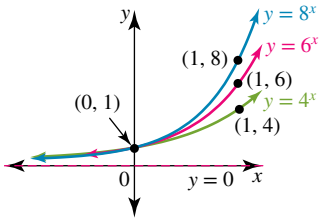
Range is R^+.

b.



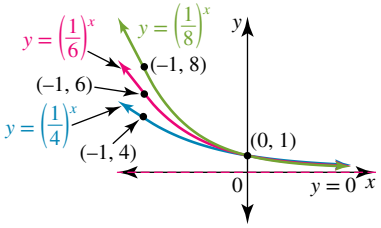
7. $y = -2 \times 3^x + 2$ and $a = -2, b = 2$

8. a. i.



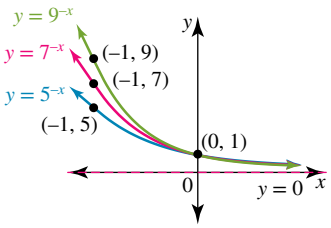
ii. For $x > 0$, as the base increases, the steepness of the graph increases.

b. i.



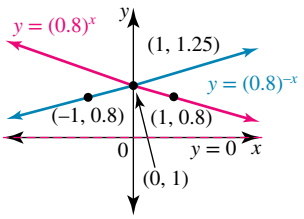
ii. The rules for the graphs can be expressed as $y = 4^{-x}$, $y = 6^{-x}$ and $y = 8^{-x}$.

9. a. i.



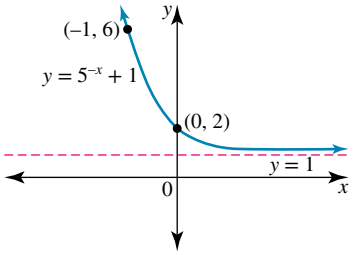
ii. As the base increases, the decrease of the graph is steeper for $x < 0$.

b. **i.**

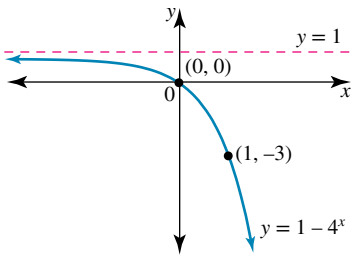


ii. The graphs of $y = (0.8)^{-x}$ and $y = (1.25)^x$ are the same. The graph of $y = (0.8)^{-x}$ is the reflection in the y-axis of the graph $y = (0.8)^x$.

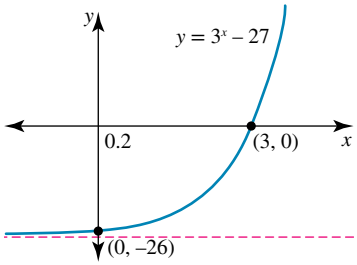
10. a.



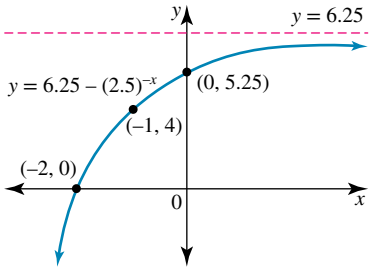
b.



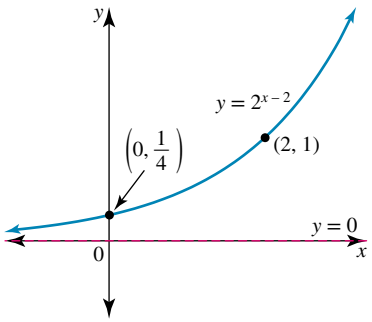
c.



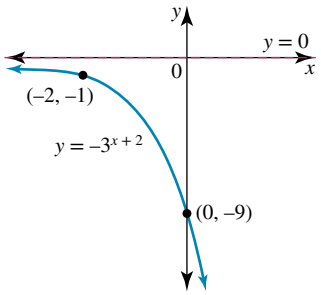
d.



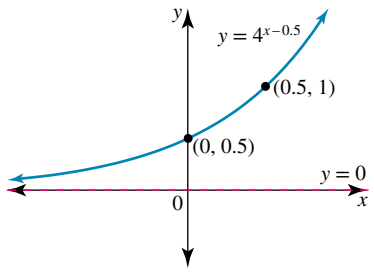
11. a.



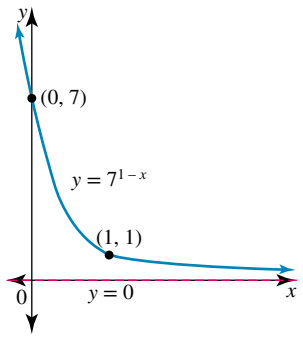
b.



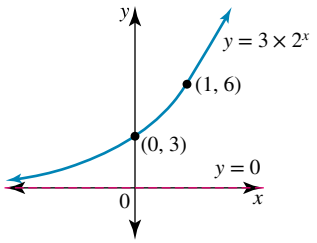
c.



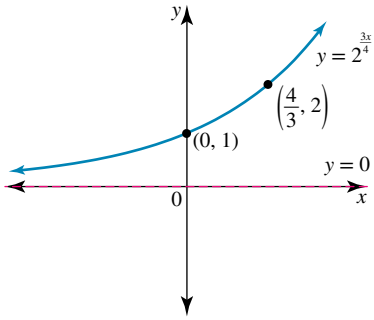
d.



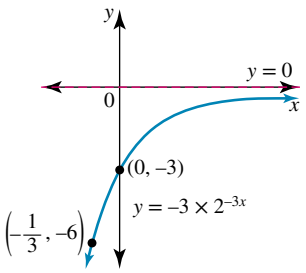
12. a.



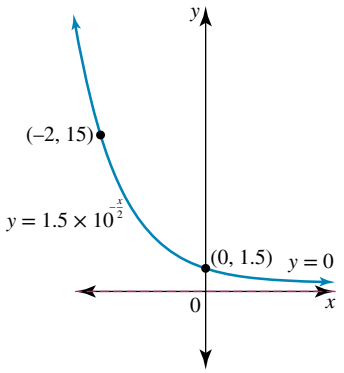
b.



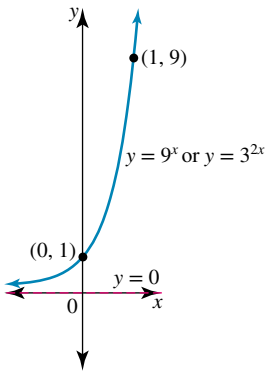
c.



d.

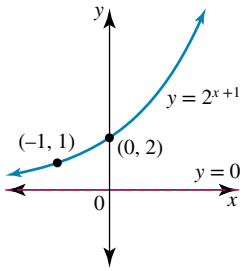


13. a.



b. i. $y = 2^{x+1}$

ii.



14. a. $y = 2 \times 10^x + 3$

b. $y = 4 \times 3^{2x}$; asymptote at $y = 0$

c. $y = 6 - 2 \times 3^{1-x}$

d. $y = 6 - 6 \times 3^{-x}$

15. a. one

b. three

c. one

d. no intersection

e. one at (1, 6)

f. The two curves are identical and therefore have an infinite number of intersections. The co-ordinates of the points of intersection are of the form $(t, 2^{2t-1})$, $t \in R$.

16. (−0.77, 0.59), (2, 4), (4, 16)

17. y_1: (1.17, 0), (0, 31) (1.0185, 10)
y_2: (0.17, 0), (0, 11) (0.0185, 10)

18. Student investigation; check your answers with your teacher.

Exercise 9.3 Logarithmic functions

1. a. $\log_2 8 = 3$ **b.** $\log_3 243 = 5$
c. $\log_5 1 = 0$ **d.** $\log_{10} 0.01 = -2$
e. $\log_b a = n$ **f.** $\log_2 \frac{1}{16} = -4$

2. a. $16 = 4^2$ **b.** $1\,000\,000 = 10^6$
c. $\frac{1}{2} = 2^{-1}$ **d.** $27 = 3^3$
e. $625 = 5^4$ **f.** $128 = 2^7$
g. $\frac{1}{9} = 3^{-2}$ **h.** $a = b^x$

3. a. 4 **b.** 4 **c.** 3
d. −2 **e.** 3 **f.** −5
g. −2 **h.** −5 **i.** 5
j. −6 **k.** undefined **l.** 5

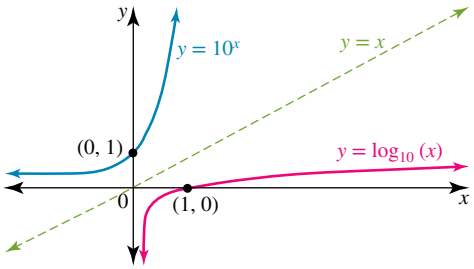
4. a. $\log_2 80$ **b.** $\log_3 105$ **c.** 2
d. $\log_6 56$ **e.** 2 **f.** 1
g. $\log_5 12.5$ **h.** $\log_2 3$ **i.** $\log_4 5$
j. $-2 \log_{10} 2$ **k.** $2 \log_3 2$ **l.** $\log_2 3$
m. $\log_3 20$ **n.** $\log_4 2$

5. a. $\log_{10} 250$ **b.** $\log_2 1728$ **c.** $\log_3 4$
d. $\log_5 3$ **e.** $\log_{10} \frac{1}{4}$ **f.** $\log_3 2$
g. −1 **h.** $\log_2 (x^4 - 4x^3)$ **i.** $\log_3 64$
j. $\log_{10} \frac{(x+3)^2}{x-2}$

6. a. $\frac{2}{3}$ **b.** 2 **c.** 2 **d.** $\frac{3}{2}$ **e.** $\frac{9}{4}$
f. 2 **g.** 3 **h.** 6 **i.** 3 **j.** 6

7. a. 4 **b.** 5 **c.** $\log_5 \frac{8}{25}$
d. $\log_{10} (100x^3)$ **e.** $\log_2 \frac{25}{8}$ **f.** $\log_3 4$
g. $\log_6 9$ **h.** $\log_{10} \left(\sqrt{10}\, x^6\right)$

8. a. $y = 10^x$



b. The points (10, 1), (100, 2), (1000, 3) lie on the logarithm graph.

With $m = 1000$, $n = 100$, the logarithm law is

$$\log_{10}(1000) - \log_{10}(100) = \log_{10}\left(\frac{1000}{100}\right)$$

$\therefore \log_{10}(1000) - \log_{10}(100) = \log_{10}(10)$

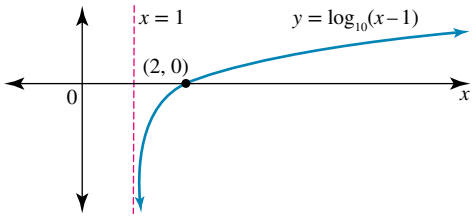
This means the difference between the y-coordinates of the points (1000, 3), (100, 2) should equal the y-coordinate of the point (10, 1).

This does hold since $3 - 2 = 1$.

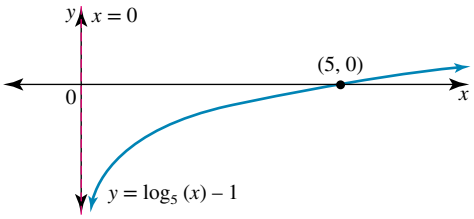
9. a. $2x$ **b.** 0.001

10. $\frac{1}{3} \times 2^x$

11. a.



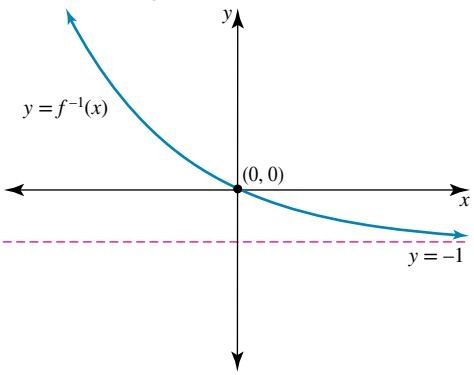
b.



c. i. $b = 1$

ii. Domain R ; range $(-1, \infty)$; $y = 2^{-x} - 1$

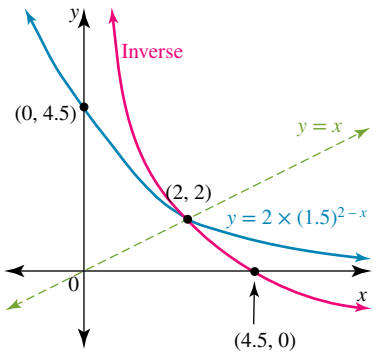
iii.



12. a. $x = 2$

b. $y = 4.5$

c, d.



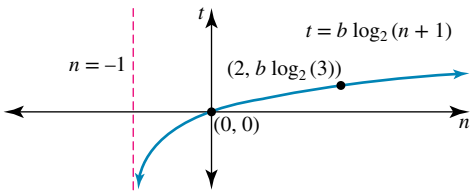
e. $y = 2 - \log_{1.5}\left(\frac{x}{2}\right)$

f. $x = 2$

13. a. i. 8 **ii.** 6 **iii.** $\frac{1}{2}$ **iv.** 5

b. i. x **ii.** x^3 **iii.** $3x$ **iv.** x

14.

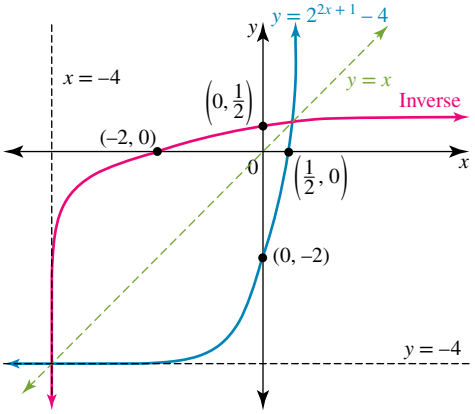


15. a. $y = 2^{2x+1} - 4$

b. $y = \frac{1}{2}\log_2(x + 4) - \frac{1}{2}$

c. $x = -4$; y -intercept $\left(0, \frac{1}{2}\right)$; x -intercept $(-2, 0)$

d. 2 points of intersection



e. $k = 14$

f. Inverse has equation $y = \frac{1}{2}\log_2(x + 4) - \frac{1}{2}$

Substitute the point $(14, \log_2(3))$

$\text{LHS} = \log_2(3)$

$$\text{RHS} = \frac{1}{2}\log_2(14 + 4) - \frac{1}{2}$$

$$= \frac{1}{2}\log_2(18) - \frac{1}{2}$$

$$= \frac{1}{2}\log_2(3^2 \times 2) - \frac{1}{2}$$

$$= \frac{1}{2}[\log_2(3^2) + \log_2(2) - 1]$$

$$= \frac{1}{2}[2\log_2(3) + 1 - 1]$$

$$= \log_2(3)$$

$= \text{LHS}$

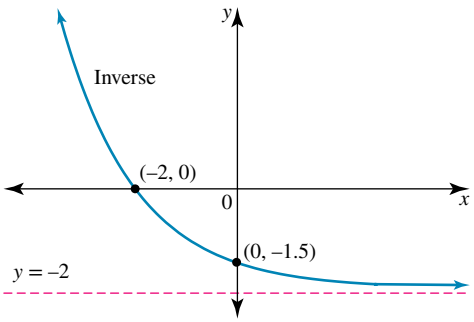
The point $(14, \log_2(3))$ lies on the inverse function.

16. a. $y = 14\log_7\left(\frac{x}{2}\right)$

b. i. $y = -4\log_3(x) + 4$ **ii.** $(0, 3)$

c. i. $y = -\log_2(x + 2) - 1$

ii. $y = 2^{-(x+1)} - 2$

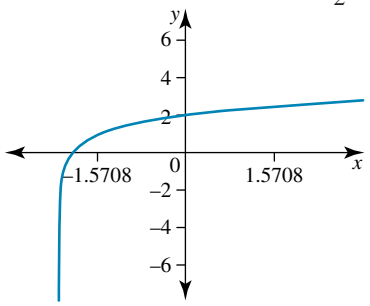


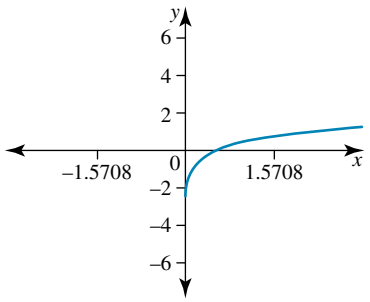
d. **i.** $d_f = \left(-\frac{9}{4}, \infty\right)$; $d_g = (-\infty, 20)$

ii. $x = -\frac{9}{4}$, x = 20

iii. f: (−2, 0), (0, 2); g: (10, 0), $\left(0, \frac{1}{2}\right)$

iv.



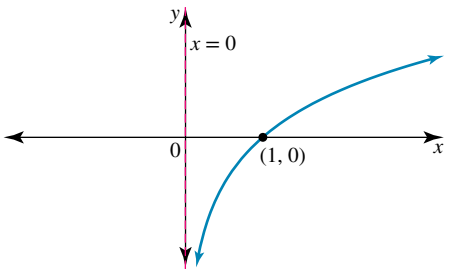


17. (0.4712, 4.632), (2, 2) and (4.632, 0.4712)

18. **a.** **i.** Should not be the same

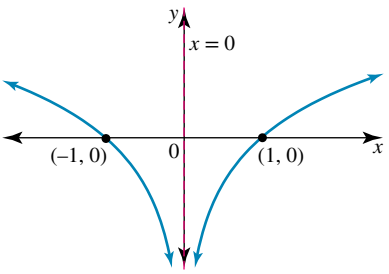
ii. $x = 1.\dot{3}$

b.



The domain is R^+, the range is R and the correspondence is one-to-one.

c.



The domain is $R\backslash\{0\}$, the range is R and the correspondence is many-to-one.

d. The right hand branch of the graph of $y = \log_3\left(x^2\right)$ is identical to that of the graph of $y = 2\log_3(x)$.
For $x > 0$, $\log_3\left(x^2\right) = 2\log(x)$.

If $x < 0$, only the domain of $y = \log_3\left(x^2\right)$ includes these values.
The logarithm law $\log_a\left(m^p\right) = p\log_a(m)$ holds for any $m > 0$, so this has not been contradicted by the graphs in parts **b** and **c**: they are identical graphs for $x > 0$.

Exercise 9.4 Modelling with exponential functions

1. **a.** $Q(0) = Q_0$ **b.** $k = 0.004$ **c.** 5.3 kg

2. **a.** 42 emails per day

b. 16 weeks

3. $y = 100x^{2.5}$

4. $y = 10^{0.3x}$

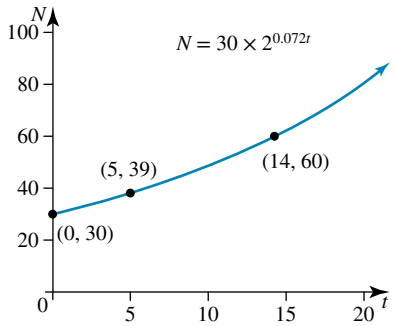
5. **a.** $k = 0.2$ **b.** 10 years

6. **a.** 30

b. approximately 39

c. approximately 14 days

d.



e. 25 days

7. **a.** **i.** $A = 2000\ (1.0025)^{12t}$

ii. $2030.19

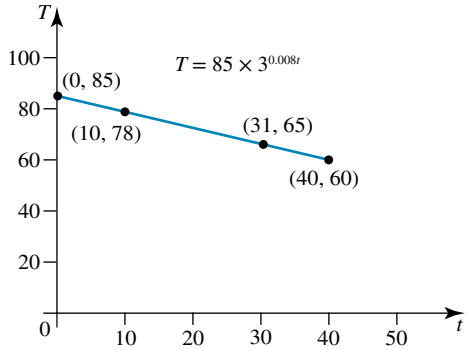
iii. 7.45 years

b. 5.6 %

8. **a.** 7 degrees

b. 30.5 minutes

c.



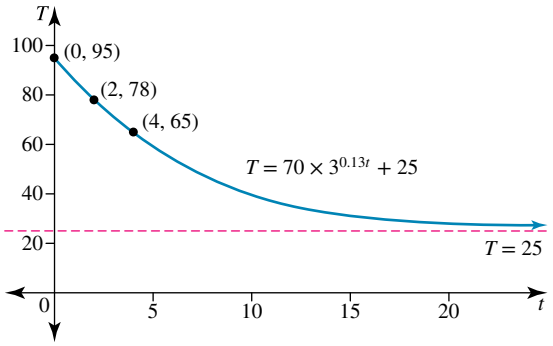
d. The asymptote for the graph of $T = 8.5 \times 3^{-0.008t}$ is $T = 0$.
This model therefore predicts that the temperature will approach zero degrees. This makes the model unrealistic, particularly in Brisbane!

9. **a.** $a = 70$

b. 77.6 degrees

c. 4 minutes

d. 25 degrees

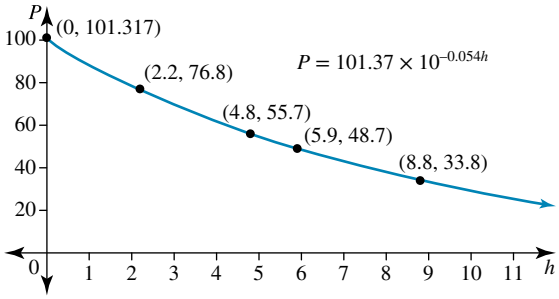


10. a. $k = 0.054$

b. 101.317

c. 55.71 k Pa; 76.80 k Pa

d.



11. a. $15 = D_0 \times 10^{15t}$ and $75 = D_0 \times 10^{18t}$

b. Correct to three decimal places, $k = \frac{1}{3}\log(5) = 0.233$ and $D_0 = 3 \times 5^{-4} = 0.005$.

c. 229 birds per square kilometre

d. Unlikely to be reduced below 39 birds per square kilometre

12. a. $k = \frac{1}{5730}$ **b.** 1540 years old

13. a. i. $y = \frac{\sqrt{x}}{10}$ **ii.** $y = 2^{-\frac{x}{4}}$

b. i. 13; 7

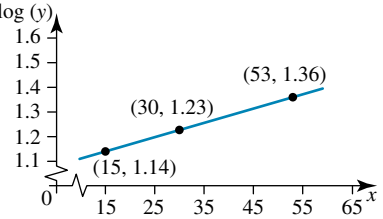
ii. 1×10^{-2}; 1×10^{-6}

iii. 10^4

iv. The solution becomes less acidic by a factor of 10.

14. a. 1.14, 1.23 and 1.36

b.



c. Sample responses can be found in the worked solutions in the online resources.

d. Sample responses can be found in the worked solutions in the online resources.

e. 50.2 years

f. yes

15. a. i. $y = 4 \times 1.38^x$

ii. $y = 4.3 + 6.2\log_e x$

b. exponential model is better

16. a. i. 16 **ii.** 51.5

b. i. 80.2

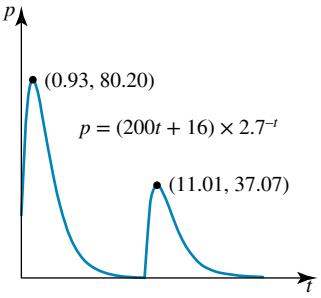
ii. 55.6 seconds

iii. 7.08 and 0.10

c. greatest after 10 minutes

d. i. $a = 0.10$

ii.



Exercise 9.5 Solving equations with indices

1. a. 2 **b.** 0 **c.** 3 **d.** 4 **e.** −1
f. −2 **g.** 6 **h.** 2 **i.** 2 **j.** 3

2. $x = \frac{3}{10}$

3. $x = 1$

4. $x = \frac{36}{35}$

5. a. i. $5 = \log_2 32$

ii. $\frac{3}{2} = \log_4 8$

iii. $-3 = \log_{10}(0.001)$

b. i. $2^4 = 16$

ii. $9^{\frac{1}{2}} = 3$

iii. $10^{-1} = 0.1$

6. a. $p + q$ **b.** $3q$ **c.** $2p + q$
d. $p - q$ **e.** $2q - 4p$ **f.** $\frac{3}{4}qp$

7. a. $y = 100x$ **b.** $y = 2^{2x} \times x^{-4}$
c. $y = 2^{3x}$ **d.** $y = \log_{10} x + 2$
e. $y = \frac{1}{x}$ **f.** $y = \sqrt{x}$, $x > 0$

8. a. $\log_2 5$ or $\log_2 9$ **b.** $\log_5\left(\sqrt{5} - 2\right)$
c. 0 or $\log_9 2$ **d.** 2
e. $\frac{1}{4}$ or 16 **f.** 1 or 999

9. a. $\log_7(15) = 1.392$ **b.** $\frac{5\log(3)}{\log(4) - 2\log(3)} = -6.774$

10. a. 3.459 **b.** −0.737 **c.** 2.727 **d.** 0.483
e. 1.292 **f.** −3.080 **g.** 2.255 **h.** 0.262
i. 0.661 **j.** 0.431 **k.** −0.423 **l.** 2.138

11. a. −3 **b.** $\frac{1}{5}$ **c.** 0.30
d. −1.10 **e.** 5 **f.** $\frac{1}{5}$

12. a. $\frac{1}{\log_{10} 2}$

b. i. 1.205 **ii.** −1.292 **iii.** 0.2823

c. i. $x \leq 2.096$ **ii.** $x < 0.5693$

d. i. 125 **ii.** 7

13. **a.** 0.14 **b.** 0.41 **c.** 5.14
d. −0.65 **e.** 0.08 **f.** 1.88

14. **a.** 1.574 **b.** $\frac{\sqrt{33}-5}{2}$

15. **a.** $\frac{\ln 5 + \ln 2}{\ln 5} + \log 5$ **b.** 1

16. Slide rule construction and results should be checked with your teacher.

9.6 Review: exam practice

1. D

2. D

3. A

4. C

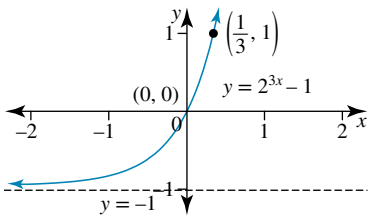
5. B

6. B

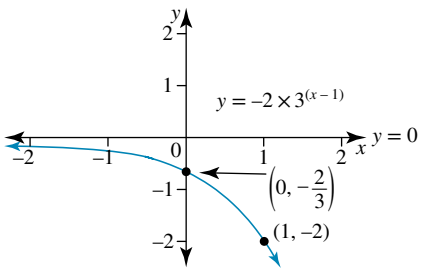
7. **a.** 3 **b.** 5

8. **a.** 2 **b.** 0 **c.** 3

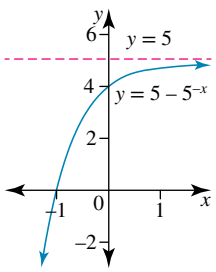
9. **a.**



b.

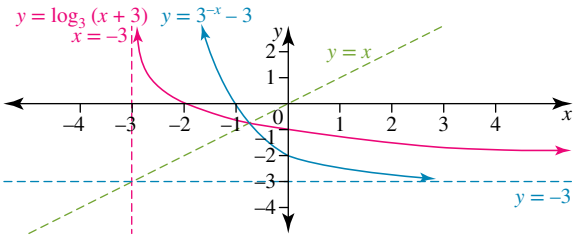


c.



10. **a.** Reflection in the x-axis and a horizontal translation 3 units to the left; asymptote: $x = -3$

b. $y = 3^{-x} - 3$; domain $= R$; range $= (-3, \infty)$



c. Dilation of 5 from x-axis, reflection in y-axis, horizontal translation 1 unit to right; asymptote: $y = 0$

d. $f^{-1}(x) = 1 - \log_7 \frac{x}{5}$

11. **a.** 120

b. **i.** 145 **ii.** 176 **iii.** 213

12. **a.** $P(t) = 10\,000\ (2^t)$

b. **i.** 80 000 **ii.** 640 000

c. 3.32 months

13. **a.** $\frac{3}{5}$ **b.** 3

c. 3 **d.** $\frac{1}{10}$ or 10

14. **a.** 12 000 **b.** 11.32 days

c. 4 days **d.** 4 more days

15. 23 months, $L = C = 31$

16. **a.** 1×10^{-5} watts/m^2 **b.** 73 dB

17. **a.** Sample responses can be found in the worked solutions in the online resources.

b. 1

18. **a.** $b = -1$, $c = 2$

b. **i.** (3, 6) **ii.** (34, 6) **iii.** (3, 1)

c. 77.5 square units

19. $p(10) \cong 4$; $p(30) \cong 9$

20. **a.** 100 **b.** $k \approx 0.02$

c. 50 degrees **d.** 10.39 am

TOPIC 1 Exponential functions 2
TOPIC 2 The logarithmic function 1

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



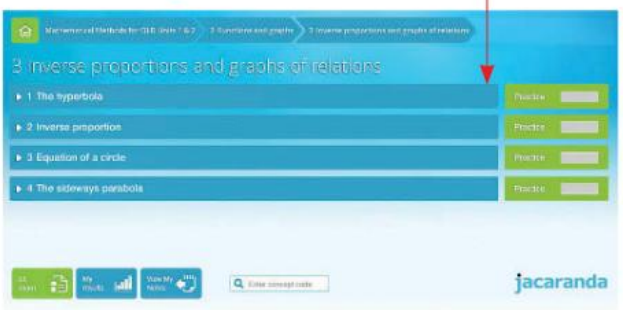
- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



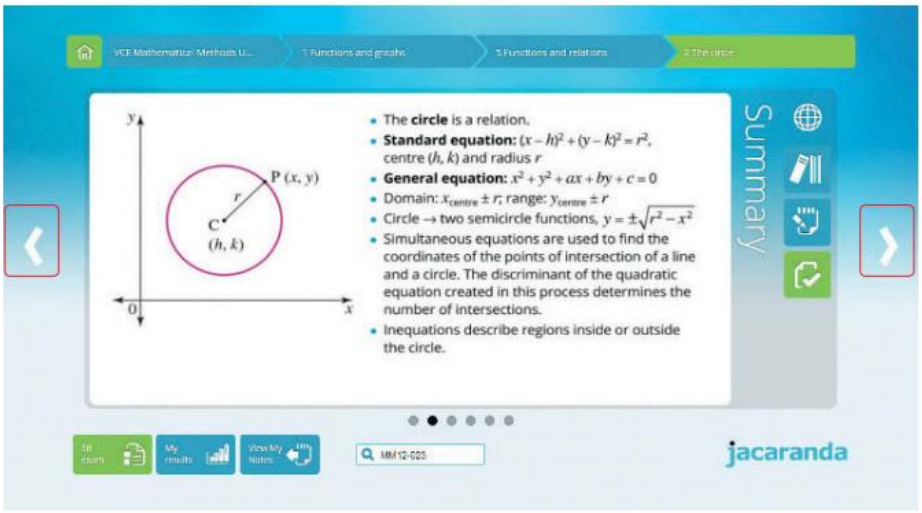
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 10
Trigonometric functions

10.1 Overview

10.1.1 Introduction



Trigonometry is the study of the relationships between lengths and angles of geometrical figures and the graphs of any recurring cycle. Early study of triangles can be traced back as far as the second millennium BCE. Through the centuries, mathematicians from India, Babylonia, Egypt, Greece, China and Islamic countries further advanced an understanding and application of trigonometric ratios. Astronomers described the solar system and both solar and lunar eclipses. The Great Pyramids of Egypt were built using a form of trigonometry. By the eighteenth-century Leonhard Euler's work (1748), amongst others including Sir Isaac Newton, contributed to what we describe as modern trigonometry. The functions covered in this chapter are sine, cosine and tangent.

Some careers that rely on this area of mathematics are in meteorology, geology, physics engineering, architecture, surveying and radiology.

MRI and CT scan machines were developed using trigonometry, as were video games. After studying this chapter you will probably notice charts and machines where you recognise the graphs of the sine and cosine functions.

Geologists measure the slopes of landforms using trigonometry. From these calculations they can reveal how mountains formed and where to locate economically profitable minerals, such as gold.

LEARNING SEQUENCE

10.1 Overview
10.2 Trigonometry review
10.3 Radian measure
10.4 Unit circle definitions
10.5 Exact values and symmetry properties
10.6 Graphs of the sine, cosine and tangent functions
10.7 Transformations of sine and cosine graphs
10.8 Solving trigonometric equations
10.9 Modelling with trigonometric functions
10.10 Review: exam practice

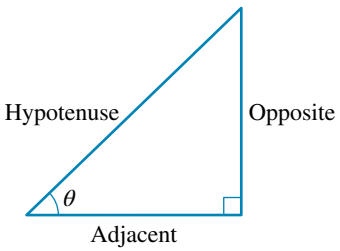
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 Trigonometry review

The process of calculating all side lengths and all angle magnitudes of a triangle is called **solving the triangle**. Here we review the use of trigonometry to solve right-angled triangles.

10.2.1 Right-angled triangles

The hypotenuse is the longest side of a right-angled triangle and it lies opposite the 90° angle, the largest angle in the triangle. The other two sides are labelled relative to one of the other angles in the triangle, an example of which is shown in the diagram.



It is likely that the trigonometric ratios of sine, cosine and tangent, possibly together with Pythagoras' theorem, will be required to solve a right-angled triangle.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ and } \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}},$$

usually remembered as SOH, CAH, TOA.

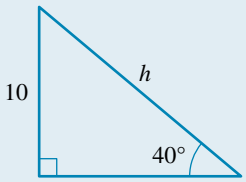
These ratios cannot be applied to triangles which do not have a right angle. However, isosceles and equilateral triangles can easily be divided into two right-angled triangles by dropping a perpendicular from the vertex between a pair of equal sides to the midpoint of the opposite side.

By and large, as first recommended by the French mathematician François Viète in the sixteenth century, decimal notation has been adopted for magnitudes of angles rather than the sexagesimal system of degrees and minutes; although, even today we still may see written, for example, either 15°24′ or 15.4° for the magnitude of an angle.

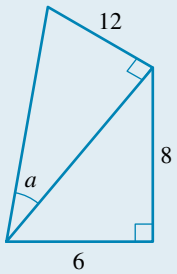
WORKED EXAMPLE 1

Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

a.



b.



THINK

a. 1. Choose the appropriate trigonometric ratio.

2. Rearrange to make the required side the subject and evaluate, checking the calculator is in degree mode.

WRITE

a. Relative to the angle, the sides marked are the opposite and the hypotenuse.

$$\sin(40°) = \frac{10}{h}$$

$$h = \frac{10}{\sin(40°)}$$

$$= 15.56 \text{ to 2 decimal places}$$

b. 1. Obtain the hypotenuse length of the lower triangle.

b. From Pythagoras' theorem the sides 6, 8, 10 form a Pythagorean triple, so the hypotenuse is 10.

2. In the upper triangle choose the appropriate trigonometric ratio.

The opposite and adjacent sides to the angle $a°$ are now known.

$$\tan(a) = \frac{12}{10}$$

3. Rearrange to make the required angle the subject and evaluate.

$$\tan(a) = 1.2$$
$$\therefore a = \tan^{-1}(1.2)$$
$$= 50.19° \text{ to 2 decimal places}$$

TI | THINK

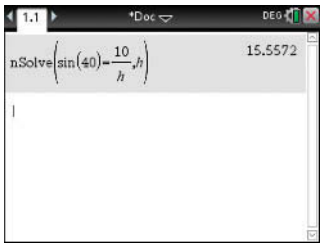
a. 1. Write a trigonometric ratio that can be solved for h using the information given.

2. Put the Calculator in DEGREE mode.
On a Calculator page, press MENU then select
3: Algebra
1: Numerical Solve
Complete the entry line as
nSolve $\left(\sin(40) = \frac{10}{h}, h\right)$
then press ENTER.

3. The answer appears on the screen.

WRITE

$\sin(40) = \frac{10}{h}$



$h = 15.56$ (2 decimal places)

CASIO | THINK

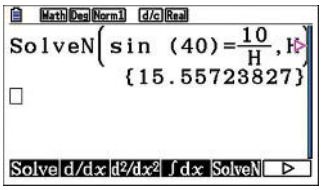
a. 1. Write a trigonometric ratio that can be solved for h using the information given.

2. On a Run-Matrix screen, press OPTN and select CALC by pressing F4, then select SolveN by pressing F5.
Complete the entry line as Solve
N $\left(\sin(40) = \frac{10}{H}, H\right)$
then press EXE.

3. The answer appears on the screen.

WRITE

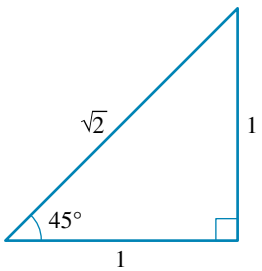
$\sin(40) = \frac{10}{h}$

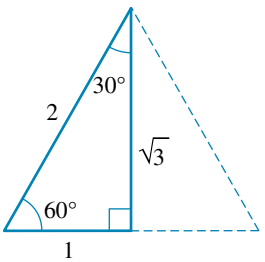


$h = 15.56$ (2 decimal places)

10.2.2 Exact values for trigonometric ratios of 30°, 45°, 60°

By considering the isosceles right-angled triangle with equal sides of one unit, the trigonometric ratios for 45° can be obtained. Using Pythagoras' theorem, the hypotenuse of this triangle will be $\sqrt{1^2 + 1^2} = \sqrt{2}$ units.





The equilateral triangle with the side length of two units can be divided in half to form a right-angled triangle containing the 60° and the 30° angles. This right-angled triangle has a hypotenuse of 2 units and the side divided in half has length $\frac{1}{2} \times 2 = 1$ unit. The third side is found using Pythagoras' theorem: $\sqrt{2^2 - 1^2} = \sqrt{3}$ units.

The exact values for trigonometric ratios of 30°, 45°, 60° can be calculated from these triangles using SOH, CAH, TOA. Alternatively, these values can be displayed in a table and committed to memory.

θ	30°	45°	60°
sin (θ)	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos (θ)	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan (θ)	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

As a memory aid, notice the sine values in the table are in the order $\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$. The cosine values reverse this order, while the tangent values are the sine values divided by the cosine values.

For other angles, a calculator, or other technology, is required. It is essential to set the calculator mode to degree in order to evaluate a trigonometric ratio involving angles in degree measure.

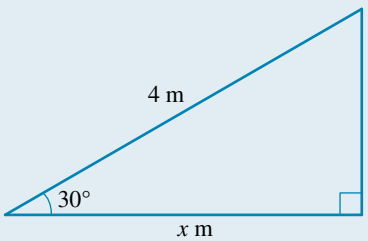
WORKED EXAMPLE 2

A ladder of length 4 metres leans against a fence. If the ladder is inclined (precariously) at 30° to the ground, how far exactly is the foot of the ladder from the fence?

THINK

1. Draw a diagram showing the given information.

WRITE



Let the distance of the ladder from the fence be x m.

2. Choose the appropriate trigonometric ratio.

Relative to the angle, the sides marked are the adjacent and the hypotenuse.

$\cos(30°) = \frac{x}{4}$

3. Calculate the required length using the exact value for the trigonometric ratio.

$x = 4\cos(30°)$

$= 4 \times \frac{\sqrt{3}}{2}$

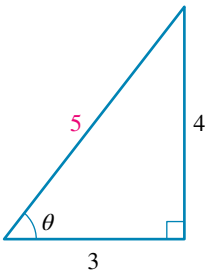
$= 2\sqrt{3}$

4. State the answer.

The foot of the ladder is $2\sqrt{3}$ metres from the fence.

10.2.3 Deducing one trigonometric ratio from another

Given the sine, cosine or tangent value of some unspecified angle, it is possible to obtain the exact value of the other trigonometric ratios of that angle using Pythagoras' theorem.



One common example is that given $\tan(\theta) = \frac{4}{3}$ it is possible to deduce that $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ without evaluating θ. The reason for this is that $\tan(\theta) = \frac{4}{3}$ means that the sides opposite and adjacent to the angle θ in a right-angled triangle would be in the ratio 4 : 3.

Labelling these sides 4 and 3 respectively and using Pythagoras' theorem (or recognising the Pythagorean triad '3, 4, 5') leads to the hypotenuse being 5 and hence the ratios $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ are obtained.

WORKED EXAMPLE 3

A line segment AB is inclined at a degrees to the horizontal, where $\tan(a) = \frac{1}{3}$.

a. Deduce the exact value of $\sin(a)$.

b. Calculate the vertical height of B above the horizontal through A if the length of AB is $\sqrt{5}$ cm.

THINK

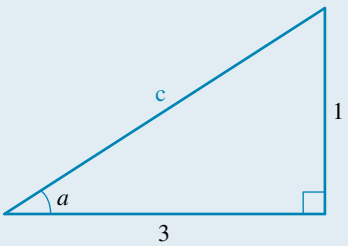
a. 1. Draw a right-angled triangle with two sides in the given ratio and calculate the third side.

2. State the required trigonometric ratio.

b. 1. Draw the diagram showing the given information.

WRITE

a. $\tan(a) = \frac{1}{3} \Rightarrow$ sides opposite and adjacent to angle a are in the ratio 1: 3.



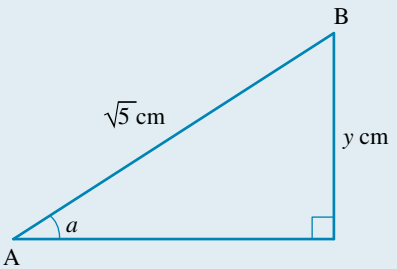
Using Pythagoras' theorem:

$c^2 = 1^2 + 3^2$

$\therefore c = \sqrt{10}$

$\sin(a) = \frac{1}{\sqrt{10}}$

b. Let the vertical height be y cm.



2. Choose the appropriate trigonometric ratio and calculate the required length.

$$\sin(a) = \frac{y}{\sqrt{5}}$$
$$y = \sqrt{5}\ \sin(a)$$
$$= \sqrt{5} \times \frac{1}{\sqrt{10}} \text{ as } \sin(a) = \frac{1}{\sqrt{10}}$$
$$= \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

3. State the answer.

The vertical height of B above the horizontal through A is $\frac{\sqrt{2}}{2}$ cm.

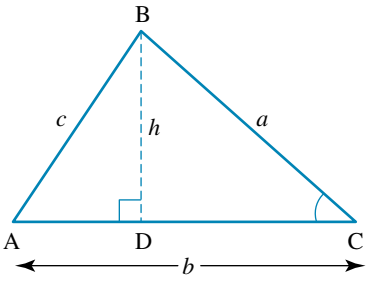
10.2.4 Area of a triangle

The formula for calculating the area of a right-angled triangle

$$\textbf{Area} = \frac{1}{2}\ \textbf{(base)} \times \textbf{(height)}$$

For a triangle that is not right-angled, if two sides and the angle included between these two sides are known, it is also possible to calculate the area of the triangle from that given information.

Consider the triangle ABC shown, where the convention of labelling the sides opposite the angles A, B and C with lower case letters a, b and c respectively has been adopted in the diagram.



In triangle ABC construct the perpendicular height, h, from B to a point D on AC. As this is not necessarily an isosceles triangle, D is not the midpoint of AC.

In the right-angled triangle BCD, $\sin(C) = \frac{h}{a} \Rightarrow h = a\ \sin(C)$.

This means the height of triangle ABC is $a\ \sin(C)$ and its base is b.

The area of the triangle ABC can now be calculated.

$$\text{Area} = \frac{1}{2}\ \text{(base)} \times \text{(height)}$$
$$= \frac{1}{2}\, b \times a\ \sin\ (C)$$
$$= \frac{1}{2}\, ab\ \sin\ (C)$$

The formula for the area of the triangle ABC, $A_\Delta = \frac{1}{2}ab\ \sin(C)$, is expressed in terms of two of its sides and the angle included between them.

Alternatively, using the height as $c\ \sin(A)$ from the right-angled triangle ABD on the left of the diagram, the area formula becomes $A_\Delta = \frac{1}{2}bc\ \sin(A)$.

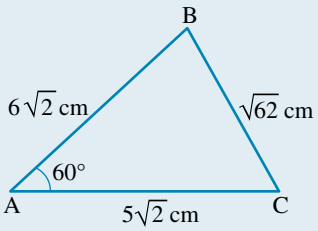
It can also be shown that the area is $A_\Delta = \frac{1}{2}ac\ \sin(B)$.

Hence, the area of a triangle is $\frac{1}{2}$ × (product of two sides) × (sine of the angle included between the two given sides).

Area of a triangle: $A_\Delta = \frac{1}{2}ab \sin(C)$

WORKED EXAMPLE 4

Calculate the exact area of the triangle ABC for which $a = \sqrt{62}$, $b = 5\sqrt{2}$, $c = 6\sqrt{2}$ cm, and $A = 60°$.

THINK	WRITE
1. Draw a diagram showing the given information. *Note:* The naming convention for labelling the angles and the sides opposite them with upper- and lower-case letters is commonly used.	
2. State the two sides and the angle included between them.	The given angle A is included between the sides b and c.
3. State the appropriate area formula and substitute the known values.	The area formula is: $A_\Delta = \frac{1}{2}bc \sin(A), b = 5\sqrt{2}, c = 6\sqrt{2}, A = 60°$ $\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \sin(60°)$
4. Evaluate, using the exact value for the trigonometric ratio.	$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$ $= \frac{1}{2} \times 30 \times 2 \times \frac{\sqrt{3}}{2}$ $= 15\sqrt{3}$
5. State the answer.	The area of the triangle is $15\sqrt{3}$ cm^2.

on Resources

Interactivity: Trigonometric ratios (int-2577)

studyon

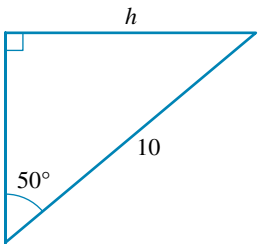
Units 1 & 2 > Area 7 > Sequence 1 > Concept 1

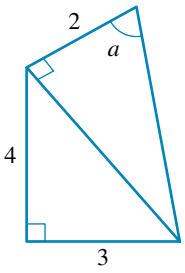
Trigonometric ratios Summary screen and practice questions

Exercise 10.2 Trigonometry review

Technology free

1. What are the exact values of the following?
   a. $\sin(45°)$
   b. $\tan(30°)$
   c. $\cos(60°)$
   d. $\tan(45°) + \cos(30°) - \sin(60°)$
2. **WE1** Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

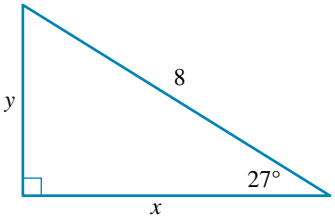
   a.
   

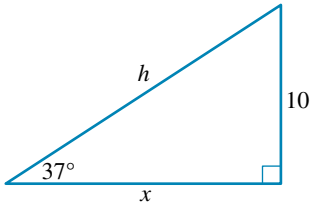
   b.
   

3. **WE2** A ladder of length 4 metres leans against a fence. If the ladder is inclined at 45° to the horizontal ground, how far exactly is the foot of the ladder from the fence?
4. Evaluate $\dfrac{\cos(30°)\sin(45°)}{\tan(45°) + \tan(60°)}$, expressing the answer in exact form with a rational denominator.
5. For an acute angle θ, obtain the following trigonometric ratios without evaluating θ.
   a. Given $\tan(\theta) = \dfrac{\sqrt{3}}{2}$, form the exact value of $\sin(\theta)$.
   b. Given $\cos(\theta) = \dfrac{5}{6}$, form the exact value of $\tan(\theta)$.
   c. Given $\sin(\theta) = \dfrac{\sqrt{5}}{3}$, form the exact value of $\cos(\theta)$.
6. **WE3** A line segment AB is inclined at a degrees to the horizontal, where $\tan(a) = \dfrac{2}{3}$.
   a. Deduce the exact value of $\cos(a)$.
   b. Calculate the run of AB along the horizontal through A if the length of AB is 26 cm.
7. A right-angled triangle contains an angle θ where $\sin(\theta) = \dfrac{3}{5}$. If the longest side of the triangle is 60 cm, calculate the exact length of the shortest side.
8. **WE4** Calculate the exact area of the triangle ABC for which $a = 10$, $b = 6\sqrt{2}$, $c = 2\sqrt{13}$ cm and $C = 45°$.

Technology active

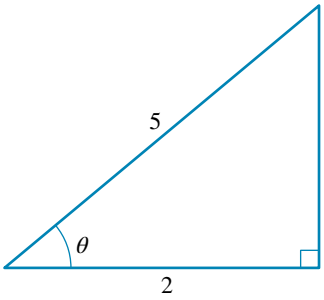
9. Calculate the values of the unknown marked sides correct to 2 decimal places.

   a.
   

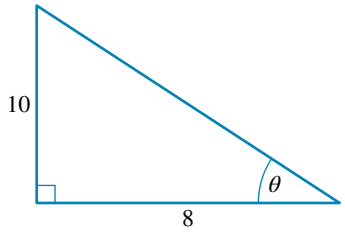
   b.
   

10. Calculate the value of angle θ, correct to 2 decimal places.

a.



b.



11. In order to check the electricity supply, a technician uses a ladder to reach the top of an electricity pole. The ladder reaches 5 metres up the pole and its inclination to the horizontal ground is 54°.



a. Calculate the length of the ladder to 2 decimal places.

b. If the foot of the ladder is moved 0.5 metres closer to the pole, calculate its new inclination to the ground and the new vertical height it reaches up the electricity pole, both to 1 decimal place.

12. The distances shown on a map are called projections. They give the horizontal distances between two places without taking into consideration the slope of the line connecting the two places. If a map gives the projection as 25 km between two points which actually lie on a slope of 16°, what is the true distance between the points?

13. The two legs of a builder's ladder are of length 2 metres. The ladder is placed on horizontal ground with the distance between its two feet of 0.75 metres. Calculate the magnitude of the angle between the legs of the ladder.



14. Triangle ABC has angles such that $\angle CAB = 60°$ and $\angle ABC = 45°$. The perpendicular distance from C to AB is 18 cm. Calculate the exact lengths of each of its sides.

15. A cube of edge length a units rests on a horizontal table. Calculate:

a. the length of the diagonal of the cube in terms of a

b. the inclination of the diagonal to the horizontal, to 2 decimal places.

16. AB is a diameter of a circle and C is a point on the circumference of the circle such that $\angle CBA = 68°$ and CB = 3.8 cm.

a. Calculate the length of the radius of the circle to 2 decimal places.

b. Calculate the shortest distance of CB from the centre of the circle, to 1 decimal place.

17. a. An isosceles triangle ABC has sides BC and AC of equal length 5 cm. If the angle enclosed between the equal sides is 20°, calculate:

i. the area of the triangle to 3 decimal places

ii. the length of the third side AB to 3 decimal places.

b. An equilateral triangle has a vertical height of 10 cm. Calculate the exact perimeter and area of the triangle.

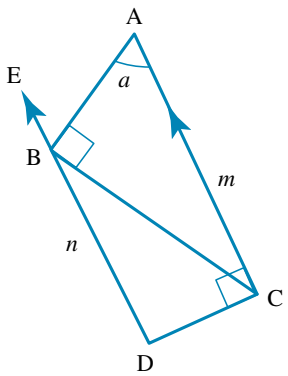
18. Horses graze over a triangular area XYZ where Y is 4 km east of X and Z is 3 km from Y on a bearing of N 20° W. Over what area, correct to 2 decimal places, can the horses graze?



19. A lookout tower is 100 metres in height. From the top of this tower, the angle of depression of the top of a second tower stood on the same level ground is 30°; from the bottom of the lookout tower, the angle of elevation to the top of the second tower is 45°. Calculate the height of the second tower and its horizontal distance from the lookout tower, expressing both measurements to 1 decimal place.



20. In the diagram, angles ABC and ACD are right angles and DE is parallel to CA. Angle BAC is a degrees and the length measures of AC and BD are m and n, respectively.



a. Show that $n = m \sin^2(a)$, where $\sin^2(a)$ is the notation for the square of $\sin(a)$.

b. If angle EBA is 60° and CD has length measure of $4\sqrt{3}$, calculate the values of a, m and n.

10.3 Radian measure

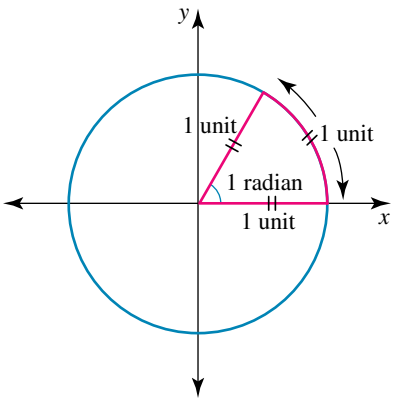
Measurements of angles up to now have been given in degree measure. An alternative to degree measure is **radian measure**. This alternative can be more efficient for certain calculations that involve circles, and it is essential for the study of trigonometric functions.

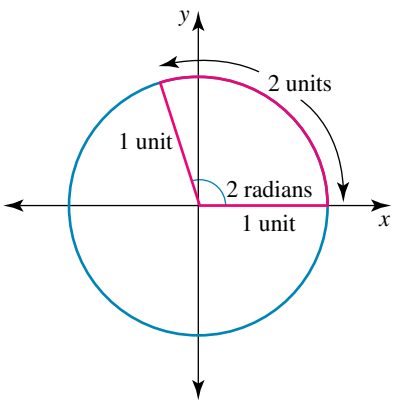
10.3.1 Definition of radian measure

Radian measure is defined in relation to the length of an **arc** of a circle. An arc is a part of the circumference of a circle.

One radian is the measure of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

In particular, an arc of 1 unit subtends an angle of one radian at the centre of a **unit circle**, a circle with radius 1 unit and, conventionally, a centre at the origin.





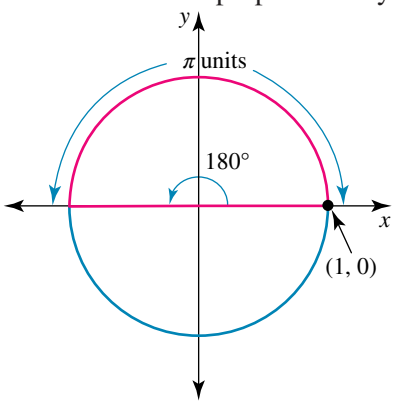
Doubling the arc length to 2 units doubles the angle to 2 radians. This illustrates the direct proportionality between the arc length and the angle in a circle of fixed radius.

The diagram suggests an angle of one radian will be a little less than 60° since the sector containing the angle has one 'edge' curved and therefore is not a true equilateral triangle.

The degree equivalent for 1 radian can be found by considering the angle subtended by an arc which is half the circumference. The circumference of a circle is given by $2\pi r$ so the circumference of a unit circle is 2π.

In a unit circle, an arc of π units subtends an angle of π radians at the centre. But we know this angle to be 180°.

This gives the relationship between radian and degree measure.



π radians $= 180°$

Hence, 1 radian equals $\frac{180°}{\pi}$, which is approximately 57.3°; 1° equals $\frac{\pi}{180}$ radians, which is approximately 0.0175 radians.

From these relationships it is possible to convert from radians to degrees and vice versa.

To convert radians to degrees, multiply by $\frac{180}{\pi}$.

To convert degrees to radians, multiply by $\frac{\pi}{180}$.

Radians are often expressed in terms of π, perhaps not surprisingly, since a radian is a circular measure and π is so closely related to the circle.

Notation

π radians can be written as π^c, where c stands for circular measure. However, linking radian measure with the length of an arc, a real number, has such importance that the symbol c is usually omitted. Instead, the onus is on degree measure to always include the degree sign in order not to be mistaken for radian measure.

WORKED EXAMPLE 5

a. Convert 30° to radian measure.

b. Convert $\frac{4\pi^c}{3}$ to degree measure.

c. Convert $\frac{\pi}{4}$ to degree measure and hence state the value of $\sin\left(\frac{\pi}{4}\right)$.

THINK

a. Convert degrees to radians.

WRITE

a. To convert degrees to radians, multiply by $\frac{\pi}{180}$.

$30° = 30 \times \frac{\pi}{180}$

$= \cancel{30} \times \frac{\pi}{\cancel{180}_6}$

$= \frac{\pi}{6}$

b. Convert radians to degrees.
Note: The degree sign must be used.

b. To convert radians to degrees, multiply by $\frac{180}{\pi}$.

$$\frac{4\pi^c}{3} = \left(\frac{4\pi}{3} \times \frac{180}{\pi}\right)^\circ$$

$$= \left(\frac{4\cancel{\pi}}{3} \times \frac{180^{60}}{\cancel{\pi}}\right)^\circ$$

$$= 240^\circ$$

c. 1. Convert radians to degrees.

c. $\frac{\pi}{4} = \frac{\cancel{\pi}}{4} \times \frac{180^\circ}{\cancel{\pi}}$

$$= 45^\circ$$

2. Calculate the trigonometric value.

$$\sin\left(\frac{\pi}{4}\right) = \sin(45^\circ)$$

$$= \frac{\sqrt{2}}{2}$$

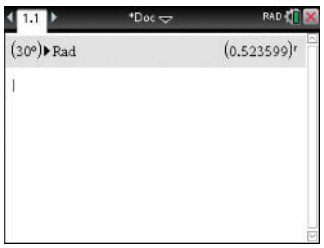
TI | THINK

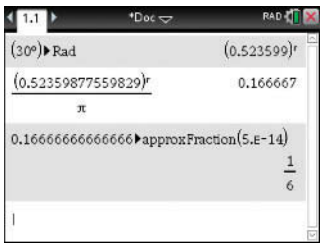
a. 1. On a Calculator page, complete the entry line as 30°. Press the Catalogue button and select ▷ Rad, then press ENTER.
Note: The degree symbol can be found by pressing the π button.

2. To express the answer as a multiple of π, complete the next entry line as $\frac{ans}{\pi}$, then press ENTER.
Press Menu then select
2: Number
2: Approximate to Fraction
then press ENTER

3. The answer appears on the screen.

WRITE





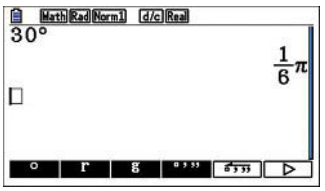
$30^\circ = \frac{\pi^c}{6}$

CASIO | THINK

a. 1. Put the Calculator in Radian mode.
On the Run-Matrix screen, complete the entry line as 30°,
then press EXE.
Note: The degree symbol can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting ANGLE by pressing F5.

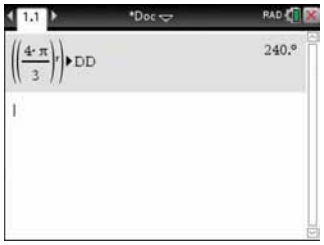
2. The answer appears on the screen.

WRITE



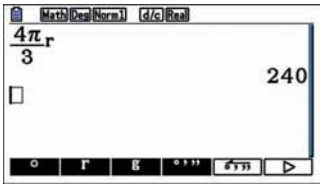
$30^\circ = \frac{\pi^c}{6}$

b. 1. On a Calculator page, complete the entry line as $\frac{4\pi}{3}$r
Press the Catalogue button and select ▷ DD, then press ENTER.
Note: The radian symbol can be found by pressing the π button.



2. The answer appears on the screen. $\frac{4\pi^c}{3} = 240°$

b. 1. Put the Calculator in Degree mode.
On the Run-Matrix screen, complete the entry line as $\frac{4\pi}{3}$r, then press EXE.
Note: The radian symbol can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting ANGLE by pressing F5.

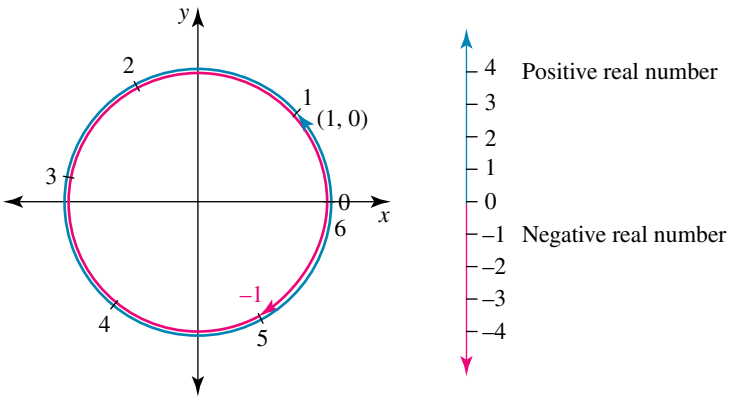


2. The answer appears on the screen. $\frac{4\pi^c}{3} = 240°$

10.3.2 Extended angle measure

By continuing to rotate around the circumference of the unit circle, larger angles are formed from arcs which are multiples of the circumference. For instance, an angle of 3π radians is formed from an arc of length 3π units created by one and a half revolutions of the unit circle: $3\pi = 2\pi + \pi$. This angle, in degrees, equals $360° + 180° = 540°$ and its endpoint on the circumference of the circle is in the same position as that of $180°$ or π^c; this is the case with any other angle which is a multiple of 2π added to π^c.

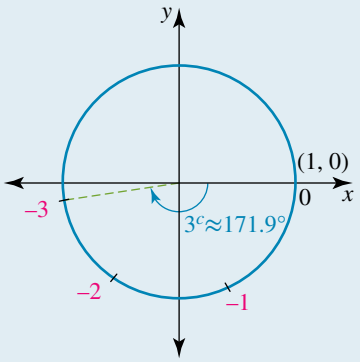
What is important here is that this process can continue indefinitely so that any real number, the arc length, can be associated with a radian measure. The real number line can be wrapped around the circumference of the unit circle so that the real number θ corresponds to the angle θ in radian measure. By convention, the positive reals wrap around the circumference anticlockwise while the negative reals wrap clockwise, with the number zero placed at the point $(1, 0)$ on the unit circle. The wrapping of the real number line around the circumference results in many numbers being placed in the same position on the unit circle's circumference.



All trigonometric functions have **periodicity**, that is, at established intervals the function repeats itself in a regular pattern.

WORKED EXAMPLE 6

a. Convert -3^c to degree measure.
b. Draw a unit circle diagram to show the position the real number -3 is mapped to when the real number line is wrapped around the circumference of the unit circle.

THINK	WRITE
a. Convert radians to degrees.	**a.** $-3^c = -\left(3 \times \frac{180}{\pi}\right)^\circ$ As the π can't be cancelled, a calculator is used to evaluate. $-3^c \approx -171.9°$
b. 1. State how the wrapping of the number line is made.	**b.** The number zero is placed at the point $(1, 0)$ and the negative number line is wrapped clockwise around the circumference of the unit circle through an angle of $171.9°$ so that the number -3 is its endpoint.
2. Draw the unit circle diagram and mark the position of the number.	

10.3.3 Using radians in calculations

From the definition of a radian, for any circle of radius r, an angle of 1^c is subtended at the centre of the circle by an arc of length r. So, if the angle at the centre of this circle is θ^c, then the length of the arc subtending this angle must be $\theta \times r$.

This gives a formula for calculating the length of an arc.

$$l = r\theta$$

In the formula l is the arc length and θ is the angle, in radians, subtended by the arc at the centre of the circle of radius r.

Any angles given in degree measure will need to be converted to radian measure to use the arc length formulas.

Some calculations may require recall of the geometry properties of the angles in a circle, such as the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.

Sectors

A **sector** is part of the area of a circle. In the figure the shaded area is a sector.

The area of the sector is proportional to the angle θ.

$$A_{\text{sector}} = \pi r^2 \times \frac{\theta}{360^\circ}$$

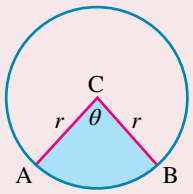
Or

$$\pi r^2 \times \frac{\theta}{2\pi}$$

$$= \frac{r^2\theta}{2}$$

$$A_{\text{sector}} = \frac{r^2\theta}{2}$$

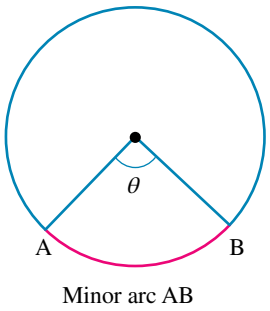
if θ is in radians



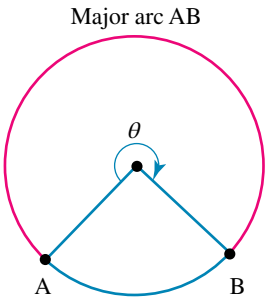
Major and minor arcs

For a minor arc, $\theta < \pi$ and for a major arc $\theta > \pi$, with the sum of the minor and major arc angles totalling 2π if the major and minor arcs have their endpoints on the same chord.

To calculate the length of the major arc, the reflex angle with $\theta > \pi$ should be used in the arc length formula. Alternatively, the sum of the minor and major arc lengths gives the circumference of the circle, so the length of the major arc could be calculated as the difference between the lengths of the circumference and the minor arc.



Minor arc AB



Trigonometric ratios of angles expressed in radians

Problems in trigonometry may be encountered where angles are given in radian mode and their sine, cosine or tangent value is required to solve the problem. A calculator, or other technology, can be set on radian or 'rad' mode and the required trigonometric ratio evaluated directly without the need to convert the angle to degrees.

Care must be taken to ensure the calculator is set to the appropriate degree or radian mode to match the measure in which the angle is expressed. Care is also needed with written presentation: if the angle is measured in degrees, the degree symbol must be given; if there is no degree sign then it is assumed the measurement is in radians.

WORKED EXAMPLE 7

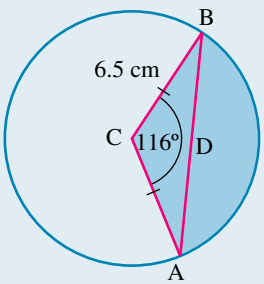
a. An arc subtends an angle of 56° at the centre of a circle of radius 10 cm. Calculate the length of the arc to 2 decimal places.

b. Calculate, in degrees, the magnitude of the angle that an arc of length 20π cm subtends at the centre of a circle of radius 15 cm.

THINK	WRITE
a. 1. The angle is given in degrees so convert it to radian measure.	**a.** $\theta^\circ = 56^\circ$ $\theta^c = 56 \times \frac{\pi}{180}$ $= \frac{14\pi}{45}$
2. Calculate the arc length.	$l = r\theta,\ r = 10,\ \theta = \frac{14\pi}{45}$ $l = 10 \times \frac{14\pi}{45}$ $= \frac{28\pi}{9}$ ≈ 9.77 The arc length is 9.77 cm (to 2 decimal places).
b. 1. Calculate the angle at the centre of the circle subtended by the arc.	**b.** $l = r\theta,\ r = 15,\ l = 20\pi$ $15\,\theta = 20\pi$ $\therefore \theta = \frac{20\pi}{15}$ $= \frac{4\pi}{3}$ The angle is $\frac{4\pi}{3}$ radians.
2. Convert the angle from radians to degrees.	In degree measure: $\theta^\circ = \frac{4\pi}{3} \times \frac{180^\circ}{\pi}$ $= 240^\circ$ The magnitude of the angle is 240°.

WORKED EXAMPLE 8

For the circle in this figure, which has a radius of 6.5 cm, calculate the area of the sector (answer should be rounded to 2 decimal places).



THINK	WRITE
1. Since the angle is measured in degrees, use the formula for the area of a sector in degrees. Determine the quantities r and θ.	$A_{\text{sector}} = \pi r^2 \times \frac{\theta}{360^\circ}$ $r = 6.5$ $\theta = 116^\circ$
2. Substitute the quantities into the formula and simplicity.	$A_{\text{sector}} = \pi(6.5)^2 \times \frac{116^\circ}{360^\circ}$ $= 42.7748$
3. Write the answer rounded to 2 decimal places.	$A_{\text{sector}} = 42.77\ \text{cm}^2$

Exercise 10.3 Radian measure

Technology free

1. a. Copy, complete and learn the following table by heart.

Degrees	30°	45°	60°
Radians			

b. Copy, complete and learn the following table by heart.

Degrees	0°	90°	180°	270°	360°
Radians					

2. Convert the following to degrees.

a. $\frac{\pi^c}{5}$ **b.** $\frac{2\pi^c}{3}$ **c.** $\frac{5\pi}{12}$ **d.** $\frac{11\pi}{6}$ **e.** $\frac{7\pi}{9}$ **f.** $\frac{9\pi}{2}$

3. Convert the following to radian measure.

a. 40° **b.** 150° **c.** 225° **d.** 300° **e.** 315° **f.** 720°

4. WE5 **a.** Convert 60° to radian measure.

b. Convert $\frac{3\pi^c}{4}$ to degree measure.

c. Convert $\frac{\pi}{6}$ to degree measure and hence state the value of $\tan\left(\frac{\pi}{6}\right)$.

5. The real number line is wrapped around the circumference of the unit circle. Give two positive and two negative real numbers which lie in the same position as the following numbers.

a. 0 **b.** -1

6. WE6 **a.** Convert 1.8^c to degree measure.

b. Draw a unit circle diagram to show the position the real number 1.8 is mapped to when the real number line is wrapped around the circumference of the unit circle.

7. a. For each of the following, draw a unit circle diagram to show the position of the angle and the arc which subtends the angle.

i. An angle of 2 radians **ii.** An angle of -2 radians

iii. An angle of $-\frac{\pi}{2}$

b. For each of the following, draw a unit circle diagram with the real number line wrapped around its circumference to show the position of the number and the associated angle subtended at the centre of the circle.

i. The number 4 **ii.** The number -1

iii. The number $\frac{7\pi}{3}$

8. WE7 **a.** An arc subtends an angle of 75° at the centre of a circle of radius 8 cm. Calculate the length of the arc, to 2 decimal places.

b. Calculate, in degrees, the magnitude of the angle that an arc of length 12π cm subtends at the centre of a circle of radius 10 cm.

Technology active

9. Express $145°12'$ in radian measure, correct to 2 decimal places.

10. a. Express in radian measure, to 3 decimal places.

i. 3° **ii.** 112°15′ **iii.** 215.36°

b. Express in degree measure to 3 decimal places.

i. 3^c **ii.** 2.3π

c. Rewrite $\left\{1.5^c,\ 50°,\ \frac{\pi^c}{7}\right\}$ with the magnitudes of the angles ordered from smallest to largest.

11. Calculate the exact lengths of the following arcs.

a. The arc which subtends an angle of 150° at the centre of a circle of radius 12 cm

b. The arc which subtends an angle of $\frac{2\pi^c}{9}$ at the circumference of a circle of radius π cm

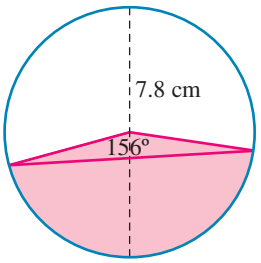
c. The major arc with endpoints on a chord which subtends an angle of 60° at the centre of a circle of radius 3 cm

12. An arc of length 6 mm subtends an angle of 0.5^c at the centre of a circle. What the radius of the circle?

13. A circle has a radius of 6 cm. An arc of length $\frac{3\pi}{4}$ cm forms part of the circumference of this circle. Find the angle the arc subtends at the centre of the circle:

a. in radian measure **b.** in degree measure.

14. WE8 For the circle in the figure at right, calculate the area of the sector. Your answer should be rounded to 3 decimal places.



15. A circle has a radius of 5.4 metres. A sector with an angle of 1.6 radians at the centre of the circle is created. Calculate the area of the sector.

16. a. A ball on the end of a rope of length 2.5 metres swings through an arc of 75 cm.
Through what angle, to the nearest tenth of a degree, does the ball swing?

b. A fixed point on the rim of a wheel of radius 3 metres rolls along horizontal ground at a speed of 2 m/s. After 5 seconds, calculate the angle the point has rotated through and express the answer in both degrees and radians.



c. An analogue wristwatch has a minute hand of length 11 mm. Calculate, to 2 decimal places, the arc length the minute hand traverses between 9.45 am and 9.50 am.

d. An arc of length 4 cm subtends an angle of 22.5° at the circumference of a circle. Calculate the area of the circle correct to 1 decimal place.

17. Evaluate the following to 3 decimal places.

a. $\tan(1.2)$

b. $\tan(1.2°)$

18. a. Calculate the following to 3 decimal places.

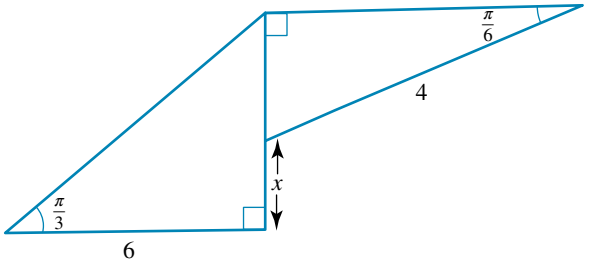
i. $\tan(1^c)$ **ii.** $\cos\left(\frac{2\pi}{7}\right)$ **iii.** $\sin(1.46°)$

b. Complete the following table with exact values.

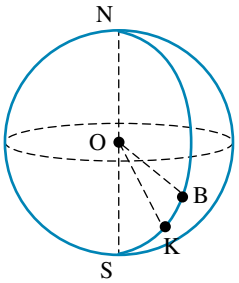
θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$			
$\cos(\theta)$			
$\tan(\theta)$			

19. a. Calculate the area of triangle ABC in which $a = 2\sqrt{2}$, $c = 2$, $B = \frac{\pi}{4}$.

b. Calculate the exact value of x in the following diagram.



20. a. The Western Australian towns of Broome(B) and Karonie(K) both lie on approximately the same longitude. Broome is approximately 1490 km due north of Karonie (the distance being measured along the meridian). When the sun is directly over Karonie, it is 13.4° south of Broome. Use this information to estimate the radius of the Earth.



This method dates back to Eratosthenes in 250 BCE, although he certainly didn't use these Australian towns to calculate his results.

b. A ship sailing due east along the equator from the Galapagos Islands to Ecuador travels a distance of 600 nautical miles. If the ship's longitude changes from 90° W to 80° W during this journey, estimate the radius of the Earth, given that 1 nautical mile is approximately 1.85 km.

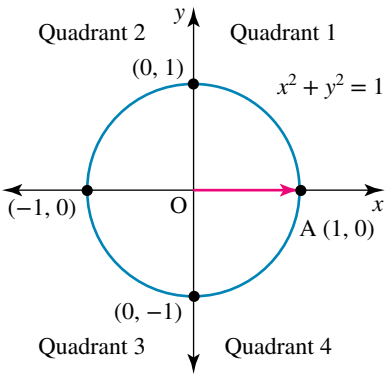
c. Taking the radius of the earth as 6370 km, calculate the distance, to the nearest kilometre, along the meridian between place A, located 20° S, 110° E, and place B, located 34° N, 110° E.



10.4 Unit circle definitions

With the introduction of radian measure, we encountered positive and negative angles of any size and then associated them with the wrapping of the real number line around the circumference of a unit circle. Before applying this wrapping to define the sine, cosine and tangent functions, we first consider the conventions for angle rotations and the positions of the endpoints of these rotations.

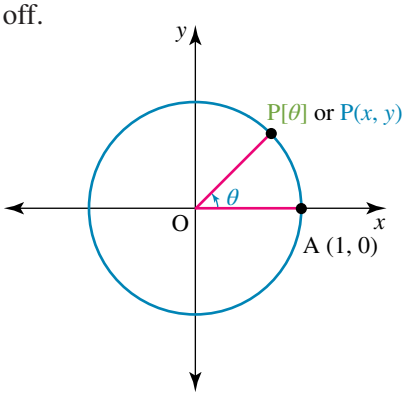
10.4.1 Trigonometric points

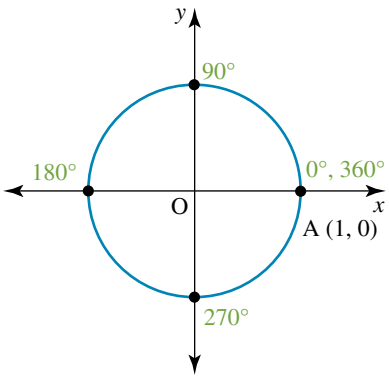


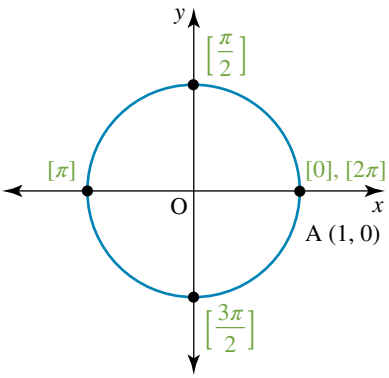
The unit circle has centre $(0, 0)$ and radius 1 unit. Its Cartesian equation is $x^2 + y^2 = 1$.

The coordinate axes divide the Cartesian plane into four quadrants. The points on the circle which lie on the boundaries between the quadrants are the endpoints of the horizontal and vertical diameters. These **boundary points** have coordinates $(-1, 0)$, $(1, 0)$ on the horizontal axis and $(0, -1)$, $(0, 1)$ on the vertical axis.

A rotation starts with an initial ray OA, where A is the point $(1, 0)$ and O $(0, 0)$. Angles are created by rotating the initial ray anticlockwise for positive angles and clockwise for negative angles. If the point on the circumference the ray reaches after a rotation of θ is P, then $\angle AOP = \theta$ and P is called the **trigonometric point** $[\theta]$. The angle of rotation θ may be measured in radian or degree measure. In radian measure, the value of θ corresponds to the length of the arc AP of the unit circle the rotation cuts off.







The point P$[\theta]$ has Cartesian coordinates (x, y) where:

- $x > 0$, $y > 0$ if P is in quadrant 1, $0 < \theta < \dfrac{\pi}{2}$
- $x < 0, y > 0$ if P is in quadrant 2, $\dfrac{\pi}{2} < \theta < \pi$
- $x < 0, y < 0$ if P is in quadrant 3, $\pi < \theta < \dfrac{3\pi}{2}$
- $x > 0, y < 0$ if P is in quadrant 4, $\dfrac{3\pi}{2} < \theta < 2\pi$.

Continued rotation, anticlockwise or clockwise, can be used to form other values for θ greater than 2π, or values less than 0, respectively. No trigonometric point has a unique θ value.

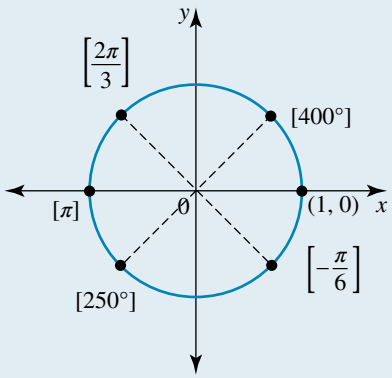
The angle θ is said to lie in the quadrant in which its endpoint P lies.

WORKED EXAMPLE 9

a. Give a trigonometric value, using radian measure, of the point P on the unit circle which lies on the boundary between the quadrants 2 and 3.

b. Identify the quadrants the following angles would lie in: 250°, 400°, $\frac{2\pi^c}{3}$, $-\frac{\pi^c}{6}$.

c. Give two other trigonometric points, Q and R, one with a negative angle and one with a positive angle respectively, which would have the same position as the point P [250°].

THINK	WRITE
a. 1. State the Cartesian coordinates of the required point.	**a.** The point $(-1, 0)$ lies on the boundary of quadrants 2 and 3.
2. Give a trigonometric value of this point. *Note:* Other values are possible.	An anticlockwise rotation of 180° or π^c from the point $(1, 0)$ would have its endpoint at $(-1, 0)$. The point P has the trigonometric value $[\pi]$.
b. 1. Explain how the quadrant is determined.	**b.** For positive angles, rotate anticlockwise from $(1, 0)$; for negative angles rotate clockwise from $(1, 0)$. The position of the endpoint of the rotation determines the quadrant.
2. Identify the quadrant the endpoint of the rotation would lie in for each of the given angles.	Rotating anticlockwise 250° from $(1, 0)$ ends in quadrant 3; rotating anticlockwise from $(1, 0)$ through 400° would end in quadrant 1; rotating anticlockwise from $(1, 0)$ by $\frac{2\pi}{3}$ would end in quadrant 2; rotating clockwise from $(1, 0)$ by $\frac{\pi}{6}$ would end in quadrant 4.
	
3. State the answer.	The angle 250° lies in quadrant 3, 400° in quadrant 1, $\frac{2\pi^c}{3}$ in quadrant 2, and $-\frac{\pi^c}{6}$ in quadrant 4.

c. 1. Identify a possible trigonometric point Q.

c. A rotation of 110° in the clockwise direction from $(1, 0)$ would end in the same position as $P[250°]$. Therefore, the trigonometric point could be Q$[-110°]$.

2. Identify a possible trigonometric point R.

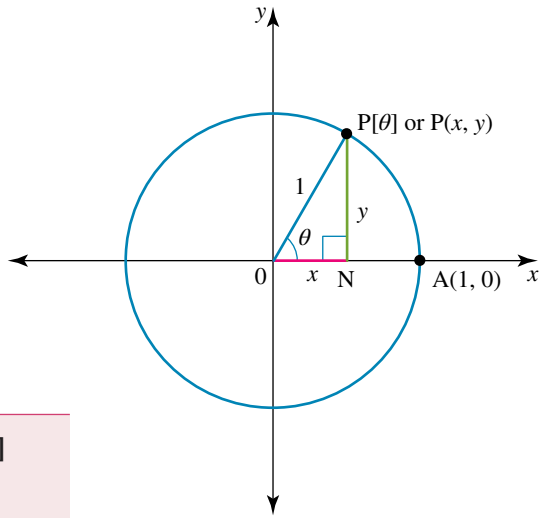
A full anticlockwise revolution of 360° plus another anticlockwise rotation of 250° would end in the same position as P[250°].
Therefore, the trigonometric point could be R[610°].

10.4.2 Unit circle definitions of the sine, cosine and tangent functions

Consider the unit circle and trigonometric point P$[\theta]$ with Cartesian coordinates (x, y) on its circumference. In the triangle ONP, $\angle NOP = \theta = \angle AOP$, ON $= x$ and NP $= y$.

As the triangle ONP is right-angled, $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$. This enables the following definitions to be given.

For a rotation from the point $(1, 0)$ of any angle θ with endpoint $P[\theta]$ on the unit circle:



$\cos(\theta)$ is the x-coordinate of the trigonometric point $P[\theta]$
$\sin(\theta)$ is the y-coordinate of the trigonometric point $P[\theta]$

Substituting these coordinate values into the equation of the circle gives:

$$x^2 + y^2 = 1$$
$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$$
$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

This relationship, $\cos^2(\theta) + \sin^2(\theta) = 1$, is known as the **Pythagorean identity**. It is a true statement for any value of θ.

The Pythagorean identity may be rearranged to give either $\cos^2(\theta) = 1 - \sin^2(\theta)$ or $\sin^2(\theta) = 1 - \cos^2(\theta)$.

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$\sin^2(\theta) = 1 - \cos^2(\theta)$$
$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

The importance of these definitions is that they enable sine and cosine functions to be defined for any real number θ. With θ measured in radians, the trigonometric point $[\theta]$ also marks the position the real number θ is mapped to when the number line is wrapped around the circumference of the unit circle, with zero placed at the point $(1, 0)$. This relationship enables the sine or cosine of a real number θ to be evaluated as the sine or cosine of the angle of rotation of θ radians in a unit circle: $\sin(\theta) = \sin(\theta^c)$ and $\cos(\theta) = \cos(\theta^c)$.

The sine and cosine functions are

$f: R \to R, f(\theta) = \sin(\theta)$ and $f: R \to R, f(\theta) = \cos(\theta)$.

They are **trigonometric functions**, also referred to as **circular functions**. The use of parentheses in writing $\sin(\theta)$ or $\cos(\theta)$ emphasises their functionality.

The mapping has a many-to-one correspondence as many values of θ are mapped to the one trigonometric point. The functions have a **period** of 2π since rotations of θ and of $2\pi + \theta$ have the same endpoint on the circumference of the unit circle. The cosine and sine values repeat after each complete revolution around the unit circle.

For $f(\theta) = \sin(\theta)$, the image of a number such as 4 is $f(4) = \sin(4) = \sin(4^c)$. This is evaluated as the y-coordinate of the trigonometric point [4] on the unit circle.

The values of a function for which $f(t) = \cos(t)$, where t is a real number, can be evaluated through the relation $\cos(t) = \cos(t^c)$ as t will be mapped to the trigonometric point $[t]$ on the unit circle.

The sine and cosine functions are periodic functions which have applications in contexts which may have nothing to do with angles, as we shall study in later chapters.

WORKED EXAMPLE 10

a. Calculate the Cartesian coordinates of the trigonometric point $P\left[\frac{\pi}{3}\right]$ and show the position of this point on a unit circle diagram.

b. Illustrate $\cos(330°)$ and $\sin(2)$ on a unit circle diagram.

c. Use the Cartesian coordinates of the trigonometric point $[\pi]$ to obtain the values of $\sin(\pi)$ and $\cos(\pi)$.

d. If $f(\theta) = \cos(\theta)$, evaluate $f(0)$.

THINK

a. 1. State the value of θ.

2. Calculate the exact Cartesian coordinates.
Note: The exact values for sine and cosine of $\frac{\pi^c}{3}$, or 60°, need to be known.

WRITE

a. $P\left[\frac{\pi}{3}\right]$

This is the trigonometric point with $\theta = \frac{\pi}{3}$.

The Cartesian coordinates are:

$x = \cos(\theta)$ $\quad$ $y = \sin(\theta)$

$= \cos\left(\frac{\pi}{3}\right)$ $\quad$ $= \sin\left(\frac{\pi}{3}\right)$

$= \cos\left(\frac{\pi}{3}\right)^c$ $\quad$ $= \sin\left(\frac{\pi}{3}\right)^c$

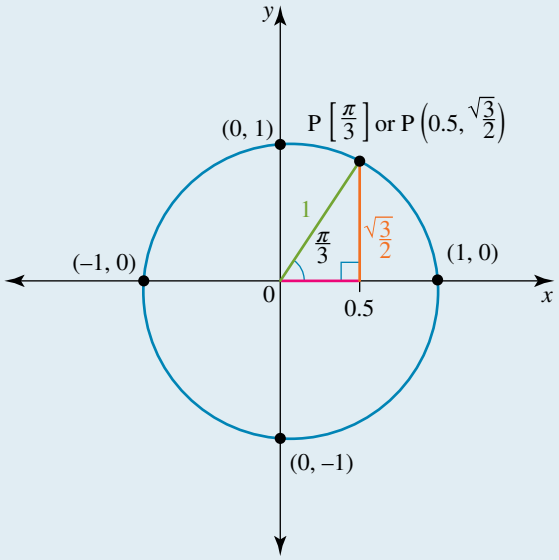
$= \cos(60°)$ $\quad$ $= \sin(60°)$

$= \frac{1}{2}$ $\quad$ $= \frac{\sqrt{3}}{2}$

Therefore, P has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

3. Show the position of the given point on a unit circle diagram.

$P\left[\frac{\pi}{3}\right]$ or $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ lies in quadrant 1 on the circumference of the unit circle.



b. 1. Identify the trigonometric point and which of its Cartesian coordinates gives the first value.

2. State the quadrant in which the trigonometric point lies.

3. Identify the trigonometric point and which of its Cartesian coordinates gives the second value.

4. State the quadrant in which the trigonometric point lies.

5. Draw a unit circle showing the two trigonometric points and construct the line segments which illustrate the x- and y-coordinates of each point.

b. cos (330°):

The value of cos (330°) is given by the x-coordinate of the trigonometric point [330°].

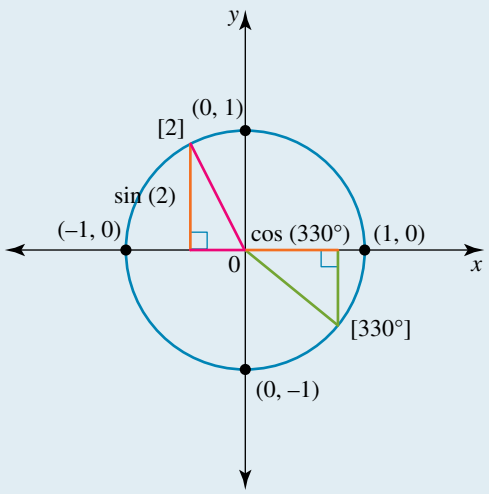
The trigonometric point [330°] lies in quadrant 4.

sin (2):

The value of sin (2) is given by the y-coordinate of the trigonometric point [2].

As $\frac{\pi}{2} \approx 1.57 < 2 < \pi \approx 3.14$, the trigonometric point [2] lies in quadrant 2.

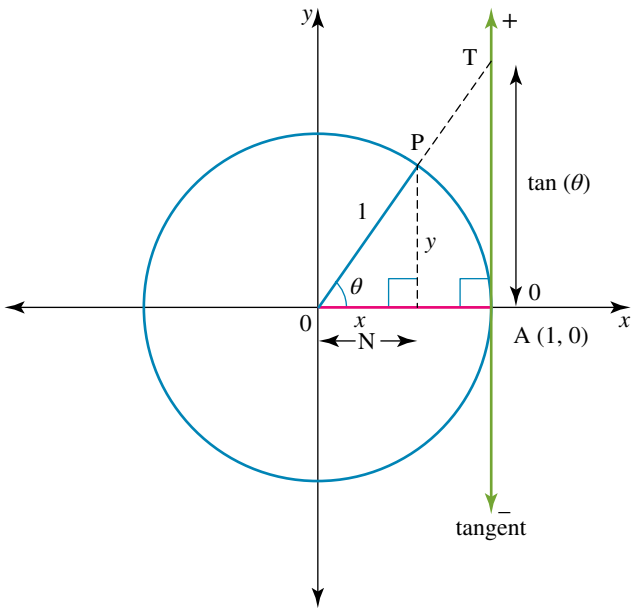
For each of the points on the unit circle diagram, the horizontal line segment gives the x-coordinate and the vertical line segment gives the y-coordinate.



6. Label the line segments which represent the appropriate coordinate for each point.	The value of cos (330°) is the length measure of the horizontal line segment. The value of sin(2) is the length measure of the vertical line segment. The line segments illustrating these values are highlighted in orange on the diagram.
c. 1. State the Cartesian coordinates of the given point.	**c.** An anticlockwise rotation of π from $(1, 0)$ gives the endpoint $(-1, 0)$. The trigonometric point $[\pi]$ is the Cartesian point $(-1, 0)$.
2. State the required values.	The point $(-1, 0)$ has $x = -1$, $y = 0$. Since $x = \cos(\theta)$, $\cos(\pi) = x$ $= -1$ Since $y = \sin(\theta)$, $\sin(\pi) = y$ $= 0$
d. 1. Substitute the given value in the function rule.	**d.** $f(\theta) = \cos(\theta)$ $\therefore f(0) = \cos(0)$
2. Identify the trigonometric point and state its Cartesian coordinates.	The trigonometric point [0] has Cartesian coordinates $(1, 0)$.
3. Evaluate the required value of the function.	The value of cos (0) is given by the x-coordinate of the point $(1, 0)$. $\therefore \cos(0) = 1$ $\therefore f(0) = 1$

10.4.3 Unit circle definition of the tangent function

Consider again the unit circle with centre O(0, 0) containing the points A(1, 0) and the trigonometric point P[θ] on its circumference. A tangent line to the circle is drawn at point A. The radius OP is extended to intersect the tangent line at point T.



For any point P[θ] on the unit circle, tan(θ) is defined as the length of the intercept AT that the extended ray OP cuts off on the tangent drawn to the unit circle at the point A(1, 0).

Intercepts that lie above the x-axis give positive tangent values; intercepts that lie below the x-axis give negative tangent values.

Unlike the sine and cosine functions, there are values of θ for which $\tan(\theta)$ is undefined. These occur when OP is vertical and therefore parallel to the tangent line through A(1, 0); these two vertical lines cannot intersect no matter how far OP is extended. The values of $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$, for instance, are undefined.

The value of $\tan(\theta)$ can be calculated from the coordinates (x, y) of the point P $[\theta]$, provided the x-coordinate is not zero.

Using the ratio of sides of the similar triangles ONP and OAT:

$$\frac{\text{AT}}{\text{OA}} = \frac{\text{NP}}{\text{ON}}$$
$$\frac{\tan(\theta)}{1} = \frac{y}{x}$$

Hence:
$\tan(\theta) = \frac{y}{x}$, $x \neq 0$, where (x, y) are the coordinates of the trigonometric point P $[\theta]$.
Since $x = \cos(\theta), y = \sin(\theta)$, this can be expressed as the relationship:

$$\boldsymbol{\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}}$$

10.4.4 Domains and ranges of the trigonometric functions

The domain and range of the unit circle require $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ so $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$.

Since θ can be any real number, this means that the function f where f is either sine or cosine has domain R and range $[-1, 1]$.

Unlike the sine and cosine functions, the domain of the tangent function is not the set of real numbers R since $\tan(\theta)$ is not defined for any value of θ which is an odd multiple of $\frac{\pi}{2}$. Excluding these values, intercepts of any size may be cut off on the tangent line so $\tan(\theta) \in R$.

This means that the function f where f is tangent has domain $R \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$ and range R.

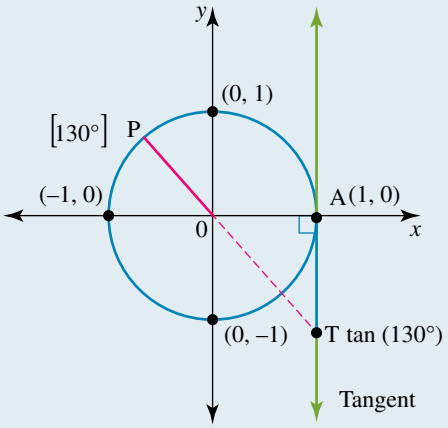
The domain of the tangent function can be written as $R \setminus \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$ and the tangent function as the mapping $f: R \setminus \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\} \to R, f(\theta) = \tan(\theta)$.

WORKED EXAMPLE 11

a. Illustrate $\tan(130°)$ on a unit circle diagram and use a calculator to evaluate $\tan(130°)$ to 3 decimal places.
b. Use the Cartesian coordinates of the trigonometric point P $[\pi]$ to obtain the value of $\tan(\pi)$.

THINK	WRITE
a. 1. State the quadrant in which the angle lies.	**a.** 130° lies in the second quadrant.

2. Draw the unit circle with the tangent at the point A(1, 0).
Note: The tangent line is always drawn at this point (1, 0).



3. Extend PO until it reaches the tangent line.

Let T be the point where the extended radius PO intersects the tangent drawn at A.
The intercept AT is tan (130°).

4. State whether the required value is positive, zero or negative.

The intercept lies below the x-axis, which shows that tan (130°) is negative.

5. Calculate the required value.

The value of tan (130°) = −1.192, correct to 3 decimal places.

b. 1. Identify the trigonometric point and state its Cartesian coordinates.

b. The trigonometric point P [π] is the endpoint of a rotation of π^c or 180°. It is the Cartesian point P(−1, 0).

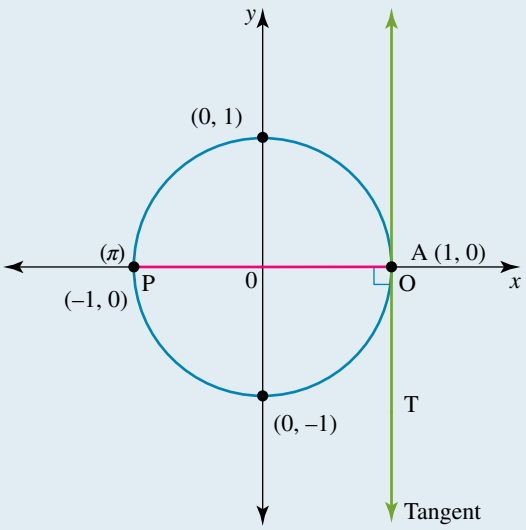
2. Calculate the required value.

The point (−1, 0) has $x = -1$, $y = 0$.
since $\tan(\theta) = \frac{y}{x}$,

$$\tan(\pi) = \frac{0}{-1}$$
$$= 0$$

3. Check the answer using the unit circle diagram.

Check:
PO is horizontal and runs along the x-axis. Extending PO, it intersects the tangent at the point A. This means the intercept is 0, which means tan (π) = 0.



TI | THINK

a. 1. Put the Calculator in Degree mode.
On a Calculator page, complete the entry line as tan(130) then press ENTER.

2. The answer appears on the screen.

WRITE



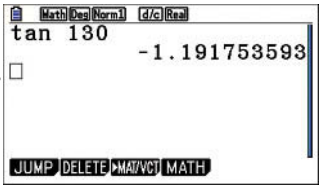
$\tan(130°) = -1.192$ (3 decimal places)

CASIO | THINK

a. 1. Put the Calculator in Degree mode.
On a Run-Matrix screen, complete the entry line as tan 130 then press EXE.

2. The answer appears on the screen.

WRITE



$\tan(130°) = -1.192$ (3 decimal places)

Resources

Interactivity: The unit circle (int-2582)

studyon

Units 1 & 2 > Area 7 > Sequence 1 > Concept 3

The unit circle Summary screen and practice questions

Exercise 10.4 Unit circle definitions

Technology free

1. Identify the quadrant each of the following would lie in.
   - **a.** 24°
   - **b.** 240°
   - **c.** 123°
   - **d.** 365°
   - **e.** −50°
   - **f.** −120°
2. Identify the quadrant in which each of the following lies.
   - **a.** 585°
   - **b.** $\frac{11\pi}{12}$
   - **c.** -18π
   - **d.** $\frac{7\pi}{4}$
3. **a.** The trigonometric point $[\theta]$ lies on the boundary between quadrants 1 and 2.
   - **i.** What are the Cartesian coordinates of the trigonometric point $[\theta]$?
   - **ii.** State the value of $\sin(\theta)$.

   b. The trigonometric point $[\alpha]$ lies on the boundary between quadrants 2 and 3.
   - **i.** What are the Cartesian coordinates of the trigonometric point $[\alpha]$?
   - **ii.** State the value of $\cos(\alpha)$.

   c. The trigonometric point $[\beta]$ lies on the boundary between quadrants 3 and 4.
   - **i.** What are the Cartesian coordinates of the trigonometric point $[\beta]$?
   - **ii.** State the value of $\tan(\beta)$.

   d. The trigonometric point $[v]$ lies on the boundary between quadrants 4 and 1.
   - **i.** What are the Cartesian coordinates of the trigonometric point $[v]$?
   - **ii.** State the value of $\sin(v)$, $\cos(v)$ and $\tan(v)$.
4. WE9 **a.** Give a trigonometric value, using radian measure, of the point P on the unit circle which lies on the boundary between the quadrants 1 and 2.

   b. Identify the quadrants the following angles would lie in: 120°, −400°, $\frac{4\pi^c}{3}$, $\frac{\pi^c}{4}$.

   c. Give two other trigonometric points, Q with a negative angle and R with a positive angle, which would have the same position as the point P [120°].

5. **WE10** **a.** Calculate the Cartesian coordinates of the trigonometric point P $\left[\frac{\pi}{6}\right]$ and show the position of this point on a unit circle diagram.
   b. Illustrate $\sin(225°)$ and $\cos(1)$ on a unit circle diagram.
   c. Use the Cartesian coordinates of the trigonometric point $\left[-\frac{\pi}{2}\right]$ to obtain the values of $\sin\left(-\frac{\pi}{2}\right)$ and $\cos\left(-\frac{\pi}{2}\right)$.
   d. If $f(\theta) = \sin(\theta)$, evaluate $f(0)$.
6. **a.** For function $f(t) = \sin(t)$, evaluate $f\left(\frac{3\pi}{2}\right)$.
   b. For function $g(t) = \cos(t)$, evaluate $g(4\pi)$.
   c. For function $h(t) = \tan(t)$, evaluate $h(-\pi)$.
   d. For function $k(t) = \sin(t) + \cos(t)$, evaluate $k(6.5\pi)$.
7. Identify the quadrants where:
   a. $\sin(\theta)$ is always positive
   b. $\cos(\theta)$ is always positive.
8. **a.** Calculate the Cartesian coordinates of the trigonometric point P $\left[\frac{\pi}{4}\right]$.
   b. Express the Cartesian point P$(0, -1)$ as two different trigonometric points, one with a positive value for θ and one with a negative value for θ.
9. Illustrate the following on a unit circle diagram.
   a. $\cos(40°)$ **b.** $\sin(165°)$ **c.** $\cos(-60°)$ **d.** $\sin(-90°)$
10. Illustrate the following on a unit circle diagram.
   a. $\sin\left(\frac{5\pi}{3}\right)$ **b.** $\cos\left(\frac{3\pi}{5}\right)$ **c.** $\cos(5\pi)$ **d.** $\sin\left(-\frac{2\pi}{3}\right)$
11. Illustrate the following on a unit circle diagram.
   a. $\tan(315°)$ **b.** $\tan\left(\frac{5\pi}{6}\right)$ **c.** $\tan\left(\frac{4\pi}{3}\right)$ **d.** $\tan(-300°)$
12. **a.** The trigonometric point P$[\theta]$ has Cartesian coordinates $(-0.8, 0.6)$. State the quadrant in which P lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
   b. The trigonometric point Q$[\theta]$ has Cartesian coordinates $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. State the quadrant in which Q lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
   c. For the trigonometric point R$[\theta]$ with Cartesian coordinates $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, state the quadrant in which R lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
   d. The Cartesian coordinates of the trigonometric point S$[\theta]$ are $(0, 1)$. Describe the position of S and state the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
13. By locating the appropriate trigonometric point and its corresponding Cartesian coordinates, obtain the exact values of the following.
   a. $\cos(0)$ **b.** $\sin\left(\frac{\pi}{2}\right)$
   c. $\tan(\pi)$ **d.** $\cos\left(\frac{3\pi}{2}\right)$
   e. $\sin(2\pi)$ **f.** $\cos\left(\frac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\frac{11\pi}{2}\right)$

14. Consider $\left\{\tan(-3\pi), \tan\left(\frac{5\pi}{2}\right), \tan(-90°), \tan\left(\frac{3\pi}{4}\right), \tan(780°)\right\}$.

a. Which elements in the set are not defined?

b. Which elements have negative values?

15. Consider O, the centre of the unit circle, and the trigonometric points P $\left[\frac{3\pi}{10}\right]$ and Q $\left[\frac{2\pi}{5}\right]$ on its circumference.

a. Sketch the unit circle showing these points.

b. How many radians are contained in the angle $\angle$QOP?

c. Express each of the trigonometric points P and Q with a negative θ value.

d. Express each of the trigonometric points P and Q with a larger positive value for θ than the given values P $\left[\frac{3\pi}{10}\right]$ and Q $\left[\frac{2\pi}{5}\right]$.

Technology active

16. WE11 **a.** Illustrate $\tan(230°)$ on a unit circle diagram and use a calculator to evaluate $\tan(230°)$ to 3 decimal places.

b. Use the Cartesian coordinates of the trigonometric point P $[2\pi]$ to obtain the value of $\tan(2\pi)$.

17. **a.** On a unit circle diagram show the trigonometric point P [2] and the line segments $\sin(2)$, $\cos(2)$ and $\tan(2)$. Label them with their length measures expressed to 2 decimal places.

b. State the Cartesian coordinates of P to 2 decimal places.

18. On a unit circle diagram show the trigonometric points A [0] and P $[\theta]$ where θ is acute, and show the line segments $\sin(\theta)$ and $\tan(\theta)$. By comparing the lengths of the line segments with the length of the arc AP, explain why $\sin(\theta) < \theta < \tan(\theta)$ for acute θ.



19. **a.** Use technology to obtain the exact Cartesian coordinates of the trigonometric points P $\left[\frac{7\pi}{4}\right]$ and Q $\left[\frac{\pi}{4}\right]$, and describe the relative position of the points P and Q on the unit circle.

b. Use technology to obtain the Cartesian coordinates of the trigonometric points R $\left[\frac{4\pi}{5}\right]$ and S $\left[\frac{\pi}{5}\right]$. Describe the relative position of these points on the unit circle.

c. Give the sine, cosine and tangent values of:

i. $\frac{7\pi}{4}$ and $\frac{\pi}{4}$, and compare the values

ii. $\frac{4\pi}{5}$ and $\frac{\pi}{5}$, and compare the values.

20. Use technology to calculate the exact value of the following.

a. $\cos^2\left(\frac{7\pi}{6}\right) + \sin^2\left(\frac{7\pi}{6}\right)$

b. $\cos\left(\frac{7\pi}{6}\right) + \sin\left(\frac{7\pi}{6}\right)$

c. $\sin^2\left(\frac{7}{6}\right) + \cos^2\left(\frac{7}{6}\right)$

d. $\sin^2(76°) + \cos^2(76°)$

e. $\sin^2(t) + \cos^2(t)$; explain the result with reference to the unit circle

10.5 Exact values and symmetry properties

There are relationships between the coordinates and associated trigonometric values of trigonometric points placed in symmetric positions in each of the four quadrants. These will now be investigated.

10.5.1 The signs of the sine, cosine and tangent values in the four quadrants

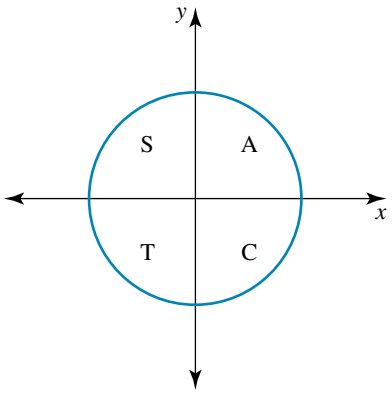
The definitions $\cos(\theta) = x, \sin(\theta) = y, \tan(\theta) = \frac{y}{x}$ where (x, y) are the Cartesian coordinates of the trigonometric point $[\theta]$ have been established.

If θ lies in the first quadrant, All of the trigonometric values will be positive, since $x > 0, y > 0$.

If θ lies in the second quadrant, only the sine value will be positive, since $x < 0, y > 0$.

If θ lies in the third quadrant, only the tangent value will be positive, since $x < 0, y < 0$.

If θ lies in the fourth quadrant, only the cosine value will be positive, since $x > 0, y < 0$.



This is illustrated in the diagram shown.

There are several mnemonics for remembering the allocation of signs in this diagram: we shall use 'CAST' and refer to the diagram as the CAST diagram.

on Resources

Interactivity: All Sin Cos Tan (int-2583)

10.5.2 The sine, cosine and tangent values at the boundaries of the quadrants

The points which do not lie within a quadrant are the four coordinate axes intercepts of the unit circle. These are called the boundary points. Since the Cartesian coordinates and the trigonometric positions of these points are known, the boundary values can be summarised by the following table.

Boundary point	**(1, 0)**	**(0, 1)**	**(−1, 0)**	**(0, −1)**	**(1, 0)**
θ radians	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
θ degrees	0°	90°	180°	270°	360°
$\sin(\theta)$	0	1	0	−1	0
$\cos(\theta)$	1	0	−1	0	1
$\tan(\theta)$	0	undefined	0	undefined	0

Other values of θ could be used for the boundary points, including negative values.

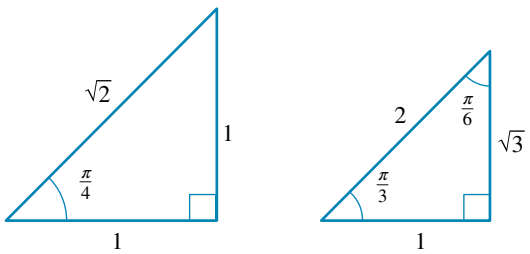
WORKED EXAMPLE 12

a. Identify the quadrant(s) where both $\cos(\theta)$ and $\sin(\theta)$ are negative.
b. If $f(\theta) = \cos(\theta)$, evaluate $f(-6\pi)$.

THINK	WRITE
a. 1. Refer to the CAST diagram.	**a.** $\cos(\theta) = x, \sin(\theta) = y$ The quadrant where both x and y are negative is quadrant 3.
b. 1. Substitute the given value in the function rule.	**b.** $f(\theta) = \cos(\theta)$ $\therefore f(-6\pi) = \cos(-6\pi)$
2. Identify the Cartesian coordinates of the trigonometric point.	A clockwise rotation of 6π from $(1, 0)$ shows that the trigonometric point $[-6\pi]$ is the boundary point with coordinates $(1, 0)$.
3. Evaluate the required value of the function.	The x-coordinate of the boundary point gives the cosine value. $\cos(-6\pi) = 1$ $\therefore f(-6\pi) = 1$

Exact trigonometric values of $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$

As the exact trigonometric ratios are known for angles of 30°, 45° and 60°, these give the trigonometric ratios for $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ respectively. A summary of these is given with the angles in each triangle expressed in radian measure. The values should be memorised.

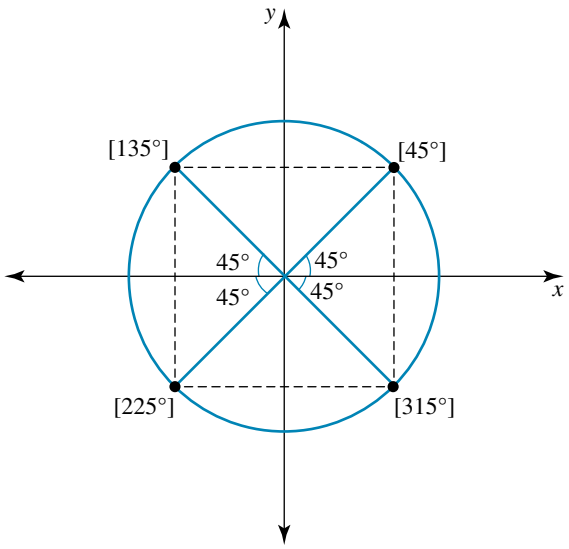


θ	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{4}$ or 45°	$\frac{\pi}{3}$ or 60°
$\sin(\theta)$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

These values can be used to calculate the exact trigonometric values for other angles which lie in positions symmetric to these first-quadrant angles.

10.5.3 Trigonometric points symmetric to $[\theta]$ where $\theta \in \left\{30°, 45°, 60°, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$

The symmetrical points to [45°] are shown in the diagram.



Each radius of the circle drawn to each of the points makes an acute angle of 45° with either the positive or the negative x-axis. The symmetric points to [45°] are the endpoints of a rotation which is 45° short of, or 45° beyond, the horizontal x-axis. The calculations $180° - 45°$, $180° + 45°$ and $360° - 45°$ give the symmetric trigonometric points [135°], [225°] and [315°], respectively.

Comparisons between the coordinates of these trigonometric points with those of the first quadrant point [45°] enable the trigonometric values of these non-acute angles to be calculated from those of the acute angle 45°.

Consider the y-coordinate of each point.

As the y-coordinates of the trigonometric points [135°] and [45°] are the same, $\sin(135°) = \sin(45°)$. Similarly, the y-coordinates of the trigonometric points [225°] and [315°] are the same, but both are the negative of the y-coordinate of [45°]. Hence, $\sin(225°) = \sin(315°) = -\sin(45°)$. This gives the following exact sine values.

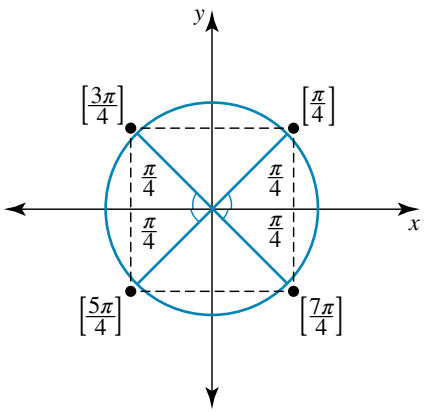
$$\sin(45°) = \frac{\sqrt{2}}{2};\ \sin(135°) = \frac{\sqrt{2}}{2};\ \sin(225°) = -\frac{\sqrt{2}}{2};\ \sin(315°) = -\frac{\sqrt{2}}{2}$$

Now consider the x-coordinate of each point.

As the x-coordinates of the trigonometric points [315°] and [45°] are the same, $\cos(315°) = \cos(45°)$. Similarly, the x-coordinates of the trigonometric points [135°] and [225°] are the same but both are the negative of the x-coordinate of [45°]. Hence, $\cos(135°) = \cos(225°) = -\cos(45°)$. This gives the following exact cosine values.

$$\cos(45°) = \frac{\sqrt{2}}{2};\ \cos(135°) = -\frac{\sqrt{2}}{2};\ \cos(225°) = -\frac{\sqrt{2}}{2};\ \cos(315°) = \frac{\sqrt{2}}{2}$$

Either by considering the intercepts cut off on the vertical tangent drawn at $(1, 0)$ or by using $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$, you will find that the corresponding relationships for the four points are $\tan(225°) = \tan(45°)$ and $\tan(135°) = \tan(315°) = -\tan(45°)$. Hence the exact tangent values are:



$$\tan(45°) = 1;\ \tan(135°) = -1;\ \tan(225°) = 1;\ \tan(315°) = -1.$$

The relationships between the Cartesian coordinates of [45°] and each of [135°], [225°] and [315°] enable the trigonometric values of 135°, 225° and 315° to be calculated from those of 45°.

If, instead of degree measure, the radian measure of $\frac{\pi}{4}$ is used, the symmetric points to $\left[\frac{\pi}{4}\right]$ are the endpoints of rotations which lie $\frac{\pi}{4}$ short of, or $\frac{\pi}{4}$ beyond, the horizontal x-axis. The positions of the symmetric points are calculated as $\pi-\frac{\pi}{4}, \pi+\frac{\pi}{4}, 2\pi-\frac{\pi}{4}$, giving the symmetric trigonometric points $\left[\frac{3\pi}{4}\right]$, $\left[\frac{5\pi}{4}\right]$, $\left[\frac{7\pi}{4}\right]$ respectively.

By comparing the Cartesian coordinates of the symmetric points with those of the first quadrant point $\left[\frac{\pi}{4}\right]$, it is possible to obtain results such as the following selection.

Second quadrant	Third quadrant	Fourth quadrant
$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$	$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right)$	$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$
$= -\frac{\sqrt{2}}{2}$	$= 1$	$= -\frac{\sqrt{2}}{2}$

A similar approach is used to generate symmetric points to the first quadrant points $\left[\frac{\pi}{6}\right]$ and $\left[\frac{\pi}{3}\right]$.

WORKED EXAMPLE 13

Calculate the exact values of the following.

a. $\cos\left(\frac{5\pi}{3}\right)$ **b.** $\sin\left(\frac{7\pi}{6}\right)$ **c.** $\tan(-30°)$

THINK

a. 1. State the quadrant in which the trigonometric point lies.

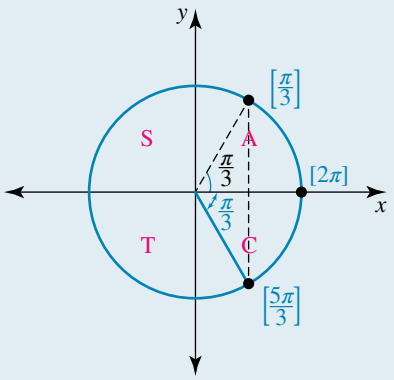
2. Identify the first-quadrant symmetric point.

3. Compare the coordinates of the symmetric points and obtain the required value.
Note: Check the +/− sign follows the CAST diagram rule.

WRITE

a. $\cos\left(\frac{5\pi}{3}\right)$

As $\frac{5\pi}{3} = \frac{5}{3}\pi = 1\frac{2}{3}\pi$,
the point $\left[\frac{5\pi}{3}\right]$ lies in quadrant 4.



Since $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, the rotation of $\frac{5\pi}{3}$ stops short of the x-axis by $\frac{\pi}{3}$. The points $\left[\frac{\pi}{3}\right]$ and $\left[\frac{5\pi}{3}\right]$ are symmetric.

The x-coordinates of the symmetric points are equal.

$$\cos\left(\frac{5\pi}{3}\right) = +\cos\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2}$$

Check: cosine is positive in quadrant 4.

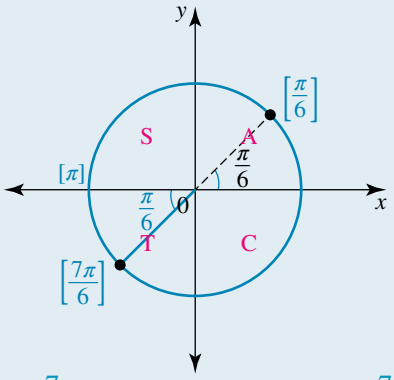
b. 1. State the quadrant in which the trigonometric point lies.

b. $\sin\left(\frac{7\pi}{6}\right)$

$\frac{7\pi}{6} = 1\frac{1}{6}\pi$

The point lies in quadrant 3.

2. Identify the first-quadrant symmetric point.



As $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$, the rotation of $\frac{7\pi}{6}$ goes beyond the x-axis by $\frac{\pi}{6}$. The points $\left[\frac{\pi}{6}\right]$ and $\left[\frac{7\pi}{6}\right]$ are symmetric.

3. Compare the coordinates of the symmetric points and obtain the required value.

The y-coordinate of $\left[\frac{7\pi}{6}\right]$ is the negative of that of $\left[\frac{\pi}{6}\right]$ in the first quadrant.

$\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$

$= -\frac{1}{2}$

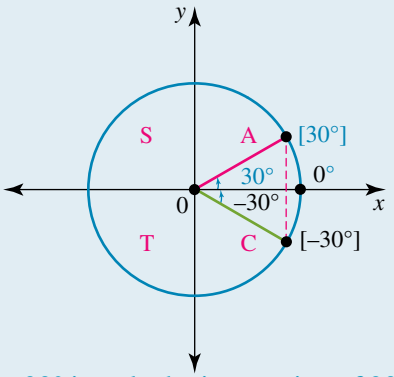
Check: sine is negative in quadrant 3.

c. 1. State the quadrant in which the trigonometric point lies.

c. $\tan(-30°)$

$[-30°]$ lies in quadrant 4.

2. Identify the first-quadrant symmetric point.



$-30°$ is a clockwise rotation of $30°$ from the horizontal so the symmetric point in the first quadrant is $[30°]$.

3. Compare the coordinates of the symmetric points and obtain the required value.

Note: Alternatively, consider the intercepts that would be cut off on the vertical tangent at $(1, 0)$.

The points $[30°]$ and $[-30°]$ have the same x-coordinates but opposite y-coordinates. The tangent value is negative in quadrant 4.

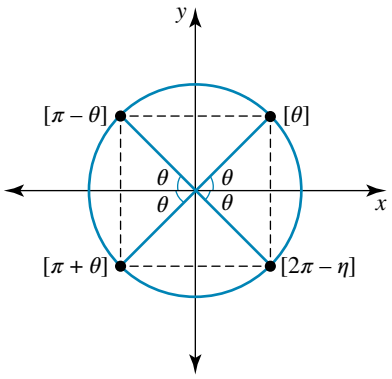
$\tan(-30°) = -\tan(30°)$

$= -\frac{\sqrt{3}}{3}$

10.5.4 Symmetry properties

The **symmetry properties** give the relationships between the trigonometric values in quadrants 2, 3, 4 and that of the first quadrant value, called the base, with which they are symmetric. The symmetry properties are simply a generalisation of what was covered for the bases $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

For any real number θ where $0 < \theta < \frac{\pi}{2}$, the trigonometric point $[\theta]$ lies in the first quadrant. The other quadrant values can be expressed in terms of the base θ, since the symmetric values will either be θ short of, or θ beyond, the horizontal x-axis.



The symmetric points to $[\theta]$ are:

- second quadrant $[\pi - \theta]$
- third quadrant $[\pi + \theta]$
- fourth quadrant $[2\pi - \theta]$.

Comparing the Cartesian coordinates with those of the first-quadrant base leads to the following general statements.

The symmetry properties for the second quadrant are:

$$\sin(\pi - \theta) = \sin(\theta)$$
$$\cos(\pi - \theta) = -\cos(\theta)$$
$$\tan(\pi - \theta) = -\tan(\theta).$$

The symmetry properties for the third quadrant are:

$$\sin(\pi + \theta) = -\sin(\theta)$$
$$\cos(\pi + \theta) = -\cos(\theta)$$
$$\tan(\pi + \theta) = \tan(\theta).$$

The symmetry properties for the fourth quadrant are:

$$\sin(2\pi - \theta) = -\sin(\theta)$$
$$\cos(2\pi - \theta) = \cos(\theta)$$
$$\tan(2\pi - \theta) = -\tan(\theta).$$

Other forms for the symmetric points

The rotation assigned to a point is not unique. With rotations or repeated revolutions, other values are always possible. However, the symmetry properties apply no matter how the points are described.

The trigonometric point $[2\pi + \theta]$ would lie in the first quadrant where all ratios are positive. Hence:

$$\sin(2\pi + \theta) = \sin(\theta)$$
$$\cos(2\pi + \theta) = \cos(\theta)$$
$$\tan(2\pi + \theta) = \tan(\theta).$$

The trigonometric point $[-\theta]$ would lie in the fourth quadrant where only cosine is positive. Hence:

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$
$$\tan(-\theta) = -\tan(\theta).$$

For negative rotations, the points symmetric to $[\theta]$ could be given as:

- fourth quadrant $[-\theta]$
- third quadrant $[-\pi + \theta]$
- second quadrant $[-\pi - \theta]$
- first quadrant $[-2\pi + \theta]$.

Using symmetry properties to calculate values of trigonometric functions

Trigonometric values are either the same as, or the negative of, the associated trigonometric values of the first-quadrant base; the sign is determined by the CAST diagram.

The base involved is identified by noting the rotation needed to reach the x-axis and determining how far short of or how far beyond this the symmetric point is. It is important to emphasise that for the points to be symmetric this is always measured from the horizontal and not the vertical axis.

To calculate a value of a trigonometric function, follow these steps.

- **Locate the quadrant in which the trigonometric point lies**
- **Identify the first-quadrant base with which the trigonometric point is symmetric**
- **Compare the coordinates of the trigonometric point with the coordinates of the base point or use the CAST diagram rule to form the sign in the first instance**
- **Evaluate the required value exactly if there is a known exact value involving the base.**

With practice, the symmetry properties allow us to recognise, for example, that $\sin\left(\frac{8\pi}{7}\right) = -\sin\left(\frac{\pi}{7}\right)$ because $\frac{8\pi}{7} = \pi + \frac{\pi}{7}$ and sine is negative in the third quadrant. Recognition of the symmetry properties is very important and we should aim to be able to apply these quickly. For example, to evaluate $\cos\left(\frac{3\pi}{4}\right)$ think: 'Second quadrant; cosine is negative; base is $\frac{\pi}{4}$,' and write the following.

$$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$$
$$= -\frac{\sqrt{2}}{2}$$

WORKED EXAMPLE 14

a. Identify the symmetric points to [20°]. At which of these points is the tangent value the same as tan (20°)?

b. Express $\sin\left(\frac{6\pi}{5}\right)$ in terms of a first-quadrant value.

c. If $\cos(\theta) = 0.6$, give the values of $\cos(\pi - \theta)$ and $\cos(2\pi - \theta)$.

d. Calculate the exact value of the following.

i. $\tan\left(\frac{7\pi}{6}\right)$ **ii. $\sin\left(\frac{11\pi}{3}\right)$**

THINK	WRITE
a. 1. Calculate the symmetric points to the given point.	a. Symmetric points to [20°] will be ±20° from the x-axis. The points are: second quadrant [180° − 20°] = [160°] third quadrant [180° + 20°] = [200°] fourth quadrant [360° − 20°] = [340°].

2. Identify the quadrant.

The point [20°] is in the first quadrant so tan (20°) is positive. As tangent is also positive in the third quadrant, tan (200°) = tan (20°).

3. State the required point.

The tangent value at the trigonometric point [200°] has the same value as tan (20°).

b. 1. Express the trigonometric value in the appropriate quadrant form.

b. $\frac{6\pi}{5}$ is in the third quadrant.

$$\sin\left(\frac{6\pi}{5}\right) = \sin\left(\pi + \frac{\pi}{5}\right)$$

2. Apply the symmetry property for that quadrant.

$$\therefore \sin\left(\frac{6\pi}{5}\right) = -\sin\left(\frac{\pi}{5}\right)$$

c. 1. Use the symmetry property for the appropriate quadrant.

c. $(\pi - \theta)$ is second quadrant form.

$\therefore \cos(\pi - \theta) = -\cos(\theta)$

2. State the answer.

Since $\cos(\theta) = 0.6$, $\cos(\pi - \theta) = -0.6$.

3. Use the symmetry property for the appropriate quadrant.

$2\pi - \theta$ is fourth quadrant form.

$\therefore \cos(2\pi - \theta) = \cos(\theta)$

4. State the answer.

Since $\cos(\theta) = 0.6$, $\cos(2\pi - \theta) = 0.6$.

d. i. 1. Express the trigonometric value in an appropriate quadrant form. Apply the symmetry property and evaluate.

d. i. $\frac{7\pi}{6}$ is in the third quadrant so $\tan\left(\frac{7\pi}{6}\right)$ is positive.

$$\tan\left(\frac{7\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right)$$
$$= \tan\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{3}$$

$$\therefore \tan\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

ii. 1. Express the trigonometric value in an appropriate quadrant form.

ii. $\frac{11\pi}{3}$ is in quadrant 4.

$$\sin\left(\frac{11\pi}{3}\right) = \sin\left(4\pi - \frac{\pi}{3}\right)$$

2. Apply the symmetry property and evaluate.

$$= \sin\left(2\pi - \frac{\pi}{3}\right)$$
$$= -\sin\left(\frac{\pi}{3}\right)$$
$$= -\frac{\sqrt{3}}{2}$$

$$\therefore \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

on Resources

Interactivity: Symmetry points & quadrants (int-2584)

Exercise 10.5 Exact values and symmetry properties

Technology free

1. State the value of each of the following. (It may help to show the boundary points on a diagram.)
   a. $\cos(4\pi)$ b. $\tan(9\pi)$ c. $\sin(7\pi)$
   d. $\sin\left(\frac{13\pi}{2}\right)$ e. $\cos\left(-\frac{9\pi}{2}\right)$ f. $\tan(-20\pi)$
2. Identify the quadrant(s), or boundaries, for which the following apply.
   a. $\cos(\theta) > 0, \sin(\theta) < 0$ b. $\tan(\theta) > 0, \cos(\theta) > 0$
   c. $\sin(\theta) > 0, \cos(\theta) < 0$ d. $\cos(\theta) = 0$
   e. $\cos(\theta) = 0, \sin(\theta) > 0$ f. $\sin(\theta) = 0, \cos(\theta) < 0$
3. Determine positions for the points in quadrants 2, 3 and 4 that are symmetric to the trigonometric point $[\theta]$ for which the value of θ is:
   a. $\frac{\pi}{3}$ b. $\frac{\pi}{6}$ c. $\frac{\pi}{4}$
   d. $\frac{\pi}{5}$ e. $\frac{3\pi}{8}$ f. 1.
4. Calculate the exact values of the following.
   a. $\cos(120°)$ b. $\tan(225°)$ c. $\sin(330°)$
   d. $\tan(-60°)$ e. $\cos(-315°)$ f. $\sin(510°)$
5. Calculate the exact values of the following.
   a. $\sin\left(\frac{3\pi}{4}\right)$ b. $\tan\left(\frac{2\pi}{3}\right)$ c. $\cos\left(\frac{5\pi}{6}\right)$
   d. $\cos\left(\frac{4\pi}{3}\right)$ e. $\tan\left(\frac{7\pi}{6}\right)$ f. $\sin\left(\frac{11\pi}{6}\right)$
6. Calculate the exact values of the following.
   a. $\cos\left(-\frac{\pi}{4}\right)$ b. $\sin\left(-\frac{\pi}{3}\right)$ c. $\tan\left(-\frac{5\pi}{6}\right)$
   d. $\sin\left(\frac{8\pi}{3}\right)$ e. $\cos\left(\frac{9\pi}{4}\right)$ f. $\tan\left(\frac{23\pi}{6}\right)$
7. **WE12** a. Identify the quadrant(s) where $\cos(\theta)$ is negative and $\tan(\theta)$ is positive.
   b. If $f(\theta) = \tan(\theta)$, evaluate $f(4\pi)$.
8. If $f(t) = \sin(\pi t)$, evaluate $f(2.5)$.
9. **WE13** Calculate the exact values of the following.
   a. $\sin\left(\frac{4\pi}{3}\right)$ b. $\tan\left(\frac{5\pi}{6}\right)$ c. $\cos(-30°)$
10. Calculate the exact values of $\sin\left(-\frac{5\pi}{4}\right)$, $\cos\left(-\frac{5\pi}{4}\right)$ and $\tan\left(-\frac{5\pi}{4}\right)$.

11. If $\cos(\theta) = 0.2$, use the symmetry properties to write down the value of the following.

a. $\cos(\pi - \theta)$ **b.** $\cos(\pi + \theta)$

c. $\cos(-\theta)$ **d.** $\cos(2\pi + \theta)$

12. If $\sin(t) = 0.9$ and $\tan(x) = 4$, calculate the value of the following.

a. $\tan(-x)$ **b.** $\sin(\pi - t)$

c. $\tan(2\pi - t)$ **d.** $\sin(-t) + \tan(\pi + x)$

13. Given $\cos(\theta) = 0.91$, $\sin(t) = 0.43$ and $\tan(x) = 0.47$, use the symmetry properties to obtain the values of the following.

a. $\cos(\pi + \theta)$ **b.** $\sin(\pi - t)$

c. $\tan(2\pi - x)$ **d.** $\cos(-\theta)$

e. $\sin(-t)$ **f.** $\tan(2\pi + x)$

14. If $\sin(\theta) = p$, express the following in terms of p.

a. $\sin(2\pi - \theta)$ **b.** $\sin(3\pi - \theta)$

c. $\sin(-\pi + \theta)$ **d.** $\sin(\theta + 4\pi)$

15. Calculate the exact values of the following.

a. $\cos\left(\frac{7\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right)$ **b.** $2\sin\left(\frac{7\pi}{4}\right) + 4\sin\left(\frac{5\pi}{6}\right)$

c. $\sqrt{3}\tan\left(\frac{5\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)$ **d.** $\sin\left(\frac{8\pi}{9}\right) + \sin\left(\frac{10\pi}{9}\right)$

e. $2\cos^2\left(-\frac{5\pi}{4}\right) - 1$ **f.** $\frac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

16. WE14 **a.** Identify the symmetric points to $[75°]$. At which of these points is the cosine value the same as $\cos(75°)$?

b. Express $\tan\left(\frac{6\pi}{7}\right)$ in terms of a first quadrant value.

c. If $\sin(\theta) = 0.8$, give the values of $\sin(\pi - \theta)$ and $\sin(2\pi - \theta)$.

d. Calculate the exact value of the following.

i. $\cos\left(\frac{5\pi}{4}\right)$

ii. $\sin\left(\frac{25\pi}{6}\right)$

17. Given $\cos(\theta) = p$, express the following in terms of p.

a. $\cos(-\theta)$ **b.** $\cos(5\pi + \theta)$

18. a. Verify that $\sin^2\left(\frac{5\pi}{4}\right) + \cos^2\left(\frac{5\pi}{4}\right) = 1$.

b. Explain, with the aid of a unit circle diagram, why $\cos(-\theta) = \cos(\theta)$ is true for $\theta = \frac{5\pi}{6}$.

c. The point $[\phi]$ lies in the second quadrant and has Cartesian coordinates $(-0.5, 0.87)$. Show this on a diagram and give the values of $\sin(\pi + \phi)$, $\cos(\pi + \phi)$ and $\tan(\pi + \phi)$.

d. Simplify $\sin(-\pi + t) + \sin(-3\pi - t) + \sin(t + 6\pi)$.

e. Use the unit circle to give two values of an angle A for which $\sin(A) = \sin(144°)$.

f. With the aid of the unit circle, give three values of B for which $\sin(B) = -\sin\left(\frac{2\pi}{11}\right)$.

Technology active

19. **a.** Identify the quadrant in which the point P[4.2] lies.
b. Calculate the Cartesian coordinates of point P[4.2] to 2 decimal places.
c. Identify the trigonometric positions, to 4 decimal places, of the points in the other three quadrants which are symmetric to the point P[4.2].

20. Consider the point Q[θ], $\tan(\theta) = 5$.
a. In which two quadrants could Q lie?
b. Determine, to 4 decimal places, the value of θ for each of the two points.
c. Calculate the Cartesian coordinates of the two points.

10.6 Graphs of the sine, cosine and tangent functions

As the two functions sine and cosine are closely related, we shall initially consider their graphs together.

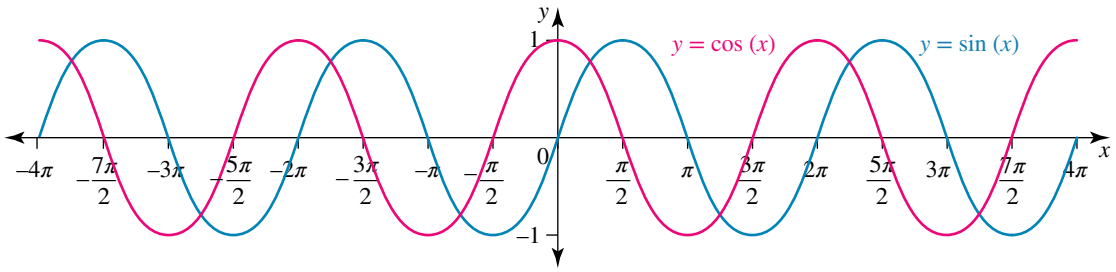
10.6.1 The graphs of $y = \sin(x)$ and $y = \cos(x)$

The functions sine and cosine are both periodic and have many-to-one correspondences, which means values repeat after every revolution around the unit circle. This means both functions have a period of 2π since $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.

The graphs of $y = \sin(x)$ and $y = \cos(x)$ can be plotted using the boundary values from continued rotations, clockwise and anticlockwise, around the unit circle.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x)$	0	1	0	-1	0
$\cos(x)$	1	0	-1	0	1

The diagram shows four cycles of the graphs drawn on the domain $[-4\pi, 4\pi]$. The graphs continue to repeat their wavelike pattern over their maximal domain R; the interval, or **period**, of each repetition is 2π.



The first observation that strikes us about these graphs is how remarkably similar they are: a horizontal translation of $\frac{\pi}{2}$ to the right will transform the graph of $y = \cos(x)$ into the graph of $y = \sin(x)$, while a horizontal translation of $\frac{\pi}{2}$ to the left transforms the graph of $y = \sin(x)$ into the graph of $y = \cos(x)$.

Recalling our knowledge of transformations of graphs, this observation can be expressed as follows.

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$
$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

The two functions are said to be '**out of phase**' by $\frac{\pi}{2}$ or to have a **phase difference** or **phase shift** of $\frac{\pi}{2}$.

Both graphs oscillate up and down one unit from the x-axis. The x-axis is the **equilibrium** or **mean position** and the distance the graphs oscillate up and down from this mean position to a maximum or minimum point is called the **amplitude**.

The graphs keep repeating this cycle of oscillations up and down from the equilibrium position, with the amplitude measuring half the vertical distance between maximum and minimum points and the period measuring the horizontal distance between successive maximum points or between successive minimum points.

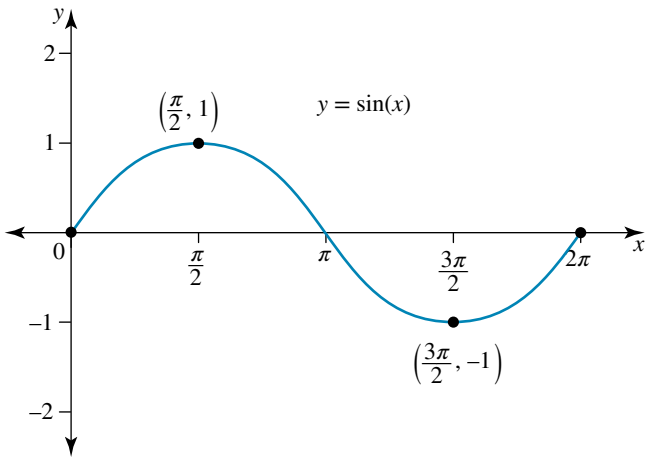
on Resources

- **Interactivity:** Graph plotter: Sine and cosine (int-2976)
- **Interactivity:** The unit circle: Sine and cosine graphs (int-6551)

10.6.2 One cycle of the graph of $y = \sin(x)$

The basic graph of $y = \sin(x)$ has the domain $[0, 2\pi]$, which restricts the graph to one cycle.

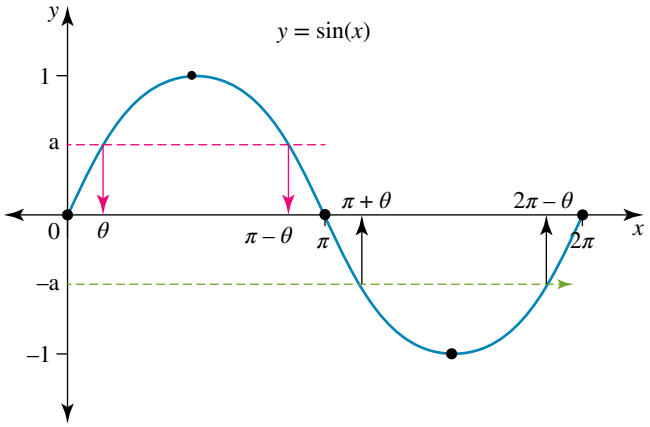
The graph of the function $f: [0, 2\pi] \to R, f(x) = \sin(x)$ is shown.



Key features of the graph of $y = \sin(x)$:

- **Equilibrium position is the x-axis, the line with equation $y = 0$.**
- **Amplitude is 1 unit.**
- **Period is 2π units.**
- **Domain is $[0, 2\pi]$.**
- **Range is $[-1, 1]$.**
- **The x-intercepts occur at $x = 0, \pi, 2\pi$.**
- **Type of correspondence is many-to-one.**

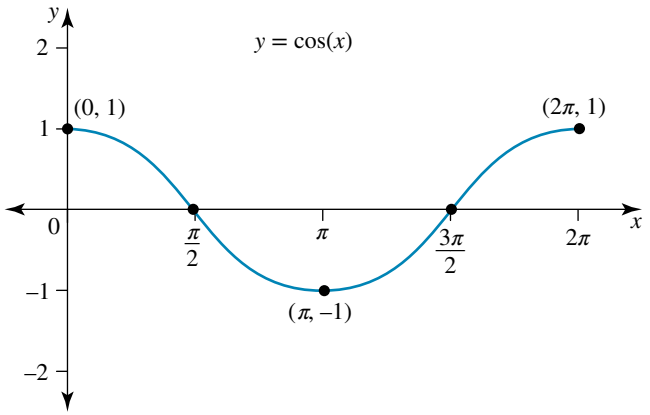
The graph lies above the x-axis for $x \in (0, \pi)$ and below for $x \in (\pi, 2\pi)$, which matches the quadrant signs of sine given in the CAST diagram. The symmetry properties of sine are displayed in its graph as $\sin(\pi - \theta) = \sin(\theta)$ and $\sin(\pi + \theta) = \sin(2\pi - \theta) = -\sin(\theta)$.



10.6.3 One cycle of the graph of $y = \cos(x)$

The basic graph of $y = \cos(x)$ has the domain $[0, 2\pi]$, which restricts the graph to one cycle.

The graph of the function $f: [0, 2\pi] \to R, f(x) = \cos(x)$ is shown.



Key features of the graph of $y = \cos \pi(x)$:

- **Equilibrium position is the x-axis, the line with equation $y = 0$.**
- **Amplitude is 1 unit.**
- **Period is 2π units.**
- **Domain is $[0, 2\pi]$.**
- **Range is $[-1, 1]$.**
- **The x-intercepts occur at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.**
- **Type of correspondence is many-to-one.**

The graph of $y = \cos(x)$ has the same amplitude, period, equilibrium (or mean) position, domain, range and type of correspondence as the graph of $y = \sin(x)$.

10.6.4 Guide to sketching the graphs on extended domains

There is a pattern of 5 points to the shape of the basic sine and cosine graphs created by the division of the period into four equal intervals.

For $y = \sin(x)$: first point starts at the equilibrium; the second point at $\frac{1}{4}$ of the period, reaches up one amplitude to the maximum point; the third point, at $\frac{1}{2}$ of the period, is back at equilibrium; the fourth point at $\frac{3}{4}$ of the period goes down one amplitude to the minimum point; the fifth point at the end of the period interval returns back to equilibrium.

In other words:

equilibrium → range maximum → equilibrium → range minimum → equilibrium.

For $y = \cos(x)$: the pattern for one cycle is summarised as:

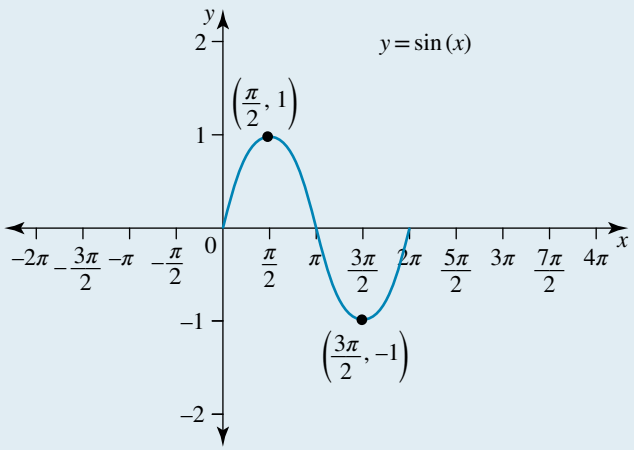
range maximum → equilibrium → range minimum → equilibrium → range maximum.

This pattern only needs to be continued in order to sketch the graph of $y = \sin(x)$ or $y = \cos(x)$ on a domain other than $[0, 2\pi]$.

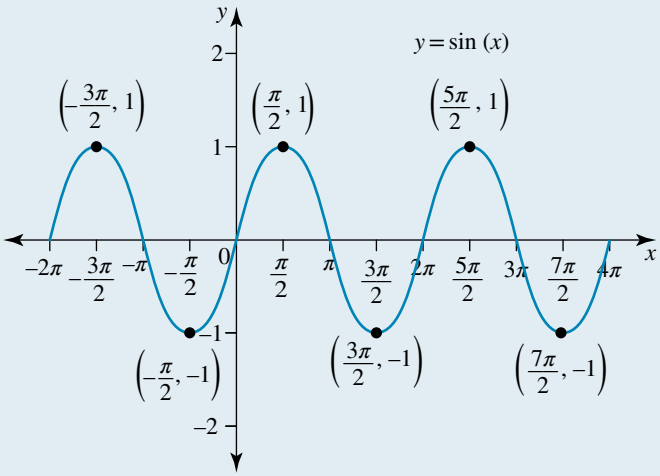
WORKED EXAMPLE 15

Sketch the graph of $y = \sin(x)$ over the domain $[-2\pi, 4\pi]$ and state the number of cycles of the sine function drawn.

THINK	WRITE
1. Draw the graph of the function over $[0, 2\pi]$.	The basic graph of $y = \sin(x)$ over the domain $[0, 2\pi]$ is drawn.



2. Extend the pattern to cover the domain specified.	The pattern is extended for one cycle in the negative direction and one further cycle in the positive direction to cover the domain $[-2\pi, 4\pi]$.



3. State the number of cycles of the function that are shown in the graph.

Altogether, 3 cycles of the sine function have been drawn.

TI | THINK

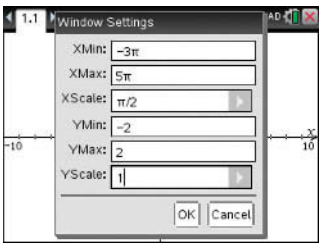
1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: -3π
XMax: 5π
XScale: $\pi/2$
YMin: -2
YMax: 2
YScale: 1
then select OK.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.

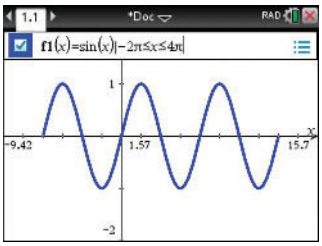
2. Complete the entry line for function 1 as
$f1(x) = \sin(x) | -2\pi \leq x \leq 4\pi$
then press ENTER.

3. Identify the coordinates of the x-intercepts from the graph.

4. Identify the coordinates of the maximums from the graph.

WRITE





The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, $(3\pi, 0)$ and $(4\pi, 0)$.
Be sure to mark these points when sketching the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has maximums at $\left(-\frac{3\pi}{2}, 1\right)$, $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{5\pi}{2}, 1\right)$.
Be sure to mark these points when sketching the graph.

CASIO | THINK

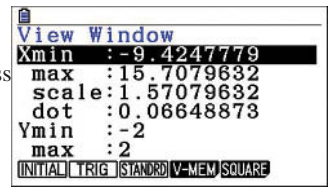
1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: -3π
max: 5π
scale: $\pi/2$
Ymin: -2
max: 2
scale: 1
then press EXIT.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.

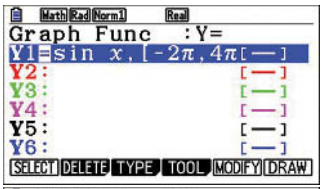
2. Complete the entry line for y1 as
$y1 = \sin x, [-2\pi, 4\pi]$
then press EXE.
Select DRAW by pressing F6.

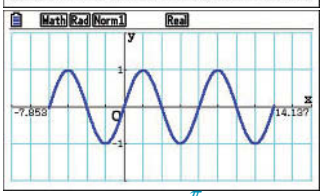
3. Identify the coordinates of the x-intercepts from the graph.

4. Identify the coordinates of the maximums from the graph.

WRITE







The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, $(3\pi, 0)$ and $(4\pi, 0)$.
Be sure to mark these points when sketching the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has maximums at $\left(-\frac{3\pi}{2}, 1\right)$, $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{5\pi}{2}, 1\right)$.
Be sure to mark these points when sketching the graph.

5. Identify the coordinates of the minimums from the graph.	The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has minimums at $\left(-\frac{\pi}{2}, -1\right)$, $\left(\frac{3\pi}{2}, -1\right)$ and $\left(\frac{7\pi}{2}, -1\right)$. Be sure to mark these points when sketching the graph.
6. Count the number of cycles of the sine function drawn.	3 cycles are drawn.

5. Identify the coordinates of the minimums from the graph.	The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has minimums at $\left(-\frac{\pi}{2}, -1\right)$, $\left(\frac{3\pi}{2}, -1\right)$ and $\left(\frac{7\pi}{2}, -1\right)$. Be sure to mark these points when sketching the graph.
6. Count the number of cycles of the sine function drawn.	3 cycles are drawn.

10.6.5 The graph of $y = \tan(x)$

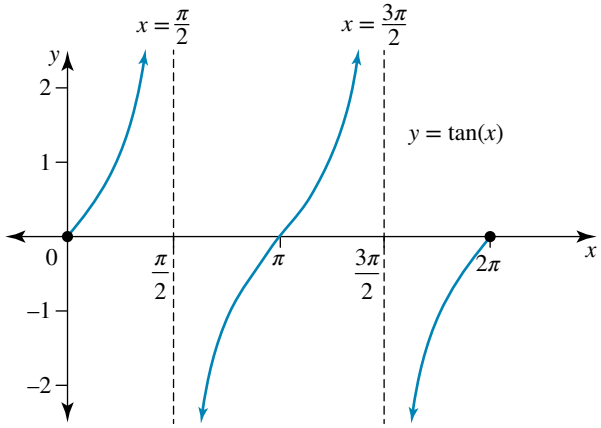
The graph of $y = \tan(x)$ has a distinct shape quite different to that of the wave shape of the sine and cosine graphs. As values such as $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$ are undefined, a key feature of the graph of $y = \tan(x)$ is the presence of vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

The relationship $\tan(x) = \frac{\sin(x)}{\cos(x)}$ shows that:

- **$\tan(x)$ will be undefined whenever $\cos(x) = 0$**
- **$\tan(x) = 0$ whenever $\sin(x) = 0$.**

The graph below shows $y = \tan(x)$ over the domain $[0, 2\pi]$.
The key features of the graph of $y = \tan(x)$ are:

- Vertical asymptotes at $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ for the domain $[0, 2\pi]$. For extended domains this pattern continues with asymptotes at $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.
- Period π. Two cycles are completed over the domain $[0, 2\pi]$.
- Range is R.
- x-intercepts occur at $x = 0, \pi, 2\pi$ for the domain $[0, 2\pi]$. For extended domains this pattern continues with x-intercepts at $x = n\pi, n \in Z$.
- Mean position is $y = 0$.
- The asymptotes are one period apart.
- The x-intercepts are one period apart.



Unlike the sine and cosine graphs, 'amplitude' has no meaning for the tangent graph. As for any graph, the x-intercepts of the tangent graph are the solutions to the equation formed when $y = 0$.

WORKED EXAMPLE 16

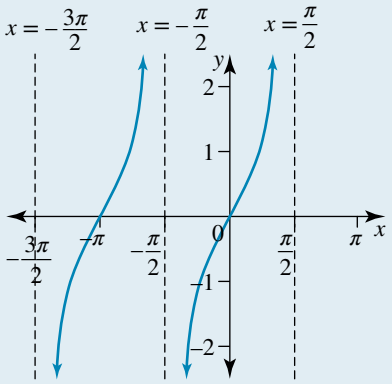
Sketch the graph of $y = \tan(x)$ for $x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.

THINK

1. State the equations of the asymptotes.
2. State where the graph cuts the x-axis.
   Note: The x-intercepts could be found by solving the equation $\tan(x) = 0, x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.
3. Sketch the graph.

WRITE

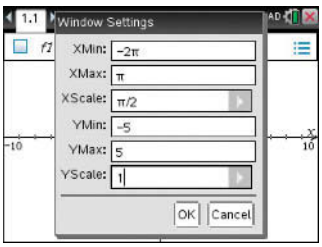
Period: the period of $y = \tan(x)$ is π.
Asymptotes: The graph has an asymptote at $x = \frac{\pi}{2}$.
As the asymptotes are a period apart, for the domain $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$ there is an asymptote at $x = \frac{\pi}{2} - \pi \Rightarrow x = -\frac{\pi}{2}$ and another at $x = -\frac{\pi}{2} - \pi = -\frac{3\pi}{2}$.
The x-intercepts occur midway between the asymptotes.
$(-\pi, 0)$ and $(0, 0)$ are the x-intercepts.



TI | THINK

1. On a Graphs page, set the Graphing Angle to Radian.
   Press MENU then select
   4: Window/Zoom
   1: Window Settings …
   Complete the fields as
   XMin: -2π
   XMax: π
   XScale: $\pi/2$
   YMin: -5
   YMax: 5
   YScale: 1
   then select OK.

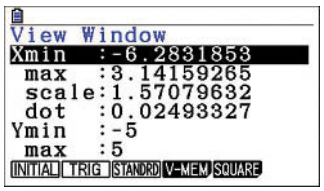
WRITE



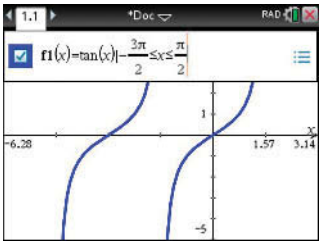
CASIO | THINK

1. Put the Calculator in Radian mode.
   On a Graph screen, press SHIFT then F3 to open the V_WIN.
   Complete the fields as
   Xmin: -2π
   max: π
   scale: $\pi/2$
   Ymin: -5
   max: 5
   scale: 1
   then press EXIT.

WRITE



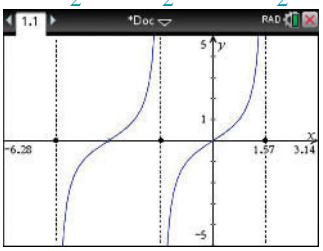
2. Complete the entry line for function 1 as
$f1(x) = \tan(x)| - \frac{3\pi}{2} \leq x \leq \frac{\pi}{2}$
then press ENTER.



3. Identify the positions of the vertical asymptotes.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has vertical asymptotes at $x = -\frac{3\pi}{2}, x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

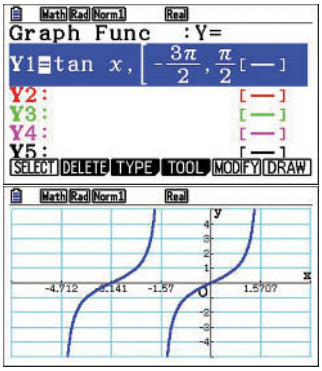
4. To draw the vertical asymptotes, press MENU then select
8: Geometry
4: Construction
1: Perpendicular
Click on the x-axis then click on the point $\left(-\frac{3\pi}{2}, 0\right)$. Repeat this step to draw the other vertical asymptotes.
Note: To change the style of the vertical lines, place the cursor on the line and press CTRL then MENU, then select 3: Attributes.



5. Read the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-\pi, 0)$ and $(0, 0)$.

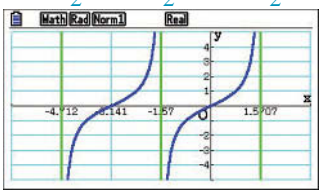
2. Complete the entry line for y1 as
$y1 = \tan x, \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$
then press EXE.
Select DRAW by pressing F6.



3. Identify the positions of the vertical asymptotes.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has vertical asymptotes at $x = -\frac{3\pi}{2}, x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

4. To draw the vertical asymptotes, select Sketch by pressing SHIFT then F4. Press F6 to scroll across to more menu options, then select Vertical by pressing F4.
Use the left/right arrows to position the vertical line, then press EXE. Repeat this step to draw the other vertical asymptotes.



5. Read the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-\pi, 0)$ and $(0, 0)$.

on Resources

Interactivity: Graph plotter: Tangent (int-2978)

studyon

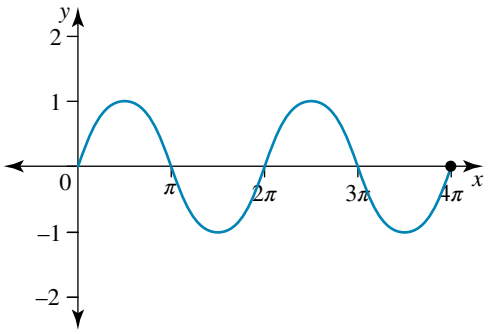
Units 1 & 2 > Area 7 > Sequence 1 > Concepts 5 & 6

Sine and cosine graphs Summary screen and practice questions
The tangent function Summary screen and practice questions

Exercise 10.6 Graphs of the sine, cosine and tangent functions

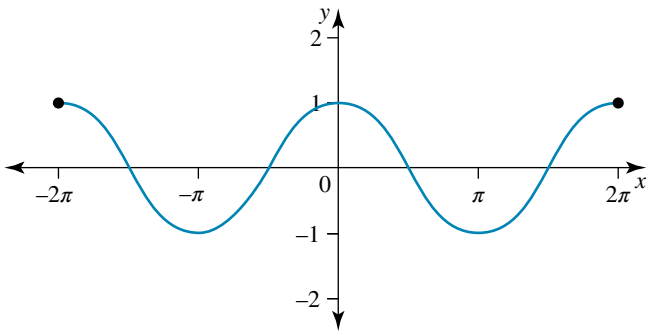
Technology free

1. Consider the graph of the function $y = f(x)$ shown.

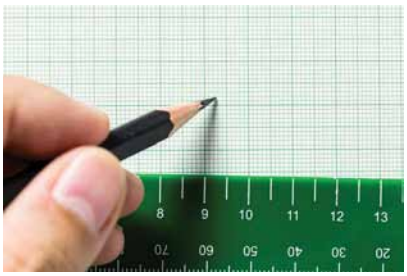


   a. State the domain and range of the graph.
   b. State the appropriate equation for the function shown.
   c. Write down the coordinates of each of the four turning points.
   d. Identify the period and the amplitude of the graph.
   e. Give the equation of the mean (or equilibrium) position.
   f. For what values of x is $f(x) > 0$?

2. Consider the graph of the function $y = g(x)$ shown.



   a. State the domain and range of the graph.
   b. State the appropriate equation for the function shown.
   c. Write down the coordinates of each of the minimum turning points.
   d. Identify the period and the amplitude of the graph and state the equation of its mean (or equilibrium) position.
   e. Write down the coordinates of the x-intercepts of the graph.
   f. For what values of x is $g(x) < 0$?



3. Sketch the graphs of $y = \sin(x)$ and $y = \cos(x)$ over the given domain interval.
   a. $y = \sin(x), 0 \leq x \leq 6\pi$
   b. $y = \cos(x), -4\pi \leq x \leq 2\pi$
   c. $y = \cos(x), -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$
   d. $y = \sin(x), -\frac{3\pi}{2} \leq x \leq \frac{\pi}{2}$

4. **WE 15** Sketch the graph of $y = \cos(x)$ over the domain $[-2\pi, 4\pi]$ and state the number of cycles of the cosine function drawn.

5. a. State the number of maximum turning points on the graph of the function $f: [-4\pi, 0] \to R, f(x) = \sin(x)$.
   b. State the number of minimum turning points of the graph of the function $f: [0, 14\pi] \to R, f(x) = \cos(x)$.

6. Sketch the graph of $y = \cos(x), -4\pi \leq x \leq 5\pi$ and state the number of cycles of the cosine function drawn.
7. State the number of intersections that the graphs of the following make with the x-axis.
   a. $y = \cos(x), 0 \leq x \leq \frac{7\pi}{2}$
   b. $y = \sin(x), -2\pi \leq x \leq 4\pi$
   c. $y = \sin(x), 0 \leq x \leq 20\pi$
   d. $y = \cos(x), \pi \leq x \leq 4\pi$
8. On the same set of axes, sketch the graphs of $y = \cos(x)$ and $y = \sin(x)$ over the domain $[0, 2\pi]$ and shade the region $\{(x, y) : \sin(x) \geq \cos(x), x \in [0, 2\pi]\}$.
9. a. The graph of the function $f: [0, a] \to R, f(x) = \cos(x)$ has 10 intersections with the x-axis. What is the smallest value possible for a?
   b. The graph of the function $f: [b, 5\pi] \to R, f(x) = \sin(x)$ has 7 turning points. If $f(b) = 0$, what is the value of b?
10. If the graph of the function $f: [-c, c] \to R, f(x) = \sin(x)$ covers 2.5 periods of the sine function, what must the value of c be?
11. Draw one cycle of the cosine graph over $[0, 2\pi]$ and give the values of x in this interval for which $\cos(x) < 0$.
12. Explain how the graph in question **11** illustrates what the CAST diagram says about the sign of the cosine function.
13. Sketch the following over the given interval.
   a. $y = \tan(x), x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
   b. $y = \tan(x), x \in (-\pi, 0)$
   c. $y = \tan(x), x \in \left(0, \frac{5\pi}{2}\right)$
14. **WE16** Sketch the graph of $y = \tan(x)$ for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
15. Consider the function defined by $y = \tan(x), -\pi \leq x \leq \pi$.
   a. Sketch the graph over the given interval.
   b. Obtain the x-coordinates of the points on the graph for which the y-coordinate is $-\sqrt{3}$.
16. Use the answers to question **15** to solve the inequation $\tan(x) + \sqrt{3} < 0$ for $x \in [-\pi, \pi]$.

10.7 Transformations of sine and cosine graphs

The shapes of the graphs of the two functions, sine and cosine, are now familiar. Each graph has a wavelike pattern of period 2π and the pair are 'out of phase' with each other by $\frac{\pi}{2}$. We now consider the effect of transformations on the shape and on the properties of these basic sine and cosine graphs.

10.7.1 Transformations of the sine and cosine graphs

The following transformations are applied to the graph of $y = f(x) \to y = Af(B(x + C)) + D$

- dilation of factor $|A|$ from the x-axis parallel to the y-axis
- reflection in the x-axis if $A < 0$
- dilation of factor $\left|\frac{1}{B}\right|, B < 0$ from the y-axis parallel to the x-axis
- horizontal translation of C units to the left if $C > 0$ or to the right if $C < 0$
- vertical translation of D units up if $D > 0$ or down if $D < 0$

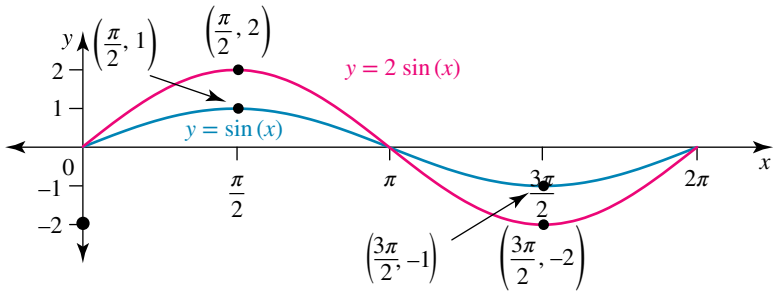
We can therefore infer that the graph of $y = A\ \sin(B(x + C)) + D$ is the image of the basic $y = \sin(x)$ graph after the same set of transformations are applied.

However, the period, amplitude and equilibrium, or mean, position of a trigonometric graph are such key features that we shall consider each transformation in order to interpret its effect on each of these features.

10.7.2 Amplitude changes

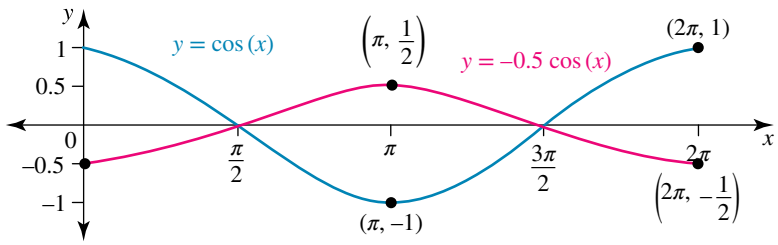
Consider the graphs of $y = \sin(x)$ and $y = 2\sin(x)$.

Comparison of the graph of $y = 2\sin(x)$ with the graph of $y = \sin(x)$ shows the dilation of factor 2 from the x-axis affects the amplitude, but neither the period nor the equilibrium position is altered.



Consider the graphs of $y = \cos(x)$ and $y = -\frac{1}{2}\cos(x)$.

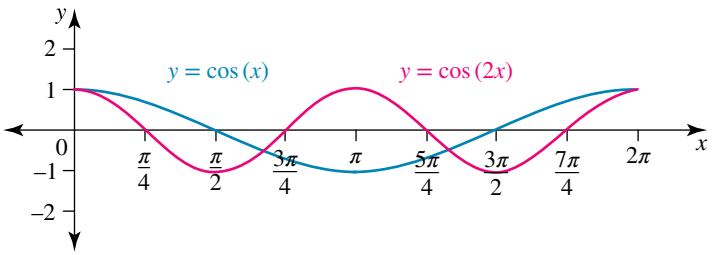
Comparison of the graph of $y = -\frac{1}{2}\cos(x)$ where $A = -\frac{1}{2}$ with the graph of $y = \cos(x)$ shows the dilation factor affecting the amplitude is $\frac{1}{2}$ and the graph of $y = -\frac{1}{2}\cos(x)$ is reflected in the x-axis.



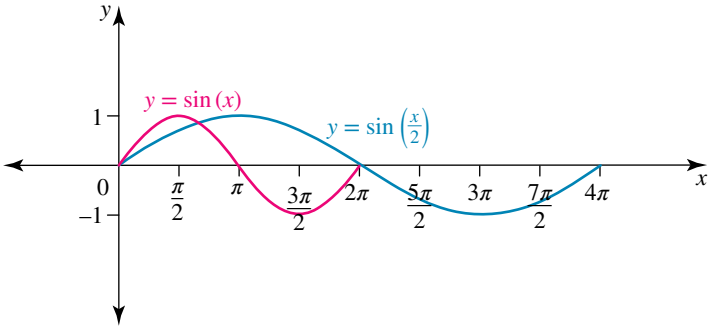
- **The graphs of $y = A\ \sin(x)$ and $y = A\ \cos(x)$ have amplitude A, if $A > 0$.**
- **If $A < 0$, the graph is reflected in the x-axis, (inverted) and the amplitude is the positive part of A (or $|A|$).**

10.7.3 Period changes

Comparison of the graph of $y = \cos(2x)$ with the graph of $y = \cos(x)$ shows the dilation factor of $\frac{1}{2}$ from the y-axis affects the period: it halves the period. The period of $y = \cos(x)$ is 2π while the period of $y = \cos(2x)$ is $\frac{1}{2}$ of 2π; that is, $y = \cos(2x)$ has a period of $\frac{2\pi}{2} = \pi$. Neither the amplitude nor the equilibrium position has been altered.



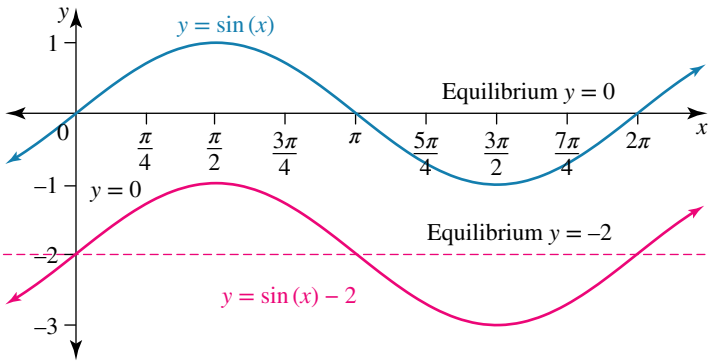
Comparison of one cycle of the graph of $y = \sin\dfrac{x}{2}$ with one cycle of the graph of $y = \sin(x)$ shows the dilation factor of 2 from the y-axis doubles the period. The period of the graph of $y = \sin\left(\dfrac{x}{2}\right)$ is $\dfrac{2\pi}{\frac{1}{2}} = 2 \times 2\pi = 4\pi$.



- **The graphs of $y = \sin(Bx)$ and $y = \cos(Bx)$ have period $\dfrac{2\pi}{|B|}$.**

10.7.4 Equilibrium (or mean) position changes

Comparison of the graph of $y = \sin(x) - 2$ with the graph of $y = \sin(x)$ shows that vertical translation affects the equilibrium position. The graph of $y = \sin(x) - 2$ oscillates about the line $y = -2$, so its range is $[-3, -1]$. Neither the period nor the amplitude is affected.



- **The graphs of $y = \sin(x) + D$ and $y = \cos(x) + D$ both oscillate about the equilibrium (or mean) position $y = D$.**
- **The range of both graphs is $[D - 1, D + 1]$ since the amplitude is 1.**

Summary of amplitude, period and equilibrium changes

The graphs of $y = A\sin(Bx) + D$ and $y = A\cos(Bx) + D$ have:

- **amplitude $|A|$; the graphs are reflected in the x-axis (inverted) if $A < 0$**
- **period $\frac{2\pi}{|B|}$**
- **equilibrium or mean position $y = D$**
- **range $[D - A, D + A]$.**

The oscillation about the equilibrium position of the graph of $y = A\sin(Bx) + D$ always starts at the equilibrium with the pattern, for each period divided into quarters, of:

- equilibrium → range maximum → equilibrium → range minimum → equilibrium if $A > 0$, or: equilibrium → range minimum → equilibrium → range maximum → equilibrium if $A < 0$.

The oscillation about the equilibrium position of the graph of $y = A\cos(Bx) + D$ either starts from its maximum or minimum point with the pattern:

- range maximum → equilibrium → range minimum → equilibrium → range maximum if $A > 0$, or: range minimum → equilibrium → range maximum → equilibrium → range minimum if $A < 0$.

When sketching the graphs, any intercepts with the x-axis are usually obtained by solving the trigonometric equation $A\sin(Bx) + D = 0$ or $A\cos(Bx) + D = 0$.

WORKED EXAMPLE 17

Sketch the graphs of the following functions.

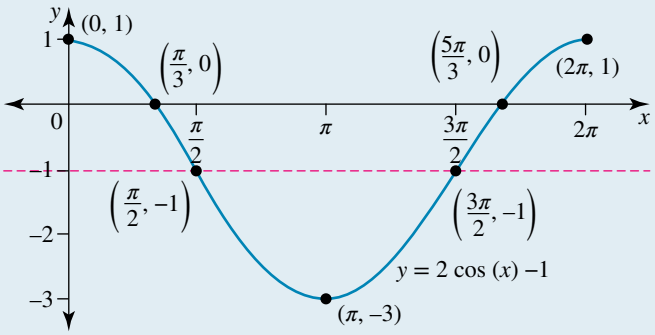
a. $y = 2\cos(x) - 1,\ 0 \le x \le 2\pi$

b. $y = 4 - 2\sin(3x),\ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

THINK	WRITE
a. 1. State the period, amplitude and equilibrium position by comparing the equation with $y = a\cos(nx) + k$.	**a.** $y = 2\cos(x) - 1,\ 0 \le x \le 2\pi$ $A = 2,\ B = 1,\ D = -1$ Amplitude 2, period 2π, equilibrium position $y = -1$
2. Determine the range and whether there will be x-intercepts.	The graph oscillates between $y = -1 - 2 = -3$, and $y = -1 + 2 = 1$, so it has range $[-3, 1]$. It will have x-intercepts.
3. Calculate the x-intercepts.	x-intercepts: let $y = 0$ $2\cos(x) - 1 = 0$ $\therefore \cos(x) = \frac{1}{2}$ Base $\frac{\pi}{3}$, quadrants 1 and 4 $x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$ $= \frac{\pi}{3}, \frac{5\pi}{3}$ x-intercepts are $\left(\frac{\pi}{3}, 0\right), \left(\frac{5\pi}{3}, 0\right)$.

4. Scale the axes by marking $\frac{\pi}{2}$ period intervals on the x-axis. Mark the equilibrium position and endpoints of the range on the y-axis. Then plot the graph using its pattern.

Period is 2π so the scale on the x-axis is in multiples of $\frac{\pi}{2}$.
Since $a > 0$, graph starts at range maximum at its y-intercept $(0, 1)$.



5. Label all key features of the graph including the maximum and minimum points.

The maximum points are $(0, 1)$ and $(2\pi, 1)$. The minimum point is $(\pi, -3)$.

b. 1. State the information the equation provides by comparing the equation with $y = A \sin(Bx) + D$.
Note: The amplitude is always a positive value.

b. $y = 4 - 2 \sin(3x)$, $-\frac{\pi}{2} \le x \le \frac{3\pi}{2}$

$y = -2 \sin(3x) + 4$, domain $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$A = -2$, $B = 3$, $D = 4$

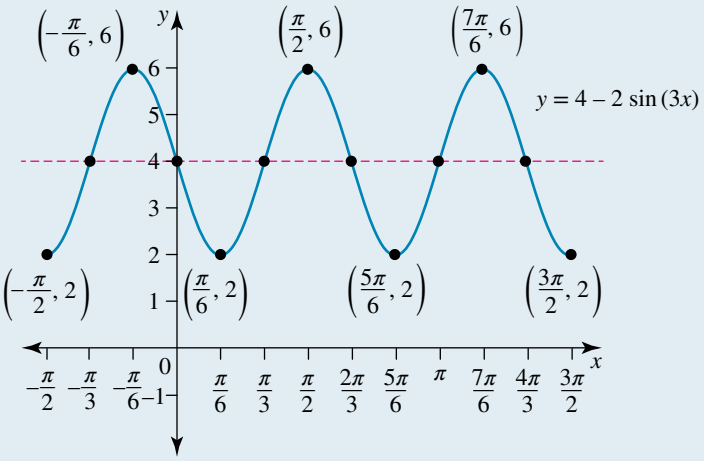
Amplitude 2; graph is inverted; period $\frac{2\pi}{3}$; equilibrium $y = 4$

2. Determine the range and whether there will be x-intercepts.

The graph oscillates between $y = 4 - 2 = 2$ and $y = 4 + 2 = 6$, so its range is $[2, 6]$.
There are no x-intercepts.

3. Scale the axes and extend the $\frac{1}{4}$-period intervals on the x-axis to cover the domain. Mark the equilibrium position and endpoints of the range on the y-axis. Then plot the graph using its pattern and continue the pattern over the given domain.

Dividing the period of $\frac{2\pi}{3}$ into four gives a horizontal scale of $\frac{\pi}{6}$. The first cycle of the graph starts at its equilibrium position at its y-intercept $(0, 4)$ and decreases as $A < 0$.

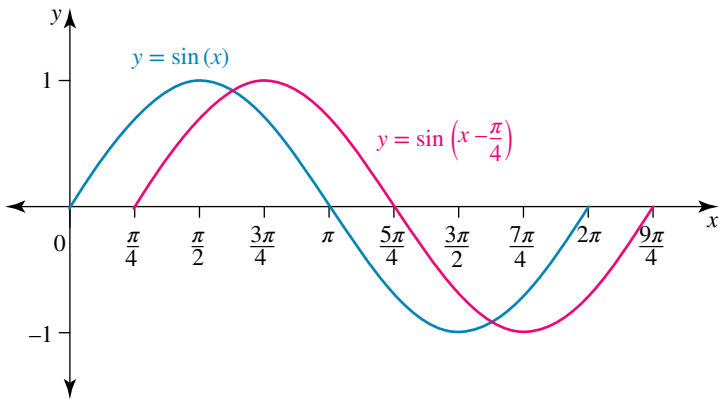


4. Label all key features of the graph including the maximum and minimum points.
Note: Successive maximum points are one period apart, as are the successive minimum points.

Maximum points are $\left(-\frac{\pi}{6}, 6\right)$, $\left(\frac{\pi}{2}, 6\right)$ and $\left(\frac{7\pi}{6}, 6\right)$.
Minimum points are $\left(-\frac{\pi}{2}, 2\right)$, $\left(\frac{\pi}{6}, 2\right)$, $\left(\frac{5\pi}{6}, 2\right)$ and $\left(\frac{3\pi}{2}, 2\right)$.

10.7.5 Phase changes

Horizontal translations of the sine and cosine graphs do not affect the period, amplitude or equilibrium, as one cycle of each of the graphs of $y = \sin(x)$ and $y = \sin\left(x - \frac{\pi}{4}\right)$ illustrate.



The horizontal translation causes the two graphs to be 'out of phase' by $\frac{\pi}{4}$.

- The graph of $y = \sin(x + C)$ has a phase shift of $-C$ from the graph of $y = \sin(x)$.
- The graph of $y = \cos(x + C)$ has a phase shift of $-C$ from the graph of $y = \cos(x)$.

10.7.6 The graphs of $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

The features of the graphs of $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$ are:

- **period $\frac{2\pi}{|B|}$**
- **amplitude $|A|$, inverted if $A < 0$**
- **equilibrium at $y = D$ oscillating between $y = D \pm A$**
- **phase shift of C from the graph of $y = A\sin(Bx)$ or $y = A\cos(Bx)$.**

Horizontal translation of the 5 key points that create the pattern for the graph of either $y = A\sin(Bx)$ or $y = A\cos(Bx)$ will enable one cycle of the graph with the phase shift to be sketched. This transformed graph may be extended to fit a given domain, with its rule used to calculate the coordinates of endpoints.

WORKED EXAMPLE 18

a. **Sketch the graph of $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$, $0 \leq x \leq 2\pi$.**

b. **State the period, amplitude, range and phase shift factor for the graph of $y = -2\sin\left(4x - \frac{\pi}{3}\right) + 5$.**

THINK

a. 1. Identify the key features of the graph from the given equation.

2. Sketch one cycle of the graph without the horizontal translation.

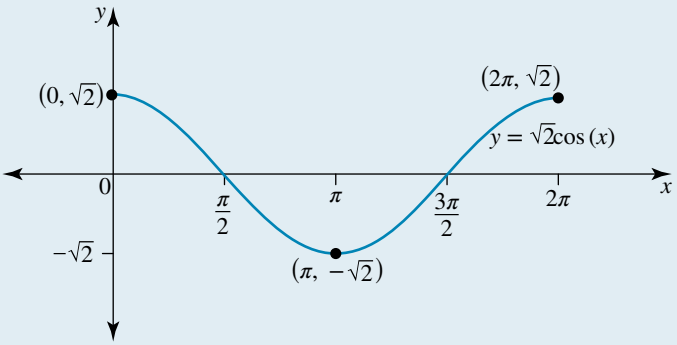
3. Sketch the required graph using horizontal translation.

WRITE

$y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$, $0 \leq x \leq 2\pi$

Period 2π; amplitude $\sqrt{2}$; equilibrium position $y = 0$; horizontal translation of $\frac{\pi}{4}$ to the right; domain $[0, 2\pi]$.

Sketching the graph of $y = \sqrt{2}\cos(x)$ using the pattern gives:



The key points of the graph of $y = \sqrt{2}\cos(x)$ become, under a horizontal translation of $\frac{\pi}{4}$ units to the right:

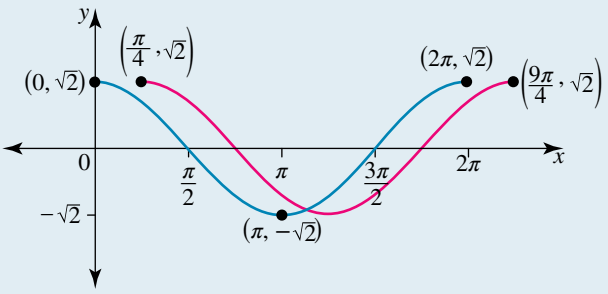
$\left(0, \sqrt{2}\right) \rightarrow \left(\frac{\pi}{4}, \sqrt{2}\right)$

$\left(\frac{\pi}{2}, 0\right) \rightarrow \left(\frac{3\pi}{4}, 0\right)$

$\left(\pi, -\sqrt{2}\right) \rightarrow \left(\frac{5\pi}{4}, -\sqrt{2}\right)$

$\left(\frac{3\pi}{2}, 0\right) \rightarrow \left(\frac{7\pi}{4}, 0\right)$

$\left(2\pi, \sqrt{2}\right) \rightarrow \left(\frac{9\pi}{4}, \sqrt{2}\right)$



4. Calculate the endpoints of the domain.

The translated graph is not on the required domain.
Endpoints for the domain $[0, 2\pi]$:

$$y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

When $x = 0$,

$$y = \sqrt{2}\cos\left(-\frac{\pi}{4}\right)$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$= 1$$

When $x = 2\pi$,

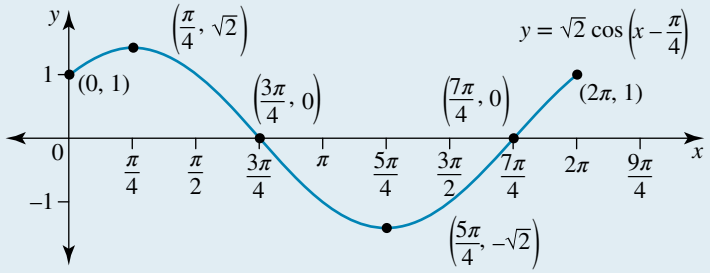
$$y = \sqrt{2}\cos\left(2\pi - \frac{\pi}{4}\right)$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}\right)$$

$$= 1$$

Endpoints are $(0,\ 1)$ and $(2\pi,\ 1)$.

5. Sketch the graph on the required domain.
Note: As the graph covers one full cycle, the endpoints should have the same y-coordinates.



b. 1. Express the equation in the form $y = A\sin(B(x + C)) + D$.

$$y = -2\sin\left(4x + \frac{\pi}{3}\right) + 5$$

$$= -2\sin\left(4\left(x + \frac{\pi}{12}\right)\right) + 5$$

$$A = -2,\ B = 4,\ C = \frac{\pi}{12},\ D = 5$$

2. Calculate the required information.

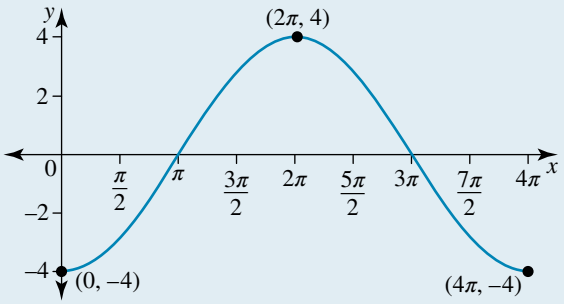
Period is $\frac{2\pi}{n} = \frac{2\pi}{4}$, so the period is $\frac{\pi}{2}$. Amplitude is 2. (graph inverted)
Graph oscillates between $y = 5 - 2 = 3$ and $y = 5 + 2 = 7$, so the range is $[3,\ 7]$.
Phase shift factor from $y = -2\sin(4x)$ is $-\frac{\pi}{12}$.

10.7.7 Forming the equation of a sine or cosine graph

The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is the same as the graph of $y = \cos(x)$ since sine and cosine have a phase difference of $\frac{\pi}{2}$. This means that it is possible for the equation of the graph to be expressed in terms of either function. This is true for all sine and cosine graphs so their equations are not uniquely expressed. Given the choice, it is simpler to choose the form which does not require a phase shift.

WORKED EXAMPLE 19

Determine two possible equations for the following graph.



THINK	WRITE
1. Identify the key features of the given graph.	The graph has a period of 4π, amplitude 4, and the equilibrium position is $y = 0$.
2. Form a possible equation for the graph that does not involve any horizontal translation.	The graph could be an inverted cosine graph. A possible cosine equation for the graph could be $y = A\cos(Bx) + D$ with $A = -4$ and $D = 0$ $\therefore y = -4\cos(Bx)$ The period is $\frac{2\pi}{B}$. From the diagram the period is 4π. $\frac{2\pi}{B} = 4\pi$ $\frac{2\pi}{4\pi} = B$ $B = \frac{1}{2}$ Therefore, a possible equation is $y = -4\cos\left(\frac{1}{2}x\right)$.
3. Form a possible equation for the graph that does involve a horizontal translation. *Note:* Other equations for the graph are possible by considering other phase shifts.	Alternatively, the graph could be a sine function that has been horizontally translated π units to the right. This sine graph is not inverted so A is positive. A possible sine equation could be: $y = A\sin(B(x + C)) + D$ with $A = 4$, $C = -\pi$, $D = 0$ $\therefore y = 4\sin(B(x - \pi))$. The graph has the same period of 4π, so $B = \frac{1}{2}$. Therefore, a possible equation is $y = 4\sin\left(\frac{1}{2}(x - \pi)\right)$.

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Units 1 & 2 > Area 7 > Sequence 1 > Concept 7

Transformations of sine and cosine graphs Summary screen and practice questions

Exercise 10.7 Transformations of sine and cosine graphs

Technology free

1. State the period and amplitude of the following.

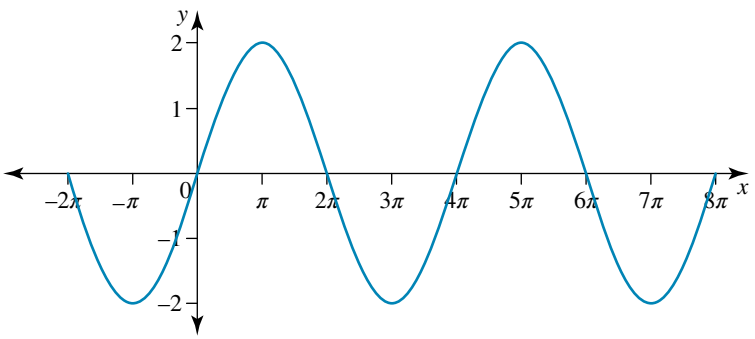
   a. $y = 6\cos(2x)$

   b. $y = -7\cos\left(\frac{x}{2}\right)$

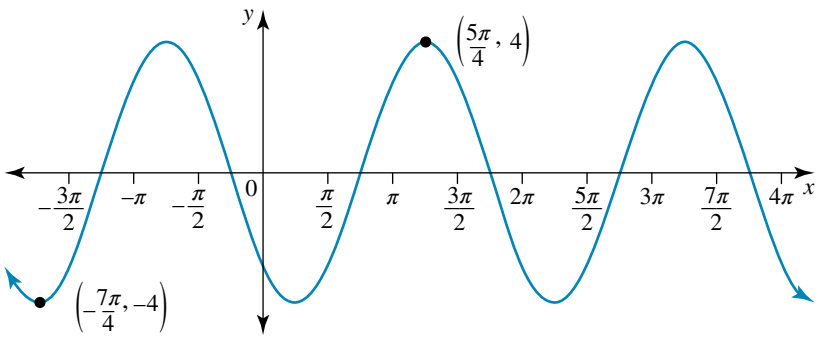
   c. $y = -\frac{3}{5}\sin\left(\frac{3x}{5}\right)$

   d. $y = \sin\left(\frac{6\pi x}{7}\right)$

   e.



   f.



2. State the amplitude, range and period, then sketch the graph of one complete period for the following functions.

   a. $y = \frac{1}{4}\sin(4x)$

   b. $y = \frac{1}{2}\sin\left(\frac{1}{4}x\right)$

   c. $y = -\sin(3x)$

   d. $y = 3\cos(2x)$

   e. $y = -6\cos\left(\frac{2x}{5}\right)$

   f. $y = \frac{1}{3}\cos(5\pi x)$.

3. Sketch the following graphs over the given domains.

   a. $y = 3\cos(2x),\ 0 \le x \le 2\pi$

   b. $y = 2\sin\left(\frac{1}{2}x\right),\ 0 \le x \le 4\pi$

   c. $y = -5\sin(4x),\ 0 \le x \le 2\pi$

   d. $y = -\cos(\pi x),\ 0 \le x \le 4$

4. Sketch each of the following over the domain specified and state the range.

   a. $y = \sin(x) + 3,\ 0 \le x \le 2\pi$

   b. $y = \cos(x) - 1,\ 0 \le x \le 2\pi$

   c. $y = \cos(x) + 2, -\pi \le x \le \pi$

   d. $y = 4 - \sin(x),\ -\pi \le x \le 2\pi$

5. **WE17** Sketch the graphs of the following functions.

   a. $y = 2\sin(x) + 1,\ 0 \le x \le 2\pi$

   b. $y = 4 - 3\cos(2x),\ -\pi \le x \le 2\pi$

6. Sketch the graph of $y = f(x)$ for the function $f{:}[0, 12] \to R, f(x) = \sin\left(\frac{\pi x}{6}\right)$.

7. State the period, mean position, amplitude and range, then sketch the graph showing one complete cycle of each of the following functions.

a. $y = 4\sin(2x) + 5$

b. $y = -2\sin(3x) + 2$

c. $y = \frac{3}{2} - \frac{1}{2}\cos\left(\frac{x}{2}\right)$

d. $y = 2\cos(\pi x) - \sqrt{3}$

8. Sketch the following graphs over the given domains and state the ranges of each.

a. $y = 2\cos(2x) - 2,\ 0 \leq x \leq 2\pi$

b. $y = 2\sin(x) + \sqrt{3},\ 0 \leq x \leq 2\pi$

c. $y = 3\sin\left(\frac{x}{2}\right) + 5,\ -2\pi \leq x \leq 2\pi$

d. $y = -4 - \cos(3x),\ 0 \leq x \leq 2\pi$

e. $y = 1 - 2\sin(2x) - \pi \leq x \leq 2\pi$

f. $y = 2[1 - 3\cos(x)],\ 0° \leq x \leq 360°$

9. a. Give the range of $f: R \to R,\ f(x) = 3 + 2\sin(5x)$.

b. What is the minimum value of the function $f: [0,\ 2\pi] \to f(x) = 10\cos(2x) - 4$?

c. What is the maximum value of the function $f: [0,\ 2\pi] \to R,\ f(x) = 56 - 12\sin(x)$ and for what value of x does the maximum occur?

d. Describe the sequence of transformations that must be applied for the following.

i. $\sin(x) \to 3 + 2\sin(5x)$

ii. $\cos(x) \to 10\cos(2x) - 4$

iii. $\sin(x) \to 56 - 12\sin(x)$

10. WE18 **a.** Sketch the graph of $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right),\ 0 \leq x \leq 2\pi$.

b. State the period, amplitude, range and phase shift factor for the graph of $y = -3\cos\left(2\pi + \frac{\pi}{4}\right) + 1$.

11. a. i. Sketch one cycle of each of $y = \cos(x)$ and $y = \cos\left(x + \frac{\pi}{6}\right)$ on the same axes.

ii. Sketch one cycle of each of $y = \cos(x)$ and $y = \cos(x - \pi)$ on a second set of axes.

b. i. Sketch one cycle of each of $y = \sin(x)$ and $y = \sin\left(x + \frac{3\pi}{4}\right)$ on the same axes.

ii. Sketch one cycle of each of $y = \sin(x)$ and $y = \sin\left(x + \frac{3\pi}{2}\right)$ on a second set of axes.

12. Sketch the following graphs for $0 \leq x \leq 2\pi$.

a. $y = 2\sin\left(x - \frac{\pi}{4}\right)$

b. $y = -4\sin\left(x + \frac{\pi}{6}\right)$

c. $y = \cos\left(2\left(x + \frac{\pi}{3}\right)\right)$

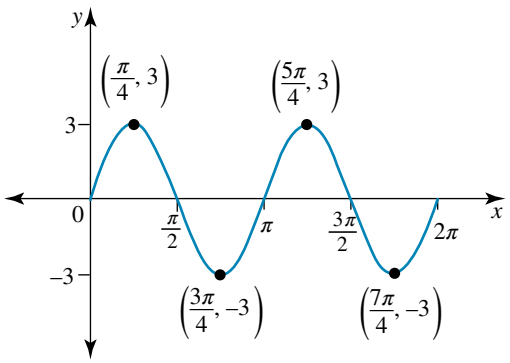
d. $y = \cos\left(2x - \frac{\pi}{2}\right)$

e. $y = \cos\left(x + \frac{\pi}{2}\right) + 2$

f. $y = 3 - 3\sin(2x - 4\pi)$

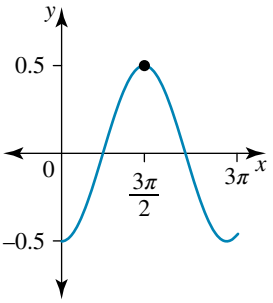
13. Sketch the graph of $y = \sin\left(2x - \frac{\pi}{3}\right),\ 0 \leq x \leq \pi$.

14. a.



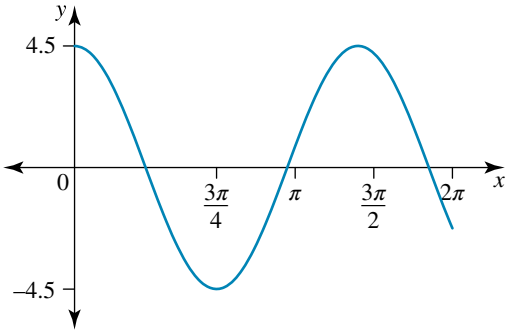
The equation of the graph shown is of the form $y = A\sin(Bx)$. Determine the values of A and B and hence state the equation of the graph.

b.



The equation of the graph shown is of the form $y = A\cos(Bx)$. Determine the values of A and B and hence state the equation of the graph.

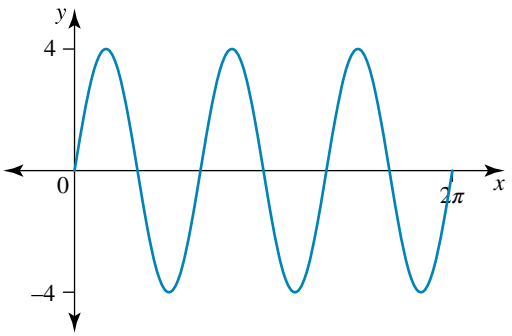
c.



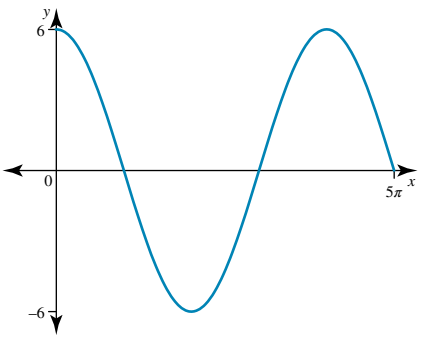
Determine the equation of the graph shown in the form $y = A\cos(Bx)$.

15. Determine the equation of each of the following graphs, given that the equation of each is either of the form $y = A\cos(Bx)$ or $y = A\sin(Bx)$.

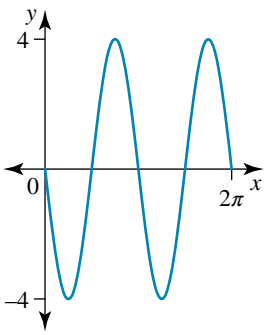
a.



b.

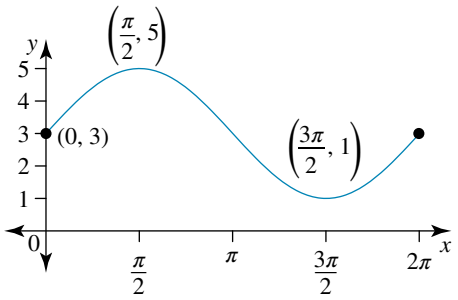


c.

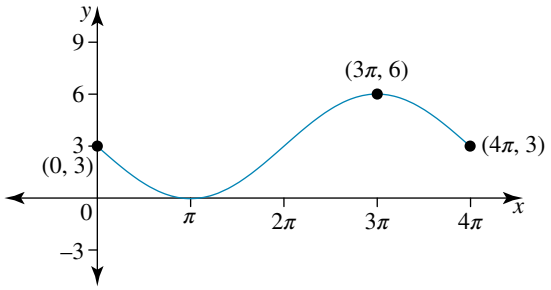


16. The following graphs are either of the form $y = A\cos(Bx) + D$ or the form $y = A\sin(Bx) + D$. Determine the equation for each graph.

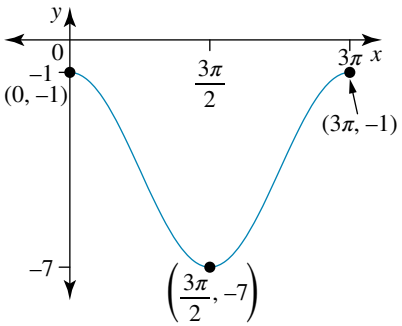
a.



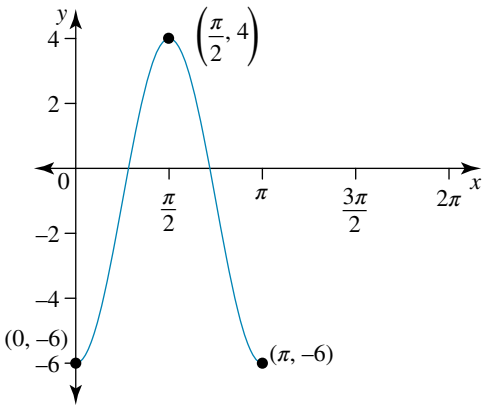
b.



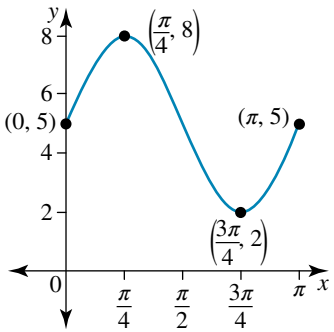
c.



d.

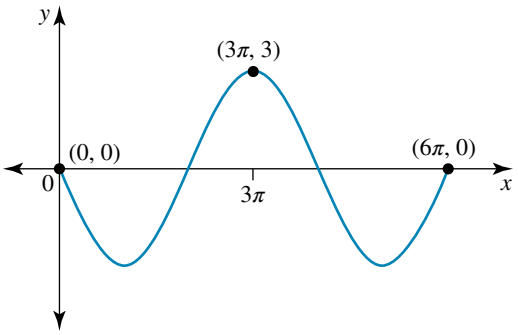


17. Determine a possible equation for the following graph.

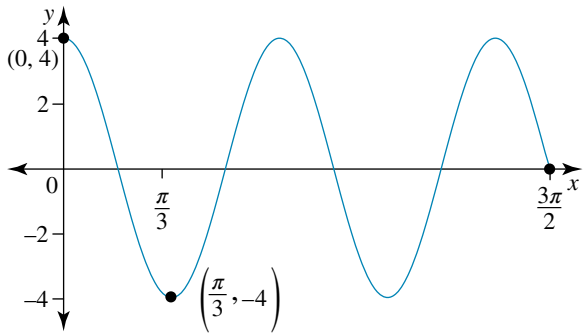


18. In parts **a**–**d**, obtain a possible equation for each of the given graphs.

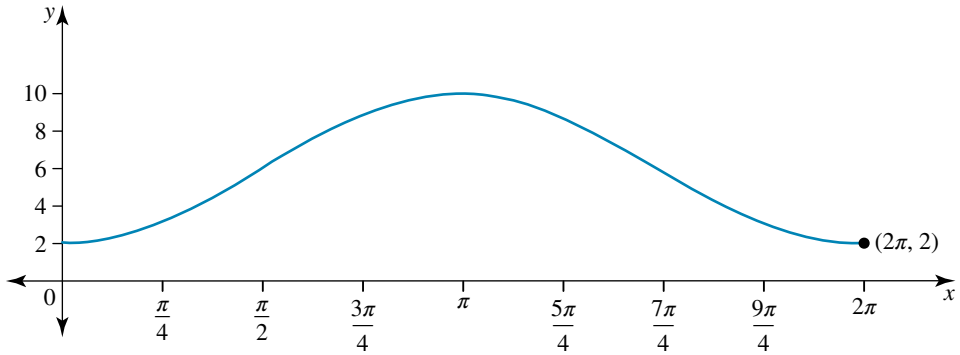
a.



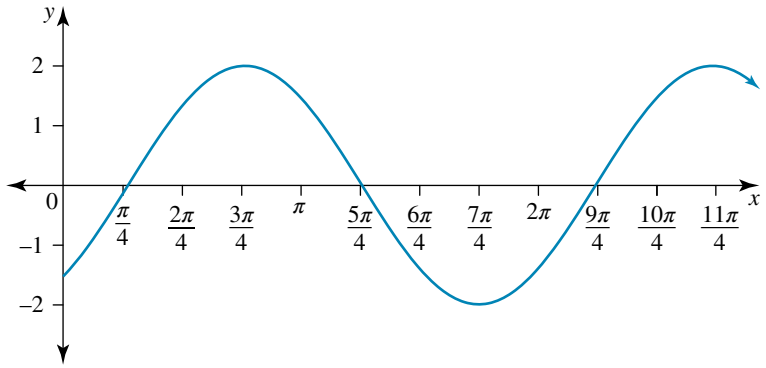
b.



c.



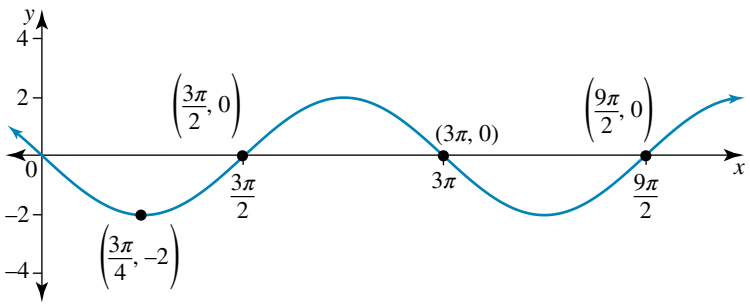
d.



e. Give an alternative equation for the graph shown in part **d**.

f. Use the symmetry properties to give an alternative equation for $y = \cos(-x)$ and for $y = \sin(-x)$.

19. **WE19** Determine two possible equations for the following graph.



20. A function has the rule $f(x) = a\ \sin(bx) + c$ and range of [5, 9].

$f(x) = f\left(x + \frac{2\pi}{3}\right)$ and $\frac{2\pi}{3}$ is the smallest positive value for which this relationship holds.

a. State the period of the function.

b. Obtain possible values for the positive constants a, b and c.

c. Sketch one cycle of the graph of $y = f(x)$, stating its domain, D.

d. A second function has the rule $g(x) = a\ \cos(bx) + c$ where a, b and c have the same values as those of $y = f(x)$. Sketch one cycle of the graph of $y = g(x), x \in D$, on the same axes as the graph of $y = f(x)$.

e. Obtain the coordinates of any points of intersection of the graphs of $y = f(x)$ and $y = g(x)$.

f. Give the values of x which are solutions to the inequation $f(x) \geq g(x)$ where $x \in D$.

10.8 Solving trigonometric equations

The symmetry properties of trigonometric functions can be used to obtain solutions to equations of the form $f(x) = a$ where f is sine, cosine or tangent. If f is sine or cosine, then $-1 \leq a \leq 1$ and, if f is tangent, then $a \in R$.

Once the appropriate base value of the first quadrant is known, symmetric points in any other quadrant can be obtained. However, there are many values, generated by both positive and negative rotations, which can form these symmetric quadrant points. Consequently, the solution of a trigonometric equation such as $\sin(x) = a$, $x \in R$ would have infinite solutions. We shall consider trigonometric equations in which a subset of R is specified as the domain in order to have a finite number of solutions.

10.8.1 Solving trigonometric equations on finite domains

To solve the basic type of equation $\sin(x) = a$, $0 \leq x \leq 2\pi$:

- Identify the quadrants in which solutions lie from the sign of a.
  - If $a > 0$, x must lie in quadrants 1 and 2 where sine is positive.
  - If $a < 0$, x must be in quadrants 3 and 4 where sine is negative.
- Obtain the base value, or first-quadrant value, by solving $\sin(x) = a$ if $a > 0$ or ignoring the negative sign if $a < 0$ (to ensure the first-quadrant value is obtained).
  - This may require recognition of an exact value ratio or it may require the use of a calculator.
- Once obtained, use the base value to generate the values for the quadrants required from their symmetric forms.

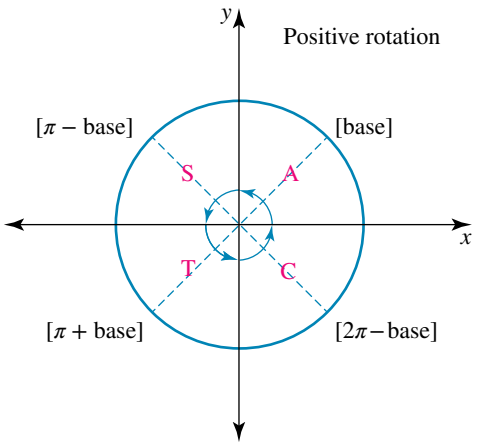
The basic equations $\cos(x) = a$ or $\tan(x) = a$, $0 \leq x \leq 2\pi$ are solved in a similar manner, with the sign of a determining the quadrants in which solutions lie.

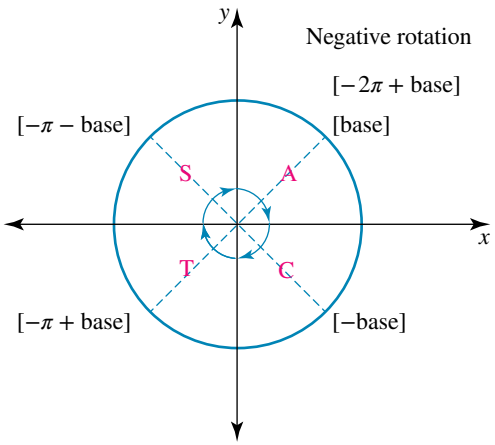
For $\cos(x) = a$: if $a > 0$, x must lie in quadrants 1 and 4 where cosine is positive; if $a < 0$, x must be in quadrants 2 and 3 where cosine is negative.

For $\tan(x) = a$: if $a > 0$, x must lie in quadrants 1 and 3 where tangent is positive; if $a < 0$, x must be in quadrants 2 and 4 where tangent is negative.

10.8.2 Symmetric forms

For one positive and one negative rotation, the symmetric points to the first-quadrant base are shown in the diagrams.





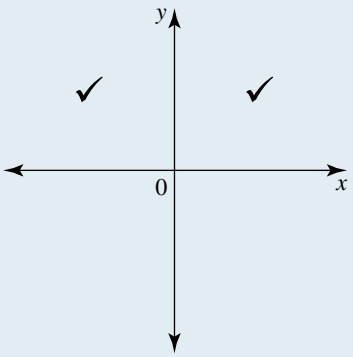
WORKED EXAMPLE 20

Solve the following equations to obtain exact values for x.

a. $\sin(x) = \frac{\sqrt{3}}{2}, 0 \leq x \leq 2\pi$

b. $\sqrt{2}\cos(x) + 1 = 0, 0 \leq x \leq 2\pi$

c. $\sqrt{3} - 3\tan(x) = 0, -2\pi \leq x \leq 2\pi$

THINK	WRITE
a. 1. Identify the quadrants in which the solutions lie.	**a.** $\sin(x) = \frac{\sqrt{3}}{2}, 0 \leq x \leq 2\pi$ Sine is positive in quadrants 1 and 2. 
2. Use knowledge of exact values to state the first-quadrant base.	Base is $\frac{\pi}{3}$ since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.
3. Generate the solutions using the appropriate quadrant forms.	Since $x \in [0, 2\pi]$ there will be two positive solutions, one from quadrant 1 and one from quadrant 2. $\therefore x = \frac{\pi}{3}$ or $x = \pi - \frac{\pi}{3}$
4. Calculate the solutions from their quadrant forms.	$\therefore x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

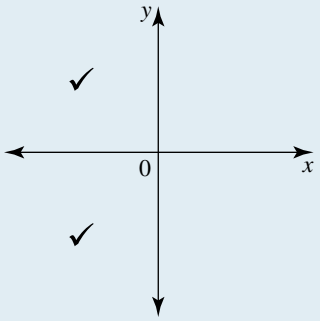
b. 1. Rearrange the equation so the trigonometric function is isolated on one side.

b. $\sqrt{2}\cos(x) + 1 = 0,\ 0 \le x \le 2\pi$

$$\sqrt{2}\cos(x) = -1$$
$$\cos(x) = -\frac{1}{\sqrt{2}}$$

2. Identify the quadrants in which the solutions lie.

Cosine is negative in quadrants 2 and 3.



3. Identify the base.
Note: The negative sign is ignored in identifying the base since the base is the first-quadrant value.

Since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, the base is $\frac{\pi}{4}$.

4. Generate the solutions using the appropriate quadrant forms.

Since $x \in [0, 2\pi]$ there will be two positive solutions.
$\therefore x = \pi - \frac{\pi}{4}$ or $x = \pi + \frac{\pi}{4}$

5. Calculate the solutions from their quadrant forms.

$\therefore x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

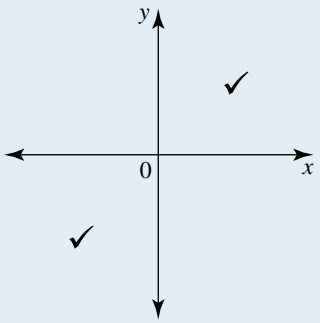
c. 1. Rearrange the equation so the trigonometric function is isolated on one side.

c. $\sqrt{3} - 3\tan(x) = 0,\ -2\pi \le x \le 2\pi$

$\therefore \tan(x) = \frac{\sqrt{3}}{3}$

2. Identify the quadrants in which the solutions lie.

Tangent is positive in quadrants 1 and 3.



3. Identify the base.

Base is $\frac{\pi}{6}$ since $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$.

4. Generate the solutions using the appropriate quadrant forms.

Since $-2\pi \le x \le 2\pi$, there will be 4 solutions, two from a positive rotation and two from a negative rotation.
$x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ or $x = -\pi + \frac{\pi}{6}, = -2\pi + \frac{\pi}{6}$

5. Calculate the solutions from their quadrant forms.

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{11\pi}{6}$

10.8.3 Trigonometric equations with boundary value solutions

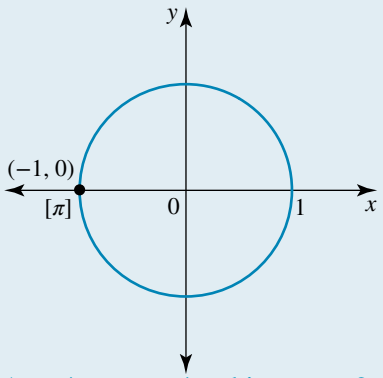
Recognition of exact trigonometric values allows us to identify the base for solving trigonometric equations to obtain exact solutions. However, there are also exact trigonometric values for boundary points. These need to be recognised should they appear in an equation. The simplest strategy to solve trigonometric equations involving boundary values is to use a unit circle diagram to generate the solutions. The domain for the equation determines the number of rotations required around the unit circle. It is not appropriate to consider quadrant forms to generate solutions, since boundary points lie between two quadrants.

Using technology

When bases are not recognisable from exact values, calculators are needed to identify the base. Whether the calculator, or other technology, is set on radian mode or degree mode is determined by the given equation. For example, if $\sin(x) = -0.7$, $0 \leq x \leq 2\pi$, the base is calculated as $\sin^{-1}(0.7)$ in radian mode. However for $\sin(x) = -0.7$, $0° \leq x \leq 360°$, degree mode is used when calculating the base as $\sin^{-1}(0.7)$. The degree of accuracy required for the answer is usually specified in the question; if not, express answers rounded to 2 decimal places.

WORKED EXAMPLE 21

a. Solve for x, $3\cos(x) + 3 = 0$, $-4\pi \leq x \leq 4\pi$.
b. Solve for x to 2 decimal places, $\sin(x) = -0.75$, $0 \leq x \leq 4\pi$.
c. Solve for x, to 1 decimal place, $\tan(x°) + 12.5 = 0$, $-180° \leq x° \leq 180°$.

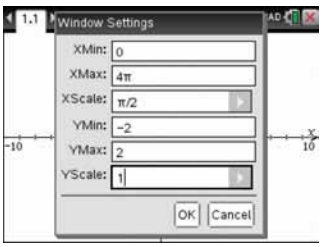
THINK	WRITE
a. 1. Express the equation with the trigonometric function as subject.	**a.** $3\cos(x) + 3 = 0$, $-4\pi \leq x \leq 4\pi$ $\therefore \cos(x) = -1$
2. Identify any boundary points.	-1 is a boundary value since $\cos(\pi) = -1$. The boundary point $[\pi]$ has Cartesian coordinates $(-1, 0)$.
3. Use a unit circle to generate the solutions.	 As $-4\pi \leq x \leq 4\pi$, this means 2 anticlockwise revolutions and 2 clockwise revolutions around the circle are required, with each revolution generating one solution. The solutions are: $x = \pi, 3\pi$ and $= -\pi, -3\pi$ $\therefore x = \pm\pi, \pm 3\pi$
b. 1. Identify the quadrants in which the solutions lie.	**b.** $\sin(x) = -0.75$, $0 \leq x \leq 4\pi$ Sine is negative in quadrants 3 and 4.

2. Calculate the base.	Base is $\sin^{-1}(0.75)$. Using radian mode, $\sin^{-1}(0.75) = 0.848$ to 3 decimal places.
3. Generate the solutions using the appropriate quadrant forms.	Since $x \in [0, 4\pi]$ there will be four positive solutions from two anticlockwise rotations. $x = \pi + 0.848,\ 2\pi - 0.848$ or $x = 3\pi + 0.848,\ 4\pi - 0.848$
4. Calculate the solutions to the required accuracy. *Note:* If the base is left as $\sin^{-1}(0.75)$ then the solutions such as $x = \pi + \sin^{-1}(0.75)$ could be calculated on radian mode in one step.	$\therefore x = 3.99,\ 5.44,\ 10.27,\ 11.72$ (correct to 2 decimal places)
c. 1. Identify the quadrants in which the solutions lie.	**c.** $\tan(x°) + 12.5 = 0,\ -180° \leq x° \leq 180°$ $\tan(x°) = -12.5$ Tangent is negative in quadrants 2 and 4.
2. Calculate the base.	Base is $\tan^{-1}(12.5)$. Using degree mode, $\tan^{-1}(12.5) = 85.43°$ to 2 decimal places.
3. Generate the solutions using the appropriate quadrant forms.	Since $-180° \leq x° \leq 180°$, a clockwise rotation of 180° gives one negative solution in quadrant 4 and an anticlockwise rotation of 180° gives one positive solution in quadrant 2. $x° = -85.43°$ or $x° = 180° - 85.43°$
4. Calculate the solutions to the required accuracy.	$\therefore x = -85.4,\ 94.6$ (correct to 1 decimal place)

TI | THINK

b. 1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: 0
XMax: 4π
XScale: $\pi/2$
YMin: -2
YMax: 2
YScale: 1
then select OK.

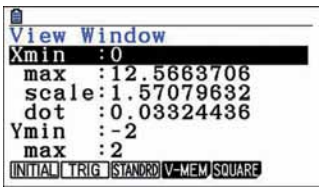
WRITE



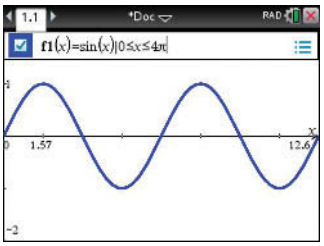
CASIO | THINK

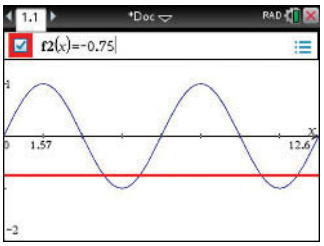
b. 1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: 0
max: 4π
scale: $\pi/2$
Ymin: -2
max: 2
scale: 1
then press EXIT.

WRITE

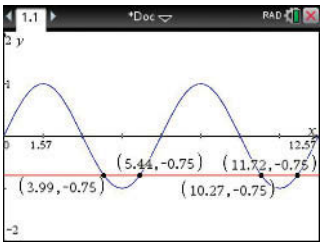


2. Complete the entry line for function 1 as $f1(x) = \sin(x)|0 \le x \le 4\pi$ then press ENTER.
Complete the entry line for function 2 as $f2(x) = -0.75$ then press ENTER.





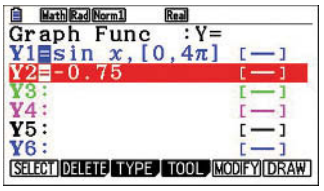
3. To find the points of intersection, press MENU then select
6: Analyze Graph
4: Intersection
Move the cursor to the left of the point of intersection when prompted for the lower bound, then press ENTER. Move the cursor to the right of the point of intersection when prompted for the upper bound, then press ENTER.
Repeat this step to find the other points of intersection.

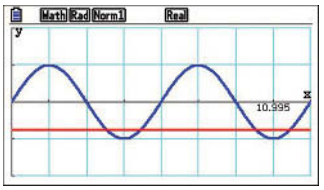


4. The answers appear on the screen.

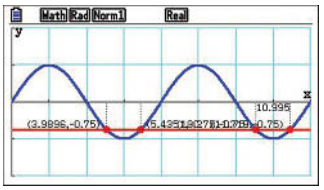
The x-coordinates of the points of intersection of the graphs $f1(x) = \sin(x)$ and $f2(x) = -0.75$ represent the solutions to the equation $\sin(x) = -0.75$ over the domain $[0, 4\pi]$.
$\therefore x = 3.99, 5.44, 10.27, 11.72$ (correct to 2 decimal places)

2. Complete the entry line for y1 as $y1 = \sin x, [0, 4\pi]$ then press EXE.
Complete the entry line for y2 as $y2 = -0.75$ then press EXE.
Select DRAW by pressing F6.





3. To find the points of intersection, select G-Solv by pressing SHIFT then F5, then select INTSECT by pressing F5. With the cursor on the first point of intersection, press EXE. Use the left/right arrows to move to the next point of intersection and mark it on the graph by pressing EXE.
Repeat this step to find the other points of intersection.



4. The answers appear on the screen.

The x-coordinates of the points of intersection of the graphs $y1 = \sin x$ and $y2 = -0.75$ represent the solutions to the equation $\sin x = -0.75$ over the domain $[0, 4\pi]$.
$\therefore x = 3.99, 5.44, 10.27, 11.72$ (correct to 2 decimal places)

10.8.4 Further types of trigonometric equations

Trigonometric equations may require algebraic techniques or the use of relationships between the functions before they can be reduced to the basic form $f(x) = a$, where f is either sin, cos or tan.

- **Equations of the form $\sin(x) = a\cos(x)$ can be converted to $\tan(x) = a$ by dividing both sides of the equation by $\cos(x)$, $\cos(x) \neq 0$.**
- **Equations of the form $\sin^2(x) = a$ can be converted to $\sin(x) = \pm\sqrt{a}$ by taking the square roots of both sides of the equation.**
- **Equations of the form $\sin^2(x) + b\sin(x) + c = 0$ can be converted to standard quadratic equations by using the substitution $u = \sin(x)$.**

Since $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$, neither $\sin(x)$ nor $\cos(x)$ can have values greater than 1 or smaller than -1. This may have implications requiring the rejection of some steps when working with sine or cosine trigonometric equations. As $\tan(x) \in R$, there is no restriction on the values the tangent function can take.

WORKED EXAMPLE 22

Solve for x, where $0 \leq x \leq 2\pi$.

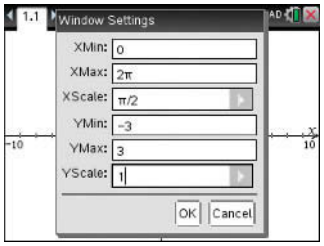
a. $\sqrt{3}\sin(x) = \cos(x)$ **b.** $\cos^2(x) + \cos(x) - 2 = 0$

THINK	WRITE
a. 1. Reduce the equation to one trigonometric function.	**a.** $\sqrt{3}\sin(x) = \cos(x), 0 \leq x \leq 2\pi$ Divide both sides by $\cos(x)$. $\frac{\sqrt{3}\sin(x)}{\cos(x)} = \frac{\cos(x)}{\cos(x)}$ $\therefore \sqrt{3}\tan(x) = 1$ $\therefore \tan(x) = \frac{1}{\sqrt{3}}$
2. Calculate the solutions.	Tangent is positive in quadrants 1 and 3. Base is $\frac{\pi}{6}$. $x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ $= \frac{\pi}{6}, \quad \frac{7\pi}{6}$
b. 1. Use substitution to form a quadratic equation.	**b.** $\cos^2(x) + \cos(x) - 2 = 0, 0 \leq x \leq 2\pi$ Let $a = \cos(x)$. The equation becomes $a^2 + a - 2 = 0$.
2. Solve the quadratic equation.	$(a + 2)(a - 1) = 0$ $\therefore a = -2$ or $a = 1$
3. Substitute back for the trigonometric function.	Since $a = \cos(x)$, $\cos(x) = -2$ or $\cos(x) = 1$. Reject $\cos(x) = -2$ since $-1 \leq \cos(x) \leq 1$. $\therefore \cos(x) = 1$
4. Solve the remaining trigonometric equation.	$\cos(x) = 1, \quad 0 \leq x \leq 2\pi$ Boundary value since $\cos(0) = 1$ $\therefore x = 0, 2\pi$

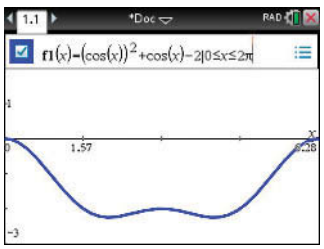
TI | THINK

b. 1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: 0
XMax: 2π
XScale: $\pi/2$
YMin: -3
YMax: 3
YScale: 1
then select OK.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be read easily from the graph.

WRITE



2. Complete the entry line for function 1 as $f1(x) = (\cos(x))^2 + \cos(x) - 2|0 \le x \le 2\pi$ then press ENTER.



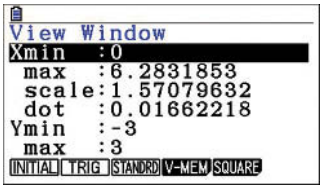
3. The answer appears on the screen.

The x-intercepts of the graph represent the solutions to the equation $(\cos(x))^2 + \cos(x) - 2 = 0$.
$\therefore x = 0$ and 2π

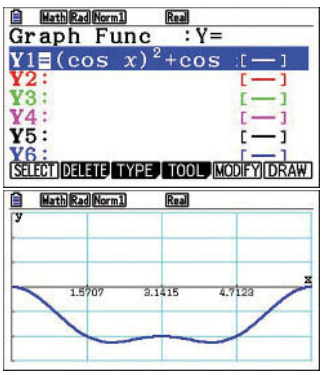
CASIO | THINK

b. 1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: 0
max: 2π
scale: $\pi/2$
Ymin: -3
max: 3
scale: 1
then press EXIT.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be read easily from the graph.

WRITE



2. Complete the entry line for y1 as $y1 = (\cos x)^2 + \cos x - 2$, $[0, 2\pi]$
then press EXE.
Select DRAW by pressing F6.



3. The answer appears on the screen.

The x-intercepts of the graph represent the solutions to the equation $(\cos(x))^2 + \cos(x) - 2 = 0$.
$\therefore x = 0$ and 2π

10.8.5 Solving trigonometric equations which require a change of domain

Equations such as $\sin(2x) = 1$, $0 \le x \le 2\pi$ can be expressed in the basic form by the substitution of $\theta = 2x$. The accompanying domain must be changed to be the domain for θ. This requires the endpoints of the domain for x to be multiplied by 2. Hence, $0 \le x \le 2\pi \Rightarrow 2 \times 0 \le 2x \le 2 \times 2\pi$ gives the domain requirement for θ as $0 \le \theta \le 4\pi$.

This allows the equation to be written as $\sin(\theta) = 1$, $0 \le \theta \le 4\pi$.

This equation can then be solved to give $\theta = \frac{\pi}{2}, \frac{5\pi}{2}$.

Substituting back for x gives $2x = \frac{\pi}{2}, \frac{5\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$. The solutions are in the domain specified for x.

The change of domain ensures all possible solutions are obtained.

However, in practice, it is quite common not to formally introduce the pronumeral substitution for equations such as $\sin(2x) = 1,\ 0 \leq x \leq 2\pi$.

With the domain change, the equation can be written as $\sin(2x) = 1,\ 0 \leq 2x \leq 4\pi$ and the equation solved for x as follows.

$$\sin(2x) = 1,\ 0 \leq 2x \leq 4\pi$$

$$\therefore 2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$$

WORKED EXAMPLE 23

a. Solve $\cos(3x) = -\frac{1}{2}$ for x, $0 \leq x \leq 2\pi$.

b. Use substitution to solve the equation $\tan\left(2x - \frac{\pi}{4}\right) = -1$, $0 \leq x \leq \pi$.

THINK	WRITE
a. 1. Change the domain to be that for the given multiple of the variable.	**a.** $\cos(3x) = -\frac{1}{2},\ 0 \leq x \leq 2\pi$ Multiply the endpoints of the domain of x by 3 $\therefore \cos(3x) = -\frac{1}{2},\ 0 \leq 3x \leq 6\pi$
2. Solve the equation for $3x$. *Note:* Alternatively, substitute $\theta = 3x$ and solve for θ.	Cosine is negative in quadrants 2 and 3. Base is $\frac{\pi}{3}$. As $3x \in [0, 6\pi]$, each of the three revolutions will generate 2 solutions, giving a total of 6 values for $3x$. $3x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 5\pi - \frac{\pi}{3}, 5\pi + \frac{\pi}{3}$ $= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}$
3. Calculate the solutions for x.	Divide each of the 6 values by 3 to obtain the solutions for x. $x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$
b. 1. State the substitution required to express the equation in basic form.	**b.** $\tan\left(2x - \frac{\pi}{4}\right) = -1, \quad 0 \leq x \leq \pi$ Let $\theta = 2x - \frac{\pi}{4}$.
2. Change the domain of the equation to that of the new variable.	For the domain change: $0 \leq x \leq \pi$ $\therefore 0 \leq 2x \leq 2\pi$ $\therefore -\frac{\pi}{4} \leq 2x - \frac{\pi}{4} \leq 2\pi - \frac{\pi}{4}$ $\therefore -\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$
3. State the equation in terms of θ.	The equation becomes $\tan(\theta) = -1,\ -\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$.

4. Solve the equation for θ.

Tangent is negative in quadrants 2 and 4.
Base is $\frac{\pi}{4}$.
$\theta = -\frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$
$= -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

5. Substitute back in terms of x.

$\therefore 2x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

6. Calculate the solutions for x.

Add $\frac{\pi}{4}$ to each value.
$\therefore 2x = 0, \pi, 2\pi$
Divide by 2.
$\therefore x = 0, \frac{\pi}{2}, \pi$

studyon

Units 1 & 2 > Area 7 > Sequence 1 > Concept 8

Solving trigonometric equations Summary screen and practice questions

Exercise 10.8 Solving trigonometric equations

Technology free

1. Solve for x, given $0 \leq x \leq 2\pi$.
   a. $\cos(x) = \frac{1}{\sqrt{2}}$ b. $\sin(x) = -\frac{1}{\sqrt{2}}$ c. $\tan(x) = -\frac{1}{\sqrt{3}}$
   d. $2\sqrt{3}\cos(x) + 3 = 0$ e. $4 - 8\sin(x) = 0$ f. $2\sqrt{2}\tan(x) = \sqrt{24}$
2. Determine the exact solutions for $\theta \in [-2\pi, 2\pi]$ for which:
   a. $\tan(\theta) = 1$ b. $\cos(\theta) = -0.5$ c. $1 + 2\sin(\theta) = 0$.
3. a. Determine the solutions to the equation $3\tan(x) + 3\sqrt{3} = 0$ over the domain $x \in [0, 3\pi]$.
   b. For $0 \leq t \leq 4\pi$, find the exact solutions to $10\sin(t) - 3 = 2$.
   c. Calculate the exact values of v which satisfy $4\sqrt{2}\cos(v) = \sqrt{2}\cos(v) + 3, -\pi \leq v \leq 5\pi$.
4. WE20 Solve the following equations to obtain exact values for x.
   a. $\sin(x) = \frac{1}{2}, 0 \leq x \leq 2\pi$
   b. $\sqrt{3} - 2\cos(x) = 0, 0 \leq x \leq 2\pi$
   c. $4 + 4\tan(x) = 0, -2\pi \leq x \leq 2\pi$
5. Consider the equation $\cos(\theta) = -\frac{1}{2}, -180° \leq \theta \leq 180°$.
   a. How many solutions for θ does the equation have?
   b. Calculate the solutions of the equation.
6. Solve for $a°$, given $0° \leq a° \leq 360°$.
   a. $\sqrt{3} + 2\sin(a°) = 0$ b. $\tan(a°) = 1$
   c. $6 + 8\cos(a°) = 2$ d. $4(2 + \sin(a°)) = 11 - 2\sin(a°)$

7. Obtain all values for t, $t \in [-\pi, 4\pi]$, for which:

a. $\tan(t) = 0$ **b.** $\cos(t) = 0$ **c.** $\sin(t) = -1$

d. $\cos(t) = 1$ **e.** $\sin(t) = 1$ **f.** $\tan(t) = 1$.

Technology active

8. a. WE21 Solve $1 - \sin(x) = 0$, $-4\pi \leq x \leq 4\pi$ for x.

b. Solve $\tan(x) = 0.75$, $0 \leq x \leq 4\pi$ for x, to 2 decimal places.

c. Solve $4\cos(x°) + 1 = 0$, $-180° \leq x° \leq 180°$ for x, to 1 decimal place.

9. Solve the following equations, where possible, to obtain the values of the pronumerals.

a. $4\sin(a) + 3 = 5, -2\pi < a < 0$

b. $6\tan(b) - 1 = 11, -\frac{\pi}{2} < b < 0$

c. $8\cos(c) - 7 = 1, -\frac{9\pi}{2} < c < 0$

d. $\frac{9}{\tan(d)} - 9 = 0, 0 < d \leq \frac{5\pi}{12}$

e. $2\cos(e) = 1, -\frac{\pi}{6} \leq e \leq \frac{13\pi}{6}$

f. $\sin(f) = -\cos(150°), -360° \leq f \leq 360°$

10. Consider the function $f: [0, 2] \to R, f(x) = \cos(\pi x)$.

a. Calculate $f(0)$. **b.** Obtain $\{x : f(x) = 0\}$.

11. WE22 Solve the following for x, given $0 \leq x \leq 2\pi$.

a. $\sqrt{3}\sin(x) = 3\cos(x)$ **b.** $\sin^2(x) - 5\sin(x) + 4 = 0$

12. Solve $\cos^2(x) = \frac{3}{4}$, $0 \leq x \leq 2\pi$ for x.

13. Solve for x, where $0 \leq x \leq 2\pi$.

a. $\sin(x) = \sqrt{3}\cos(x)$ **b.** $\sin(x) = -\frac{\cos(x)}{\sqrt{3}}$

c. $\sin(2x) + \cos(2x) = 0$ **d.** $\frac{3\sin(x)}{8} = \frac{\cos(x)}{2}$

e. $\sin^2(x) = \cos^2(x)$ **f.** $\cos(x)(\cos(x) - \sin(x)) = 0$

14. Solve for x, where $0 \leq x \leq 2\pi$.

a. $\sin^2(x) = \frac{1}{2}$ **b.** $2\cos^2(x) + 3\cos(x) = 0$

c. $2\sin^2(x) - \sin(x) - 1 = 0$ **d.** $\tan^2(x) + 2\tan(x) - 3 = 0$

e. $\sin^2(x) + 2\sin(x) + 1 = 0$ **f.** $\cos^2(x) - 9 = 0$

15. Solve $\sin\left(\frac{x}{2}\right) = \sqrt{3}\cos\left(\frac{x}{2}\right)$, $0 \leq x \leq 2\pi$ for x.

16. a. WE23 Solve $\sin(2x) = \frac{1}{\sqrt{2}}$, $0 \leq x \leq 2\pi$ for x.

b. Use a substitution to solve the equation $\cos\left(2x + \frac{\pi}{6}\right) = 0$, $0 \leq x \leq \frac{3\pi}{2}$.

17. Solve for $\theta, 0 \leq \theta \leq 2\pi$.

a. $\sqrt{3}\tan(3\theta) + 1 = 0$ **b.** $2\sqrt{3}\sin\left(\frac{3\theta}{2}\right) - 3 = 0$

c. $4\cos^2(-\theta) = 2$ **d.** $\sin\left(2\theta + \frac{\pi}{4}\right) = 0$

18. Use technology to calculate the values of θ, correct to 2 decimal places, which satisfy the following conditions.

a. $2 + 3\cos(\theta) = 0$, $0 \leq \theta \leq 2\pi$ **b.** $\tan(\theta) = \frac{1}{\sqrt{2}}, -2\pi \leq \theta \leq 3\pi$

c. $5\sin(\theta°) + 4 = 0, -270° \leq \theta° \leq 270°$ **d.** $\cos^2(\theta°) = 0.04, 0° \leq \theta \leq 360°$

19. **a.** How many solutions are there to the equation $\cos(x) = -0.3, -\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$?



b. Calculate the sum of the solutions to $\sin(x) = 0.2$, $0 \leq x \leq 2\pi$.

c. If $x = 0.4$ is a solution of the equation $\tan(x) = c$, $0 \leq x \leq 3\pi$, obtain the other possible solutions.

d. Set up a trigonometric equation, the solution of which gives the angle of inclination of the following two lines with the positive direction of the x-axis.

i. $y = -3x$ **ii.** $y = \sqrt{3}x$

e. Give the exact angle in each case for part **d**.

f. Earlier, θ, the obtuse angle of inclination with the x-axis of a line with a negative gradient m, was calculated from the rule $\theta = 180° - \tan^{-1}(|m|)$ where $|m|$ gave the positive part of m. Explain this rule by reference to the solution of a trigonometric equation.

20. Consider the function $f: [0, 2\pi] \rightarrow R$, $f(x) = a\sin(x)$.

a. If $f\left(\frac{\pi}{6}\right) = 4$, calculate the value of a.

b. Use the answer to part **a** to detemine, where possible, any values of x, to 2 decimal places, for which the following apply.

i. $f(x) = 3$ **ii.** $f(x) = 8$ **iii.** $f(x) = 10$

10.9 Modelling with trigonometric functions

Phenomena that are cyclical in nature can often be modelled by a sine or cosine function.

Examples of periodic phenomena include sound waves, ocean tides and ovulation cycles. Trigonometric models may be able to approximate things like the movement in the value of the All Ordinaries Index of the stock market or fluctuations in temperature or the vibrations of violin strings about a mean position.

10.9.1 Maximum and minimum values

As $-1 \leq \sin(x) \leq 1$ and $-1 \leq \cos(x) \leq 1$, the maximum value of both $\sin(x)$ and $\cos(x)$ is 1 and the minimum value of both functions is -1. This can be used to calculate, for example, the maximum value of $y = 2\sin(x) + 4$ by substituting 1 for $\sin(x)$:

$$y_{max} = 2 \times 1 + 4 \Rightarrow y_{max} = 6.$$

The minimum value can be calculated as:

$$y_{min} = 2 \times (-1) + 4 \Rightarrow y_{min} = 2.$$

To calculate the maximum value of $y = 5 - 3\cos(2x)$ the largest negative value of $\cos(2x)$ would be substituted for $\cos(2x)$. Thus:

$$y_{max} = 5 - 3 \times (-1) \Rightarrow y_{max} = 8.$$

The minimum value can be calculated by substituting the largest positive value of $\cos(2x)$:

$$y_{\mathbf{min}} = 5 - 3 \times 1 \Rightarrow y_{min} = 2.$$

Alternatively, identifying the equilibrium position and amplitude enables the range to be calculated. For $y = 5 - 3\cos(2x)$, with amplitude 3 and equilibrium at $y = 5$, the range is calculated from $y = 5 - 3 = 2$ to $y = 5 + 3 = 8$, giving a range of $[2, 8]$. This also shows the maximum and minimum values.

WORKED EXAMPLE 24

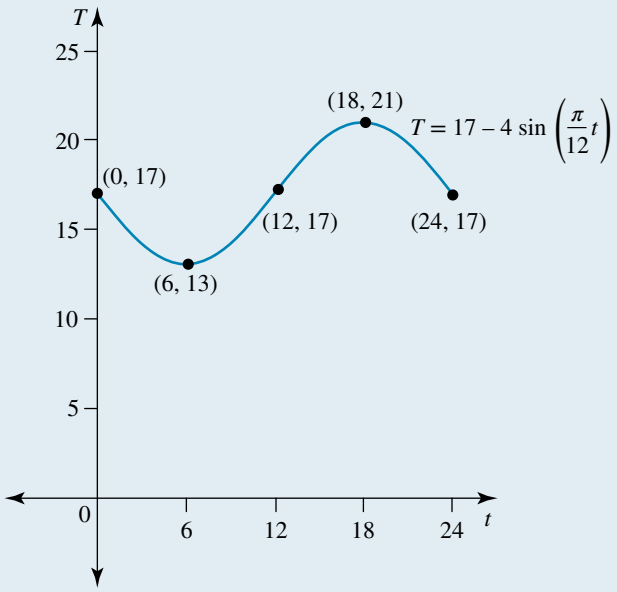
The temperature, T °C, during one day in April is given by $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ where t is the time in hours after midnight.

a. What was the temperature at midnight?
b. What was the minimum temperature during the day and at what time did it occur?
c. Over what interval did the temperature vary that day?
d. State the period and sketch the graph of the temperature for $t \in [0, 24]$.
e. If the temperature was below k degrees for 2.4 hours, obtain the value of k to 1 decimal place.

THINK	WRITE
a. State the value of t and use it to calculate the required temperature.	**a.** At midnight, $t = 0$. Substitute $t = 0$ into $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$. $T = 17 - 4\sin(0)$ $= 17$ The temperature at midnight was 17°.
b. 1. State the condition on the value of the trigonometric function for the minimum value of T to occur.	**b.** $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ The minimum value occurs when $\sin\left(\frac{\pi}{12}t\right) = 1$.
2. Calculate the minimum temperature.	$T_{\min} = 17 - 4 \times 1$ $= 13$ The minimum temperature was 13°.
3. Calculate the time when the minimum temperature occurred.	The minimum temperature occurs when $\sin\left(\frac{\pi}{12}t\right) = 1$ Solving this equation, $\frac{\pi}{12}t = \frac{\pi}{2}$ $t = 6$ The minimum temperature occurred at 6 am.
c. Use the equilibrium position and amplitude to calculate the range of temperatures.	**c.** $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ Amplitude 4, equilibrium $T = 17$ Range is $[17 - 4,\ 17 + 4] = [13,\ 21]$. Therefore, the temperature varied between 13 °C and 21 °C.
d. 1. Calculate the period.	**d.** $2\pi \div \frac{\pi}{12} = 2\pi \times \frac{12}{\pi}$ $= 24$ The period is 24 hours.

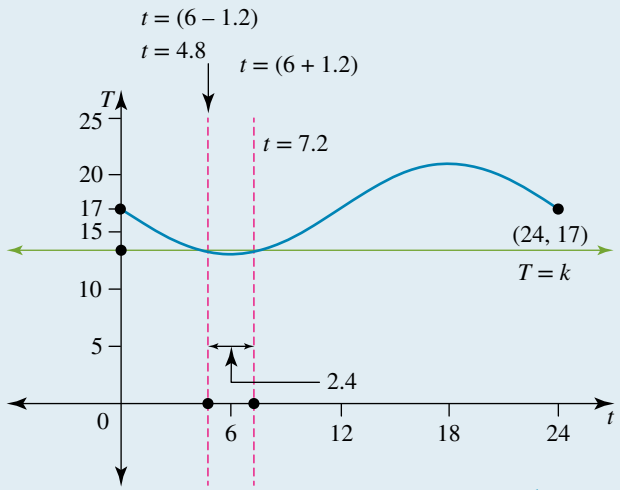
2. Sketch the graph.

Dividing the period into quarters gives a horizontal scale of 6.



e. 1. Use the symmetry of the curve to deduce the endpoints of the interval involved.

e. $T < k$ for an interval of 2.4 hours.
For T to be less than the value for a time interval of 2.4 hours, the 2.4-hour interval is symmetric about the minimum point.



The endpoints of the t interval must each be $\frac{1}{2} \times 2.4$ from the minimum point where $t = 6$. The endpoints of the interval occur at $t = 6 \pm 1.2$.
$\therefore T = k$ when $t = 4.8$ or 7.2.

2. Calculate the required value.

Substituting either endpoint into the temperature model will give the value of k.

If $t = 4.8$:

$$T = 17 - 4\sin\left(\frac{\pi}{12} \times 4.8\right)$$

$$= 17 - 4\sin(0.4\pi)$$

$$= 13.2$$

Therefore, $k = 13.2$ and the temperature is below 13.2 °C for 2.4 hours.

on Resources

Interactivity: Oscillation (int-2977)

Exercise 10.9 Modelling with trigonometric functions

Technology active

1. A child plays with a yo-yo attached to the end of an elastic string. The yo-yo rises and falls about its rest position so that its vertical distance, y cm, above its rest position at time t seconds is given by $y = -40\cos(t)$.



a. Sketch the graph of $y = -40\cos(t)$ showing two complete cycles.

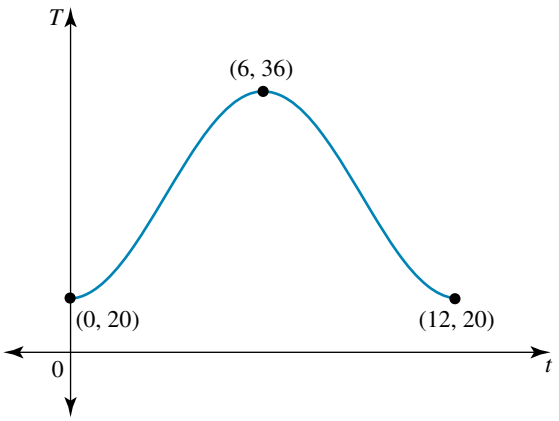
b. What is the greatest distance the yo-yo falls below its rest position?

c. At what times does the yo-yo return to its rest position during the two cycles?

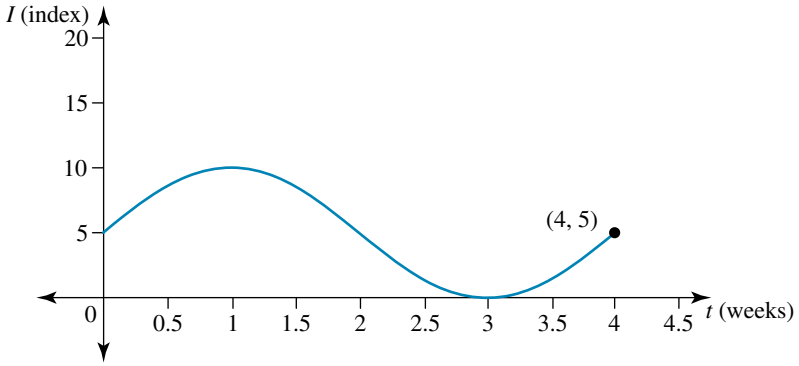
d. After how many seconds does the yo-yo first reach a height of 20 cm above its rest position?

2. The temperature from 8 am to 10 pm on a day in February is shown.

If T is the temperature in degrees Celsius t hours from 8 am, form the equation of the temperature model and use this to calculate the times during the day when the temperature exceeded 30 degrees.



3. Emotional ups and downs are measured by a wellbeing index which ranges from 0 to 10 in increasing levels of happiness. A graph of this index over a four-week cycle is shown.



   a. Express the relationship between the wellbeing index I and the time t in terms of a trigonometric equation.
   b. A person with a wellbeing index of 6 or higher is considered to experience a high level of happiness. For what percentage of the four-week cycle does the model predict this feeling would last?

4. **WE24** During one day in October the temperature T °C is given by $T = 19 - 3\sin\left(\frac{\pi}{12}t\right)$ where t is the time in hours after midnight.
   a. What was the temperature at midnight?
   b. What was the maximum temperature during the day and at what time did it occur?
   c. Over what interval did the temperature vary that day?
   d. State the period and sketch the graph of the temperature for $t \in [0, 24]$.
   e. If the temperature was below k degrees for 3 hours find, to 1 decimal place, the value of k.

5. John is a keen amateur share trader who keeps careful records showing the fluctuation in share prices. One share in particular, Zentium, appears to have been following a sinusoidal shape with a period of two weeks over the last five weeks. Its share price has fluctuated between 12 and 15 cents, with its initial price at the start of the observations at its peak.



   a. Given the Zentium share price can be modelled by $p = a\cos(nt) + b$ where p is the price in cents of the share t weeks after the start of the recorded observations, determine the values of a, n and b.
   b. Sketch the graph of the share price over the last five weeks and state the price of the shares at the end of the five weeks.
   c. When John decides to purchase the share, its price is 12.75 cents and rising. He plans to sell it immediately once it reaches 15 cents. According to the model, how many days will it be before he sells the share? Round your answer to the nearest day. Assume 7 days trading week applies.
   d. If John buys 10 000 shares at 12.75 cents, sells them at 15 cents and incurs brokerage costs of 1% when buying and selling, how much profit does he make?

6. The height, h metres, of the tide above mean sea level is given by $h = 4\sin\left(\frac{\pi(t-2)}{6}\right)$, where t is the time in hours since midnight.



a. How far below mean sea level was the tide at 1 am?

b. State the high tide level and show that this first occurs at 5 am.

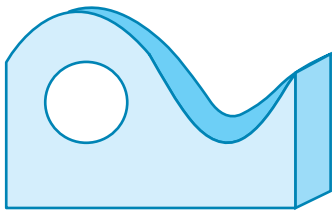
c. How many hours are there between high tide and the following low tide?

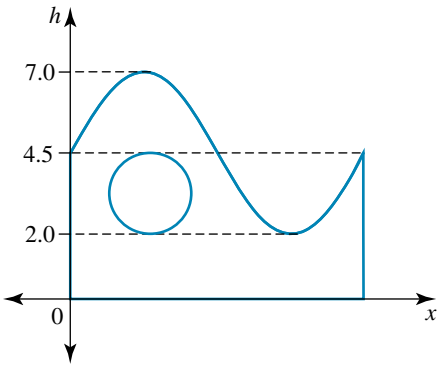
d. Sketch the graph of h versus t for $t \in [0, 12]$.

e. What is the height of the tide predicted to be at 2 pm?

f. How much higher than low tide level is the tide at 11.30 am? Give the answer to 2 decimal places.

7. The diagram shows the cross-section of a sticky tape holder with a circular hole through its side. It is bounded by a curve with the equation $h = a\sin\left(\frac{\pi}{5}x\right) + b$, where h cm is the height of the curve above the base of the holder at distance x cm along the base.





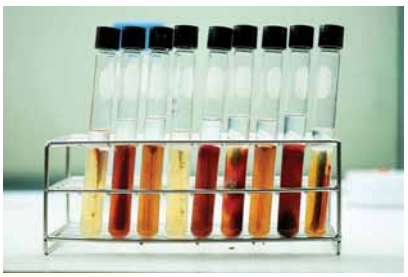
a. Use the measurements given on the graph to state the values of a and b and hence write down the equation of the bounding curve.

b. Calculate the length of the base of the holder.

c. If the centre of the circular hole lies directly below the highest point of the curve, what are the coordinates of the centre?

d. Using the symmetry of the curve, calculate the cross-sectional area lightly shaded in the diagram, to 1 decimal place.

8. In a laboratory, an organism is being grown in a test tube. To help increase the rate at which the organism grows, the test tube is placed in an incubator where the temperature T°C varies according to the rule $T = 30 - \cos\left(\frac{\pi}{12}t\right)$, where t is the time in minutes since the test tube has been placed in the incubator.



a. State the range of temperature in the incubator.

b. How many minutes after the test tube is placed in the incubator does the temperature reach its greatest value?

c. Sketch the graph of the temperature against time for the first half hour.

d. How many cycles are completed in 1 hour?

e. If the organism is kept in the incubator for 2.5 hours, what is the temperature at the end of this time?

f. Express the rule for the temperature in terms of a sine function.

9. During a particular day in a Mediterranean city, the temperature inside an office building between 10 am and 7.30 pm fluctuates so that t hours after 10 am, the temperature T °C is given by $T = 19 + 6\sin\left(\frac{\pi t}{6}\right)$.

a. i. State the maximum temperature and the time it occurs.

ii. State the minimum temperature and the time it occurs.

b. i. What is the temperature in the building at 11.30 am? Answer to 1 decimal place.

ii. What is the temperature in the building at 7.30 pm? Answer to 1 decimal place.

c. Sketch the graph of the temperature against time from 10 am and 7.30 pm.

d. When the temperature reaches 24°, an air conditioner in the boardroom is switched on and it is switched off when the temperature in the rest of the building falls below 24°. For how long is the air conditioner on in the boardroom?

e. The office workers who work the shift between 11.30 am and 7.30 pm complain that the temperature becomes too cool towards the end of their shift. If management agrees that heating can be used for the coldest two-hour period of their shift, at what time and at what temperature would the heating be switched on? Express the temperature in both exact form and to 1 decimal place.

10. The height above ground level of a particular carriage on a Ferris wheel is given by $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$ where h is the height in metres above ground level after t seconds.



a. How far above the ground is the carriage initially?

b. After one minute, how high will the carriage be?

c. How many revolutions will the Ferris wheel complete in a four-minute time interval?

d. Sketch the graph of h against t for the first four minutes.

e. For how long, to the nearest second, in one revolution, is the carriage higher than 12 metres above the ground?

f. The carriage is attached by strong wire radial spokes to the centre of the circular wheel. Calculate the length of a radial spoke.

11. A person sunbathing at a position P on a beachfront observes the waves wash onto the beach in such a way that after t minutes, the distance p metres of the end of the water's wave from P is given by $p = 3\sin(n\pi t) + 5$.

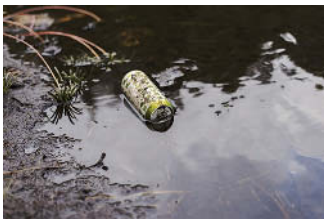
a. What is the closest distance the water reaches to the sunbather at P?

b. Over a one-hour interval, the sunbather counts 40 complete waves that have washed onto the beach. Calculate the value of n.

c. At some time later in the day, the distance p metres of the end of the water's wave from P is given by $p = a\sin(4\pi t) + 5$. If the water just reaches the sunbather who is still at P, deduce the value of a and determine how many times in half an hour the water reaches the sunbather at P.

d. In which of the two models of the wave motion, $p = 3\sin(n\pi t) + 5$ or $p = a\sin(4\pi t) + 5$, is the number of waves per minute greater?

12. A discarded drink can floating in the waters of a creek oscillates vertically up and down 20 cm about its equilibrium position. Its vertical displacement, x metres, from its equilibrium position after t seconds is given by $x = a\sin(bt)$. Initially the can moved vertically downwards from its mean position and it took 1.5 seconds to rise from its lowest point to its highest point.



a. Determine the values of a and b and state the equation for the vertical displacement.

b. Sketch one cycle of the motion.

c. Calculate the shortest time, T seconds, for the can's displacement to be one half the value of its amplitude.

d. What is the total distance the can moved in one cycle of the motion?

13. The intensity, I, of sound emitted by a device is given by $I = 4\sin(t) - 3\cos(t)$ where t is the number of hours the device has been operating.

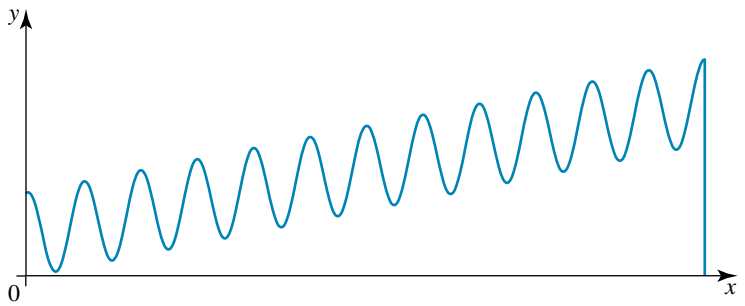
a. Use the $I - t$ graph to obtain the maximum intensity the device produces.

b. State, to 2 decimal places, the first value of t for which $I = 0$.

c. Express the equation of the $I - t$ graph in the form $I = a\sin(t + b)$.

d. Express the equation of the $I - t$ graph in the form $I = a\cos(t + b)$.

14. The teeth of a tree saw can be approximated by the function $y = x + 4 + 4\cos(6x)$, $0 \leq x \leq 4\pi$, where y cm is the vertical height of the teeth at a horizontal distance x cm from the end of the saw.



Each peak of the graph represents one of the teeth.

a. How many teeth does the saw have?

b. Exactly how far apart are the successive peaks of the teeth?

c. The horizontal length of the saw is 4π cm. What is the greatest width measurement of this saw?

d. Give the equation of the linear function which will touch each of the teeth of the saw.

10.10 Review: exam practice

A summary of this chapter is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC In a rectangle ABCD, the angle CAD is 27° and the side AD is 3 cm. The length of the diagonal AC in cm is closest to:

A. 6.61 **B.** 5.89 **C.** 3.37 **D.** 2.67

2. MC The exact value of $\sin(45°) + \tan(30°) \times \cos(60°)$ is:

A. $\frac{3}{2}$ **B.** $\frac{\sqrt{2} + 1}{2}$ **C.** $\frac{3\sqrt{2} + \sqrt{3}}{6}$ **D.** $\frac{3\sqrt{2} + 2\sqrt{3}}{12}$

3. Express in degree measure.

a. $\frac{11\pi^c}{9}$ b. $-3.5\pi^c$

4. An arc subtends an angle of 114° at the centre of a circle of radius 6 2 cm.
Find:

a. the length of the arc

b. the area of the sector with subtended angle 114°.

5. MC Consider the unit circle diagram at right.

a. A possible value of θ for the trigonometric point $P[\theta]$ is:

A. $\frac{\pi}{5}$

B. $\frac{4\pi}{5}$

C. $\frac{6\pi}{5}$

D. $\frac{7\pi}{10}$

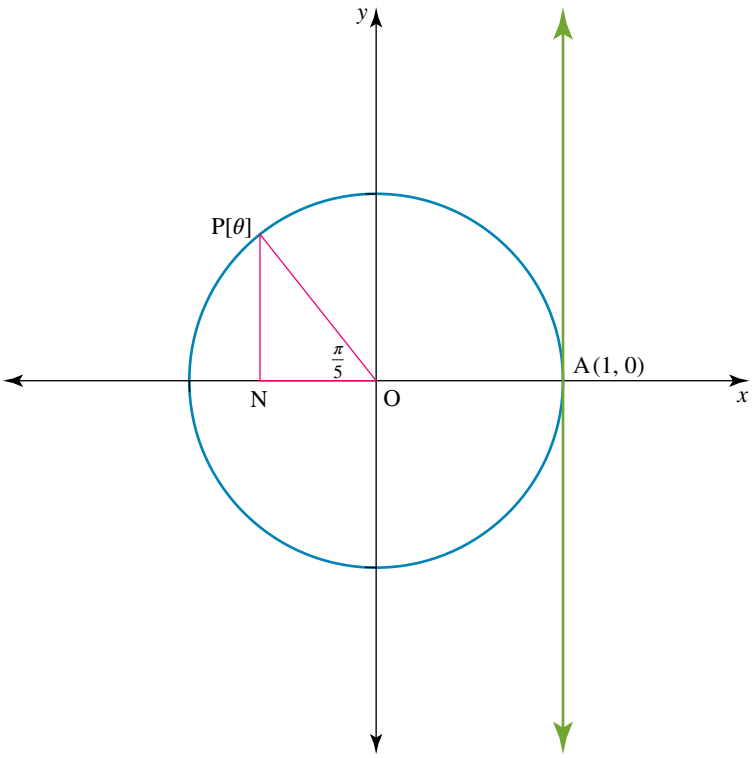
b. The value of $\sin(\theta)$ is given by the length of the line segment:

A. OP

B. PA

C. ON

D. NP



6. MC In which quadrant(s) is $\sin(\theta) < 0$ and $\cos(\theta) > 0$?

A. First quadrant **B.** Second quadrant

C. Third quadrant **D.** Fourth quadrant

E. Both the second and fourth quadrants

7. If $\cos(t) = 0.6$, evaluate the following.

a. $\cos(-t)$ **b.** $\cos(\pi + t)$

c. $\cos(3\pi - t)$ **d.** $\cos(-2\pi + t)$

8. MC The range of the graph of $y = 2 + 4\cos\left(\frac{3x}{2}\right)$ is:

A. R **B.** $[2, 6]$ **C.** $[-2, 6]$ **D.** $[-6, 2]$

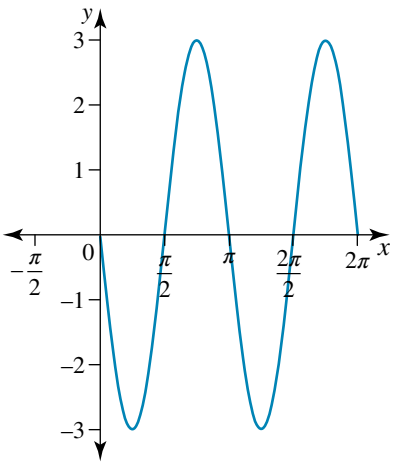
9. MC A possible equation for the given graph could be:

A. $y = -3\cos(2x)$

B. $y = 3\cos\left(\frac{x}{2}\right)$

C. $y = -3\sin(2x)$

D. $y = -3\sin\left(\frac{x}{2}\right)$



10. Sketch the following graphs over the given domain.

a. $y = \sin(x),\ 0 \leq x \leq \dfrac{7\pi}{2}$

b. $y = \cos(x),\ -\pi \leq x < 3\pi$

c. $y = \tan(\theta)$ for the domain $-2\pi \leq \theta \leq 2\pi$

Sketch the following graphs, stating the range and identifying the key features.

d. $y = 5 - 3\cos(x),\ 0 \leq x \leq 2\pi$

e. $y = 5\sin(3x),\ 0 \leq x \leq 2\pi$

f. $y = 1 + 2\cos\left(\dfrac{x}{2}\right),\ -2\pi \leq x \leq 3\pi$

g. $y = -\sin\left(x + \dfrac{\pi}{6}\right),\ 0 \leq x \leq 2\pi$

11. MC The number of solutions to the equation $\cos(x) = 0,\ -\dfrac{\pi}{2} \leq x \leq \dfrac{5\pi}{2}$ is:

A. 1 **B.** 2 **C.** 3 **D.** 4

12. MC The solutions to the equation $\sin(x) = \dfrac{1}{2},\ 0 \leq x \leq 2\pi$ are:

A. $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ **B.** $x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ **C.** $x = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ **D.** $x = \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

Complex familiar

13. Evaluate $\cos\left(\dfrac{11\pi}{6}\right) - \tan\left(\dfrac{11\pi}{3}\right) + \sin\left(-\dfrac{11\pi}{4}\right)$ exactly.

14. a. Given $\tan(x) = \dfrac{5}{2},\ x \in \left(\pi, \dfrac{3\pi}{2}\right)$, obtain the exact values for $\cos(x)$ and $\sin(x)$.

b. Given $\sin(y) = -\dfrac{\sqrt{5}}{3},\ y \in \left(\dfrac{3\pi}{2}, 2\pi\right)$, obtain the exact values for $\cos(y)$ and $\tan(y)$.

15. If $\cos(\theta) = p$, express the following in terms of p.

a. $\sin\left(\dfrac{\pi}{2} - \theta\right)$ **b.** $\sin\left(\dfrac{3\pi}{2} + \theta\right)$

c. $\sin^2(\theta)$ **d.** $\cos(5\pi - \theta)$

16. Solve the following equations for x.

a. $\sqrt{6}\cos(x) = -\sqrt{3},\ 0 \leq x \leq 2\pi$

b. $2 - 2\cos(x) = 0,\ 0 \leq x \leq 4\pi$

c. $2\sin(x) = \sqrt{3},\ -2\pi \leq x \leq 2\pi$

d. $\sqrt{5}\sin(x) = \sqrt{5}\cos(x),\ 0 \leq x \leq 2\pi$

e. $8\sin(3x) + 4\sqrt{2} = 0,\ -\pi \leq x \leq \pi$

f. $\tan(2x°) = -2,\ 0° \leq x \leq 270°$

g. $2\sin^2(x) - \sin(x) - 3 = 0,\ 0 \leq x \leq 2\pi$

h. $2\sin^2(x) - 3\cos(x) - 3 = 0,\ 0 \leq x \leq 2\pi$

Complex unfamiliar

17. **a.** Show on a diagram the positions of the real numbers -2 and $\frac{3\pi}{4}$ when the real number line is wrapped around the circumference of a unit circle.

b. Let P be the trigonometric point $\left[\frac{3\pi}{4}\right]$. Give the exact Cartesian coordinates of point P.

c. Let Q be the trigonometric point $[-2]$. Give the Cartesian coordinates of Q to 2 decimal places.

d. R is the trigonometric point $[\theta]$ where $0 < \theta < \frac{\pi}{2}$. The points P and R are symmetric points. What is the exact value of θ?

e. **i.** Determine the exact value for the angle POQ where O is the centre of the unit circle.

ii. Express the angle POQ in degrees, correct to 2 decimal places.

f. Give another real number which would be mapped to the same position as each of the numbers -2 and $\frac{3\pi}{4}$ when the real number line is wrapped around the circumference of a unit circle.

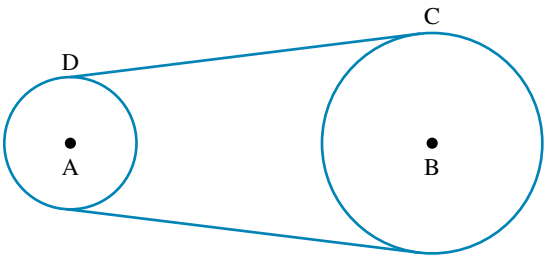
18. A real estate agent has three land sites for sale. All three sites are triangular in shape.

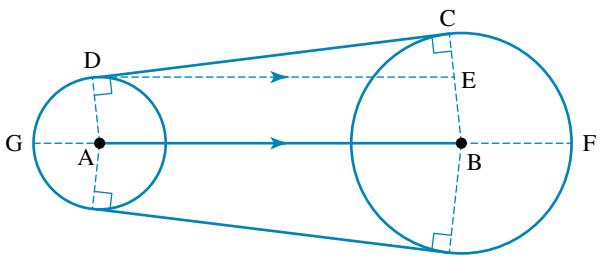


a. The first site is in the shape of an equilateral triangle of side length $\sqrt{12}$ km. Calculate the exact area of this site.

b. The second site is a triangle ABC where $A = 40°$, $a = 50$ m, $B = 25°$ and $b = 78$ m, using the naming convention. Calculate the area of this site, correct to the nearest square metre.

19. Two circular pulleys, with radii 3 cm and 8 cm, respectively, have their centres 13 cm apart.





Calculate:

a. the length of DC

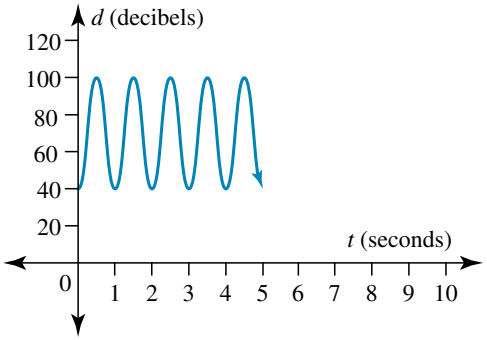
b. the angle CED in radians

c. the angle EBF in radians

d. the angle GAD in radians

e. the length of the belt required to pass tightly around the pulleys, giving the answer to 1 decimal place.

20. Sound waves emitted from an amplifier during a band practice session are illustrated over a 5-second interval, with the vertical axis measuring the sound in decibels.





a. State the period of the sound waves.

b. State the amplitude of the sound waves.

c. Give the values of a, b and c for which the equation of the graph can be expressed in the form $d = a\cos(bt) + c$ where d is the number of decibels of the sound after t seconds.

d. Normal speech range is 50–60 decibels. For what percentage of the first second did the sound level fit this category?

e. During an interval of 10 minutes of the band practice session, how often was the sound at its greatest level?

f. Sound under 80 decibels is considered to be a safe level. Calculate the percentage of time during the band practice when the level of the sound was potentially damaging to anyone present with unprotected hearing.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 10 Trigonometric functions

Exercise 10.2 Trigonometry review

1. **a.** $\frac{\sqrt{2}}{2}$ **b.** $\frac{\sqrt{3}}{3}$ **c.** $\frac{1}{2}$ **d.** 1

2. **a.** $h = 7.66$ **b.** $a \approx 68.20$

3. $2\sqrt{2}$ metres

4. $\frac{3\sqrt{2} - \sqrt{6}}{8}$

5. **a.** $\frac{\sqrt{21}}{7}$ **b.** $\frac{\sqrt{11}}{5}$ **c.** $\frac{2}{3}$

6. **a.** $\cos(a) = \frac{3}{\sqrt{13}}$ **b.** $6\sqrt{13}$ cm

7. 36 cm

8. 30 cm^2

9. **a.** $x = 7.13,\ y = 3.63$ **b.** $x = 13.27,\ h = 16.62$

10. **a.** 66.42° **b.** 51.34°

11. **a.** 6.18 metres **b.** 59.5°, 5.3 metres

12. 26.007 km

13. 21.6°

14. AC $= 12\sqrt{3}$ cm; BC $= 18\sqrt{2}$ cm; AB $= 18 + 6\sqrt{3}$ cm

15. **a.** $\sqrt{3}a$ units **b.** 35.26°

16. **a.** 5.07 cm **b.** 4.7 cm

17. **a.** **i.** 4.275 cm^2 **ii.** 1.736 cm **b.** $20\sqrt{3}$ cm; $\frac{100\sqrt{3}}{3}\text{cm}^2$

18. 5.64 km^2

19. Height 63.4 metres; distance 63.4 metres

20. **a.** Sample responses can be found in the worked solutions in the online resources.

b. $a = 60°;\ m = 16;\ n = 12$

Exercise 10.3 Radian measure

1. **a.**

Degrees	30°	45°	60°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

b.

Degrees	0°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

2. **a.** 36° **b.** 120° **c.** 75° **d.** 330° **e.** 140° **f.** 810°

3. **a.** $\frac{2\pi}{9}$ **b.** $\frac{5\pi}{6}$ **c.** $\frac{5\pi}{4}$ **d.** $\frac{5\pi}{3}$ **e.** $\frac{7\pi}{4}$ **f.** 4π

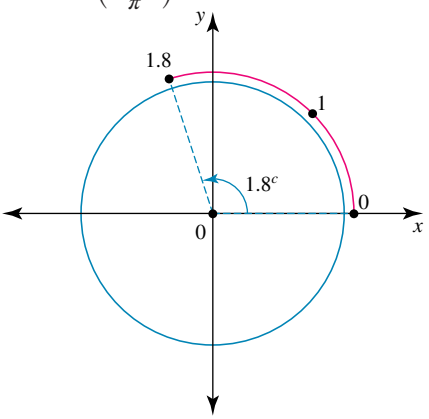
4. **a.** $\frac{\pi}{3}$ **b.** 135° **c.** 30°; $\frac{\sqrt{3}}{3}$

5. **a.** $-4\pi,\ -2\pi,\ 0,\ 2\pi,\ 4\pi$

b. $-1 - 4\pi, -1 - 2\pi, -1, -1 + 2\pi, -1 + 4\pi$

6. **a.** $\approx 103.1°\ \left(\frac{324°}{\pi}\right)$

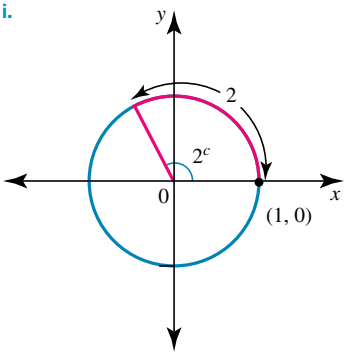
b.



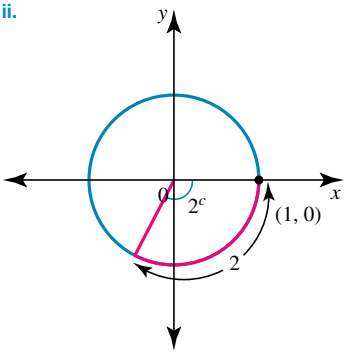
7. **a.** See figure at bottom of the page.

b. See figure at bottom of the page.

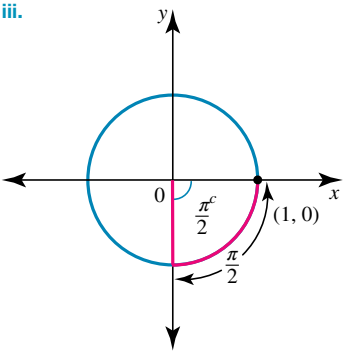
7. **a.** **i.**



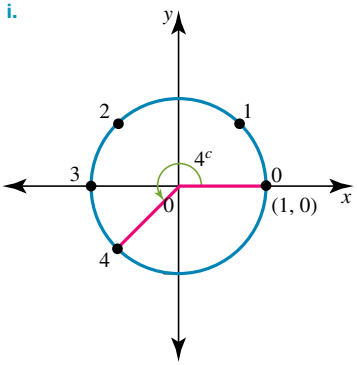
ii.



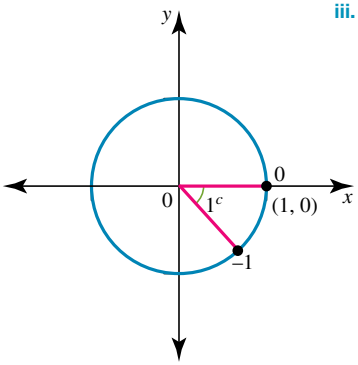
iii.



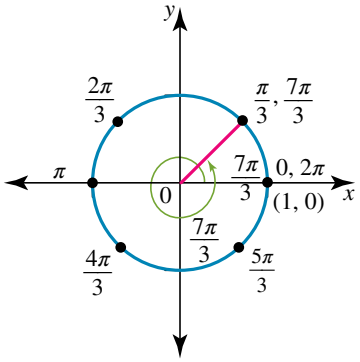
b. **i.**



ii.



iii.



8. **a.** 10.47 cm **b.** 216°

9. 2.53

10. **a.** **i.** 0.052 **ii.** 1.959 **iii.** 3.759

b. **i.** 171.887° **ii.** 414°

c. $\left\{\frac{\pi}{7}, 50°, 1.5\right\}$

11. **a.** 10π cm **b.** $\frac{4\pi^2}{9}$ cm **c.** 5π cm

12. 12 mm

13. **a.** $\frac{\pi}{8}$ **b.** 22.5°

14. 20.706 cm^2

15. 23.33 m^2

16. **a.** 17.2° **b.** 191° or $\frac{10}{3}$

c. 5.76 mm **d.** 81.5 cm^2

17. **a.** 2.572 **b.** 0.021

18. **a.** **i.** 1.557 **ii.** 0.623 **iii.** 0.025

b.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

19. **a.** 2 $units^2$ **b.** $6\sqrt{3} - 2$

20. **a.** 6371 km **b.** 6360 km **c.** 6004 km

Exercise 10.4 Unit circle definitions

1. **a.** Quadrant 1 **b.** Quadrant 3 **c.** Quadrant 2
d. Quadrant 1 **e.** Quadrant 4 **f.** Quadrant 3

2. **a.** Quadrant 3
b. Quadrant 2
c. Boundary of quadrant 1 and quadrant 4
d. Quadrant 4

3. **a.** **i.** (0, 1) **ii.** $\sin(\theta) = 1$
b. **i.** (−1, 0) **ii.** $\cos(\alpha) = -1$
c. **i.** (0, −1) **ii.** $\tan(\beta)$ is undefined.
d. **i.** (1, 0)
ii. $\sin(v) = 0$, $\cos(v) = 1$, $\tan(v) = 0$

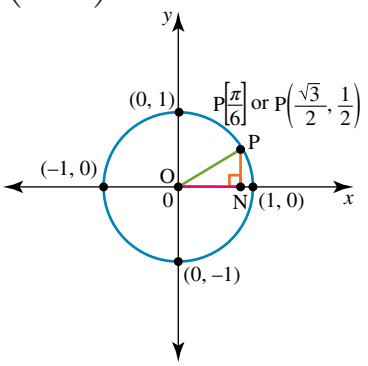
4. **a.** P$\left[\frac{\pi}{2}\right]$

b.

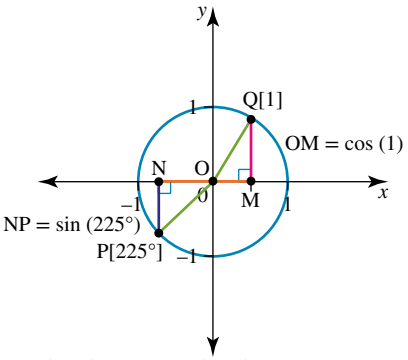
120°	$-400° = -360° - 40°$	$\frac{4\pi}{3} = \pi + \frac{1}{3}\pi$	$\frac{\pi}{4}$
quadrant 2	quadrant 4	quadrant 3	quadrant 1

c. Q[− 240°]; R[480°]

5. **a.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



b.



c. $\cos\left(-\frac{\pi}{2}\right) = 0$; $\sin\left(-\frac{\pi}{2}\right) = -1$

d. $f(0) = 0$

6. **a.** $f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

b. $g(4\pi) = \cos(4\pi) = 1$

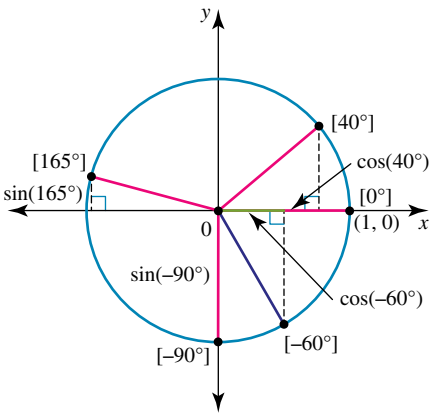
c. $h(-\pi) = \tan(-\pi) = 0$

d. $k(6.5\pi) = \sin\left(\frac{13\pi}{2}\right) + \cos\left(\frac{13\pi}{2}\right) = 1.$

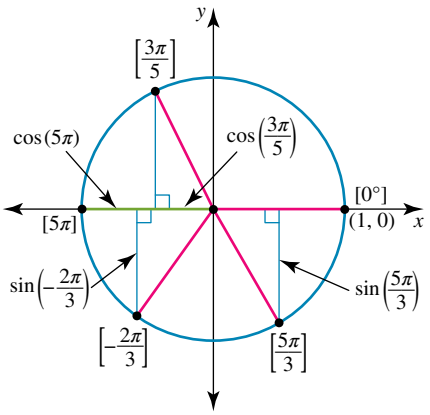
7. **a.** First and second quadrants **b.** First and fourth quadrants

8. **a.** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **b.** $\left[\frac{3\pi}{2}\right]$ or $\left[-\frac{\pi}{2}\right]$

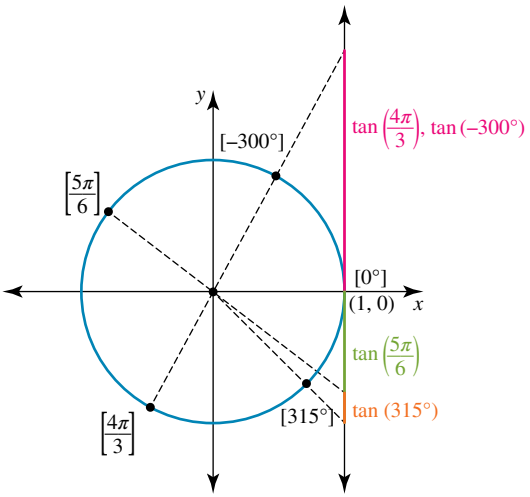
9.



10.



11.



12. a. Quadrant 2;
$\sin(\theta) = 0.6,\ \cos(\theta) = -0.8,\ \tan(\theta) = -0.75$

b. Quadrant 4;
$\sin(\theta) = -\frac{\sqrt{2}}{2},\ \cos(\theta) = \frac{\sqrt{2}}{2},\ \tan(\theta) = -1$

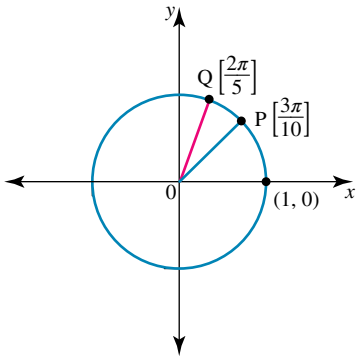
c. Quadrant 1; $\sin(\theta) = \frac{1}{\sqrt{5}}, \cos(\theta) = \frac{2}{\sqrt{5}}, \tan(\theta) = \frac{1}{2}$

d. Boundary between quadrant 1 and quadrant 2;
$\sin(\theta) = 1,\ \cos(\theta) = 0,\ \tan(\theta) =$ undefined

13. a. 1 **b.** 1 **c.** 0
d. 0 **e.** 0 **f.** −1

14. a. $\tan\left(\frac{5\pi}{2}\right)$; tan (−90°) **b.** $\tan\left(\frac{3\pi}{4}\right)$

15. a.

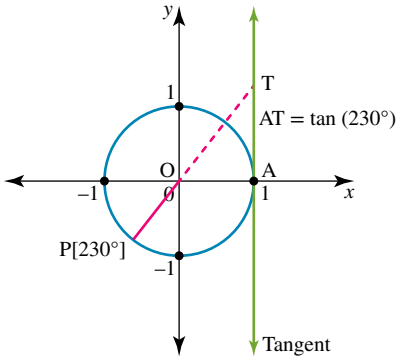


b. $\frac{\pi^c}{10}$

c. $P\left[-\frac{17\pi}{10}\right]$; $Q\left[-\frac{8\pi}{5}\right]$ (other answers are possible)

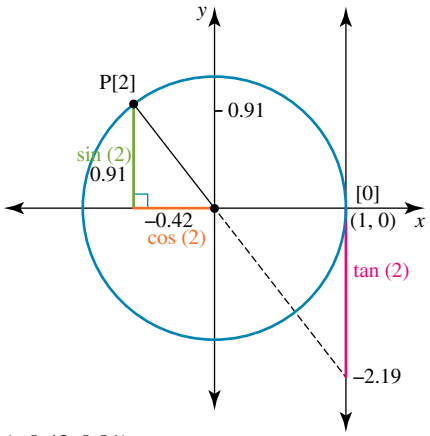
d. $P\left[\frac{23\pi}{10}\right]$; $Q\left[\frac{12\pi}{5}\right]$ (other answers are possible)

16. a. $\tan(230°) \approx 1.192$



b. $\tan(2\pi) = 0$

17. a.



b. (−0.42, 0.91)

18. Sample responses can be found in the worked solutions in the online resources.

19. a. $P\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ and $Q\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ are reflections in the x-axis.

b. $R\left(\frac{-(\sqrt{5}+1)}{4}, \frac{\sqrt{2(-\sqrt{5}+5)}}{4}\right)$ and
$S\left(\frac{\sqrt{5}+1}{4}, \frac{\sqrt{2(-\sqrt{5}+5)}}{4}\right)$ are reflections in the y-axis.

c. i. $\sin\left(\frac{7\pi}{4}\right) = \frac{-\sqrt{2}}{2}, \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{7\pi}{4}\right) = -1$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{\pi}{4}\right) = 1$

$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right),$

$\tan\left(\frac{7\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right)$

ii. $\sin\left(\frac{4\pi}{5}\right) = \frac{\sqrt{2(-\sqrt{5}+5)}}{4}$,
$\cos\left(\frac{4\pi}{5}\right) = -\frac{(\sqrt{5}+1)}{4}$,
$\tan\left(\frac{4\pi}{5}\right) = -\sqrt{-2\sqrt{5}+5}$

$\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{2(-\sqrt{5}+5)}}{4}, \cos\left(\frac{\pi}{5}\right) = \frac{(\sqrt{5}+1)}{4}$,
$\tan\left(\frac{\pi}{5}\right) = \sqrt{-2\sqrt{5}+5}$

$\sin\left(\frac{4\pi}{5}\right) = \sin\left(\frac{\pi}{5}\right)$, $\cos\left(\frac{4\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right)$,
$\tan\left(\frac{4\pi}{5}\right) = -\tan\left(\frac{\pi}{5}\right)$

20. a. 1
b. $-\frac{\sqrt{3}}{2} - \frac{1}{2}$
c. 1
d. 1
e. 1; Sample responses can be found in the worked solutions in the online resources.

Exercise 10.5 Exact values and symmetry properties

1. a. 1 b. 0 c. 0
d. 1 e. 0 f. 0

2. a. Fourth
b. First
c. Second
d. Boundary between quadrants 1 and 2, and boundary between quadrants 3 and 4
e. Boundary between quadrants 1 and 2
f. Boundary between quadrants 2 and 3

3.

	Quadrant 2	Quadrant 3	Quadrant 4
a.	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
b.	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$
c.	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
d.	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{9\pi}{5}$
e.	$\frac{5\pi}{8}$	$\frac{11\pi}{8}$	$\frac{13\pi}{8}$
f.	$\pi - 1$	$\pi + 1$	$2\pi - 1$

4. a. $-\frac{1}{2}$ b. 1 c. $-\frac{1}{2}$
d. $-\sqrt{3}$ e. $\frac{\sqrt{2}}{2}$ f. $\frac{1}{2}$

5. a. $\frac{\sqrt{2}}{2}$ b. $-\sqrt{3}$ c. $-\frac{\sqrt{3}}{2}$
d. $-\frac{1}{2}$ e. $\frac{\sqrt{3}}{3}$ f. $-\frac{1}{2}$

6. a. $\frac{\sqrt{2}}{2}$ b. $-\frac{\sqrt{3}}{2}$ c. $\frac{\sqrt{3}}{3}$
d. $\frac{\sqrt{3}}{2}$ e. $\frac{\sqrt{2}}{2}$ f. $-\frac{\sqrt{3}}{3}$

7. a. Third quadrant b. 0

8. 1

9. a. $-\frac{\sqrt{3}}{2}$ b. $-\frac{\sqrt{3}}{3}$ c. $\frac{\sqrt{3}}{2}$

10. $\sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $\cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$;
$\tan\left(-\frac{5\pi}{4}\right) = -1$

11. a. −0.2 b. −0.2
c. 0.2 d. 0.2

12. a. −4 b. 0.9
c. −4 d. 3.1

13. a. −0.91 b. 0.43
c. −0.47 d. 0.91
e. −0.43 f. 0.47

14. a. $-p$ b. p c. $-p$ d. p

15. a. $\frac{-\left(\sqrt{3}+1\right)}{2}$ b. $2-\sqrt{2}$ c. $2\sqrt{3}$
d. 0 e. 0 f. −2

16. a. [105°]; [255°]; [285°]; cos (285°) = cos (75°); [285°]
b. $-\tan\left(\frac{\pi}{7}\right)$
c. $\sin(\pi-\theta) = 0.8$, $\sin(2\pi-\theta) = -0.8$
d. i. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ ii. $\sin\left(\frac{25\pi}{6}\right) = \frac{1}{2}$

17. a. $\cos(-\theta) = p$ b. $\cos(5\pi+\theta) = -p$

18. a. Sample responses can be found in the worked solutions in the online resources.
b. Same x-coordinates
c. $\sin(\pi+\phi) = -0.87, \cos(\pi+\phi) = 0.5, \tan(\pi+\phi) = -1.74$
d. sin (t)
e. $A = 36°$ or $-216°$ (other answers are possible)
f. $B = \frac{13\pi}{11}$ or $\frac{20\pi}{11}$ or $-\frac{2\pi}{11}$ (other answers are possible)

19. a. Third quadrant
b. (−0.49, −0.87)
c. Quadrant 1, $\theta = 1.0587$; quadrant 2, $\theta = 2.0829$; quadrant 4, $\theta = 5.2244$

20. a. Quadrant 1 or quadrant 3
b. Quadrant 1, $\theta = 1.3734$; quadrant 3, $\theta = 4.5150$
c. Quadrant 1: $\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$; quadrant 3: $\left(-\frac{1}{\sqrt{26}}, -\frac{5}{\sqrt{26}}\right)$

Exercise 10.6 Graphs of the sine, cosine and tangent functions

1. a. Domain $[0, 4\pi]$, range $[-1, 1]$

b. $y = \sin(x)$

c. $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 1\right), \left(\frac{7\pi}{2}, -1\right)$.

d. Period 2π, amplitude 1

e. $y = 0$ (the x-axis)

f. $x \in (0, \pi) \cup (2\pi, 3\pi)$.

2. a. Domain $[-2\pi, 2\pi]$, range $[-1, 1]$

b. $y = \cos(x)$

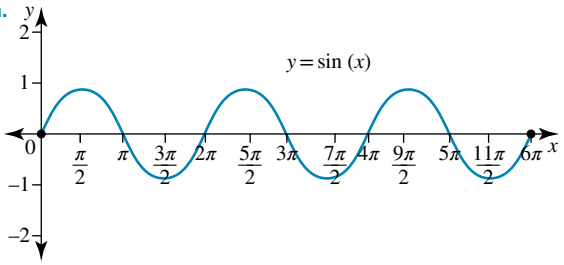
c. $(-\pi, -1), (\pi, -1)$

d. Period 2π, amplitude 1, mean position $y = 0$ (the x-axis)

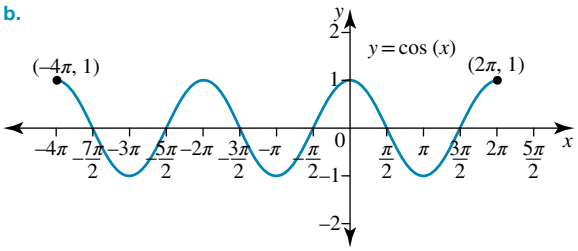
e. $\left(-\frac{3\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

f. $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

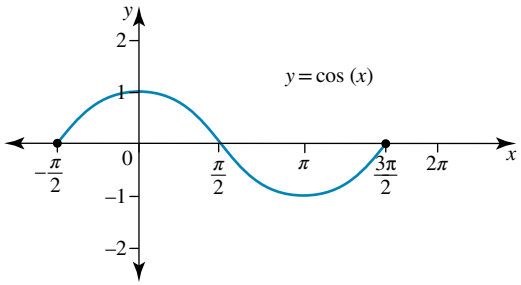
3. a.



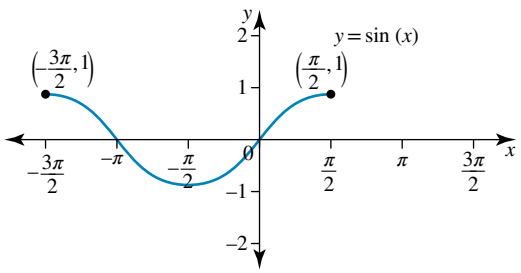
b.



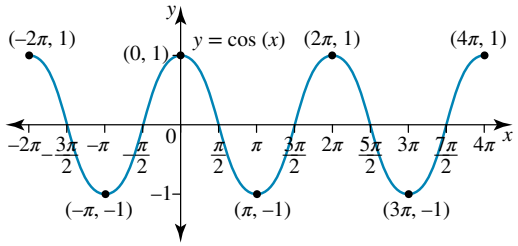
c.



d.



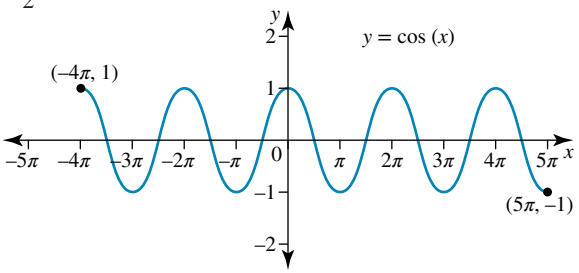
4. Three cycles



5. a. 2 maximum turning points

b. 7 minimum turning points

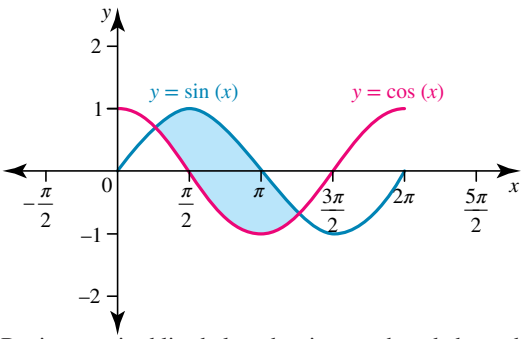
6. $4\frac{1}{2}$ cycles



7. a. 4 **b.** 7

c. 21 **d.** 3

8.



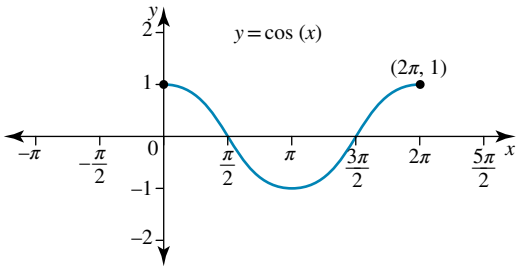
Region required lies below the sine graph and above the cosine graph between their points of intersection.

9. a. $a = \frac{19\pi}{4}$

b. $b = -2\pi$

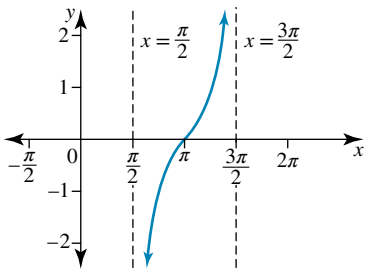
10. $c = \frac{5\pi}{2}$

11. $\frac{\pi}{2} < x < \frac{3\pi}{2}$

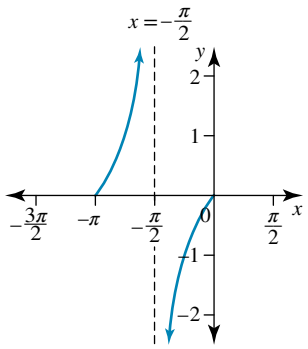


12. Sample responses can be found in the worked solutions in the online resources.

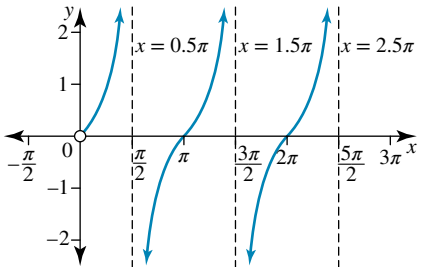
13. a.



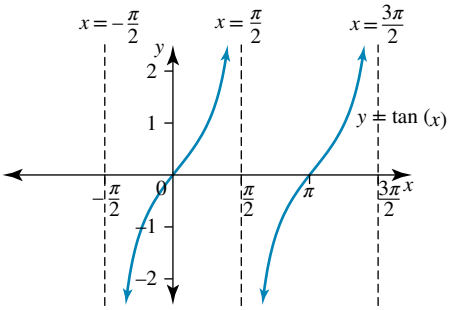
b.



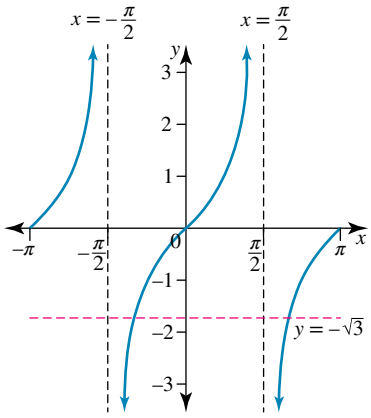
c.



14.



15. a.



b. $x = -\frac{\pi}{3}, \frac{2\pi}{3}$

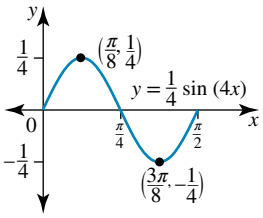
16. $x \in \left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

Exercise 10.7 Transformations of sine and cosine graphs

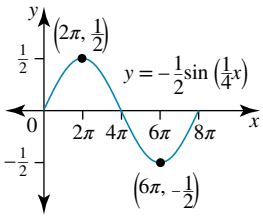
1.

	Amplitude	Period
a.	6	π
b.	7	4π
c.	$\frac{3}{5}$	$\frac{10\pi}{3}$
d.	1	$\frac{7}{3}$
e.	2	4π
f.	4	2π

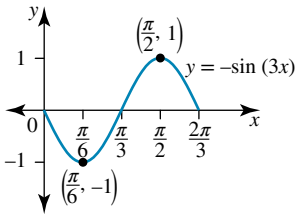
2. a. amplitude $\frac{1}{4}$, range $\left[-\frac{1}{4}, \frac{1}{4}\right]$, period $\frac{\pi}{2}$



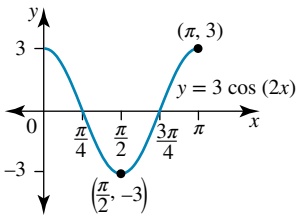
b. amplitude $\frac{1}{2}$, range $\left[-\frac{1}{2}, \frac{1}{2}\right]$, period 8π



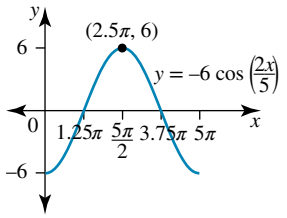
c. amplitude 1, range $[-1, 1]$, period $\frac{2\pi}{3}$.



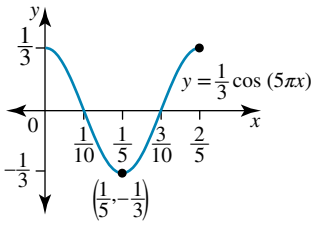
d. amplitude 3, range $[-3, 3]$, period π



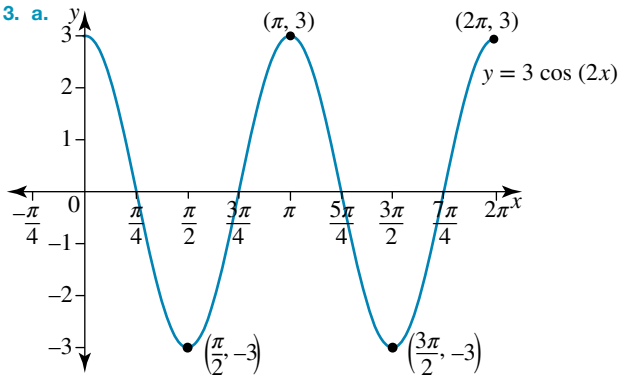
e. amplitude 6, range $[-6, 6]$, period 5π



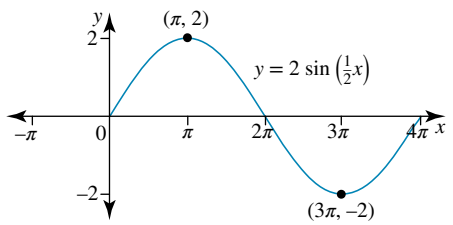
f. amplitude $\frac{1}{3}$, range $\left[-\frac{1}{3}, \frac{1}{3}\right]$, period 0.4



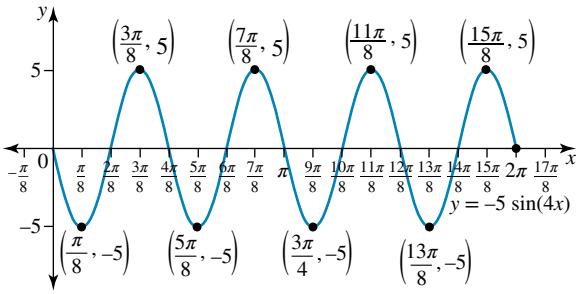
3. a.



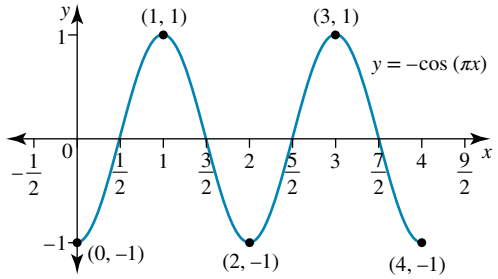
b.



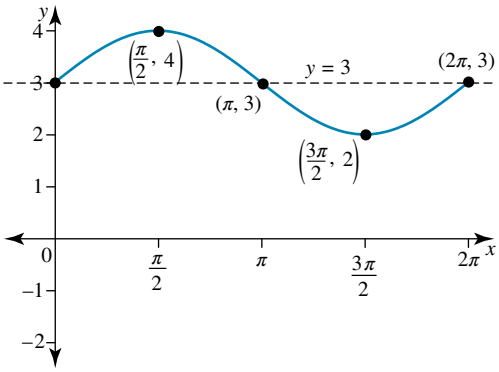
c.



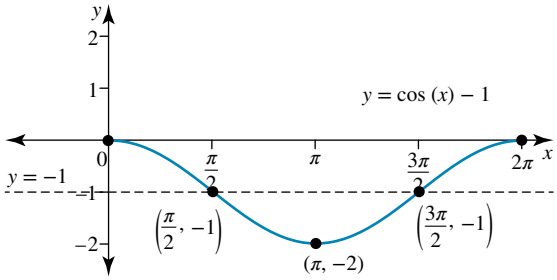
d.



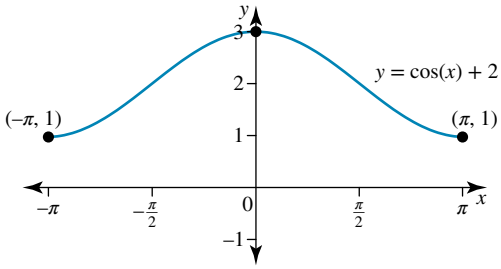
4. a. $[2, 4]$



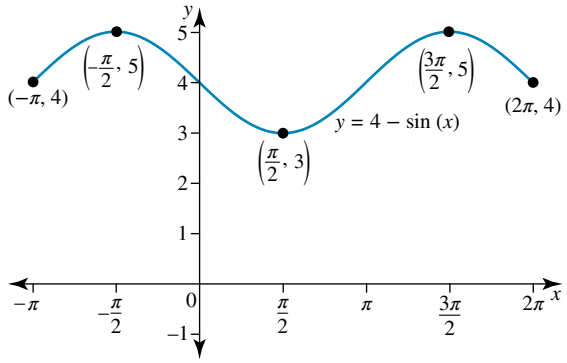
b. $[-2, 0]$



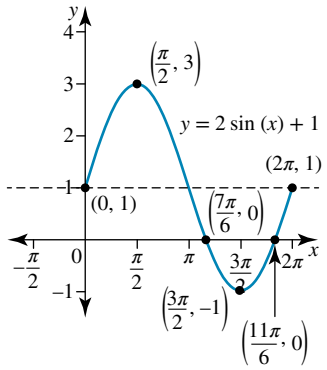
c. $[1, 3]$



d. $[3, 5]$

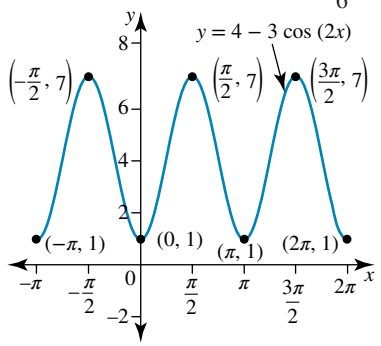


5. a.



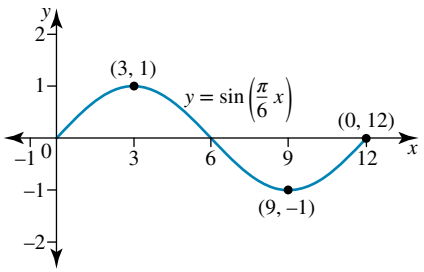
Period 2π; amplitude 1; equilibrium position $y = 1$; range $[-1, 3]$; x-intercepts at $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

b.



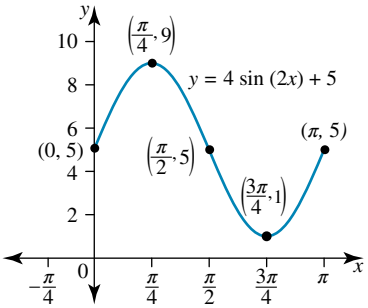
Period π; amplitude 3; inverted; equilibrium $y = 4$; range $[1, 7]$

6.

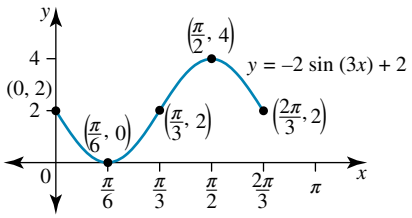


Period 12; amplitude 1; equilibrium $y = 0$; range $[-1, 1]$; domain $[0, 12]$

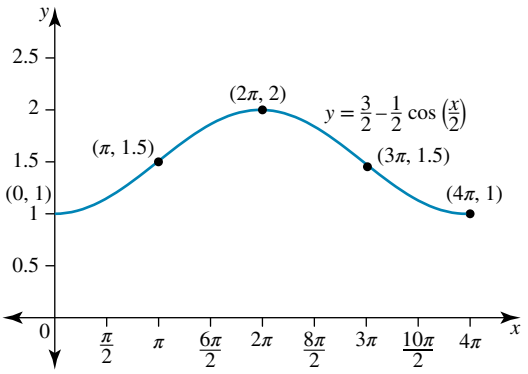
7. a. period π, mean $y = 5$, amplitude 4, range $[1, 9]$



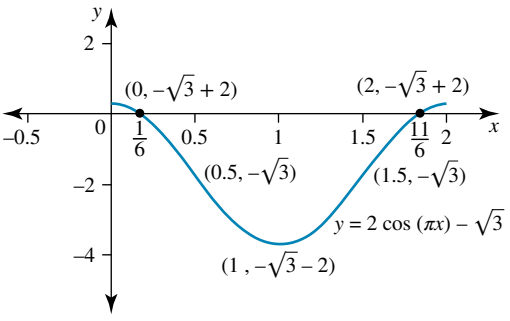
b. period $\frac{2\pi}{3}$, mean $y = 2$, amplitude 2, range $[0, 4]$.



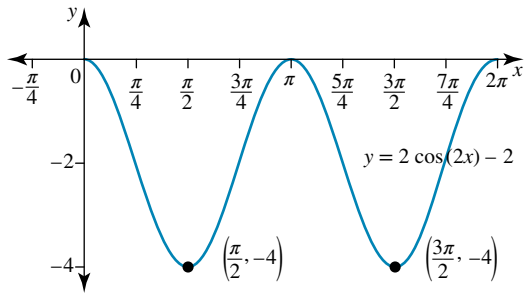
c. period 4π, mean $y = \frac{3}{2}$, amplitude $\frac{1}{2}$, range $[1, 2]$



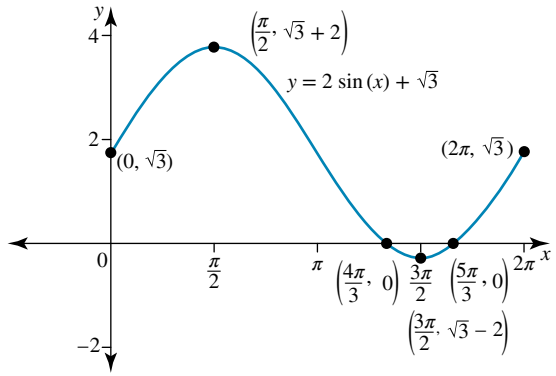
d. period 2, mean $y = -\sqrt{3}$, amplitude 2, range $[-\sqrt{3} - 2, -\sqrt{3} + 2]$



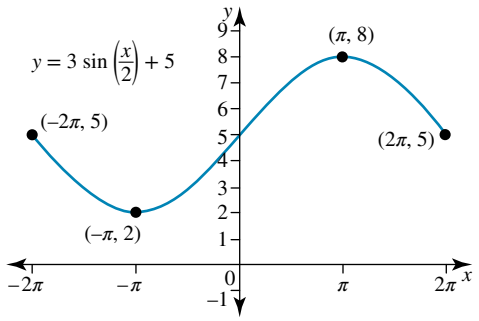
8. a. Range $[-4, 0]$



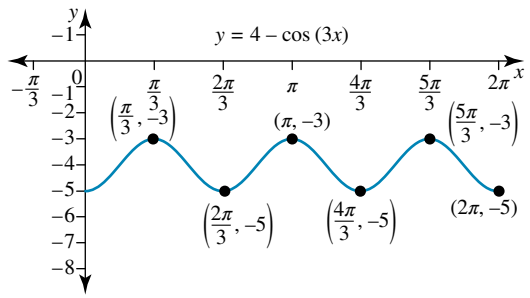
b. Range $[\sqrt{3} - 2, \sqrt{3} + 2]$



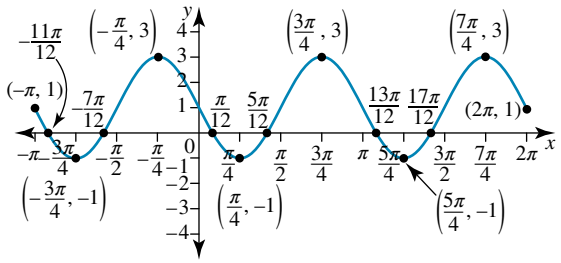
c. Range $[2, 8]$



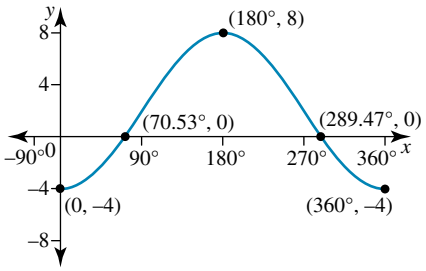
d. Range $[-5, -3]$



e. Range $[-1, 3]$



f. Range $[-4, 8]$



9. a. $[1, 5]$

b. -14

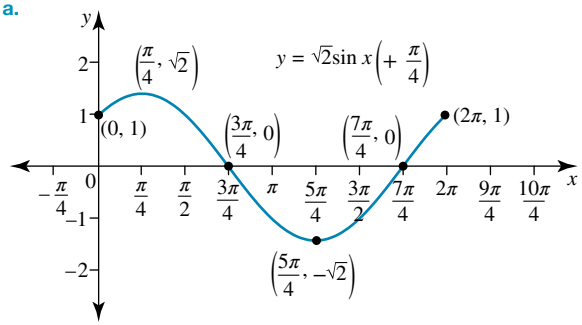
c. 68; $x = \frac{3\pi}{2}$

d. i. Dilation factor $\frac{1}{5}$ from y-axis; dilation factor 2 from x-axis; vertical translation up 3

ii. Dilation factor $\frac{1}{2}$ from y-axis; dilation factor 10 from x-axis; vertical translation down 4

iii. Dilation factor 12 from x-axis; reflection in x-axis; vertical translation up 56

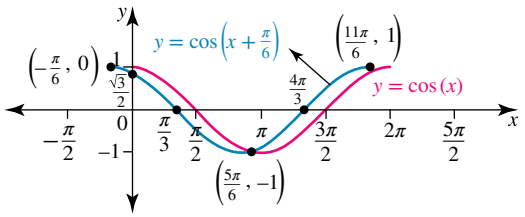
10. a.



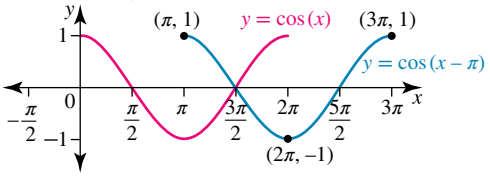
Period 2π; amplitude $\sqrt{2}$; phase shift $-\frac{\pi}{4}$ from $y = \sqrt{2}\sin(x)$; endpoints $(0, 1)$, $(2\pi, 1)$; range $[-\sqrt{2}, \sqrt{2}]$

b. Period π; amplitude 3; phase shift factor $-\frac{\pi}{8}$; range $[-2, 4]$

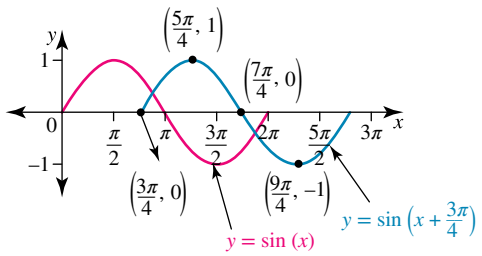
11. a. i.



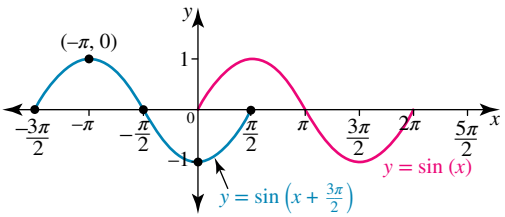
ii.



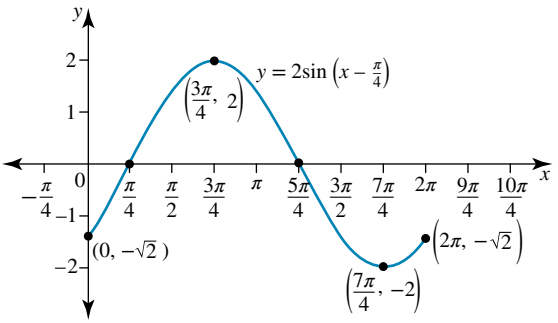
b. i.



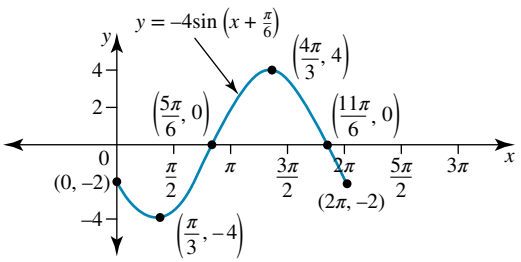
ii.



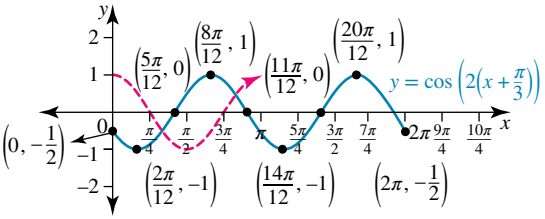
12. a.



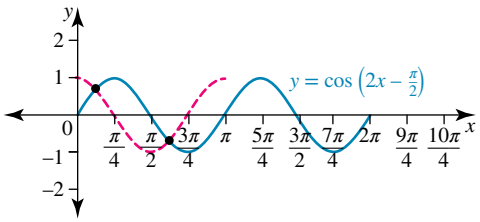
b.



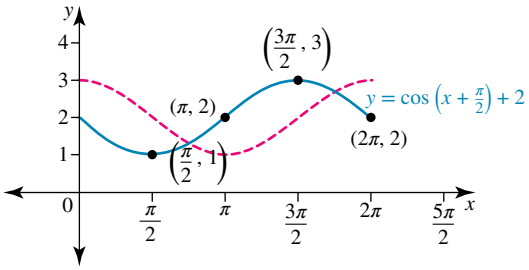
c.



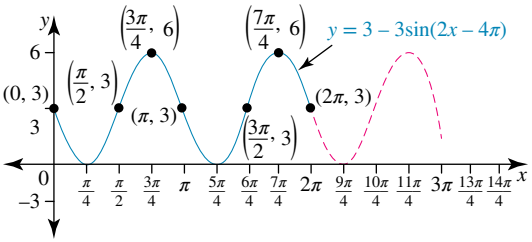
d.



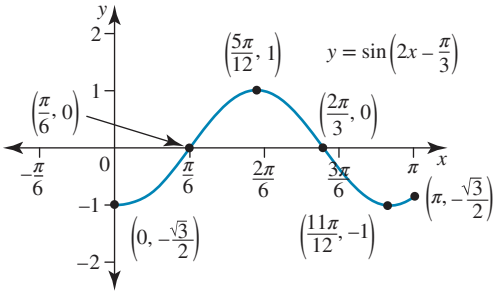
e.



f.



13.



Period π; amplitude 1; phase $\frac{\pi}{6}$; endpoints $\left(0, -\frac{\sqrt{3}}{2}\right), \left(\pi, -\frac{\sqrt{3}}{2}\right)$

14. a. $A = 3,\ B = 2,\ y = 3\sin(2x)$

b. $A = -0.5,\ B = \frac{2}{3},\ y = -0.5\cos\left(\frac{2x}{3}\right)$

c. $A = 4.5,\ B = \frac{4}{3},\ y = 4.5\cos\left(\frac{4}{3}x\right)$

15. a. $y = 4\sin(3x)$ **b.** $y = 6\cos\left(\frac{x}{2}\right)$

c. $y = -4\sin(2x)$

16. a. $y = 2\sin(x) + 3$ **b.** $y = -3\sin\left(\frac{x}{2}\right) + 3$

c. $y = 3\cos\left(\frac{2x}{3}\right) - 4$ **d.** $y = -5\cos(2x) - 1$

17. $y = 3\sin(2x) + 5$ (other answers are possible).

18. a. $y = -3\sin\left(\frac{x}{2}\right)$ **b.** $y = 4\cos(3x)$

c. $y = -4\cos(x) + 6$ **d.** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

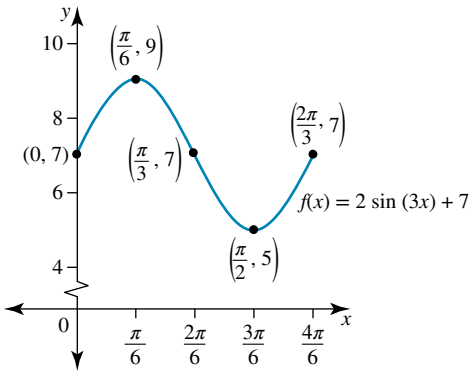
e. $y = 2\cos\left(x - \frac{3\pi}{4}\right)$ **f.** $y = \cos(x)$; $y = -\sin(x)$

19. $y = -2\sin\left(\frac{2x}{3}\right)$ or $y = -2\cos\left(\frac{2x}{3} - \frac{\pi}{2}\right)$ or $y = 2\sin\left(\frac{2x}{3} - \pi\right)$. Other answers are possible.

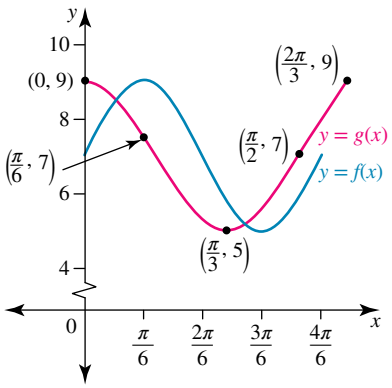
20. a. $\frac{2\pi}{3}$

b. $a = 2; b = 3; c = 7$

c. $D = \left[0, \frac{2\pi}{3}\right]$



d.



e. $\left(\frac{\pi}{12}, 7 + \sqrt{2}\right), \left(\frac{5\pi}{12}, 7 - \sqrt{2}\right)$

f. $\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$

Exercise 10.8 Solving trigonometric equations

1. a. $\frac{\pi}{4}, \frac{7\pi}{4}$ **b.** $\frac{5\pi}{4}, \frac{7\pi}{4}$ **c.** $\frac{5\pi}{6}, \frac{11\pi}{6}$

d. $\frac{5\pi}{6}, \frac{7\pi}{6}$ **e.** $\frac{\pi}{6}, \frac{5\pi}{6}$ **f.** $\frac{\pi}{3}, \frac{4\pi}{3}$

2. a. $\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

b. $\theta = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

c. $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3. a. $x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$

b. $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

c. $v = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

4. a. $\frac{\pi}{6}, \frac{5\pi}{6}$

b. $\frac{\pi}{6}, \frac{11\pi}{6}$

c. $-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

5. a. 2 solutions **b.** $\pm 120°$

6. a. $240°, 300°$ **b.** $45°, 225°$

c. $120°, 240°$ **d.** $30°, 150°$

7. a. $-\pi, 0, \pi, 2\pi, 3\pi, 4\pi$ **b.** $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

c. $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$ **d.** $0, 2\pi, 4\pi$

e. $\frac{\pi}{2}, \frac{5\pi}{2}$ **f.** $-\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

8. a. $-\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$

b. $0.64, 3.79, 6.93, 10.07$

c. $x° = \pm 104.5°$

9. a. $-\frac{11\pi}{6}, -\frac{7\pi}{6}$ **b.** No solution

c. $-4\pi, -2\pi$ **d.** $\frac{\pi}{4}$

e. $\frac{\pi}{3}, \frac{5\pi}{3}$ **f.** $-300, -240, 60, 120$

10. a. 1 **b.** $\left\{\frac{1}{2}, \frac{3}{2}\right\}$

11. a. $\frac{\pi}{3}, \frac{4\pi}{3}$ **b.** $\frac{\pi}{2}$

12. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

13. a. $\frac{\pi}{3}, \frac{4\pi}{3}$ **b.** $\frac{5\pi}{6}, \frac{11\pi}{6}$

c. $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$ **d.** $0.93, 4.07$

e. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **f.** $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}$

14. a. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **b.** $\frac{\pi}{2}, \frac{3\pi}{2}$

c. $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **d.** $\frac{\pi}{4}, \frac{5\pi}{4}, 1.89, 5.03$

e. $\frac{3\pi}{2}$ **f.** No solution

15. $x = \frac{2\pi}{3}$

16. a. $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$ **b.** $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$

17. a. $\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}$

b. $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

c. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

d. $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

18. a. $2.30, 3.98$

b. $-5.67, -2.53, 0.62, 3.76, 6.90$

c. $-53.13, -126.87, 233.13$

d. $78.46, 101.54, 258.46, 281.54$

19. a. Four solutions

b. π

c. $\pi + 0.4, 2\pi + 0.4$

d. i. $\tan(\theta) = -3, 0° < \theta < 180°$

ii. $\tan(\theta) = \sqrt{3}, 0° < \theta < 180°$

e. **i.** $\theta = 180° - \tan^{-1}(3)$ **ii.** $\theta = 60°$

f. θ is the second quadrant solution to $\tan(\theta) = m$, $m < 0$.

20. a. $a = 8$

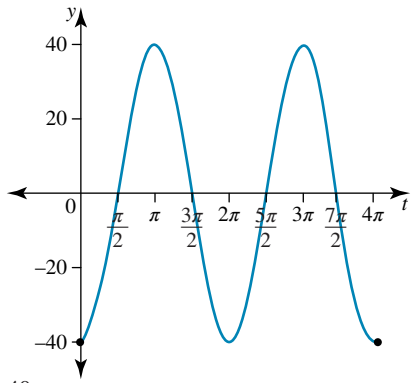
b. **i.** 0.38, 2.76

ii. 1.57

iii. No solution

Exercise 10.9 Modelling with trigonometric functions

1. a.



b. 40 cm

c. $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

d. $\frac{2\pi}{3}$ seconds ≈ 2.1 seconds

2. $T = 28 - 8\cos\left(\frac{\pi}{6}t\right)$; between 11.29 am and 4.31 pm

3. a. $I = 5\sin\left(\frac{\pi}{2}t\right) + 5$

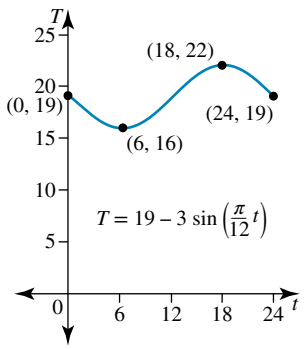
b. 44%

4. a. 19°

b. 22° at 6 pm

c. Between 16° and 22°

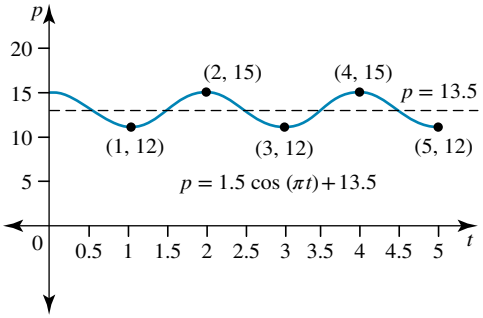
d. Period is 24 hours.



e. $k = 16.2$

5. a. $a = 1.5; n = \pi; b = 13.5$

b. 12 cents



c. 5 days

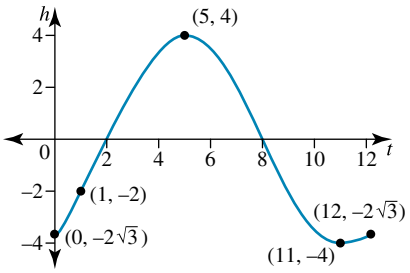
d. \$197.25

6. a. 2 metres below mean sea level

b. 4 metres above mean sea level; Sample responses can be found in the worked solutions in the online resources.

c. 6 hours

d.



e. At mean sea level

f. Risen by 0.14 metres

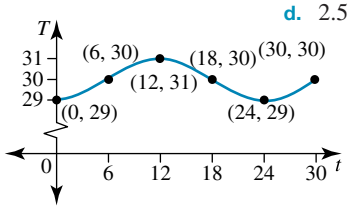
7. a. $a = 2.5; b = 4.5; h = 2.5\sin\left(\frac{\pi}{5}x\right) + 4.5$ **b.** 10 cm

c. (2.5, 3.25) **d.** 40.1 cm^2

8. a. Between 29° and 31° **b.** 12 minutes

d. 2.5 cycles

c.

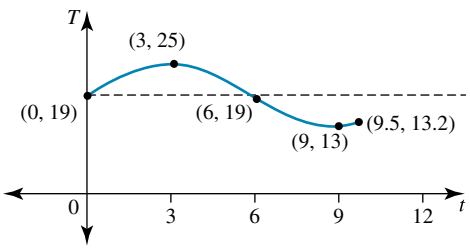


e. 30°C **f.** $T = \sin\left(\frac{\pi}{12}(t-6)\right) + 30$

9. a. **i.** 25° at 1 pm **ii.** 13° at 7 pm

b. **i.** 23.2 °C **ii.** 13.2 °C

c.



d. 2.24 hours

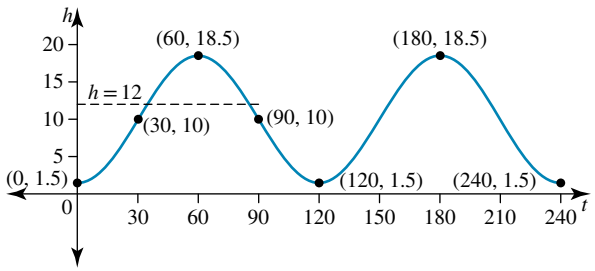
e. $\left(19 - 3\sqrt{2}\right)^{\circ} \approx 14.8°$ at 5.30 pm

10. a. 1.5 metres

b. 18.5 metres

c. 2

d.



e. 51 seconds

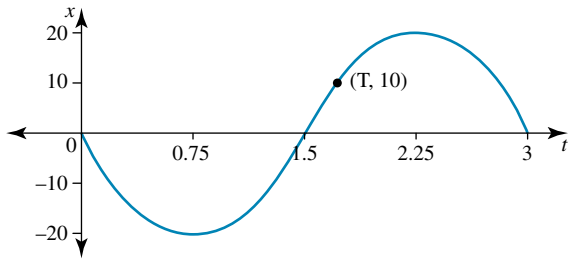
f. 8.5 metres

11. a. 2 metres **b.** $n = \frac{4}{3}$

c. $a = 5$; 60 times **d.** $p = a\sin(4\pi t) + 5$

12. a. $a = -20, b = \frac{2\pi}{3}, x = -20\sin\left(\frac{2\pi}{3}t\right)$

b.



c. $T = 1.75$ seconds

d. 80 cm

13. a. 5 units **b.** $t = 0.64$

c. $I = 5\sin(t - 0.64)$ **d.** $I = 5\cos(t - 2.21)$

14. a. 13 (includes the single-sided teeth at the ends)

b. $\frac{\pi}{3}$ cm (≈ 1.047)

c. $(4\pi + 8)$ cm

d. $y = x + 8$

10.10 Review: exam practice

1. C

2. C

3. a. 220° **b.** −630°

4. a. 12.34 cm **b.** 38.24 cm^2

5. a. B **b.** D

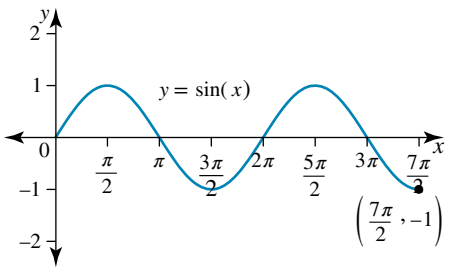
6. D

7. a. 0.6 **b.** −0.6 **c.** −0.6 **d.** 0.6

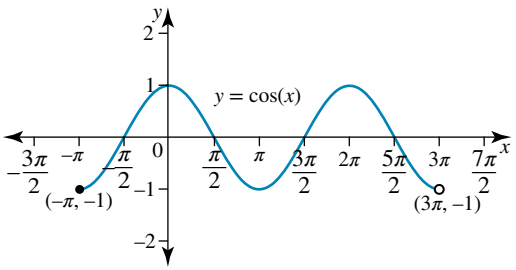
8. C

9. C

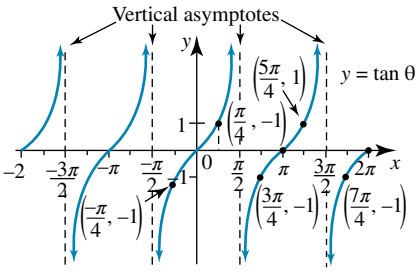
10. a.



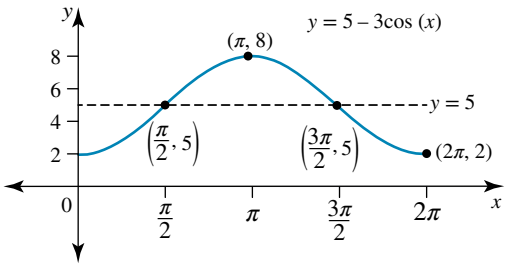
b.



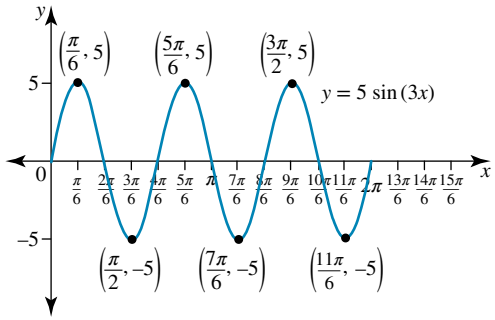
c.



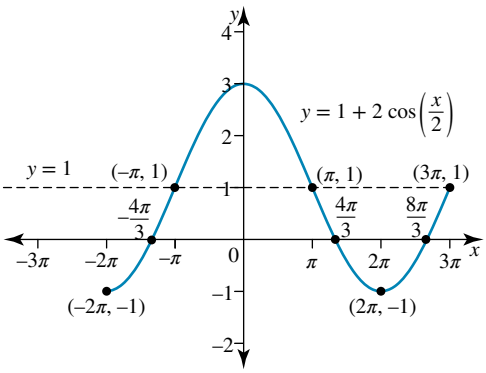
d. Period 2π; amplitude 3; range $[2, 8]$



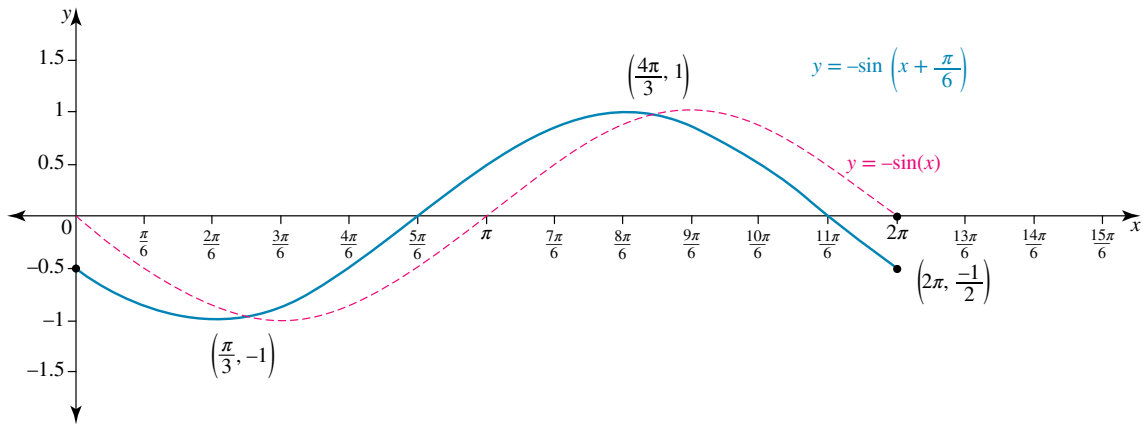
e. Period $\frac{2\pi}{3}$; amplitude 5; range $[-5, 5]$



f. Period 4π; amplitude 2; range $[-1, 3]$



g. Period 2π; amplitude 1; range $[-1, 1]$



11. D

12. A

13. $\dfrac{3\sqrt{3} - \sqrt{2}}{2}$

14. a. $\cos(x) = -\dfrac{2}{\sqrt{29}}$; $\sin(x) = -\dfrac{5}{\sqrt{29}}$

b. $\cos(y) = \dfrac{2}{3}$; $\tan(y) = -\dfrac{\sqrt{5}}{2}$

15. a. p **b.** $-p$ **c.** $1 - p^2$ **d.** $-p$

16. a. $\dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

b. $0, 2\pi, 4\pi$

c. $\dfrac{-5\pi}{3}, \dfrac{-4\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

d. $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

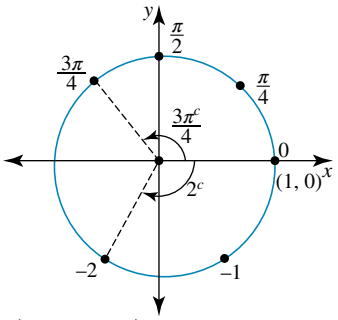
e. $-\dfrac{11\pi}{12} - \dfrac{3\pi}{4}, -\dfrac{\pi}{4}, -\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

f. 58.28, 148.28, 238.28

g. $\dfrac{3\pi}{2}$

h. $\dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}$

17. a.



b. $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

c. $(-0.42, -0.91)$

d. $\theta = \dfrac{\pi}{4}$

e. i. $\left(\dfrac{5\pi}{4} - 2\right)$ radians

ii. 110.41°

f. $-2\pi - 2, \dfrac{11\pi}{4}$ (other answers are possible)

18. a. $3\sqrt{3}$ km^2

b. 1767 m^2

19. a. 12 cm **b.** 1.176^C **c.** 1.966^C **d.** 1.176^C

e. 62.5 cm

20. a. 1 second

b. 30 decibels

c. $a = -30$, $b = 2\pi$, $c = 70$, $d = 70 - 30\cos(2\pi t)$

d. 12.4%

e. 600

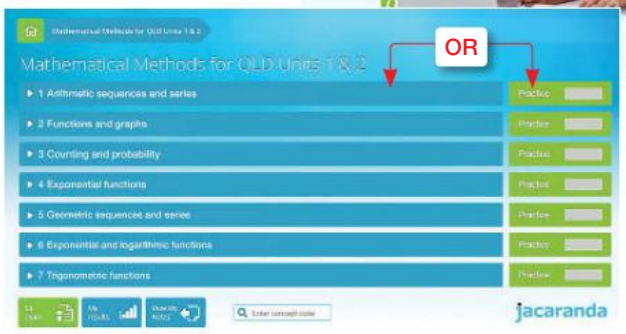
f. 39.2%

TOPIC 3 Trigonometric functions 1

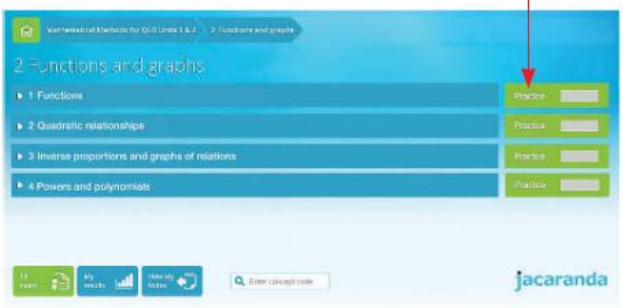
- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



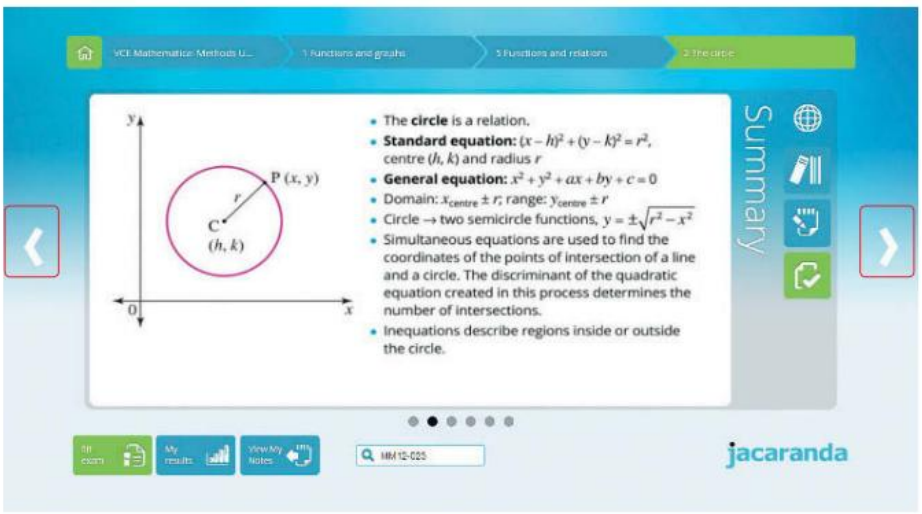
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 11
Rates of change

11.1 Overview

11.1.1 Introduction



A rate, or rate of change, indicates how one quantity varies in relation to another. A key clue that you are dealing with a rate is the word 'per' in the units, for example, metres per second. Speed is a rate and tells you how your distance has changed over a period of time. But how is it telling you the speed right now? How can you find how something is changing at one point? If the model developed is linear then the rate is constant, it always changes by the same amount, so that's easy. But not all models are linear, and so determining how they are changing is not so straightforward.

Two men, Sir Isaac Newton and Gottfried Leibniz, both claimed to have discovered the solution to this issue in the mid-1600s (both accused the other of stealing their idea and fought bitterly right up until their deaths). And they did it with an idea that doesn't sound like good mathematics at all. What if, instead of working the rate out at one point you use two, but make those points so close together that no one notices the difference. And from their work was born the current field of study called calculus.

Any time something changes its size, location, or any other characteristic, you can model that change to better understand or replicate it. From improving the performance of a Formula 1 engine to calculating the volume of concrete needed for a bridge, from keeping a refrigerator at the right temperature to determining the optimal amount of fuel to have for a rocket launch, understanding change is a part of life.

Proportional–integral–derivative (PID) controllers are simple microprocessors that use calculus principles to control a wide range of modern devices including coffee machines, air conditioners and elevators.

LEARNING SEQUENCE

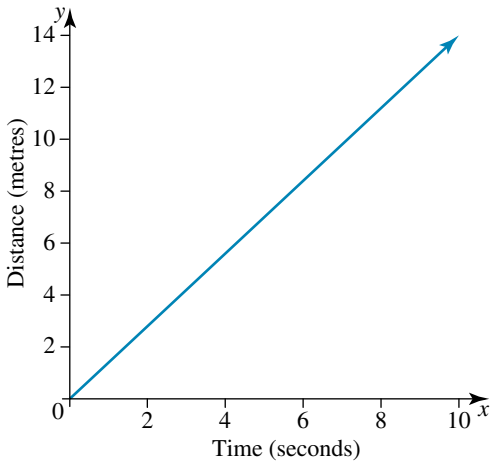
11.1 Overview
11.2 Exploring rates of change
11.3 The difference quotient
11.4 Differentiating simple functions
11.5 Interpreting the derivative
11.6 Review: exam practice

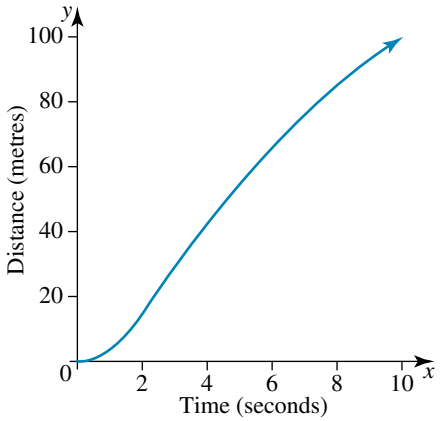
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.2 Exploring rates of change

11.2.1 Constant and variable rates of change

It is useful for us to consider change as either variable or constant. A situation with a constant rate of change could be a person walking at a constant speed. They start walking at 5 km/h and end walking at 5 km/h and stay at 5 km/h the whole time in-between. But a sprinter's speed is a variable rate of change. They initially rush up to a high speed and then gradually lose speed as they head to the finish. Graphs of the distance versus time for these two situations could look like the graphs below.





As can be seen, constant rates have a linear function while variable rates have a non-linear function. The rate of change is given by the gradient of the function. For linear functions the gradient is always the same and calculated by $m = \frac{\text{rise}}{\text{run}}$. For variable rates of change we need to work out what the gradient is at any particular point on the graph. There are two ways we can consider the rate of change, or gradient, of a function when it is changing.

11.2.2 Average rates of change

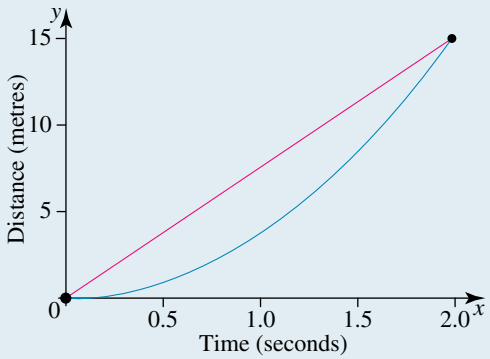
One way is to work out what the average gradient is, or the average rate of change, between two points. Basically, we work out the equivalent constant rate of change that would give the same result as the variable rate of change. Consider the graph of our sprinter shown above. Over the course of the race he covered 100 m in 10 seconds. Average rate of change:

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{100 - 0}{10 - 0} \\ &= 10 \text{ m/s}. \end{aligned}$$

We can therefore conclude that on average he ran at 10 m/s. In other words, if he had run at a constant speed of 10 m/s he would have ended at the same distance in the same time.

WORKED EXAMPLE 1

Calculate the average rate of change for the sprinter over the first 2 seconds.



THINK

1. From the graph identify the distance the sprinter covered in the first 2 seconds.
2. Calculate the gradient and use it to state the average rate of change. Include appropriate units.
3. Write the answer.

WRITE

The sprinter covered 15 m in the first 2 seconds.

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{15 - 0}{2 - 0}$$
$$= 7.5 \text{ m/s}$$

The average rate of change over the first 2 seconds is 7.5 m/s.

We can also determine the average rate of change from a table or a function.

WORKED EXAMPLE 2

Find the average rate of change of height between $t = 1$ and $t = 3$ from the table below.

t (min)	0	1	2	3	4	5
d (m)	20	60	90	130	140	145

THINK

Calculate the average rate of change of height with respect to time by considering the change in each quantity. When the time changes from $t = 1$ min to $t = 3$ min, the height changes from 60 m to 130 m.

WRITE

$$\text{Average rate of change of height} = \frac{\text{change in height}}{\text{change in time}}$$
$$= \frac{(130 - 60)\text{ m}}{(3 - 1)\text{ min}}$$
$$= \frac{70\text{ m}}{2\text{ min}}$$
$$= 35\text{ m/min}$$

WORKED EXAMPLE 3

Over a period of 6 hours, the temperature of a room is described by the function $T(h) = h^2 - 4h + 22$ where T is the temperature in degrees Celsius after h hours.

a. What is the initial temperature of the room?

b. Sketch the graph of the function over the given time interval.

c. Draw a chord between the points where $h = 1$ and $h = 5$.

d. What is the gradient of this chord?

e. What is the average rate of change of temperature between $h = 1$ and $h = 5$?

THINK

a. Initial temperature is the temperature at the start of the time period. Substitute $h = 0$ into the function to find $T(0)$.

b. Use a graphics calculator or rewrite the function in turning point form.

c. 1. Find the required points:
$T(1) = 1^2 - 4(1) + 22 = 19$
$T(5) = 5^2 - 4(5) + 22 = 27$.

2. Indicate the points (1, 19) and (5, 27) on the graph and join with a straight line.

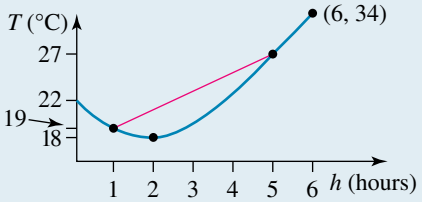
d. Use gradient $= \frac{\text{rise}}{\text{run}}$ and the points (1, 19) and (5, 27).

e. Use the gradient to state the average rate of change. Include appropriate units.

WRITE

a. When $h = 0$, $T(0) = 0 - 0 + 22$
$= 22$
Initial temperature is 22 °C.

b. $T(h) = h^2 - 4h + 22$
$= (h^2 - 4h + 4) - 4 + 22$
$= (h - 2)^2 + 18$
Turning point of parabola is (2, 18).

c 

d. Gradient $= \frac{27 - 19}{5 - 1}$
$= \frac{8}{4}$
$= 2$

e. Average rate of change is 2 °C/h.

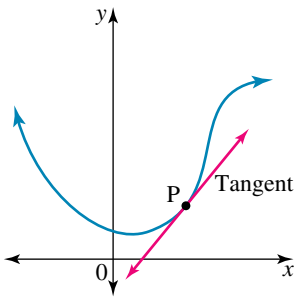
Resources

Interactivity: Rate of change (int-5960)

11.2.3 Instantaneous rates of change

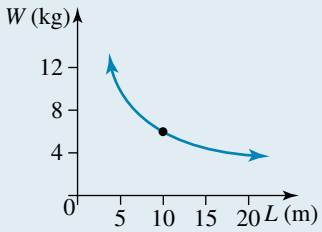
The other way to determine the gradient of a function with a variable rate of change is to calculate it at a specific point. We cannot simply use the gradient equation because there is only one point. The rate of change at a point is the gradient of the tangent at that point.

If we construct a tangent line then we can select points on it and use them to find the gradient of the tangent. The difficulty is working out exactly where to position the tangent.



WORKED EXAMPLE 4

a. Use the following graph to find the gradient of the tangent at the point where $L = 10$.
b. Hence, find the rate of change of weight, W, with respect to length, L, when $L = 10$.



THINK

a. 1. Draw in the required tangent.

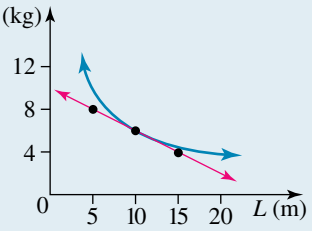
2. To find the gradient of the tangent, choose a convenient interval (between the points where $L = 5$ and $L = 15$).

3. Use gradient $= \frac{\text{rise}}{\text{run}}$.

b. Use the gradient to state the rate of change. Include appropriate units.

WRITE

a.



$$\text{Gradient} = \frac{4 - 8}{15 - 5}$$
$$= \frac{-4}{10}$$
$$= -0.4$$

b. Rate $= -0.4$ kg/m
The weight is decreasing with respect to length at a rate of 0.4 kg/m.

studyon

Units 1 & 2 > Area 8 > Sequence 1 > Concept 1

Rates of change Summary screen and practice questions

Exercise 11.2 Exploring rates of change

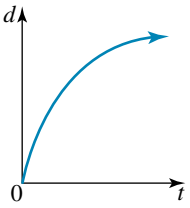
Technology free

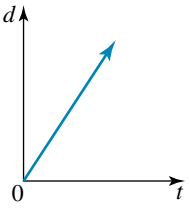
1. Identify the following rates as constant or variable:
- **a.** a person's pulse rate when running 3 km
- **b.** the rate of growth of Australia's population
- **c.** a person's pulse rate when lying down
- **d.** the daily hire rate of a certain car
- **e.** the rate of growth of a baby
- **f.** the rate of temperature change during the day

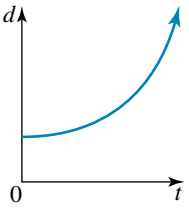


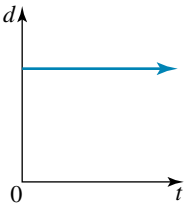
g. the commission rate of pay of a salesperson
h. the rate at which the Earth spins on its axis
i. the rate at which students arrive at school in the morning
j. the rate at which water runs into a bath when the tap is left on
k. the number of hours of daylight per day.

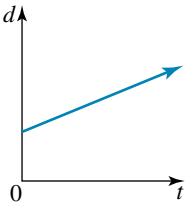
2. Identify whether the following graphs show constant or variable rates of change.

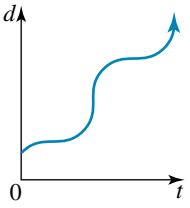
a. 

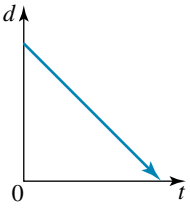
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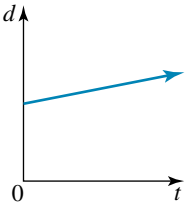
c. 

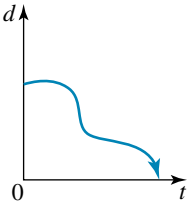
d. 

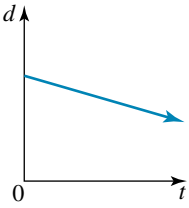
e. 

f. 

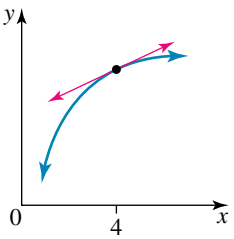
g. 

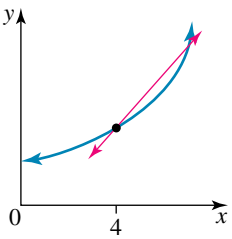
h. 

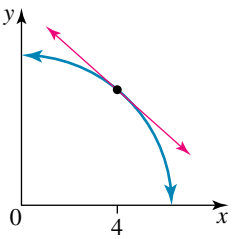
i. 

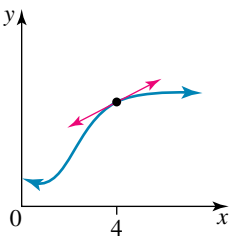
j. 

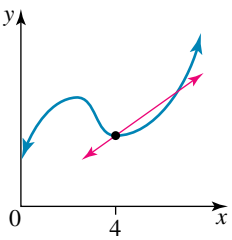
3. Identify which of the following graphs have a tangent drawn at the point where $x = 4$.

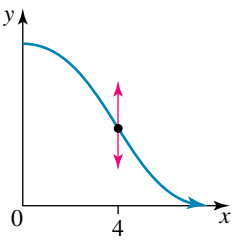
a. 

b. 

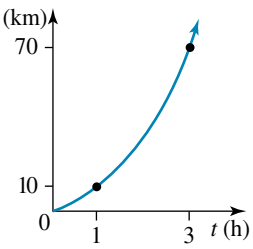
c. 

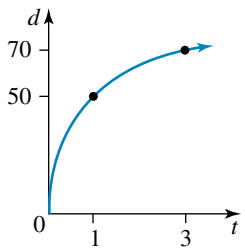
d. 

e. 

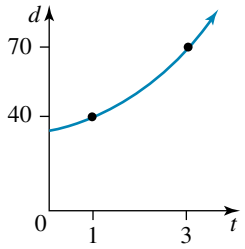
f. 

4. **WE1** For each of the distance–time graphs below:
i. draw a chord to the graph for the interval $t = 1$ to $t = 3$
ii. calculate the gradient of this chord
iii. hence, determine the average speed from $t = 1$ to $t = 3$.

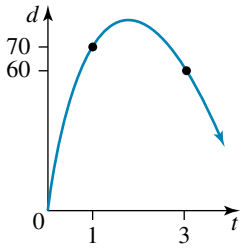
a. 

b. 

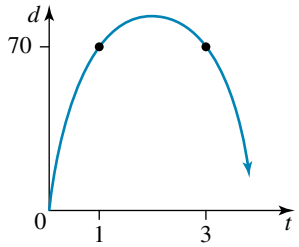
c.



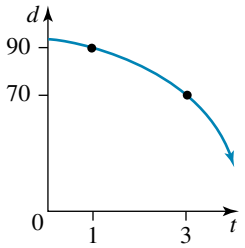
d.



e.

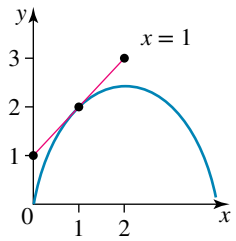


f.

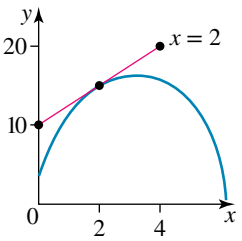


5. **WE4** In each of the following graphs calculate an approximate value for the gradient of the tangent at the point indicated.

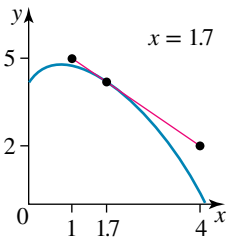
a.



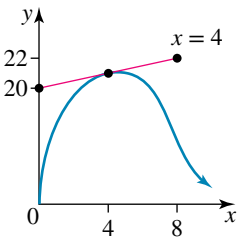
b.



c.



d.



6. The total number of people at the zoo at various times of the day is shown in the table below.

	am		pm					
***T* (time of day)**	10.00	11.00	12.00	1.00	2.00	3.00	4.00	5.00
***N* (number of people at the zoo)**	0	200	360	510	540	550	550	550

a. Plot the graph of N versus T.

b. Draw chords to the graph for the interval:

i. 10.00 am to 1.00 pm

ii. 1.00 pm to 3.00 pm

iii. 3.00 pm to 5.00 pm.

c. Calculate the gradient of each of these chords.

d. Determine the average rate of change from:

i. 10.00 am to 1.00 pm

ii. 1.00 pm to 3.00 pm

iii. 3.00 pm to 5.00 pm.

e. Briefly describe what these rates suggest about the number of people attending the zoo during the course of the day.



7. WE2 The height, h metres, reached by a balloon released from ground level after t minutes, is shown in the table below.

t (min)	0	2	4	6	8	10
h (m)	0	220	360	450	480	490

   a. Without drawing the graph, calculate the average rate of change of height with respect to time between:
      i. $t = 0$ and $t = 2$
      ii. $t = 2$ and $t = 4$
      iii. $t = 4$ and $t = 6$
      iv. $t = 6$ and $t = 8$
      v. $t = 8$ and $t = 10$.
   b. Is the average rate of change for each 2-minute interval increasing or decreasing?

8. The following tables of values show distance travelled, d km, at various times, t hours. Decide whether the rate of change of distance with respect to time appears 'constant' or 'variable'. Justify your answer.

   a.

t	0	1	2	3	4
d	0	5	10	15	20

   b.

t	0	1	2	3	4
d	0	15	20	45	80

   c.

t	0	1	2	3	4
d	0	6	12	18	24

   d.

t	0	1	2	3	4
d	0	4	12	24	40

   e.

t	0	1	2	3	4
d	0	1.5	4	8.5	11

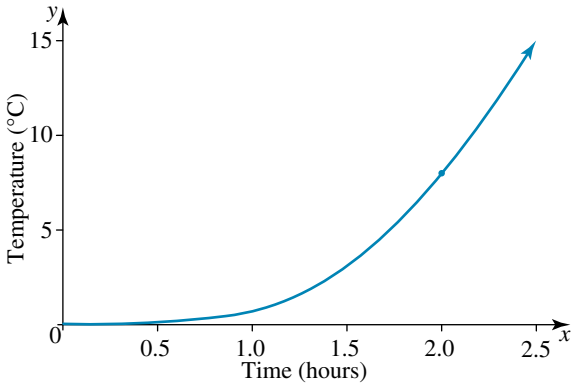
   f.

t	1	2	3	4	5
d	6	9	13	16	20

9. Describe each of the rates below as either an instantaneous rate (I) or an average rate (A).
   a. Bill walked 12 kilometres in 2 hours.
   b. An aircraft leaves the runway at 270 km/h.
   c. A household used 560 litres of water in one day.
   d. The pulse rate of a runner as he crosses the finish line
   e. A gas heater raises the temperature of a room by 10 °C in half an hour.
   f. A baby put on 300 g in one week.
   g. A road drops 20 m over a distance of 100 m.
   h. Halfway along a flying fox, Jill is travelling at 40 km/h.
   i. The maximum speed of a power drill is 320 revolutions per minute.
   j. Water flows through a fire hose at 60 litres per minute.

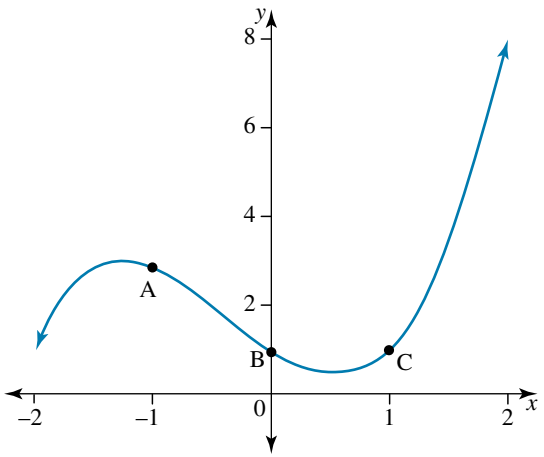


10. Murray calculates that the rate of change at $t = 2$ for the graph shown below is 4 °C/hour.
   a. Determine what error he has made.
   b. Calculate a more reasonable value.

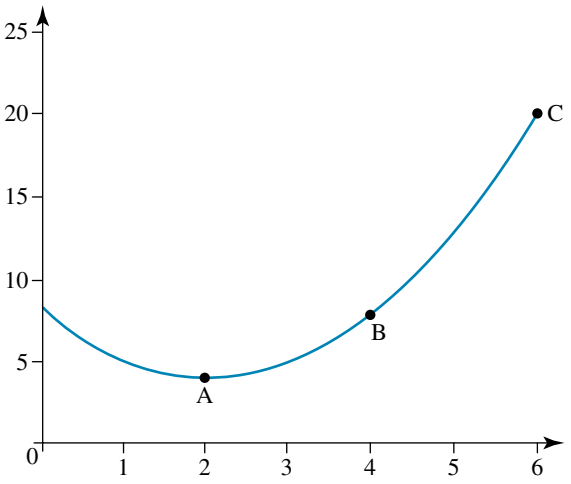


11. Calculate the average rates of change between Points A and B, B and C, and A and C for the graphs below. Draw a conclusion as to a general relationship that exists between the three gradients.

a.

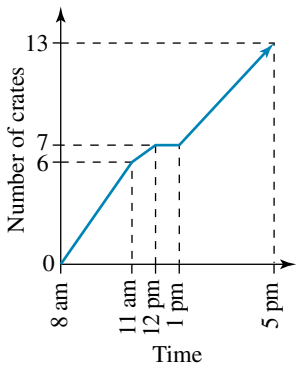


b.



12. The number of crates of fruit picked by a fruit-picker over the course of a day is shown in the graph.





If the fruit-picker is paid $12 per crate, answer the following.

a. What is the rate of pay per hour in the first 3 hours?

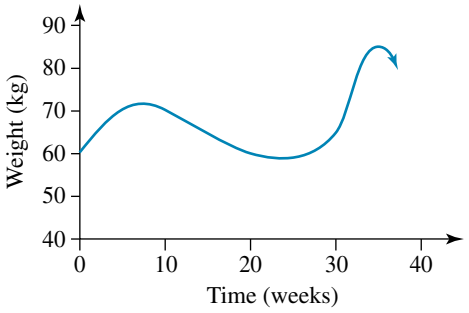
b. Explain what probably happened between 12.00 pm and 1.00 pm.

c. What is the rate of pay per hour in the last 4 hours?

d. Suggest two possible reasons why the line is not as steep in the afternoon.

e. How much is earned for the day?

13. The weight of a person over a 40-week period is illustrated in the graph at right.



- **a.** Estimate the maximum weight and the time at which it occurs.
- **b.** Estimate the average rate of change of weight between week 10 and week 20.
- **c.** Estimate when the person was experiencing their greatest increase in weight.
- **d.** Discuss whether in this situation average or instantaneous rates of change of weight would be of more significance to the person.

Technology active

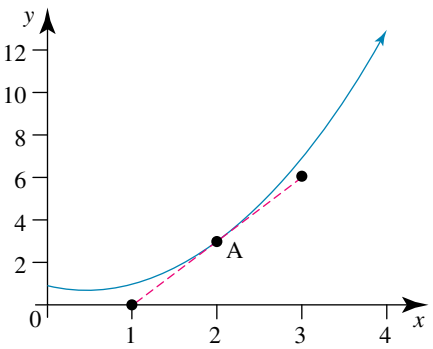
14. **WE3** The temperature of an iron rod placed in a furnace is described by the function $T(t) = t^2 + 20$ between $t = 0$ and $t = 10$.

T represents the temperature of the rod in degrees Celsius and t is the time in seconds.

- **a.** What is the initial temperature of the iron rod?
- **b.** Sketch the graph of the function over the given time interval.
- **c.** Draw a chord between the points where $t = 2$ and $t = 8$.
- **d.** What is the gradient of this chord?
- **e.** What is the average rate of change of temperature between $t = 2$ and $t = 8$?

15. Jasmine and Jesse drive from Abingdon to Boulia at a speed of 90 km/h, and from Boulia to Clarenvale at an average speed of 100 km/h. If the total time taken for the journey from Abingdon to Clarenvale was 2 hours, and the average speed for the entire journey was 96 km/h, how far is it from Abingdon to Boulia?

16. **a.** Use the tangent provided to calculate the instantaneous rate of change at point A ($x = 2$) of the function $f(x) = x^2 - x + 1$.



- **b.** Investigate the relationship between the instantaneous rate of change at $x = 2$ and the average rate of change by calculating the average rates of change to complete the table below.

1.8 to 2	1.9 to 2	2	2 to 2.1	2 to 2.2

Explain your observations.

- **c.** Generate a method for calculating a close approximate value for the instantaneous rate of change at any value. Test your method on the following functions.
  - **i.** $f(x) = 3x^2 + x - 1$ at $x = 3$
  - **ii.** $f(x) = x^3 - 0.5x^2 + 3.2x - 1.7$ at $x = 5$

11.3 The difference quotient

11.3.1 The limit



In mathematics it is important to understand the concept of a limit. This concept is especially important in the study of calculus. In everyday life we use the term *limit* to describe a restriction put on a quantity. For example, the legal blood alcohol concentration limit for a driver is normally 0.05 g/100 mL. As the number of standard alcoholic drinks consumed in 1 hour approaches 2, the average adult male's blood alcohol concentration approaches 0.05. Likewise, some time after a celebration, a person who has been drinking heavily at an earlier time may have a blood alcohol concentration which is approaching the legal limit of 0.05 from a higher level as the number of drinks not yet metabolised by their body approaches 2. We could say that as the number of standard drinks remaining in the body approaches 2 the blood alcohol concentration approaches 0.05. In essence the blood alcohol concentration is a function, say $f(x)$, of the number of drinks, x, remaining in the body.

WORKED EXAMPLE 5

Add the following series of numbers and state what value it is approaching.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$

THINK	WRITE
1. Add the first 2 terms.	Sum of first 2 terms is $\frac{3}{4}$ $(= 0.750)$.
2. Add the first 3 terms.	Sum of first 3 terms is $\frac{7}{8}$ $(= 0.875)$.
3. Add the first 4 terms.	Sum of first 4 terms is $\frac{15}{16}$ (≈ 0.938).
4. Add the first 5 terms.	Sum of first 5 terms is $\frac{31}{32}$ (≈ 0.969).
5. Add the first 6 terms.	Sum of first 6 terms is $\frac{63}{64}$ (≈ 0.984).
6. Give the upper limit.	The sum is approaching 1.

Expressing limits in mathematical language, we say that a limit can be used to describe the behaviour of a function, $f(x)$, as the independent variable, x, approaches a certain value, say a. In some cases the function will not be defined at a. As an example, the notation:

$$\lim_{x \to t} f(x) = 10t$$

is read as 'the limit of $f(x)$ as x approaches t is equal to $10t$'.

WORKED EXAMPLE 6

By investigating the behaviour of the function $f(x) = x + 3$ in the vicinity of $x = 2$, show that $\lim_{x \to 2} f(x) = 5$.

THINK

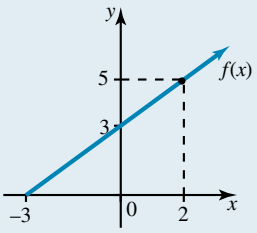
1. Create a table of values for x and $f(x)$ in the vicinity of $x = 2$.

WRITE

x	1.95	1.99	1.995	2	2.005	2.01	2.05
$f(x)$	4.95	4.99	4.995	5	5.005	5.01	5.05

2. Consider the values taken by $f(x)$ as x approaches 2.

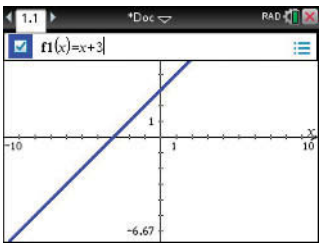
As x approaches 2 from the left and the right, $f(x)$ approaches a value of 5.
So $\lim_{x \to 2} f(x) = 5$.



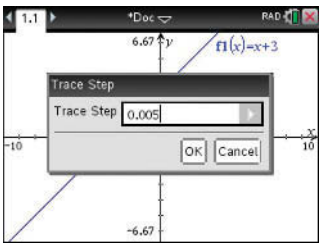
TI | THINK

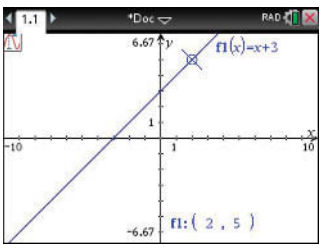
1. On a Graphs page, complete the entry line for function 1 as $f1(x) = x + 3$ then press ENTER.

WRITE



2. Press MENU then select
   5: Trace
   3: Trace Step …
   Set the Trace Step to 0.005 then select OK.
   Press MENU then select
   5: Trace
   1: Graph Trace.
   Type '1.95' and press ENTER, then use the right arrow to investigate the behavior of the function as x approaches 2 from the left.
   Type '2.05' and press ENTER, then use the left arrow to investigate the behavior of the function as x approaches 2 from the right.

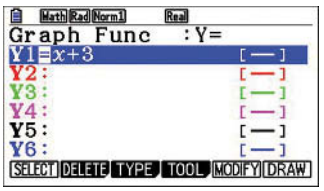


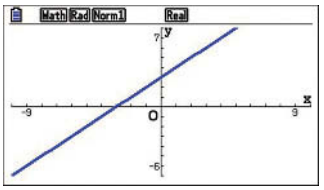


CASIO | THINK

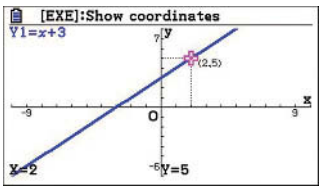
1. On a Graph screen, complete the entry line for Y1 as $Y1 = x + 3$ then press EXE. Select DRAW by pressing F6.

WRITE





2. Select Trace by pressing SHIFT then F1. Use the right arrow to investigate the behavior of the function as x approaches 2 from the left.
   Use the left arrow to investigate the behavior of the function as x approaches 2 from the right.



3. The answer appears on the screen.

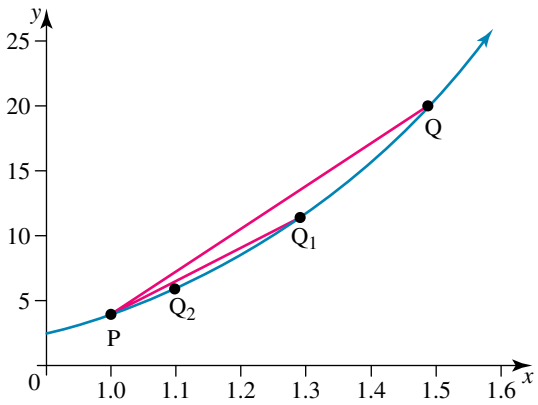
As x approaches 2 from the left and the right, $f(x)$ approaches a value of 5. So $\lim_{x \to 2} f(x) = 5$.

3. The answer appears on the screen.

As x approaches 2 from the left and the right, $f(x)$ approaches a value of 5. So $\lim_{x \to 2} f(x) = 5$.

11.3.2 The gradient as a limit

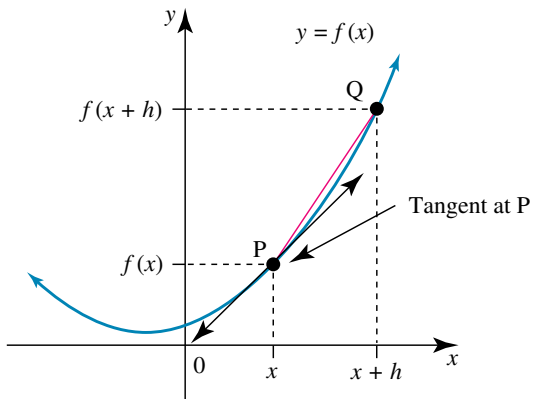
We can use this same approach to get a better idea of the value of our gradient, or instantaneous rate of change, using the average rate of change. Consider the graph shown. As the distance between two points P and Q gets smaller and smaller, $Q \to Q_1 \to Q_2$, we can see that the gradient between the points gets closer and closer to the gradient of the tangent at P, the instantaneous gradient.



To represent this situation, we make the difference between the x-values of P and Q approach 0. It is important at this point to reiterate that we are not making the difference 0, that would make them the same point. We use the letter h to represent this small change in the x-value.

Therefore, for any function $f(x)$, P is the point $(x, f(x))$ and Q is the point $(x + h, f(x + h))$. So, if we use our gradient formula to find the gradient between our close points we get the following.

$$\begin{aligned} \text{PQ} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$



The closer h gets to 0 the closer the two points are to each other and the more accurate our value for the instantaneous rate of change is.

WORKED EXAMPLE 7

For the function $f(x) = x^2 + 2x + 1$:

a. Complete the following table of values.

x	1.1	1.01	1.001	1.0001	1
y					

b. Using $x = 1$ as the initial point, state the value of h for each other point.

c. Calculate the average rate of change from $x = 1$ to:

i. $x = 1.1$ **ii. $x = 1.01$** **iii. $x = 1.001$** **iv. $x = 1.0001$.**

d. Predict the gradient of the tangent to the curve at $x = 1$.

THINK	WRITE
a. Substitute $x = 1.1, 1.01, 1.001, 1.0001, 1$ into $f(x) = x^2 + 2x + 1$.	$f(1.1) = (1.1)^2 + 2(1.1) + 1$ $= 4.41$ $f(1.01) = (1.01)^2 + 2(1.01) + 1$ $= 4.0401$ $f(1.001) = (1.001)^2 + 2(1.001) + 1$ $= 4.004$ $f(1.0001) = (1.0001)^2 + 2(1.0001) + 1$ $= 4.0004$ $f(1) = (1)^2 + 2(1) + 1$ $= 4$
b. Subtract $x = 1$ from each x-value.	$h_1 = 1.1 - 1$ $= 0.1$ $h_2 = 1.01 - 1$ $= 0.01$ $h_3 = 1.001 - 1$ $= 0.001$ $h_4 = 1.0001 - 1$ $= 0.0001$
c. Substitute the values into the gradient formula.	$m_1 = \frac{f(1.1) - f(1)}{1.1 - 1}$ $= \frac{4.41 - 4}{0.1}$ $= 4.1$ $m_2 = \frac{f(1.01) - f(1)}{1.01 - 1}$ $= \frac{4.0401 - 4}{0.01}$ $= 4.01$ $m_3 = \frac{f(1.001) - f(1)}{1.001 - 1}$ $= \frac{4.004 - 4}{0.001}$ $= 4.001$ $m_4 = \frac{f(1.0001) - f(1)}{1.0001 - 1}$ $= \frac{4.0004 - 4}{0.0001}$ $= 4.0001$
d. State the answer.	The gradient at $x = 1$ appears to be approaching 4.

How close is close?

When we say close we really mean it. So far we've used differences like 0.0001 as 'close', but to get a reliable result that's not close enough. When we say that 'h approaches 0', how close to 0 do we need to get? Your chance of winning lotto is about 0.000000006 or 6×10^{-9}; that's not close enough. The probability of correctly guessing a random atom in the universe is about 10^{-82}; that's still too big. To say that h approaches 0 is to say that any number that can be written, regardless of how small, is still too big.

Exercise 11.3 The difference quotient

Technology free

1. **WE5** Add the following series of numbers and state what value it is approaching.

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$$

2. The diagram at right shows regular polygons with 3, 4 and 5 sides. As the number of sides gets very large ($\rightarrow \infty$), what shape emerges?



$n = 3$ $n = 4$ $n = 5$

3. **a.** Find the value of $\frac{1}{n}$ as n gets infinitely large.
   b. Write this using limit notation.
4. **WE6** By investigating the behaviour of the function $f(x) = x + 5$ in the vicinity of $x = 3$ show that $\lim_{x \to 3} f(x) = 8$.
5. **WE7** **a.** If $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ and n represents the number of terms to be summed in the series, copy and complete the following table:

n	1	2	3	4	5	6	10
S	1	$1\frac{1}{2}$					

   b. **MC** Which of the following is equal to $\lim_{n \to \infty} S$?

   A. 1.75 **B.** 1.95 **C.** 2 **D.** 1

6. For the equation $y = \frac{x^2 - 4}{x - 2}$:
   a. calculate y at $x = 2$
   b. investigate the behaviour of the function at values near $x = 2$
   c. predict the $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
   d. comment on the difference between your results in **a** and **c**.
7. **a.** Determine the value being approached by the sequence $\sum_{n=0}^{\infty} \frac{1}{n!}$.
   b. Predict the $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ using large values of n.
   c. Comment on your results.

8. For $f(x) = x^2 + 3x$:

a. calculate the gradient $m = \dfrac{f(x+h) - f(x)}{h}$ for $x = 1$ and $h = 0.1,\ 0.01,\ 0.001$

b. use these results to predict the gradient at $x = 1$.

9. For $f(x) = \dfrac{x^2 - 5x}{x}$:

a. state the value, a, for which the function is undefined

b. examine the $\lim\limits_{x \to a} \dfrac{x^2 - 5x}{x}$ using a table of values approaching a.

10. The temperature of a meteor, T °C, t minutes after reaching the Earth's atmosphere is $T = 2t^3 + 5t^2 + 200$ where $t \in [0,\ 10]$.

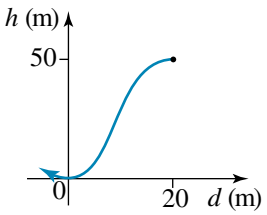
Estimate the rate of change of the temperature of the meteor after:

a. 1 minute

b. 10 minutes.



11. The height, h metres, of a roller-coaster is given by $h = -\dfrac{1}{80}d^3 + \dfrac{3}{8}d^2$ where d is the horizontal distance from 0, in metres.



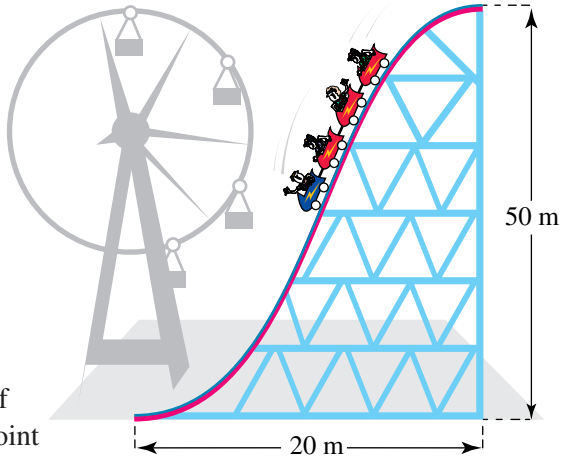
Estimate the gradient of the curve describing the path of the roller-coaster, accurate to 2 decimal places, at the point where d is:

a. 10 metres

b. 15 metres

c. 20 metres

d. 0 metres.





12. The gradient, m, of a function of the form $f(x) = ax^2$ is 3 at the point where $x = 2$. Simplify the equation $m = \dfrac{f(x+h) - f(x)}{h}$ and then choose an extremely small value for h to determine a likely value for a.

Technology active

13. For the function $f(x) = x^3 - 3x^2 + 2$:

a. Complete the following table of values.

x	2.6	2.51	2.501	2.5001	2.5
y					

b. Using $x = 2.5$ as the initial point, state the value of h for each other point.

c. Calculate the average rate of change from $x = 2.5$ to:

i. $x = 2.6$ **ii.** $x = 2.51$ **iii.** $x = 2.501$ **iv.** $x = 2.5001$.

d. Predict the gradient of the tangent to the curve at $x = 2.5$.

14. For the function $f(x) = 0.5x^2 - 3x + 1$:

a. Complete the following table of values.

x	2.1	2.01	2.001	2.0001	2
y					

b. Using $x = 2$ as the initial point state the value of h for each other point.

c. Calculate the average rate of change from $x = 2$ to:

i. $x = 2.1$ **ii.** $x = 2.01$ **iii.** $x = 2.001$ **iv.** $x = 2.0001$.

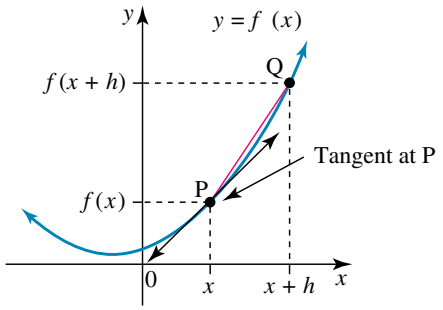
d. Predict the gradient of the tangent to the curve at $x = 2$.

11.4 Differentiating simple functions

We have seen so far that we can get an approximation for the value of the gradient at a point by finding the average gradient between two points that are very close. To improve the accuracy of our result we need to get our points so close that there's so little difference between them we can't tell the difference. Only then can we be comfortable that we've found the gradient at a point, the instantaneous rate of change.

To represent this situation, we make the difference between the x-values of P and Q, known as h, approach 0. It is important at this point to reiterate that we are not making the difference 0, that would make them the same point. There is still a difference; h is just infinitesimally small. As we have considered previously, for any function $f(x)$, P is the point $(x, f(x))$ and Q is the point $(x + h, f(x + h))$. So, if we use our gradient formula to find the gradient between our extremely close points we get:

$$\begin{aligned} \text{PQ} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$



Now, h is getting extremely close to 0, without equalling 0. Mathematically, we write this as $\lim_{h \to 0}$, which is read as 'the limit as h approaches 0'. We can therefore say that the gradient of the tangent at P is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \; h \neq 0,$$

where $f'(x)$ denotes the gradient of a tangent at any point, x, on the graph of $f(x)$.

That is, $f'(x)$ is the gradient function of $f(x)$.

The process of finding the gradient function $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is called *differentiation from first principles*, or *finding the derivative from first principles*.

Instead of writing a function as, for example, $f(x) = x^2 + 1$ we may write it as $y = x^2 + 1$. If writing a function in the form $y = \ldots$ then the derivative is typically written as $\frac{dy}{dx}$ (Leibniz notation). This notation is read as 'the change in y with respect to x'. The derivative of y may sometimes be written as y', similar to the function notation.

We can use differentiation from first principles to derive the function for the gradient at any point.

WORKED EXAMPLE 8

Calculate the derivative of $x^2 - 2x$ using first principles.

THINK	WRITE
1. Define $f(x)$.	$f(x) = x^2 - 2x$
2. The derivative is equal to: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
3. Simplify the numerator $f(x+h) - f(x)$.	$f(x+h) - f(x)$ $= (x+h)^2 - 2(x+h) - (x^2 - 2x)$ $= x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x$ $= 2xh + h^2 - 2h$
4. Factorise the numerator $f(x+h) - f(x)$.	$= h(2x + h - 2)$
5. Simplify $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ by cancelling the common factor of h.	$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $= \lim_{h \to 0} \frac{h(2x + h - 2)}{h}$ $= \lim_{h \to 0} (2x + h - 2),\ h \neq 0$
6. Evaluate the limit by substituting $h = 0$.	$= 2x - 2.$

WORKED EXAMPLE 9

If $g(x) = 2x^2 + 5x - 2$, calculate:
a. the gradient function using first principles
b. the value(s) of x where the gradient equals 0.

THINK	WRITE
a. 1. Let $g(x) = 2x^2 + 5x - 2$.	a. $g(x) = 2x^2 + 5x - 2$
2. The derivative is equal to: $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$.	$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

3. Simplify the numerator $g(x+h) - g(x)$.

$$\begin{aligned} & g(x+h) - g(x) \\ &= 2(x+h)^2 + 5(x+h) - 2 - (2x^2 + 5x - 2) \\ &= 2(x^2 + 2xh + h^2) + 5x + 5h - 2 - 2x^2 - 5x + 2 \\ &= 2x^2 + 4xh + 2h^2 + 5x + 5h - 2 - 2x^2 - 5x + 2 \\ &= 4xh + 2h^2 + 5h \end{aligned}$$

4. Factorise the numerator $g(x+h) - g(x)$.

$$= h(4x + 2h + 5)$$

5. Simplify $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$ by cancelling the common factor of h.

$$\begin{aligned} & \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h + 5)}{h} \\ &= \lim_{h \to 0} (4x + 2h + 5), \ h \neq 0 \end{aligned}$$

6. Evaluate the limit by substituting $h = 0$.

$$= 4x + 5$$

So $g'(x) = 4x + 5$.

b. Solve $g'(x) = 0$.

b. $g'(x) = 0$

$$\begin{aligned} 4x + 5 &= 0 \\ 4x &= -5 \\ x &= -\frac{5}{4} \end{aligned}$$

So the gradient equals 0 when $x = -\frac{5}{4}$.

studyon

Units 1 & 2 > Area 8 > Sequence 1 > Concept 2

Differentiating simple functions from first principles Summary screen and practice questions

Exercise 11.4 Differentiating simple functions

Technology free

1. Calculate the derivative of the following using first principles.
   a. $5x - 7$ **b.** $x^2 + 10x$ **c.** $x^2 - 8x$ **d.** $x^3 + 2x$
2. WE8 **a.** If $f(x) = x^3 - 8$, calculate the gradient function using first principles.
   b. Hence, determine the value(s) of x where the gradient function is equal to 12.
3. WE9 If $g(x) = x^2 - 6x$ calculate:
   a. the gradient function using first principles
   b. the value(s) of x where the gradient equals 0.
4. Derive the following functions and calculate where their gradient is equal to -1.
   a. $7x + 5$ **b.** $x^2 + 4x$ **c.** $x^2 - 3x + 2$ **d.** $x^3 - 5$.
5. Describe the relationship between the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ and the derivative formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
6. Sketch an appropriate function and use it to support a reasonable explanation as to why using an h-value of 0.001 would not be close enough to 0 to get a good approximation of the instantaneous gradient.

7. For the function $f(x) = 4x - x^2$:
   a. graph the function
   b. identify the value of x where the gradient is equal to 0
   c. determine the derivative of y using the method of first principles
   d. calculate the derivative at the value identified in **b**.
8. Identify which of the following denote the gradient at any point on a graph? (One or more answers.)
   a. $f'(x)$
   b. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
   c. $\lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}$
   d. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$
   e. $\frac{f(x+h) - f(x)}{h}$
9. **MC** The most accurate method for calculating the gradient when $x = 3$ for the function $f(x) = x^2 + 2x$ is by:
   A. sketching the graph and drawing a tangent at $x = 3$ to calculate the gradient
   B. calculating the gradient of the secant to the curve joining the points where $x = 3$ and $x = 3.1$
   C. calculating the derivative using first principles and evaluating at $x = 3$
   D. guessing.
10. If the denominator of a rational expression is 0 then the expression is undefined. Explain why the gradient function $\frac{f(x+h) - f(x)}{h}$ isn't undefined as h approaches 0.
11. For the function $p(d) = 2 \times 10^4 d^3 - 4 \times 10^5 d^2 + 7 \times 10^6$:
    a. using technology calculate the derivative of $p(d)$
    b. graph the functions of $p(d)$ and $p'(d)$ and compare them.
12. Apply the method of differentiation by first principles to determine the derivative function of $f(x) = \frac{1}{x}$.
13. Examine the functions and their derivatives shown in the table below.

Function	Derivative function
$f(x) = 9$	$f'(x) = 0$
$f(x) = 4x$	$f'(x) = 4$
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = 5x^3$	$f'(x) = 15x^2$
$f(x) = 4x^6$	$f'(x) = 24x^5$

    a. Describe, in everyday language or mathematically, a rule for calculating the derivative without using the method of first principles.
    b. Test your rule by applying it to the following functions. Confirm by differentiating using first principles.
       i. $f(x) = 9x - 3$
       ii. $f(x) = x^2 - 8x + 2$

Technology active

14. Confirm your solutions to **1a–d** using technology.

11.5 Interpreting the derivative

11.5.1 Interpreting the derivative as the instantaneous rate of change

It is important to remember what it is we are determining when we calculate the derivative. The derivative is the function that expresses the instantaneous rate of change at every point on the original function. Many quantities you may be familiar with are defined as a rate of change.

WORKED EXAMPLE 10

Electrical current, i, is the flow of electrical charge in a circuit. The current, measured in amperes, is equal to the change in charge, q measured in coulombs, over the change in time, t, in seconds.

In other words, $i(t) = \frac{dq}{dt}$.

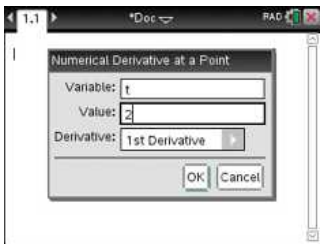
The charge on a metal plate is changing according to the function $q(t) = 9 - t^2, 0 \leq t \leq 3$. Calculate the current flowing across the plate at $t = 2$ seconds.

THINK	WRITE
1. Recognise that current is the derivative of charge with respect to time.	
2. Calculate the derivative function $i(t)$.	$i(t) = q'(t)$ $= \lim_{h \to 0} \frac{9 - (t+h)^2 - (9 - t^2)}{h}$ $= \lim_{h \to 0} \frac{9 - t^2 - 2th - h^2 - 9 + t^2}{h}$ $= \lim_{h \to 0} \frac{-2th - h^2}{h}$ $= \lim_{h \to 0} -2t - h$ $= -2t$
3. Calculate the current at $t = 2$ seconds.	$i(2) = -2(2)$ $= -4$
4. Write the answer. Include appropriate units.	The current at $t = 2$ s is -4 amperes.

TI | THINK

1. On a Calculator page, press MENU then select 4: Calculus
   1: Numerical Derivative at a Point …
   Complete the fields as
   Variable: t
   Value: 2
   Derivative: 1st Derivative
   then select OK. Complete the entry line as
   $\frac{d}{dt}(9 - t^2) \mid t = 2$
   then press ENTER.
2. The answer appears on the screen.

WRITE



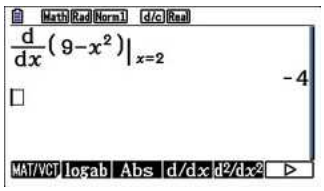


The current at $t = 2$ s is -4 A.

CASIO | THINK

1. On a Run-Matrix screen, select MATH by pressing F4, then select d/dx by pressing F4.
   Complete the entry line as
   $\frac{d}{dx}(9 - x^2) \mid x = 2$
   then press EXE.
2. The answer appears on the screen.

WRITE



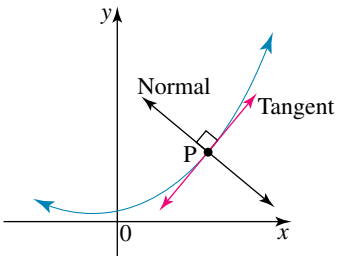
The current at $t = 2$ s is -4 A.

11.5.2 Interpreting the derivative as the gradient of a tangent line

Tangents and normals

As mentioned earlier in this chapter the derivative $f'(x)$ is actually the gradient function. This means that the value of the gradient at any particular point on a curve is equal to the numerical value of the derivative at that point.

Recall that if the gradient of a tangent to a curve at point P is m_T, then the normal, m_N, is a straight line perpendicular (at right angles) to the tangent such that $m_N = -\frac{1}{m_T}$ and passing through the point P as shown at right.



Also recall that the equation of a straight line is given by $y - y_1 = m_T(x - x_1)$ where (x_1, y_1) is the point P, above, and m_T is the gradient.

So now, rather than using a tangent line to find an approximate instantaneous gradient, we can use the instantaneous rate of change to determine the equation of the tangent line.

WORKED EXAMPLE 11

a. Determine the equation of the tangent to the curve $f(x) = x^2 + 6x - 8$ at the point where the gradient has a value of 8.

b. Hence, determine the equation of the normal at this point.

THINK

a. 1. Determine the gradient function of the curve, $f'(x)$.

2. Determine x_1, the value of x where $f'(x) = 8$; that is, solve $2x + 6 = 8$.

3. Find $f(x_1)$ to determine the value of y_1.

WRITE

a. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^2 + 6(x+h) - 8 - (x^2 + 6x - 8)}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 6x + 6h - 8 - x^2 - 6x + 8}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2 + 6h}{h}$

$= \lim_{h \to 0} 2x + h + 6$

$= 2x + 6$

For gradient $= 8$

$2x + 6 = 8$

$2x = 2$

$x = 1$ So $x_1 = 1$.

$y_1 = f(x_1)$

$= f(1)$

$= (1)^2 + 6(1) - 8$

$= -1$

4. Simplify the equation $y - y_1 = m_T(x - x_1)$ to find the equation of the tangent.

b. 1. Find the gradient of the normal using $m_N = -\frac{1}{m_T}$.

2. Simplify the equation $y - y_1 = m_N(x - x_1)$ to find the equation of the normal.

The equation of the tangent at the point $(1, -1)$ is

$$y - (-1) = 8(x - 1)$$
$$y + 1 = 8x - 8$$
$$y = 8x - 9$$

b. $m_N = -\frac{1}{8}$

The equation of the normal at the point $(1, -1)$ is

$$y - (-1) = -\frac{1}{8}(x - 1)$$
$$y + 1 = -\frac{(x - 1)}{8}$$
$$8y + 8 = -x + 1$$
$$x + 8y + 7 = 0$$

studyon

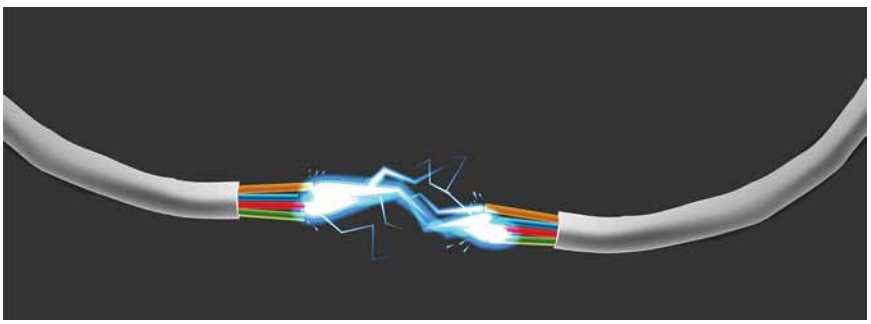
Units 1 & 2 > Area 8 > Sequence 1 > Concept 3

Interpreting the derivative Summary screen and practice questions

Exercise 11.5 Interpreting the derivative

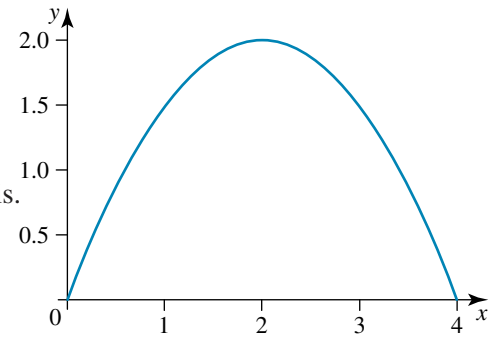
Technology free

1. **WE10** The charge across a metal wire varies according to the function $q(t) = 0.1t + 0.6$. Calculate the current at $t = 3$ s.



2. Determine the equation of the line tangent to the function $f(x) = x^2 - 3x$ at $x = 4$.
3. **WE11** **a.** Determine the equation of the tangent to the curve $f(x) = x^2 - 6$ at the point where the gradient has a value of 6.
   b. Hence, determine the equation of the normal at this point.

4. The graph of the function $y = 2x - 0.5x^2$ is shown.



   a. Calculate the gradient of the tangent at $x = 1$.
   b. Sketch the tangent line on the graph and determine the point where the tangent crosses the y-axis.

5. Determine the equation of the normal to the curve $y = 2x^2 - 2x + 5$ at the point where the curve crosses the y-axis.

6. Electrical power, P, of a circuit is the rate at which electrical energy is transferred. It is measured in watts, W, and calculated by the formula $P = \frac{dw}{dt}$, where w is the energy, in joules, and t is time, in seconds.
   a. Given $w = 0.1t^2 + 0.3t + 0.2$, derive the power function.
   b. Calculate the power output at $t = 6$.
   c. Sketch the energy and power functions on the same axes.
   d. Comment on the relationship between the two functions.

7. The sequence $1, 4, 9, 16, \ldots$ is defined by the equation $a_n = n^2$, where n is a positive integer and a_n the nth value.
   a. Derive $\frac{da}{dn}$, using first principles.
   b. What is the gradient at $n = 1, 2, 3, 4$?
   c. What is the change between the values a_1, a_2, a_3, a_4?
   d. Explain the difference between the gradients in **b** and change in values in **c**.

8. Assume an oil spill from an oil tanker is circular and remains that way.
   a. Write down a relationship between the area of the spill, A m^2, and the radius, r metres.
   b. Determine the rate of change of A with respect to the radius, r.
   c. Calculate the rate of change of A when the radius is:
      i. 10 m ii. 50 m iii. 100 m.
   d. Is the area increasing more rapidly as the radius increases? Why?

9. The height of a ball thrown in the air is given by the function $h(t) = 8t - t^2$. The rate of change of height with respect to time is the velocity.



   a. Derive the function.
   b. Calculate the rate of change at $t = 0, 2, 4, 6$ and 8.
   c. What is the velocity of the ball at $t = 0, 2, 4, 6$ and 8?
   d. Describe the flight of the ball.

10. A balloon is inflated so that its volume, V cm^3, at any time, t seconds later is:

$$V = -\frac{8}{5}t^2 + 24t.$$

   a. Calculate the average rate of change between $t = 0$ and $t = 10$.
   b. Calculate the rate of change of volume when:
      i. $t = 0$ ii. $t = 5$ iii. $t = 10$.
   c. Compare your results from **a** and **b**.

11. A section of roller-coaster track is defined by the function $y = 4x - x^2 + 8, 0 \leq x \leq 4$. The section that joins at $x = 4$ is a straight piece that has to have the same gradient so the pieces join smoothly. What is the equation of the straight piece that joins at $x = 4$?

Technology active

12. A bushfire burns out A hectares of land t hours after it started according to the rule:

$$A = 90t^2 - 3t^3.$$

a. At what rate, in hectares per hour, is the fire spreading at any time, t?

b. What is the rate when t equals:

i. 0 **ii.** 4 **iii.** 8 **iv.** 10

v. 12 **vi.** 16 **vii.** 20?

c. Briefly explain how the rate of burning changes during the first 20 hours.

d. Why isn't there a negative rate of change in the first 20 hours?

e. What happens after 20 hours?

f. After how long is the rate of change equal to 756 hectares per hour?

13. A severe hailstone can produce hailstones with a radius of over 1 cm that fall to the ground such that $d(t) = 4.9t^2$, where d is the distance fallen, in metres, and t is the time, in seconds. The maximum velocity an object falling can reach is called the terminal velocity and is given by the formula $v = \sqrt{\dfrac{2mg}{\rho A C_d}}$, where m is the mass (use 0.22 g), g the acceleration due to gravity (use 980 cm/s^2), ρ the density (for ice use 0.9167), A the projected area (which for a sphere is πr^2) and C_d the coefficient of drag (which is 0.48 for a sphere). Assuming a hailstone is a sphere with $r = 1$ cm, how long does it take and how far has it fallen before it reaches terminal velocity?



14. The charge across a metal surface is recorded each millisecond and recorded in the table below.

t	1	2	3	4	5	6
q	−0.21	−0.015	0.103	0.133	0.089	−0.035

a. Develop a quadratic model for the charge function $q(t)$.

b. Calculate the current at any time, t.

c. Calculate the initial current, at $t = 0$.

d. What is the maximum positive charge reached by the surface?

11.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Which one of the following is a constant rate?
   - **a.** The number of people entering a zoo per hour
   - **b.** The number of days it rains in Brisbane per year
   - **c.** The hourly rate of pay of a tutor
   - **d.** The number of crates of fruit picked per hour by a fruit-picker
   - **e.** The number of patients visiting a doctor per day
2. For the table below, calculate the average rate of change of H between $t = 2$ and $t = 5$.

t(h)	0	1	2	3	4	5	6
H(m)	0	20	40	70	120	190	280

3. **a.** Use the table below to plot the graph of M versus t.

t	0	1	2	3	4
M	−9	−7	−1	9	23

   - **b.** Is the rate of change of M constant or variable? Explain your answer.
   - **c.** Estimate the gradient when $t = 2$.
4. The amount of substance, A kg, in a container at any time, t hours, is $A = t^2 - 3t + 4$, $t \in [0, 5]$.
   - **a.** Calculate the average rate of change during the first 4 hours.
   - **b.** At what rate is the amount changing after 4 hours?
5. **MC** The rate of change of a polynomial $f(x)$ when $x = 3$ is closest to:
   - **A.** $\dfrac{f(3.1) - f(3)}{3.1 - 3}$
   - **B.** $\dfrac{f(3.001) - f(3)}{3.001 - 3}$
   - **C.** $\dfrac{f(3.5) - f(3)}{3.5 - 3}$
   - **D.** $\dfrac{f(3.000\,01) - f(3)}{3.000\,01 - 3}$
6. **MC** The $\lim_{x \to 5}(3x - 7)$ is:
   - **A.** −7 **B.** undefined **C.** 15 **D.** 8
7. **MC** The gradient of the tangent to the curve $f(x)$ at $x = 5$ is:
   - **A.** $\lim_{x \to 0} \dfrac{f(x + h) - f(5)}{h}$
   - **B.** $\lim_{h \to 0} \dfrac{f(5)}{h}$
   - **C.** $\lim_{h \to 0} \dfrac{f(h)}{h}$
   - **D.** $\lim_{h \to 0} \dfrac{f(5 + h) - f(5)}{h}$
8. Find $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$; that is, find $f'(x)$ for $f(x) = 2x + 3$.
9. **MC** If $f(x) = x^2 - 2x$ then $\lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ equals:
   - **A.** $2x - 2$ **B.** $2x + h$ **C.** $2x$ **D.** -2
10. Determine the equation of the line tangent to the function $f(x) = 2x^2 + x$ at $x = 3$.
11. Find the equation of the normal to the curve $y = 6 - x^2$ at the point where the curve crosses the x-axis.

12. Voltage, v, is the difference in electrical potential energy between two points per unit of charge. It is measured in volts, V, and is equal to the change in energy (work done) over the change in charge, $v(q) = \frac{dw}{dq}$.

a. Given $w = 0.2q^2 - 0.1q + 0.5$, derive the voltage function.

b. Calculate the voltage at $q = 2.5$. Use technology of your choice to answer this question.

Complex familiar

13. The height, h metres, of a bird in flight is shown in the table below.



t(s)	0	2	4	6	8	10	12
h(m)	20	18	8	2	4	5	5

a. Calculate the average rate of change of height with respect to time over each 2-second interval.

b. Identify whether the rate of change is increasing or decreasing for each interval.

14. The height, h metres, of a golf ball above the ground at any time, t seconds after it is hit down the fairway, is given by the function $h(t) = 5 + 12t - t^2$.

a. Use a table of values to predict the rate of change of height when t equals:

i. 4 seconds **ii.** 6 seconds **iii.** 10 seconds.

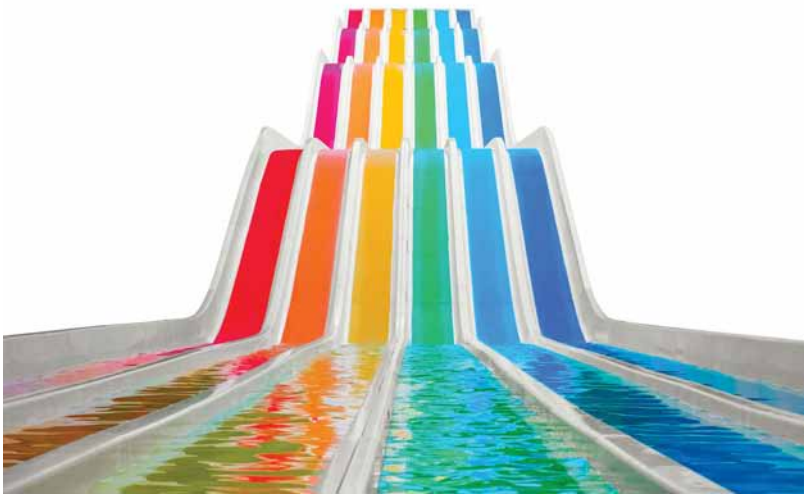
b. Briefly describe the change of height of the golf ball within the first 12 seconds.

15. Differentiate $f(x) = 5 + 4x - 3x^2$ using first principles.

16. The first part of a children's water slide is defined by the function $y = 0.5x^2 - 2x + 2, 0 \leq x \leq 1.5$, where y is the height above ground level, in metres, and x is the distance from the start of the slide, in metres. The second section that joins at $x = 1.5$ is a straight piece that has the same gradient, so the sections join smoothly.

a. What is the equation of the straight piece that joins at $x = 1.5$?

b. What is the horizontal length of the slide?



Complex unfamiliar

17. A vending machine at a railway station dispenses cans of soft drink. It has a capacity of 600 cans when full.



On a particular day:

- The machine is half full at 6.00 am.
- In the next four hours drinks are dispensed at a constant rate of 15 per hour.
- At 10.00 am the machine is filled.
- In the next four hours the machine dispenses an average of 60 cans per hour at a steadily increasing rate.
- Finally, from 2.00 pm to 6.00 pm an average of 30 cans per hour are dispensed at a steadily decreasing rate.

a. Sketch a graph showing the number of cans in the machine over the period from 6.00 am to 6.00 pm.

b. Find the number of cans in the machine at 6.00 pm.

c. Estimate at what time in the afternoon the machine is half full.

d. Estimate the rate at which cans are dispensed at 1.00 pm.

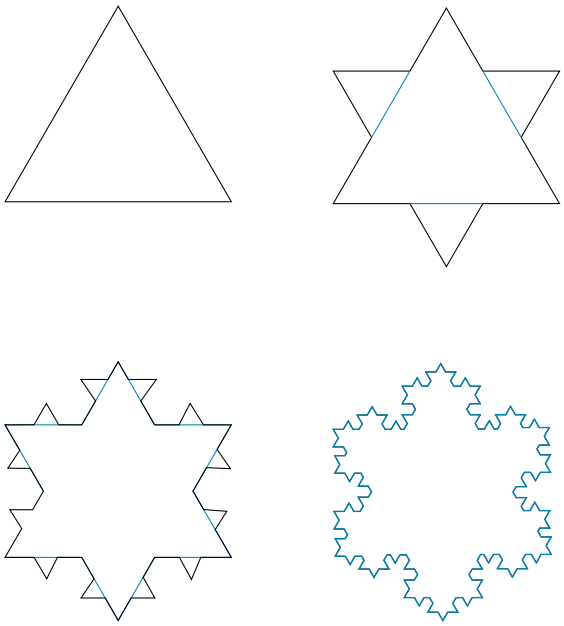
18. The Koch snowflake is a simple example of a fractal. Starting with an equilateral triangle, the steps for constructing a Koch snowflake are as follows.

i. Divide each side into three segments of equal length.

ii. Draw an equilateral triangle pointing outward from the middle segment.

iii. Remove the line segment that is the base of the triangle from step 2.

The first four iterations of the Koch snowflake look like this:



With each new iteration the number of triangles added is given by $t_n = \frac{3}{4} \times 4^n$, and the area of each new triangle is $a_n = \frac{A_0}{9^n}$, where A_0 is the original area of the triangle. Determine an expression for the area of the Koch snowflake, A_n, at each iteration, n, and calculate the area of a Koch snowflake as n approaches infinity.

19. The concentration (x mg/L) of a tranquiliser in a patient's bloodstream at any time t hours after it is administered is given by the following rule.

$$x = \frac{2t}{t^2 + 1}, \ t \geq 0$$

The tranquiliser is only effective if the concentration is at least 0.5 mg/L.

a. Determine the concentration at:

i. $t = 0$ **ii.** $t = 1$ **iii.** $t = 2$ **iv.** $t = 4$.

b. With the aid of a graphics calculator, sketch the graph for the concentration function.

c. Find the exact length of time that the tranquiliser is effective.

d. Determine the average rate at which the tranquiliser is absorbed into the bloodstream from $t = 1$ to $t = 3$ hours. Explain why the rate is negative.

e. Determine the instantaneous rate at which the tranquiliser is absorbed into the bloodstream at $t = 0.5$ and $t = 2$ hours. (Give answers to 2 decimal places.)

f. Verify that the rate at which the tranquiliser is absorbed into the bloodstream is 0 at $t = 1$. Hence, give the time and concentration when the concentration is at a maximum.

20. Coulomb's Law relates the force acting between two charged particles to the amount of charge on the particles and the distance between the particles. It is given by the formula $F = k_e \frac{q_1 q_2}{d^2}$, where F is the force, in Newtons (N), q_1 and q_2 are the charges on the two particles, in Coulombs (C), d is the distance in metres (m) between the particles, and k_e is Coulomb's Constant (about 8.99×10^9 Nm^2C^{-2}). For an experiment, the two particles have the same charge and the distance is kept constant at 1 cm. The charge on the particles is increased linearly, according to the formula $q = 2 \times 10^{-7} t$ where t is the time, in seconds (s). Determine the time at which the force is changing at 200 N/s.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 11 Rates of change

Exercise 11.2 Exploring rates of change

1. **a.** variable **b.** variable
c. constant **d.** constant
e. variable **f.** variable
g. constant **h.** constant
i. variable **j.** constant
k. variable

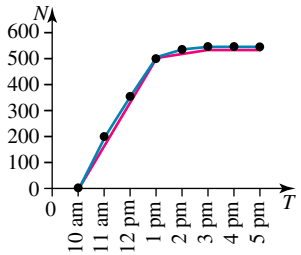
2. **a.** variable **b.** constant
c. variable **d.** constant
e. constant **f.** variable
g. constant **h.** constant
i. variable **j.** constant

3. a, c, d

4. **a.** **ii.** 30 **iii.** 30 km/h
b. **ii.** 10 **iii.** 10 km/h
c. **ii.** 15 **iii.** 15 km/h
d. **ii.** −5 **iii.** −5 km/h
e. **ii.** 0 **iii.** 0 km/h
f. **ii.** −10 **iii.** −10 km/h

5. **a.** 1 **b.** 2.5
c. −1 **d.** 0.25

6. **a.** and **b.**



c. **i.** 170
ii. 20
iii. 0

d. **i.** 170 people/h
ii. 20 people/h
iii. 0 people/h

e. Most people arrive in the morning, few in the middle of the day and nobody later in the afternoon.

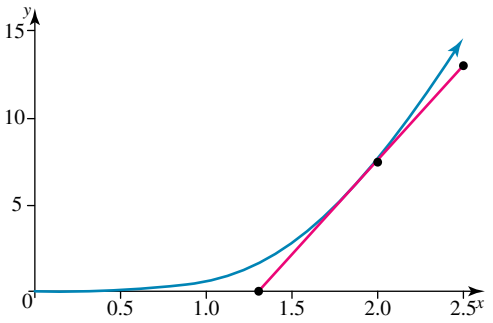
7. **a.** **i.** 110 m/min
ii. 70 m/min
iii. 45 m/min
iv. 15 m/min
v. 5 m/min

b. Decreasing

8. **a**, **c**: Constant — the increase in distance over each time interval is constant. **b**, **d**, **e**, **f**: Variable — the increase in distance over each interval is not constant.

9. **a.** A **b.** I **c.** A **d.** I
e. A **f.** A **g.** A **h.** I
i. I **j.** I

10. **a.** He's calculated the average rate of change.
b. 11.4 °C/hour.

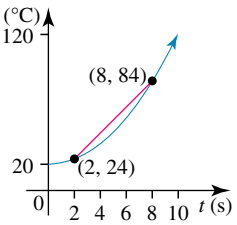


11. **a.** −2, 0, −1
b. 2, 6, 4
The average rate of change over AC is the average of the rates of change from AB and BC.

12. **a.** $24/h
b. Took a lunch break
c. $18/h
d. The picker is getting tired or there is less fruit to find.
e. $156

13. **a.** 85 kg at 35 weeks
b. −1 kg/week
c. around 31 weeks
d. more likely average change as weight loss is usually a long-term process

14. **a.** 20 °C
b, c



d. 10
e. 10 °C/s

15. 72 km

16. **a.** 3
b. 2.8, 2.9, 3.1, 3.2 (can't calculate average rate of change at $x = 2$)
c. **i.** 19 **ii.** 73.2

Exercise 11.3 The difference quotient

1. 3/2

2. circle

3. **a.** approaches 0 **b.** $\lim_{n \to \infty} \frac{1}{n} = 0$

4. Solutions will vary but should include a table similar to the following.

x	2.95	2.99	2.995	3	3.005	3.01	3.05
$f(x)$	7.95	7.99	7.995	8	8.005	8.01	8.05

5. a.

n	1	2	3	4	5	6	10
s	1	$1\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{7}{8}$	$1\frac{15}{16}$	$1\frac{31}{32}$	$1\frac{511}{512}$

b. C

6. a. undefined

b. 3.99, 3.999, 4.001, 4.01

c. 4

d. y doesn't have a value at $x = 2$ because the denominator would equal zero. But for any value not exactly zero we can see the value approaches 4.

7. a. e

b. e

c. The both approach the same value: e

8. a. 5.1, 5.01, 5.001 **b.** 5

9. a. 0 **b.** -5

10. a. 16 °C/min **b.** 700 °C/min

11. a. 3.75 **b.** 2.81

c. 0 **d.** 0

12. $\frac{3}{4}$

13. a. See table at the bottom of the page.*

b. 0.1, 0.01, 0.001, 0.0001, 0

c. i. 4.21

ii. 3.7951

iii. 3.7545

iv. 3.750 45

d. 3.75

14. a. See table at the bottom of the page.*

b. 0.1, 0.01, 0.001, 0.0001, 0

c. i. -0.95,

ii. -0.995,

iii. -0.9995,

iv. $-0.999\,95$

d. -1

Exercise 11.4 Differentiating simple functions

1. a. 5 **b.** $2x + 10$

c. $2x - 8$ **d.** $3x^2 + 2$

2. a. $3x^2$ **b.** -2, 2

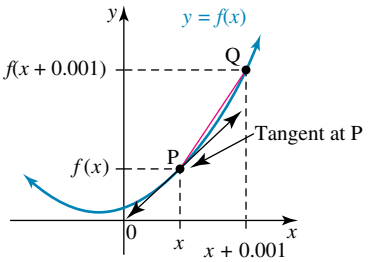
3. a. $2x - 6$ **b.** 3

4. a. Never **b.** $-\frac{5}{2}$

c. 1 **d.** Never

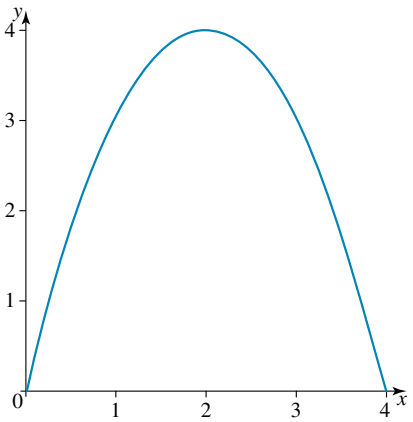
5. The derivative formula is the gradient formula applied to two points infinitesimally close to each other.

6.



The gradient of the secant between x and $x + 0.001$ is not the same as the gradient at x- therefore an h-value of 0.001 is not close enough to 0 to get a good approximation of the instantaneous gradient.

7. a.



b. 2 **c.** $4 - 2x$ **d.** 0

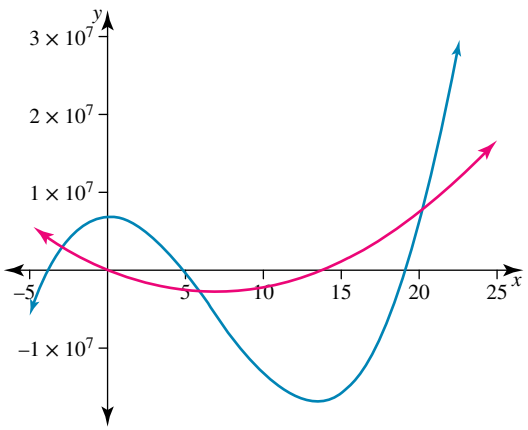
8. a, b, d

9. C

10. Because h never equals 0 so the points are extremely close but not the same point.

11. a. $6 \times 10^4 d^2 - 8 \times 10^5 d$

b.



12. $f'(x) = -\frac{1}{x^2}$

13. a. $f'(x) = nax^{n-1}$

b. i. 9 **ii.** $2x - 8$

***13.**

x	2.6	2.51	2.501	2.5001	2.5
y	-0.704	$-1.087\,05$	$-1.121\,25$	$-1.124\,62$	-1.125

***14.**

x	2.1	2.01	2.001	2.0001	2
y	-3.095	$-3.009\,95$	-3.001	-3.0001	-3

14. **a.** 5 **b.** $2x + 10$
c. $2x - 8$ **d.** $3x^2 + 2$

Exercise 11.5 Interpreting the derivative

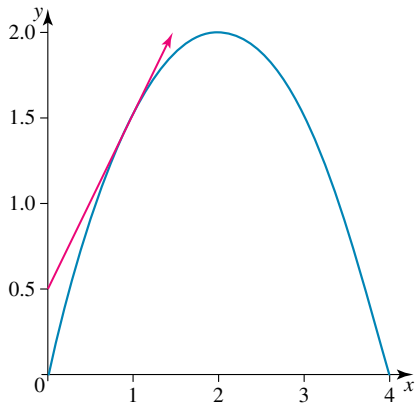
1. 0.1

2. $y = 5x - 16$

3. **a.** $y = 6x - 15$ **b.** $y = -\frac{1}{6}x + \frac{7}{2}$

4. **a.** 1

b. 0.5

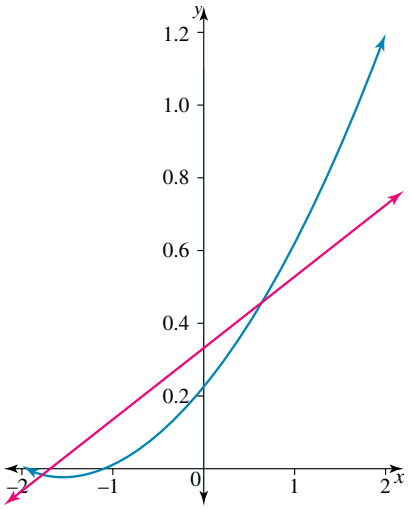


5. $y = \frac{1}{2}x + 5$

6. **a.** $0.2t + 0.3$

b. 1.5 W

c.



d. Derivative function equals 0 when the functions gradient is 0.

7. **a.** $2n$

b. 2, 4, 6, 8

c. 3, 5, 7

d. **b** are instantaneous rates of change at the points, **c** are average rates of change between the points.

8. **a.** $A = \pi r^2$

b. $\frac{dA}{dr} = 2\pi r$

c. **i.** 20π m^2/m **ii.** 100π m^2/m **iii.** 200π m^2/m

d. Yes, because $\frac{dA}{dr}$ is increasing

9. **a.** $8 - 2t$

b. 8, 4, 0, −4, −8

c. same as **b**

d. Slows travelling up and then stops and then drops and gets faster as it falls.

10. **a.** 8 cm^3/s

b. 24, 8, −8

c. Initially the rate is faster than the average and towards the end is negative, indicating the balloon was deflated. The average rate of change was equal to the instantaneous rate of change at $t = 5$.

11. $y = 16 - 4x$

12. **a.** $\frac{dA}{dt} = 180t - 9t^2$ hectares/hour

b. **i.** 0 **ii.** 576
iii. 864 **iv.** 900
v. 864 **vi.** 576
vii. 0 (all hectares/hour)

c. The fire spreads at an increasing rate in the first 10 hours, then at a decreasing rate in the next 10 hours.

d. The fire is spreading, the area burnt out by a fire does not decrease.

e. The fire stops spreading; that is, the fire is put out or contained to the area already burnt.

f. $t = 6$ and $t = 14$ hours.

13. 1.802 s (3 dp), 15.915 m (3 dp)

14. **a.** $-0.04005t^2 + 0.3151t - 0.4847$

b. $0.3151 - 0.0801t$

c. 0.3151

d. $0.135q$

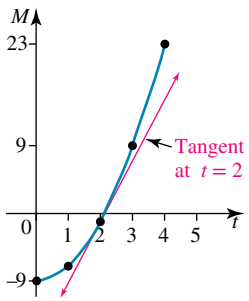
11.6 Review: exam practice

Simple familiar

1. **a.** variable **b.** variable **c.** constant
d. variable **e.** variable

2. 50 m/h

3. **a.**



b. Variable, as the graph is not a straight line.

c. Approx. 8

4. **a.** 1 kg/h **b.** 5 kg/h

5. D

6. D

7. D

8. 2

9. A

10. $y = 13x - 18$

11. $\frac{1}{2\sqrt{6}}x - \frac{1}{2}$ and $-\frac{1}{2\sqrt{6}}x - \frac{1}{2}$

12. **a.** $0.4q - 0.1$

b. 0.9

Complex familiar

13. a. $-1, -5, -3, 1, 0.5, 0$

b. decreasing, decreasing, decreasing, increasing, increasing, neither

14. a. **i.** 4 m/s **ii.** 0 m/s **iii.** −8 m/s

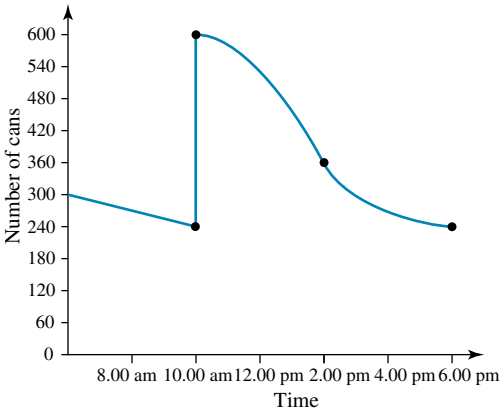
b. The height increases over the first 6 seconds and then decreases from then on.

15. $4 - 6x$

16. a. $0.875 - 0.5x$ **b.** 1.75

Complex unfamiliar

17. a.



b. 240

c. Approx. 3 pm

d. Approx. 90 cans/hour

18. $\frac{8}{5}A_0$

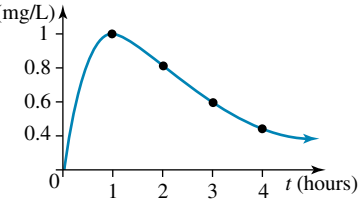
19. a. **i.** 0 mg/L

ii. 1 mg/L

iii. 0.8 mg/L

iv. 0.471 mg/L

b.



c. $2\sqrt{3}$ hours

d. −0.2 mg/L; it is negative because the concentration is decreasing.

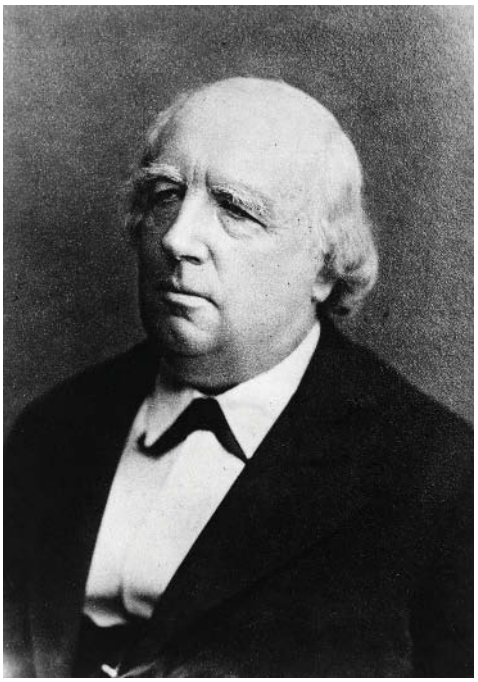
e. 0.96 mg/L at $t = 0.5$ hours, −0.24 mg/L at $t = 2$ hours

f. The maximum concentration is 1 mg/L after 1 hour.

20. 27.81 s

CHAPTER 12
Properties and applications of derivatives

12.1 Overview



In the previous chapter we introduced the process of differentiation to help determine how a quantity is changing at a specific point. As we move forward in the study of calculus it is worth noting that we have only considered the idea of a limit from an intuitive approach and not tried to rigorously define it or prove it. This is consistent with the historical development of calculus, and many other concepts in mathematics for that matter. Calculus techniques were being used effectively to solve problems for well over a century, despite rather ambiguous definitions for many foundational concepts. In the 1800s, Karl Weierstrass, who in modern parlance might be called a university drop-out, developed an acceptably rigorous definition of a limit.

Differentiation of a function, by first principles, is a tedious process. For the developers of calculus, there were no computers to simplify the differentiation process and so written techniques were developed to solve for an instantaneous rate of change. As such, rules have been developed for a range of function types to simplify the differentiation process. These rules are all provable using the method of differentiation by first principles and provide faster resolution of the derivative.

We will continue our study of differentiation by looking at some of the rules that make differentiation easier and allow us to quickly, and at much less risk of error, determine the instantaneous rate of change or gradient of a function at a point.

LEARNING SEQUENCE

12.1 Overview
12.2 Differentiation by formula
12.3 The derivative as a function
12.4 Properties of the derivative
12.5 Differentiation of power and polynomial functions
12.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

12.2 Differentiation by formula

If we look at a set of functions and their derivatives we can see a pattern that will allow us to simplify the process of differentiating.

Function	Derivative function
$f(x) = 9$	$f'(x) = 0$
$f(x) = 4x$	$f'(x) = 4$
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = 5x^3$	$f'(x) = 15x^2$
$f(x) = 4x^6$	$f'(x) = 24x^5$
$f(x) = 3x^{10}$	$f'(x) = 30x^9$

We should be able to identify that any constants become 0 and linear functions become constants when differentiated. Higher-order powers reduce their power by 1 and their coefficient is multiplied by the original power.

For any polynomial function, and indeed any function that can be written as a power, we can apply the following rules.

Rule 1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
Rule 2. If $f(x) = ax^n$, then $f'(x) = nax^{n-1}$.
Rule 3. If $f(x) = c$, then $f'(x) = 0$ (where c is constant).

WORKED EXAMPLE 1

Differentiate each of the following.

a. $y = x^8$ **b.** $y = 3x^2$ **c.** $y = 7x$ **d.** $y = \frac{3}{5}x^2$

THINK

a. Write the expression for y. Apply rule 1 to find the derivative.

b. Apply rule 2.

c. Apply rule 2.
Remember that $x^0 = 1$.

WRITE

a. $y = x^8$

$\frac{dy}{dx} = 8x^{8-1}$

$= 8x^7$

b. $y = 3x^2$

$\frac{dy}{dx} = 2(3x^{2-1})$

$= 6x$

c. $y = 7x$

$\frac{dy}{dx} = 7x^{1-1}$

$= 7x^0$

$= 7$

d. Apply rule 2.

d. $y = \frac{3}{5}x^2$

$\frac{dy}{dx} = 2\left(\frac{3}{5}x^{2-1}\right)$

$= \frac{6}{5}x$

We can prove that for $f(x) = x^n$, for any positive integer n, $f'(x) = nx^{n-1}$ as follows.

$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$	Use differentiation by first principles.
$= \lim_{h \to 0} \frac{x^n + hnx^{n-1} + h^2(...) - x^n}{h}$	Apply the binomial theorem. We have recognised though that from the third term onwards there will be at least an h term after factorising, so regardless of what else is in (…) it will approach 0.
$= \lim_{h \to 0} \frac{h(nx^{n-1} + h(...))}{h}$	Factorise the rational expression.
$= \lim_{h \to 0} nx^{n-1} + h(...)$	As h approaches 0, h(…) approaches 0.
$= nx^{n-1}$	

The second derivative of a function is determined by calculating the derivative of the derivative. It is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$. This represents the rate of change of the rate of change. An example of this is acceleration, which is the rate of change of speed, which is itself the rate of change of distance. The third derivative is denoted by $f'''(x)$ or $\frac{d^3y}{dx^3}$ and so forth if we needed to go further.

WORKED EXAMPLE 2

Determine the second derivative of the following functions.

a. $f(x) = 4x^5$ **b.** $y = 8x$

THINK	WRITE
a. 1. Differentiate the function.	$f'(x) = 5 \times 4x^{5-1}$
	$= 20x^4$
2. Differentiate the derivative.	$f''(x) = 4 \times 20x^{4-1}$
	$= 80x^3$
b. 1. Differentiate the function.	$\frac{dy}{dx} = 8x^{1-1}$
	$= 8$
2. Differentiate the derivative.	$\frac{d^2y}{dx^2} = 0$

WORKED EXAMPLE 3

The shockwave from a nuclear blast spreads out at ground level in a circular manner.

a. Write down a relationship between the area of ground, A km^2, over which the shockwave passes and its radius, r km.

b. Determine the rate of change of A with respect to r.

c. Calculate the rate of change of A when the radius is 2 km.

d. What is the rate of change of A when the area covered is 314 km^2?



THINK

a. State the formula for the area of a circle.

b. Differentiate $A(r)$.

c. Substitute $r = 2$ into $A'(r)$.
Note: The units for the rate of change of A (km^2) with respect to r (km) are km^2 per km or km^2/km.

d. 1. Substitute $A = 314$ into the area function $A(r)$ and solve for r.

2. Find the rate of change when $r = 10$.

WRITE

a. $A(r) = \pi r^2$

b. $A'(r) = 2\pi r$

c. $A'(2) = 2(3.14)(2)$
$= 12.56$
Rate of change of A when the radius is 2 km is 12.56 km^2/km.

d. $A(r) = \pi r^2$
$314 = 3.14r^2$
$r^2 = \dfrac{314}{3.14}$
$= 100$
$r = 10$ since $r > 0$

$A'(10) = 2\pi(10)$
$= 62.8$
Rate of change of A when area is 314 km^2 is 62.8 km^2/km.

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 1

Differentiation by formula Summary screen and practice questions

Exercise 12.2 Differentiation by formula

Technology free

1. WE1 Differentiate each of the following.

a. $y = x^6$ **b.** $y = 7x^2$ **c.** $y = -5x$ **d.** $y = \dfrac{2}{3}x^2$

2. WE2 Determine the second derivative of the following functions.

a. $f(x) = 4x^3$ **b.** $g(x) = \dfrac{x^5}{4}$ **c.** $h(t) = -\sqrt{2}\,t$ **d.** $i(t) = 0.004t^8$

3. Match the following functions to their derivatives.

$p = -6w$	$\frac{dp}{dw} = -6w^2$
$p = -6$	$\frac{dp}{dw} = -6$
$p = -2w^3$	$\frac{dp}{dw} = 0$
$p = -3w^2$	$\frac{dp}{dw} = -6w$

4. Calculate the value of the derivative at $x = -3$ for the following functions.

a. $y = 0.5x^3$ **b.** $r = \frac{x^2}{3}$ **c.** $w = 4x$ **d.** $q = \frac{2x^2}{\sqrt{2}}$

5. Calculate the value of the second derivative at $x = 5$ for the following functions.

a. $y = x^2$ **b.** $y = -\frac{4}{3}x^4$ **c.** $t = \frac{x^3}{0.003}$ **d.** $v = 4 \times 10^{-6}x^5$

6. a. Derive $f'(x)$ of the function $f(x) = 4x$.

b. What does this result tell us about the gradient of the function $f(x)$?

c. Derive $f''(x)$ of the function.

d. What does this result tell us about the gradient of the derivative function $f''(x)$?

7. a. Graph the function $f(x) = 0.5x^2$ over the interval $-3 \leq x \leq 3$.

b. Calculate the gradient of the function at $x = -2, 0, 2$.

c. Plot the points on the graph and sketch the gradient function.

d. Describe how the slope of $f(x) = 0.5x^2$ changes over the interval $-3 \leq x \leq 3$.

8. Identify and correct any errors made in the following derivative calculation.

$$f(x) = 4.1 \times 10^6 x$$
$$= 6 \times 4.1 \times 10^5 x$$
$$f'(x) = 2.46 \times 10^6 x$$

9. The power rule tells us that if $f(x) = ax^n$ then $f'(x) = nax^{n-1}$ and the second derivative $f''(x)$ is the derivative of the derivative. Construct the general power rule for finding the second derivative of a function of the form $f(x) = ax^n$.

10. The instantaneous rate of change of a function is given by $f'(x) = 6x$. What could the function be?

11. A rectangular fish tank has a square base with its height being equal to half the length of its base.

a. Express the length and width of the base in terms of its height, h.

b. Hence, express the volume, $V\,\text{m}^3$, in terms of the height, h, only.

c. Find the rate of change of V when:

i. $h = 1\,\text{m}$ **ii.** $h = 2\,\text{m}$ **iii.** $h = 3\,\text{m}$.

12. For the triangular package shown find:

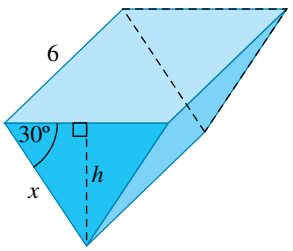
a. x in terms of h

b. the volume, V, as a function of h only

c. the rate of change of V when

i. $h = 0.5\,\text{m}$

ii. $h = 1\,\text{m}$.



13. The distance of a spanner falling from its starting point is given by $d(t) = 4.9t^2$. If the spanner falls from the top of the Sydney Harbour Bridge, a height of 134 m, what is its speed (rate of change of distance) when it hits the water? What is the acceleration (rate of change of speed) of the spanner?



14. Prove, using first principles, that for $f(x) = x^{-n}$, for any positive integer n, $f'(x) = -nx^{-n-1}$.

Technology active

15. **WE3** Water falling from a leaking roof spreads out on the ground in a circular manner.
    a. Write down the relationship between the area of ground, $A\,\text{cm}^2$, over which the water has spread and its radius, $r\,\text{cm}$.
    b. Find the rate of change of A with respect to r.
    c. Find the rate of change of A when the radius is 5 cm.
    d. What is the rate of change of A when the area covered is $30\,\text{cm}^2$?
16. A spherical balloon is being inflated.
    a. Express the volume of the balloon, $V\,\text{m}^3$, as a function of the radius, r metres.
    b. Find the rate of change of V with respect to r.
    c. Find the rate of change when the radius is:
        i. 0.1 m
        ii. 0.2 m
        iii. 0.3 m.



12.3 The derivative as a function

12.3.1 Derivative notation

There are a number of different notations that are used to represent differentiation:

i. Leibniz's notation denotes the derivative of y as $\frac{dy}{dx}$, the second derivative as $\frac{d^2y}{dx^2}$ and so forth.
ii. Newton's notation was to place a dot above the variable. So, the derivative of y is $\dot{y}$, the second derivative is $\ddot{y}$ and so forth.
iii. Euler's notation was to use the operator D. So, the derivative of y would be Dy, the second derivative is D^2y and so forth.
iv. But probably the most commonly used notation in modern mathematics is Lagrange's notation. This uses a prime mark to indicate the derivative, such that the derivative of y is y', the second derivative is y'' and so forth.

Lagrange's notation is particularly helpful because it works well with the standard function notation. In function notation the derivative of $f(x)$ is denoted by $f'(x)$. It is also easier to type on a keyboard than either Newton's and Leibniz's, which helped improve its adoption more broadly.

With function notation to identify the value of the derivative at a specific value for x we write $f'(value)$. So, the value of the derivative at $x = -3$ is written as $f'(-3)$.

WORKED EXAMPLE 4

Calculate each of the following.

a. the gradient function for $f(x) = 4x^3$

b. the instantaneous gradient at $x = 3$

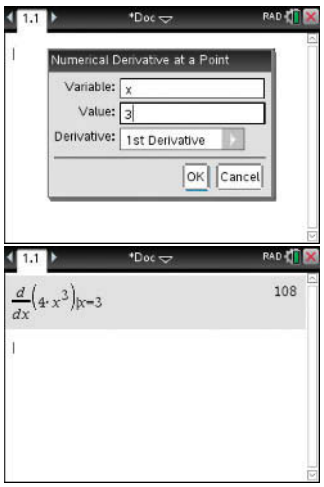
THINK	WRITE
a. 1. Use the rule $f'(x) = nax^{n-1}$.	$f'(x) = 3 \times 4x^{3-1}$
2. Solve for the derivative function.	$= 12x^2$
b. 1. Substitute into the derivative function for $x = 3$.	$f'(3) = 12(3)^2$
	$= 108$
2. Calculate the derivative and use it to state the instantaneous gradient.	The instantaneous gradient at $x = 3$ is 108.

TI | THINK

b. 1. On a Calculator page, press MENU then select
4: Calculus
1: Numerical Derivative at a Point …
Complete the fields as
Variable: x
Value: 3
Derivative: 1st Derivative
then select OK.
Complete the entry line as
$\frac{d}{dx}(4x^3) \mid x = 3$
then press ENTER.

2. The answer appears on the screen.

WRITE



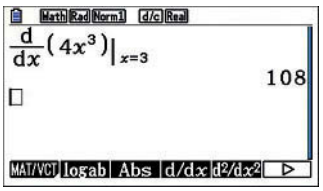
The instantaneous gradient at $x = 3$ is 108.

CASIO | THINK

b. 1. On a Run-Matrix screen, select MATH by pressing F4, then select d/dx by pressing F4.
Complete the entry line as
$\frac{d}{dx}(4x^3) \mid x = 3$
then press EXE.

2. The answer appears on the screen.

WRITE



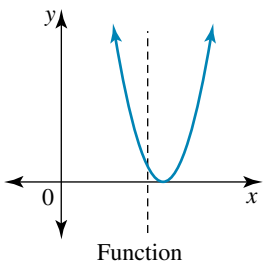
The instantaneous gradient at $x = 3$ is 108.

12.3.2 Differentiability of a function

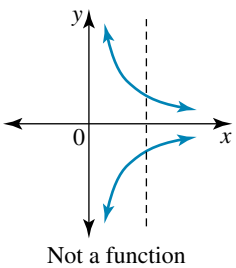
To be differentiable (able to be derived) an equation must be a continuous function across the domain.

We have previously considered what defines a function. In short, a function occurs when for each x-value there is only one possible y-value. Different inputs can produce the same output, but for every input there can't be two possible outputs.

Graphically this can be tested by the vertical line test, where any vertical line drawn on the graph cannot cross the equation more than once.



Function

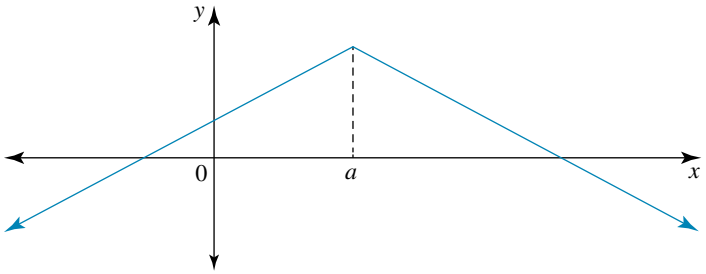


Not a function

While a function must be continuous at a point in order to find the gradient of its tangent, that condition is necessary but not sufficient. The derivative function has a limit definition, and as a limit exists only if the limit from the left equals the limit from the right, to test whether a continuous function is **differentiable** at a point $x = a$ in its domain, the derivative from the left of $x = a$ must equal the derivative from the right of $x = a$.

A function f is differentiable at $x = a$ if:
- **f is continuous at $x = a$; and**
- **$f'(a^-) = f'(a^+)$.**

This means that only **smoothly continuously functions** are differentiable at $x = a$: their graph contains no 'sharp' points at $x = a$. Polynomials are smoothly continuous functions but a hybrid function, for example, may not always join its ends smoothly. An example of a hybrid function which is continuous but not differentiable at $x = a$ is shown in the diagram below.



For this hybrid function, the line on the left of a has a positive gradient but the line on the right of a has a negative gradient; $\frac{dy}{dx}|_{x=a^-} \neq \frac{dy}{dx}|_{x=a^+}$. The 'sharp' point at $x = a$ is clearly visible and it is impossible to calculate the gradient at the graph point; therefore, there is no derivative. However, we cannot always rely on the graph to determine if a continuous function is differentiable because visually it may not always be possible to judge whether the join is smooth or not.

WORKED EXAMPLE 5

The function defined as $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$ is continuous at $x = 1$.

a. Test whether the function is differentiable at $x = 1$.
b. Give the rule for $f'(x)$ stating its domain and sketch the graph of $y = f'(x)$.

THINK

a. 1. Calculate the derivative from the left and the derivative from the right at the given value of x.

WRITE/DRAW

a. $f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Derivative from the left of $x = 1$:

$f(x) = x$

$f'(x) = 1$

$\therefore f'(1^-) = 1$

Derivative from the right of $x = 1$:

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$\therefore f'(1^+) = 2 \times 1$$

$$= 2$$

2. State whether the function is differentiable at the given value of x.

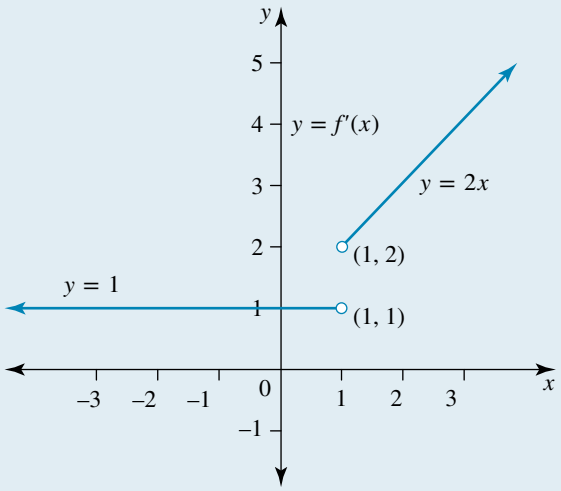
Since the derivative from the left does not equal the derivative from the right, the function is not differentiable at $x = 1$.

b. 1. State the rule for the derivative in the form of a hybrid function.

b. Each branch of this function is a polynomial so the derivative of the function is:

$$f'(x) = \begin{cases} 1, & x < 1 \\ 2x, & x > 1 \end{cases} \text{ with domain } R \setminus \{1\}.$$

2. Sketch the graph.



on Resources

Interactivity: Graph of derivative function (int-5961)

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 2

Differentiability of a function Summary screen and practice questions

Exercise 12.3 The derivative as a function

Technology free

1. Complete the table of the following derivative notations.

First derivative	Second derivative	Third derivative
$f'(x)$		
		$\frac{d^3y}{dx^3}$
	$\ddot{r}$	
		D^3t

2. WE4 For each function below, calculate:

i. the gradient function for $f(x)$

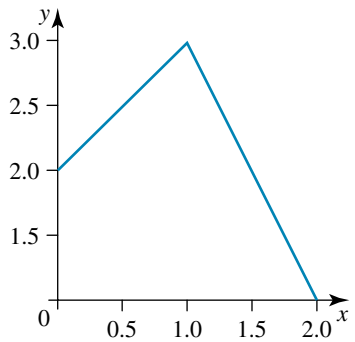
ii. the instantaneous gradient at $x = 2$.

a. $f(x) = x^2$ **b.** $f(x) = \frac{x}{6}$ **c.** $f(x) = 3x^5$ **d.** $f(x) = 9$

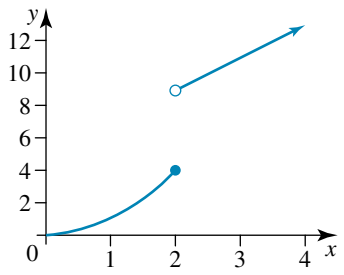
3. Calculate $f''(-1)$ for each of the functions from question **2**.

4. Do the following graphs represent continuous functions? Justify your response.

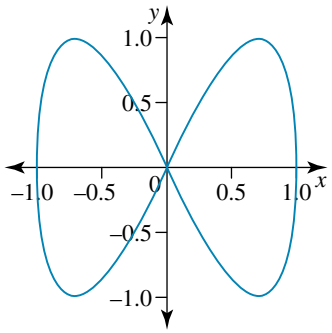
a.



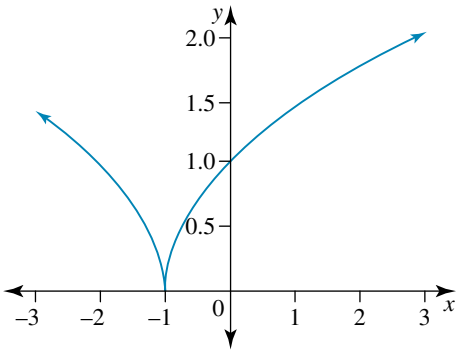
b.



c.



d.



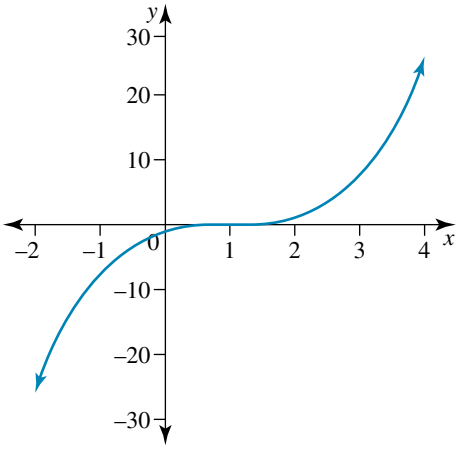
5. By first factorising the numerator, simplify the following rational functions, stating the value for which the function does not exist (is discontinuous).

a. $f(x) = \frac{x^2 + 3x}{x}$ **b.** $f(x) = \frac{6x - 18}{x - 3}$ **c.** $f(x) = \frac{x^2 - 5x}{x}$ **d.** $f(x) = \frac{x^2 + 5x + 4}{x + 4}$

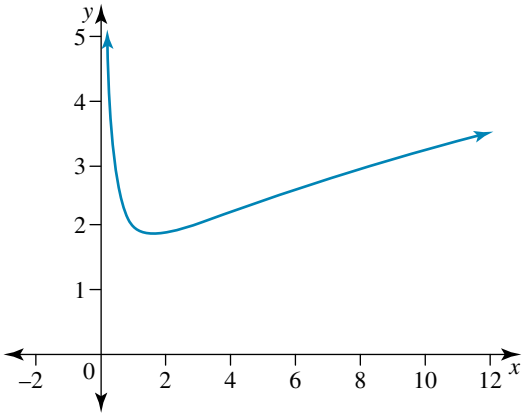
e. $f(x) = \frac{x^2 - 7x + 6}{x - 6}$ **f.** $f(x) = \frac{x^3 + 8}{x + 2}$ **g.** $f(x) = \frac{x^2 + 3x - 4}{x - 1}$ **h.** $f(x) = \frac{x^3 - 27}{x - 3}$

6. Classify the following functions as differentiable or not differentiable over the shown domains. If they are not differentiable identify a domain they are differentiable over.

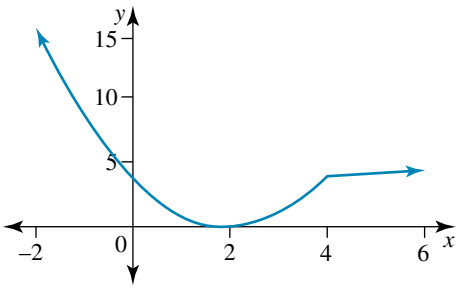
a.



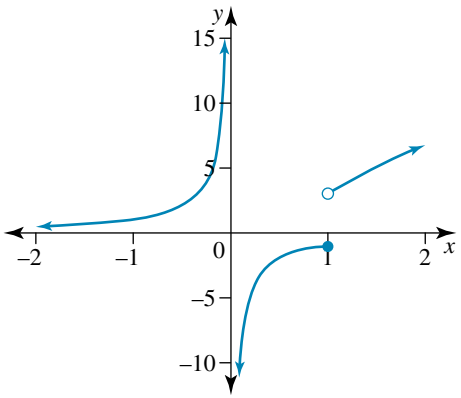
b.



c.



d.



7. Classify the following functions as differentiable or not differentiable over the shown domains.

a. $y = x^3 - 4x + 1, (-\infty, \infty)$

b. $y = \sqrt{x-2} + 4, [-5, 5]$

c. $y = \dfrac{x}{x^2 - 4}, [-3, 3]$

d. $y = \begin{cases} 3x - 2 & x \leq 2 \\ x^2 & x > 2 \end{cases}$

8. The local police are looking at implementing a speed monitoring system where they record number plates when drivers enter and exit the motorway. If their average speed is over a certain speed then they will be fined because they must have been driving over that speed at some point on the trip. Given the Mean Value Theorem in question **14**, mathematically defend the reasoning of the police department. Are there ways for drivers to cheat the system?



9. Consider the function with rule $g(x) = \dfrac{x^2 - x}{x}$.

a. State the domain over which the function is differentiable.

b. Calculate $\lim_{x \to 0} g(x)$.

c. Explain why the function is not continuous at $x = 0$.

d. Sketch the graph of $y = g(x)$.

10. WE5 The function defined as:

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$$

is continuous at $x = 1$.

a. Test whether the function is differentiable at $x = 1$.

b. Give the rule for $f'(x)$ stating its domain and sketch the graph of $y = f'(x)$.

11. Consider the function defined by the following rule.

$$f(x) = \begin{cases} 3 - 2x, & x < 0 \\ x^2 + 3, & x \geq 0 \end{cases}$$

Determine whether the function is differentiable at $x = 0$.

12. Determine a linear function of the form $y = ax + b$ that could be used to form a differentiable hybrid function with $y = \dfrac{1}{2x}$, joining at $x = 3$.

13. Determine the value of a and b so that:

$$f(x) = \begin{cases} ax^2, & x \leq 2 \\ 4x + b, & x > 2 \end{cases}$$

is smoothly continuous at $x = 2$.

Technology active

14. The Mean Value Theorem tells us that if $f(x)$ is continuous over $[a, b]$ and differentiable over (a, b) then there exists a number, c, where $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

For the function $f(x)$ over the domain $[2, 5]$ calculate the point c if:

a. $f(x) = 5x^4$ **b.** $f(x) = 0.4x^3$.

12.4 Properties of the derivative

We can further expand our usage of the basic rule for differentiation of a function of the form $f(x) = ax^n$ by utilising a property of differentiation that if $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$. This allows us to use the power rule for differentiation more broadly.

WORKED EXAMPLE 6

Differentiate the following.

a. $f(x) = 3x^2 - 4x + 1$ **b.** $f(x) = \dfrac{3}{8}x^5 - \dfrac{x^3}{4} + 9x - 2$

THINK

a. 1. Consider each term as a separate function and derive each individually.

b. 1. Consider each term as a separate function and derive each individually.

WRITE

$f'(x) = 2 \times 3x^{2-1} - 4x^{1-1} + 0$

$= 6x - 4$

$f'(x) = 5 \times \dfrac{3}{8}x^{5-1} - 3\dfrac{x^{3-1}}{4} + 9x^{1-1}$

$= \dfrac{15}{8}x^4 - \dfrac{3}{4}x^2 + 9$

Sometimes it will be necessary to change the way a function is written so it is in an appropriate form to apply the rules.

WORKED EXAMPLE 7

Derive $f'(x)$ if $f(x) = 3x(x - 2)$.

THINK	WRITE
1. Write down $f(x)$.	$f(x) = 3x(x - 2)$
2. Expand the brackets.	$f(x) = 3x^2 - 6x$
3. Differentiate by rule.	$f'(x) = 6x - 6$

WORKED EXAMPLE 8

If $g(x) = \dfrac{4x^3 + 3x^2}{x}$, derive $g'(x)$ by first simplifying $g(x)$.

THINK	WRITE
1. Factorise the numerator because at this stage we can only differentiate a constant denominator.	$g(x) = \dfrac{4x^3 + 3x^2}{x}$ $= \dfrac{x^2(4x + 3)}{x}$
2. Simplify $g(x)$.	$= x(4x + 3),\ x \neq 0$
3. Expand the brackets.	$= 4x^2 + 3x$
4. Differentiate $g(x)$ by rule.	$g'(x) = 2 \times 4x^{2-1} + 3x^{1-1}$
5. Recognise that as the function is not defined at $x = 0$ it is not differentiable at $x = 0$.	$g'(x) = 8x + 3, x \neq 0$

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 3

Properties of a derivative Summary screen and practice questions

Exercise 12.4 Properties of the derivative

Technology free

1. WE6 Differentiate the following.
   a. $y = x^6 + 3x^2 - 4$
   b. $y = 5x^4 - 7x^3 + 6x$
   c. $y = x^{11} - 3x^6 + 4x^5 + 3x^2$
   d. $y = 10x^5 - 3x^4 + 2x^3 - 8x$
   e. $y = 6$
   f. $y = 3x^4 + 5x^4$
   g. $\dfrac{4x^3}{5} + \dfrac{6}{7}$
   h. $\dfrac{x^2}{2} + \dfrac{9x}{4} + 3$
   i. $3.4x^3 - 0.68x^2 + 1.92x - 9.37$
   j. $5.61 \times 10^7 x^5 - 3.98 \times 10^9 x^3 - 1.06 \times 10^{12} x^2$

2. **WE7** Find $f'(x)$ for each of the following.
   a. $f(x) = x(x + 3)$
   b. $f(x) = 3x(2x - 5)$
   c. $f(x) = (x + 4)^2$
   d. $f(x) = 9(8 - 3x)^2$
   e. $f(x) = (x + 2)^3$
   f. $f(x) = (2x - 5)^3$
3. **WE8** Determine $g'(x)$ by first simplifying $g(x)$.
   a. $g(x) = \dfrac{x^3 + 5x}{x}$
   b. $g(x) = \dfrac{8x^2 - 6x}{2x}$
   c. $g(x) = \dfrac{3x^3 + 2x^2 - 5x}{x}$
   d. $g(x) = \dfrac{5x^4 + x^3 + 7x^2}{x^2}$
4. Calculate the second derivative at $x = 4$ of:
   a. $y = 4x^3 - 2x^2 + x$
   b. $z = x^4 - 9x^2 + 4$
   c. $f(x) = \dfrac{5}{6}x(2 - 3x^2)$
   d. $g(x) = (3x - 1)^3$.
5. Determine $r'(6)$ by first simplifying $r(x)$ for each of the following functions:
   a. $r(x) = \dfrac{(x + 1)(x - 2)}{x - 2}$
   b. $r(x) = \dfrac{x^2 - 4x - 12}{x - 6}$
   c. $r(x) = \dfrac{x^3 + 27}{x + 3}$
   d. $r(x) = \dfrac{(x - 1)(4x^2 - 11x - 3)}{x - 3}$.
6. Calculate $f'(\sqrt{2})$ for each of the following functions.
   a. $f(x) = x^2 + \sqrt{2}x$
   b. $f(x) = \dfrac{x^3}{6} - \sqrt{3}x^2 + x$
   c. $f(x) = (1 - \sqrt{3})(x^2 + x + 2)$
   d. $f(x) = \dfrac{2\sqrt{2}x^3 + 3\sqrt{2}x^2}{x}$
7. Andrew throws a ball to Jasmine according to the path $y = 3x - x^2 + 1$, where x is the horizontal distance (in metres) from Andrew and y the height (in metres) above the ground.
   a. Plot the function.
   b. What will be the gradient of the function when the ball reaches its maximum height?
   c. Determine the horizontal distance, x, when the maximum height is reached using differentiation.
   d. Calculate the maximum height reached by the ball.
8. a. Calculate the x-intercepts of the parabola $y = x^2 - 5x + 6$.
   b. Calculate the gradient of the parabola at the points where it crosses the x-axis.
   c. Determine the value of x for which the gradient of the parabola is:
      i. 0
      ii. 7
      iii. -3.
9. Confirm for $f(x) = x^2 + 3x - 1$ and $g(x) = 5x^3 + 2x^2 - 9x$ that $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$.
10. The gradient of a polynomial function at its turning points is 0. We can use this fact to locate any turning points by calculating where the gradient is 0.
    a. Solve for the exact values of x for which $f(x) = \dfrac{x^3}{3} - 4x^2 + 12x$ has a gradient of 0.
    b. Infer where the turning points (x, y) of the function $f(x)$ are located.
11. John says that the gradient of any quadratic function $y = ax^2 + bx + c$ is always changing at a constant rate. Is he correct? Justify your response.

12. Under emergency breaking, a truck's distance from when it started applying its breaks is given by the formula $d_t(t) = ut - 2.5t^2$, while for a car it is given by $d_c(t) = ut - 6t^2$, where u is the speed of the vehicle, in m/s, when it starts breaking. The vehicles stop when their speeds (the derivative of distance) reach 0. How much longer does it take for a truck to stop than a car if both are travelling at 60 km/h.



13. **a.** Expand the following turning point form quadratics, derive them and then factorise.

i. $f(x) = 3(x-4)^2 + 1$ **ii.** $g(x) = 9(x+1)^2 - 4$

b. Compare the relationship between the functions above and their derivatives to $f(x) = ax^n$ and its derivative $f(x) = nax^{n-1}$.

c. Make a conjecture as to a possible rule for the derivative of any function of the form $f(x) = a(x-h)^n + k$.

d. Test your rule on the following functions and confirm the results using technology.

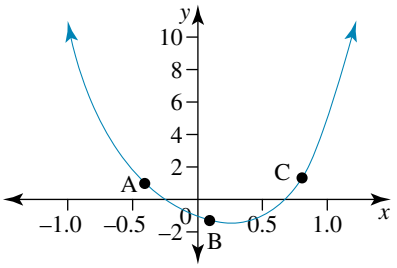
i. $f(x) = 2(x+1)^3 - 7$ **ii.** $g(x) = 5(x-2)^4 + 8$

Technology active

14. Use the graph below of the function $y = 5x^4 + 4x^2 - 3x - 1$ to determine the following:

a. the average rate of change between points A and B and between points B and C

b. the instantaneous rate of change at points A, B and C.



15. In business contexts the 'marginal cost' is the rate of change of cost at a given value. If the cost of producing surfboards is given by the function $C(n) = 0.14n^3 - 0.36n^2 + 0.18n + 330$, where n is the number of surfboards produced, what is the marginal cost for producing 60 surfboards?

16. Harbin, in north-eastern China, is famous for the massive ice sculptures that are built there each winter as part of its annual ice festival. The average temperatures for each month in Harbin are as follows.



J	F	M	A	M	J	J	A	S	O	N	D
−18 °C	−14 °C	−3 °C	7 °C	15 °C	20 °C	23 °C	21 °C	15 °C	6 °C	−5 °C	−15 °C

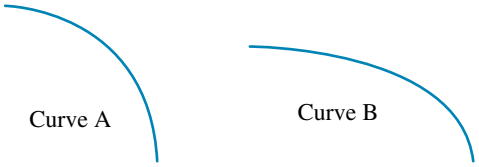
a. Using appropriate technology develop a quadratic function to model the temperature data.

b. Calculate the rate of change of average temperature:

i. in April **ii.** in September **iii.** from April to September.

17. The curvature (κ) of a function is a measure of the degree to which it deviates from being straight. The tighter the curve or the more sharply that it bends, the greater the curvature. Curve A below has a higher curvature than Curve B. The units of curvature are $length^{-1}$ $\left(\text{i.e. } \text{m}^{-1}, \text{cm}^{-1}\right)$.





The curvature of a straight line is zero, it has no curve.

The curvature at any point along the curve, y, is given by the equation: $\kappa = \dfrac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$, where y' is the derivative and y'' the second derivative.

a. Determine the curvature of the functions $f(x) = 0.5x^2 + 4x$ and $g(x) = 0.3x^3 - 0.8x^2$ at $x = 0,\ 2$ and 4.

b. Which function has the tightest curve from $0 \leq x \leq 4$?

c. Confirm your result visually by graphing the functions.

18. A survey team is conducting a feasibility study for the positioning of an overland chairlift system. They set up a straight survey line and measured the elevation (height above sea level) at horizontal intervals of 500 m. At this stage of the study, an approximate model for the cross-section is required, rather than a detailed survey of all undulations of the terrain. A table of the data is given below.

Horizontal distance (×100 m)	0	1	2	3	4	5	6	7	8	9
Elevation (metres)	640	292	218	240	305	356	362	328	300	330

Using technology develop a higher order polynomial model of the data. Locate the peaks of the terrain using differentiation. If the chairlift system is to have stations located on each peak, what is the maximum slope that the chairlift will have to operate across?

12.5 Differentiation of power and polynomial functions

So far we have only used $f'(x) = nax^{n-1}$ for cases where n is a positive integer. But this rule also holds true for any other power functions, and henceforth will be referred to as the power rule.

A **power function** is of the form $y = x^n$ where n is a rational number. The hyperbola $y = \dfrac{1}{x} = x^{-1}$ and the square root function $y = \sqrt{x} = x^{\frac{1}{2}}$ are examples of power functions that we have already studied.

WORKED EXAMPLE 9

Differentiate each of the following.

a. $f(x) = x^{-3}$

b. $f(x) = \dfrac{1}{x^7}$

c. $f(x) = x^{\frac{1}{3}}$

d. $f(x) = \dfrac{4}{\sqrt{x}}$

THINK	WRITE
a. 1. Write down $f(x)$.	**a.** $f(x) = x^{-3}$
2. Differentiate by rule 1.	$f'(x) = -3x^{-3-1}$ $= -3x^{-4}, x \neq 0$
b. 1. Write down $f(x)$.	**b.** $f(x) = \dfrac{1}{x^7}$
2. Bring the x term to the numerator using the index laws, as we can only differentiate a constant denominator.	$= 1x^{-7}, x \neq 0$
3. Differentiate by rule 1.	$f'(x) = -7\left(1x^{-7-1}\right)$ $= -7x^{-8}$
4. Express answer with a positive index to follow style of $f(x)$.	$= -\dfrac{7}{x^8}, x \neq 0$
c. 1. Write down $f(x)$.	**c.** $f(x) = x^{\frac{1}{3}}$
2. Differentiate by rule 1.	$f'(x) = \dfrac{1}{3}\left(x^{\frac{1}{3}-1}\right)$ $= \dfrac{x^{-\frac{2}{3}}}{3}$
3. Express answer with a positive index.	$= \dfrac{1}{3x^{\frac{2}{3}}}$
d. 1. Write down $f(x)$.	**d.** $f(x) = \dfrac{4}{\sqrt{x}}$
2. Convert x to index from.	$f(x) = \dfrac{4}{x^{\frac{1}{2}}}$
3. Bring the x term to the numerator using the index laws.	$f(x) = 4x^{-\frac{1}{2}}$
4. Differentiate by rule 2.	$f'(x) = -\dfrac{1}{2}\left(4x^{-\frac{1}{2}-1}\right)$ $= -2x^{-\frac{3}{2}}$

5. Express with a positive index. $= -\frac{2}{x^{\frac{3}{2}}}$

6. Express the power of x back surd (square root) from. $= -\frac{2}{\sqrt{x^3}}, x > 0$

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 4

Differentiation of power and polynomial functions Summary screen and practice questions

Exercise 12.5 Differentiation of power and polynomial functions

Technology free

1. **WE9** Differentiate each of the following.

a. x^{-4} b. x^{-7} c. $3x^{-4}$ d. $5x^{-8}$

e. $-4x^{-6}$ f. $-3x^{-5}$ g. $\frac{1}{x^4}$ h. $\frac{1}{x^9}$

i. $\frac{5}{x^3}$ j. $\frac{10}{x^6}$ k. $2x^{\frac{1}{2}}$ l. $x^{\frac{2}{3}}$

m. $4x^{\frac{1}{4}}$ n. $3x^{\frac{2}{5}}$ o. $\sqrt{x}$ p. $\frac{1}{\sqrt{x}}$

q. $4\sqrt{x}$ r. $\sqrt[3]{x}$ s. $\frac{2}{\sqrt[3]{x}}$

2. Differentiate each of the following.

a. $x^2 + \sqrt{x}$ b. $3x^{-2} + \frac{5}{2x} - x$

c. $\sqrt[5]{x} + 4\sqrt[3]{x}$ d. $9 \times 10^{-4}x^{-3} + 4 \times 10^{-3}x^{-4}$

3. If $y = \frac{4 - 3x + 7x^4}{x^4}$, calculate $\frac{dy}{dx}$ and state its domain.

4. Calculate $f'(x)$, expressing the answer with positive indices, if:

a. $f(x) = \frac{3x^2 + 5x - 9}{3x^2}$ b. $f(x) = \left(\frac{x}{5} + \frac{5}{x}\right)^2$

c. $f(x) = \sqrt[5]{x^2} + \sqrt{5x} + \frac{1}{\sqrt{x}}$ d. $f(x) = 2x^{\frac{3}{4}}(4 + x - 3x^2)$.

5. A function f is defined as $f: [0, \infty) \to R, f(x) = 4 - \sqrt{x}$.

a. Define the derivative function $f'(x)$.

b. Obtain the gradient of the graph of $y = f(x)$ as the graph cuts the x-axis.

c. Calculate the gradient of the graph of $y = f(x)$ when $x = 0.0001$ and when $x = 10^{-10}$.

d. What happens to the gradient as $x \to 0$?

6. Consider the hyperbola defined by $y = 1 - \frac{3}{x}$.

a. At its x-intercept, the gradient of the tangent is g. Calculate the value of g.

b. Calculate the coordinates of the other point where the gradient of the tangent is g.

c. Sketch the hyperbola showing both tangents.

d. Express the gradients of the hyperbola at the points where $x = 10$ and $x = 10^3$ in scientific notation; describe what is happening to the tangent to the curve as $x \to \infty$.

Technology active

7. **a.** Sketch the graph of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{5}{3}}$ using technology and find the coordinates of the points of intersection.

b. Compare the gradients of the tangents to each curve at the points of intersection.

8. Danielle calculates a derivative as follows.

$$\begin{aligned} y &= \frac{1}{2x^2} \\ &= 2x^{-2} \\ \frac{dy}{dx} &= -2 \times 2x^{-2-1} \\ &= -4x^{-3} \\ &= -\frac{4}{x^3} \end{aligned}$$

Identify the error(s) she has made and correctly calculate the derivative.

9. The weekly profit, P (hundreds of dollars), of a factory is given by $P = 4.5n - n^{\frac{3}{2}}$, where n is the number of employees.

a. Find $\frac{dP}{dn}$.

b. Hence, find the rate of change of profit, in dollars per employee, if the number of employees is:

i. 4 **ii.** 16 **iii.** 25.

c. Find n when the rate of change is zero.

10. Given the following function and derivative pairs, formulate a possible general rule for the derivative of any function of the form $f(x) = (x + a)^n$.

$f(x) = (x + 3)^2$	$f'(x) = 2(x + 3)$
$f(x) = (x - 5)^{\frac{3}{2}}$	$f'(x) = \frac{3}{2}(x - 5)^{\frac{1}{2}}$
$f(x) = (x + 2)^{\frac{2}{3}}$	$f'(x) = \frac{2}{3}(x + 2)^{-\frac{1}{3}}$
$f(x) = \left(x + \frac{1}{5}\right)^{-\frac{3}{4}}$	$f'(x) = -\frac{3}{4}\left(x + \frac{1}{5}\right)^{-\frac{7}{4}}$

11. The height of a magnolia tree, in metres, is modelled by $h = 0.5 + \sqrt{t}$ where h is the height t years after the tree was planted.



a. How tall was the tree when it was planted?

b. At what rate is the tree growing 4 years after it was planted?

c. When will the tree be 3 metres tall?

d. What will be the average rate of growth of the tree over the time period from planting to a height of 3 metres?

12. On a warm day in a garden, water in a bird bath evaporates in such a way that the volume, V mL, at time t hours is given by:

$$V = \frac{60t + 2}{3t},\ t > 0.$$



a. Show that $\frac{dV}{dt} < 0$.

b. At what rate is the water evaporating after 2 hours?

c. Sketch the graph of $V = \frac{60t + 2}{3t}$ for $t \in \left[\frac{1}{3}, 2\right]$.

d. Calculate the gradient of the chord joining the endpoints of the graph for $t \in \left[\frac{1}{3}, 2\right]$ and explain what the value of the value this gradient measures.

13. Calculate $f'(6)$ for the following functions, accurate to 2 decimal places.

a. $f(x) = 4\sqrt{x} - 13x$ **b.** $f(x) = 2x^{-\frac{1}{2}} - 3x^{\frac{2}{3}} - x$

14. Calculate $g''(2)$ for the functions, accurate to 3 decimal places.

a. $g(x) = 2\sqrt{x} + x^3 - 5x$ **b.** $g(x) = 3x^{\frac{3}{4}} + 1.2x^{\frac{5}{2}} + 4.1x^{-\frac{1}{2}}$

15. **a.** Graph the function $y = \sqrt{x} + \frac{1}{x}$ using technology.

b. Compare the behaviour of $y = \sqrt{x} + \frac{1}{x}$ to the functions $y = \sqrt{x}$ and $y = \frac{1}{x}$ as $x \to 0$ and $x \to \infty$.

c. Derive the gradient function of y.

d. Calculate the domain and range of y.

16. At a chocolate factory, sugar is added to the chocolate mix according to the function $m(t) = 800\sqrt{t} + t^3, 0 \leq t \leq 10$, where m is the mass of chocolate (in grams) and t is the time (in seconds). Calculate the time at which the flow rate is equal to the average flow rate over the whole 10 seconds.



17. The cost of producing a product is given by the function $C(n) = 0.005n^3 + 0.06n^2 - 5n + 300$, where n is the number of items produced.

a. Develop the function for the average cost per item and then determine where the gradient of the average cost function is equal to 0.

b. Explain what this value represents.

18. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. Air is being pumped into a bubble at a constant rate.

a. Define the equation that expresses how the radius of the bubble changes with respect to the volume.

b. What is the radius of the bubble at which the bubble's radius is changing by less than $0.025\ \text{cm}/\text{cm}^3$.

12.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Differentiate the following functions.

a. $f(x) = 3x^5$

b. $h(t) = 5\sqrt{3}\,t^{-2}$

c. $y = 3x^2 - x + 1$

d. $p(t) = 5t^2 - 2\sqrt{t}$

2. Calculate the value of $f'(5)$ for the functions.

a. $f(x) = 5x^4$

b. $f(x) = \frac{2}{3}x^2 - x + 1$

3. Determine the second derivative of the following functions.

a. $g(x) = \frac{x^{15}}{5}$

b. $i(t) = 1.4 \times 10^{-6}t^3 + 9.3 \times 10^{-5}t^2$

c. $a(v) = 4.9v^{-3} + 1.2v^{-1} + 6.7$

d. $y(t) = 2\sqrt{t} - \sqrt[3]{t^4}$

4. Calculate $f''(2)$ for each of the following functions.

a. $f(x) = 5x^3$

b. $f(x) = \frac{x^2}{4} + 3x - 2$

c. $f(x) = 6x^{\frac{1}{2}} + 4x^{\frac{2}{3}}$

d. $f(x) = 9 \times 10^4\sqrt{x}$

5. Identify whether the following functions are differentiable over the domain $[0, 10]$. Justify your response.

a. $y = x^2 - x + 3$

b. $y = \frac{x}{x - 2}$

6. The function $f(x) = \begin{cases} 4x + 6 & x \leq 3 \\ x^3 - 9 & x > 3 \end{cases}$ is continuous at $x = 3$. Test whether the function is differentiable at $x = 3$.

7. Determine $f'(x)$ for the function $f(x) = (3x - 2)^3$.

8. Determine $g'(x)$ by first simplifying the function $g(x) = \frac{4x^3 + 7x^2 - 2x}{x}$.

9. Determine $f''(-3)$ for the function $f(x) = \frac{x^4 - 7x^3 + 12x^2}{x - 3}$.

10. For the function $f(x) = x - \frac{1}{x}$, $x \neq 0$:

a. State the domain of the function.

b. Form the rule for its gradient function, stating its domain.

c. Calculate the gradient of the tangent to the curve at the point $(1, 0)$.

d. Find the coordinates of the points on the curve where the tangent has a gradient of 5.

11. For the function $f(x) = x^2 + \frac{2}{x}$, $x \neq 0$:

a. Evaluate $f'(2)$.

b. Determine the coordinates of the point for which $f'(x) = 0$.

c. Calculate the exact values of x for which $f'(x) = -4$.

12. The monthly profit, P (thousands of dollars), of an online retailer is given by $P = 5n + n^{\frac{5}{3}} - 0.5n^2$, where n is the number of employees.

a. Determine $\dfrac{dP}{dn}$.

b. Calculate when the gradient of the function will be zero. Use technology of your choice to answer this question.

Complex familiar

13. A function is defined by the following rule.

$$f(x) = \begin{cases} (2-x)^2 & x < 4 \\ 2 + \sqrt{x} & x \geq 4 \end{cases}$$

a. Calculate, if it exists, $\lim\limits_{x \to 4} f(x)$.

b. Explain whether the function is continuous at $x = 4$.

c. Determine whether the function is differentiable at $x = 4$.

d. Calculate the rule for $f'(x)$.

e. Evaluate, if possible, $f'(0)$.

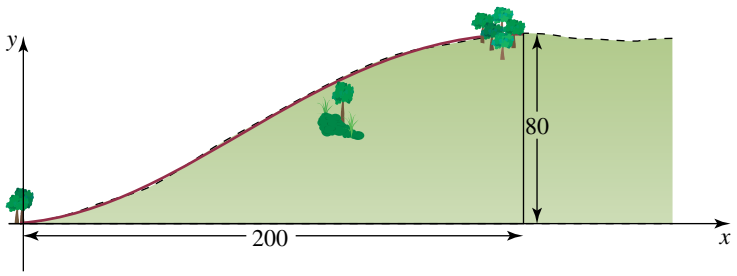
f. calculate the values of x for which $f'(x) < 0$.

14. Determine the values of a and b so that

$$f(x) = \begin{cases} 2x + a & x \leq 1 \\ bx^3 & x > 1 \end{cases}$$

is smoothly continuous at $x = 1$.

15. A new estate is to be established on the side of a hill.



Regulations will not allow houses to be built on slopes where the gradient is greater than 0.45. If the equation of the cross-section of the hill is:

$$y = -0.000\,02x^3 + 0.006x^2.$$

Determine:

a. the gradient of the slope $\dfrac{dy}{dx}$

b. the gradient of the slope when x equals

i. 160 **ii.** 100

iii. 40 **iv.** 20

c. the values of x where the gradient is 0.45

d. the range of heights for which houses cannot be built on the hill.

16. A mountain trail can be modelled by the curve with equation $y = 1.8 + 0.16x - 0.005x^4$, where x and y are, respectively, the horizontal and vertical distances measured in kilometres, $0 < x < 3$.



a. Find the gradient at the beginning and end of the trail.

b. Calculate the point where the gradient is 0.

c. Hence, state the maximum height of the path.

Complex unfamiliar

17. A stone dropped into a pond creates a ripple that increases in radius 6 cm/s.

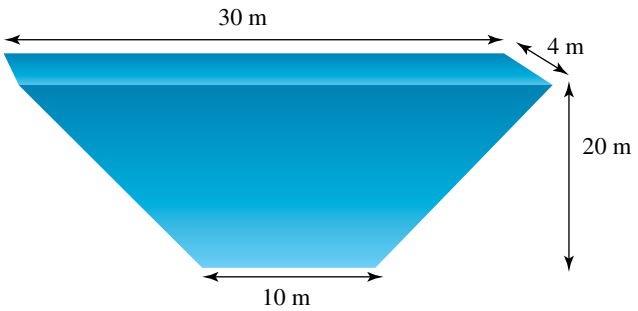
a. Determine the equation for the rate of change of area with respect to time.

b. Calculate when the area is increasing at 50 cm^2/s.



18. A container in the shape of an inverted right cone of radius 5 cm and depth 10 cm is being filled with water. When the depth of water is h cm, the radius of the water level is r cm.
At what rate, with respect to the depth of water, is the volume of water changing when its depth is 3 cm?

19. A small farm dam is in the approximate shape of a trapezoidal prism, with a depth of 4 m, length of 20 m, a width of 30 m at the top and 10 m at the base.



a. Determine the equation for the rate of change of volume with respect to height.

b. Calculate the rate at which the volume is changing when the dam is:

i. one-quarter full

ii. half full.

20. A company's income each week is $\$(500n + 1800\sqrt{n} - 10n^2)$, where n is the number of employees. If each employee is paid \$750 per week, determine the optimum number of employees for the company to make the most profit. What is the profit that will be generated with the optimum number of employees?

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Units 1 & 2 Sit chapter test

Answers

Chapter 12 Properties and applications of derivatives

Exercise 12.2 Differentiation by formula

1. **a.** $6x^5$ **b.** $14x$ **c.** -5 **d.** $\frac{4}{3}x$

2. **a.** $y = 24x$ **b.** $5x^3$ **c.** 0 **d.** $0.224t^6$

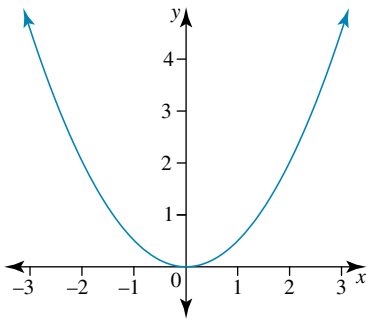
3. $p = -6w$ $\frac{dp}{dw} = -6$; $p = -6$ $\frac{dp}{dw} = 0$;
$p = -2w^3$ $\frac{dp}{dw} = -6w^2$; $p = -3w^2$ $\frac{dp}{dw} = -6w$

4. **a.** 13.5 **b.** -2 **c.** 4 **d.** $-6\sqrt{2}$

5. **a.** 2 **b.** -400 **c.** 10 000 **d.** 0.01

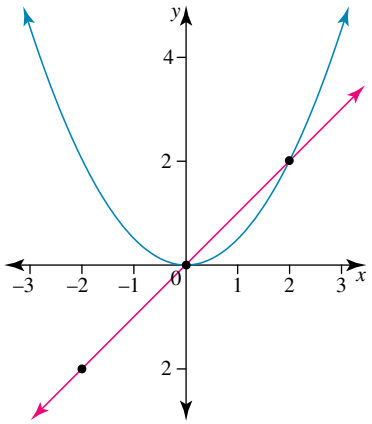
6. **a.** 4
b. It is constant for all x values, so it is a straight line.
c. 0
d. There is no change in the gradient, it is constant.

7. **a.**



b. -2, 0, 2

c.



d. The slope changes constantly from negative to positive.

8. $f'(x) = 4.1 \times 10^6 x^{1-1}$
$= 4.1 \times 10^6$

9. $f'(x) = (n-1)nax^{n-2}$

10. One possibility is $f(x) = 3x^2$.

11. **a.** Length $= 2h$, width $= 2h$
b. $V = 4h^3$
c. **i.** $12\ \text{m}^3/\text{m}$ **ii.** $48\ \text{m}^3/\text{m}$ **iii.** $108\ \text{m}^3/\text{m}$

12. **a.** $x = 2h$ **b.** $V = 6\sqrt{3}h^2$
c. **i.** $\frac{dV}{dh} = 6\sqrt{3}$ **ii.** $\frac{dV}{dh} = 12\sqrt{3}$

13. 51.25 m/s (2 dp); $9.8\ \text{m/s}^2$ (gravity)

14. Sample responses can be found in the worked solutions in the online resources.

15. **a.** $A = \pi r^2$ **b.** $\frac{dA}{dr} = 2\pi r$
c. $10\pi\ \text{cm}^2/\text{cm}$ **d.** 19.416 (3 dp)

16. **a.** $V = \frac{4}{3}\pi r^3$ **b.** $\frac{dV}{dr} = 4\pi r^2$
c. **i.** $0.04\pi\ \text{m}^3/\text{m}$
ii. $0.16\pi\ \text{m}^3/\text{m}$
iii. $0.36\pi\ \text{m}^3/\text{m}$

Exercise 12.3 The derivative as a function

1.

First derivative	Second derivative	Third derivative
$f'(x)$	$f''(x)$	$f'''(x)$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
$\dot{r}$	$\ddot{r}$	$\dddot{r}$
Dt	D^2t	D^3t

2. **a.** **i.** $f'(x) = 2x$, **ii.** 4
b. **i.** $f'(x) = \frac{1}{6}$, **ii.** $\frac{1}{6}$
c. **i.** $f'(x) = 15x^4$, **ii.** 240
d. **i.** $f'(x) = 0$, **ii.** 0

3. **a.** 2 **b.** 0 **c.** -60 **d.** 0

4. **a.** No, sharp point at $x = 1$
b. No, discontinuous at $x = 2$
c. No, not a function
d. No, sharp point at $x = -1$

5. **a.** $f(x) = x + 3,\ x \neq 0$
b. $f(x) = 6,\ x \neq 3$
c. $f(x) = x - 5,\ x \neq 0$
d. $f(x) = x + 1,\ x \neq -4$
e. $f(x) = x - 1,\ x \neq 6$
f. $f(x) = x^2 - 2x + 4,\ x \neq -2$
g. $f(x) = x + 4,\ x \neq 1$
h. $f(x) = x^2 + 3x + 9,\ x \neq 3$

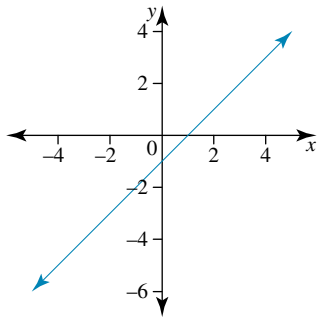
6. **a.** Differentiable
b. Differentiable over $(0, \infty)$
c. Differentiable over $x \in R\backslash\{4\}$
d. Differentiable over $x \in R\backslash\{0, 1\}$

7. **a.** Differentiable
b. Not differentiable
c. Not differentiable
d. Not differentiable

8. The statement is true if the speed of the vehicle is a continuous differentiable function, which is an appropriate assumption. Drivers could speed in some sections, but then slow in others to keep their average lower, even though they were speeding.
If their average is above the limit there is no way to cheat that.

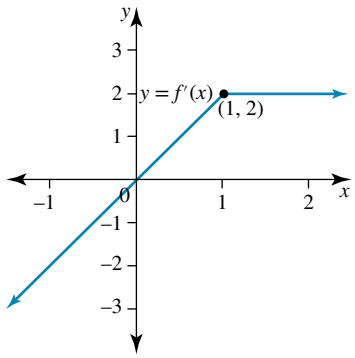
9. **a.** $x \in R\backslash\{0\}$
b. -1
c. The denominator cannot be 0 as the function is undefined at that value.

d.



10. a. Yes

b. $f'(x) = \begin{cases} 2x & x \leq 1 \\ 2 & x > 1 \end{cases}$



11. No

12. $-\frac{1}{18}x + \frac{1}{3}$

13. $a = 1, b = -4$

14. a. 3.702 (3 dp) **b.** 3.606 (3 dp)

Exercise 12.4 Properties of the derivative

1. a. $\frac{dy}{dx} = 6x^5 + 6x$

b. $\frac{dy}{dx} = 20x^3 - 21x^2 + 6$

c. $\frac{dy}{dx} = 11x^{10} - 18x^5 + 20x^4 + 6x$

d. $\frac{dy}{dx} = 50x^4 - 12x^3 + 6x^2 - 8$

e. $\frac{dy}{dx} = 0$ **f.** $\frac{dy}{dx} = 32x^3$

g. $\frac{12x^2}{5}$ **h.** $x + \frac{9}{4}$

i. $10.2x^2 - 1.36x + 1.92$

j. $2.805 \times 10^8 x^4 - 1.194 \times 10^{10} x^2 - 2.12 \times 10^{12} x$

2. a. $2x + 3$ **b.** $12x - 15$

c. $2x + 8$ **d.** $-432 + 162x$

e. $3x^2 + 12x + 12$ **f.** $24x^2 - 120x + 150$

3. a. $2x, x \neq 0$ **b.** $4, x \neq 0$

c. $6x + 2, x \neq 0$ **d.** $10x + 1, x \neq 0$

4. a. 92 **b.** 174

c. -60 **d.** 594

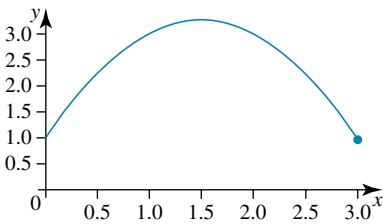
5. a. 1 **b.** undefined

c. 9 **d.** 45

6. a. $3\sqrt{2}$ **b.** $2 - 2\sqrt{6}$

c. $(1 - \sqrt{3})(1 + 2\sqrt{2})$ **d.** $8 + 3\sqrt{2}$

7. a.



b. 0

c. 1.5 m

d. 3.25 m

8. a. $x = 2, x = 3$

b. At $x = 2$ gradient $= -1$, at $x = 3$ gradient $= 1$

c. **i.** $x = 2\frac{1}{2}$ **ii.** $x = 6$ **iii.** $x = 1$

9. Sample responses can be found in the worked solutions in the online resources.

10. a. $x = 2, 6$ **b.** $\left(2, \frac{32}{3}\right), (6, 0)$

11. Yes. The derivative is a linear function of the form $2ax + b$, which is the gradient. The derivative of the gradient is the rate of change of the gradient, which is $2a$.

12. 7 seconds

13. a. **i.** $6(x - 4)$ **ii.** $18(x + 1)$

b. The x and the $(x - h)$ are treated similarly

c. $f'(x) = na(x - h)^{n-1}$

d. **i.** $f'(x) = 6(x + 1)^2$ **ii.** $g'(x) = 20(x - 2)^3$

14. a. -4.455 and 3.525 **b.** $-7.48, -2.18, 13.64$

15. \$1468.98

16. a. $-1.3012x^2 + 17.7478x - 40.5455$

b. **i.** 7.34 **ii.** -5.67 **iii.** 1.6

17. a. 0.0143, 0.0044, 0.0019, 1.6, 1.6008, 0.0107

b. $g(x)$

c. Sample responses can be found in the worked solutions in the online resources.

18. Maximum slope is -0.4896.

Exercise 12.5 Differentiation of power and polynomial functions

1. a. $-4x^{-5}$ **b.** $-7x^{-8}$ **c.** $-12x^{-5}$

d. $-40x^{-9}$ **e.** $24x^{-7}$ **f.** $15x^{-6}$

g. $-\frac{4}{x^5}$ **h.** $-\frac{9}{x^{10}}$ **i.** $-\frac{15}{x^4}$

j. $-\frac{60}{x^7}$ **k.** $x^{-\frac{1}{2}}$ **l.** $\frac{2}{3}x^{-\frac{1}{3}}$

m. $x^{-\frac{3}{4}}$ **n.** $\frac{6}{5}x^{-\frac{3}{5}}$ **o.** $\frac{1}{2\sqrt{x}}$

p. $-\frac{1}{2\sqrt{x^3}}$ **q.** $\frac{2}{\sqrt{x}}$ **r.** $-\frac{1}{3\sqrt[3]{x^2}}$

s. $-\frac{2}{3\sqrt[3]{x^4}}$

2. a. $2x + \dfrac{1}{2\sqrt{x}}$ **b.** $-6x^{-3} - \dfrac{5}{2x^2} - 1$

c. $\dfrac{1}{5\sqrt[5]{x^4}} + \dfrac{4}{3\sqrt[3]{x^2}}$

d. $-2.7 \times 10^{-3}x^{-4} - 1.6 \times 10^{-2}x^{-5}$

3. $-16x^{-5} + 9x^{-4}, x \in \mathbb{R}\backslash\{0\}$

4. a. $\dfrac{6}{x^3} - \dfrac{5}{3x^2}$ **b.** $\dfrac{2x}{25} - \dfrac{50}{x^3}$

c. $\dfrac{2}{5\sqrt[5]{x^3}} + \dfrac{\sqrt{5}}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x^3}}$ **d.** $-\dfrac{33}{2}x^{\frac{7}{4}} + \dfrac{7}{2}x^{\frac{3}{4}} + \dfrac{6}{x^{\frac{1}{4}}}$

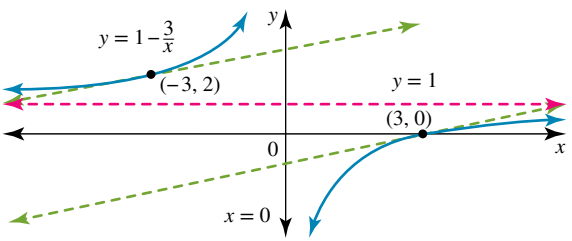
5. a. $f'(x) = \dfrac{-1}{2\sqrt{x}}$ **b.** $-\dfrac{1}{8}$

c. $-50, -50\,000$ **d.** approaches $-\infty$

6. a. $g = \dfrac{1}{3}$

b. $(-3, 2)$

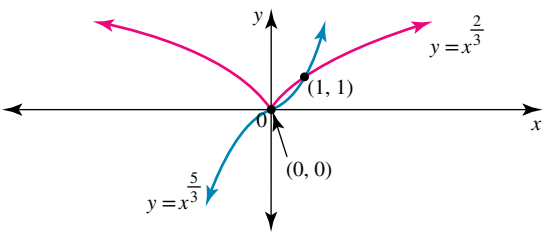
c.



Asymptotes $x = 0$, $y = 1$

d. 3×10^{-2}, 3×10^{-6}, tangent approaches the horizontal asymptote.

7. a.



$(0, 0)$ and $(1, 1)$

b. At $(1, 1)$, gradient of $y = x^{\frac{2}{3}}$ is $\dfrac{2}{3}$ and gradient of $y = x^{\frac{5}{3}}$ is $\dfrac{5}{3}$ so $y = x^{\frac{2}{3}}$ is steeper; at $(0, 0)$, gradient of $y = x^{\frac{2}{3}}$ is undefined and gradient of $y = x^{\frac{2}{3}}$ is zero.

8. $\dfrac{1}{2x^2} = \dfrac{1}{2}x^{-2}$, therefore the solution is $-\dfrac{1}{x^3}$

9. a. $\dfrac{\mathrm{d}p}{\mathrm{d}n} = 4.5 - 1.5n^{\frac{1}{2}}$

b. **i.** \$37.50 **ii.** −\$9.38 **iii.** −\$12.00

c. $n = 9$

10. $f'(x) = n(x + a)^{n-1}$

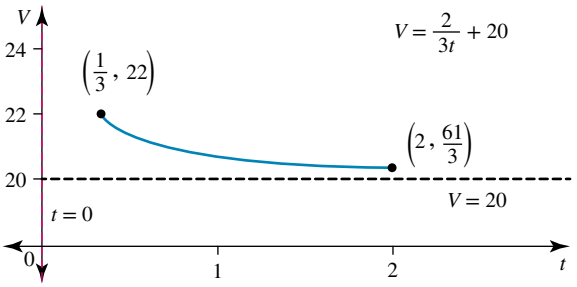
11. a. 0.5 m **b.** 0.25 m/year

c. 6.25 years **d.** 0.4 m/year

12. a. $\dfrac{dV}{dt} = -\dfrac{2}{3t^2}$ which is < 0 for $t > 0$.

b. $\dfrac{1}{6}$ ml/hour

c.



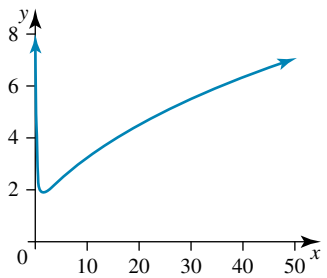
Part of hyperbola with asymptotes at $V = 20$, $t = 0$; endpoints $\left(\frac{1}{3}, 22\right)$, $\left(2, \frac{61}{3}\right)$

d. -1, average rate of evaporation

13. a. -12.18 **b.** -2.17

14. a. 11.823 **b.** 6.671

15. a.



b. as $x \to 0$, y behaves like $\dfrac{1}{x}$ and as $x \to \infty$, y behaves like $\sqrt{x}$

c. $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{x^2}$

d. $x > 0, y \geq 2^{\frac{1}{3}} + 2^{\frac{-2}{3}}$

16. 1.323 s and 8.477 s

17. a. 29.196

b. This is the number of items to be produced to have the minimum average cost per item.

18. a. $\dfrac{dr}{dV} = \sqrt[3]{\dfrac{1}{36\pi V^2}}$ **b.** 1.784 m

12.6 Review: exam practice

simple familiar

1. a. $15x^4$ **b.** $-10\sqrt{3}\,t^{-3}$

c. $6x - 1$ **d.** $10t - \dfrac{1}{\sqrt{t}}$

2. a. 2500 **b.** $\dfrac{17}{3}$

3. a. $42x^{13}$ **b.** $8.4 \times 10^{-6}t + 1.86 \times 10^{-4}$

c. $\dfrac{58.8}{v^5} + \dfrac{2.4}{v^3}$ **d.** $-\dfrac{1}{2\sqrt{t^3}} - \dfrac{4}{9\sqrt[3]{t^2}}$

4. a. 60 **b.** 0.5

c. $-\frac{3}{4\sqrt{2}} - \frac{2}{9}2^{\frac{2}{3}}$ **d.** $-5625\sqrt{2}$

5. a. Yes, smoothly continuous across the domain

b. No, discontinuity at $x = 2$

6. No

7. $81x^2 - 108x + 36$

8. $8x + 7, x \neq 0$

9. -26

10. a. $R\backslash\{0\}$ **b.** $f\prime(x) = 1 + \frac{1}{x^2}$

c. 2 **d.** $\left(-\frac{1}{2}, \frac{3}{2}\right), \left(\frac{1}{2}, -\frac{3}{2}\right)$

11. a. 3.5

b. $(1, 3)$

c. $x = -1, \frac{-1 \pm \sqrt{5}}{2}$

12. a. $-n + \frac{5}{3}n^{\frac{2}{3}} + 5$

b. 15.25 (2 dp)

Complex familier

13. a. 4

b. Continuous as $f(4) = \lim_{x \to 4} f(x)$

c. Not differentiable as $f'(4^-) \neq f'(4^+)$

d. $f\prime(x) = \begin{cases} -4 + 2x, & x < 4 \\ \frac{1}{2\sqrt{x}}, & x > 4 \end{cases}$

e. -4

f. $x < 2$

14. $a = -\frac{4}{3}, b = \frac{2}{3}$

15. a. $\frac{dy}{dx} = -0.000\,06x^2 + 0.012x$

b. **i.** 0.384 **ii.** 0.6 **iii.** 0.384 **iv.** 0.216

c. $x = 50$ and $x = 150$

d. $12.5 < y < 67.5$

16. a. Gradient is 0.16 at the beginning and -0.38 at the end.

b. $(2, 2.04)$

c. Maximum height is 2.04 km.

Complex unfamilier

17. a. $\frac{dA}{dt} = 72\pi t$ **b.** $t = 0.22\text{s}$ (2 dp)

18. $\frac{9\pi}{4}\text{cm}^3/\text{cm}$

19. a. $200 + 100h$

b. **i.** $300\,\text{m}^3/\text{m}$ **ii.** $400\,\text{m}^3/\text{m}$

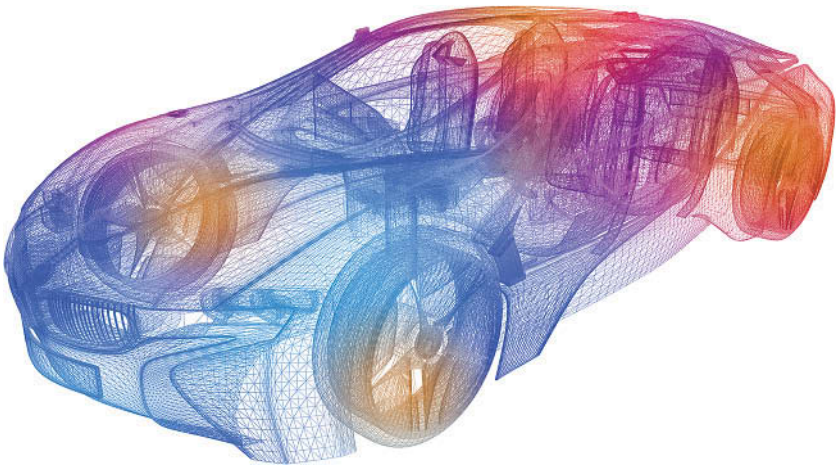
20. \$2549.08 with 6 employees

CHAPTER 13
Applications of derivatives

13.1 Overview

13.1.1 Introduction

In chapters 11 and 12 we saw that differential calculus can be applied to any situation where we are dealing with change. This chapter introduces the application of derivatives to four common fields: tangents, kinematics, curve sketching and optimisation.



These fields of study were the driving forces behind the development of calculus by its originators, Sir Isaac Newton and Gottfried Leibniz. Newton was interested in problems relating to motion, or kinematics, while Leibniz was interested in solving the tangent problem. This led both Newton and Leibniz to confront the problem of infinitesimals (infinitely small quantities) and their application to the study of change. Scientists, mathematicians and engineers continue to use infinitesimals in their fields of study. They are used to help formulate equations for the relationships between all forms of energy, also known as the study of thermodynamics. For example, the internal combustion engine in a car can be explained using such equations.

For curve sketching and optimisation alike, differentiation is all about identifying where the stationary points of a function lie. In curve sketching we are interested in determining where stationary points are and what their nature is, while in optimisation we are specifically interested in locating the stationary points that are maxima and minima.

LEARNING SEQUENCE

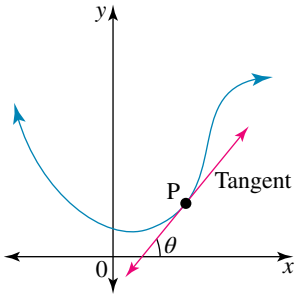
13.1 Overview
13.2 Gradient and equation of a tangent
13.3 Displacement–time graphs
13.4 Sketching curves using derivatives
13.5 Modelling optimisation problems
13.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

13.2 Gradient and equation of a tangent

13.2.1 Tangents

From chapter 11, recall that the gradient of a curve at a point is equal to the gradient of the tangent line to the curve at that point. The equation of the tangent function can be determined using the equation for a straight line, $y - y_1 = m(x - x_1)$. Also recall that the angle the tangent makes with the positive x-axis is given by $\theta = \tan^{-1}(m)$.



WORKED EXAMPLE 1

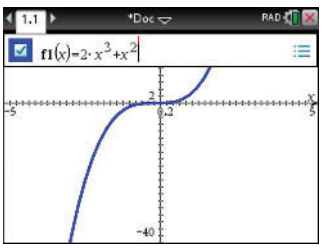
Form the equation of the tangent to the curve $y = 2x^3 + x^2$ at the point on the curve where $x = -2$.

THINK	WRITE
1. Obtain the coordinates of the point of contact of the tangent with the curve by substituting $x = -2$ into the equation of the curve.	$y = 2x^3 + x^2$ When $x = -2$, $y = 2(-2)^3 + (-2)^2$ $= -12$ The point of contact is $(-2, -12)$.
2. Calculate the gradient of the tangent at the point of contact by finding the derivative of the function at the point $x = -2$.	$y = 2x^3 + x^2$ $\therefore \frac{dy}{dx} = 6x^2 + 2x$ When $x = -2$, $\frac{dy}{dx} = 6(-2)^2 + 2(-2)$ $= 20$ The gradient of the tangent is 20.
3. Form the equation of the tangent line.	Equation of the tangent: $y - y_1 = m(x - x_1), m = 20, (x_1, y_1) = (-2, -12)$ $\therefore y + 12 = 20(x + 2)$ $\therefore y = 20x + 28$ The equation of the tangent is $y = 20x + 28$.

TI | THINK

1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 2x^3 + x^2$ then press ENTER.

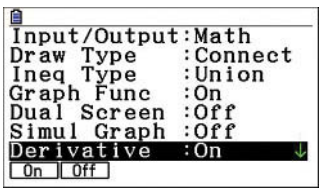
WRITE



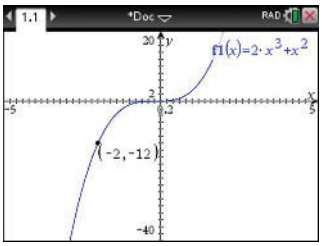
CASIO | THINK

1. On a Graph screen, press SHIFT then MENU to access the set-up menu. Change the Derivative Mode to On, then press EXIT.

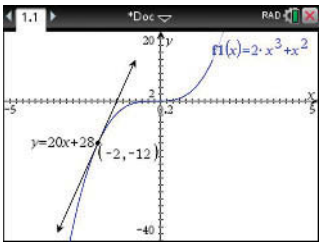
WRITE



2. To find the coordinates of the point where $x = -2$, press MENU then select
5: Trace
1: Graph Trace
Type '−2' then press ENTER twice.



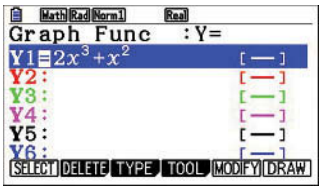
3. To draw the tangent to the curve at $x = -2$, press MENU then select
8: Geometry
1: Points & Lines
7: Tangent
Click on the curve then click on the point $(-2, -12)$.

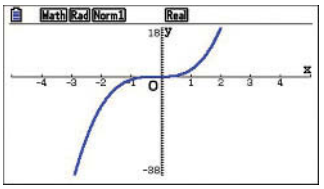


4. The answer appears on the screen.

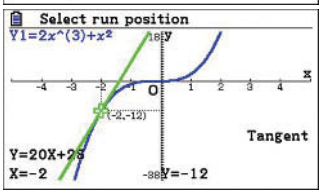
The equation of the tangent is $y = 20x + 28$.

2. Complete the entry line for Y1 as $y1 = 2x^3 + x^2$ then press EXE. Select DRAW by pressing F6.





3. To draw the tangent to the curve at $x = -2$, select SKETCH by pressing SHIFT then F4, then select Tangent by pressing F2.
Type '−2 ' then press EXE twice.



4. The answer appears on the screen.

The equation of the tangent is $y = 20x + 28$.

Resources

Interactivity: Equations of tangents (int-5962)

WORKED EXAMPLE 2

The path of a car can be modelled by the function $f(x) = -0.002x^3 + 3x, 0 \leq x \leq 40$. The car is rounding a corner when it slides off the road at the point where $x = 25$. Assuming that the car slides off the road in a straight line:

a. Calculate the gradient of the tangent to the curve at the point where $x = 25$.

b. Calculate the angle at which the vehicle slides, to the nearest degree, with respect to the positive x-axis.

c. Determine the equation of the tangent to the curve at the point where $x = 25$.

d. Determine if the car will hit a large tree located at the point (40, 32) as it slides off the road.

THINK	WRITE
a. 1. Find the instantaneous rate of change of $f(x)$ at the point where $x = 25$.	$f'(x) = -0.006x^2 + 3$ $f'(25) = -0.006(25)^2 + 3$ $= -0.75$
2. The gradient of the tangent to $f(x)$ at the point where $x = 25$ is equal to the instantaneous rate of change at $x = 25$.	The gradient of the tangent, m_T, at $x = 25$ is -0.75.

b. 1. The angle is related to the gradient by $m = \tan\theta$.

$\theta = \tan^{-1}(m_T)$
$= \tan^{-1}(-0.75)$
$= -37°$

c. 1. Find the coordinates of the point where $x = 25$.

$f(x) = -0.002(25)^3 + 3(25)$
$= 43.75$
$\rightarrow (25, 43.75)$

2. Determine the equation of the tangent at $x = 25$ by substituting the values of the gradient and the known point at (25, 43.75) into $y - y_1 = m(x - x_1)$.

$y - 43.75 = -0.75(x - 25)$
$y = -0.75x + 18.75 + 43.75$
$y = 62.5 - 0.75x$

d. 1. The car will only hit the tree if the tree is on its path. Since the car is sliding along the path given by the tangent to the curve at $x = 25$, substitute $x = 40$ into the equation of the tangent.

$y = 62.5 - 0.75(40)$
$= 32.5$

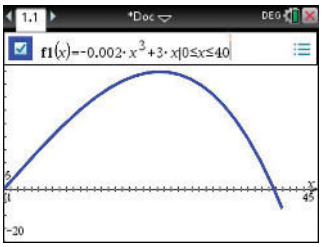
2. Compare the coordinates of the car (40, 32.5) and the coordinates of the tree (40, 32).

The point (40, 32) does not lie on a tangent (see d1), so the car will just miss the tree.

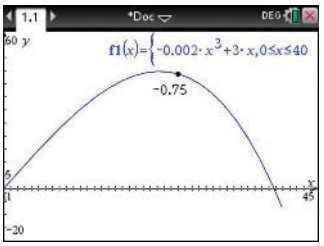
TI | THINK

a. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = -0.002x^3 + 3x|0 \le x \le 40$ then press ENTER.

WRITE



2. Press MENU then select
6: Analyze Graph
5: dy/dx
Type '25' then press ENTER.



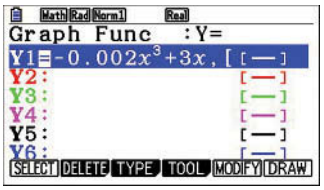
3. The answer appears on the screen.

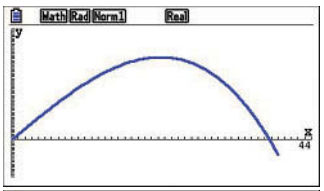
The gradient of the tangent of the curve at $x = 25$ is -0.75.

CASIO | THINK

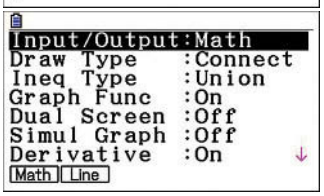
a. 1. On a Graph screen, complete the entry line for Y1 as $y1 = -0.002x^3 + 3x, [0, 40]$ then press EXE. Select DRAW by pressing F6.

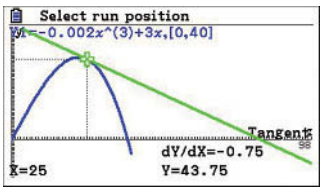
WRITE





2. Press SHIFT then MENU to access the set-up menu. Change the Derivative Mode to On, then press EXIT. Select SKETCH by pressing SHIFT then F4, then select Tangent by pressing F2. Type '−2' then press EXE.

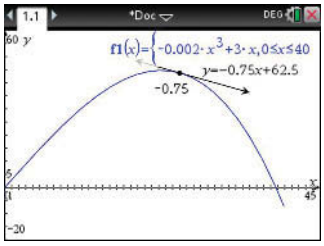




3. The answer appears on the screen.

The gradient of the tangent of the curve at $x = 25$ is -0.75.

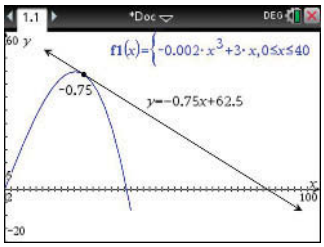
c. 1. To draw the tangent to the curve at $x = 25$, press MENU then select
8: Geometry
1: Points & Lines
7: Tangent
Click on the curve then click on the point on the curve at $x = -25$.



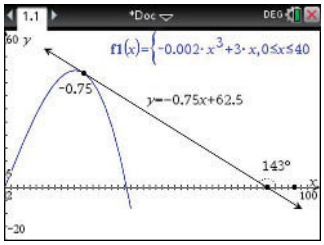
2. The answer appears on the screen.

The equation of the tangent to the curve at $x = 25$ is $y = -0.75x + 62.5$.

b. 1. Click and hold on the end of the tangent line until the arrow head turns into a hand, then drag the tangent line until it reaches the x-axis.



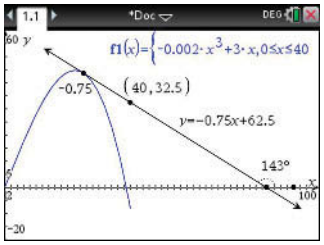
2. To find the angle between the tangent and the positive x-axis, press MENU then select
8: Geometry
3: Measurement
4: Angle.
Click on the point on the tangent at $x = 25$, then click on the point where the tangent meets the x-axis, then click on a point on the x-axis to the right of the tangent.
Note: Press MENU then select 9: Settings to ensure that the Graphing Angle is set to Degree.



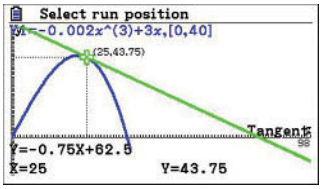
3. The answer appears on the screen.

The vehicle slides at an angle of 143°(−37°) with respect to the positive x-axis.

d. 1. Press MENU then select
8: Geometry
1: Points & Lines
2: Point On
Click on the tangent line, then click a point on the tangent line close to where $x = 40$. Place the cursor on the newly created point then press CRTL then MENU, then select
7: Coordinates and Equations.
Double-click on the x-coordinate of the point and change the value to 40, then press ENTER.



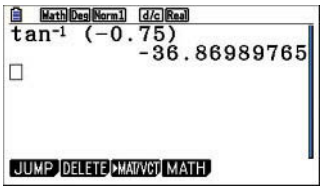
c. 1. To draw the tangent to the curve at $x = 25$, select SKETCH by pressing SHIFT then F4, then select Tangent by pressing F2.
Type '−2' then press EXE twice.



2. The answer appears on the screen.

The equation of the tangent to the curve at $x = 25$ is $y = -0.75x + 62.5$.

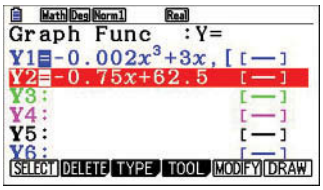
b. 1. On a Run-Matrix screen, complete the entry line as $\tan^{-1}(-0.75)$ then press EXE.

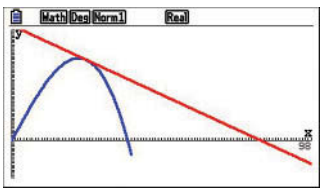


2. The answer appears on the screen.

The vehicle slides at an angle of −37° with respect to the positive x-axis.

d. 1. On the Graph screen, complete the entry line for Y2 as $y2 = -0.75x + 62.5$ then press EXE.
Select DRAW by pressing F6.

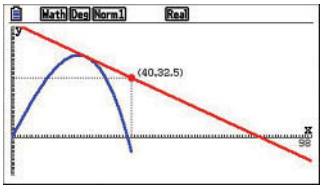




2. The answer appears on the screen.

Since the vehicle will pass through the point (40, 32.5) and not (40, 32), the vehicle will just miss the tree.

2. Select Trace by pressing SHIFT then F1. Use the up/down arrows to move the cursor to the graph of y2, then type '40' and press EXE twice.
3. The answer appears on the screen.

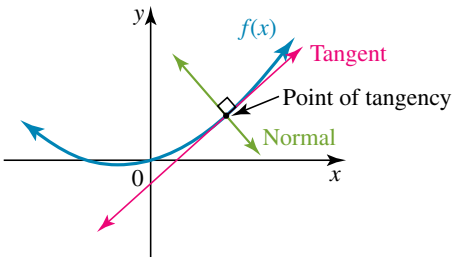


Since the vehicle will pass through the point (40, 32.5)) and not (40, 32), the vehicle will just miss the tree.

13.2.2 Normals

Recall that the **normal** is a line that is perpendicular to the tangent at a point of tangency. As such, the gradient of the normal is the negative reciprocal of the gradient of the tangent at the same point:

$$m_N = \frac{-1}{m_T}.$$



WORKED EXAMPLE 3

Determine the equation of the normal to the graph of $y = x^2 + x$ at $x = 2$.

THINK

1. Find $\frac{dy}{dx}$.
2. Find the gradient of the curve when $x = 2$.
3. The gradient of the normal is $m_N = \frac{-1}{m_T}$.
4. Evaluate y when $x = 2$.
5. Substitute values of m_N and (2, 6) into $y - y_1 = m_N(x - x_1)$.

WRITE

$\frac{dy}{dx} = 2x + 1$

When $x = 2$:

$\frac{dy}{dx} = 2(2) + 1$

$\frac{dy}{dx} = 5$

$\therefore m_T = 5$

$m_N = -\frac{1}{5}$

$y = (2)^2 + (2)$

$= 6$

$y - 6 = -\frac{1}{5}(x - 2)$

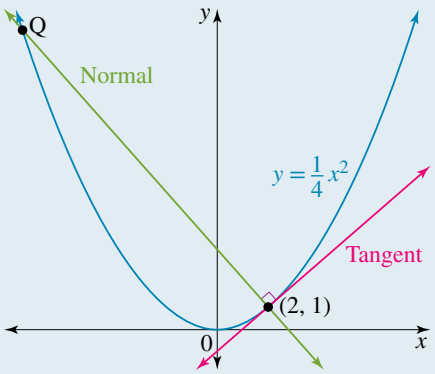
$y = -\frac{1}{5}x + \frac{32}{5}$

WORKED EXAMPLE 4

At the point (2, 1) on the curve $y = \frac{1}{4}x^2$, a line is drawn perpendicular to the tangent to the curve . This line meets the curve $y = \frac{1}{4}x^2$ again at the point Q.

a. Calculate the coordinates of the point Q.

b. Calculate the magnitude of the angle that the line passing through Q and the point (2, 1) makes with the positive direction of the x-axis.



THINK

a. 1. Calculate the gradient of the tangent at the given point.

2. Calculate the gradient of the line perpendicular to the tangent.

3. Form the equation of the perpendicular line.

4. Use simultaneous equations to calculate the coordinates of Q.

WRITE

a. $y = \frac{1}{4}x^2$

$\frac{dy}{dx} = \frac{1}{2}x$

At the point (2, 1),

$\frac{dy}{dx} = \frac{1}{2} \times 2$

$= 1$

The gradient of the tangent at the point (2, 1) is 1.

For perpendicular lines, $m_1 m_2 = -1$.

Since the gradient of the tangent is 1, the gradient of the line perpendicular to the tangent is -1.

Equation of the line perpendicular to the tangent:

$y - y_1 = m(x - x_1)$, $m = -1, (x_1, y_1) = (2, 1)$

$y - 1 = -(x - 2)$

$\therefore y = -x + 3$

Point Q lies on the line $y = -x + 3$ and the curve $y = \frac{1}{4}x^2$.

At Q:

$\frac{1}{4}x^2 = -x + 3$

$x^2 = -4x + 12$

$x^2 + 4x - 12 = 0$

$(x + 6)(x - 2) = 0$

$\therefore x = -6, x = 2$

$x = 2$ is the x-coordinate of the given point. Therefore, the x-coordinate of Q is $x = -6$.
Substitute $x = -6$ into $y = -x + 3$:

$$y = -(-6) + 3$$
$$= 9$$

Point Q has coordinates $(-6, 9)$.

b. Calculate the angle of inclination required.

b. For the angle of inclination, $m = \tan\theta$.
As the gradient of the line passing through Q and the point $(2,\ 1)$ is -1, $\tan\theta = -1$.
Since the gradient is negative, the required angle is obtuse.
The second quadrant solution is
$\theta = 180° - \tan^{-1}(1)$

$= 180° - 45°$
$= 135°$

The angle made with the positive direction of the x-axis is 135°.

studyon

Units 1 & 2 > Area 8 > Sequence 3 > Concept 1

Gradient and equation of a tangent Summary screen and practice questions

Exercise 13.2 Gradient and equation of a tangent

Technology free

1. Calculate the gradient of the tangent to the curve at $x = 3$ for the following functions.
   a. $f(x) = x^2 - 4x + 1$
   b. $f(x) = 2x^3 - 8x^2 + x$
   c. $f(x) = x^3 - 2x - \frac{4}{x}$
   d. $f(x) = 3\sqrt{x}$
2. Calculate the gradient of the normal to the curve at $x = 3$ for the functions given in question 1.
3. **WE1** Form the equation of the tangent to the curve $y = 5x - \frac{1}{3}x^3$ at the point on the curve where $x = 3$.
4. Form the equation of the tangent to the curve at the given point.
   a. $y = 2x^2 - 7x + 3; (0, 3)$
   b. $y = 5 - 8x - 3x^2; (-1, 10)$
   c. $y = \frac{1}{2}x^3; (2, 4)$
   d. $y = \frac{1}{3}x^3 - 2x^2 + 3x + 5; (3, 5)$
   e. $y = \frac{6}{x} + 9; \left(-\frac{1}{2}, -3\right)$
   f. $y = 38 - 2x^{\frac{3}{4}}; (81,\ -16)$
5. **WE3** Determine the equation of the normal to the graph of $y = x^3 + 2x^2 - 3x + 1$ at $x = -2$.
6. The tangent to the curve $y = ax^2 + b$ at the point where $x = 1$ has the equation $y = 2x + 3$. Find the values of a and b.

Technology active

7. Determine the equation of the tangent to the curve described by $y = \sqrt{3}x^3 + \dfrac{x}{\sqrt{2}} + 1$ at $x = 2.8$.

8. For the function $f(x) = 4x^2 - 3x + \dfrac{2}{x}$:

a. Calculate the angle to the positive x-axis made by the tangent to the curve at $x = 7$.

b. Calculate the angle to the positive x-axis made by the normal line at $x = -5$.

9. For the function $f(x) = 0.05x^3 - 0.4x^2 + x$:

a. **i.** Calculate the angle between the tangent to the curve at $x = -3$ and the positive x-axis.

ii. Calculate the angle between the normal to the curve at $x = -3$ and the positive x-axis.

b. What do you notice about your answers from part **a**?

10. The function $y = -3x^2 + 4x + 5\sqrt{x}$ gives the path of a missile travelling through the air, where y is the vertical height above the ground in metres and x is the horizontal distance travelled in metres.



a. Calculate the gradient of the curve at $x = 0$.

b. Calculate the gradient of the tangent line as $x \to 0$.

c. Calculate the angle of the tangent to the positive x-axis as $x \to 0$.

d. Explain why there is no tangent at $x = 0$.

11. For each of the following functions calculate:

a. the average rate of change between $x = 2$ and $x = 4$

b. the value of x at which the gradient of the tangent is equal to the value found in part a

i. $f(x) = x^2 + 4x - 3$

ii. $f(x) = 3.2x - 1.8x^2$

iii. $f(x) = 190x^3 + 460x - 345$

iv. $f(x) = \dfrac{0.21}{x} + 4\sqrt{x} + 0.04x$.

c. Draw a conclusion about the relationship between the average rate of change between two points and tangent to the curve with gradient equal to the average rate of change.

12. **WE2** The function $y = -3x^2 + 4x + 5$ gives the height, y metres, and horizontal distance, x metres, of a particle travelling through the air relative to the position from which it is launched.

a. Calculate the gradient of the tangent to the curve at the point where $x = 0$.

b. Calculate the angle to the horizontal at which the particle is initially launched. Give your answer to the nearest degree.

c. Determine the equation of the tangent to the curve at the point where $x = 0$.

d. The particle is now launched in a vacuum rather than in the air, and travels in a straight line from its launching position. It is launched from the same initial position and at the same angle to the horizontal. Does it pass through the point $(10, 50)$?

13. A pirate ship is making a sharp turn as it attempts to get into position to fire upon an English frigate. The path of the pirate ship is given by the function $y = 0.01x^3 - 0.3x^2, 0 \leq x \leq 30$, where x is the distance in metres north of the ship's initial position, and y is the distance in metres east of the ship's initial position. The first gunshot is fired at the English frigate at an angle of 90 degrees to the side of the ship, when the ship is 15 m east from its initial position.

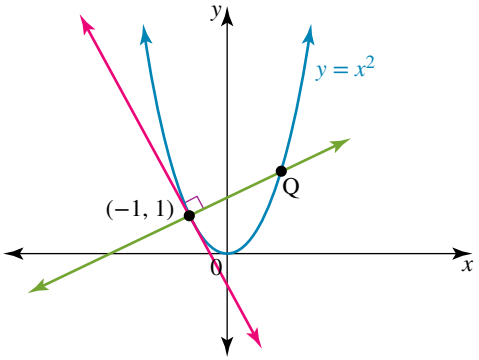


a. Determine the equation of the path of the gunshot.

b. If the frigate is at the position $(20, -31.530)$ at the time of the gunshot, will it be hit?

14. A small landing vehicle is being dropped from a height of 100 m to test its survivability. Its height is modelled by the function $h(t) = 100 - 4.9t^2$. After 2 seconds, a series of booster rockets are activated that cause the vehicle to continue to fall at a constant rate.
    a. Calculate the rate at which the landing vehicle is falling 2 seconds after being dropped.
    b. Determine the equation of the tangent to the curve at $t = 2$.
    c. When will the landing vehicle reach the ground?
    d. How much longer did it take the landing vehicle to reach the ground compared to how long it would take if it didn't have the boosters?

15. **WE4** At the point $(-1, 1)$ on the curve $y = x^2$ a line is drawn perpendicular to the tangent to the curve at that point. This line meets the curve $y = x^2$ again at the point Q.
    a. Calculate the coordinates of the point Q.
    b. Calculate the magnitude of the angle that the line through Q and the point $(-1, 1)$ makes with the positive direction of the x-axis.



16. Consider the curve with equation $y = \frac{1}{3}x(x + 4)(x - 4)$.
    a. Sketch the curve and draw a tangent to the curve at the point where $x = 3$.
    b. Form the equation of this tangent.
    c. i. The tangent meets the curve again at a point P. Show that the x-coordinate of the point P satisfies the equation $x^3 - 27x + 54 = 0$.
    ii. Explain why $(x - 3)^2$ must be a factor of this equation and hence calculate the coordinates of P.
    d. Show that the tangents to the curve at the points where $x = \pm 4$ are parallel.
    e. i. For $a \in R\backslash\{0\}$, show that the tangents to the curve $y = x(x + a)(x - a)$ at the points where $x = \pm a$ are parallel.
    ii. Calculate the coordinates, in terms of a, of the points of intersection of the tangent at $x = 0$ with each of the tangents at $x = -a$ and $x = a$.

17. Determine the equations of the tangents to the curve $y = -\frac{4}{x} - 1$ at the point(s):
    a. where the tangent is inclined at 45° to the positive direction of the x-axis
    b. where the tangent is perpendicular to the line $2y + 8x = 5$
    c. where the parabola $y = x^2 + 2x - 8$ touches the curve $y = -\frac{4}{x} - 1$; draw a sketch of the two curves showing the common tangent.

13.3 Displacement–time graphs

In this section we will consider only objects moving in straight lines, either right and left or up and down. Motion in a straight line is known as **rectilinear motion.**



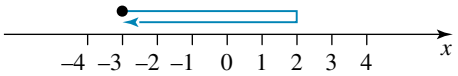
13.3.1 Definitions

1. Position, x, describes where an object is or was.
2. Distance, d, is how far an object has travelled.
3. **Displacement**, s, describes the change in an object's position; that is,
   displacement = change in position = final position − initial position
   or $s = x_{final} - x_{initial}$.
4. Speed $= \dfrac{\text{distance}}{\text{time taken}}$ or speed $= \dfrac{d}{t}$.

5. **Velocity** is the rate of change of position with respect to time, so

$$\text{velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{\text{displacement}}{\text{time taken}}; \text{ that is, } v = \frac{s}{t}.$$

Consider an object that begins at the −3-metre mark on a number line, moves 5 metres to the right, then 5 metres to the left, taking 2 seconds in total to do so.



Then, for this example we have:

1. Position (initially and finally) = −3 m
2. Distance = 10 m
3. Displacement = 0 m
4. Speed = $\frac{10\text{ m}}{2\text{ s}} = 5$ m/s
5. Velocity = $\frac{0\text{ m}}{2\text{ s}} = 0$ m/s

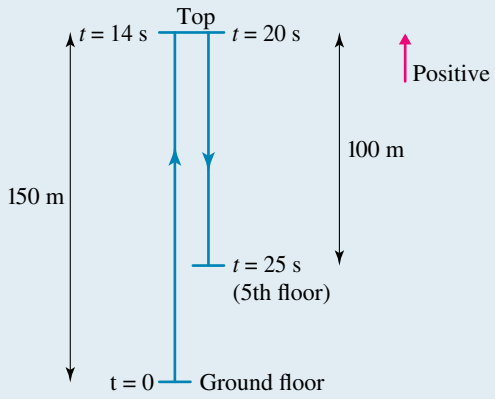
Distance and speed are (technically) always positive. Displacement and velocity can be either positive or negative, depending on the direction of motion.

(*Note:* Some texts use displacement and position interchangeably, perhaps assuming displacement from a fixed origin.)

WORKED EXAMPLE 5

Consider a lift starting from the ground floor moving up to the top floor, stopping, then coming down to the 5th floor at the times shown on the diagram. Determine:

a. the total distance travelled by the lift
b. the displacement of the lift after 25 s
c. the average speed of the lift
d. the average velocity of the lift.



THINK

a. Add the distance travelled up (150 m) to the distance travelled down (100 m).

b. At $t = 0$ s, position of the lift is 0 m. At $t = 25$ s, position is +50 m.

c. Average speed = $\frac{\text{total distance travelled}}{\text{time taken}}$

d. Average velocity = $\frac{\text{change in position}}{\text{change in time}}$

WRITE

a. Total distance = 150 m + 100 m
= 250 m

b. Displacement = change in position
= +50 − 0
= +50 m

c. Average speed = $\frac{250\text{ m}}{25\text{ s}}$
= 10 m/s

d. Average velocity = $\frac{(+50 - 0)\text{ m}}{25\text{ s}}$
= +2 m/s

WORKED EXAMPLE 6

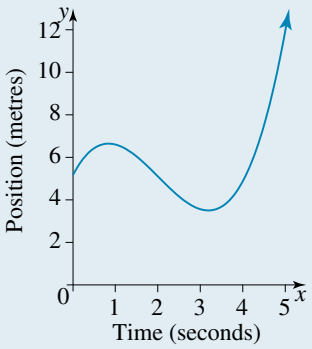
The graph below shows the position, x, versus time, t, of a particle travelling horizontally according to the function $x(t) = \frac{t^3}{2} - 3t^2 + 4t + 5$, where x is the horizontal distance travelled in metres to the right of the particle's starting point and t is the time in seconds since the particle started moving.

a. In what direction did the particle move initially?

b. What was the average velocity of the particle over the first 2 seconds?

c. At what speed was the particle travelling after 3 seconds?

d. What was its acceleration after 3 seconds?



THINK	WRITE
a. 1. Looking at the graph it can be seen that initially the particle travels from 5 m towards 6 m.	The particle initially travels to the right of the observation point.
b. 1. Calculate the position of the particle at $t = 0$ and $t = 2$.	$x(0) = \frac{0^3}{2} - 3(0)^2 + 4(0) + 5$ $= 5$ $(0, 5)$ $x(2) = \frac{2^3}{2} - 3(2)^2 + 4(2) + 5$ $= 5$ $(2, 5)$
2. Calculate the gradient between the two points.	$m = \frac{x(2) - x(0)}{2 - 0}$ $= \frac{5 - 5}{2 - 0}$ $= 0$
3. Write the answer. Include appropriate units.	The average velocity over the first 2 seconds is 0 m/s.
c. 1. The instantaneous velocity function is the derivative of the displacement function, so find the derivative of $x(t)$.	$v(t) = x'(t) = \frac{3t^2}{2} - 6t + 4$
2. Calculate the instantaneous velocity at $t = 3$ s.	$v(3) = \frac{3(3)^2}{2} - 6(3) + 4$ $= \frac{27}{2} - 18 + 4$ $= -\frac{1}{2}$
3. Write the answer. Include appropriate units.	The particle is travelling at 0.5 m/s at $t = 3$ s. (*Hint:* We do not need to include the direction as speed is a scalar quantity.)

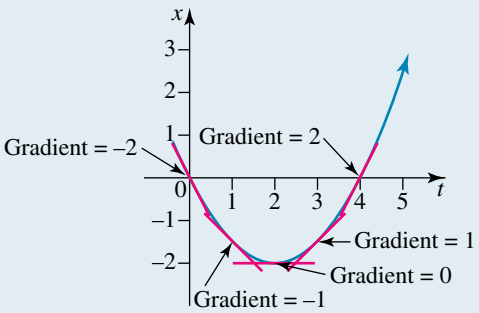
d. 1.	The instantaneous acceleration is the derivative of the velocity function, so calculate the derivative of $v(t)$.	$a(t) = v'(t)$ $= 3t - 6$
2.	Calculate the acceleration at $t = 3$ s.	$a(3) = 3(3) - 6$ $= 3$
3.	Write the answer. Include appropriate units.	The acceleration of the particle at $t = 3$ s is 3 m/s^2.

The gradient of a position-time graph gives the velocity because velocity is the rate of change of position with respect to time.

Therefore, by measuring the gradient of a position-time graph at various points, a velocity-time graph can be derived.

WORKED EXAMPLE 7

The position–time graph for a particle moving in a straight line is shown below.
The gradient of the curve at various times is indicated on the graph.
Use this information to draw a velocity–time graph for the particle.

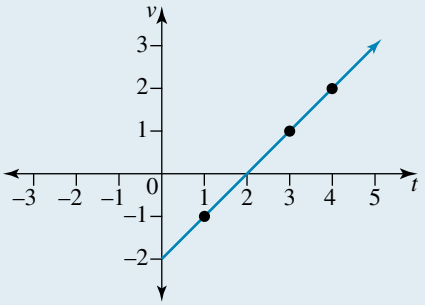


THINK

1. Set up a table of corresponding velocity and time values from the graph.
2. Use the table of values to plot the velocity–time graph for $t \geq 0$.

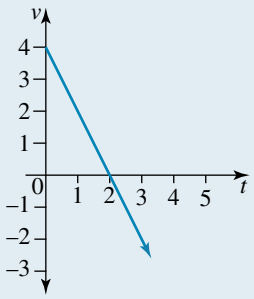
WRITE

t	0	1	2	3	4
v	–2	–1	0	1	2



WORKED EXAMPLE 8

The velocity–time graph for a particle moving in a straight line and starting at the origin is shown in the diagram. Sketch the corresponding position–time graph.

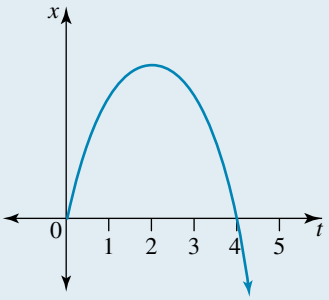


THINK

1. Set up a table of corresponding velocity and time values. (These velocity values represent the gradient of the position–time curve at the given times.)
2. Set up the axes for the position–time graph.
3. Draw in a curve starting at (0, 0) with a gradient of 4 decreasing to a gradient of 0 at $t = 2$ (turning point). From $t = 2$ to $t = 4$, the gradient changes from 0 to −4. This means the curve will become steeper but with a negative slope.

WRITE

t	0	1	2	3	4
v	4	2	0	−2	−4



studyon

Units 1 & 2 > Area 8 > Sequence 3 > Concept 2

Displacement–time graphs Summary screen and practice questions

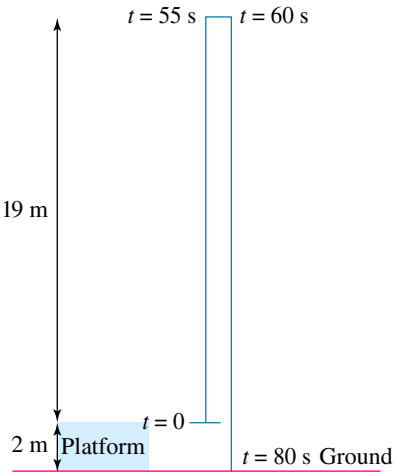
Exercise 13.3 Displacement–time graphs

Technology free

1. Match the correct description (A, B, C or D) to each of the quantities (**a**, **b**, **c** or **d**) below.

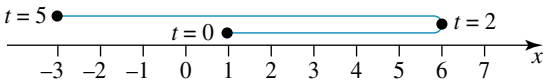
Quantity	Description
a Distance	A Rate of change of displacement with respect to time
b Displacement	B Position relative to origin
c Speed	C Length travelled
d Velocity	D Distance travelled with respect to time

2. **WE5** A parachute ride takes people in a basket vertically up in the air from a platform 2 metres above the ground, then 'drops' them back to the ground. Use the illustration showing the position of the parachute basket at various times to determine:



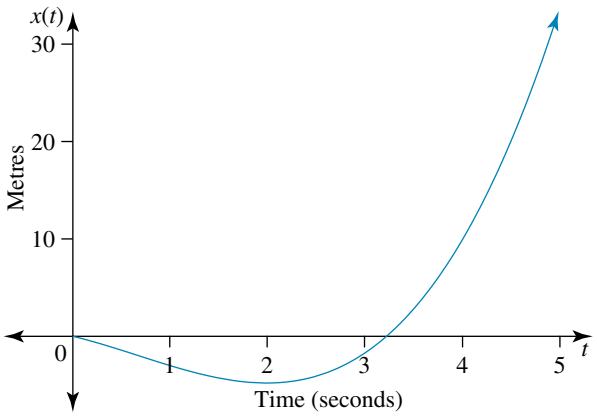
- **a.** the total distance travelled by the parachute basket during a ride
- **b.** the displacement of the parachute basket after 80 s
- **c.** the average speed of the parachute basket during the ride
- **d.** the average velocity of the parachute basket during the ride.

3. Consider the position and direction, at various times, of a particle travelling in a straight line as indicated at right.



- **a.** Where does the particle start?
- **b.** Where does the particle finish?
- **c.** In which direction does the particle move initially?
- **d.** When does the particle change direction?
- **e.** **MC** The total distance travelled in the first 5 seconds is:
  A. 4 m **B.** 13 m **C.** 9 m **D.** 14 m
- **f.** **MC** The displacement of the particle after 5 seconds is:
  A. −3 m **B.** 14 m **C.** 4 m **D.** −4 m
- **g.** **MC** The average speed in the first 2 seconds is:
  A. 3 m/s **B.** −2.5 m/s **C.** 6 m/s **D.** 2.5 m/s
- **h.** **MC** The average velocity between $t = 2$ and $t = 5$ is:
  A. 3 m/s **B.** −2 m/s **C.** −3 m/s **D.** 2 m/s
- **i.** **MC** The instantaneous speed when $t = 2$ is:
  A. 2.5 m/s **B.** 0 m/s **C.** 3 m/s **D.** 2.8 m/s

4. **WE6** The graph at right shows the position, x, versus time, t, of a particle travelling vertically according to the function $x(t) = 0.6t^3 - 1.2t^2 - 2.4t$, where x is the distance in metres above ground level and t is the time in seconds since the particle started moving.

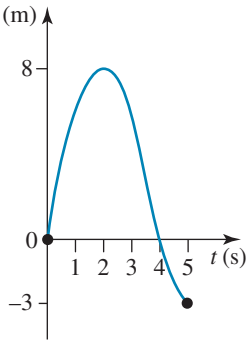


- **a.** In what direction did the particle move initially?
- **b.** What was the average velocity of the particle over the first 4 seconds?
- **c.** How fast was the particle travelling after 1 second?
- **d.** What was its acceleration after 2 seconds?

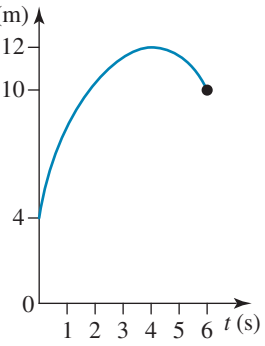
5. The following position–time graphs show the journey of a particle travelling in a straight line. For each graph determine:

- **i.** where the journey started
- **ii.** in which direction the particle moved initially
- **iii.** when and where the particle changed direction
- **iv.** when and where the particle finished its journey.

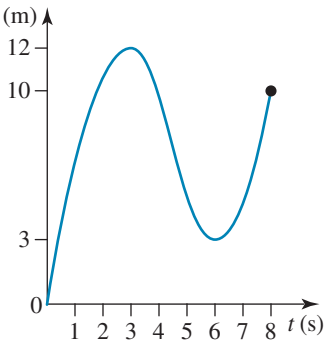
a.



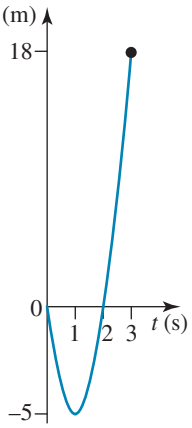
b.



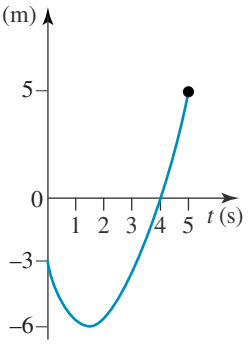
c.



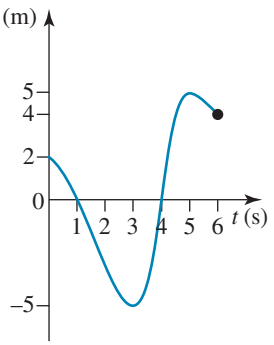
d.



e.



f.

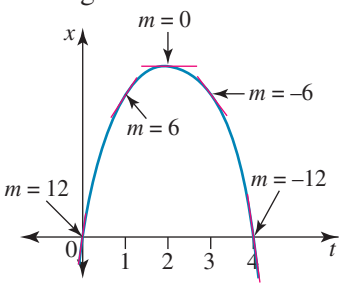


6. For each position function of a particle given below, sketch the position–time graph. In each case explain:

- **i.** where the particle started its journey
- **ii.** in which direction it moved initially
- **iii.** whether the particle changed its direction and, if so, when and where that happened
- **iv.** where the particle finished its journey.

a. $x(t) = 2t, t \in [0, 5]$
b. $x(t) = 3t - 2, t \in [0, 6]$
c. $x(t) = t^2 - 2t, t \in [0, 5]$
d. $x(t) = 2t - t^2, t \in [0, 4]$
e. $x(t) = t^2 - 4t + 4, t \in [0, 5]$
f. $x(t) = t^2 + t - 12, t \in [0, 5]$

7. **WE7** The position–time graph for a particle moving in a straight line is shown at right. The gradient of the curve at various times is indicated on the graph. Use this information to draw a velocity–time graph for the particle.



8. **a.** Plot the position–time graph for $x(t) = 4t - t^2$.

b. Calculate the gradient at:

- **i.** $t = 0$
- **ii.** $t = 1$
- **iii.** $t = 2$
- **iv.** $t = 3$
- **v.** $t = 4$.

c. Hence, give the instantaneous rate of change of position with respect to time (that is, velocity) at:

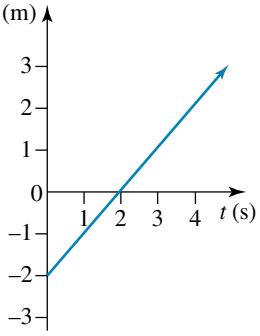
- **i.** $t = 0$
- **ii.** $t = 1$
- **iii.** $t = 2$
- **iv.** $t = 3$
- **v.** $t = 4$.

d. Sketch the velocity–time graph from $t = 0$ to $t = 5$.

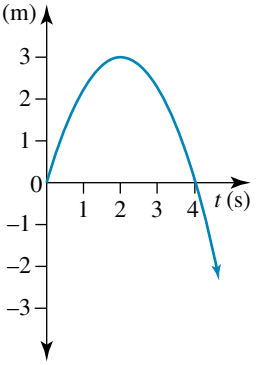
9. Match the following position–time graphs with the corresponding velocity–time graphs below.

Position–time graphs

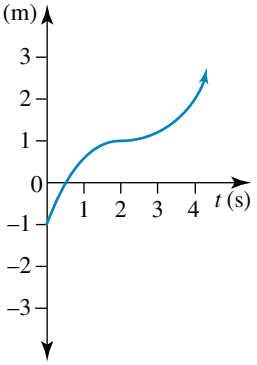
a.



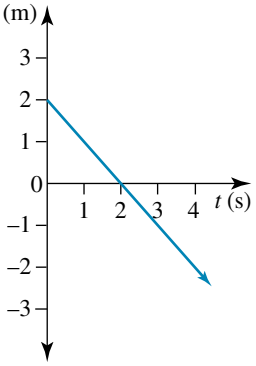
b.



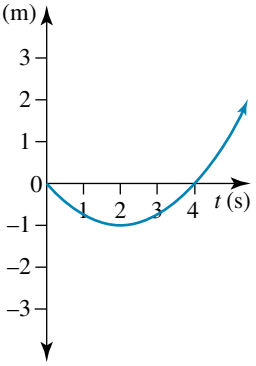
c.



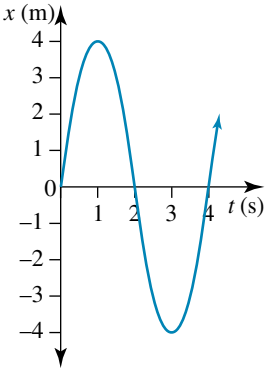
d.



e.

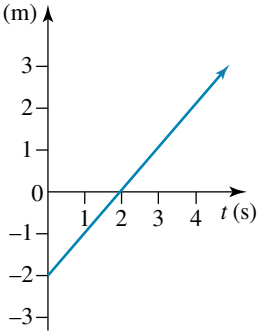


f.

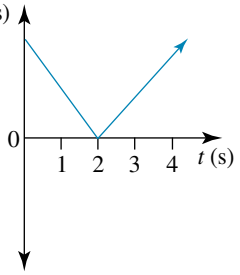


Velocity–time graphs

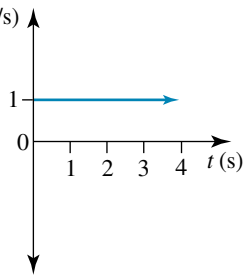
A.



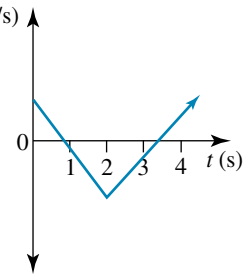
B.

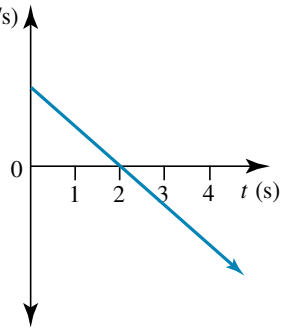


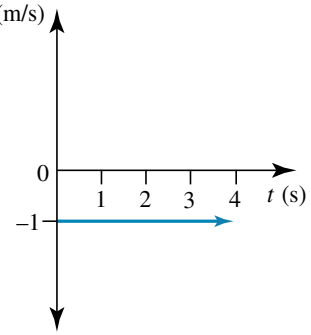
C.



D.



E. 

F. 

10. The position, x cm, relative to a fixed origin of a particle moving in a straight line at time t seconds is $x = 5t - 10$, $t \geq 0$.

a. Give its initial position and its position after 3 seconds.

b. Calculate the distance travelled in the first 3 seconds.

c. Show the particle is moving with a constant velocity.

d. Sketch the $x - t$ and $v - t$ graphs and explain their relationship.

11. Relative to a fixed origin, the position of a particle moving in a straight line at time t seconds, $t \geq 0$, is given by $x = 6t - t^2$, where x is the displacement in metres.

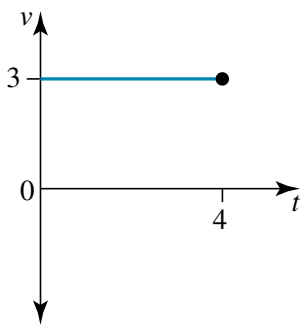
a. Write down expressions for its velocity and its acceleration at time t.

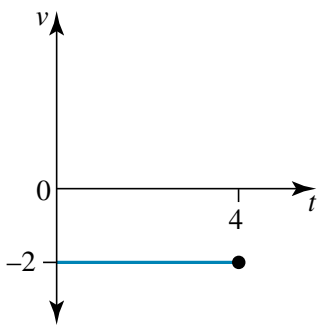
b. Sketch the three motion graphs showing displacement, velocity and acceleration versus time and describe their shapes.

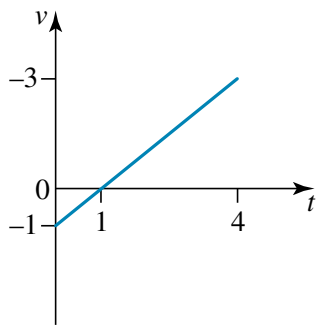
c. Use the graphs to determine when the velocity is zero; find the value of x at that time.

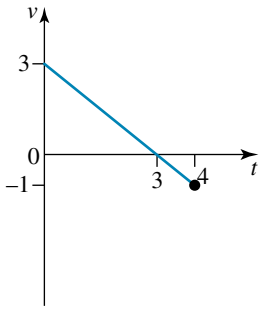
d. Use the graphs to determine when the displacement is increasing and what the sign of the velocity is for that time interval.

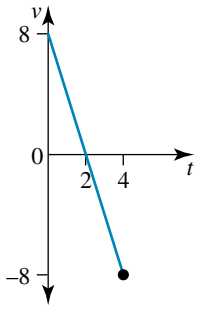
12. **WE8** For each velocity–time graph shown below, sketch a position–time graph, given that the particle starts at the origin.

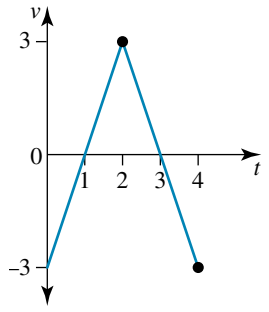
a. 

b. 

c. 

d. 

e. 

f. 

13. The velocities for a particle starting at the origin are given as a function of time below. Sketch a position–time graph for each using $t \in [0, 4]$.

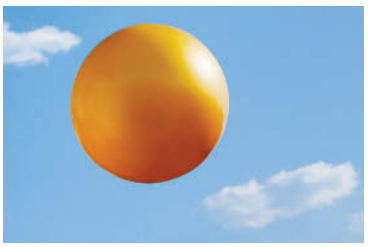
a. $v = t + 2$

b. $v = 2 - t$

c. $v = 3t$

d. $v = -t$

14. A ball is projected vertically upwards from the top of a building 25 m high. Its position relative to the ground is given by the equation $x = 25 + 20t - 5t^2$ where t is the time in seconds.



Sketch a position–time graph for the ball and hence find:

a. the greatest height reached

b. when the ball reaches the ground

c. when the velocity of the ball is zero

d. an estimate for the velocity at which the ball is initially projected.

15. A parachutist jumps from an aircraft and freefalls for 6 seconds.

a. If the parachutist falls y metres in t seconds where $y = 5t^2$, find the average speed of the parachutist between:

i. $t = 0$ and $t = 3$

ii. $t = 3$ and $t = 6$.

b. What is the speed of the parachutist after 6 seconds of freefall?

c. When the parachute is released (6 seconds after freefall), the speed of the parachutist is reduced by 2 m/s every second until a speed of 4 m/s is reached. How long after jumping from the aircraft does it take the parachutist to reach a speed of 4 m/s?



Technology active

16. The position of a particle from its starting point when moving in a straight line is given by the function $x(t) = 2.1 \times 10^{-2}\sqrt{t} - 1.36 \times 10^{-4}t^2$, where x is in metres and t is in seconds.

a. Graph the position–time function for $0 \leq t \leq 30$.

b. Graph the velocity–time function for $0 \leq t \leq 30$.

c. Graph the acceleration–time function for $0 \leq t \leq 30$.

d. Comment on the behaviour of the particle during the first 30 seconds of motion.

13.4 Sketching curves using derivatives

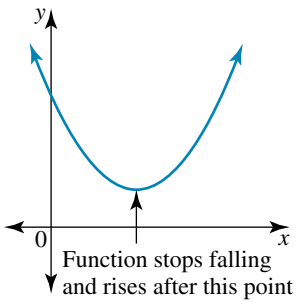
13.4.1 Sketching curves with the first derivative test

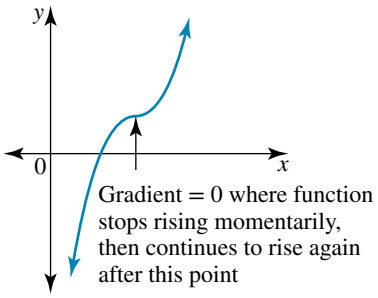
When the graphs of polynomial functions are being sketched, four main characteristics should be featured:

1. the basic shape (whenever possible)
2. the y-intercept
3. the x-intercept
4. the stationary points.

Stationary points

A *stationary point* is a point on a graph where the function momentarily stops rising or falling; that is, it is a point where the gradient is zero.





The *stationary point* (or turning point) of a quadratic function can be found by completing a perfect square in the form $y = (x + c)^2 + k$ to obtain $(-c, d)$, but for cubics, quartics or higher-degree polynomials there is no similar procedure. Differentiation enables stationary points to be found for any polynomial function where the rule is known.

The gradient of a function $f(x)$ is $f'(x)$.

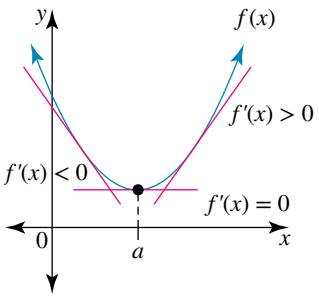
Stationary points occur wherever the gradient is zero.

$f(x)$ has stationary points when $f'(x) = 0$
or
y has stationary points when $\frac{dy}{dx} = 0$.

The solution of $f'(x) = 0$ gives the x-value or values where stationary points occur.

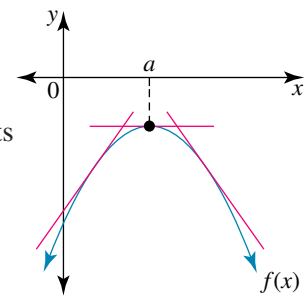
If $f'(a) = 0$, a stationary point occurs when $x = a$ and $y = f(a)$.
So the coordinate of the stationary point is $(a, f(a))$.

Types of stationary points



There are four types of stationary point.

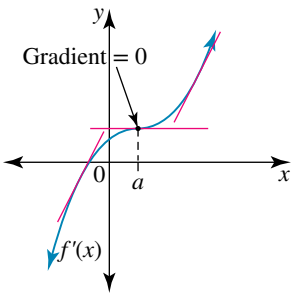
1. A **local minimum** turning point at $x = a$.
   If $x < a$, then $f'(x) < 0$ (immediately to the left of $x = a$, the gradient is negative).
   If $x = a$, then $f'(x) = 0$ (at $x = a$ the gradient is zero).
   If $x > a$, then $f'(x) > 0$ (immediately to the right of $x = a$, the gradient is positive).
2. A **local maximum** turning point at $x = a$.
   If $x < a$, then $f'(x) > 0$.
   If $x = a$, then $f'(x) = 0$.
   If $x > a$, then $f'(x) < 0$.



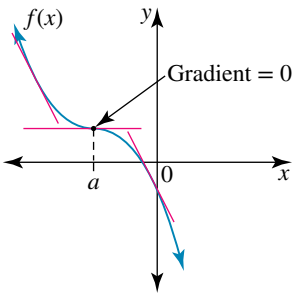
The two cases (1 and 2) can be called 'turning points' because the gradients each side of the stationary point are opposite in sign (that is, the graph turns).

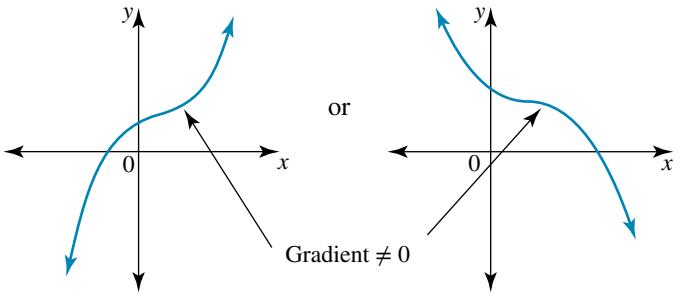
The term 'local turning point at $x = a$' implies 'in the vicinity of $x = a$', as polynomials can have more than one stationary point.

3. A positive **stationary point of horizontal inflection** at $x = a$.
If $x < a$, then $f'(x) > 0$.
If $x = a$, then $f'(x) = 0$.
If $x > a$, then $f'(x) > 0$.
That is, the gradient is positive either side of the stationary point.



4. A negative stationary point of horizontal inflection at $x = a$.
If $x < a$, then $f'(x) < 0$.
If $x = a$, then $f'(x) = 0$.
If $x > a$, then $f'(x) < 0$.
In cases 3 and 4 above the word 'stationary' implies that the gradient is zero.
Not all points of inflection are stationary points.





When determining the nature of stationary points it is helpful to complete a 'gradient table', which shows the sign of the gradient either side of any stationary points. This is known as the *first derivative* test.

Gradient tables are demonstrated in the examples that follow.

on Resources

Interactivity: Stationary points (int-5963)

WORKED EXAMPLE 9

a. Determine the stationary points for $y = x^3 + 6x^2 - 15x + 2$.
b. Determine the nature (or type) of each stationary point.
c. Sketch the graph.

THINK

a. 1. Write the equation.
2. Find $\frac{dy}{dx}$.

WRITE

a. $y = x^3 + 6x^2 - 15x + 2$
$\frac{dy}{dx} = 3x^2 + 12x - 15$

3. Solve for x if $\frac{dy}{dx} = 0$ to locate stationary points.

For stationary points,

$$\frac{dy}{dx} = 0$$

$$3x^2 + 12x - 15 = 0$$

$$3(x^2 + 4x - 5) = 0$$

$$3(x + 5)(x - 1) = 0$$

$$(x + 5)(x - 1) = 0$$

$$x = -5 \text{ or } x = 1$$

4. Substitute the x solutions into the equation and evaluate to find the corresponding y-values.

When $x = -5$, $y = (-5)^3 + 6(-5)^2 - 15(-5) + 2$

$= 102$

When $x = 1$, $y = (1)^3 + 6(1)^2 - 15(1) + 2$

$= -6$

5. State the stationary points.

The stationary points are $(-5, 102)$ and $(1, -6)$

b. 1. For each stationary point find $\frac{dy}{dx}$ immediately to the left and right to determine the nature of the stationary points. We will use $x = -6$, $x = 0$ and $x = 2$.

b. Nature: if $x = -6$, $\frac{dy}{dx} = 3(-6)^2 + 12(-6) - 15$

$= 21$ (that is, positive).

If $x = 0$, $\frac{dy}{dx} = 3(0)^2 + 12(0) - 15$

$= -15$ (that is, negative).

If $x = 2$, $\frac{dy}{dx} = 3(2)^2 + 12(2) - 15$

$= 21$ (that is, positive).

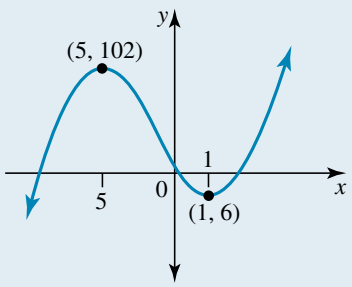
2. Complete a gradient table and state the type of each stationary point from the definitions on the previous page.

Gradient table:

x	-6	-5	0	1	2
$\frac{dy}{dx}$	+	0	−	0	+
Slope	/	−	\	−	/

Therefore $(-5, 102)$ is a local maximum turning point and $(1, -6)$ is a local minimum turning point.

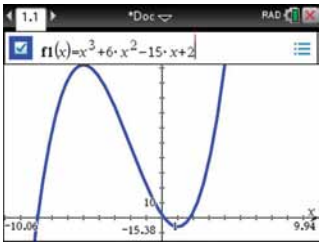
c. Sketch the graph using the stationary points.



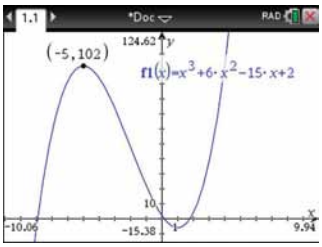
TI | THINK

WRITE

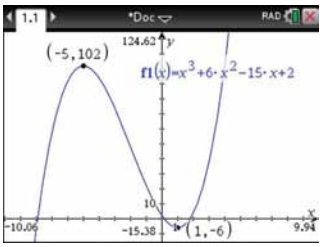
c. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = x^3 + 6x^2 - 15x + 2$ then press ENTER.



a. 1. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



2. To find the minimum, press MENU then select
6: Analyze Graph
2: Minimum
Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.



3. The answer appears on the screen.

There are stationary points at $(-5, 102)$ and $(1, -6)$.

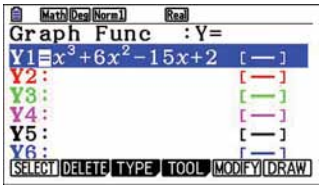
b. 1. The answer appears on the screen.

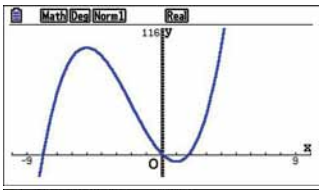
$(-5, 102)$ is a local maximum and $(1, -6)$ is a local minimum.

CASIO | THINK

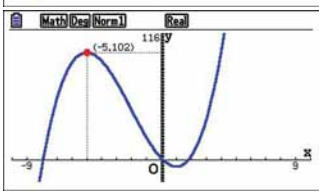
WRITE

c. 1. On a Graph screen, complete the entry line for Y1 as $y1 = x^3 + 6x^2 - 15x + 2$ then press EXE. Select DRAW by pressing F6.

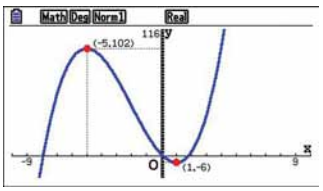




a. 1. To find the maximum, select G-Solv by pressing SHIFT F5, then select MAX by pressing F2. Press EXE.



2. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3. Press EXE.



3. The answer appears on the screen.

There are stationary points at $(-5, 102)$ a nd $(1, -6)$.

b. 1. The answer appears on the screen.

$(-5, 102)$ is a local maximum and $(1, -6)$ is a local minimum.

WORKED EXAMPLE 10

If $f(x) = x^3 + 4x^2 - 3x - 7$:

a. sketch the graph of $f'(x)$

b. state the values of x where $f(x)$ is:

i. increasing **ii. decreasing.**

THINK

a. 1. Write the rule for $f(x)$.

2. Differentiate $f(x)$ to find $f'(x)$.

WRITE

a. $f(x) = x^3 + 4x^2 - 3x - 7$

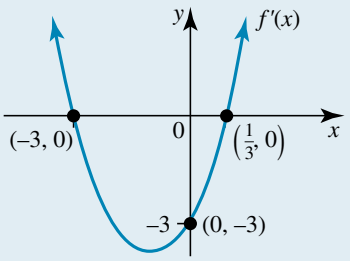
$f'(x) = 3x^2 + 8x - 3$

3. Solve $f'(x) = 0$ to find the x-intercepts of $f'(x)$.

x-intercepts: When $f'(x) = 0$,
$3x^2 + 8x - 3 = 0$
$(3x - 1)(x + 3) = 0$
$x = \frac{1}{3}$ or -3
The x-intercepts of $f'(x)$ are $\left(\frac{1}{3}, 0\right)$ and $(-3, 0)$.

4. Evaluate $f'(0)$ to find the y-intercept of $f'(x)$.

y-intercept: When $x = 0$,
$f'(0) = -3$
so the y-intercept of $f'(x)$ is $(0, -3)$.



5. Sketch the graph of $f'(x)$ (an upright parabola).

b. i. By inspecting the graph of $f'(x)$ deduce where $f'(x)$ is positive (that is, above the x-axis).

b. i. $f'(x) > 0$ where $x < -3$ and $x > \frac{1}{3}$
so $f(x)$ is increasing where $x < -3$ and $x > \frac{1}{3}$.

ii. By inspecting the graph of $f'(x)$, deduce where $f'(x)$ is negative (that is, below the x-axis).

ii. $f'(x) < 0$ where $-3 < x < \frac{1}{3}$
so $f(x)$ is decreasing where $-3 < x < \frac{1}{3}$.

13.4.2 Sketching curves with the second derivative test

The second derivative of a function can sometimes be used as an alternative method when examining the nature of any stationary points. Whilst it is often faster than using the first derivative test, the second derivative test can be inconclusive, in which case the first derivative test needs to be undertaken anyway.

The second derivative test
For a function $f(x)$, if c is a point such that $f'(c) = 0$ then $f(x)$ has a local maximum at c if $f''(c) < 0$ and a local minimum at c if $f''(c) > 0$.
If $f''(c) = 0$. then the test is inconclusive.

WORKED EXAMPLE 11

Apply the second derivative test to the following functions to determine the nature of any stationary points.

a. $f(x) = 3x^2 + 4x - 1$

b. $g(x) = x^3 - 3x^2 + 3x - 1$

c. $h(x) = \frac{x^4}{4} + x^3 - 3x^2 - 8x$

THINK	WRITE
a. 1. Find the first derivative of $f(x)$.	$f'(x) = 6x + 4$
2. Determine any stationary points where $f'(x) = 0$.	Let $f'(x) = 0$: $0 = 6x + 4$ $x = -\frac{4}{6}$ $x = -\frac{2}{3}$ $f\left(-\frac{2}{3}\right) = 3 \times \left(-\frac{2}{3}\right)^2 + 4 \times \left(-\frac{2}{3}\right) - 1$ $= -\frac{7}{3}$ There is a stationary point at $\left(-\frac{2}{3}, -\frac{7}{3}\right)$.
3. Find the second derivative of $f(x)$.	$f''(x) = 6$
4. State the nature of the stationary point.	As $f''(x) > 0$, the point $\left(-\frac{2}{3}, -\frac{7}{3}\right)$ is a local minimum.
b. 1. Find the first derivative of $g(x)$.	$g'(x) = 3x^2 - 6x + 3$
2. Determine any stationary points where $g'(x) = 0$.	Let $g'(x) = 0$: $0 = 3x^2 - 6x + 3$ $= x^2 - 2x + 1$ $= (x - 1)^2$ $x = 1$ $g(1) = (1)^3 - 3(1)^2 + 3(1) - 1$ $= 0$ There is a stationary point at $(1, 0)$.
3. Find the second derivative of $g(x)$.	$g''(x) = 6x - 6$
4. Determine the value of $g''(x)$ at any stationary points.	At $x = 1$: $g''(1) = 6(1) - 6$ $= 0$
5. State the nature of the stationary point.	As $g''(x) = 0$, the result is inconclusive and the first derivative test will need to be used.
c. 1. Find the derivative of $h(x)$.	$h'(x) = x^3 + 3x^2 - 6x - 8$
2. Determine any stationary points where $h'(x) = 0$.	Let $h'(x) = 0$: $0 = x^3 + 3x^2 - 6x - 8$ $x = -4, -1, 2$

$$h(-4) = \frac{(-4)^4}{4} + (-4)^3 - 3(-4)^2 - 8(-4)$$
$$= -16$$
$$h(-1) = \frac{(-1)^4}{4} + (-1)^3 - 3(-1)^2 - 8(-1)$$
$$= \frac{17}{4}$$
$$h(2) = \frac{(2)^4}{4} + (2)^3 - 3(2)^2 - 8(2)$$
$$= -16$$

There are stationary points at $(-4, -16)$, $\left(-1, \frac{17}{4}\right)$, $(2, -16)$.

3. Find the second derivative of $h(x)$.

$$h''(x) = 3x^2 + 6x - 6$$

4. Determine the values of $h''(x)$ at any stationary points.

$$h''(-4) = 3(-4)^2 + 6(-4) - 6$$
$$= 18$$
$$h''(-1) = 3(-1)^2 + 6(-1) - 6$$
$$= -9$$
$$h''(2) = 3(2)^2 + 6(2) - 6$$
$$= 18$$

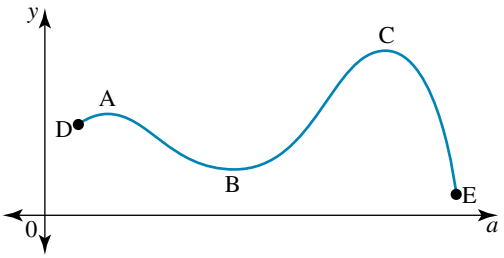
5. State the nature of the stationary points.

At $x = -4$: as $h''(-4) > 0$ the point $(-4, -16)$ is a minimum.
At $x = -1$: as $h''(-1) < 0$ the point $(-1, -9)$ is a maximum.
At $x = 2$: as $h''(2) > 0$ the point $(2, -16)$ is a minimum.

13.4.3 Global maxima and minima

Local and global maxima and minima

The diagram shows the graph of a function sketched over a domain with endpoints D and E.



There are three turning points: A and C are maximum turning points, and B is a minimum turning point.

The y-coordinate of point A is greater than those of its neighbours, so A is a local maximum point. At point C, not only is its y-coordinate greater than those of its neighbours, it is greater than that of any other point on the graph. For this reason, C is called the **global** or **absolute maximum** point.

The **global** or **absolute minimum** point is the point whose y-coordinate is smaller than any others on the graph. For this function, point E, an endpoint of the domain, is the global or absolute minimum point. Point B is a local minimum point; it is not the global minimum point.

Global maximums and global minimums may not exist for all functions. For example, a cubic function on its maximal domain may have one local maximum turning point and one local minimum turning point, but there is neither a global maximum nor a global minimum point since as $x \to \pm\infty$, $y \to \pm\infty$ (assuming a positive coefficient of x^3).

If a differentiable function has a global maximum or a global minimum value, then this will either occur at a turning point or at an endpoint of the domain. The y-coordinate of such a point gives the value of the global maximum or the global minimum.

Definitions

- **A function $y = f(x)$ has a global maximum $f(a)$ if $f(a) \geq f(x)$ for all x-values in its domain.**
- **A function $y = f(x)$ has a global minimum $f(a)$ if $f(a) \leq f(x)$ for all x-values in its domain.**
- **A function $y = f(x)$ has a local maximum $f(x_0)$ if $f(x_0) \geq f(x)$ for all x-values in the neighbourhood of x_0.**
- **A function $y = f(x)$ has a local minimum $f(x_0)$ if $f(x_0) \leq f(x)$ for all x-values in the neighbourhood of x_0.**

WORKED EXAMPLE 12

A function defined on a restricted domain has the rule $y = \frac{x}{2} + \frac{2}{x}$, $x \in \left[\frac{1}{4}, 4\right]$.

a. Specify the coordinates of the endpoints of the domain.
b. Obtain the coordinates of any stationary point and determine its nature.
c. Sketch the graph of the function.
d. State the global maximum and the global minimum values of the function, if they exist.

THINK	WRITE
a. Use the given domain to calculate the coordinates of the endpoints.	**a.** $y = \frac{x}{2} + \frac{2}{x}$ For the domain, $\frac{1}{4} \leq x \leq 4$. Substitute each of the end values of the domain in the function's rule. Left endpoint: When $x = \frac{1}{4}$, $y = \frac{x}{2} + \frac{2}{x}$ $= \frac{1}{8} + 8$ $= 8\frac{1}{8}$ Right endpoint: When $x = 4$, $y = 2 + \frac{1}{2}$ $= 2\frac{1}{2}$ Endpoints are $\left(\frac{1}{4}, \frac{65}{8}\right), \left(4, \frac{5}{2}\right)$.
b. 1. Calculate the derivative of the function.	**b.** $y = \frac{x}{2} + \frac{2}{x}$ $= \frac{x}{2} + 2x^{-1}$ $\frac{dy}{dx} = \frac{1}{2} - 2x^{-2}$ $= \frac{1}{2} - \frac{2}{x^2}$
2. Calculate the coordinates of any stationary point.	At a stationary point, $\frac{dy}{dx} = 0$, so: $\frac{1}{2} - \frac{2}{x^2} = 0$ $\frac{1}{2} = \frac{2}{x^2}$ $x^2 = 4$ $x = \pm 2$ $x = 2, x \in \left[\frac{1}{4}, 4\right]$ When $x = 2$, $y = \frac{2}{2} + \frac{2}{2}$ $= 2$ $(2, 2)$ is a stationary point.

3. Test the gradient at two selected points either side of the stationary point.

x	1	2	3
$\frac{dy}{dx}$	$\frac{1}{2}-\frac{2}{1}=-\frac{3}{2}$	0	$\frac{1}{2}-\frac{2}{9}=\frac{5}{18}$
Slope	\	—	/

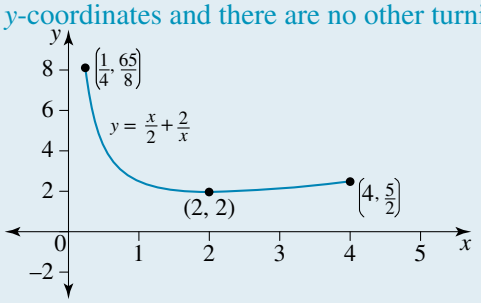
4. State the nature of the stationary point.

The gradient changes from negative to zero to positive about the stationary point.
The point (2, 2) is a minimum turning point.

c. Calculate any intercepts with the coordinate axes.

c. There is no y-intercept since $x = 0$ is not in the given domain, nor is $y = \frac{x}{2} + \frac{2}{x}$ defined at $x = 0$.
There is no x-intercept since the endpoints and the minimum turning point all have positive y-coordinates and there are no other turning points.

2. Sketch the graph using the three known points.



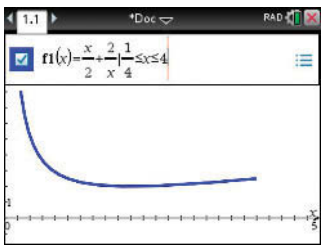
d. Examine the graph and the y-coordinates to identify the global extrema.

d. The function has a global maximum of $8\frac{1}{8}$ at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

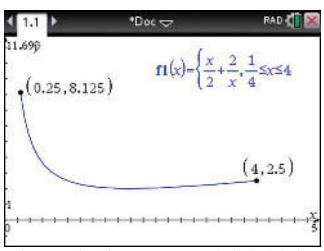
TI | THINK

c. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = \frac{x}{2} + \frac{2}{x} \mid \frac{1}{4} \le x \le 4$ then press ENTER.

WRITE



a. 1. To find the endpoints, press MENU then select
5: Trace
1: Graph Trace
Type '0.25' then press ENTER twice. Type '4' then press ENTER twice.



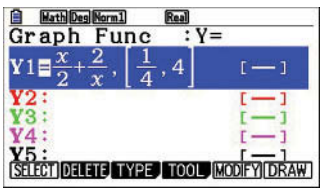
2. The answer appears on the screen.

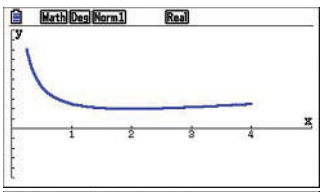
The endpoints are (−.25, 8.125) and(4, 2.5).

CASIO | THINK

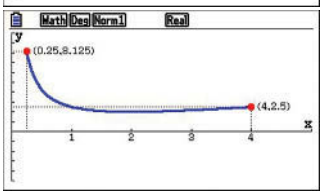
c. 1. On a Graph screen, complete the entry line for Y1 as $y1 = \frac{x}{2} + \frac{2}{x}, \left[\frac{1}{4}, 4\right]$ then press EXE.
Select DRAW by pressing F6.

WRITE





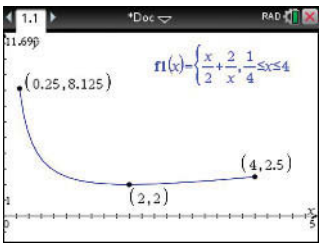
a. 1. To find the endpoints, select Trace by pressing SHIFT then F1. Type '0.25' then press EXE twice. Type '4' then press EXE twice.



2. The answer appears on the screen.

The endpoints are (−.25, 8.125) and (4, 2.5).

b. 1. To find the minimum, press MENU then select
6: Analyze Graph
2: Minimum
Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.



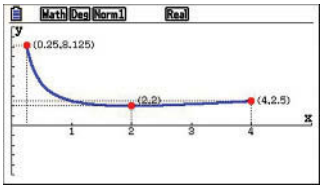
2. The answer appears on the screen.

There is a local minimum at (2, 2).

d. 1. The answer appears on the screen.

The function has a global maximum of 8.125 at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

b. 1. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3. Press EXE.



2. The answer appears on the screen.

There is a local minimum at (2, 2).

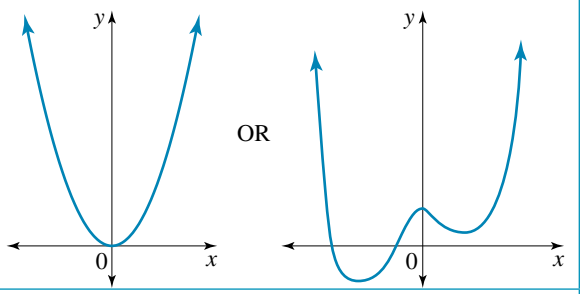
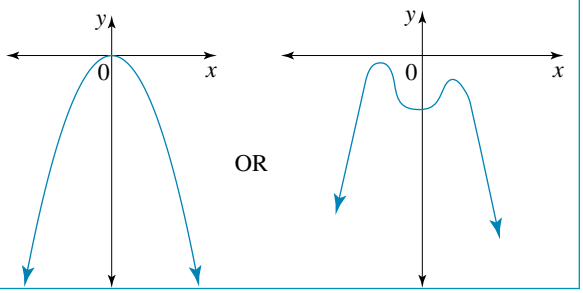
d. 1. The answer appears on the screen.

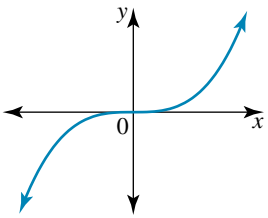
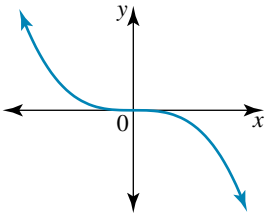
The function has a global maximum of 8.125 at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

13.4.4 End behaviour of a function

The end behaviour of a **function** indicates what the function does as it approaches positive and negative infinity.

For a polynomial function the term with the largest degree has the biggest impact as $x \to \pm\infty$. The coefficient of this term is called the **leading coefficient**. We use the degree of the polynomial and the sign of the leading coefficient to determine the end behaviour of a function.

Degree	Leading coefficient	End behaviour of the function	Example Graphs of the functions
Even	Positive	$f(x) \to \infty$ as $x \to -\infty$ $f(x) \to \infty$ as $x \to \infty$	
Even	Negative	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to \infty$	

Odd	Positive	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to \infty$ as $x \to \infty$	
Odd	Negative	$f(x) \to \infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to \infty$	

We examine end behaviour when sketching functions because it gives us further information about the shape of the graph.

studyon

Units 1 & 2 > Area 8 > Sequence 3 > Concept 3

Sketching curves with local maximum and minimum turning points Summary screen and practice questions

Exercise 13.4 Sketching curves using derivatives

Technology free

1. Determine the stationary points and their nature for each of the following functions.
   a. $y = 8 - x^2$ b. $f(x) = x^3 - 3x$ c. $g(x) = 2x^2 - 8x$
   d. $f(x) = 4x - 2x^2 - x^3$ e. $g(x) = 4x^3 - 3x^4$ f. $y = x^2(x + 3)$
2. WE9 a. Determine the stationary points for $y = x^3 + 6x^2 - 15x + 2$.
   b. Determine the nature of each stationary point.
   c. Examine the end behaviours of the function.
   d. Sketch the graph.
3. WE10 If $h(x) = x^4 + 4x^3 + 4x^2$:
   a. Sketch the graph of $h'(x)$.
   b. State the values of x where $h(x)$ is:
      i. increasing
      ii. decreasing.
4. WE11 Apply the second derivative test to the following functions to determine the nature of any stationary points.
   a. $y = 5 - 6x + x^2$ b. $f(x) = x^3 + 8$ c. $y = -x^2 - x + 6$
   d. $y = 3x^4 - 8x^3 + 6x^2 + 5$ e. $g(x) = x(x^2 - 27)$ f. $y = x^3 + 4x^2 - 3x - 2$
   g. $h(x) = 12 - x^3$ h. $g(x) = x^3(x - 4)$
5. Describe the graph of $g(x) = x^4 - 4x^2$ by examining its stationary points and end behaviour.
6. MC If $f'(x) < 0$ where $x > 2$ and $f'(x) > 0$ where $x < 2$, then $x = 2$, $f(x)$ has a:
   A. local minimum B. local maximum
   C. point of inflection D. discontinuous point.

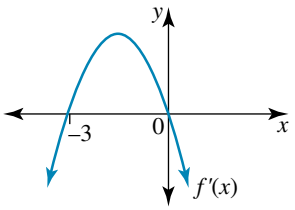
7. **MC** The graph of $y = x^4 + x^3$ has:
   - **A.** a local maximum where $x = 0$
   - **B.** a local minimum where $x = 0$
   - **C.** a local minimum where $x = -\frac{3}{4}$
   - **D.** a local maximum where $x = -\frac{3}{4}$.

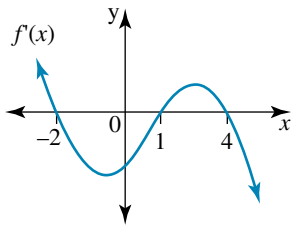
8. The curve $y = ax^2 + bx + c$ contains the point $(0, 5)$ and has a stationary point at $(2, -14)$. Calculate the values of a, b and c.

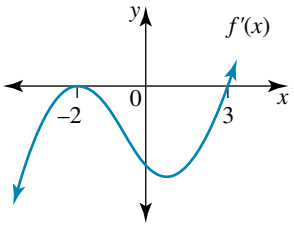
9. **a.** What is the greatest and least number of turning points a cubic function can have?
   b. Show that $y = 3x^3 + 6x^2 + 4x + 6$ has one stationary point and determine its nature.
   c. Determine the values of k so the graph of $y = 3x^3 + 6x^2 + kx + 6$ will have no stationary points.
   d. If a cubic function has exactly one stationary point, explain why it is not possible for that stationary point to be a maximum turning point. What type of stationary point must it be?
   e. State the degree of the gradient function of a cubic function and use this to explain whether it is possible for the graph of a cubic function to have two stationary points: one a stationary point of inflection and the other a maximum turning point.
   f. Show that the line through the turning points of the cubic function $y = xa^2 - x^3$ must pass through the origin for any real positive constant a.

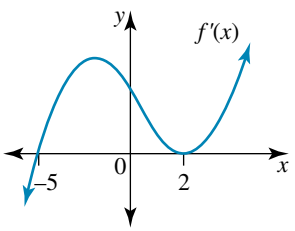
10. The curve $y = x^3 + ax^2 + bx - 11$ has stationary points when $x = 2$ and $x = 4$.
    a. Calculate a and b.
    b. Determine the coordinates of the stationary points and their nature.

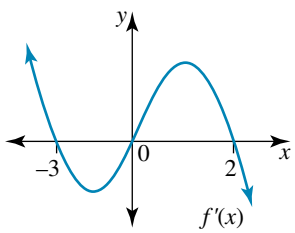
11. The graphs of $f'(x)$ are shown below. Obtain all values of x for which $f(x)$ has stationary points and state their nature.

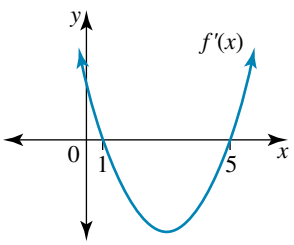
a.


b.


c.


d.


e.


f.


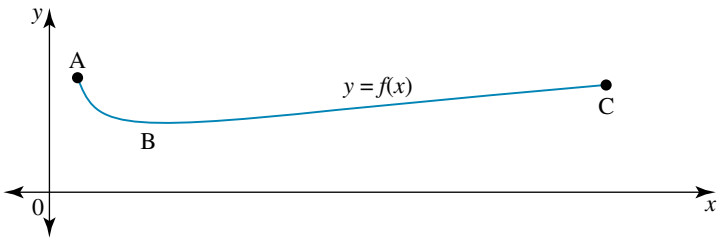
12. **WE12** A function defined on a restricted domain has the rule:
    $y = \frac{1}{16}x^2 + \frac{1}{x}$, $x \in \left[\frac{1}{4}, 4\right]$.
    a. Specify the coordinates of the endpoints of the domain.
    b. Obtain the coordinates of any stationary point and determine its nature.
    c. Sketch the graph of the function.
    d. State the global maximum and global minimum values of the function, if they exist.

13. The graph of $f(x) = 2\sqrt{x} + \frac{1}{x}$, $0.25 \leq x \leq 5$ is shown below.

a. Determine the coordinates of the endpoints A and C and the stationary point B.

b. At which point does the global maximum occur?

c. State the global maximum and global minimum values.



Technology active

14. a. Give the coordinates and state the type of any stationary points on the graph of $f(x) = -0.625x^3 + 7.5x^2 - 20x$, expressing answers to 2 decimal places.

b. Sketch $y = f'(x)$ and state the coordinates of its turning point.

c. What does the behaviour of $y = f'(x)$ at its turning point tell us about the behaviour of $y = f(x)$ at the point with the same x-coordinate?

15. a. Use technology to sketch the function $y = 2x^5 + 6x^4 + 4x^3$, examining end behaviours and identifying all intercepts and stationary points.

b. Use technology to sketch the function over the domain $[-10, 10]$ and comment on the importance of examining all points of interest.

16. a. Demonstrate that the function $y = 2x^3 - x^2 + x - 1$ has no turning points.

b. The second derivative of a function will be equal to zero at any point of inflection. Use the second derivative to determine the coordinates of the point of inflection.

c. Calculate the value of the gradient at selected x-values either side of the point of inflection.

d. Use your answers from part *c* to explain the behaviour of the gradient of this function at the point of inflection.

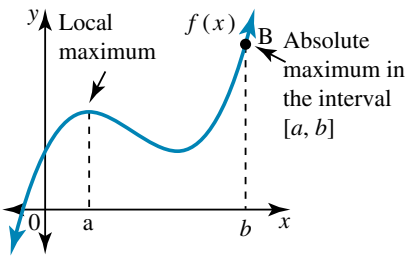
13.5 Modelling optimisation problems

There are many practical situations where it is necessary to determine the maximum or minimum value of a function.

When solving maximum or minimum problems (to obtain the value(s) of x) it should be verified that it is in fact a maximum or minimum by checking the sign of the derivative to left and right of the turning point.

In the case of cubic and higher order polynomials, the local maximum (or minimum) may or may not be the highest (or lowest) value of the function in a given domain.

An example where the local maximum, found by solving $f'(x) = 0$, is *not* the largest value of $f(x)$ in the domain $[a, b]$ is shown. Here, B is the point where $f(x)$ is greatest in this domain, and is called the *absolute maximum* for the interval.



13.5.1 When the rule for the function is known

WORKED EXAMPLE 13

A cricket fielder throws a ball so that the equation of its path is:

$$y = 1.5 + x - 0.02x^2$$

where x (metres) is the horizontal distance travelled by the ball and y (metres) is the vertical height reached.

a. Determine the value of x for which the maximum height is reached and verify that it is a maximum.

b. What is the maximum height reached?

THINK	WRITE
a. 1. Write the equation of the path.	**a.** $y = 1.5 + x - 0.02x^2$
2. Find the derivative $\frac{dy}{dx}$.	$\frac{dy}{dx} = 1 - 0.04x$
3. Solve the equation $\frac{dy}{dx} = 0$ to find the value of x for which height is a maximum.	For stationary points: $\frac{dy}{dx} = 0$ $1 - 0.04x = 0$ $-0.04x = -1$ $x = 25$
4. Determine the nature of this stationary point at $x = 25$ by evaluating $\frac{dy}{dx}$ to the left and right, say, at $x = 24$ and at $x = 26$.	When $x = 24$, $\frac{dy}{dx} = 1 - 0.04\,(24)$ $= 0.04$ When $x = 26$, $\frac{dy}{dx} = 1 - 0.04\,(26)$ $= -0.04$ Zero gradient Positive gradient Negative gradient
5. Since the gradient changes from positive to negative as we move from left to right in the vicinity of $x = 25$, the stationary point is a local maximum.	The stationary point is a local maximum.
b. Substitute $x = 25$ into $y = 1.5 + x - 0.02x^2$ to find the corresponding y-value (maximum height).	**b.** When $x = 25$, $y = 1.5 + 25 - 0.02\,(25)^2$ $= 14$ So the maximum height reached is 14 m.

13.5.2 When the rule for the function is not known

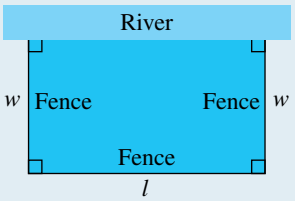
If the rule is not given directly then the following steps should be followed:

1. Draw a diagram if necessary and write an equation linking the given information.
2. Identify the quantity to be maximised or minimised.

3. Express this quantity as a function of one variable only (often this will be x).
4. Differentiate, set the derivative equal to zero, and solve.
5. Determine, in the case of more than one value, which one represents the maximum or minimum value.
6. For some functions, a maximum or minimum may occur at the extreme points of the domain so check these also.
7. Sketch a graph of the function if it helps to answer the question, noting any restrictions on the domain.
8. Answer the question that is being asked, in words.

WORKED EXAMPLE 14

A farmer wishes to fence off a rectangular paddock on a straight stretch of river so that only 3 sides of fencing are required. Determine the largest possible area of the paddock if 240 metres of fencing is available.

THINK	WRITE
1. Draw a diagram to represent the situation, using labels to represent the variables for length and width and write an equation involving the given information.	Let w = width l = length P = perimeter  $P = l + 2w = 240$ [1]
2. Write a rule for the area, A, of the paddock in terms of length, l, and width, w.	$A = l \times w$ [2]
3. Express the length, l, of the rectangle in terms of the width, w, using equation [1].	$l + 2w = 240$ [3] $l = 240 - 2w$
4. Express the quantity, A, as a function of one variable, w, by substituting [3] into [2].	Substituting [3] into [2]: $A(w) = (240 - 2w)w$ $= 240w - 2w^2$
5. Solve $A'(w) = 0$.	$A'(w) = 240 - 4w$ For stationary points: $A'(w) = 0$ $240 - 4w = 0$ $240 = 4w$ $w = 60$
6. Test to see if the stationary point is a maximum or minimum value for the area by evaluating the second derivative, $A''(w)$, at $w = 60$.	$A''(w) = -4$ $A''(w)$ is negative for all so the stationary point is a local maximum
7. Find the maximum area of the paddock by substituting $w = 60$ into the function for area.	The stationary point is a local maximum. The area of the paddock is a maximum when $w = 60$. $A(60) = (240 - 2 \times 60) \times 60$ $= 7200 \text{ m}^2$

Exercise 13.5 Modelling optimisation problems

Technology active



1. **WE13** A golfer hits the ball so that the equation of its path is:

$$y = 1.2 + x - 0.025x^2$$

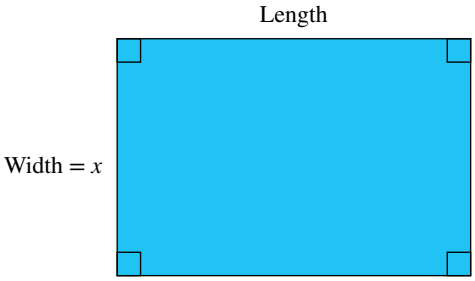
where x (metres) is the horizontal distance travelled by the ball and y (metres) is the vertical height reached.
   a. Determine the value of x for which the maximum height is reached (and verify that it is a maximum).
   b. What is the maximum height reached?



2. If the volume of water, V litres, in a family's hot water tank t minutes after the shower is turned on is given by the rule $V = 200 - 1.2t^2 + 0.08t^3$, where $0 < t < 15$:
   a. What is the time when the volume is a minimum (that is, during the length of time the shower is on)?
   b. Verify that it is a minimum by checking the sign of the derivative.
   c. Calculate the minimum volume.
   d. Calculate the value of t when the tank is back to its intial volume.
3. A ball is thrown into the air so that its height, h metres, above the ground at time t seconds after being thrown is given by the function:

$$h(t) = 1 + 15t - 5t^2.$$

   a. Determine the greatest height reached by the ball and the value of t for which it occurs.
   b. Verify that it is a maximum.
4. **WE14** A gardener wishes to fence off a rectangular vegetable patch against her back fence so that only 3 sides of new fencing are required. Determine the largest possible area of the vegetable patch if she has 16 m of fencing material available.
5. The sum of two numbers is 16.
   a. By letting one number be x, find an expression for the other number.
   b. Obtain an expression for the product of the two numbers, P.
   c. Hence, find the numbers if P is a maximum.
   d. Verify that it is a maximum.
6. The rectangle at right has a perimeter of 20 cm.
   a. If the width is x cm, write an expression for the length.
   b. Write an expression for the area, A, in terms of x only.
   c. What is the value of x required for maximum area?
   d. Determine the dimensions of the rectangle for maximum area.
   e. Hence, calculate the maximum area.



7. A farmer wishes to create a rectangular pen to contain as much area as possible using 60 metres of fencing.

a. Write expressions for the dimensions (length and width) of the pen.

b. Hence, calculate the maximum area.

8. The cost of producing a particular toaster is $\$(250 + 1.2n^2)$ where n is the number produced each day. If the toasters are sold for \$60 each:



a. Write an expression for the profit, P, dollars.

b. How many toasters should be produced each day for maximum profit?

c. Hence, determine the maximum daily profit possible.

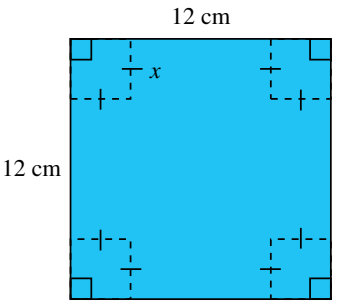
9. A company's income each week is $\$(800 + 1000n - 20n^2)$ where n is the number of employees. The company spends \$760 per employee for wages and materials.

a. Write an expression for the company weekly profit, P dollars.

b. Determine the number of employees required for maximum profit and hence calculate the maximum weekly profit.

10. The sum of two numbers is 10. Calculate the numbers if the sum of their squares is to be a minimum.

11. A square has four equal squares cut out of the corners as shown at right. It is then folded to form an open rectangular box.



a. What is the range of possible values for x?

b. In terms of x find expressions for the:

i. height **ii.** length

iii. width of the box.

c. Write an expression for the volume, V (in terms of x only).

d. What is the maximum possible volume of the box?

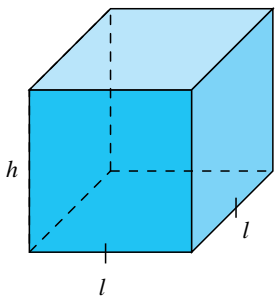
12. The base and sides of a shirt box are to be made from a rectangular sheet of cardboard measuring 50 cm × 40 cm. Determine:

a. the dimensions of the box required for maximum volume

b. the maximum volume.

(Give answers correct to 2 decimal places.)

13. The volume of the square-based box shown at right is $256\,\text{cm}^3$.



a. Write h in terms of l.

If the box has an open top determine:

b. the surface area, A, in terms of l only

c. the dimensions of the box if the surface area is to be a minimum

d. the minimum area. (*Hint:* $\frac{1}{l} = l^{-1}$)

14. A closed, square-based box of volume $1000\,\text{cm}^3$ is to be constructed using the minimum amount of metal sheet possible. Calculate its dimensions.

15. The cost of flying an aircraft on a 900 km journey is $1600 + \frac{1}{100}v^2$ dollars per hour, where v is the speed of the aircraft in km/h. Calculate:



a. the cost, C dollars, of the journey if $v = 300$ km/h

b. the cost, C dollars, of the journey in terms of v (*Hint:* time = distance ÷ speed)

c. the most economical speed and minimum cost.

16. Calculate the maximum volume of a cylinder that can be enclosed by a cone that is 10 cm high and has a radius of 5 cm.

13.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Determine the equation of the tangent line to the function $y = 2x^2 - 3x + 1$ at $x = 3$.
2. For the function $y = 0.5x^4 - x^2 - 4$:
   a. Calculate the angle between the tangent to the curve at $x = 0.8$ and the positive x-axis.
   b. Calculate the angle between the normal to the curve at $x = -1.2$ and the positive x-axis.
3. Determine the equation of the normal line to the function $y = x^3 + 7x^2 - 2x + 3$ at $x = -2$.
4. For the position–time function $x(t) = 2t^2 - 4t + 1$:
   a. Calculate the gradient at
      i. $t = 0$ ii. $t = 2$ iii. $t = 4$.
   b. Sketch the velocity–time graph from $t = 0$ to $t = 4$.
5. For the position–time function $x(t) = t^3 + \frac{t^2}{3} - \frac{3}{4}t + 2$ determine:
   a. the velocity–time function
   b. the acceleration–time function.
6. The position of a particle after t seconds is given by $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1$, $t \geq 0$.
   a. Determine its initial position and initial velocity.
   b. Calculate the distance travelled before it changes its direction of motion.
   c. What is its acceleration at the instant it changes direction?
7. a. Use the first derivative test to determine the nature of the stationary points for each of the following functions.
      i. $f(x) = 2x^3 + 6x^2$ ii. $g(x) = -x^3 + 4x^2 + 3x - 12$
      iii. $h(x) = 9x^3 - 117x + 108$ iv. $p(x) = x^3 + 2x$
      v. $x^4 - 6x^2 + 8$ vi. $y = 2x(x + 1)^3$
   b. Use the second derivative test to confirm your results from part a.
8. Determine the stationary points of $f(x) = x^3 + x^2 - x + 4$ and justify their nature.
9. Consider the function defined by $f(x) = x^3 + 3x^2 + 8$.
   a. Show that $(-2, 12)$ is a stationary point of the function.
   b. Determine the nature of this stationary point.
   c. Give the coordinates of the other stationary point.
   d. Justify the nature of the second stationary point.
10. The cost in dollars of employing n people per hour in a small distribution centre is modelled by $C = n^3 - 10n^2 - 32n + 400$, $5 \leq n \leq 10$.
    Calculate the number of people who should be employed in order to minimise the cost and justify your answer.
11. A batsman opening the innings in a cricket match strikes the ball so that its height y metres above the ground after it has travelled a horizontal distance x metres is given by $y = 0.0001x^2(625 - x^2)$.
    a. Calculate, to 2 decimal places, the greatest height the ball reaches and justify the maximum nature.
    b. Determine how far the ball travels horizontally before it strikes the ground.



12. A rectangular vegetable garden patch uses part of a back fence as the length of once side. There are 40 meters of fencing available for enclosing the other three sides of the vegetable garden.



a. Draw a diagram of the garden and express the area in terms of the width (the width being the length of the sides perpendicular to the back fence).

b. Use calculus to obtain the dimensions of the garden for maximum area. And hence state the maximum area.

Complex familiar

13. For the function $f(x) = 0.8x^2 + 0.4x - 3$, calculate the value of x at which the gradient of the tangent is equal to the average rate of change between $x = -1$ and $x = 2$.

14. A ball is thrown vertically upwards into the air so that after t seconds, its height h metres above the ground is $h = 40t - 5t^2$.

a. At what rate is its height changing after 2 seconds?

b. Calculate its velocity when $t = 3$.

c. At what time is its velocity -10 m/s and in what direction is the ball then travelling?

d. When is its velocity zero?

e. What is the greatest height the ball reaches?

f. At what time and with what speed does the ball strike the ground?

15. Sketch the function $y = x^4 + 2x^3 - 2x - 1$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.

16. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length x cm out of the four corners and folding the flaps up.

a. Express the volume as a function of x.

b. Determine the dimensions of the box with greatest volume and give this maximum volume to the nearest whole number.

Complex unfamiliar



17. A skier is following a trail that curves according to the function $y = -0.09x^2 + 2x$, $0 \leq x \leq 30$ when she loses control and slides off the trail at the point where $x = 15$. Assuming that the skier slides off the path in a straight line, determine if the skier is likely to hit a large tree located at the point $(20, 6.25)$.

18. A particle P moving in a straight line has displacement, x metres, from a fixed origin O of $x_P(t) = t^3 - 12t^2 + 45t - 34$ for time t seconds.

a. At what time (s) is the particle stationary?

b. Over what time interval is the velocity negative?

c. When is its acceleration negative?

A second particle Q also travels in a straight line with its position from O at time t seconds given by $x_Q(t) = -12t^2 + 54t - 44$.

d. At what time are P and Q travelling with the same velocity?

e. At what times do P and Q have the same displacement from O?

19. The point $(2, -54)$ is a stationary point of the curve $y = x^3 + bx^2 + cx - 26$.

a. Calculate the values of b and c.

b. Obtain the coordinates of the other stationary point.

c. Identify where the curve intersects each of the coordinate axes.

d. Sketch the curve and label all key points with their coordinates.

20. The city of Prague has an excellent transport system. Shirley is holidaying in Prague and has spent the day walking in the countryside. Relative to a fixed origin, with measurements in kilometres, Shirley is at the point S $(4, 0)$. She intends to catch a tram back to her hotel in the heart of the city. Looking at her map, Shirley notices the tram route follows a path that could be modelled by the curve $y = \sqrt{x}$.



a. Draw a diagram showing the tram route and Shirley's position, and calculate how far directly north of Shirley (in the direction of the y-axis) the tram route is.

Being a smart mathematician, Shirley realises she can calculate how far it is to the closest point on that tram route. She calculates a function W, the square of the distance from the point S$(4, 0)$ to the point T(x, y) on the curve $y = \sqrt{x}$.

b. Write down an expression for the distance TS and hence show that $W = x^2 - 7x + 16$.

c. Use calculus to obtain the value of x for which W is minimised.

d. Obtain the coordinates of T, the closest point on the tram route to Shirley.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 13 Applications of derivatives

Exercise 13.2 Gradient and equation of a tangent

1. **a.** 2 **b.** 7 **c.** $\frac{229}{9}$ **d.** $\frac{\sqrt{3}}{2}$

2. **a.** $-\frac{1}{2}$ **b.** $-\frac{1}{7}$
c. $-\frac{9}{229}$ **d.** $-\frac{2}{\sqrt{3}}$ or $\frac{-2\sqrt{3}}{3}$

3. $y = -4x + 18$

4. **a.** $y = -7x + 3$ **b.** $y = -2x + 8$
c. $y = 6x - 8$ **d.** $y = 5$
e. $y = -24x - 15$ **f.** $2y + x = 49$

5. $y = 5 - x$

6. $a = 1$ and $b = 4$

7. $y = -0.024x + 41.069$

8. **a.** 88.9° **b.** 1.33°

9. **a.** **i.** 78.11° **ii.** −11.89°
b. The difference is 90°.

10. **a.** undefined **b.** ∞ **c.** 90°
d. the derivative of the function is undefined at $x = 0$.

11. **a.** **i.** 10 **ii.** −7.6 **iii.** 5780 **iv.** 1.185
b. **i.** 3 **ii.** 3
iii. −3.055, 3.055 **iv.** 0.284, 2.923
c. There is always a point between the two points over which the average was calculated where the gradient of the tangent is equal to the average rate of change.

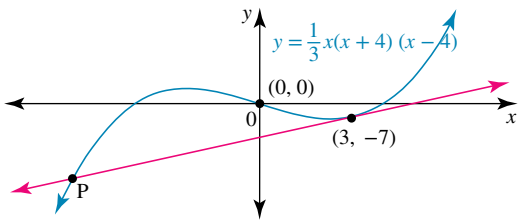
12. **a.** 4 **b.** 76°
c. $y = 4x + 5$ **d.** No

13. **a.** $y = 0.444x - 40.417$ **b.** Yes

14. **a.** −19.6 m/s **b.** $y = -19.6t + 119.6$
c. 6.102s **d.** 1.585s

15. **a.** $\left(\frac{3}{2}, \frac{9}{4}\right)$ **b.** 26.6°

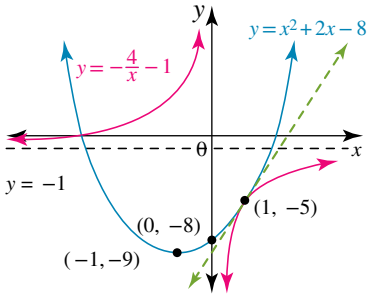
16. **a.**



b. $y = \frac{11}{3}x - 18$
c. **i.** Sample responses can be found in the worked solutions in the online resources.
ii. (−6, −40)
d. Same gradient
e. **i.** Same gradient
ii. $\left(-\frac{2a}{3}, \frac{2a^3}{3}\right), \left(\frac{2a}{3}, -\frac{2a^3}{3}\right)$

17. **a.** $y = x - 5,\ y = x + 3$
b. $4y - x + 12 = 0,\ 4y - x - 4 = 0$

c. $y = 4x - 9$;



Exercise 13.3 Displacement–time graphs

1. **a.** C **b.** B **c.** D **d.** A

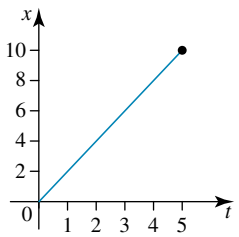
2. **a.** 40 m
b. −2 m (or 2 m below the platform)
c. 0.5 m/s
d. 0.025 m/s (or 0.025 m/s downwards)

3. **a.** $x = 1$ **b.** $x = -3$ **c.** right
d. $t = 2$ **e.** D **f.** D
g. D **h.** C **i.** B

4. **a.** Downward **b.** 2.4 m/s
c. −3 m/s **d.** 4.8 m/s^2

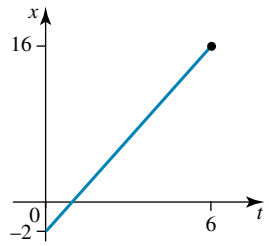
5. **a.** **i.** $x = 0$ **ii.** Right
iii. $t = 2,\ x = 8$ **iv.** $t = 5,\ x = -3$
b. **i.** $x = 4$ **ii.** Right
iii. $t = 4,\ x = 12$ **iv.** $t = 6,\ x = 10$
c. **i.** $x = 0$
ii. Right
iii. $t = 3,\ x = 12$ and $t = 6,\ x = 3$
iv. $t = 8,\ x = 10$
d. **i.** $x = 0$ **ii.** Left
iii. $t = 1,\ x = -5$ **iv.** $t = 3,\ x = 18$
e. **i.** $x = -3$ **ii.** Left
iii. $t = 1\frac{1}{2},\ x = -6$ **iv.** $t = 5,\ x = 5$
f. **i.** $x = 2$
ii. Left
iii. $t = 3,\ x = -5$ and $t = 5,\ x = 5$
iv. $t = 6,\ x = 4$

6. **a.**



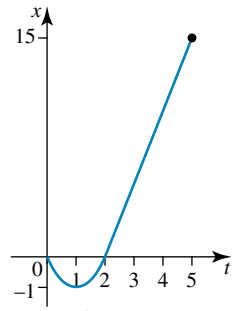
i. $x = 0$
ii. Right
iii. No
iv. $x = 10$

b.



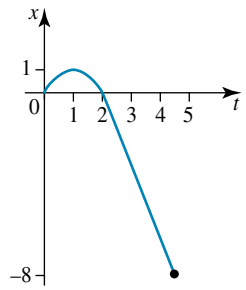
i. $x = -2$
ii. Right
iii. No
iv. $x = 16$

c.



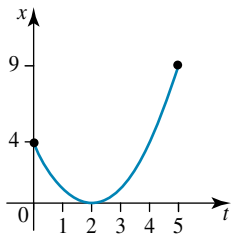
i. $x = 0$
ii. Left
iii. Yes, $t = 1, x = -1$
iv. $x = 15$

d.



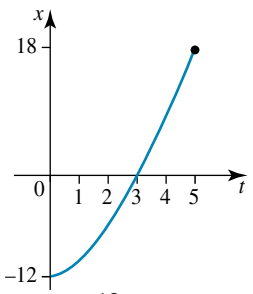
i. $x = 0$
ii. Right
iii. Yes, $t = 1, x = 1$
iv. $x = -8$

e.



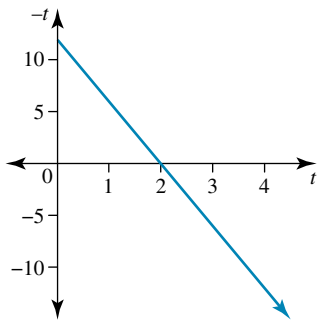
i. $x = 4$
ii. Left
iii. Yes, $t = 2, x = 0$
iv. $x = 9$

f.

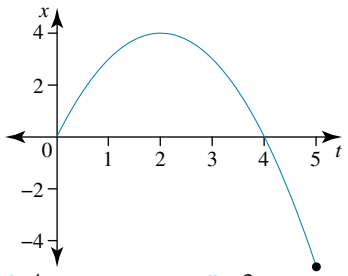


i. $x = -12$
ii. Right
iii. No
iv. $x = 18$

7.



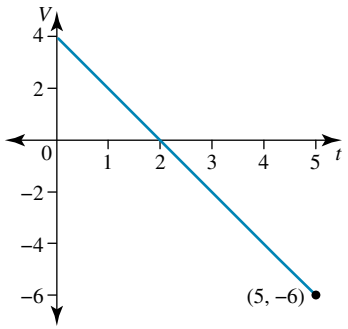
8. a.



b. **i.** 4 **ii.** 2 **iii.** 0
iv. –2 **v.** –4

c. **i.** 4 m/s **ii.** 2 m/s **iii.** 0 m/s
iv. –2 m/s **v.** –4 m/s

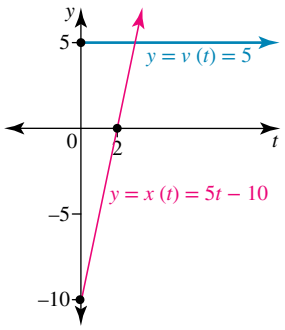
d.



9. a. C **b.** E **c.** B **d.** F **e.** A **f.** D

10. a. 10 cm to left of origin, 5 cm to right of origin
b. 15 cm
c. $v = 5$ cm/s

d.



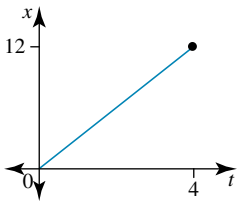
11. a. $v = 6 - 2t,\ a = -2$

b. Displacement graph is quadratic with maximum turning point when $t = 3$; velocity graph is linear with t-intercept at $t = 3$; acceleration graph is horizontal with constant value of -2.

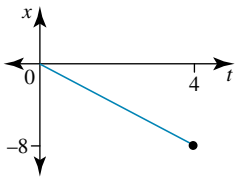
c. $v = 0 \Rightarrow t = 3,\ t = 3 \Rightarrow x = 9$

d. Displacement increases for $0 < t < 3$ and $v > 0$.

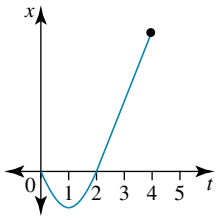
12. a.



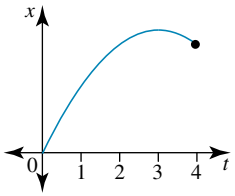
b.



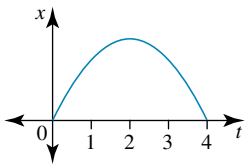
c.



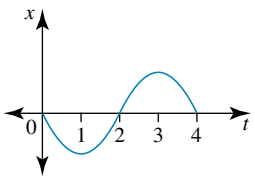
d.



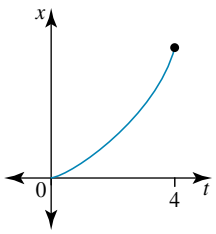
e.



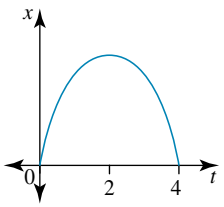
f.



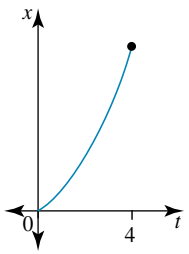
13. a.



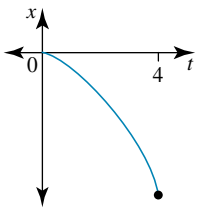
b.



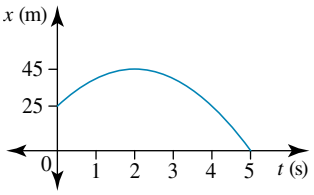
c.



d.



14.

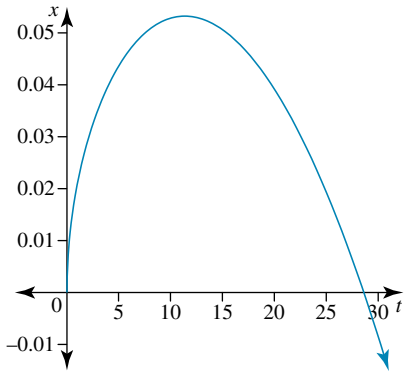


a. 45 m **b.** $t = 5$ **c.** $t = 2$ **d.** 20 m/s

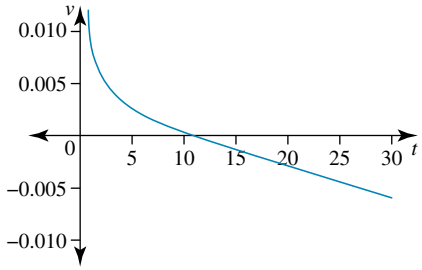
15. a. i. 15 m/s **ii.** 45 m/s

b. 60 m/s **c.** 34 seconds

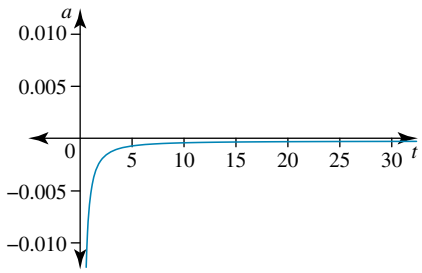
16. a.



b.



c.



d. The particle decelerates quickly initially and continues to decelerate throughout but less so over time. It starts with a high positive velocity, reaching its farthest distance around $t = 11$ before increasing in speed as it heads back towards the starting point.

Exercise 13.4 Sketching curves using derivatives

1. a. (0, 8) maximum

b. $(-1, 2)$ maximum and $(1, -2)$ minimum

c. $(2, -8)$ minimum

d. $(-2, -8)$ minimum and $\left(\frac{2}{3}, \frac{40}{27}\right)$ maximum

e. (0, 0) Point of horizontal inflection and (1, 1) maximum

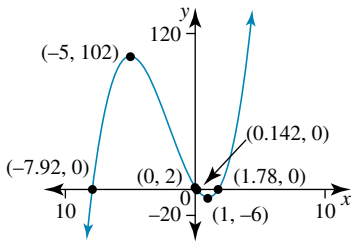
f. $(-2, 4)$ maximum and (0, 0) minimum

2. a. $(1, -6)$ and $(-5, 102)$

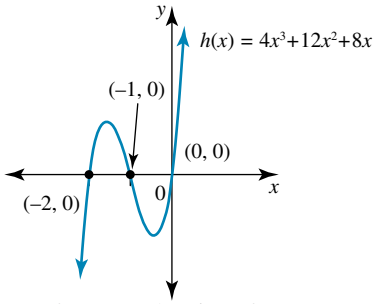
b. $(1, -6)$ is a local minimum and $(-5, 102)$ is a local maximum.

c. As $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$.

d.



3. a.



b. i. $-2 < x < -1$ and $x > 0$

ii. $x < -2$ and $-1 < x < 0$

4. a. $(3, -4)$ is a minimum.

b. $(0, 8)$ is inconclusive

c. $\left(-\frac{1}{2}, 6\frac{3}{4}\right)$ is a maximum.

d. $(0, 5)$ is a minimum and $(1, 6)$ is inconclusive

e. $(-3, 54)$ is a maximum and $(3, -54)$ is a minimum

f. $(-3, 16)$ is a maximum and $\left(\frac{1}{3}, -2\frac{14}{27}\right)$ is a minimum

g. $(0, 12)$ is inconclusive

h. $(0, 0)$ is inconclusive and $(3, -27)$ is a minimum

5. Local minimums at $(-\sqrt{2}, -4)$ and $(\sqrt{2}, -4)$ and a maximum at $(0, 0)$. $f(x) \to \infty$ as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$.

6. B

7. C

8. $a = \frac{19}{4}, b = -19, c = 5$

9. a. $(2, 0)$

b. Sample responses can be found in the worked solutions in the online resources; stationary point of inflection.

c. $k > 4$

d. Sample responses can be found in the worked solutions in the online resources; stationary point of inflection

e. Degree 2; not possible; sample responses can be found in the worked solutions in the online resources.

f. Origin lies on the line $y = -\frac{2a^2x}{3}$.

10. a. $a = -9$, $b = 24$

b. $(2, 9)$ is a maximum turning point; $(4, 5)$ is a minimum turning point.

11. a. $x = -3$ a local min., $x = 0$ a local max.

b. $x = -2$ a local max., $x = 1$ a local min., $x = 4$ a local max.

c. $x = -2$ a negative point of inflection, $x = 3$ a local min.

d. $x = -5$ a local min., $x = 2$ a positive point of inflection

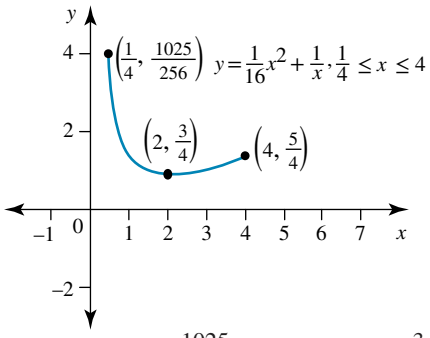
e. $x = -3$ a local max., $x = 0$ a local min., $x = 2$ a local max.

f. $x = 1$ a local max., $x = 5$ a local min.

12. a. $\left(\frac{1}{4}, \frac{1025}{256}\right), \left(4, \frac{5}{4}\right)$

b. $\left(2, \frac{3}{4}\right)$ is a minimum turning point.

c.



d. Global maximum $\frac{1025}{256}$; global minimum $\frac{3}{4}$

13. a. A $(0.25, 5)$, B$(1, 3)$, C$(5, 2\sqrt{5} + 0.2)$

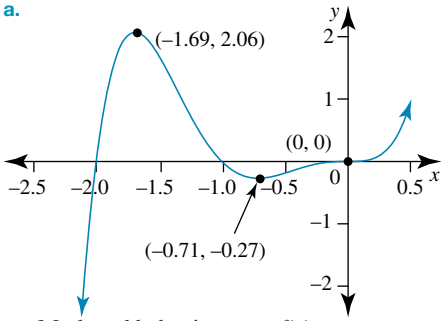
b. A **c.** 5, 3

14. a. Maximum turning point $(6.31,\ 15.40)$; minimum turning point $(1.69, -15.40)$

b. $(4,\ 10)$

c. Gradient of the curve is greatest at the point $(4,\ 0)$.

15. a.



Mark end behaviours as: $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. Mark stationary points at $(0, 0), (-1.69, 2.06), (-0.71, -0.27)$

b. At a domain of $[-10,\ 10]$ it is difficult to recognise that there are any stationary points.

16. a. $f'(x) \neq 0$ for all real x.

b. $\left(\frac{1}{6}, -\frac{23}{27}\right)$

c.

–0.133	–0.033	0.067	0.167	0.267	0.367	0.467
1.373	1.073	0.893	0.833	0.893	1.073	1.373

d. The gradient is always increasing but reaches a non-zero minimum value at $x = \frac{1}{6}$. As the gradient never reaches zero it is not a point of horizontal inflection.

Exercise 13.5 Modelling optimisation problems

1. a. $x = 20$ m, $y'(19) > 0$ and $y'(21) < 0$ (a maximum)

b. $y = 11.2$ m

2. a. $t = 10$ min **b.** $V'(10)$ is a minimum

c. $V = 160$ litres **d.** $t = 15$ min

3. a. $h = 12.25$ m (when $t = 1.5$ s)

b. $h'(1) = 5$ and $h'(2) = -5$ (a maximum)

4. 32 m^2

5. a. $16 - x$

b. $P = x(16 - x)$

c. Both numbers are 8.

d. $P'(7) = 2$, $P'(9) = -2$, (a maximum)

6. **a.** $10 - x$ **b.** $A = x(10 - x)$ **c.** $x = 5$
d. Length and with $= 5$ cm
e. 25 cm^2

7. **a.** Length and with $= 15$ cm **b.** 225 m^2

8. **a.** $P = 60n - 250 - 1.2n^2$
b. 25 **c.** \$500

9. **a.** $P = 800 + 240n - 20n^2$ **b.** $n = 6$, $p = \$1520$

10. Both numbers are 5.

11. **a.** $x \in (0, 6)$ or $0 < x < 6$
b. **i.** x **ii.** $12 - 2x$ **iii.** $12 - 2x$
c. $V = x(12-2x)(12-2x)$ **d.** 128 cm^3

12. **a.** 7.36 cm by 25.28 cm by 35.28 cm
b. 6564.23 cm^3

13. **a.** $h = \dfrac{256}{l^2}$ **b.** $A = l^2 + \dfrac{1024}{l}$
c. 8 cm $\times$ 8 cm $\times$ 4 cm **d.** 192 cm^2

14. 10 cm $\times$ 10 cm $\times$ 10 cm

15. **a.** \$7500 **b.** $C = \dfrac{1\,440\,000}{v} + 9v$
c. $v = 400$ km/h and $C = \$7200$

16. $\dfrac{1000\pi}{27}$ or 116.355 cm^3

13.6 Review: exam practice

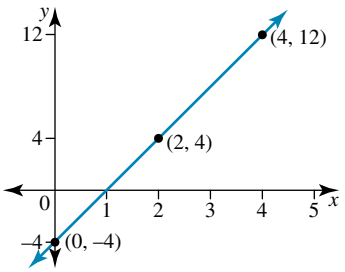
Simple familiar

1. $y = 9x - 17$

2. **a.** $-29.9°$ **b.** $43.4°$

3. $y = \dfrac{1}{18}x + \dfrac{244}{9}$

4. **a.** **i.** -4 **ii.** 4 **iii.** 12
b.



5. **a.** $v(t) = 3t^2 + \dfrac{2}{3}t - \dfrac{3}{4}$ **b.** $a(t) = 6t + \dfrac{2}{3}$

6. **a.** 1 metre to right of origin; 8 m/s
b. $16\dfrac{2}{3}$ metres **c.** -6 m/s^2

7. **a.** **i.** Maximum at $x = -2$ and minimum at $x = 0$
ii. Minimum at $x = -\dfrac{1}{3}$ and maximum at $x = 3$
iii. Maximum at $x = -\sqrt{\dfrac{13}{3}}$ and minimum at $x = \sqrt{\dfrac{13}{3}}$
iv. None
v. Minimum at $x = -\sqrt{3}$, maximum at $x = 0$ and minimum at $x = \sqrt{3}$
vi. Point of horizontal inflection at $x = -1$ and minimum at $x = -\dfrac{1}{4}$
b. Sample responses can be found in the worked solutions in the online resources.

8. Maximum at $(-1, 5)$ and minimum at $\left(\dfrac{1}{3}, \dfrac{103}{27}\right)$

9. **a.** Sample responses can be found in the worked solutions in the online resources.
b. Maximum **c.** (0, 8) **d.** Minimum

10. 8 people

11. **a.** 9.77 m **b.** 25 m

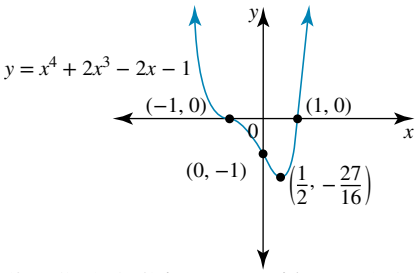
12. **a.** $A = 40x - 2x^2$
b. width $= 10$ m; length $= 20$ m, maximum area $= 200$ m^2

Complex familiar

13. $x = 0.5$

14. **a.** 20 m/s upwards **b.** 10 m/s upwards
c. 5 seconds, travelling downward
d. 4 seconds **e.** 80 m
f. 8 seconds, 40 m/s

15.



(0, −1), (±1, 0) intercepts with axes; (−1, 0) is a stationary point of inflection; $\left(\dfrac{1}{2}, -\dfrac{27}{16}\right)$ is a minimum turning point.

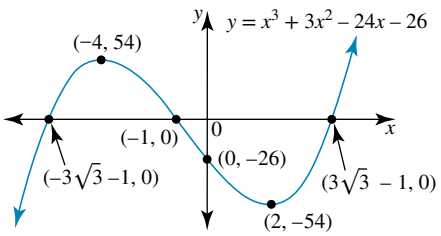
16. **a.** $V = 240x - 64x^2 + 4x^3$
b. Length 15.14 cm; width 7.14 cm; height 2.43 cm; volume 263 cm^3

Complex unfamiliar

17. If the skier continues in that direction they will reach the point (20, 6.25), so they are likely to hit the tree at (20, 6.25).

18. **a.** 3 seconds and 5 seconds
b. $t \in (3, 5)$
c. $t \in [0, 4)$ **d.** $\sqrt{3}$ seconds
e. 2 seconds and $(\sqrt{6} - 1)$ seconds

19. **a.** $b = 3$, $c = -24$ **b.** $(-4, 54)$
c. $(0, -26)$, $(-1 \pm 3\sqrt{3}, 0)$, $(-1, 0)$
d.



key points are $(-1 - 3\sqrt{3}, 0)$, $(-4, 54)$, $(-1, 0)$, $(0, -26)$, $(2, -54)$, $(-1 + 3\sqrt{3}, 0)$.

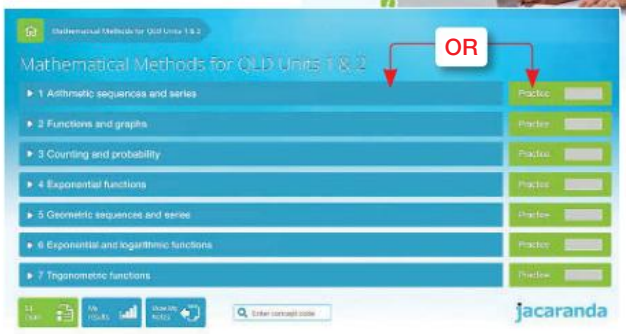
20. **a.** 2 km due north
b. TS $= \sqrt{(x-4)^2 + (\sqrt{x})^2}$; Sample responses can be found in the worked solutions in the online resources.
c. $x = 3.5$
d. T$\left(\dfrac{7}{2}, \dfrac{\sqrt{14}}{2}\right)$

TOPIC 4 Introduction to differential calculus

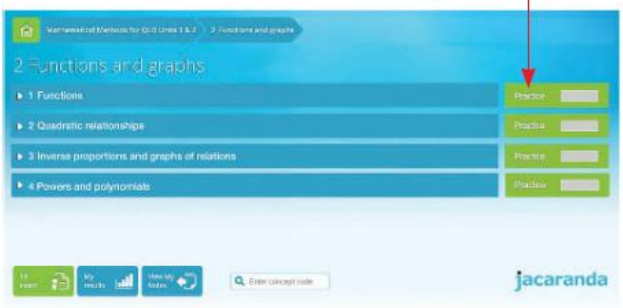
- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



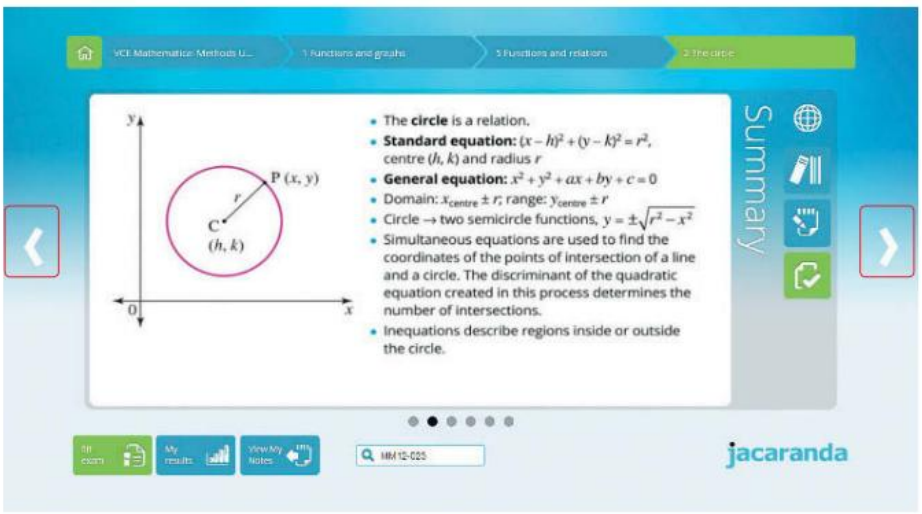
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.

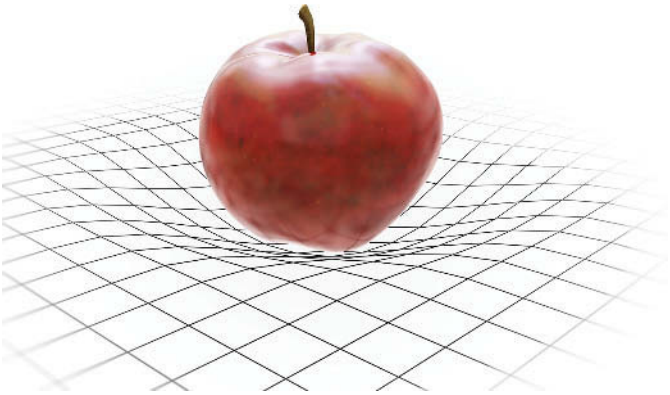


- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 14
Differentiation rules

14.1 Overview



While there is some debate, discovery of the power rule is credied to Leibniz. The first known use of the chain rule was also by Leibniz. He used it to solve functions of the form $\sqrt{a + bz + cz^2}$.

As we have seen in the previous chapter, there are many functions that can be differentiated using the power rule after they have been manipulated in some manner (e.g. expanded or simplified). For many functions the process of manipulation is quite time consuming and can increase the likelihood of errors been made during calculations. There are also many functions that cannot be differentiated using the power rule. As such, we need other methods to deal with a wider variety of differentiable functions and in this chapter, we will examine three more differentiation rules.

Given that y, u and v are all differentiable functions then:

- **the product rule applies to functions of the form $y = uv$,**
- **the quotient rule applies to functions of the form $y = \frac{u}{v}$; and**
- **the chain rule applies to composite functions of the form $y = u(v)$.**

LEARNING SEQUENCE

14.1 Overview
14.2 The product rule
14.3 The quotient rule
14.4 The chain rule
14.5 Applications of the product, quotient and chain rules
14.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

14.2 The product rule

Any function that is a product of two simpler functions, for example,

$$f(x) = (x + 2)(x - 5)$$

can be differentiated using the product rule of differentiation.

The product rule is stated as follows.

$$\text{If } y = uv \text{ then } \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \text{ or similarly } y' = uv' + vu'.$$

Or if $h(x) = f(x) \times g(x)$ then $h'(x) = f(x)g'(x) + f'(x)g(x)$.

WORKED EXAMPLE 1

If $y = (3x - 1)(x^2 + 4x + 3)$ is expressed as $y = uv$, determine:

a. u **and** v

b. $\frac{du}{dx}$ **and** $\frac{dv}{dx}$

c. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

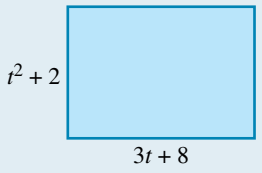
THINK	WRITE
a. 1. Write the equation.	**a.** $y = (3x - 1)(x^2 + 4x + 3)$
2. Identify u and v, two functions of x which are multiplied together.	Let $u = 3x - 1$ and $v = x^2 + 4x + 3$.
b. 1. Differentiate u with respect to x.	**b.** $\frac{du}{dx} = 3$
2. Differentiate v with respect to x.	$\frac{dv}{dx} = 2x + 4$
c. 1. Apply the product rule to find $\frac{dy}{dx}$.	**c.** $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
	$= (3x - 1)(2x + 4) + (x^2 + 4x + 3) \times 3$
2. Expand and simplify where possible.	$= 6x^2 + 10x - 4 + 3x^2 + 12x + 9$
	$= 9x^2 + 22x + 5$

WORKED EXAMPLE 2

The sides of a rectangle, in millimetres, are varying with time, in seconds, as shown.

a. State the function that expresses the area, A, of the rectangle at any time, t.

b. Calculate the rate of change of area against time at $t = 2$ seconds.



THINK	WRITE
a. 1. Area of a rectangle = length × width.	$A = l \times w$ $= (3t + 8)(t^2 + 2)$
b. 1. Apply the product rule to find the derivative.	Let $u = 3t + 8$ and $v = t^2 + 2$ $A' = uv' + vu'$ $= 2t(3t + 8) + 3(t^2 + 2)$ $= 6t^2 + 16t + 3t^2 + 6$ $= 9t^2 + 16t + 6$
2. Solve the derivative at $t = 2$.	$A'(2) = 9(2)^2 + 16(2) + 6$ $= 74$
3. State the answer.	The area is changing at a rate of 74 mm/s at $t = 2$.

studyon

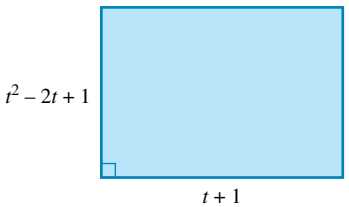
Units 1 & 2 > Area 9 > Sequence 1 > Concept 1

The product rule Summary screen and practice questions

Exercise 14.2 The product rule

Technology free

1. **WE1** If $y = (x + 3)(2x^2 - 5x)$ is expressed as $y = u \times v$, determine:
   - **a.** u and v
   - **b.** $\frac{du}{dx}$ and $\frac{dv}{dx}$
   - **c.** $\frac{dy}{dx}$ using the product rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.
2. For each of the following functions:
   - **i.** identify $f(x)$ and $g(x)$
   - **ii.** derive $f'(x)$ and $g'(x)$
   - **iii.** calculate the derivative.
   - **a.** $h(x) = (x + 2)(x - 3)$
   - **b.** $h(x) = 3x^2(x^2 - 4x + 1)$
   - **c.** $k(x) = x^{-1}(x + 2)$
   - **d.** $p(x) = (\sqrt{x} + 3x)(x^2 - 4)$
3. **WE2** The sides of a rectangle, in millimetres, are varying with time, in seconds, as shown.
   - **a.** State the function that expresses the area of the rectangle at any time (t).
   - **b.** Calculate the rate of change of area at $t = 5$ s.



4. Differentiate $2\sqrt{x}(4 - x)$, using the product rule.
5. Calculate $f'(x)$ given $f(x) = 4x^{-1}(3 + x^2)$, using the product rule.
6. Given $f(x) = 2x^2(x - x^2)$, use the product rule to determine the coordinates where $f'(x) = 0$.
7. Given that if $f(x) = (x + a)^n$ then $f'(x) = n(x + a)^{n-1}$, determine the derivative of:
   - **a.** $x^2(x + 1)^3$
   - **b.** $x^3(x + 1)^2$
   - **c.** $\sqrt{x}(x + 1)^5$
   - **d.** $x^{\frac{3}{2}}(x - 2)^3$
   - **e.** $x(x - 1)^{-2}$
   - **f.** $x\sqrt{x + 1}$.

8. Determine the equation of the tangent and the normal to the line $y = (x^2 - 2)(4 - 3x)$ at $x = 2$.
9. The position of a particle from its starting point when moving in a straight line is given by the function $x(t) = (t + 2)(2t - t^2)$, where x is on metres and t is in seconds.
   a. Graph the position–time function over $0 \leq t \leq 2$.
   b. Graph the velocity–time function over $0 \leq t \leq 2$.
   c. Graph the acceleration–time function over $0 \leq t \leq 2$.
   d. Comment on the behaviour of the particle during the first 2 seconds.



10. Expand two terms and then apply the product rule to find the derivatives of:
    a. $y = x(2x - 5)(3x - 1)$
    b. $y = (x - 2)(2x + 1)(3x + 3)$.
    Use these results to confirm that for functions of the form $y = uvw$, the product rule states that $y' = u'vw + uv'w + uvw'$.
11. Apply the product rule to the functions below to develop a general rule for functions of the form $y = (ax + b)^n$.
    a. $(3x - 2)^2$ b. $(4x - 1)^2$ c. $(5x + 2)^3$ d. $(-3x + 2)^3$
12. Sketch the function $y = (x^2 + 1)(x + 2)$, examining end behaviours and identifying all intercepts and stationary points.

Technology active

13. The revenue generated (in $) by selling a product is calculated using the formula $R(x) = x\rho(x)$, where x is the number sold and ρ is the demand function, which indicates the price per item. The number of items sold is often closely linked to price. If the price of an item is lower, than more items are likely to be sold.

    A manufacturer knows that its demand function varies linearly (i.e. $\rho(x) = ax + b$) with 1250 units being sold if the price is $500 per unit and 1500 units if the price is $400.



    a. Determine the revenue function R.
    The marginal revenue of a product indicates the rate of change in total revenue with respect to the quantity demanded. It is calculated by finding the derivative of the revenue function.
    b. Using the product rule, determine the marginal revenue for this product at $x = 50$.
14. Plot the function $y = (2x + 1)(x^2 - 3x)$ and its derivative on the same axes and compare the two graphs.

14.3 The quotient rule

The quotient rule is used to differentiate functions which are rational expressions (that is, one function divided by another). For example:

$$f(x) = \frac{x^2 - 6x + 3}{5x + 2}.$$

The quotient rule is stated as follows.

$$\text{If } y = \frac{u}{v} \text{ then } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \text{ or similarly } \mid y' = \frac{vu' - uv'}{v^2}.$$

Or if $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(x) = \dfrac{g(x) \times f'(x) - f(x) \times g'(x)}{(g(x))^2}$.

WORKED EXAMPLE 3

If $y = \dfrac{3-x}{x^2+4x}$ is expressed as $y = \dfrac{u}{v}$, determine:

a. u and v **b. $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x}$** **c. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{v}\right)$.**

THINK	WRITE
a. 1. Write the equation.	**a.** $y = \dfrac{3-x}{x^2+4x}$
2. Identify u and v.	Let $u = 3 - x$ and $v = x^2 + 4x$.
b. 1. Differentiate u with respect to x.	**b.** $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -1$
2. Differentiate v with respect to x.	$\dfrac{\mathrm{d}v}{\mathrm{d}x} = 2x + 4$
c. 1. Apply the quotient rule to obtain $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.	**c.** $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
	$= \dfrac{(x^2+4x) \times -1 - (3-x)(2x+4)}{(x^2+4x)^2}$
2. Simplify $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ where possible, factorising the final answer where appropriate.	$= \dfrac{-x^2 - 4x - 12 - 2x + 2x^2}{(x^2+4x)^2}$
	$= \dfrac{x^2 - 6x - 12}{(x^2+4x)^2}$
	$= \dfrac{x^2 - 6x - 12}{x^2(x+4)^2}$

WORKED EXAMPLE 4

Sketch the function $y = \dfrac{x^2+4}{x-2}$, identifying all asymptotes, intercepts and stationary points.

THINK	WRITE
1. Identify asymptotes.	Vertical asymptote is at $x = 2$ (oblique asymptote $y = x + 2$).

2. Find the x-intercept and the y-intercept: y-intercept at $x = 0$.

$$y = \frac{(0)^2 + 4}{(0) - 2}$$
$$= -\frac{4}{2}$$
$$= -2$$

x-intercept at $y = 0$

$$0 = \frac{x^2 + 4}{x - 2}$$
$$= x^2 + 4, x \neq 2$$
$$x = \pm\sqrt{-4}$$

which has no real solutions, so it doesn't have any x-intercepts.

The stationary points at $y' = 0$.

$$u = x^2 + 4 \quad v = x - 2$$
$$u' = 2x \qquad v' = 1$$

3. Determine the derivative.

$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{2x(x-2) - 1(x^2+4)}{(x-2)^2}$$
$$= \frac{2x^2 - 4x - x^2 - 4}{(x-2)^2}$$
$$= \frac{x^2 - 4x - 4}{(x-2)^2}$$

4. Solve for $y' = 0$.

$$0 = \frac{x^2 - 4x - 4}{(x-2)^2}$$
$$= x^2 - 4x - 4, x \neq 2$$
$$= (x-2)^2 - 8$$
$$= (x-2)^2 - (\sqrt{8})^2$$
$$= (x - 2 - \sqrt{8})(x - 2 + \sqrt{8})$$
$$= (x - (2 + 2\sqrt{2}))(x - (2 - 2\sqrt{2}))$$
$$x = 2 + 2\sqrt{2}, 2 - 2\sqrt{2}$$
$$\approx -0.828, 4.828$$

5. Identify the type of stationary points, using first derivative test.

x	-1	-0.828	0	4.828	5
y'	0.11	0	-1	0	0.11
slope	/	—	\	—	/

6. Calculate the y-values of the stationary points.

At $x = 2 + 2\sqrt{2}$

$$y = \frac{(2 + 2\sqrt{2})^2 + 4}{2 + 2\sqrt{2} - 2}$$
$$= \frac{12 + 8\sqrt{2} + 4}{2\sqrt{2}}$$
$$= 4 + 4\sqrt{2}$$
$$\approx 9.657$$

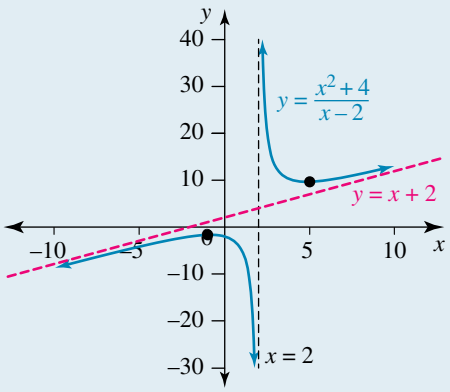
Local minimum at $(2 + 2\sqrt{2}, 4 + 4\sqrt{2})$

At $x = 2 - 2\sqrt{2}$

$$y = \frac{(2-2\sqrt{2})^2 + 4}{2 - 2\sqrt{2} - 2}$$
$$= \frac{12 - 8\sqrt{2} + 4}{-2\sqrt{2}}$$
$$= 4 - 4\sqrt{2}$$
$$\approx -1.657$$

Local maximum at $(2-2\sqrt{2}, 4-4\sqrt{2})$

7. Sketch the function.



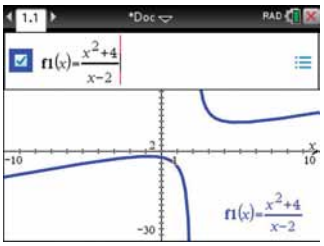
TI | THINK

1. On a Graphs page, complete the entry line for function 1 as

$f1(x) = \frac{x^2+4}{x-2}$

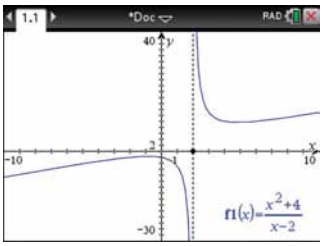
then press ENTER.

WRITE



2. To draw the vertical asymptote, press MENU then select
8: Geometry
4: Construction
1: Perpendicular.
Click on the x-axis, then click on the point on the x-axis at $x = 2$.
Note: The style can be changed to dashed by placing the cursor on the asymptote, pressing SHIFT then MENU, then selecting 3: Attributes. Press the down arrow then the right arrow twice to change the line style to dashed, then press ENTER.

WRITE



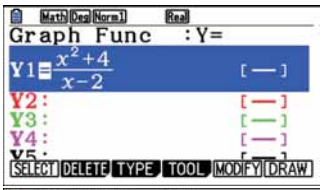
CASIO | THINK

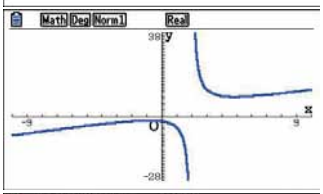
1. On a Graph screen, complete the entry line for Y1 as

$y1 = \frac{x^2+4}{x-2}$

then press EXE. Select DRAW by pressing F6.

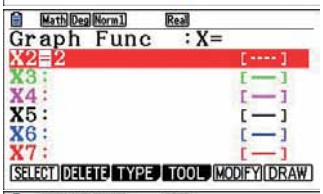
WRITE

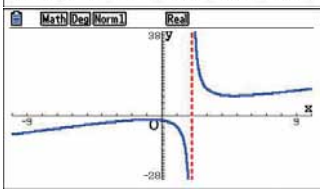




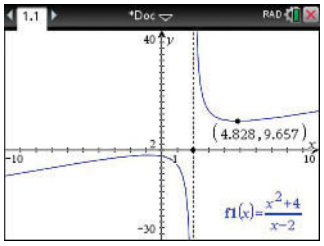
2. To draw the vertical asymptote, return to the function editor screen and change the TYPE to X = by pressing F3 then F4. Complete the entry line for X2 as
X2 = 2
then press EXE. Select DRAW by pressing F6.
Note: The style can be changed to dashed by selecting TOOL then STYLE by pressing F4 then F1.

WRITE

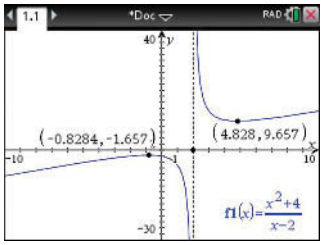




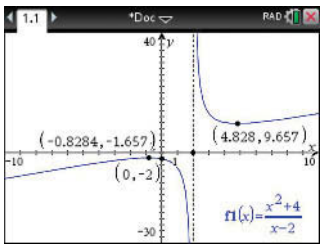
3. To find the minimum, press MENU then select
6: Analyze Graph
2: Minimum
Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.



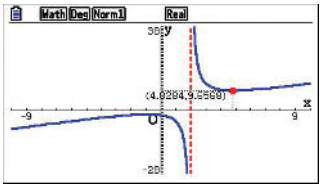
4. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



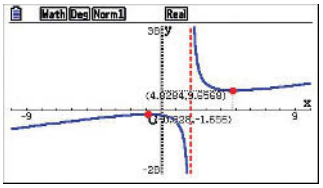
5. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



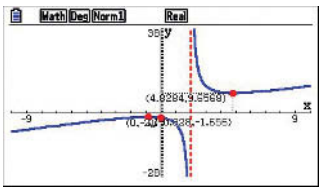
3. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3.
Press EXE.



4. To find the maximum, select G-Solv by pressing SHIFT F5, then select MAX by pressing F2.
Press EXE.



5. To find the y-intercept, select G-Solv by pressing F5, then select Y-ICEPT by pressing F4.
Press EXE.



studyon

Units 1 & 2 | Area 9 | Sequence 1 | Concept 2

The quotient rule Summary screen and practice questions

Exercise 14.3 The quotient rule

Technology free

1. WE3 If $y = \frac{x+3}{x+7}$ is expressed as $y = \frac{u}{v}$, determine:
   a. u and v
   b. $\frac{du}{dx}$ and $\frac{dv}{dx}$
   c. $\frac{d}{dx}\left(\frac{u}{v}\right)$ using the quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
2. If $h(x) = \frac{x^2+2x}{5-x}$ is expressed as $h(x) = \frac{f(x)}{g(x)}$, determine:
   a. $f(x)$ and $g(x)$
   b. $f'(x)$ and $g'(x)$
   c. $h'(x)$ using the quotient rule.
3. Determine the derivative of $\frac{x+1}{x^2-1}$.

4. Calculate $f'(x)$, given $f(x) = \dfrac{2x^2 + 3x - 1}{5 - 2x}$.
5. Determine the derivative of each of the following.
   a. $\dfrac{2x}{x^2 - 4x}$ b. $\dfrac{x^2 + 7x + 6}{3x + 2}$ c. $\dfrac{4x - 7}{10 - x}$ d. $\dfrac{5 - x^2}{x^{\frac{3}{2}}}$
6. Explain how you could derive the following rational functions without using the quotient rule (or first principles).
   a. $\dfrac{7 + x}{x}$ b. $\dfrac{x^3 - 3x^2 + 4}{x^2}$ c. $\dfrac{x^2 + 5x + 6}{x + 2}$ d. $\dfrac{x^2 - 16}{x + 4}$
7. Graph the derivative of $\dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$.
8. Determine the equation of the tangent and the normal to the line $y = \dfrac{1}{x^2 - 9}$ at $x = -1$.
9. Compare the derivative of $\dfrac{8 - 9x^2}{x^2}$ using the quotient rule and the derivative of $\dfrac{8}{x^2} - 9$ using the power rule. Comment on the result.
10. The position of a particle from its starting point, when moving in a straight line, is given by the function $x(t) = \dfrac{t + 2}{t + 1}$, where x is in metres and t is in seconds.
    a. Graph the position–time function over $0 \leq t \leq 2$.
    b. Graph the velocity–time function over $0 \leq t \leq 2$.
    c. Graph the acceleration–time function over $0 \leq t \leq 2$.
    d. Comment on the behaviour of the particle during the first 2 seconds.
11. Differentiate the function $\dfrac{(2x - 3)(3x + 4)}{x - 2}$ using the product and quotient rules.
12. **WE4** Sketch the function $y = \dfrac{x^2 + 1}{x + 2}$, identifying asymptotes, intercepts and stationary points.
13. Calculate the minimum volume possible for a cylinder with a height given by $h = \dfrac{2}{r - 3}, r > 3$.



14. The average profit of a product is the profit generated per unit sold and is calculated using the formula $AP(x) = \dfrac{P(x)}{x}$, where $P(x)$ is the profit and x is the number of units sold.
    The profit generated by a certain product is given by the function $P(x) = 2x^2 + 12x + 4$.
    a. Determine the average profit function for the product.
    b. Determine the rate of change of average profit at $x = 100$.

Technology active

15. The yearly average cost function for a large Australian gold mine is given by the function $AvgC(m) = \dfrac{700m^3 - 1.8 \times 10^6 m}{m + 10^5} + 6 \times 10^6$, where m is the mass of gold in tonnes, t, and AvgC is the cost, in $ per tonne. What is the optimal amount of gold required to mine in a year to have the lowest average cost?



16. Claire has been trialling a new fly poison she developed. She has many flies in a sealed enclosure that she then releases the poison into. She then monitors the number of flies that are alive at a certain number of minutes. Assume the fly population is modelled by a hyperbolic function of the form $y = \frac{a}{x - b}$.

Time (min)	0	2	4	7
Number of flies	340	109	65	40

a. Determine the values of a and b using technology to fit a function.
b. Determine the gradient function for the fly population.

14.4 The chain rule

A function which can be expressed as a composition of two simpler functions is called a composite function. For example, $y = (x + 3)^2$ can be expressed as $y = u^2$ where $u = x + 3$.

That is, to obtain y from x, the first function to be performed is to add 3 to x ($u = x + 3$), then this function has to be 'squared' ($y = u^2$).

Or if, say, $x = 1$, to obtain y first calculate $1 + 3(= 4)$, then secondly 'square' the result, 4^2, giving $y = 16$.

Composite functions can be differentiated using the chain rule. For example, using the previous function, $y = (x + 3)^2$:

Let $u = x + 3$, so $y = u^2$.

Then $\frac{du}{dx} = 1$ and $\frac{dy}{du} = 2u$.

But we require $\frac{dy}{dx} : \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. This is known as the chain rule.

It is known as the chain rule because u provides the 'link' between y and x.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Now $\frac{dy}{dx} = 2u \times 1$

$= 2(x + 3) \times 1$ (replacing u with $x + 3$)

$= 2(x + 3)$

The chain rule is used when it is necessary to differentiate a 'function of a function' as above.

In function or notations if $y = f(g(x))$, then $y' = f'(y(x))g'(x)$.

WORKED EXAMPLE 5

If $y = (3x - 2)^3$ is expressed as $y = u^n$, find:

a. $\frac{dy}{du}$ **b.** $\frac{du}{dx}$ **and hence** **c.** $\frac{dy}{dx}$.

THINK

a. 1. Write the equation.

2. Express y as a function of u.

WRITE

a. $y = (3x - 2)^3$

Let $y = u^3$ where $u = 3x - 2$.

3. Differentiate y with respect to u.	$\frac{dy}{du} = 3u^2$
b. 1. Express u as a function of x.	**b.** $u = 3x - 2$
2. Differentiate u with respect to x.	$\frac{du}{dx} = 3$
c. 1. Calculate $\frac{dy}{dx}$ using the chain rule.	**c.** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
	$= 3u^2 \times 3$
	$= 9u^2$
2. Replace u as a function of x.	$= 9(3x - 2)^2$

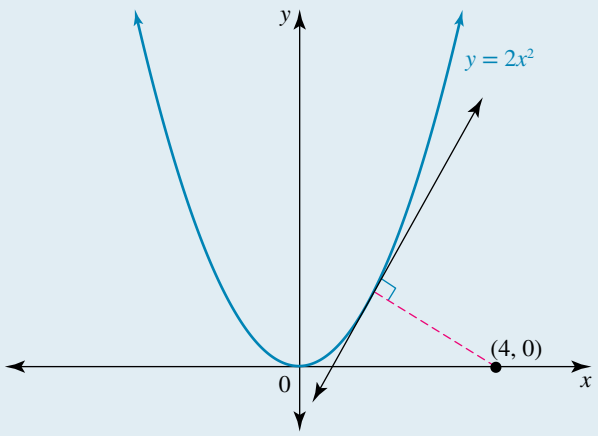
WORKED EXAMPLE 6

If $f(x) = \dfrac{1}{\sqrt{2x^2 - 3x}}$ find $f'(x)$.

THINK	WRITE
1. Write the equation.	$f(x) = \dfrac{1}{\sqrt{2x^2 - 3x}}$
2. Express $f(x)$ in index form, that is, as $y = [g(x)]^n$.	$y = (2x^2 - 3x)^{-\frac{1}{2}}$
3. Express y as a function of u.	Let $y = u^{-\frac{1}{2}}$ where $u = 2x^2 - 3x$.
4. Differentiate y with respect to u.	$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$
5. Express u as a function of x.	$u = 2x^2 - 3x$
6. Differentiate u with respect to x.	$\frac{du}{dx} = 4x - 3$
7. Find $f'(x)$ using the chain rule.	$\frac{dy}{dx} = f'(x) = -\frac{1}{2}u^{-\frac{3}{2}} \times (4x - 3)$
	$= -\frac{1}{2}(2x^2 - 3x)^{-\frac{3}{2}}(4x - 3)$
8. Replace u as a function of x and simplify.	$= \dfrac{-(4x - 3)}{2(2x^2 - 3x)^{\frac{3}{2}}}$
	$= \dfrac{3 - 4x}{2\sqrt{(2x^2 - 3x)^3}}$

WORKED EXAMPLE 7

Find the minimum distance from the curve $y = 2x^2$ to the point $(4, 0)$, correct to 2 decimal places. You do not need to justify your answer.

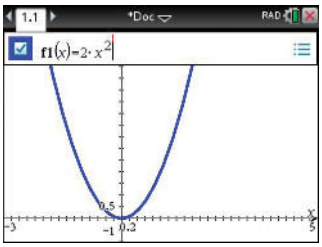


THINK	WRITE
1. Let P be the point on the curve such that the distance from P to the point $(4, 0)$ is a minimum.	$P = (x, y)$
2. Write the formula for the distance between the two points.	$\text{d}(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(x - 4)^2 + (y - 0)^2}$ $= \sqrt{(x - 4)^2 + y^2}$
3. Express the distance between the two points as a function of x only.	$y = 2x^2$ $\therefore \text{d}(x) = \sqrt{(x - 4)^2 + (2x^2)^2}$ $= (x^2 - 8x + 16 + 4x^4)^{\frac{1}{2}}$
4. Differentiate $\text{d}(x)$.	$\text{d}'(x) = \frac{1}{2} \times (4x^4 + x^2 - 8x + 16)^{-\frac{1}{2}} \times (16x^3 + 2x - 8)$ $= \frac{16x^3 + 2x - 8}{2\sqrt{4x^4 + x^2 - 8x + 16}}$ $= \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$
5. Solve $\text{d}'(x) = 0$ using technology.	$0 = \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$ $0 = 8x^3 + x - 4$ $x = 0.741$
6. Evaluate $\text{d}(0.741)$.	$\text{d}(0.741) = \sqrt{(0.741)^2 - 8(0.741) + 16 + 4(0.741)^4}$ $= 3.439$
7. Write the answer.	The minimum distance is 3.44 units.

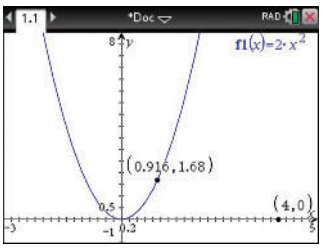
TI | THINK

WRITE

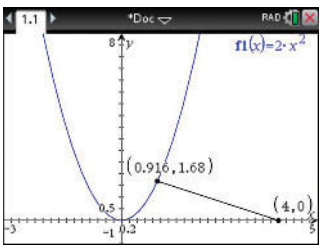
1. On a Graphs page, complete the entry line for function 1 as
   $f1(x) = 2x^2$
   then press ENTER.



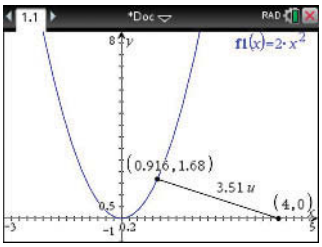
2. Press MENU then select
   8: Geometry
   1: Points & Lines
   2: Point On
   Click on the graph of function 1 then click again on the graph to create a point on the curve. Click on the x-axis and then click on the point at $(4, 0)$ to create another point.



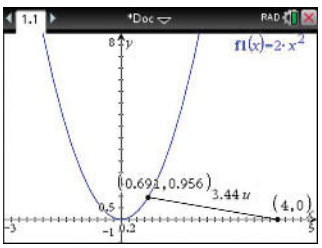
3. Press MENU then select
   8: Geometry
   1: Points & Lines
   5: Segment
   Click on the point created on the graph of function 1 and then click on the point at $(4, 0)$ to create a line segment between the two points.



4. Press MENU then select
   8: Geometry
   3: Measurement
   1: Length
   Click on the line segment joining the two points, then click next to the line segment to display the lenth of the line segment.



5. Click and drag the point on the graph of function 1, finding the position that results in the length of the line segment being as small as possible.



6. The answer appears on the screen.

   The minimum distance is 3.44 units.

CASIO | THINK

WRITE

1. Let P be the point on the curve such that the distance from P to the point $(4, 0)$ is a minimum.

   $P = (x, y)$

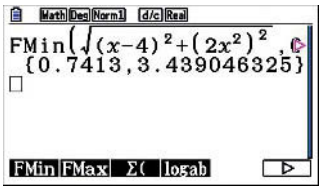
2. Write the formula for the distance between the two points.

   $$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
   $$= \sqrt{(x - 4)^2 + (y - 0)^2}$$
   $$= \sqrt{(x - 4)^2 + y^2}$$

3. Express the distance between the two points as a function of x only.

   $y = 2x^2$
   $\therefore d(x) = \sqrt{(x - 4)^2 + (2x^2)^2}$

4. On a Run-Matrix screen, press OPTN then select CALC by pressing F4, press F6 to scroll across to more menu options, then select FMin by pressing F1. Complete the entry line as
   $\text{FMin}\left(\sqrt{(x - 4)^2 + (2x^2)^2}, 0, 10, 5\right)$
   then press EXE.



5. The answer appears on the screen.

   The minimum distance is 3.44 units.

Exercise 14.4 The chain rule

Technology free

1. WE5 If each of the following composite functions is expressed as $y = u^n$, calculate:

i. $\frac{dy}{du}$ ii. $\frac{du}{dx}$ and hence iii. $\frac{dy}{dx}$.

a. $y = (3x + 2)^2$ **b.** $y = (7 - x)^3$ **c.** $y = \frac{1}{2x - 5}$ **d.** $y = \frac{1}{(4 - 2x)^4}$

e. $y = \sqrt{5x + 2}$ **f.** $y = \frac{3}{\sqrt{3x - 2}}$ **g.** $y = 3(2x^2 + 5x)^5$ **h.** $y = (4x - 3x^2)^{-2}$

i. $y = \left(x + \frac{1}{x}\right)^6$ **j.** $y = 4(5 - 6x)^{-4}$

2. Use the chain rule to calculate the derivative of the following.

a. $y = (8x + 3)^4$ **b.** $y = (2x - 5)^3$ **c.** $f(x) = (4 - 3x)^5$ **d.** $y = \sqrt{3x^2 - 4}$

e. $f(x) = (x^2 - 4x)^{\frac{1}{3}}$ **f.** $g(x) = (2x^3 + x)^{-2}$ **g.** $g(x) = \left(x - \frac{1}{x}\right)^6$ **h.** $y = (x^2 - 3x)^{-1}$

3. WE6 If $f(x) = \frac{1}{\sqrt{4x + 7}}$, calculate $f'(x)$.

4. Calculate the derivative of:

a. $f(x) = (x^2 + 5x)^8$ **b.** $y = (x^3 - 2x)^2$

c. $f(x) = (x^3 + 2x^2 - 7)^{\frac{1}{5}}$ **d.** $y = (2x^4 - 3x^2 + 1)^{\frac{3}{2}}$.

5. Match the following functions to their derivatives.

a.	$(3x - 2)^3$	A $3(x - 2)^2$
b.	$3(3x - 2)^2$	B $9(x - 2)^2$
c.	$3(x - 2)^3$	C $9(3x - 2)^2$
d.	$(x - 2)^3$	D $18(3x - 2)$

6. The length of a snake, L cm, at time t weeks after it is born is modelled as:

$$L = 12 + 6t + 0.01(20 - t)^2,\ 0 \leq t \leq 20.$$

Find:

a. the length at

i. birth ii. 20 weeks

b. R, the rate of growth, at any time, t

c. the maximum and minimum growth rate.



7. Use the chain rule to calculate the derivative of the following. (*Hint:* Simplify first using index notation and the laws of indices.)

a. $y = \dfrac{\sqrt{6x-5}}{6x-5}$ **b.** $f(x) = \dfrac{(x^2+2)^2}{\sqrt{x^2+2}}$

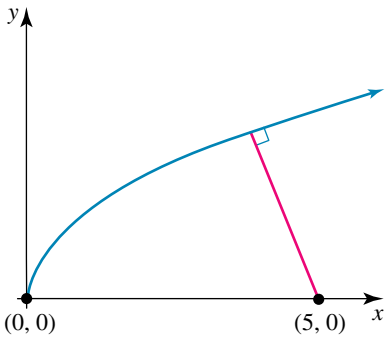
8. If $f(x) = \sqrt{x^2-2x+1}$, calculate:

a. $f(3)$ **b.** $f'(x)$ **c.** $f'(3)$ **d.** $f'(x)$ when $x = 2$.

9. Differentiate the function $f(x) = 2(5x+1)^3$ using the chain rule and by applying the product rule. Confirm they produce the same result.

10. Compare and contrast the terms of a function and its derivative when the chain rule has been applied.

11. Find the minimum distance from the line $y = 2\sqrt{x}$ to the point $(5, 0)$.



12. The profit, $\$P$, per item that a store makes by selling n items of a certain type each day is $P = 40\sqrt{n+25} - 200 - 2n$.

a. Find the number of items that need to be sold to maximise the profit on each item.

b. What is the maximum profit per item?

c. Hence, find the total profit per day by selling this number of items.

13. A particle moves in a straight line so that its displacement from a point, O, at any time, t, is $x = \sqrt{3t^2+4}$. Find:

a. the velocity as a function of time

b. the acceleration as a function of time

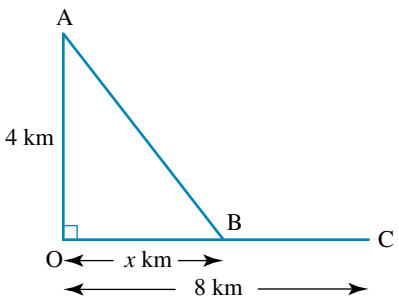
c. the velocity and acceleration when $t = 2$.

Technology active

14. **a.** If $f(x) = (2x-1)^6$, calculate $f'(3)$.

b. If $g(x) = (x^2-3x)^{-2}$, calculate $g'(-2)$.

15. A rower is in a boat 4 kilometres from the nearest point, O, on a straight beach. His destination is 8 kilometres along the beach from O. If he is able to row at 5 km/h and walk at 8 km/h, what point on the beach should he row to in order to reach his destination in the least possible time? Give your answer correct to 1 decimal place.



16. A slingshot is spinning a stone anticlockwise, according to the functions $y = \pm\sqrt{0.09 - x^2}, -0.3 \leq x \leq 0.3$. Assuming the stone travels in a straight line when released, at what point should it be released to hit a target directly in line with the x-axis a distance of 1 m to the right from the centre of rotation?



14.5 Applications of the product, quotient and chain rules

We can combine the product, quotient and chain rules to derive more complicated functions.

WORKED EXAMPLE 8

Determine the derivative of $y = x^2(3x + 4)^2$.

THINK	WRITE
1. Recognise the product rule can be applied as $y = uv$.	$u = x^2,\ v = (3x + 4)^2$ $u' = 2x,\ v' = \ldots$
2. Identify that to derive v the chain rule needs to be applied.	Let $w = 3x + 4$, $\therefore v = w^2$ $\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$ $= 2w \times 3$ $= 6(3x + 4)$
3. Complete the product rule.	$u = x^2,\ v = (3x + 4)^2$ $u' = 2x,\ v' = 6(3x + 4)$ $y' = uv' + vu'$ $= 6x^2(3x + 4) + 2x(3x + 4)^2$ $= 18x^3 + 24x^2 + 2x(3x + 4)(3x + 4)$ $= 18x^3 + 24x^2 + 2x(9x^2 + 24x + 16)$ $= 18x^3 + 24x^2 + 18x^3 + 48x^2 + 32x$ $= 36x^3 + 72x^2 + 32x$

studyon

Units 1 & 2 > Area 9 > Sequence 1 > Concept 4

Applications of differentiation rules Summary screen and practice questions

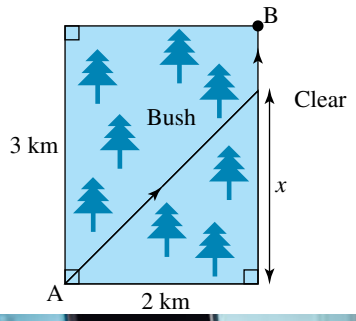
Exercise 14.5 Applications of the product, quotient and chain rules

Technology free

1. WE8 Calculate the derivative of:
   a. $x^2(x+1)^3$ b. $x^3(x+1)^2$ c. $\sqrt{x}(x+1)^5$ d. $x^{\frac{3}{2}}(x-2)^3$
   e. $x(x-1)^{-2}$ f. $x\sqrt{x+1}$.
2. Differentiate the following.
   a. $x^{-2}(2x+1)^3$ b. $2\sqrt{x}(4-x)$ c. $(x-1)^4(3-x)^{-2}$ d. $(3x-2)^2g(x)$
3. Differentiate the following.
   a. $h(x) = \dfrac{(5-x)^2}{\sqrt{5-x}}$ b. $y = \dfrac{3x-1}{2x^2-3}$ c. $h(x) = \dfrac{x-4x^2}{2\sqrt{x}}$ d. $y = \dfrac{3\sqrt{x}}{x+2}$
4. Calculate the derivative of each of the following.
   a. $x\left(x^2+1\right)^3$ b. $\dfrac{\left(x^2+1\right)^3}{x}$ c. $\dfrac{1}{(x^2-3)^5}$ d. $\dfrac{\sqrt{x}(x+1)^3}{x-1}$
5. Calculate the gradient at the stated point for each of the following functions.
   a. $y = \dfrac{2x}{x^2+1}, x=1$ b. $y = \dfrac{x+1}{\sqrt{3x+1}}, x=5$
6. For each of the following functions, determine the equation of:
   i. the tangent at the given value of x
   ii. the normal at the given value of x.
   a. $y = x^2+1, x=1$
   b. $y = (x-1)(x^2+2), x=-1$
   c. $y = \sqrt{2x+3}, x=3$
   d. $y = x(x+2)(x-1), x=-1$
7. Graph the function $3x(x-6)^3$ and its derivative. Comment on any points of significance.
8. Obtain any stationary points of the following curves and justify their nature.
   a. $y = x(x+2)^2$ b. $y = \dfrac{x^2}{x+1}$
9. Consider the function $f(x) = \dfrac{2x}{(x^2-3x)}$.
   Differentiate the function using:
   a. the quotient rule
   b. the product and chain rule
   c. the chain rule only.
   Comment on which was the most efficient method.
10. Consider the function $f(x) = (x-a)(x-b)^3$, where $a>0$, $b>0$ and $a<b$.
    a. Determine the x-intercepts.
    b. Locate the coordinates of the stationary points.
    c. State the nature of the stationary points.
    d. If one of the stationary points has coordinates $(3,-27)$, determine the values of a and b.
11. Calculate the area of the largest rectangle with its base on the x-axis that can be inscribed in the semicircle $y = \sqrt{4-x^2}$.

12. A bushwalker can walk at 5 km/h through clear land and 3 km/h through bushland. If she has to get from point A to point B following a route indicated at right, determine the value of x so that the route is covered in a minimum time.

$\left(\textit{Note}: \text{time} = \dfrac{\text{distance}}{\text{speed}}\right)$



13. A colony of viruses can be modelled by the rule

$$N(t) = \frac{2t}{(t+0.5)^2} + 0.5$$

where N hundred thousand is the number of viruses on a nutrient plate t hours after they started multiplying.



a. How many viruses were present initially?

b. Determine $N'(t)$.

c. What is the maximum number of viruses, and when will this maximum occur?

d. At what rate were the virus numbers changing after 10 hours?

14. Prove that the rectangle of largest area that can be inscribed in a circle of a fixed radius is a square.

14.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Use the product rule to determine the derivative of the following functions.

a. $h(x) = x^3(x^2 + 2x)$ **b.** $h(x) = \dfrac{2}{x}(x^3 + 7)$

2. Calculate the gradient at $x = 4$ for the function $y = 4x^2(3 - 5x)$ using the product rule.

3. Using the product rule, determine the derivative of $y = \left(2x^2 - 3 + \dfrac{1}{x}\right)\left(1 + \dfrac{3}{x}\right)$.

4. Determine the derivative of $\dfrac{x+1}{x^2-1}$.

5. Consider the curve defined by the rule $y = \dfrac{2x-1}{3x^2+1}$.

a. Determine the rule for the gradient.

b. For what value(s) of x is the gradient equal to 0.875? Give your answers correct to 4 decimal places. Use technology of your choice to answer the question.

6. The amount of chlorine in a jug of water t hours after it was filled from a tap is $C = \dfrac{20}{t+1}$, where C is in millilitres. Evaluate the rate of decrease of chlorine 9 hours after being poured.



7. Use the chain rule to determine the derivatives of the following.

a. $y = \sqrt{x^2 - 7x + 1}$ **b.** $y = (3x^2 + 2x - 1)^3$

8. Determine the derivatives of the following functions.

a. $g(x) = 3(x^2 + 1)^{-1}$ **b.** $g(x) = \sqrt{(x+1)^2 + 2}$

c. $f(x) = \sqrt{x^2 - 4x + 5}$ **d.** $f(x) = \left(x^3 - \dfrac{2}{x^2}\right)^{-2}$

9. The function h has a rule $h(x) = \sqrt{x^2 - 16}$ and the function g has the rule $g(x) = x - 3$. Calculate the gradient of the function $h(g(x))$ at the point when $x = -2$.

10. Calculate the derivative of:

a. $(4 - x^2)^3$

b. $x^2(x + 3)^4$

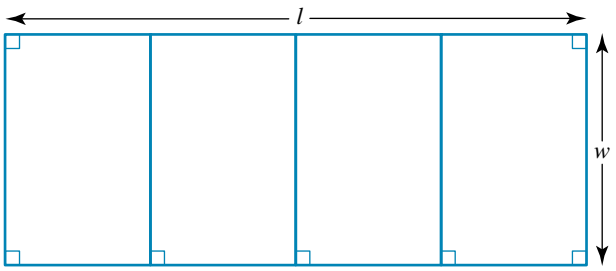
c. $\dfrac{x^3}{x^2 + 1}$.

11. Given $f(x) = 2x^2(1 - x)^3$, use calculus to determine the coordinates where $f'(x) = 0$.

12. Determine the derivative of the following function, and hence calculate the gradient at the given x-value.
$f(x) = \sqrt[3]{(3x^2 - 2)^4}$; $x = 1$.

Complex familiar

13. A pen for holding farm animals has dimensions $l \times w$ metres. This pen is to be partitioned so that there are four spaces of equal area as shown.



The farmer has 550 metres of fencing material to construct this pen.

a. Calculate the required length and width in order to maximise the area of the pen.

b. Hence, calculate the maximum area.

14. Water is being poured into a vase. The volume, V mL, of water in the vase after t seconds is given by:

$$V = \frac{2}{3}t^2(15 - t),\ 0 \leq t \leq 10.$$



a. What is the volume after 10 seconds?

b. At what rate is the water flowing into the vase at t seconds?

c. What is the rate of flow after 3 seconds?

d. When is the rate of flow the greatest, and what is the rate of flow at this time?

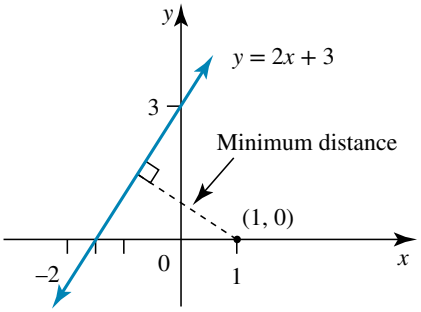
15. The volume of water, V litres, in a bath t minutes after the plug is removed is given by $V = 0.4(8 - t)^3$, $0 \leq t \leq 8$.
Use technology of your choosing to answer the question.

a. At what rate is the water leaving the bath after 3 minutes?

b. What is the average rate of change of the volume for the first 3 minutes?

c. When is the rate of water leaving the bath the greatest?

16. Find the minimum distance from the line $y = 2x + 3$ to the point $(1, 0)$.



Complex unfamiliar

17. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 16 cm by 10 cm. The box will be made by cutting equal squares of side length x cm out of the four corners and folding the flaps up.

a. Express the volume as a function of x.

b. Using the product rule, determine the dimensions of the box with greatest volume and give this maximum volume.

18. A veterinarian has administered a painkiller by injection to a sick horse.



The concentration of painkiller in the blood, y mg/L, can be defined by the rule

$$y = \frac{3t}{(4 + t^2)}$$

where t is the number of hours since the medication was administered.

a. Find $\frac{dy}{dt}$.

b. What is the maximum concentration of painkiller in the blood, and at what time is this achieved?

c. The effect of the painkiller is considerably reduced once the concentration falls below 0.5 mg/L, when a second dose needs to be given to the horse. When does this occur?

d. Find the rate of change of concentration of painkiller in the blood after one hour. Give your answer correct to 2 decimal places.

e. When is the rate of change of painkiller in the blood equal to −0.06 mg/L/hour? Give your answer correct to 2 decimal places.

19. Consider the function $f(x) = (a - x)^2(x - 2)$ where $a > 2$.

a. Find the coordinates of the stationary points.

b. State the nature of the stationary points.

c. Find the value of a if the graph of $y = f(x)$ has a turning point at $(3, 4)$.

20. Sketch the graph of $f(x) = \frac{1}{2}(2x - 3)^4(x + 1)^5$, showing all intercepts and stationary points.

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Units 1 & 2 Sit chapter test

Answers

Chapter 14 Differentiation rules

Exercise 14.2 The product rule

1. a. $u = x + 3, v = 2x^2 - 5x$

b. $\frac{du}{dx} = 1, \frac{dv}{dx} = 4x - 5$

c. $\frac{dy}{dx} = 6x^2 + 2x - 15$

2. a. i. $f(x) = x + 2, g(x) = x - 3$

ii. $f'(x) = 1, g'(x) = 1$

iii. $2x - 1$

b. i. $f(x) = 3x^2, g(x) = x^2 - 4x + 1$

ii. $f'(x) = 6x, g'(x) = 2x - 4$

iii. $6x(2x^2 - 6x + 1)$

c. i. $f(x) = x^{-1}, g(x) = x + 2$

ii. $f'(x) = -x^{-2}, g'(x) = 1$

iii. $\frac{-2}{x^2}$

d. i. $f(x) = \sqrt{x} + 3x,\ g(x) = x^2 - 4$

ii. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 3,\ g'(x) = 2x$

iii. $9x^2 + \frac{5}{2}\sqrt{x^3} - 12 - \frac{2}{\sqrt{x}}$

3. a. $A(t) = (t^2 - 2t + 1)(t + 1)$

b. 64 mm/s

4. $\frac{4}{\sqrt{x}} - 3\sqrt{x}$

5. $4 - \frac{12}{x^2}$

6. $(0, 0)$ and $\left(\frac{3}{4}, \frac{27}{128}\right)$

7. a. $x(5x + 2)(x + 1)^2$

b. $(x + 1)(5x + 3)x^2$

c. $\frac{1}{2\sqrt{x}}(x + 1)^4(11x + 1)$

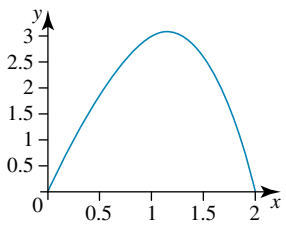
d. $\frac{3}{2}\sqrt{x}(x - 2)^2(3x - 2)$

e. $-(x + 1)(x - 1)^{-3}$

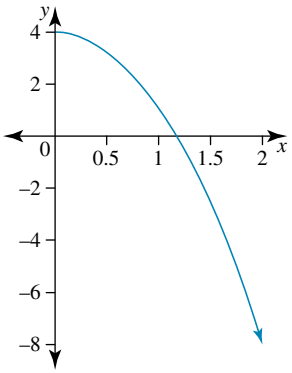
f. $\frac{1}{2\sqrt{x + 1}}(3x + 2)$

8. $y_T = -14x + 24, y_N = \frac{1}{14}x - \frac{29}{7}$

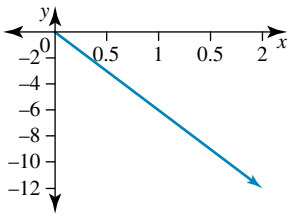
9. a.



b.



c.



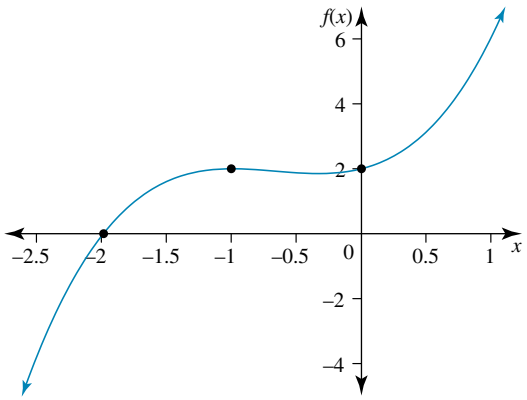
d. The particle travels away from the starting point for about 1.15 seconds (clearly seen on the velocity graph) before returning to the initial position. It starts with an initial velocity of 4 m/s and slows as it moves away from the starting point. It continues to decelerate linearly throughout the time.

10. a. $18x^2 - 34x + 5$

b. $3(6x^2 - 2x - 5)$

11. $y' = na(ax + b)^{n-1}$

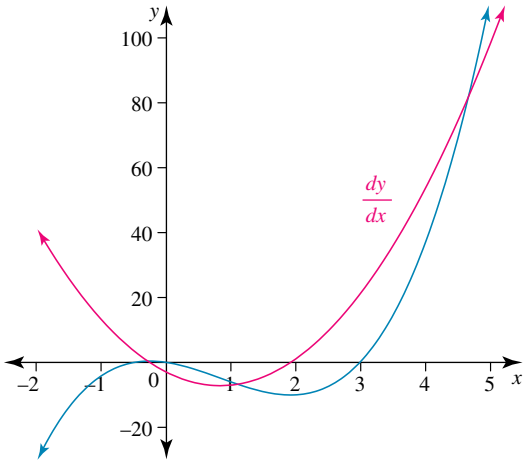
12.



13. a. $R(x) = x\left(-\frac{2}{5}x + 1000\right)$

b. \$960/unit

14. $\frac{dy}{dx} = 6x^2 - 10x - 3$



The function is a cubic and the derivative is a quadratic. The derivative crosses the x-axis when the function is at its maximum and minimum.

Exercise 14.3 The quotient rule

1. a. $u = x + 3,\ v = x + 7$

b. $\frac{du}{dx} = 1,\ \frac{dv}{dx} = 1$

c. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{4}{(x+7)^2}$

2. a. $f(x) = x^2 + 2x,\ g(x) = 5 - x$

b. $f'(x) = 2x + 2,\ g'(x) = -1$

c. $h'(x) = \frac{-x^2 + 10x + 10}{(5-x)^2}$

3. $-\frac{x^2 + 2x + 1}{(x^2 - 1)^2}$

4. $\frac{-4x^2 + 20x + 13}{(5 - 2x)^2}$

5. a. $\frac{-2}{(x-4)^2}$ **b.** $\frac{3x^2 + 4x - 4}{(3x+2)^2}$

c. $\frac{33}{(10-x)^2}$ **d.** $-\frac{(x^2+15)}{2x^{\frac{5}{2}}}$

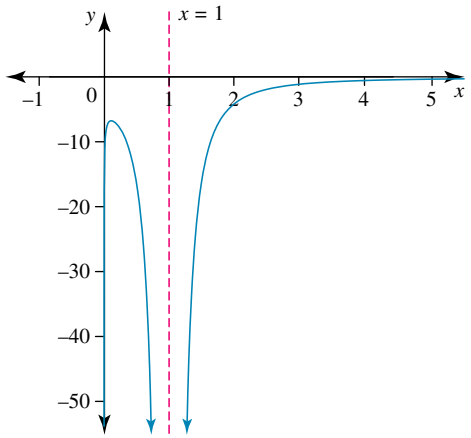
6. a. Change form to $\frac{7}{x} + 1 = 7x^{-1} + 1$, then apply the power rule.

b. Change form to $x - 3 + \frac{4}{x^2} = x - 3 + 4x^{-2}$, then apply the power rule.

c. Factorise $\frac{(x+2)(x+3)}{x+2} = x + 3, x \neq -2$, then apply the power rule.

d. Factorise using difference of two squares $\frac{(x-4)(x+4)}{x+4} = x - 4, x \neq -4$, then apply the power rule.

7.



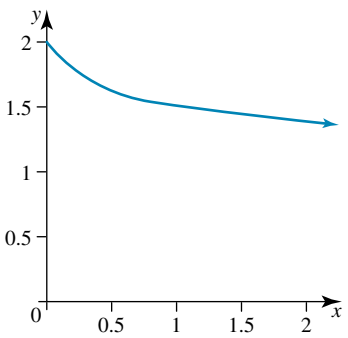
$y' = -\frac{1}{\sqrt{x}(\sqrt{x} - 1)^2}$

8. $y_T = \frac{1}{32}x + \frac{5}{32}$

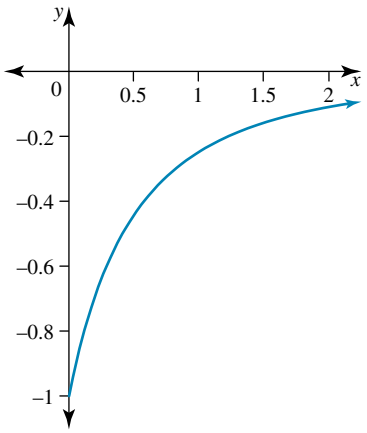
$y_N = -32x - \frac{255}{8}$

9. $y' = \frac{-16}{x^3}$; Both methods produce the same result as they are the same function represented differently.

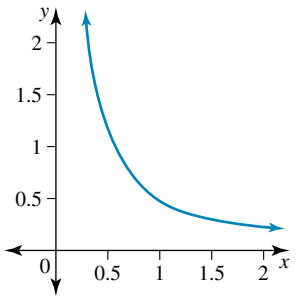
10. a.



b. $v(t) = \frac{-1}{(t+1)^2}$



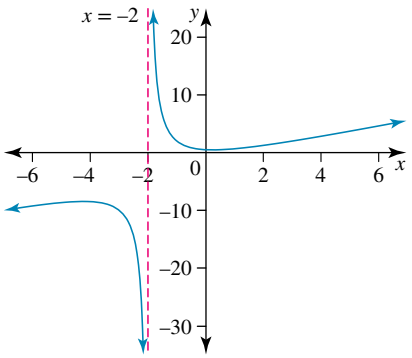
c. $a(t) = \dfrac{2}{(t+1)^3}$



d. The particle starts 2 m away from the reference point and then moves towards the reference point. It slows as it travels and is gradually decelerating less and less.

11. $\dfrac{2(3x^2 - 12x + 7)}{(x-2)^2}$

12.



13. At $r = 6$, $V = 24\pi$ units^3

14. a. $AP(x) = \dfrac{2x^2 + 12x + 4}{x}$ **b.** $\dfrac{4999}{2500}$

15. 29.27 t (2 dp)

16. A possible solution depending on technology used:

a. $a = -317.442$

$b = -0.9164$

$y = -\dfrac{317.442}{x + 0.9164}$

b. $\dfrac{317.442}{(x + 0.9164)^2}$

Exercise 14.4 The chain rule

1. a. i. $2u$ **ii.** 3 **iii.** $6(3x + 2)$

b. i. $3u^2$ **ii.** -1 **iii.** $-3(7 - x)^2$

c. i. $\dfrac{-1}{u^2}$ **ii.** 2 **iii.** $-\dfrac{2}{(2x-5)^2}$

d. i. $\dfrac{-4}{u^5}$ **ii.** -2 **iii.** $\dfrac{8}{(4-2x)^5}$

e. i. $\dfrac{1}{2\sqrt{u}}$ **ii.** 5 **iii.** $\dfrac{5}{2\sqrt{5x+2}}$

f. i. $\dfrac{-3}{2u^{\frac{3}{2}}}$ **ii.** 3 **iii.** $\dfrac{-9}{2(3x-2)^{\frac{3}{2}}}$

g. i. $15u^4$

ii. $4x + 5$

iii. $15(4x + 5)(2x^2 + 5x)^4$

h. i. $-2u^{-3}$

ii. $4 - 6x$

iii. $-4(2 - 3x)(4x - 3x^2)^{-3}$

i. i. $6u^5$

ii. $\dfrac{x^2 - 1}{x^2}$

iii. $\dfrac{6(x^2 - 1)\left(x + \frac{1}{x}\right)^5}{x^2}$

j. i. $-16u^{-5}$

ii. -6

iii. $96(5 - 6x)^{-5}$

2. a. $32(8x + 3)^3$ **b.** $6(2x - 5)^2$

c. $-15(4 - 3x)^4$ **d.** $\dfrac{3x}{\sqrt{3x^2 - 4}}$

e. $\dfrac{2}{3}(x-2)(x^2 - 4x)^{-\frac{2}{3}}$ **f.** $-2(6x^2 + 1)(2x^3 + x)^{-3}$

g. $6\left(1 + \dfrac{1}{x^2}\right)\left(x - \dfrac{1}{x}\right)^5$ **h.** $-(2x - 3)(x^2 - 3x)^{-2}$

3. $\dfrac{-2}{\sqrt{(4x+7)^3}}$

4. a. $8x^7(2x + 5)(x + 5)^7$

b. $2x(3x^2 - 2)(x^2 - 2)$

c. $\dfrac{3x^2 + 4x}{5(x^3 + 2x^2 - 7)^{\frac{4}{3}}}$

d. $3x(4x^2 - 3)\sqrt{2x^4 - 3x^2 + 1}$

5. a. C **b.** D **c.** B **d.** A

6. a. i. 16 cm **ii.** 132 cm

b. $R = 0.02t + 5.6$

c. Max. = 6, min. = 5.6

7. a. $\dfrac{-3}{(6x-5)^{\frac{3}{2}}}$ **b.** $3x\sqrt{x^2 + 2}$

8. a. 2 **b.** $\dfrac{x-1}{\sqrt{x^2 - 2x + 1}}$

c. 1 **d.** 1

9. $f'(x) = 30(5x + 1)^2$

10. The function and the derivative share whichever term is set to u. Any powers will decrease by one as per the power rule. Any coefficient in the derivative will be a multiple of the coefficient in the function, the derivative of the u term and the initial power.

11. 4

12. a. 75 **b.** \$50 **c.** \$3750

13. a. $v = \dfrac{3t}{(3t^2 + 4)^{\frac{1}{2}}} = \dfrac{3t}{\sqrt{3t^2 + 4}}$

b. $a = \dfrac{3}{\sqrt{3t^2 + 4}} - \dfrac{9t^2}{\left(\sqrt{3t^2 + 4}\right)^3} = \dfrac{12}{(3t^2 + 4)^{\frac{3}{2}}}$

c. $v = 1.5$, $a = \dfrac{3}{16}$

14. a. 37 500 **b.** $\dfrac{7}{500}$ or 0.014

15. 3.2 km to the right

16. At $x = 0.09$

Exercise 14.5 Applications of the product, quotient and chain rules

1. a. $x(5x+2)(x+1)^2$

b. $(x+1)(5x+3)x^2$

c. $\frac{1}{2\sqrt{x}}(x+1)^4(11x+1)$

d. $\frac{3}{2}\sqrt{x}(x-2)^2(3x-2)$

e. $-(x+1)(x-1)^{-3}$

f. $\frac{1}{2\sqrt{x+1}}(3x+2)$

2. a. $\frac{2(x-1)(2x+1)^2}{x^3}$

b. $\frac{4-3x}{\sqrt{x}}$

c. $\frac{2(x-5)(x-1)^3}{(x-3)^3}$

d. $(3x-2)((3x-2)g'(x)+6g(x))$

3. a. $-\frac{3\sqrt{5-x}}{2}$ **b.** $\frac{-6x^2+4x-9}{(2x^2-3)^2}$

c. $\frac{1}{4\sqrt{x}}-3\sqrt{x}$ **d.** $\frac{6-3x}{2\sqrt{x}(x+2)^2}$

4. a. $(x^2+1)^2(7x^2+1)$ **b.** $\frac{(x^2+1)^2(5x^2-1)}{x^2}$

c. $\frac{-10x}{(x^2-3)^6}$ **d.** $\frac{(x+1)^2(5x^2-8x-1)}{2\sqrt{x}(x-1)^2}$

5. a. 0 **b.** $\frac{7}{64}$

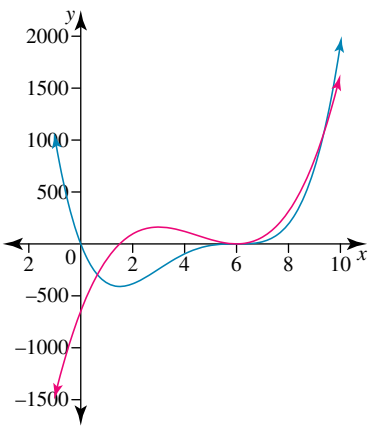
6. a. i. $y = 2x$ **ii.** $x+2y=5$

b. i. $y = 7x+1$ **ii.** $x+7y+43=0$

c. i. $3y = x+6$ **ii.** $y+3x=12$

d. i. $x+y=1$ **ii.** $y=x+3$

7.



When the function (blue) has a stationary point the derivative (orange) intercepts the x-axis. When the function's gradient is downward the derivative is below the x-axis and when the function's gradient is upward the derivative is above the x-axis.

8. a. $(-2, 0)$ maximum turning point, $\left(-\frac{2}{3}; -\frac{32}{27}\right)$ minimum turning point

b. $(-2, -4)$ maximum turning point, $(0, 0)$ minimum turning point

9. All equal $\frac{-2}{(x-3)^2}$ if fully simplified. Simplifying before deriving, as in part c results in a simpler differentiation.

10. a. $(a,0)$, $(b,0)$

b. $(b, 0)$, $\left(\frac{3a+b}{4}, \frac{-27(a-b)^4}{256}\right)$

c. $(b, 0)$ is a stationary point of inflection; $\left(\frac{3a+b}{4}, \frac{-27(a-b)^4}{256}\right)$ is a minimum turning point.

d. $a = 2$, $b = 6$

11. 4 units^2

12. 1.5 km

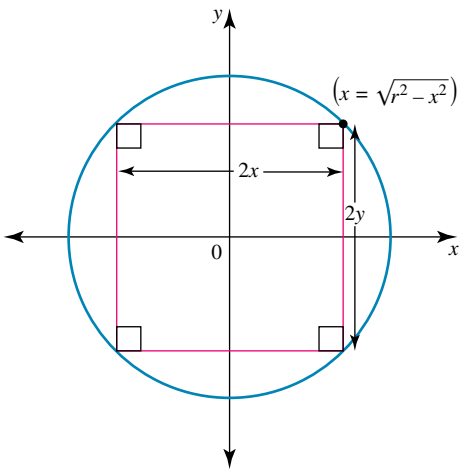
13. a. 0.5 hundred thousand or 50 000

b. $N'(t) = \frac{-2t^2+0.5}{(t+0.5)^4}$

c. $N_{\max} = 1.5$ hundred thousand or 150 000 after half an hour.

d. −1641 viruses/hour

14.



By Pythagoras:

$r^2 = x^2 + y^2$

$r^2 - x^2 = y^2$

$\sqrt{r^2 - x^2} = y,\ y > 0$

Area of rectangle is given by:

$A = (2x)(2y) = 4xy$

$A = 4x\sqrt{r^2 - x^2}$

$\frac{dA}{dx} = -\frac{8x^2}{2\sqrt{r^2-x^2}} + 4\sqrt{r^2-x^2}$

$\frac{dA}{dx} = \frac{8(r^2-x^2)-8x^2}{2\sqrt{r^2-x^2}}$

$\frac{dA}{dx} = \frac{8r^2-8x^2-8x^2}{2\sqrt{r^2-x^2}}$

$\frac{dA}{dx} = \frac{4r^2-8x^2}{\sqrt{r^2-x^2}}$

Max area occurs when $\frac{dA}{dx} = 0$.

$$\frac{4r^2 - 8x^2}{\sqrt{r^2 - x^2}} = 0$$
$$4r^2 - 8x^2 = 0$$
$$4r^2 = 8x^2$$
$$\frac{1}{2}r^2 = x^2$$
$$\frac{1}{\sqrt{2}}r = x,\ x > 0$$

Substitute $x = \frac{1}{\sqrt{2}}r$ into Pythagoras relationship.

$$r^2 - x^2 = y^2$$
$$r^2 - \frac{1}{2}r^2 = y^2$$
$$\frac{1}{2}r^2 = y^2$$
$$y = \frac{1}{\sqrt{2}}r,\ y > 0$$

The x and y values are the same, thus, the largest rectangle is a square.

14.6 Review: exam practice

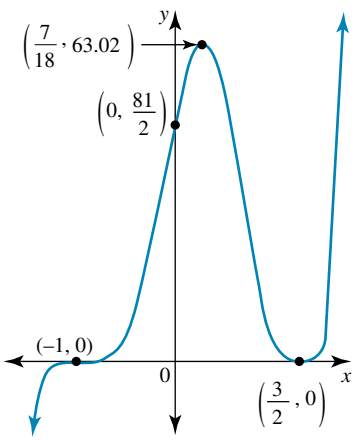
Simple familiar

1. **a.** $x^3(5x + 8)$ **b.** $4x - \frac{14}{x^2}$
2. -864
3. $4x + 6 + \frac{8}{x^2} - \frac{6}{x^3}$
4. $-\frac{1}{(x-1)^2}$
5. **a.** $\frac{-6x^2 + 6x + 2}{(3x^2 + 1)^2}$ **b.** $x = -0.1466, 0.5746$
6. 0.2 mL/h
7. **a.** $\frac{2x - 7}{2\sqrt{x^2 - 7x + 1}}$ **b.** $6(3x+1)(3x^2+2x-1)^2$
8. **a.** $-\frac{6x}{(x^2 + 1)^2}$ **b.** $\frac{x + 1}{\sqrt{x^2 + 2x + 3}}$
   c. $\frac{x - 2}{\sqrt{x^2 - 4x + 5}}$ **d.** $-\frac{6x^5 + 8}{x^3\left(x^3 - \frac{2}{x^2}\right)^3}$
9. $-\frac{5}{3}$
10. **a.** $-6x(4 - x^2)^2$ **b.** $6x(x + 1)(x + 3)^3$
    c. $\frac{x^4 + 3x^2}{(x^2 + 1)^2}$
11. $(0, 0), (1, 0), \left(\frac{2}{5}, \frac{216}{3125}\right)$
12. $f'(x) = 8x\sqrt[3]{3x^2 - 2}$, $f'(1) = 8$

Complex familiar

13. **a.** $l = 137.5$ m, $w = 55$ m
    b. $A_{max} = 7562.5\text{m}^2$
14. **a.** $V = 333\frac{1}{3}$ mL
    b. $\frac{dV}{dt} = 20t - 2t^2$
    c. $\frac{dV}{dt} = 42$ mL/s
    d. $t = 5s$, $\frac{dV}{dt} = 50$mL/s
15. **a.** $\frac{dV}{dt} = 30$L/min
    b. -51.6 L/min
    c. $t = 0$
16. $\sqrt{5}$ units

Complex unfamiliar

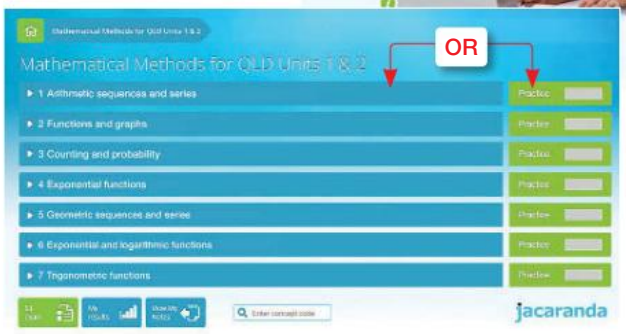
17. **a.** $V = x(16 - 2x)(10 - 2x)$
    b. $x = 2$; therefore, height is 2 cm, length is 12 cm and width is 6 cm. Volume is 144 cm^3.
18. **a.** $\frac{3(4 - t^2)}{(4 + t^2)^2}$
    b. $y_{max} = 0.75$ mg/L after 2 hrs
    c. Next dose after 5.24 hours
    d. 0.36 mg/L/h
    e. $t = 2.45$ h and $t = 6$ h
19. **a.** Stationary points $(a, 0)$, $\left(\frac{a+4}{3}, \frac{4(a-2)^3}{27}\right)$
    b. Minimum turning point $(a, 0)$, maximum turning point $\left(\frac{a+4}{3}, \frac{4(a-2)^3}{27}\right)$
    c. $a = 5$
20. 

TOPIC 5 Further differentiation and applications 1

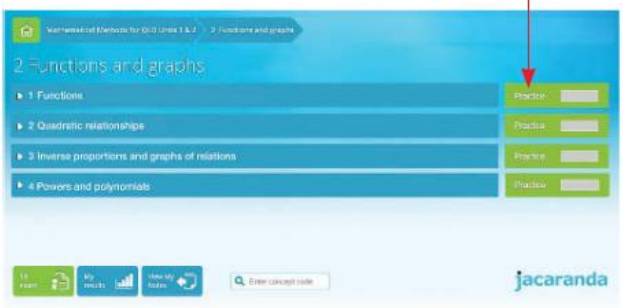
- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



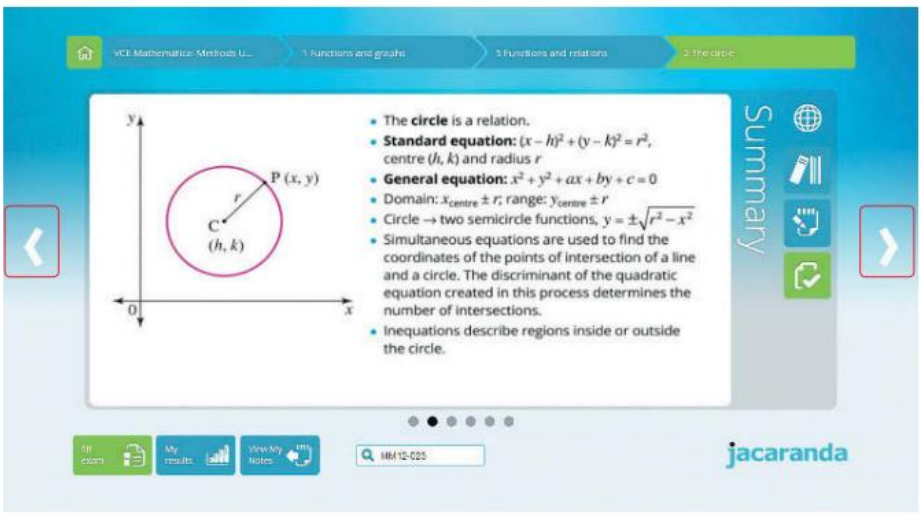
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



CHAPTER 15
Discrete random variables 1

15.1 Overview

15.1.1 Introduction



When we toss a coin, the result can be either a Head or a Tail. When a standard die is thrown, the result can only be one of the numbers from 1 to 6. On selecting a card from a deck, there are only 52 possibilities.

The tossing of a coin, the selection of a card and the rolling of a die can all be regarded as experimental trials, the outcome of which cannot be predicted with absolute certainty ahead of time. The results of such trials are referred to as random variables, a term first formally coined by the Russian mathematician Andrei Kolmogorov in his 1933 text *Grundbegriffe der wahrscheinlichkeitsrechnung* ('Basic concepts of probability').

Kolmogorov set out ideas that are now very familiar to many of us such as:

- the outcome of a single experiment is an 'elementary event'
- all elementary events form a set of all possible outcomes called the 'sample space'
- a random event is defined as a 'measurable set' in this sample space
- the probability of a random event is the 'measure' of this set.

While the sample spaces for some events are infinitely large or include an infinite number of possibilities within an interval, in this chapter we will explore the probabilities of events with sample spaces that have a countable number of elements.

LEARNING SEQUENCE

15.1 Overview
15.2 Discrete random variables
15.3 Expected values
15.4 Variance and standard deviation
15.5 Applications of discrete random variables
15.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

15.2 Discrete random variables

A **random variable** is one whose value cannot be predicted but is determined by the outcome of an experiment. For example, two dice are rolled simultaneously a number of times. The sum of the numbers appearing uppermost is recorded. The possible outcomes we could expect are {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Since the outcome may vary each time the dice are rolled, the sum of the numbers appearing uppermost is a random variable.

Random variables are expressed as capital letters (for example, X, Y, Z) and the value they can take on is represented by lowercase letters (for example, x, y, z respectively).

The above situation with dice illustrates an example of a **discrete random variable** since the possible outcomes were able to be counted. Discrete random variables generally deal with number or size.

A random variable which can take on any value is defined as a **continuous random variable**. Continuous random variables generally deal with quantities which can be measured, such as mass, height or time.

WORKED EXAMPLE 1

Which of the following represent discrete random variables?

a. The number of goals scored at a football match
b. The height of students in a Maths B class
c. Shoe sizes
d. The number of girls in a five-child family
e. The time taken to run a distance of 10 kilometres in minutes

THINK	WRITE
Determine whether the variable can be counted or needs to be measured.	
a. Goals can be counted.	**a.** Discrete
b. Height must be measured.	**b.** Continuous
c. The number of shoe sizes can be counted.	**c.** Discrete
d. The number of girls can be counted.	**d.** Discrete
e. Time must be measured.	**e.** Continuous

15.2.1 Discrete probability distributions

When we are dealing with random variables, we often need to know the probabilities associated with them.

WORKED EXAMPLE 2

Let X represent the variable 'number of Tails' obtained in three tosses. Draw up a table which displays the values the discrete random variable can assume (x) and the corresponding probabilities.

THINK	WRITE
1. List all of the possible outcomes.	HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

2. Draw up a table with two columns: one labelled 'Number of Tails', the other 'Probability'.

Number of Tails (x)	Probability (P(x))
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

3. Enter the information into the table.

The table above displays the **probability distribution** of the total number of Tails obtained in three tosses of a fair coin. Since the variable in this case is discrete, the table displays a discrete probability distribution.

In Worked example 2, we used X to denote the random variable and x the value which the random variable could take. Thus the **probability** can be denoted by $p(x)$ or $\text{P}(X = x)$. Hence, the above table could be presented as shown below.

x	0	1	2	3
$\text{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Close inspection of this table shows important characteristics that satisfy all discrete probability distributions.

1. **Each probability lies in a restricted interval $0 \leq \text{P}(X = x) \leq 1$.**
2. **The probabilities of a particular experiment sum to 1, that is:**

$$\sum \text{P}(X = x) = 1.$$

If these two characteristics are not satisfied, then there is no discrete probability distribution.

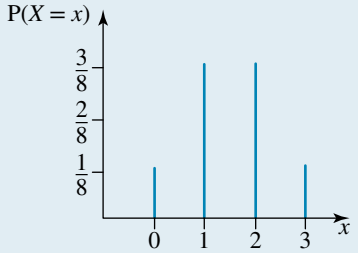
WORKED EXAMPLE 3

Draw a probability distribution graph of the outcomes in Worked example 2.

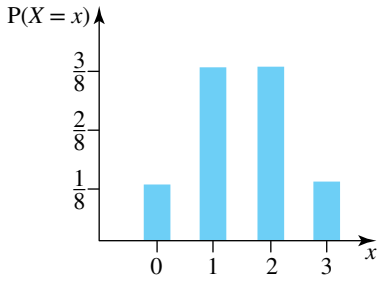
THINK

1. Draw a set of axes in the first quadrant only. Label the horizontal axis x and the vertical axis $\text{P}(X = x)$.
2. Mark graduations evenly along the horizontal and vertical axes, and label them with appropriate values.
3. Draw a straight line from each x-value to its corresponding probability.

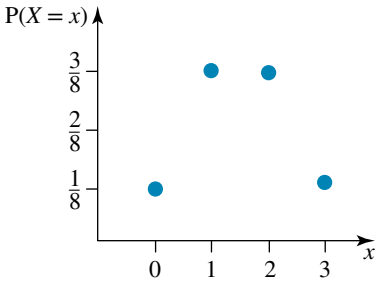
WRITE/DRAW



Note: The probabilty distribution graph may also be drawn as follows.



A column graph



A dot graph

The tossing of an unbiased die 3 times to see how many sixes are obtained is an example of a **uniform distribution**, because all of the outcomes are equally likely. Another example is finding how many Heads are obtained when a single coin is tossed n times. However, a non-uniform distribution exists when a biased coin is used, because all of the outcomes are not equally likely.

WORKED EXAMPLE 4

A motorist travels along a main road in Brisbane. In doing so they must travel through three intersections with traffic lights over a stretch of two kilometres. At each intersection the motorist will encounter either a red light or a green light (ignoring amber!). The probability that the motorist will have to stop because of a red light at any of the intersections is $\frac{2}{5}$. Let X be the number of red lights encountered by the motorist.

a. Use a tree diagram to produce a sample space for this situation.

b. Determine the probability of each outcome.

c. Find the probability distribution for this random variable.

d. Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.



THINK

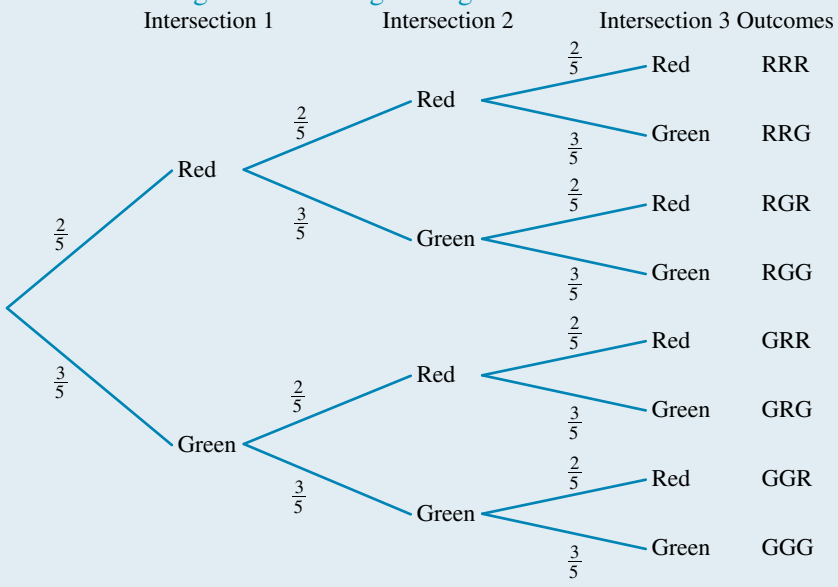
a. 1. Set up a tree diagram to show the sample space.

Note: $P(R) = \frac{2}{5}$,

$P(G) = \frac{3}{5}$.

WRITE

a. Let R = a red light and G = a green light.



2. List the event or sample space.

$\xi = \{\text{RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG}\}$

b. Calculate the probability of each outcome.

b. $\text{P(RRR)} = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{125}$

$\text{P(RRG)} = \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{12}{125}$

$\text{P(RGR)} = \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{125}$

$\text{P(RGG)} = \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{18}{125}$

$\text{P(GRR)} = \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{12}{125}$

$\text{P(GRG)} = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{18}{125}$

$\text{P(GGR)} = \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{18}{125}$

$\text{P(GGG)} = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

c. 1. Set up the probability distribution by combining the outcomes related to each possible value of x.

c. $\text{P}(X = 0) = \text{P(GGG)} = \frac{27}{125}$

$\text{P}(X = 1) = \text{P(RGG)} + \text{P(GRG)} + \text{P(GGR)}$

$= \frac{18}{125} + \frac{18}{125} + \frac{18}{125}$

$= \frac{54}{125}$

$\text{P}(X = 2) = \text{P(RRG)} + \text{P(RGR)} + \text{P(GRR)}$

$= \frac{12}{125} + \frac{12}{125} + \frac{12}{125}$

$= \frac{36}{125}$

$\text{P}(X = 3) = \text{P(RRR)} = \frac{8}{125}$

2. Enter the combined results into a table.

X = number of red lights

x	0	1	2	3
$\text{P}(X = x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

d. Test whether the two properties of a discrete random variable are obeyed.

d. Each $\text{P}(X = x)$ is such that $0 \leq \text{P}(X = x) \leq 1$ and

$\sum \text{P}(X = x) = \frac{27}{125} + \frac{54}{125} + \frac{36}{125} + \frac{8}{125}$

$= \frac{125}{125}$

$\sum_{\text{all}\,x} \text{P}(X = x) = 1$

Therefore, both properties of a discrete random distribution are obeyed.

WORKED EXAMPLE 5

a. State, giving reasons, whether each of the following represents a discrete probability distribution.

i.

x	0	2	4	6
$P(X = x)$	−0.1	0.3	0.4	0.2

ii.

x	−3	−1	4	6
$P(X = x)$	0.01	0.32	0.52	0.15

iii.

x	−1	0	1	2
$P(X = x)$	0.2	0.1	0.2	0.3

b. A random variable, X , has the following probability distribution.

x	1	2	3	4	5
$P(X = x)$	b	$2b$	$0.5b$	$0.5b$	b

Calculate the value of the constant b.

THINK

a. i. 1. Check that each probability is a value from 0 to 1.
2. If this condition is satisfied, add the probabilities together to see if they add to 1.
3. Answer the question.

ii. 1. Check that each probability is a value from 0 to 1.
2. If this condition is satisfied, add the probabilities together to see if they add to 1.
3. Answer the question.

iii. 1. Check that each probability is a value from 0 to 1.
2. If this condition is satisfied, add the probabilities together to see if they add to 1.

WRITE

a. i. Each probability does not meet the requirement $0 \leq P(X = x) \leq 1$, as $P(X = 0) = -0.1$.

As one of the probabilities is a negative value, there is no point checking the sum of the probabilities.

This is not a discrete probability distribution.

ii. Each probability does meet the requirement $0 \leq P(X = x) \leq 1$.

$$\sum P(X = x) = 0.01 + 0.32 + 0.52 + 0.15$$
$$= 1$$

Yes, this is a discrete probability function, as both of the conditions have been satisfied.

iii. Each probability does meet the requirement $0 \leq P(X = x) \leq 1$.

$$\sum P(X = x) = 0.2 + 0.1 + 0.2 + 0.3$$
$$= 0.8$$

3. Answer the question.

As the sum of the probabilities is not equal to 1, this is not a discrete probability distribution.

b. 1. As we know this is a probability distribution, we can equate the probabilities to 1.

2. Simplify.

3. Solve for b.

b. $\sum P(X = x) = 1$

$$b + 2b + 0.5b + 0.5b + b = 1$$
$$5b = 1$$
$$b = \frac{1}{5}$$

WORKED EXAMPLE 6

a. Show that the function $p(x) = \frac{1}{42}(5x + 3)$, where $x = 0,\ 1,\ 2,\ 3$ is a discrete probability function.

b. Show that the function $p(x) = \frac{1}{100}x^2(6 - x)$, where $x = 2,\ 3,\ 4,\ 5$ is a discrete probability function.

THINK

a. 1. Substitute each of the x-values into the equation and obtain the corresponding probability.

2. Simplify where possible.

3. Check whether each of the probabilities lies within the restricted interval $0 \leq P(X = x) \leq 1$.

4. Check whether the probabilities sum to 1.

5. Answer the question.

b. 1. Substitute each of the x-values into the equation and obtain the corresponding probability.

WRITE

a. When $x = 0$, $P(x) = \frac{3}{42}$

$$= \frac{1}{14}$$

When $x = 1$, $P(x) = \frac{8}{42}$

$$= \frac{4}{21}$$

When $x = 2$, $P(x) = \frac{13}{42}$

When $x = 3$, $P(x) = \frac{18}{42}$

$$= \frac{3}{7}$$

All probabilities lie between 0 and 1.

$$\frac{1}{14} + \frac{4}{21} + \frac{13}{42} + \frac{3}{7} = 1$$

Yes, this is a probability function since both requirements have been met.

b. When $x = 2$, $P(x) = \frac{16}{100}$

$$= \frac{4}{25}$$

2. Simplify where possible.	When $x = 3$, $P(x) = \frac{27}{100}$ When $x = 4$, $P(x) = \frac{32}{100}$ $= \frac{8}{25}$ When $x = 5$, $P(x) = \frac{25}{100}$ $= \frac{1}{4}$
3. Check whether each of the probabilities lies within the restricted interval $0 \leq P(X = x) \leq 1$.	All probabilities lie between 0 and 1.
4. Check whether the probabilities sum to 1.	$\frac{4}{25} + \frac{27}{100} + \frac{8}{25} + \frac{1}{4} = 1$
5. Answer the question.	Yes, this is a probability function since both requirements have been met.

Consider the case of a probability experiment where we know part of the outcome. Suppose your friend Brett comes from a family of four children. What is the probability that there are three boys in Brett's family? Because you know Brett, you know that at least one of the four children is male.

Normally, the probability distribution of four children can be represented by the table shown below.

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625

Therefore, $P(X = 3) = 0.25$ but we know that the number of males in the family is greater than 0. From the table, $P(X > 0) = 0.9375$. We can say that the probability that there are three males in the family, given that at least one is male is $\frac{0.25}{0.9375}$, is 0.266.

This is known as **conditional probability**. The rule for conditional probability is written as follows.

$$P(X = x | X > n) = \frac{P(X = x \cap X > n)}{P(X > n)}$$

WORKED EXAMPLE 7

Three balls are selected from a box containing 6 blue balls and 4 yellow balls. If the ball chosen after each selection is replaced before the next selection, find:

a. the probability distribution for the following numbers of blue balls drawn

i. 0 blue balls **ii. 1 blue ball** **iii. 2 blue balls** **iv. 3 blue balls**

b. the probability that 3 blue balls are chosen, given that at least 2 balls were blue.

THINK	WRITE
a. i. 1. Define the random variable.	**a. i.** Let X = the number of blue balls.
2. Assign values which x can take on.	$x = 0, 1, 2, 3$

3. Determine the probability of each outcome.

$$\text{P(blue)} = \frac{6}{10} \quad \text{P(yellow)} = \frac{4}{10}$$
$$= \frac{3}{5} \qquad = \frac{2}{5}$$

4. Simplify where possible.

$$\text{P}(X = 0) \Rightarrow \text{no blue, three yellow}$$
$$= \text{P(YYY)}$$
$$\text{P}(X = 0) = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$$
$$= 0.064$$

ii. Simplify where possible.

ii. $\text{P}(X = 1) \Rightarrow \text{one blue, two yellow}$
$$= \text{P(BYY)} + \text{P(YBY)} + \text{P(YYB)}$$
$$\text{P}(X = 1) = 3 \times \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}$$
$$= 0.288$$

iii. Simplify where possible.

iii. $\text{P}(X = 2) \Rightarrow \text{two blue, one yellow}$
$$= \text{P(BBY)} + \text{P(BYB)} + \text{P(YBB)}$$
$$\text{P}(X = 2) = 3 \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}$$
$$= 0.432$$

iv. Simplify where possible.

iv. $\text{P}(X = 3) \Rightarrow \text{three blue, no yellow}$
$$= \text{P(BBB)}$$
$$\text{P}(X = 3) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$$
$$= 0.216$$

5. Place all of the information in a table.

x	0	1	2	3
$\text{P}(X = x)$	0.064	0.288	0.432	0.216

6. Check that the probabilities sum to 1.

$$\sum \text{P}(X = x) = 0.064 + 0.288 + 0.432 + 0.216$$
$$= 1$$

b. 1. Define the rule for conditional probability.

b. $$\text{P}(X = 3 | X > 1) = \frac{\text{P}(X = 3 \cap X > 1)}{\text{P}(X > 1)}$$

2. Determine each of the probabilities.

$$\text{P}(X = 3 \cap X > 1) = \text{P}(X = 3)$$
$$= 0.216$$
$$\text{P}(X > 1) = 0.432 + 0.216$$
$$= 0.648$$

3. Substitute values into the rule.

$$\text{P}(X = 3 | X > 1) = \frac{0.216}{0.648}$$

4. Evaluate and simplify.

$$= 0.3333 \left(\text{or } \frac{1}{3} \right)$$

studyon

Units 1 & 2 > Area 10 > Sequence 1 > Concept 1

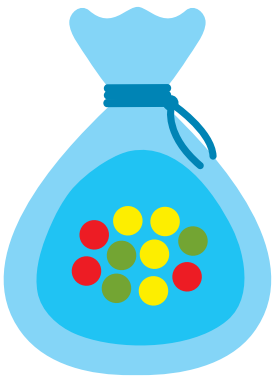
Discrete random variables Summary screen and practice questions

Exercise 15.2 Discrete random variables

Technology free



1. **WE1** Which of the following represent discrete random variables?
   - **a.** The number of people at a tennis match
   - **b.** The time taken to read this question
   - **c.** The length of the left arms of students in your class
   - **d.** The shoe sizes of twenty people
   - **e.** The weights of babies at a maternity ward
   - **f.** The number of grains in each of ten 250-gram packets of rice
   - **g.** The height of jockeys competing in a certain race
   - **h.** The number of books in Brisbane libraries.
2. **WE2, 3** **a.** If X represents the number of Heads obtained in two tosses of a coin, draw up a table which displays the values that the discrete random variable can assume and the corresponding probabilities.
   - **b.** Draw a probability distribution graph of the outcomes in part **a**.
3. A fair coin is tossed three times and a note is taken of the number of Tails.
   - **a.** List the possible outcomes.
   - **b.** List the possible values of the random variable X, representing the number of Tails obtained in the three tosses.
   - **c.** Find the probability distribution of X.
   - **d.** Find $P(X \leq 2)$.
4. **WE4** A bag contains 3 red, 3 green and 4 yellow balls. A ball is withdrawn from the bag, its colour is noted, and then the ball is returned to the bag. This process is repeated on two more occasions. Let Y be the number of green balls obtained
   - **a.** Use a tree diagram to produce the sample space for the experiment.
   - **b.** Determine the probability of each outcome of Y.
   - **c.** Determine the probability distribution for this random variable.
   - **d.** Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.



5. An unbiased die is tossed twice. Let the random variable X be the number of sixes obtained. Find the probability distribution for this discrete random variable.
6. **WE5** **a.** State, giving reasons, whether each of the following represent a discrete probability distribution.

   i.

y	3	6	9	12
$P(Y = y)$	0.2	0.3	0.3	0.2

   ii.

y	−2	−1	0	1	2
$P(Y = y)$	0.15	0.2	0.3	0.2	0.15

   - **b.** Determine the value(s) of k if the table represents a discrete probability distribution.

x	2	3	4	5	6
$P(X = x)$	$5k$	$3k - 0.1$	$2k$	k	$0.6 - 3k$

7. Draw graphs for each of the following probability distributions.

a.

x	1	2	3	4	5
$P(X = x)$	0.05	0.2	0.5	0.2	0.05

b.

x	5	10	15	20
$P(X = x)$	0.5	0.3	0.15	0.05

c.

x	2	4	6	8	10
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

d.

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.4

8. Two fair dice are rolled simultaneously, and X, the sum of the two numbers appearing uppermost, is recorded.

a. Draw up a table which displays the probability distribution of X, and find:

b. $P(X > 9)$

c. $P(X < 6)$

d. $P(4 \leq X < 6)$

e. $P(3 \leq X \leq 9)$

f. $P(X < 12)$

g. $P(6 \leq X < 10)$.

9. Two dice are weighted so that $P(2) = 0.2$, $P(1) = P(3) = P(5) = 0.1$ and $P(4) = P(6) = 0.25$. They are both rolled at the same time. Let Z be the number of even numbers obtained.



a. List the sample space.

b. List the possible values of Z and construct a probability distribution table.

c. Determine $P(Z = 1)$.

10. Each of the following tables shows a discrete probability distribution. Calculate the unknown value in each case. (Assume the unknown value is not zero.)

a.

x	2	4	6	8	10
$P(X = x)$	$3d$	$0.5-3d$	$2d$	$0.4-2d$	$d-0.05$

b.

y	-6	-3	0	3	6
$P(Y = y)$	$0.5k$	$1.5k$	$2k$	$1.5k$	$0.5k$

c.

z	1	3	5	7
$P(Z = z)$	$\frac{1}{3} - a^2$	$\frac{1}{3} - a^2$	$\frac{1}{3} - a^2$	a

11. WE6a Show that the function $p(x) = \frac{1}{90}(8x + 2)$, where $x = 0,\ 1,\ 2,\ 3,\ 4$ is a probability function.

12. WE6b Show that the function $p(x) = \frac{1}{160}x^2(x + 2)$, where $x = 1,\ 2,\ 3,\ 4$ is a probability function.

13. State, with reasons, whether the following are discrete probability distributions.

a. $p(x) = \frac{1}{7}(5 - x), x \in \{1, 3, 4\}$

b. $p(x) = \frac{x^2 - x}{40}, x \in \{-1, 1, 2, 3, 4, 5\}$

c. $p(x) = \frac{1}{15}\sqrt{x}, x \in \{1, 4, 9, 16, 25\}$

14. Find the value of a if the following is a discrete probability function.

$$p(x) = \frac{1}{a}(15 - 3x), x \in \{1, 2, 3, 4, 5\}$$

Technology active

15. **WE7** Three balls are selected from a box containing 4 red balls and 5 blue balls. If the ball chosen after each selection is replaced before the next selection, determine

a. the probability distribution for the following number of red balls drawn

i. 0 red balls **ii.** 1 red ball **iii.** 2 red balls **iv.** 3 red balls

b. the probability that three reds are chosen, given that at least one ball is red.

16. A mature British Blue female cat has just given birth to 4 kittens. Assume that there is an equally likely chance of a kitten being of either sex.

a. Use a tree diagram to list the sample space for the possible number of males and females in the litter.

b. Let X be the number of females in the litter. Construct a probability distribution table for the gender of the kittens.

c. Determine the probability that 4 females will be born.

d. Determine the probability that at least 1 female will be born.

e. Determine the probability that at most 2 females will be born.



17. Matthew likes to collect differently shaped dice. Currently he has two tetrahedrons (4 sides), an icosahedron (20 sides), two dodecahedrons (12 sides) and an octahedron (8 sides) as well as two standard six-sided cubes.

He has decided to play a game of chance using the octahedral die (with sides numbered 1 to 8) and one dodecahedral die (with sides numbered 1 to 12). He tosses the dice simultaneously and notes the number showing uppermost on both dice.

a. List the sample space for the simultaneous tossing of the two dice.

b. Let X be the number of primes obtained as a result of a toss. Determine the value of $P(X = 0)$, $P(X = 1)$ and $P(X = 2)$.

c. This particular game of chance involves tossing the two dice simultaneously on three occasions. The winner of the game must obtain two primes with each of the three tosses. Determine the probability of being a winner. Give your answer correct to 3 decimal places.



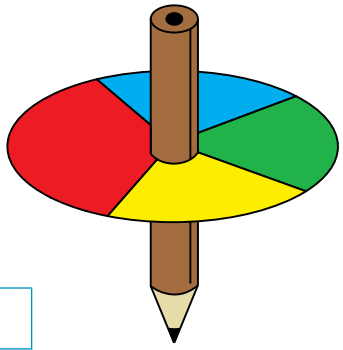
18. Diabetes is the name of a group of diseases that affect how the body uses blood glucose. If you have diabetes, it means that you have too much glucose in your blood. This can lead to serious health problems. Treatment for type 2 diabetes primarily involves monitoring your blood sugar level along with medications, insulin or both.

A new diabetes medication is to be trialled by 5 patients. From experiments that have been performed with mice, the success rate of the new medication is about 60%.

a. Let X denote the number of patients who improve their health with the new medication. Find the probability distribution.

b. The new medication will be considered a success if 68% or more of the patients improve their health. Determine $P(X = 3) + P(X = 4) + P(X = 5)$ and comment on the success of the new medication.

19. A game is played using a spinner that has been loaded so that it is more likely to land on the red side. In fact, $\text{P(red)} = \frac{2}{5}$, and $\text{P(blue)} = \text{P(green)} = \text{P(yellow)} = \frac{1}{5}$.
Each player pays $2 to play. The player spins the spinner a total of 3 times; however, once the spinner lands on the red side the game is over. If a player has a combination of any 3 colours, they win $1, but if the player has a combination of 3 colours that are all the same, they win $10. There are a total of 40 different outcomes for the game.



a. List the possible ways in which the game could end.
b. List the possible ways in which the player could win $10.
c. Suppose X equals the amount of money won by playing the game, excluding the amount the person pays to play, so $X = \{0, 1, 10\}$. Find the probability distribution. Give your answers correct to 4 decimal places.

20. A discrete random variable has the following probability distribution.

y	1	2	3	4	5
$\mathbf{P}(Y = y)$	$0.5k^2$	$0.3 - 0.2k$	0.1	$0.5k^2$	0.3

Find the value(s) of k, correct to 4 decimal places, that meet the criteria for this to be a valid probability distribution function.

15.3 Expected values

In past studies of statistics, the mean ($\tilde{x}$) was defined as the average of a set of data or values. It was determined by the rule $\tilde{x} = \frac{\sum xf}{\sum f}$, where x represented the value a variable could assume and f the frequency (that is, the number of times the variable occurred).

When dealing with discrete random variables, the mean is called the **expected value** or expectation. Since the expected value signifies the average outcome of an experiment, it could be used to determine the feasibility of a situation.

Consider the following example. John tosses two coins. If two Heads are obtained, he wins $20. If one Head is obtained, he wins $10. If no Heads are obtained, he loses $25. John must consider his options and decide whether it is in his best interest to play. Determining the expected value (that is, the average outcome) may help John in his decision making process.

Allowing X to represent the number of Heads obtained, the above information is summarised in the table below.

Outcome	TT	TH or HT	HH
x	0	1	2
$\mathbf{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Win($)	−25	10	20

$$\text{The expectation or expected gain} = \frac{1}{4} \times -25 + \frac{1}{2} \times 10 + \frac{1}{4} \times 20$$
$$= -6.25 + 5 + 5$$
$$= \$3.75$$

The average outcome or expected gain is \$3.75 per toss. This might seem appealing; however, if there is a charge of \$5 per game played, it would not be in John's best interest to participate because he would lose \$1.25 per game on average. The above game would not be considered **fair** since the cost to play does not equal the expected gain.

The expected value of a discrete random variable, X, is denoted by $\mathrm{E}(X)$ or the symbol μ (mu). It is defined as the sum of each value of X multiplied by its respective probability; that is,
$\mathrm{E}(X) = x_1\mathrm{P}(X = x_1) + x_2\mathrm{P}(X = x_2) + x_3\mathrm{P}(X = x_3) + \ldots + x_n\mathrm{P}(X = x_n)$
$= \sum x\mathrm{P}(X = x)$.
Note: **The expected value will not always assume a discrete value.**

on Resources

Interactivity expected value or mean (int-6428)

WORKED EXAMPLE 8

Determine the expected value of a random variable which has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

THINK

1. Write the rule for the expected value.
2. Substitute the values into the rule.
3. Evaluate.

WRITE

$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$

$\mathrm{E}(X) = 1 \times \frac{2}{5} + 2 \times \frac{1}{10} + 3 \times \frac{3}{10} + 4 \times \frac{1}{10} + 5 \times \frac{1}{10}$

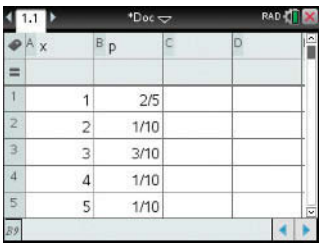
$= \frac{2}{5} + \frac{2}{10} + \frac{9}{10} + \frac{4}{10} + \frac{5}{10}$

$= 2\frac{2}{5}$

TI | THINK

1. On a Lists & Spreadsheet page, label the first column as x and the second column as p. Enter the given x values in the first column and their respective probabilities in the second column.

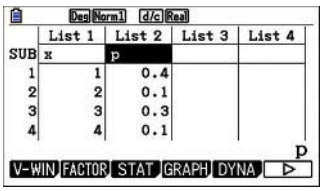
WRITE



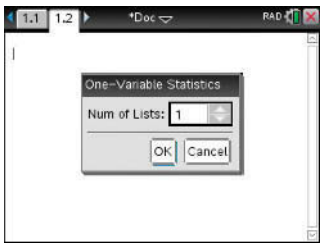
CASIO | THINK

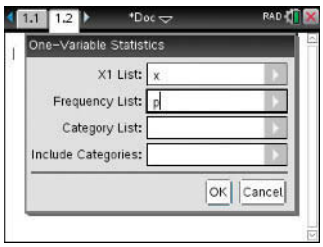
1. On a Statistics screen, label List 1 as x and List 2 as p. Enter the given x values in the first column and their respective probabilities in the second column.

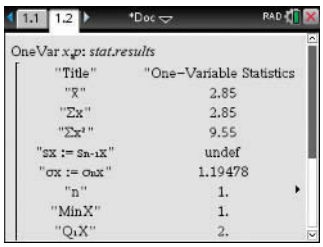
WRITE



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
Select 1 for the Num of Lists then select OK.
Compete the fields as
X1 List: *x*
Frequency List: *p*
then select OK.





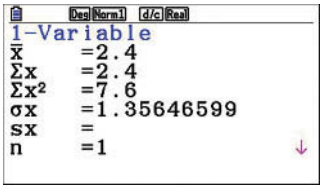


3. The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.4.

2. Select CALC by pressing F2, then select SET by pressing F6.
Complete the fields as
1Var XList: List1
1Var Freq: List2
then press EXIT.
Select 1-VAR by pressing F1.





3. The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.4.

WORKED EXAMPLE 9

Determine the unknown probability, *a*, and hence determine the expected value of a random variable which has the following probability distribution.

x	2	4	6	8	10
$P(X = x)$	0.2	0.4	a	0.1	0.1

THINK

1. Determine the unknown value of a using the knowledge that the sum of the probabilities must total 1.

WRITE

$0.2 + 0.4 + a + 0.1 + 0.1 = 1$
$0.8 + a = 1$
$a = 1 - 0.8$
$= 0.2$

2. Write the rule for the expected value.
3. Substitute the values into the rule.

$E(X) = \sum xP(X = x)$
$E(X) = 2 \times 0.2 + 4 \times 0.4 + 6 \times 0.2 + 8 \times 0.1 + 10 \times 0.1$

4. Evaluate.

$= 0.4 + 1.6 + 1.2 + 0.8 + 1$
$= 5$

WORKED EXAMPLE 10

Determine the values of a and b in the following probability distribution if $\mathrm{E}(X) = 4.29$.

x	1	2	3	4	5	6	7
$\mathrm{P}(X = x)$	0.1	0.1	a	0.3	0.2	b	0.2

THINK

1. Write an equation for the unknown values of a and b using the knowledge that the sum of the probabilities must total 1. Call this equation [1].
2. Write the rule for the expected value.
3. Substitute the values into the rule.
4. Evaluate and call this equation [2].
5. Solve equations simultaneously.

   Multiply equation [1] by 3 and call it equation [3].

   Subtract equation [3] from equation [2]. Solve for b.

   Substitute $b = 0.03$ into equation [1]. Solve for a.
6. Answer the question.

WRITE

$0.1 + 0.1 + a + 0.3 + 0.2 + b + 0.2 = 1$

$0.9 + a + b = 1$

$a + b = 1 - 0.9$

$a + b = 0.1$ [1]

$\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$

$4.29 = 1 \times 0.1 + 2 \times 0.1 + 3 \times a + 4 \times 0.3 + 5 \times 0.2 + 6 \times b + 7 \times 0.2$

$= 0.1 + 0.2 + 3a + 1.2 + 1 + 6b + 1.4$

$4.29 - 3.9 = 3a + 6b$

$3a + 6b = 0.39$ [2]

$a + b = 0.1$ [1]

$3a + 6b = 0.39$ [2]

$3 \times (a + b = 0.1)$

$3a + 3b = 0.3$ [3]

[2]−[3]:

$3b = 0.09$

$b = 0.03$

$a + 0.03 = 0.1$

$a = 0.1 - 0.03$

$= 0.07$

$a = 0.07$ and

$b = 0.03$

WORKED EXAMPLE 11

Niki and Melanie devise a gambling game based on tossing three coins simultaneously. If three Heads or three Tails are obtained, the player wins \$20. Otherwise the player loses \$5. In order to make a profit they charge each person two dollars to play.

a. What is the expected gain to the player?

b. Are Niki and Melanie expected to make a profit?

c. Is this a fair game?

THINK	WRITE
a. 1. Define the random variable. Place all of the information in a table.	**a.** Let X = the number of Heads obtained.

x	0	1	2	3
$\mathbf{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Gain (\$)	20	−5	−5	20

2. Write the rule for the expected value.

$$E(X) = \sum xP(X = x)$$

3. Substitute the values into the rule.

$$= 20 \times \frac{1}{8} + -5 \times \frac{3}{8} + -5 \times \frac{3}{8} + 20 \times \frac{1}{8}$$

4. Evaluate.

$$= \frac{20}{8} - \frac{15}{8} - \frac{15}{8} + \frac{20}{8}$$

$$= \frac{10}{8}$$

$$= \$1.25$$

5. Answer the question.

The player's expected gain per game is \$1.25; however, as each game incurs a cost of \$2, the player in fact loses 75c per game.

b. Answer question using results from **a**.

b. The girls are expected to make a profit of 75c per game.

c. Answer question using results from **a**.

c. No, this not a fair game, since the cost to play each game is more than the expected gain of each game.

WORKED EXAMPLE 12

A random variable has the following probability distribution.

x	1	2	3	4
$\mathbf{P}(X = x)$	0.25	0.26	0.14	0.35

Evaluate:

a. $\mathbf{E}(X)$ **b.** $\mathbf{E}(3X)$ **c.** $\mathbf{E}(2X - 4)$ **d.** $\mathbf{E}(X^2)$.

THINK	WRITE
a. 1. Write the rule for the expected value.	**a.** $E(X) = \sum xP(X = x)$
2. Substitute the values into the rule.	$E(X) = 1 \times 0.25 + 2 \times 0.26 + 3 \times 0.14 + 4 \times 0.35$
3. Evaluate.	$= 0.25 + 0.52 + 0.42 + 1.4$
	$= 2.59$

b. 1. Write the rule for the expected value.
2. Substitute the values into the rule.
3. Evaluate.
Notes: 1. The probability remains the same.
2. Each x-value is multiplied by 3 is because of the new function, which is $3x$.

b. $\mathrm{E}(3X) = \sum 3x\mathrm{P}(X = x)$

$\mathrm{E}(X) = (3 \times 1) \times 0.25 + (3 \times 2) \times 0.26 + (3 \times 3) \times 250.14 + (3 \times 4) \times 0.35$
$= 3 \times 0.25 + 6 \times 0.26 + 9 \times 0.14 + 12 \times 0.35$
$= 0.75 + 1.56 + 1.26 + 4.2$
$= 7.77$

c. 1. Write the rule for the expected value.
2. Substitute the values into the rule.
3. Evaluate.
Notes: 1. The probability remains the same.
2. Each x-value is multiplied by 2 and then 4 is subtracted from the result, because of the new function, which is $2x - 4$.

c. $\mathrm{E}(2X - 4) = \sum(2x - 4)\mathrm{P}(X = x)$

$= (2 \times 1 - 4) \times 0.25 + (2 \times 2 - 4) \times 0.26 + (2 \times 3 - 4) \times 0.14 + (2 \times 4 - 4) \times 0.35$
$= -2 \times 0.25 + 0 \times 0.26 + 2 \times 0.14 + 4 \times 0.35$
$= -0.5 + 0 + 0.28 + 1.4$
$= 1.18$

d. 1. Write the rule for the expected value.
2. Substitute the values into the rule.
3. Evaluate
Notes: 1. The probability remains the same.
2. Each x-value is squared because of the new function, which is x^2.

d. $\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x)$

$= (1^2) \times 0.25 + (2^2) \times 0.26 + (3^2) \times 0.14 + (4^2) \times 0.35$

$= 1 \times 0.25 + 4 \times 0.26 + 9 \times 0.14 + 16 \times 0.35$

$= 0.25 + 1.04 + 1.26 + 5.6$

$= 8.15$

Worked example 12 displays some important points that will now be investigated.

For this example,	$\mathrm{E}(X)$	$= 2.59$
from **b**	$\mathrm{E}(3X)$	$= 7.77$
note that	$3\mathrm{E}(X)$	$= 3 \times 2.59$
		$= 7.77$
from **c**	$\mathrm{E}(2X - 4)$	$= 1.18$
note that	$2\mathrm{E}(X) - 4$	$= 2 \times 2.59 - 4$
		$= 1.18$

Hence if X is a random variable and a is a constant, its expected value is defined by $\mathrm{E}(aX) = a\mathrm{E}(X)$. Furthermore, if X is a random variable where a and b are constants, then the expected value of a linear function in the form $f(X) = aX + b$ is defined by

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

If $a = 0$ then becomes

$$\begin{aligned}\mathrm{E}(aX + b) &= a\mathrm{E}(X) + b\\ \mathrm{E}(0X + b) &= 0\mathrm{E}(X) + b\\ &= b\end{aligned}$$

These rules are called *expectation theorems* and are summarised below.

$\mathbf{E}(aX) = a\mathbf{E}(X)$	**where X is a random variable and a is a constant.**
$\mathbf{E}(aX + b) = a\mathbf{E}(X) + b$	**where X is a random variable a and b are constants.**
$\mathbf{E}(b) = b$	**where b is a constant.**
$\mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y)$	**where X and Y are both random variables.**

These theorems make it easier to calculate the expected values.

WORKED EXAMPLE 13

Casey decides to apply for a job selling mobile phones. She receives a base salary of $200 per month and $15 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, x, being sold per month. What would be the expected salary Casey would receive each month?

x	50	100	150	200	250
$\mathbf{P}(X = x)$	0.48	0.32	0.1	0.06	0.04

THINK

Method 1

1. Define a random variable.
2. Write the rule for the expected salary.
3. Substitute the values into the rule.
4. Evaluate.
5. Answer the question.

WRITE

Let X = the number of mobile phones sold by Casey in a mont

$$\begin{aligned}\mathrm{E}(15X + 200) &= \sum(15x + 200)\mathrm{P}(X = x)\\ &= (15 \times 50 + 200) \times 0.48 +\\ &\quad (15 \times 100 + 200) \times 0.32 +\\ &\quad (15 \times 150 + 200) \times 0.1 +\\ &\quad (15 \times 200 + 200) \times 0.06 +\\ &\quad (15 \times 250 + 200) \times 0.04\\ &= 950 \times 0.48 + 1700 \times 0.32 +\\ &\quad 2450 \times 0.1 + 3200 \times 0.06 +\\ &\quad 3950 \times 0.04\\ &= 456 + 544 + 245 + 192 + 158\\ &= 1595\end{aligned}$$

The expected salary Casey would receive each month would be $1595.

Method 2

Using the expectation theorem:

1. Write the rule for the expected salary.
2. Substitute the values into the rule.
3. Evaluate.
4. Using the fact that $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$, find $\mathrm{E}(15X + 200)$.

$$\begin{aligned}\mathrm{E}(X) &= \sum xP(X = x)\\ &= 50 \times 0.48 + 100 \times 0.32 + 150 \times 0.1 + 200 \times 0.06 + 250 \times 0.04\\ &= 24 + 32 + 15 + 12 + 10\\ &= 93\\ \mathrm{E}(15X + 200) &= 15E(X) + 200\\ &= 15 \times 93 + 200\\ &= 1595\end{aligned}$$

Note: Using the expectation theorem is quicker because it is easier to evaluate $a\mathrm{E}(X) + b$ than $\mathrm{E}(aX + b)$.

studyon

Units 1 & 2 > Area 10 > Sequence 1 > Concept 2

Expected values Summary screen and practice questions

Exercise 15.3 Expected values

Technology active

1. WE8 Find the expected value of a random variable which has the following probability distribution.

x	1	2	3	4
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{16}$	$\frac{3}{16}$

2. Find the expected value of a random variable which has the following probability distribution.

x	−4	−2	0	2	4	6
$P(X = x)$	0.15	0.18	0.06	0.23	0.31	0.07

3. WE9 Find the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	1	3	5	7	9	11
$P(X = x)$	0.11	0.3	0.15	0.25	a	0.1

4. Find the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	−2	1	4	7	10	13
$P(X = x)$	$\frac{5}{18}$	a	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{1}{18}$	$\frac{2}{9}$

5. Find the unknown probability, b, and hence determine the expected value of a random variable which has the following probability distribution.

x	0	1	2	3	4	5
$\mathbf{P}(X = x)$	b	0.2	0.02	$3b$	0.1	0.08

6. Find the value of k, and hence determine the expected value of a random variable which has the following probability distribution.

x	4	8	12	16	20
$\mathbf{P}(X = x)$	$6k$	$2k$	k	$3k$	$8k$

7. If X represents the outcome of a fair die being rolled, find:
   a. the probability distribution of each outcome b. $\mathrm{E}(X)$.
8. Two fair dice are rolled simultaneously. If X represents the sum of the two numbers appearing uppermost, find:
   a. the probability distribution of each outcome b. $\mathrm{E}(X)$.
9. If X represents the number of Heads obtained when a fair coin is tossed twice, find:
   a. the probability distribution of each outcome b. $\mathrm{E}(X)$.
10. A fair coin is tossed 4 times. If X represents the number of Tails obtained, find:
    a. the probability distribution of each outcome b. $\mathrm{E}(X)$.
11. **WE10** Find the values of a and b of the following distribution if $\mathrm{E}(X) = 1.91$.

x	0	1	2	3	4	5	6
$\mathbf{P}(X = x)$	0.2	0.32	a	0.18	b	0.05	0.05

12. Find the values of a and b of the following distribution if $\mathrm{E}(X) = 2.41$.

x	0	1	2	3	4	5
$\mathbf{P}(X = x)$	0.2	a	0.23	0.15	b	0.12

13. **WE11** Lucas contemplates playing a new game which involves tossing three coins simultaneously. He will receive \$15 if he obtains 3 Heads,\$10 if he obtains 2 Heads and \$5 if he obtains 1 Head. However, if he obtains no Heads he must pay \$30. He must also pay \$5 for each game he plays.
    a. What is Lucas' expected gain?
    b. Should he play the game? Why?
    c. Is this a fair game? Why?



14. X is a discrete random variable with the following probability distribution.

x	-2	3	8	10	14	k
$\mathbf{P}(X = x)$	0.1	0.08	0.07	0.27	0.16	0.32

Find the value of k if the mean is 10.98.

15. A coin is biased such that the probability of obtaining a Tail is 0.6. If X represents the number of Tails in three tosses of the coin, find:

a. the probability distribution of X **b.** $\mathrm{E}(X)$.

16. **WE12** A random variable has the following probability distribution.

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{2}{15}$	$\frac{7}{15}$	$\frac{1}{3}$	$\frac{1}{15}$

Find:

a. $\mathrm{E}(X)$ **b.** $\mathrm{E}(4X)$ **c.** $\mathrm{E}(2X + 1)$ **d.** $\mathrm{E}\left(X^2\right)$.

17. Find the mean of the discrete random variable, Z, for a probability function defined by $p(z) = \frac{1}{38}\left(z^2 - 4\right), 2 \leq z \leq 5$.

Give your answer correct to 2 decimal places.

18. **WE13** Christian decides to apply for a job selling mobile phones. He receives a base salary of \$180 per month and \$12 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, x, being sold per month. What would be the expected salary Christian would receive each month?



x	50	100	150	200	250
$P(X = x)$	0.32	0.38	0.2	0.06	0.04

15.4 Variance and standard deviation

15.4.1 Introduction

The measure of spread of a random variable distribution tells us how the data is dispersed. The measure of spread is called the **variance**, and the square root of the variance gives the **standard deviation**. The variance is denoted by $\mathrm{Var}(X)$ or σ^2 (sigma squared) and is defined as follows.

$$\mathbf{Var}(X) = \sigma^2 = \mathbf{E}\left(X^2\right) - \mathbf{E}[(X)]^2$$

This may also be written as $\mathrm{Var}(X) = \sigma^2 = \mathrm{E}(X^2) - [\mu]^2$, where $\mu = \mathrm{E}(X)$.

The derivation of this rule is as follows:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X - \mu)^2 \\
&= \mathrm{E}(X^2 - 2X\mu + \mu^2) \\
&= \mathrm{E}(X^2) - \mathrm{E}(2X\mu) + \mathrm{E}(\mu^2) \\
&= \mathrm{E}(X^2) - 2\mu\mathrm{E}(X) + \mu^2
\end{aligned}$$

Since $\mathrm{E}(X) = \mu$,

$$\begin{aligned}
&= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 \\
\mathrm{Var}(x) &= \mathrm{E}(X^2) - \mu^2 \\
\mathrm{Var}(x) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2
\end{aligned}$$

The standard deviation of X is the square root of the variance of X and is denoted by SD(X) or σ.

$$\text{SD}(X) = \sigma = \sqrt{\text{Var}(X)}$$

If the standard deviation is large, the spread of the data is large. If the standard deviation is small, the data is clumped together, close to the mean.

on Resources

Interactivity: Variance and standard deviation (int-6429)

WORKED EXAMPLE 14

A **discrete random variable, X, has the following probability distribution.**

x	1	2	3	4	5
$\text{P}(X = x)$	0.15	0.25	0.3	0.2	0.1

Determine:

a. $\text{E}(X)$ **b.** $\text{Var}(X)$ **c.** $\text{SD}(X)$.

THINK

a. 1. Write the rule to find the expected value.

2. Substitute the appropriate values into the rule.

3. Simplify.

b. 1. Evaluate $\text{E}(X^2)$.

2. Write the rule for the variance.

3. Substitute in the appropriate values and evaluate.

c. 1. Write the rule for the standard deviation.

2. Substitute in the variance and evaluate.

WRITE

a. $\text{E}(X) = \sum_{\text{all } x} x\text{P}(X = x)$

$\text{E}(X) = 1(0.15) + 2(0.25) + 3(0.3) + 4(0.2) + 5(0.1)$

$\text{E}(X) = 0.15 + 0.5 + 0.9 + 0.8 + 0.5$

$= 2.85$

b. $\text{E}(X^2) = \sum_{\text{all } x} x^2\text{P}(X = x)$

$\text{E}(X^2) = 1^2(0.15) + 2^2(0.25) + 3^2(0.3) + 4^2(0.2) + 5^2(0.1)$

$= 0.15 + 1 + 2.7 + 3.2 + 2.5$

$= 9.55$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\text{Var}(X) = 9.55 - (2.85)^2$

$= 9.55 - 8.1225$

$= 1.4275$

c. $\text{SD}(X) = \sqrt{\text{Var}(X)}$

$\text{SD}(X) = \sqrt{1.4275}$

$= 1.1948$

TI | THINK

a. 1. On a Lists & Spreadsheet page, label the first column as x and the second column as p. Enter the given x values in the first column and their respective probabilities in the second column.

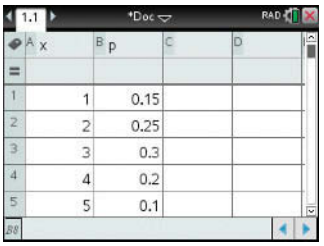
2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
Select 1 for the Num of Lists then select OK.
Compete the fields as
X1 List: x
Frequency List: p
then select OK.

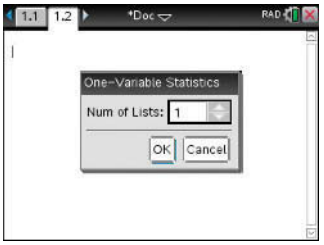
3. The answer appears on the screen.

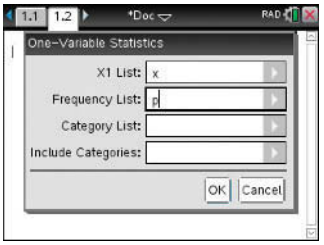
b. 1. See the Calculator page from part **a**. Scroll down to find the variance.

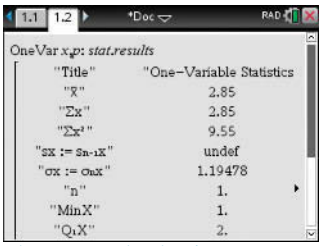
2. The answer appears on the screen.

WRITE

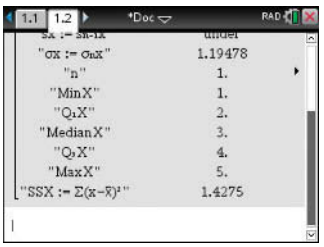








The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.85.



The variance is represented by the symbol $SSX := \Sigma(x - \bar{x})^2$ on the screen.
The variance is 1.4275.

CASIO | THINK

a. 1. On a Statistics screen, label List 1 as x and List 2 as p.
Enter the given x values in the first column and their respective probabilities in the second column.

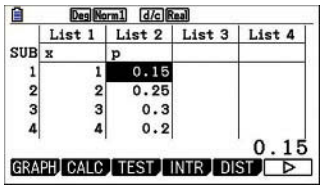
2. Select CALC by pressing F2, then select SET by pressing F6.
Complete the fields as
1Var XList: List1
1Var Freq: List2
then press EXIT.
Select 1-VAR by pressing F1.

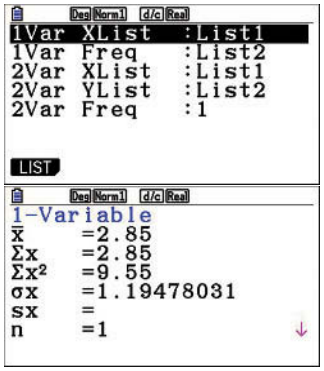
3. The answer appears on the screen.

b. 1. On the Run-Matrix screen, press OPTN and select STAT by pressing F5, then select Var by pressing F5. Select σ^2 by pressing F2.
Complete the entry line as
Variance σ^2(List 1, List 2)
then press EXE.
Note: List can be found by pressing OPTN then selecting List then List again by pressing F1 twice.

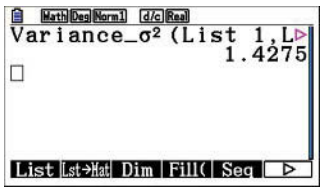
2. The answer appears on the screen.

WRITE



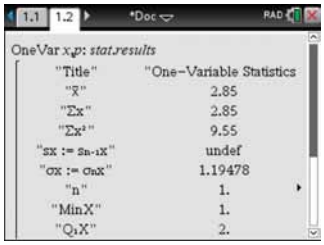


The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.85.



The variance is 1.4275.

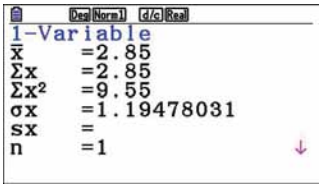
c. 1. See the Calculator page from part **a**.



2. The answer appears on the screen.

The standard deviation is represented by the symbol $\sigma x := \sigma_n x$ on the screen.
The standard deviation is 1.19478.

c. 1. Return to the Statistics screen from part **a**.



2. The answer appears on the screen.

The standard deviation is represented by the symbol σx on the screen.
The standard deviation is 1.19478.

15.4.2 Properties of the variance

The variance of a linear function has rules similar to those for the expectation of a linear function.

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

This can be proved in the following manner.

$$\begin{aligned}
\text{Var}(aX + b) &= \text{E}(aX + b)^2 - [\text{E}(aX + b)]^2 \\
&= \text{E}(a^2X^2 + 2abX + b^2) - [a\text{E}(X) + b]^2 \\
&= \text{E}(a^2X^2) + \text{E}(2abX) + \text{E}(b^2) - (a^2[\text{E}(X)]^2 + 2ab\text{E}(X) + b^2) \\
&= a^2\text{E}(X^2) + 2ab\text{E}(X) + b^2 - a^2[\text{E}(X)]^2 - 2ab\text{E}(X) - b^2 \\
&= a^2(\text{E}(X^2) - [\text{E}(X)]^2)
\end{aligned}$$

But $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$, so:

$$\textbf{Var}\,(aX + b) = a^2\textbf{Var}\,(X).$$

WORKED EXAMPLE 15

A discrete probability function is defined by the rule $p(y) = \frac{1}{12}(10 - 3y), y \in \{1, 2, 3\}$.

a. Show that the sum of the probabilities is equal to one.

b. Determine:

i. E(Y) **ii. Var(Y).**

c. Determine:

i. Var($3Y - 1$) **ii. Var($4 - 5Y$).**

THINK

a. 1. Evaluate the probabilities for the given values of y.

WRITE

a. $p(y) = \frac{1}{12}(10 - 3y)\, y \in \{1,2,3\}$

$$p(1) = \frac{1}{12}(10 - 3(1)) = \frac{7}{12}$$

$$p(2) = \frac{1}{12}(10 - 3(2)) = \frac{4}{12} = \frac{1}{3}$$

$$p(3) = \frac{1}{12}(10 - 3(3)) = \frac{1}{12}$$

2. Add the probabilities.

$$\begin{aligned}&P(Y=1)+P(Y=2)+P(Y=3)\\&=\frac{7}{12}+\frac{4}{12}+\frac{1}{12}\\&=\frac{12}{12}\\&=1\end{aligned}$$

b. i. 1. Write the rule to find the expected value.

b. i. $E(Y)=\sum_{\text{all } y} yP(Y=y)$

2. Substitute the appropriate values into the rule.

3. Simplify.

$$\begin{aligned}E(Y)&=1\left(\frac{7}{12}\right)+2\left(\frac{4}{12}\right)+3\left(\frac{1}{12}\right)\\&=\frac{7}{12}+\frac{8}{12}+\frac{3}{12}\\&=\frac{18}{12}\\&=\frac{3}{2}\end{aligned}$$

ii. 1. Evaluate $E(Y^2)$.

ii.
$$\begin{aligned}E(Y^2)&=1^2\left(\frac{7}{12}\right)+2^2\left(\frac{4}{12}\right)+3^2\left(\frac{1}{12}\right)\\&=\frac{7}{12}+\frac{16}{12}+\frac{9}{12}\\&=\frac{32}{12}\\&=\frac{8}{3}\end{aligned}$$

2. Write the rule for the variance.

3. Substitute in the appropriate values and evaluate.

$$\begin{aligned}&\text{Var}(Y)=E(Y^2)-[E(Y)]^2\\&\text{Var}(Y)=\frac{8}{3}-\left(\frac{3}{2}\right)^2\\&=\frac{8}{3}-\frac{9}{4}\\&=\frac{32-27}{12}\\&=\frac{5}{12}\end{aligned}$$

c. i. 1. Apply the property of the variance: $\text{Var}(aY+b)=a^2\text{Var}(Y)$.

c. i. $\text{Var}(3Y-1)=3^2\text{Var}(Y)$

2. Substitute in the value of $\text{Var}(Y)$ and evaluate.

$$\begin{aligned}\text{Var}(3Y-1)&=9\times\frac{5}{12}\\&=\frac{15}{4}\end{aligned}$$

ii. 1. Apply the property of the variance: $\text{Var}(aY+b)=a^2\text{Var}(Y)$.

ii. $\text{Var}(4-5Y)=(-5)^2\text{Var}(Y)$

2. Substitute in the value of $\text{Var}(Y)$ and evaluate.

$$\begin{aligned}\text{Var}(4-5Y)&=25\times\frac{5}{12}\\&=\frac{125}{12}\end{aligned}$$

Exercise 15.4 Variance and standard deviation

Technology active

1. **WE 14** Recently the large supermarket chains have been waging a price war on bread.



On a particular Tuesday, a standard loaf of bread was purchased from a number of outlets of different chains. The following table shows the probability distribution for the price of the bread, X.

x	\$1	\$2	\$3	\$4	\$5
$\mathbf{P}(X = x)$	0.3	0.15	0.4	0.1	0.05

a. Calculate the expected cost of a loaf of bread on that given Tuesday.
b. Calculate the variance and the standard deviation of that loaf of bread, correct to 2 decimal places.

2. A discrete random variable, X, has the following probability distribution.

x	-2	0	2	4	6
$\mathbf{P}(X = x)$	k	k	$2k$	$3k$	$3k$

a. Determine the value of the constant k.
b. Determine the expected value of X.
c. Determine the variance and the standard deviation of X, correct to 2 decimal places.

3. **WE 15** A discrete probability function is defined by $p(x) = \frac{x^2}{30}, x \in \{1, 2, 3, 4\}$.

Where appropriate, give your answers to the following to 2 decimal places.
a. Construct a probability distribution table and show that $\sum_{\text{all } x} \mathrm{P}(X = x) = 1$.
b. Determine:
i. $\mathrm{E}(X)$ **ii.** $\mathrm{Var}(X)$.
c. Determine:
i. $\mathrm{Var}(4X + 3)$ **ii.** $\mathrm{Var}(2 - 3X)$.

4. **a.** Find the value of the constant m if the discrete random variable Z has the probability distribution shown and $\mathrm{E}(Z) = 14.94$.

z	-7	m	23	31
$\mathbf{P}(Z = z)$	0.21	0.34	0.33	0.12

b. Find $\mathrm{Var}(Z)$ and hence find $\mathrm{Var}(2(Z - 1))$ and $\mathrm{Var}(3 - Z)$, correct to 2 decimal places.

5. For each of the following probability distributions, calculate:

i. the expected value

ii. the standard deviation, correct to 4 decimal places.

a.

x	-3	-2	-1	0	1	2	3
$\mathbf{P}(X = x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

b.

y	1	4	7	10	13
$\mathbf{P}(Y = y)$	0.15	0.2	0.3	0.2	0.15

c.

z	1	2	3	4	5	6
$\mathbf{P}(Z = z)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

6. A random variable, Y, has the following probability distribution.

y	-1	1	3	5	7
$\mathbf{P}(Y = y)$	$1 - 2c$	c^2	c^2	c^2	$1 - 2c$

a. Determine the value of the constant c.

b. Calculate $\mathrm{E}(Y)$, the mean of Y.

c. Calculate $\mathrm{Var}(Y)$, and hence find the standard deviation of Y, correct to 2 decimal places.

7. Given that $\mathrm{E}(X) = 4.5$, determine:

a. $\mathrm{E}(2X - 1)$ **b.** $\mathrm{E}(5 - X)$ **c.** $\mathrm{E}(3X + 1)$.

8. Given that $\mathrm{SD}\ (X) = \sigma = 2.5$, determine:

a. $\mathrm{Var}(6X)$ **b.** $\mathrm{Var}(2X + 3)$ **c.** $\mathrm{Var}(-X)$.

9. A discrete probability function is defined by the rule $p(x) = h(3 - x)(x + 1)$, $x = 0, 1, 2$.

a. Show that the value of h is $\frac{1}{10}$.

b. Hence, find the mean, variance and standard deviation of X. Where appropriate, give your answers to 4 decimal places.

10. A discrete probability function has the following distribution.

x	1	2	3	4	5
$\mathbf{P}(X = x)$	a	0.2	0.3	b	0.1

The expected value of the function is 2.5.

a. Calculate the values of the constants a and b

b. Hence, calculate the variance and standard deviation of X. Where appropriate, give your answers to 4 decimal places.

11. For a given discrete random variable, X, it is known that $\mathrm{E}(X) = a$ and $\mathrm{Var}(X) = 2a - 2$, where a is a constant that is greater than zero.

a. Write $\mathrm{E}(X^2)$ in terms of a.

b. If $\mathrm{E}(X^2)$ is known to be 6, calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

12. For a discrete random variable, Y, the probability function is defined by the following.

$$p(y) = \begin{cases} ny, & y \in \{1, 2, 3, 4\} \\ n(7 - y), & y \in \{5, 6\} \end{cases}$$

a. Determine the value of the constant n.

b. Calculate the expected value, the variance and the standard deviation of Y, correct to 4 decimal places.

13. Two octahedral dice (with faces numbered 1 to 8) are rolled simultaneously and the two numbers are recorded.



a. List the probability or event space and find $n(\xi)$. Let Z be the larger of the two numbers on the two dice.

b. State the probability distribution for Z.

c. Calculate the expected value and standard deviation of Z, correct to 4 decimal places.

14. A dart competition at a local sports centre allows each player to throw one dart at the board, which has a radius of 20 centimetres. The board consists of five concentric circles, each with the same width.



The inner circle has a radius of 4 cm. The probability of landing on each band is determined by the area of that band available on the board.

a. Calculate the probability of landing on each of the bands.

The outer red band is called band E, the next white band is called band D and so on until you get to the inner red circle, which is band A.

The competition costs $1 to enter and the prizes are as follows:

If a dirt hits band E, the player receives nothing.

If a dart hits band D, the player receives $1.

If a dart hits band C, the player receives $2.

If a dart hits band B, the player receives $5.

If a dart hits band A, the player receives $10.

b. If X is a discrete random variable that represents the profit in dollars for the player, construct a probability distribution table for this game.

c. Calculate:

i. the expected profit a player could make in dollars

ii. the standard deviation.

15. At a beginner's archery competition, each archer has two arrows to shoot at the target. A target is marked with ten evenly spaced concentric rings.



The following is a summary of the scoring for the beginner's competition.

Yellow – 10 points

Red – 7 points

Blue – 5 points

Black – 3 points

White – 1 point

Let X be the total score after a beginner shoots two arrows.

a. List the possible score totals.

The probability of a beginner hitting each of the rings has been calculated as follows:

P(yellow) = 0.1, P(red) = 0.2, P(blue) = 0.3, P(black) = 0.2 and P(white) = 0.2.

b. Construct a probability distribution table for the total score achieved by a beginner archer.

c. Calculate the expected score and the standard deviation for a beginner. Where appropriate, give your answers correct to 4 decimal places.

16. A discrete random variable, X, has the following probability distribution.

x	-2	-1	0	1	2	3	4
$\mathbf{P}(X = x)$	$0.5k^2$	$0.5k^2$	$k + k^2$	$4k$	$2k$	$2k + k^2$	$7k^2$

a. Determine the value of the constant k.
b. Calculate the expected value of X.
c. Calculate the standard deviation of X, correct to 4 decimal places.

15.5 Applications of discrete random variables

One important application of the expected value and standard deviation of a random variable is that approximately 95% of the distribution lies within two standard deviations of the mean.

$$\mathbf{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma \approx \mathbf{0.95}$$

WORKED EXAMPLE 16

Let Y be a discrete random variable with the following probability distribution.

y	0	1	2	3	4
$\mathbf{P}(Y = y)$	0.08	0.34	0.38	0.17	0.03

a. Determine the expected value of Y.
b. Calculate the standard deviation of Y.
c. Calculate $\mathbf{P}(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$.

THINK

a. 1. Write the rule to find the expected value.

2. Substitute the appropriate values into the rule.

3. Simplify.

b. 1. Find $E(Y^2)$.

2. Write the rule for the variance.

3. Substitute in the appropriate values and evaluate.

WRITE

a. $E(Y) = \sum_{\text{all } y} yP(Y = y)$

$E(Y) = 0(0.08) + 1(0.34) + 2(0.38) + 3(0.17) + 4(0.03)$

$E(Y) = 0 + 0.34 + 0.76 + 0.51 + 0.12$
$= 1.73$

b. $E(Y^2) = 0^2(0.08) + 1^2(0.34) + 2^2(0.38) + 3^2(0.17) + 4^2(0.03)$
$= 0 + 0.34 + 1.52 + 1.53 + 0.48$
$= 3.87$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$\text{Var}(Y) = 3.87 - 1.73^2$
$= 0.8771$

4. Write the rule for the standard deviation.

$$SD(Y) = \sqrt{Var(Y)}$$

5. Substitute in the variance and evaluate.

$$SD(Y) = \sqrt{0.8771}$$
$$= 0.9365$$

c. 1. Find $\mu - 2\sigma$.

c. $\mu - 2\sigma = 1.73 - 2(0.9365)$
$= -0.143$

2. Find $\mu + 2\sigma$.

$\mu + 2\sigma = 1.73 + 2(0.9365)$
$= 3.603$

3. Substitute the values into $P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$.

$P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$
$= P(-0.143 \leq Y \leq 3.603)$

4. Interpret this interval in the context of a discrete distribution. The smallest y-value in the distribution table is 0, so -0.143 is rounded up to 0. The largest y-value in the distribution table that is smaller than 3.603 is 3.

$P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$
$= P(0 \leq Y \leq 3)$
$= 0.08 + 0.34 + 0.38 + 0.17$
$= 0.97$

Note: This is very close to the estimated value of 0.95.

WORKED EXAMPLE 17

A biased die has a probability distribution for the outcome of the die being rolled as follows.

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.1	0.2	0.25	0.25	0.1

a. Calculate P(even number).
b. Calculate $P(X \geq 3 | X \leq 5)$.
c. Calculate $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

THINK

a. 1. State the probabilities to be added.

2. Substitute the values and simplify.

b. 1. Define the rule.

2. Find $P(X \geq 3 \cap X \leq 5)$.

3. Calculate $P(X \leq 5)$.

WRITE

a. $P(\text{even number}) = P(X = 2) + P(X = 4) + P(X = 6)$
$= 0.1 + 0.25 + 0.1$
$= 0.45$

b. $P(X \geq 3 | X \leq 5) = \dfrac{P(X \geq 3 \cap X \leq 5)}{P(X \leq 5)}$

$P(X \geq 3 \cap X \leq 5) = P(3 \leq X \leq 5)$
$= P(X = 3) + P(X = 4) + P(X = 5)$
$= 0.2 + 0.25 + 0.25$
$= 0.7$

$P(X \leq 5) = 1 - P(x = 6)$
$= 1 - 0.1$
$= 0.9$

4. Substitute the appropriate values into the formula.

$$P(X \geq 3 | X \leq 5) = \frac{P(X \geq 3 \cap X \leq 5)}{P(X \leq 5)}$$
$$= \frac{P(3 \leq X \leq 5)}{P(X \leq 5)}$$
$$= \frac{0.7}{0.9}$$

5. Evaluate and simplify.

$$= \frac{7}{9}$$

c. 1. Calculate the expected value.

c. $E(x) = 1(0.1) + 2(0.1) + 3(0.2) + 4(0.25) + 5(0.25) + 6(0.1)$
$= 0.1 + 0.2 + 0.6 + 1 + 1.25 + 0.6$
$= 3.75$

2. Calculate $E(X^2)$.

$E(X^2) = 1^2(0.1) + 2^2(0.1) + 3^2(0.2) + 4^2(0.25) + 5^2(0.25) + 6^2(0.1)$
$= 0.1 + 0.4 + 1.8 + 4 + 6.25 + 3.6$
$= 16.15$

3. Calculate the variance.

$Var(x) = E(X^2) - [E(X)]^2$
$= 16.15 - 3.75^2$
$= 2.0875$

4. Calculate the standard deviation.

$SD(X) = \sqrt{Var(X)}$
$= \sqrt{2.0875}$
$= 1.4448$

5. Calculate $\mu - 2\sigma$.

$\mu - 2\sigma = 3.75 - 2(1.4448)$
$= 0.8604$

6. Calculate $\mu + 2\sigma$.

$\mu + 2\sigma = 3.75 + 2(1.4448)$
$= 6.6396$

7. Substitute the appropriate values into $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$
$= P(0.8604 \leq X \leq 6.6396)$
$= P(1 \leq X \leq 6)$
$= 1$

8. Interpret this interval in the context of a discrete distribution. The smallest x-value in the distribution table is 1, so 0.8604 is rounded up to 1. The largest x-value in the distribution table that is smaller than 6.6396 is 6.

Note: This is very close to the estimated value of 0.95.

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Units 1 & 2 > Area 10 > Sequence 1 > Concept 4

Applications of discrete random variables Summary screen and practice questions

Exercise 15.5 Applications of discrete random variables

Technology active

1. **WE16** A discrete random variable, X, has the following probability distribution.

x	5	10	15	20	25
$\mathbf{P}(X = x)$	0.05	0.25	0.4	0.25	0.05

   a. Determine the expected value of X.
   b. Calculate the standard deviation of X, correct to 4 decimal places.
   c. Calculate $\mathrm{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

2. The number of Tails, X, when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$\mathbf{P}(X = x)$	0.012	0.093	0.243	0.315	0.214	0.1	0.023

   Calculate $\mathrm{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

3. **WE17** A financial adviser for a large company has put forward a number of options to improve the company's profitability, X (measured in hundreds of thousands of dollars). The decision to implement the options will be based on the cost of the options as well as their profitability. The company stands to make an extra profit of 1 million dollars with a probability of 0.1, an extra profit of \$750 000 with a probability of 0.3, an extra profit of 500 000 with a probability of 0.3, an extra profit of 250 000 with a probability of 0.2 and an extra profit of \$100 000 with a probability of 0.1.
   Determine:
   a. $\mathrm{P}(X \leq \$500\,000)$
   b. $\mathrm{P}(X \geq \$2000 \mid X \leq \$750\,000)$
   c. the expected profit
   d. $\mathrm{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

4. A discrete random variable, Z, can take the values 0, 1, 2, 3, 4 and 5. The probability distribution of Z is:
   $\mathrm{P}(Z = 0) = \mathrm{P}(Z = 1) = \mathrm{P}(Z = 2) = m$
   $\mathrm{P}(Z = 3) = \mathrm{P}(Z = 4) = \mathrm{P}(Z = 5) = n$
   and $\mathrm{P}(Z < 2) = 3\mathrm{P}(Z > 4)$ where m and n are constants.
   a. Determine the values of m and n.
   b. Show that the expected value of Z is $\frac{11}{5}$, and determine the variance and standard deviation for Z, correct to 4 decimal places.
   c. Calculate $\mathrm{P}(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

5. A discrete random variable, Y, has the following probability distribution.

y	1	2	d	8
$\mathbf{P}(Y = y)$	0.3	0.2	0.4	0.1

   a. Determine the value of the constant d if it is known that $\mathrm{E}(Y) = 3.5$.
   b. Determine $\mathrm{P}(Y \geq 2 | Y \leq d)$.
   c. Calculate $\mathrm{Var}(Y)$.
   d. Calculate $\mathrm{SD}(Y)$ correct to 4 decimal places.

6. A discrete random variable, Z, has the following probability distribution.

z	1	3	5	7	9
$\mathbf{P(Z = z)}$	0.2	0.15	a	b	0.05

The expected value of Z is known to be equal to 4.6.

a. Determine the values of the constants a and b.

b. Determine the variance and standard deviation of Z, correct to 4 decimal places where appropriate.

c. Evaluate:

i. $E(3Z + 2)$ **ii.** $\text{Var}(3Z + 2)$.

7. A probability distribution is such that:

$P(Z = 0) = P(Z = 1) = P(Z = 2) = P(Z = 3) = m$

$P(Z = 4) = P(Z = 5) = n$

and $P(Z \leq 3) = P(Z \geq 4)$.

a. Determine the values of the constants m and n.

b. Calculate:

i. $E(Z)$ **ii.** $\text{Var}(Z)$.

c. Calculate $P(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

8. In a random experiment the events M and N are independent events where $P(M) = 0.45$ and $P(N) = 0.48$.

a. Calculate the probability that both M and N occur.

b. Calculate the probability that neither M nor N occur.

Let Y be the discrete random variable that defines the number of times M and N occur.

$Y = 0$ if neither M and N occurs.

$Y = 1$ if only one of M and N occurs.

$Y = 2$ if both M and N occur.

c. Specify the probability distribution for Y.

d. Determine, correct to 4 decimal places where appropriate:

i. $E(Y)$

ii. $\text{Var}(Y)$

iii. $\text{SD}(Y)$.

9. A probability function is defined as $p(x) = \frac{1}{9}(4 - x)$, $x \in \{0, 1, 2\}$.

a. Construct a probability distribution table.

b. Calculate, correct to 4 decimal places where appropriate:

i. $E(X)$

ii. $\text{Var}(X)$

iii. $\text{SD}(X)$.

c. Calculate $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

10. The number of customers, X, waiting in line at a bank just before closing time has a probability distribution as follows.

x	0	1	2	3
$\mathbf{P(X = x)}$	$\frac{k^2}{4}$	$\frac{5k - 1}{12}$	$\frac{3k - 1}{12}$	$\frac{4k - 1}{12}$

a. Determine the value of the constant k.

b. Determine the expected number of customers waiting in line just before closing time.

c. Calculate the probability that the number of customers waiting in line just before closing time is no greater than $E(X)$.

11. The television show *Steal or No Steal* features 26 cases with various amounts of money ranging from 50 cents to $200 000. The contestant chooses one case and then proceeds to open the other cases. At the end of each round, the banker makes an offer to end the game. The game ends when the contestant accepts the offer or when all the other 25 cases have been opened; in the latter event, the contestant receives the amount of money in the case they first chose.



Suppose a contestant has five cases left and the amounts of $200 000, $100 000, $50 000, $15 000 and $1000 are still to be found.

a. Determine the expected amount that the banker should offer the contestant to end the game.

b. The contestant turned down the offer and opened a case containing $100 000. What would you expect the banker to offer the contestant at this stage?

12. A bookstore sells both new and secondhand books. A particular new autobiography costs $65, a good-quality used autobiography costs $30 and a worn autobiography costs $12. A new cookbook costs $54, a good-quality used cookbook costs $25 and a worn cookbook costs $15. Let X denote the total cost of buying two books (an autobiography and a cookbook). Assume that the purchases are independent of one another.



a. Construct a probability distribution table for the cost of the two textbooks if the following probabilities apply.

- The probability of buying a new autobiography is 0.4.
- The probability of buying a good-quality used autobiography is 0.3.
- The probability of buying a worn used autobiography is 0.3.
- The probability of buying a new cookbook is 0.4.
- The probability of buying a good-quality used cookbook is 0.25.
- The probability of buying a worn used cookbook is 0.35.

b. Calculate the expected cost of the two books.

13. Let X be the number of dining suites sold by the dining suite department of a large furniture outlet on any given day. The probability function for this discrete random variable is as follows.

x	0	1	2	3
$\mathbf{P}(X = x)$	0.3	0.4	0.2	0.1

The dining suite department receives a profit of $350 for every dining setting sold. The daily running costs for the sales operation of the department are $120. The net profit per day is a function of the random variable such that $y(x) = 350x - 120$ dollars.



a. Set up a probability distribution table for the net profit, $\$Y$, per day.

b. Find the expected daily profit for the dining suite department.

c. Determine $\mathrm{P}(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$.

14. A loaded six-sided die and a biased coin are tossed simultaneously. The coin is biased such that the probability of obtaining a Head is three times the probability of obtaining a Tail. The loaded die has the following probabilities for each of the numbers 1 to 6.

$$P(1) = P(2) = P(5) = \frac{1}{12}$$

$$P(3) = P(4) = P(6) = \frac{1}{4}$$

When a player tosses the coin and die simultaneously, they receive the following outcomes.

$$\overbrace{\text{1T 2T 5T}}^{\text{10 points}} \qquad \overbrace{\text{1H 2H 5H}}^{\text{5 points}} \qquad \overbrace{\text{All other results}}^{\text{1 point}}$$

Let X be the number of points scored from a simultaneous toss.

a. Construct a probability distribution table for the number of points scored.

b. Calculate the expected points received from a single toss, correct to 1 decimal place.

c. If 25 simultaneous tosses occurred, what would the expected score be, correct to 1 decimal place?

d. What is the minimum number of simultaneous tosses that would have to occur for the expected total to be a score of 100 ?

15. In a certain random experiment the events V and W are independent events.

a. If $P(V \cup W) = 0.7725$ and $P(V \cap W) = 0.227$, find $P(V)$ and $P(W)$, given $P(V) < P(W)$.

b. Calculate the probability that neither V nor W occur.

Let X be the discrete random variable that defines the number of times events V and W occur.

$X = 0$ if neither V nor W occurs.

$X = 1$ if only one of V and W occurs.

$X = 2$ if both V and W occur.

c. Specify the probability distribution for X.

d. Determine, correct to 4 decimal places where appropriate:

i. $E(X)$ **ii.** $\text{Var}(X)$ **iii.** $\text{SD}(X)$.

16. The probability distribution table for the discrete random variable, Z, is as follows.

z	1	3	5
$\mathbf{P(Z = z)}$	$\frac{k^2}{7}$	$\frac{5 - 2k}{7}$	$\frac{8 - 3k}{7}$

a. Determine the value(s) of the constant k.

b. Calculate, correct to 4 decimal places:

i. $E(Z)$ **ii.** $\text{Var}(Z)$ **iii.** $\text{SD}(Z)$.

c. Calculate $P(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

15.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC Which of the following random variables is *not* discrete?

A. The number of goals scored at a football match

B. The number of T-shirts owned by a student

C. The volume of soft drink consumed by a family over the period of a week

D. The number of customers at a department store sale

2. MC Consider the discrete probability function with the following distribution.

x	2	4	6	8	10
$\mathbf{P}(X = x)$	$2a$	$3a$	$4a$	$5a$	$6a$

The value of the constant a is:

A. 20 **B.** $\frac{1}{20}$ **C.** $\frac{1}{2}$ **D.** $\frac{1}{14}$

3. MC For a discrete random variable X with a mean of 2.1 and a variance of 1.3, the values of $\mathrm{E}(2X + 1)$ and $\mathrm{Var}(2X + 1)$ are, respectively:

A. 4.2 and 5.2 **B.** 5.2 and 6.2 **C.** 5.2 and 5.2 **D.** 4.2 and 6.2

4. MC The random variable Y has the following probability distribution.

y	-2	0	2
$\mathbf{P}(Y = y)$	$2p$	$3p$	$1 - 5p$

The mean of Y is:

A. $2 - 11p$ **B.** $2 - 14p$ **C.** $1 - 3p$ **D.** $14p$

5. MC The probability distribution for the random variable X is as follows.

x	-1	0	1	2
$\mathbf{P}(X = x)$	m	$m + n$	$3m$	$m - n$

If $\mathrm{E}(X) = 0.4$, then m and n are equal to:

A. $m = \frac{1}{6},\ n = \frac{1}{5}$ **B.** $m = \frac{1}{5},\ n = \frac{1}{6}$

C. $m = \frac{1}{6},\ n = \frac{2}{15}$ **D.** $m = \frac{2}{15},\ n = \frac{1}{6}$

6. The probability distribution of X is given by the formula, $\mathrm{P}(X = x) = \frac{x^2}{30}$, where $x = 1,\ 2,\ 3,\ 4$.

a. Write the probability distribution of X as a table.

b. Calculate the expected value of X.

7. A biased coin is tossed twice. If the probability of obtaining a Head is $\frac{3}{5}$:

a. Write the probability distribution table of the number of Heads in 2 tosses.

b. Calculate the expected number of Heads.

8. Examine the following probability distribution table.

x	4	9	16	25	36
$\mathbf{P}(X = x)$	0.16	0.21	0.35	0.08	0.2

Calculate the value of $\mathrm{P}(X \geq 10)$.

9. A game is played where two dice are rolled and the sum of the two numbers showing uppermost is recorded. If players get a sum of 7, they win $10. If they get a sum of 2 or 12, they win $5. For any other sum, they must pay $2.50. Is it a fair game?
Justify your response mathematically.
You may choose to use technology to answer question 10 and 11.

10. A discrete random variable, Z, has a probability distribution as follows.

z	1	2	3	4	5
$\mathbf{P(Z = z)}$	0.1	0.25	0.35	0.25	0.05

Calculate:

a. the expected value of Z

b. the variance of Z

c. the standard deviation of Z.

11. Maya constructed a spinner that will fall onto one of the numbers 1 to 5 with the following probability.

Number	1	2	3	4	5
Probability	0.1	0.3	0.3	0.2	0.1

Calculate the mean and the standard deviation of this distribution, correct to 2 decimal places.

12. A player rolls a fair die. If the player gets a 1 on the first roll, she rolls again and her score is the sum of the two results; otherwise, her score is the result of the first roll. The die cannot be thrown more than twice. Determine:

a. the probability distribution

b. the expected score

c. $P(X < \mu)$.

Complex familiar

13. A biased coin is tossed four times. The probability of a Head from a toss is a where $0 < a < 1$.

a. Determine, in terms of a, the probability of obtaining:

i. four Tails from four tosses

ii. one Head and three Tails from four tosses.

b. If the probability of obtaining four Heads is the same as the probability of obtaining one Head and three Tails, determine the value of a.

14. Alicia and Harry have devised a game where a biased spinner is spun. There are 4 colours on the wheel and the sectors are of varying sizes.

$$\text{P(red)} = \frac{1}{20},\ \text{P(blue)} = \text{P(green)} = 2 \times \text{P(red)}, \text{ and the other sector colour is yellow.}$$

Players have to pay $2.00 to play. If the spinner lands on yellow, players receive nothing. If the spinner lands on green or blue, players get their money back. If the spinner lands on red, players win $5.

a. Determine the probability distribution for the amount of money a person can win.

b. What is the expected amount of money a player will win each game?

15. On any given day the number of text messages, Y, received by Garisht is a discrete random variable with a distribution as follows.

y	0	2	4	6	8	10
$\mathbf{P}(Y = y)$	0.05	0.4	0.2	0.15	0.15	0.05



You may choose to use technology to answer questions **a**–**c**.

a. Determine the expected value of Y.

b. What is the probability that Garisht receives no texts on four consecutive days?

c. Garisht received text messages on Thursday and Friday. What is the probability that he received 10 text messages over these two days?

16. At Fast Eddy's Drive-In Theatre the cost is \$10 per car, plus \$3 per occupant. The variable X-represents the number of people in any car and is known to follow the probability distribution as shown below. Determine:

a. the expected cost per car

b. Fast Eddy's expected profit if 100 cars enter, and costs for wages, electricity, and so on are \$500.

x	2	3	4	5
$\mathbf{P}(X = x)$	0.4	0.2	0.3	0.1

Complex unfamiliar

17. A discrete random variable, Z, has a probability distribution as shown.

z	1	2	3	4	5	6
$\mathbf{P}(Z = z)$	$\frac{m}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2m}{5}$	$\frac{1}{10}(5-6m)$

This random variable describes the outcome of tossing a loaded die. The die is thrown twice. You may choose to use technology to answer questions **a**–**c**.

a. Prove that the chance of throwing a total of 11 is $\frac{10m - 12m^2}{25}$.

b. Find the value of m that makes this chance a maximum, and find the maximum probability.

c. Using the value of m from part **b**, determine:

i. the expected value of Z and the standard deviation of Z

ii. $\mathrm{P}(\mu - 2\sigma \leq Z \leq \mu + 2\sigma)$.

18. A discrete random variable, Z, can only take the values 0, 1, 2, 3, 4, 5, and 6. The probability distribution for Z is given by the following:

$$\mathrm{P}(Z = 0) = \mathrm{P}(Z = 2) = \mathrm{P}(Z = 4) = \mathrm{P}(Z = 6) = m$$
$$\mathrm{P}(Z = 1) = \mathrm{P}(Z = 3) = \mathrm{P}(Z = 5) = n$$

and $2\mathrm{P}(0 < Z < 2) = \mathrm{P}(3 < Z \leq 6)$, where m and n are constants.
Determine the values of m and n.

You may choose to use technology to answer questions **19** and **20**.

19. The number of passengers per car, X, entering Brisbane on a motorway on a workday morning is as follows.



x	0	1	2	3	4	5
$\mathbf{P}(X = x)$	0.37	0.22	0.21	0.1	0.05	0.05

The fees for cars at a toll booth on the motorway are as follows.

- Cars carrying no passengers: $2.50
- Cars carrying 1 or 2 passengers: $1.00
- Cars carrying more than 2 passengers: no fee

a. Determine the expected value of the toll per car.

b. What is the probability that, out of 10 cars selected at random, at least 8 cars have no passengers?

20. A random variable, X, represents the number of televisions serviced per week by a television serviceman. The probability distribution is as follows.



x	10	11	12	13	14	15	16	17	18	19	20
$\mathbf{P}(X = x)$	0.07	0.12	0.12	0.1	0.1	0.1	0.1	0.08	0.08	0.08	0.05

The serviceman is paid $20 for each television that he services plus a bonus depending on how many televisions he services a week. The bonuses are as follows:

- If less than 13 televisions are serviced, there is no bonus.
- If 13 – 16 televisions are serviced, he receives a bonus of $120.
- If more than 16 televisions are serviced, he receives a bonus of $250.

Determine the expected amount that the serviceman will be paid in a week.

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Units 1 & 2 Sit chapter test

Answers

Chapter 15: Discrete random variables 1

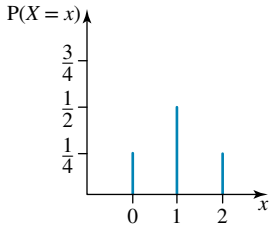
Exercise 15.2 Discrete random variables

1. a. Discrete **b.** Continuous **c.** Continuous
d. Discrete **e.** Continuous **f.** Discrete
g. Continuous **h.** Discrete

2. a.

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b.

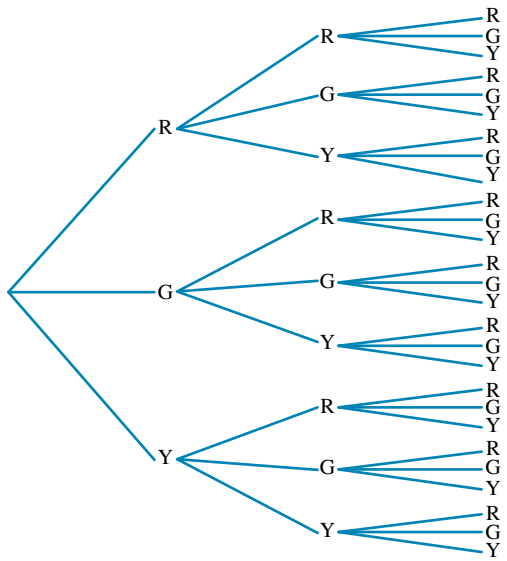


3. a. HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
b. $x = 0,\ 1,\ 2,\ 3$
c.

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d. $\frac{7}{8}$

4. a.



$\xi = \{\mathrm{RRR}, \mathrm{RRG}, \mathrm{RRY}, \mathrm{RGR}, \mathrm{RGG}, \mathrm{RGY}, \mathrm{RYR}, \mathrm{RYG}, \mathrm{RYY}, \mathrm{GRR}, \mathrm{GRG}, \mathrm{GRY}, \mathrm{GGR}, \mathrm{GGG}, \mathrm{GGY}, \mathrm{GYR}, \mathrm{GYG}, \mathrm{GYY}, \mathrm{YRR}, \mathrm{YRG}, \mathrm{YRY}, \mathrm{YGR}, \mathrm{YGG}, \mathrm{YGY}, \mathrm{YYR}, \mathrm{YYG}, \mathrm{YYY}\}$

b. $\mathrm{P}(Y = 3) = \frac{27}{1000}$; $\mathrm{P}(Y = 2) = \frac{189}{1000}$;
$\mathrm{P}(Y = 1) = \frac{441}{1000}$; $\mathrm{P}(Y = 0) = \frac{343}{1000}$

c.

y	0	1	2	3
$\mathrm{P}(Y = y)$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

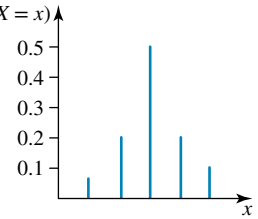
d. This is a discrete probability function.

5.

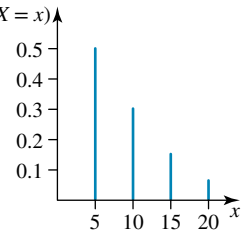
x	0	1	2
$\mathrm{P}(X = x)$	$\frac{25}{36}$	$\frac{10}{36} = \frac{5}{18}$	$\frac{1}{36}$

6. a. **i.** This is a discrete probability function.
ii. This is a discrete probability function.
b. $k = \frac{1}{16}$

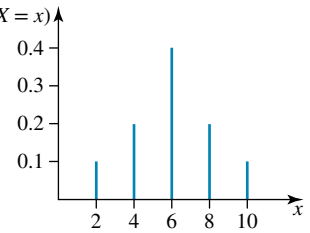
7. a.



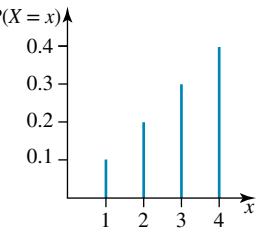
b.



c.



d.



8. a.

x	2	3	4	5	6	7	8	9	10	11	12
$\mathrm{P}(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b. $\frac{1}{6}$
c. $\frac{5}{18}$
d. $\frac{7}{36}$
e. $\frac{29}{36}$

f. $\frac{35}{36}$

g. $\frac{5}{9}$

9. a. $\xi = \{11, 12, 13, 14, 15, 16$
$21, 22, 23, 24, 25, 26$
$31, 32, 33, 34, 35, 36$
$41, 42, 43, 44, 45, 46$
$51, 52, 53, 54, 55, 56$
$61, 62, 63, 64, 65, 66\}$

b. $\mathbf{Z} = [0,\ 1,\ 2]$

z	0	1	2
$\mathbf{P(Z = z)}$	0.09	0.42	0.49

c. 0.42

10. a. $d = 0.15$ **b.** $k = \frac{1}{6}$ **c.** $a = \frac{1}{3}$

11. Sample responses can be found in the worked solutions in the online resources.

12. Sample responses can be found in the worked solutions in the online resources.

13. a. This is a discrete probability function.

b. This is not a discrete probability function.

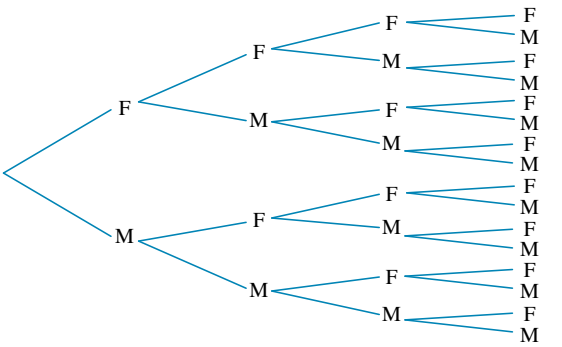
c. This is a discrete probability function.

14. $a = 30$

15. a. i. 0.1715 **ii.** 0.4115
iii. 0.3292 **iv.** 0.0878

b. 0.1060

16. a. F = female and M = male



$$\xi = \{FFFF, FFFM, FFMF, FFMM, FMFF, FMFM, FMMF, FMMM, MFFF, MFFM, MFMF, MFMM, MMFF, MMFM, MMMF, MMMM,\}$$

b.

x	0	1	2	3	4
$\mathbf{P(X = x)}$	$\frac{1}{16}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{6}{16} = \frac{3}{8}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{1}{16}$

c. $\frac{1}{16}$

d. $\frac{15}{16}$

e. $\frac{11}{16}$

17. a. $\xi = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 110, 111, 112$
$21, 22, 23, 24, 25, 26, 27, 28, 29, 210, 211, 212$
$31, 32, 33, 34, 35, 36, 37, 38, 39, 310, 311, 312$
$41, 42, 43, 44, 45, 46, 47, 48, 49, 410, 411, 412$
$51, 52, 53, 54, 55, 56, 57, 58, 59, 510, 511, 512$
$61, 62, 63, 64, 65, 66, 67, 68, 69, 610, 611, 612$
$71, 72, 73, 74, 75, 76, 77, 78, 79, 710, 711, 712$
$81, 82, 83, 84, 85, 86, 87, 88, 89, 810, 811, 812\}$

b. $\mathrm{P}(X = 0) = \frac{28}{96}$; $\mathrm{P}(X = 1) = \frac{48}{96}$; $\mathrm{P}(X = 2) = \frac{20}{96}$

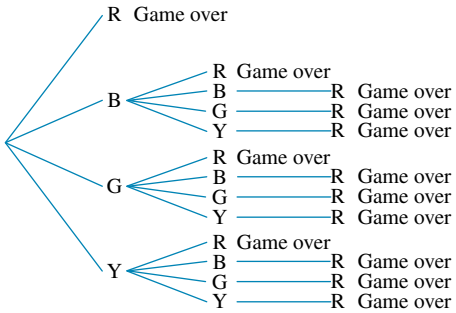
c. 0.009

18. a.

x	0	1	2	3	4	5
$\mathbf{P(X = x)}$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

b. It is a success.

19. a.



b. BBB, GGG or YYY

c.

x	\$0	\$1	\$10
$\mathbf{P(X = x)}$	0.7840	0.1920	0.0240

20. $k = -0.4568$ or 0.6568

Exercise 15.3 Expected values

1. $2\frac{7}{16}$

2. 1.16

3. $a = 0.09;\ \mathrm{E}(X) = 5.42$

4. $a = \frac{1}{18};\ \mathrm{E}(X) = 5\frac{1}{3}$

5. $b = 0.15;\ \mathrm{E}(X) = 2.39$

6. $k = 0.05;\ \mathrm{E}(X) = 13$

7. a.

x	1	2	3	4	5	6
$\mathbf{P(X = x)}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

b. $3\frac{1}{2}$

8. a.

x	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{P(X = x)}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b. 7

9. a.

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b. 1

10. a.

x	0	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

b. 2

11. $a = 0.15,\ b = 0.05$

12. $a = 0.1, b = 0.2$

13. a. \$3.75

b. No, he shouldn't play the game because his loss per game is \$1.25.

c. It is not a fair game because the expected gain is less than the initial cost of the game.

14. $k = 17$

15. a.

x	0	1	2	3
$\mathrm{P}(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

b. 1.8

16. a. $2\frac{1}{3}$ **b.** $9\frac{1}{3}$

c. $5\frac{2}{3}$ **d.** $6\frac{1}{15}$

17. 4.42

18. \$1452

Exercise 15.4 Variance and standard deviation

1. a. \$2.45

b. $\mathrm{Var}(X) = \$1.35,\ \mathrm{SD}(X) = \$1.16$

2. a. $k = \frac{1}{10}$

b. 3.2

c. $\mathrm{Var}(X) = 6.56,\ \mathrm{SD}(X) = 2.56$

3. a.

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{1}{30}$	$\frac{4}{30} = \frac{2}{15}$	$\frac{9}{30} = \frac{3}{10}$	$\frac{16}{30} = \frac{8}{15}$

b. i. 3.33 **ii.** 0.69

c. i. 11.02 **ii.** 6.20

4. a. $m = 15$

b. $\mathrm{Var}(Z) = 153.48,\ \mathrm{Var}(2(Z-1)) = 613.91,\ \mathrm{Var}(3 - Z) = 153.48$

5. a. i. $E(X) = \frac{1}{9}$

ii. $\mathrm{SD}(X) = 1.7916$

b. i. $E(Y) = 7$

ii. $\mathrm{SD}(Y) = 3.7947$

c. i. $E(Z) = \frac{19}{6}$

ii. $\mathrm{SD}(Z) = 1.3437$

6. a. $c = \frac{1}{3}$

b. $\mathrm{E}(Y) = 3$

c. $\mathrm{Var}(Y) = 11.56,\ \mathrm{SD}(Y) = 3.40$

7. a. 8 **b.** 0.5 **c.** 14.5

8. a. 225 **b.** 25 **c.** 6.25

9. a. $h = \frac{1}{10}$

b. $\mathrm{E}(X) = 1,\ \mathrm{Var}(X) = 0.6,\ \mathrm{SD}(X) = 0.7746$

10. a. $a = 0.3;\ b = 0.1$

b. $\mathrm{Var}(X) = 1.65,\ \mathrm{SD}(X) = 1.2845$

11. a. $\mathrm{E}(X^2) = a^2 + 2a - 2$

b. $\mathrm{E}(X) = 2,\ \mathrm{Var}(X) = 2$

12. a. $n = \frac{1}{13}$

b. $\mathrm{Var}(Y) = 1.7870,\ \mathrm{E}(Y) = 3.5385,\ \mathrm{SD}(Y) = 1.3368$

13. a. E = {11, 12, 13, 14, 15, 16, 17, 18
21, 22, 23, 24, 25, 26, 27, 28
31, 32, 33, 34, 35, 36, 37, 38
41, 42, 43, 44, 45, 46, 47, 48
51, 52, 53, 54, 55, 56, 57, 58
61, 62, 63, 64, 65, 66, 67, 68
71, 72, 73, 74, 75, 76, 77, 78
81, 82, 83, 84, 85, 86, 87, 88}

b.

z	1	2	3	4	5	6	7	8
$\mathrm{P}(Z = z)$	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{5}{64}$	$\frac{7}{64}$	$\frac{9}{64}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{15}{64}$

c. $\mathrm{E}(Z) = 5.8125,\ SD(Z) = 1.8781$

14. a. $\mathrm{P}(A) = \frac{1}{25};\ \mathrm{P}(B) = \frac{3}{25};\ \mathrm{P}(C) = \frac{5}{25};\ \mathrm{P}(D) = \frac{7}{25};$
$\mathrm{P}(E) = \frac{9}{25}$

b.

x	−\$1	\$0	\$1	\$4	\$9
$\mathrm{P}(X = x)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25} = \frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$

c. i. \$0.68 **ii.** \$2.29

15. a. 2, 4, 6, 8, 10, 11, 12, 13, 14, 15, 17, 20

b. See table at bottom of the page*

c. $\mathrm{E}(X) = 9.4,\ \mathrm{SD}(X) = 3.7974$

16. a. $k = 0.1$

b. $\mathrm{E}(X) = 1.695$

c. $\mathrm{SD}(X) = 1.167$

***15. b.**

x	2	4	6	8	10	11	12	13	14	15	17	20
$\mathrm{P}(X = x)$	0.04	0.08	0.16	0.2	0.17	0.04	0.12	0.04	0.04	0.06	0.04	0.01

Exercise 15.5 Applications of discrete random variables

1. **a.** 15 **b.** 4.7434 **c.** 0.9

2. 0.965

3. **a.** 0.6 **b.** $\frac{8}{9}$ **c.** \$535 000 **d.** 1

4. **a.** $m = \frac{1}{5}; n = \frac{2}{15}$
b. $\text{Var}(Z) = 2.8267,\ \text{SD}(Z) = 1.6813$
c. 1

5. **a.** $d = 5$ **b.** $\frac{2}{3}$
c. 5.25 **d.** 2.2913

6. **a.** $a = 0.35;\ b = 0.25$
b. $\text{Var}(Z) = 5.44,\ SD(Z) = 2.3324$
c. **i.** 15.8 **ii.** 48.96

7. **a.** $m = \frac{1}{8}; n = \frac{1}{4}$
b. **i.** 3 **ii.** 3
c. 1

8. **a.** 0.216
b. 0.286
c.

y	0	1	2
$\mathbf{P}(Y = y)$	0.286	0.498	0.216

d. **i.** 0.93 **ii.** 0.4971 **iii.** 0.7051

9. **a.**

x	0	1	2
$\mathbf{P}(X = x)$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$

b. **i.** $\frac{7}{9}$ **ii.** $\frac{50}{81}$ **iii.** 0.7857
c. 1

10. **a.** $k = 1$ **b.** 1.4 **c.** $\frac{7}{12}$

11. **a.** \$73 200 **b.** \$66 500

12. **a.** See table at bottom of the page*
b. \$71.70

13. **a.** Let Y be the net profit per day

Y	−\$120	\$230	\$580	\$930
$\mathbf{P}(Y = y)$	0.3	0.4	0.2	0.1

b. \$265
c. 0.9

14. **a.**

x	1	5	10
$\mathbf{P}(X = x)$	$\frac{12}{16} = \frac{3}{4}$	$\frac{3}{16}$	$\frac{1}{16}$

b. 2.3
c. 57.8
d. 44

15. **a.** $\text{P}(W) = 0.65,\ \text{P}(V) = 0.35$
b. 0.2275
c.

x	0	1	2
$\mathbf{P}(X = x)$	0.2275	0.545	0.2275

d. **i.** 1 **ii.** 0.455 **iii.** 0.6745

16. **a.** $k = 2$
b. **i.** 2.4286 **ii.** 3.1020 **iii.** 1.7613
c. 1

15.6 Review: exam practice

1. C **2.** B **3.** C **4.** B **5.** C

6. **a.**

x	1	2	3	4
$\mathbf{P}(X = x)$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{8}{15}$

b. $3\frac{1}{3}$

7. **a.**

x	0	1	2
$\mathbf{P}(X = x)$	$\frac{4}{25}$	$\frac{12}{25}$	$\frac{9}{25}$

b. 1.2

8. 0.63

9. Yes

10. **a.** 2.9 **b.** 1.09 **c.** 1.044

11. Mean = 2.9; SD = 1.14

12. **a.**

x	2	3	4	5	6	7
$\mathbf{P}(X = x)$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{1}{36}$

b. $4\frac{1}{12}$
c. $\frac{7}{12}$

13. **a.** **i.** $(1-a)^4$ **ii.** $4a(1-a)^3$ **b.** $a = \frac{1}{5}$

14. **a.** $\text{P}(X = 0) = \frac{3}{4};\ \text{P}(X = 2) = \frac{1}{5};\ \text{P}(X = 5) = \frac{1}{20}$
b. \$0.65

15. **a.** 4.2 **b.** $\frac{1}{160\,000}$ **c.** 0.1994

16. **a.** \$ 19.30/car **b.** \$1430

17. **a.** Sample responses can be found in the worked solutions in the online resources.
b. $m = \frac{5}{12}$; maximum probability $= \frac{1}{12}$
c. **i.** $\text{E}(Z) = 3.9167,\ \text{SD}(Z) = 1.6562$
ii. 1

18. $m = \frac{1}{10};\ n = \frac{1}{5}$

19. **a.** \$1.36 **b.** 0.007 14

20. \$412.10

*12. **a.**

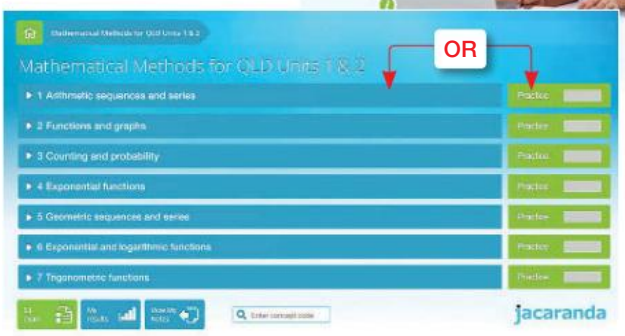
x	\$119	\$90	\$84	\$80	\$66	\$55	\$45	\$37	\$27
$\mathbf{P}(X = x)$	0.16	0.10	0.12	0.14	0.12	0.075	0.105	0.075	0.105

TOPIC 6 Discrete random variables 1

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Mathematical Methods for Queensland Units 1 & 2* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.



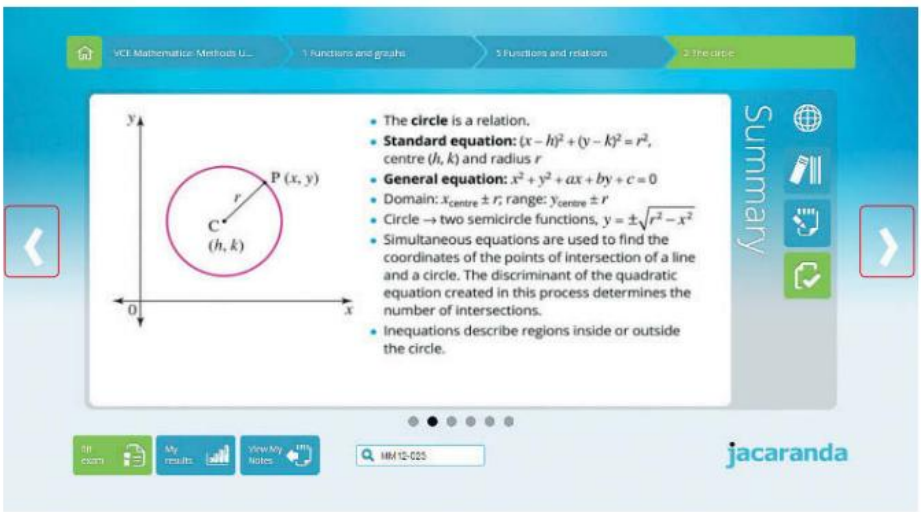
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **Sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 3

Mathematical Methods: Unit 2 examination

Unit

Unit 2: Calculus and further functions

Topic

Topic 1: Exponential functions 2
Topic 2: The logarithmic function 1
Topic 3: Trigonometric functions 1
Topic 4: Introduction to differential calculus
Topic 5: Further differentiation and applications 1
Topic 6: Discrete random variables 1

Conditions

Technique	Response type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 60 minutes Paper 2: 60 minutes	5 minutes
Resources		**Instructions**	
• QCAA formula sheet • Notes not permitted • Paper 1: Calculator not permitted • Paper 2: Non-CAS graphics calculator permitted		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.	

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	54	46	

*Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1,* Brisbane, 2018.

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.
A detailed breakdown of the examination marks summary can be found in the PDF version of this assessment instrument in your eBookPLUS.

Paper 1 — Technology free

Part A: Simple familiar — total marks: 34

Question 1 (8 marks)

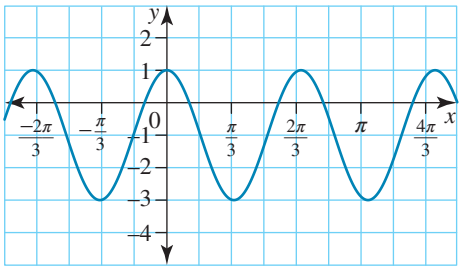
Solve the following equations for x, giving exact solutions:

a. $\log_5(2x) + \log_5(9) = \log_5(x+10) - \log_5(2)$

b. $\left(\log_2(x)\right)^2 + \log_2\left(x^6\right) = 16$

Question 2 (6 marks)

Determine the function for the graph below. Use mathematical reasoning to justify your response.



Question 3 (4 marks)

The following table represents a discrete probability function.

x	2	4	6	8	10
$P(X = x)$	0.1	k	0.4	0.2	0.1

a. Calculate the value of k.

b. Determine the expected value of a random variable which has the probability distribution shown.

Question 4 (11 marks)

Determine the derivatives of the following functions. Do not expand your solution.

a. $y = 5x^4 + 3x^2 - \frac{2}{x}$

b. $y = (2x - 3)^3$

c. $y = \frac{x^2 + 5x - 6}{x + 4}$

d. $y = \left(x^3 - 2\right)\left(\sqrt{2x + 3}\right)$

Question 5 (5 marks)

Determine exact solutions to each of the following equations over the domain $0 \leq x \leq 2\pi$.

a. $2\cos x + \sqrt{3} = 0$

b. $\sin^2 x + 3\sin x = 4$

Part B: Complex familiar — total marks: 10

Question 6 (10 marks)

a. Sketch the graph of $y = 3\sin\left(2x - \frac{\pi}{4}\right) + 1,\ 0 \leq x \leq 2\pi$.

b. Determine an exact solution for y when $x = \pi$.

Part C: Complex unfamiliar — total marks: 10

Question 7 (3 marks)

Find the values of a for which the function $\frac{1}{3}x^3 + ax^2 - 2x + 1$ is increasing at $x = 2$.
Use mathematical reasoning to justify your response.

Question 8 (7 marks)

The volume of water flowing out of a pond is modelled by the equation $V = \frac{1}{300}\left(2t^3 - \frac{t^4}{4}\right)$, where t is the time in hours. If the pond takes 6 hours to empty completely, calculate the time when the rate of flow was the greatest.
Use mathematical reasoning to justify your response.

Paper 2 — Technology active

Part A: Simple familiar — total marks: 26

Question 1 (4 marks)

The table below gives statistics for the purchases made at a market stall.

Amount spent by customer	Percentage of customers
$20	30%
$30	40%
$45	25%
$80	5%

Determine:

a. the expected earnings of the market stall on a day when it had 80 customers

b. the standard deviation.

Question 2 (6 marks)

A particle travels so that its distance D (in metres) from its origin O is modelled by the equation $D = 24 + 15t - \frac{t^2}{2}$, where t is the time in minutes after the particle has started to move.

a. Calculate the particle's distance from O when it first started to move.

b. Determine the time when the particle first reaches O. Give your answer to 2 decimal places.

c. Determine is the particle's speed when it has been moving for 3 minutes. Give your answer in m/s.

Question 3 (6 marks)

$5000 is placed in a bank account for 4 years where it earns 6% p.a. compounded annually.

a. Determine the balance of the account at the end of the 4 years.

b. Calculate how much of this balance is interest.

c. How many more years would the account take to reach a balance of $7500? Give your answer to 1 decimal place.

Question 4 (5 marks)

A farmer wishes to create a rectangular pen to contain as much area as possible using 120 metres of fencing. Determine the dimensions of the pen.

Sketch any graphs that you generate graphically in the course of your solution process.

Question 5 (5 marks)

If $f(x) = x^3 - 8$, determine $f'(x)$ using first principles.

Part B: Complex familiar — total marks: 9

Question 6 (6 marks)

The points $\left(\frac{1}{2}, 6\right)$ and $(4, 24)$ both lie on the graph of the function $y = a\log_2(bx)$. Determine the equation of the function.

Question 7 (3 marks)

The temperature, T°C, of coffee in a ceramic mug t minutes after it is poured is given by the equation $T = 70\left(4^{-0.06t}\right) + 25$.
Calculate the time taken after the coffee is poured for it to reach a temperature of 30°C. Round your answer to the nearest minute.

Part C: Complex unfamiliar — total marks: 11

Question 8 (4 marks)

Rohith is a keen fisherman. The ideal time for fishing in Rohith's favourite tidal lake is when it is more than 3 metres deep.

The depth, D metres, of water in the lake can be modelled by the equation

$$D = 5 - 4\sin\frac{\pi}{3}t$$

where t is the time in hours after midnight.

Determine the fraction of the day the lake suitable for fishing.

Question 9 (7 marks)

Determine the equations of all lines that are tangental to $y = (x-1)^3$ which are also perpendicular to the line $3y + x = 0$.

PRACTICE ASSESSMENT 4

Mathematical Methods: Units 1 & 2 examination

Topic

Unit 1: Topic 1: Arithmetic and geometric sequences and series 1
Topic 2: Functions and graphs
Topic 3: Counting and probability
Topic 4: Exponential functions 1
Topic 5: Arithmetic and geometric sequences and series 2

Unit 2: Topic 1: Exponential functions 2
Topic 2: The logarithmic function 1
Topic 3: Trigonometric functions 1
Topic 4: Introduction to differential calculus
Topic 5: Further differentiation and applications 1
Topic 6: Discrete random variables 1

Conditions

Technique	Response type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 50 minutes Paper 2: 70 minutes	5 minutes

Resources	Instructions
• QCAA formula sheet: • Notes not permitted • Paper 1: Calculator not permitted • Paper 2: Non-CAS graphics calculator permitted	• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem-solving *Assessment objective/s numbers: 1, 2, 3, 4, 5, 6	50	50	

*Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1*, Brisbane, 2018.

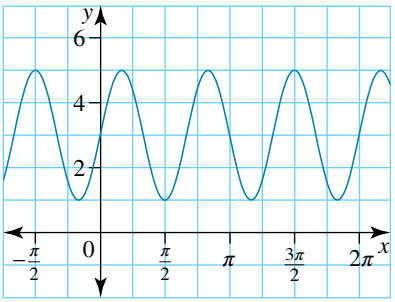
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.
A detailed breakdown of the examination marks summary can be found in the PDF version of this assessment instrument in your eBookPLUS.

Paper 1 – Technology free

Part A: Simple familiar – total marks: 30

Question 1 (3 marks)

Determine the equation describing the cosine curve below.



Question 2 (7 marks)

Differentiate the following with respect to x. Do not expand your solution.

a. $\frac{1}{3}x^4 - 2x^2 + 3x - 10$

b. $(3x - 2)\left(1 - 4x^3\right)$

c. $\dfrac{\left(x^2 + 5\right)}{(1 - x)}$

Question 3 (4 marks)

a. Sketch the hybrid function described by

$$f(x) = \begin{cases} \sqrt{4 - x^2}, & -2 \leq x < 0 \\ 3x + 1, & x \geq 0 \end{cases}$$

b. Identify the domain and the range of the function.

c. Determine the value of:

i. $f(-2)$ **ii.** $f(6)$ **iii.** $f(-4)$

Question 4 (6 marks)

Determine the value of the 5th term in each of these sequences

a. $1, -113, -323 \ldots$

b. $3, -9, 27 \ldots$

c. $t_{n+1} = 2t_n,\ t_1 = 4$

Question 5 (6 marks)

Consider the equation $y = 2(x-1)^2 - 10$.

a. Identify the graph's turning point.

b. State whether the turning point is a maximum or a minimum.

c. Determine where the graph crosses the x-axis.

d. Determine the gradient of the tangent to the curve when $x = 3$.

Question 6 (4 marks)

Solve for x in each of the following equations.

a. $-2\log_x\left(\frac{1}{100}\right) = 4$

b. $2^x \times 8^{3x-1} = 64$

Part B: Complex familiar — total marks: 12

Question 7 (7 marks)

Factorise the expression $x^3 + 2x^2 - 5x - 6$.

Question 8 (5 marks)

Determine the equation for the line that is tangent to $y = \frac{5}{x+2}$ at $x = 3$.
Give the line equation in the form $ay + bx + c = 0$.

Part C: Complex unfamiliar — total marks: 8

Question 9 (8 marks)

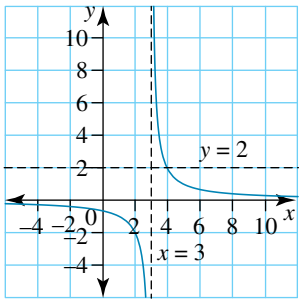
The sum of the first five terms of an arithmetic series is 45 and the sum of its first eight terms is 120. Determine the sum of its first ten terms.
Show your mathematical reasoning in full.

Paper 2 – Technology active

Part A: Simple familiar — total marks: 30

Question 1 (4 marks)

Determine the equation of the graph shown.



Question 2 (4 marks)

An archery club is made up of 20 people who use compound bows, 14 who use recurve bows and 8 who use longbows. If the leadership committee is to be made up of 6 members, calculate the probability that all 6 committee members are compound bow users.

Express your answer as a percentage to 2 decimal places.

Question 3 (6 marks)

A dolphin is diving and leaping through the water. Its displacement, D (in metres) is given by the equation $D = \frac{2}{3}x^3 - 8x^2 + 25x - 10$, where x is the time elapsed in seconds and $0 \leq x \leq 7$ seconds.

a. Determine the time(s) when the dolphin is stationary.

b. Determine the dolphin's maximum height above the pond.

c. Calculate the maximum speed of the dolphin over the time interval $0 \leq x \leq 7$ s and the time it reaches this speed.

Question 4 (6 marks)

A slice cut from a small pizza of radius 12 cm is a sector with an arc length of 12 cm as shown.

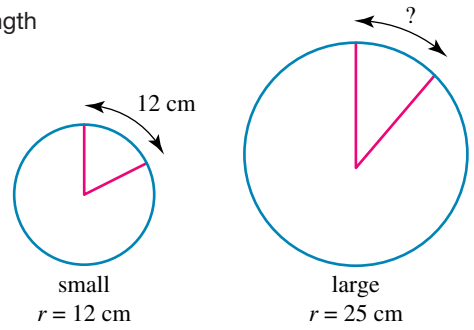
a. What angle (in radians) does the sector contain?

b. Convert your answer in part a. to degrees

c. Calculate the area of the pizza slice from the small pizza.

d. Calculate the arc length of a slice from a large pizza (radius of 25 cm) if it has the same area as the slice from the small pizza.

Round all answers to 2 decimal places.



Question 5 (5 marks)

Find the values of k and m for the following probability distribution table given that $E(X) = 2.81$.

x	0	1	2	3	4	5	6
$P(X = x)$	0.1	0.14	0.25	k	0.22	m	0.08
$E(X)$	0	0.14	0.5		0.88		0.48

Question 6 (5 marks)

The Venn diagram shown here represents the Year 11 student enrolments in three subjects at a small school.

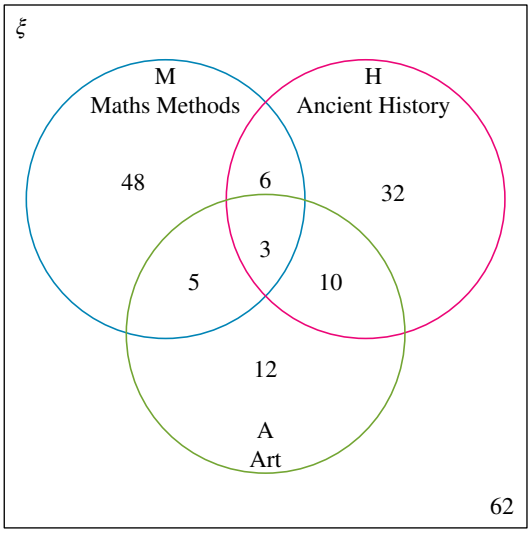
Determine the probability that:

a. a randomly selected student studies Art, Ancient History and Maths Methods.

b. a randomly selected student studies Ancient History.

c. an Art student does Ancient History but not Maths Methods.

d. a student who does Maths Methods does not do Art or Ancient History.



Leave all answers in the form of fractions.

Part B: Complex familiar — total marks: 10

Question 7 (10 marks)

The curve $y = x^4 - ax^3 + bx - 3$ crosses the x-axis at $x = -1$ and $x = 3$.

a. Calculate a and b.

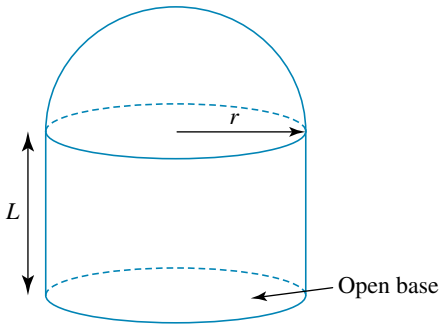
b. Determine the coordinates of the stationary points of the graph.

Part C: Complex unfamiliar — total marks: 10

Question 8 (5 marks)

A farmer wishes to make a mini-silo from sheet steel in the form of a hollow cylinder with a hemispherical top as shown.

Determine the smallest area of sheet steel that can be used to make the mini-silo if it is to have a volume of $50\,\text{m}^3$. (Round your answer to the nearest m^2.)



Question 9 (5 marks)

a. Demonstrate that the sum of the first n terms of an arithmetic sequence $S_n = \frac{n}{2}(t_1 + t_n)$ can be expressed in the form of a quadratic over n such that $S_n = an^2 + bn + c$ where $n \geq 1$.

Justify your reasoning mathematically.

b. Express a and b in terms of t_1 and the common difference d.

GLOSSARY

addition principle states that, for mutually exclusive procedures, if there are m ways of doing one procedure and n ways of doing another procedure, then there are $m + n$ ways of doing one or the other procedure

algorithm set of instructions that are repeated until a desired end is reached, such as in the process of long division of polynomials

amplitude the distance a sine or cosine graph oscillates up or down from its equilibrium, or mean, position

arc a section of the circumference of a circle. A *minor arc* is shorter than the circumference of the semicircle; a *major arc* is longer than the circumference of the semicircle

arithmetic sequence a sequence in which the difference between any two successive terms is the same

arrangements or permutation in probability, a counting technique in which order is important

asymptote a line that a graph approaches but never reaches. A horizontal asymptote shows the long-term behaviour as, for example, $x \to \infty$; a vertical asymptote may occur where a function is undefined such as at $x = 0$ for the hyperbola $y = \frac{1}{x}$

average rate of change the average rate of change of a function f over an interval $[a, b]$ is measured by $\frac{f(b) - f(a)}{b - a}$; this is the gradient of the chord joining the endpoints of the interval on the curve $y = f(x)$

axis of symmetry a line about which a graph is symmetrical

binomial coefficients or combinatoric coefficient the coefficients of the terms in the binomial expansion of $(x + y)^n$, with $\binom{n}{r}$ or ${}^nC_r$ the coefficient of the $(r + 1)$th term For r and n non-negative whole numbers, $\binom{n}{r} = \frac{n!}{r!(n - r)!}, 0 \leq r \leq n$

binomial probabilities probability with two outcomes, favourable and non-favourable, the sum of which is 1

boundary points in trigonometry, the four points $(1, 0), (0, 1), (-1, 0), (0, -1)$ on the circumference of the unit circle that lie on the boundaries between the four quadrants

box table a method of calculating the number of possible arrangements or permutations

chain rule use the chain rule to differentiate a function that is a composite function of two simpler functions; $\frac{dy}{dx} = \frac{dy}{du} = \frac{du}{dx}$

circle a relation rule $(x - a)^2 + (y - b)^2 = r^2$, where the centre is (a, b) and the radius is r

codomain the set of all y-values available for pairing with x-values to form a mapping according to a function rule $y = f(x)$

combination or selection in probability, a counting technique in which order is not important

common difference the difference between each term in an arithmetic sequence: $d = t_{n+1} - t_n$

common ratio the ratio between two consecutive terms in a geometric sequence; $r = \frac{t_{n+1}}{t_n}$

completing the square technique used to express a quadratic as a difference of two squares. To apply this technique to a monic quadratic expression in x, add the square of half the coefficient of the term in x and then subtract this. This enables $x^2 \pm mx$ to be expressed as $\left(x \pm \frac{m}{2}\right)^2 - \left(\frac{m}{2}\right)^2$

composite function a function that can be expressed as a composition of two simpler functions

compound interest interest calculated on the changing value throughout the time period of a loan or investment. Interest is added to the balance before the next interest calculation is made. The amount of the loan or investment can be calculated using $A = \left(1 + \frac{r}{100}\right)^n P$

concave down describes parabolas that open downwards

concave up describes parabolas that open upwards

conditional probability the probability of an event given that another event has occurred

conic sections a family of curves that include the hyperbola, parabola and the circle

constant term (of a polynomial or other expression) the term that does not contain the variable

continuous all values within a specified interval are permitted

continuous random variable a random variable that assumes quantities that can be measured, such as weight, time or height, in a given range

degree of a polynomial the highest power of a variable; for example, polynomials of degree 1 are linear, degree 2 are quadratic and degree 3 are cubic

dependent variable the second value, or y-value, in a set of ordered pairs

differentiable describes a function that is continuous at $x = a$ and its derivative from the left of $x = a$ equals its derivative from the right of $x = a : f'(a) = f'\left(a^+\right)$. Smoothly continuous functions are differentiable

differentiation from first principles (or finding the derivative from first principles) requires the derivative to be obtained from its limit definition, either $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

dilation a linear transformation that enlarges or reduces the size of a figure by a scale factor k parallel to either axis or both axes

dilation factor measures the amount of stretching or compression of a graph from an axis

discontinuous describes a point in a graph at which there is a break

discrete only fixed values are permitted

discrete random variable a random variable that can have only countable numbers, generally integer values

discriminant for the quadratic expression $ax^2 + bx + c$, the discriminant is $\Delta = b^2 - 4ac$

displacement measures both distance and direction from a fixed origin; represents the position of a particle relative to a fixed origin

division of polynomials can be performed using technology, long division or the inspection method: $\frac{\text{dividend}}{\text{divisor}} = \text{quotent} + \frac{\text{remainder}}{\text{divisor}}$

domain the set of all x-values of the ordered pairs (x, y) that make up a relation

element the members of a set; $a \in A$ means a is an element of, or belongs to, the set A. If a is not an element of the set A, this is written as $a \notin A$

equating coefficients if $ax^2 + bx + c \equiv 2x^2 + 5x + 7$ then $a = 2$, $b = 5$ and $c = 7$ for all values of x

equilibrium or mean position the position about which a trigonometric graph oscillates

equiprobable having an equal likelihood of occurring

event a set of outcomes that is a subset of the sample space in an experiment

expected value (or mean) a measure of the central tendency of the probability distribution of a random variable. For a discrete random variable, X, $\mathrm{E}(X) = \mu = \sum (xp(x))$

exponent an index or power; for the number $n = a^p$, the base is a and the power, or index, or exponent is p

exponential form also known as index form; a way of expressing a standard number n using a base a and an exponent, or index (power) $x : n = a^x$

exponential functions functions of the form $f : R \to R, f(x) = a^x, a \in R^+ \setminus \{1\}$

factor theorem states that for the polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$

fair a gamble with an average net result of zero

function a relation in which the set of ordered pairs (x, y) have each x-coordinate paired to a unique y-coordinate

future value the value of an asset at a time in the future, based on the original cost less depreciation

geometric sequence a pattern of numbers whose consecutive terms increase or decrease in the same ratio. Each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio

geometric series when the terms of a geometric sequence are added; calculated by $S_n = t_1 \dfrac{r^n - 1}{r - 1}$

global or absolute maximum a point on a curve where the y-coordinate is larger that that of any other point on the curve

global or absolute minimum a point on a curve where the y-coordinate is smaller than that of any other point on the curve

gradient measures the steepness of a line as the ratio $m = \dfrac{\text{rise}}{\text{run}}$. If (x_1, y_1) and (x_2, y_2) are two points on the line, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

gradient function or derivative function (of a function f) the function f' whose rule is defined as $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ for $y = f(x)$, $\dfrac{dy}{dx} = f'(x)$ where $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

hyperbola a smooth curve with two branches, formed by the intersection of a plane with both halves of a double cone

identically equal describes polynomials with the same coefficients of corresponding like terms; the $\equiv$ symbol is used to emphasise the fact that the equality must hold for all values of the variable

image a figure after a transformation; for the function $x \to f(x)$, $f(x)$ is the image of x under the mapping f

implied domain the set of x-values for which a rule has meaning

independent events events that have no effect on each other

independent variable the first value, or x-value, in a set of ordered pairs

index *see* **exponent**

index notation *see* **exponential form**

indicial equation an equation in which the unknown variable is an exponent

interval a set of numbers described by two numbers where any number that lies between those two numbers is also included in the set

interval notation a convenient alternative way of describing sets of numbers: $[a, b] = \{x: a \leq x \leq b\}$; $(a, b) = \{x: a < x < b\}$; $[a, b) = \{x: a \leq x < b\}$; $(a, b] = \{x: a < x \leq b\}$

inverse functions pairs of relations for which the rule of one can be formed from the rule of the other by interchanging x- and y-coordinates. The inverse of a one-to-one function f is given by the symbol f^{-1}

inverse proportion a relationship between two variables such that when one variable increases the other decreases in proportion so that the product is unchanged; if y is inversely proportional to x, then $y = kx$, where k is the constant of proportionality

irreducible describes a quadratic that cannot be factorised over R

iteration a method of approximation in which a value is estimated and then the process repeated using the values obtained from one stage to calculate the value of the next stage, and so on

leading coefficient (of a polynomial or other expression) the coefficient of the term containing the highest power of a variable

leading term (of a polynomial or other expression) the term containing the highest power of a variable

limit the behaviour of a function as it approaches a point, not its behaviour at that point. If $\lim\limits_{x \to a} f(x) = L$, then the function approaches the value L as x approaches a. The limit must be independent of which direction $x \to a$ so $L^- = L^+ = L$ for the limit to exist

local maximum a stationary point on a curve where the gradient is zero and the y-coordinate is greater than those of neighbouring points

local minimum a stationary point on a curve where the gradient is zero and the y-coordinate is lower than those of neighbouring points

logarithm an index or power. If $n = a^x$ then $x = \log_a(n)$ is an equivalent statement

log-log plot a graph that plots $\log(y)$ versus $\log(x)$ for each data point (x, y). If the plot shows a linear graph then there is a power law relationship between the data points

long-term behaviour refers to the effect on the y-values of a graph as the x-values approach $\pm\infty$

mapping a function that pairs each element of a given set (the domain) with one or more elements of a second set (the range)

method of bisection a procedure for improving the estimate for a root of an equation by halving the interval in which the root is known to lie

modelling uses mathematical concepts to describe the behaviour of a system, usually in the form of equations

monic for quadratics 'monic' indicates the coefficient of x^2 equals 1

multiplication principle states that, if there are m ways of doing the first procedure and for each one of these there are n ways of doing the second procedure, then there are $m \times n$ ways of doing the first and the second procedures

multiplicity if $(x - a)^2$ is a factor of a polynomial then $x = a$ is a zero of multiplicity 2 or a repeated zero; if $(x - a)^3$ is a factor then $x = a$ is a zero of multiplicity 3

mutually exclusive describes events that cannot occur simultaneously

negative definite describes a quadratic expression $ax^2 + bx + c$ if $ax^2 + bx + c < 0$ for all $x \in R$

normal a line that is perpendicular to the tangent at a point of tangency

Null Factor Law mathematical law stating that if $ab = 0$ then either $a = 0$ or $b = 0$ or both a and b equal zero. This allows equations for which the product of two or more terms equals zero to be solved.

outcome the result of an experiment

out of phase *see* phase difference

parabola (with a vertical axis of symmetry) the graph of a quadratic function; its equation may be expressed in turning point form $y = a(x - b^2) + c$ with turning point (b, c), factored form $y = a(x - b_1)(x - c_2)$ with x-intercepts at $x = b_1$, $x = c_2$, or general form $y = ax^2 + bx + c$

partial fractions fractions that sum together to form a single fraction

Pascal's triangle a triangle formed by rows of the binomial coefficients of the terms in the expansion of $(a + b)^n$, with n as the row number

period on a trigonometric graph, the length of the domain interval required to complete one full cycle. For the sine and cosine functions, the period is 2π since $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$. The tangent function has a period of π since $\tan(x + \pi) = \tan(x)$

periodicity a function that repeats itself in a regular pattern

phase difference or shift the horizontal translation of a trigonometric graph. For example, $\sin(x - a)$ has a phase shift or a phase change of a or a horizontal translation of a from $\sin(x)$; and $\sin(x)$ and $\cos(x)$ are out of phase by $\frac{\pi}{2}$ or have a phase difference of $\frac{\pi}{2}$

piece-wise function a function whose rule takes different forms for different sections of its domain

polynomial an algebraic expression in which the power of the variable is a positive whole number

positive definite describes a quadratic expression $ax^2 + bx + c$ if $ax^2 + bx + c > 0$ for all $x \in R$

power function a function with rules of the form $f(x) = x^n$, $n \in Q$

probability the long-term proportion, or relative frequency, of the occurrence of an event

probability distribution the possible values of a random variable together with the corresponding probabilities of their occurrence. These may be listed in a table or specified by a function rule. The probability values must satisfy the conditions that $\sum p(x) = 1$ and $0 \leq p(x) \leq 1$

probability table or Karnaugh maps a format for presenting all known probabilities of events in rows and columns

probability tree diagram a graphic method used to represent a sample space

product rule use the product rule to differentiate a function that is a product of two simpler functions; If $y = uv$ then $\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

proper rational function form of rational function in which the degree of the numerator is smaller than that of the denominator

Pythagorean identity states that, for any θ, $\sin^2(\theta) + \cos^2(\theta) = 1$

quadrants the four sections into which the Cartesian plane is divided by the coordinate axes

quadratically related variables variables in an equation in which one variable is expressed in terms of another raised to the power of 2

quadratic formula the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is used to find the solutions

quotient rule use the quotient rule to differentiate a function that is a quotient of two simpler functions; If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

radian measure the measure of the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle

random variable in probability theory, a variable whose values are assigned by the outcomes of a random experiment

range the set of all y-values of the ordered pairs (x, y) that make up a relation

rational function a function expressed as the quotient of two polynomials $\frac{P(x)}{Q(x)}, Q(x) \neq 0$

rational number any real number that can be expressed exactly as a fraction

rational roots solutions from quadratic equations that can be expressed as rational numbers

rational root theorem states that if a polynomial with integer coefficients has a rational zero $\frac{p}{q}$, then its constant term is divisible by p and its leading coefficient is divisible by q

rectangular hyperbola the function $f: R \setminus \{0\} \rightarrow R, f(x) = x^{-1}$

rectilinear motion motion in a straight line

recursive function (or recurrence relation) an equation that recursively defines a sequence; that is, once one or more initial terms are given, each further term of the sequence is defined as a function of the preceding terms

reflections transformations of a figure defined by the line of reflection where the image point is a mirror image of the pre-image point

relation any set of ordered pairs

relative frequency refers to actual data obtained;

$$\text{Relative frequency} = \frac{\text{number of times an event has occured}}{\text{number of trials}}$$

remainder theorem states that if the polynomial $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$

restricted domain a subset of a function's maximal domain, often due to practical limitations on the independent variable in modelling situations

roots the solutions to an equation

sample space the set of all possible outcomes in an experiment, denoted as ξ

sampling without replacement a type of sampling whereby the object is not replaced, resulting in a dependent trial

sampling with replacement a type of sampling whereby the object is replaced, resulting in an independent trial

scientific notation or standard form Expression of a number in the form $a \times 10^b$, the product of a number a, where $1 < a < 10$, and a power of 10

sector a fraction of the area of a circle, enclosed by two radii and an arc

semicircle the area with radius r and centre $(0, 0)$ described by the function $f\colon [-r, r] \to R, f(x) = \sqrt{r^2 - x^2}$

semi-log plot a graph that plots $\log(y)$ versus x for each data point (x, y)

sequence a related set of objects or events that follow each other in a particular order

set a collection of elements

sideways parabola a parabola with a horizontal axis of symmetry; a relation with the rule $y^2 = x$; the inverse of the parabola $y = x^2$

sign diagram a diagram showing where the graph of a function lies above or below the x-axis and where the function is positive or negative

significant figures the number of digits that would occur in a if the number was expressed in scientific notation as $a \times 10^b$ or as $a \times 10^{-b}$

smoothly continuously functions describes a function for which the graph has no breaks or sharp points

solving the triangle the process of calculating all side lengths and all angle magnitudes of a triangle

standard deviation a measure of spread. $\text{SD}(X) = \sigma = \sqrt{\text{Var}(X)}$. The larger the standard deviation, the more spread out the data. If the standard deviation is small, the data is clumped about the mean

stationary point of horizontal inflection a point on a curve where the tangent to the curve is horizontal and the curve changes concavity. The graph of $y = x^3$ has a stationary point of inflection at $(0,\ 0)$

stationary point of inflection a point on a curve where the tangent to the curve is horizontal and the curve changes concavity; the graph of $y = x^3$ has a stationary point of inflection at $(0, 0)$

surds roots of numbers that do not have an exact answer, so they are irrational numbers. Surds themselves are exact numbers; for example, $6 - \sqrt{6}$ or $5 - \sqrt{353}$

symmetry properties indicate the relationships between trigonometric values of symmetric points in the four quadrants. If $[\theta]$ lies in the first quadrant, the symmetric points to it could be expressed as $[\pi - \theta], [\pi + \theta], [2\pi - \theta]$. This gives, for example, the symmetric property $\cos(\pi - \theta) = -\cos(\theta)$

tangent line a line that touches a curve at a single point; for a circle, the tangent is perpendicular to the radius drawn to the point of contact

term (in a sequence) a number in a sequence

term number the position of a term in a sequence

transformations geometric operators that may change the shape and/or position of a graph

translations transformations of a figure where each point in the plane is moved a given distance in a horizontal or vertical direction

trial the number of times a probability experiment has been conducted

trigonometric (or circular) functions sine, cosine, tangent. On a unit circle, $\cos(\theta)$ s the x-coordinate of the trigonometric point; $\sin(\theta)$ is the y-coordinate of the trigonometric point $[\theta]$; $\tan(\theta)$ is the length of the intercept that the line through the origin and the trigonometric point cuts off on the tangent drawn to the unit circle at $(1, 0)$

trigonometric point on a unit circle, the endpoint of an arc, drawn from the initial point $(1, 0)$ and of length θ, where θ is any real number, is the trigonometric point $[\theta]$. The point has Cartesian coordinates $(\cos\theta, \sin\theta)$

uniform distribution all outcomes are equally likely

unit circle in trigonometry, the circle $x^2 + y^2 = 1$ with the centre at the origin and a radius of 1 unit

unit cost depreciation a method of depreciating an asset according to its use — the more it is used, the faster it will depreciate

variance a measure of the spread of a data set using the equation: $\text{Var}(X) = \sigma^2 = \text{E}\left(X^2\right) - \text{E}\left[(X)\right]^2$

velocity the rate of change of displacement with respect to time

Venn diagram a diagram that displays the union and intersection of sets

write-off value the value at which an asset is removed from the books of a company as it is considered effectively worthless; also called the 'scrap value'

zero the value of x for which $f(x) = 0$

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Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
Paper 1				
1	6			2
2	3			5
3	5			1
4	5			2
5	3			3
6	5			2
7		4		4
8		5		2
9			4	3
10			6	2
Paper 2				
1	3			6
2	4			2
3	7			2
4	5			5
5	8			3
6	6			5
7		6		5
8		5		2
9			10	2
Totals	**60**	**20**	**20**	
Percentage	**60%**	**20%**	**20%**	

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
Paper 1				
1	8			2
2	6			3
3	4			6
4	11			4, 5
5	5			3
6		10		3
7			3	4
8			7	4
Paper 2				
1	4			6
2	6			4
3	6			1
4	5			4
5	5			4
6		6		2
7		3		1, 2
8			4	3
9			7	4
Totals	**60**	**19**	**21**	
Percentage	**60%**	**19%**	**21%**	

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
Paper 1				
1	3			Unit 2, Topic 3
2	7			Unit 2, Topic 4
3	4			Unit 1, Topic 2
4	6			Unit 1, Topic 1, 5
5	6			Unit 1, Topic 2 Unit 2, Topic 4
6	4			Unit 2, Topic 1, 2
7		7		Unit 1, Topic 2
8		5		Unit 2, Topic 4, 5
9			8	Unit 1, Topic 1
Paper 2				
1	4			Unit 1, Topic 2
2	4			Unit 1, Topic 3
3	6			Unit 2, Topic 4
4	6			Unit 2, Topic 3
5	5			Unit 2, Topic 6
6	5			Unit 1, Topic 3
7		10		Unit 1, Topic 2
8			5	Unit 2, Topic 4
9			5	Unit 1, Topic 1, 2
Totals	**60**	**22**	**18**	
Percentage	**60%**	**22%**	**18%**	