CAMBRIDGE

SPECIALIST MATHEMATICS

UNITS 3 & 4

SECOND EDITION

CAMBRIDGE SENIOR MATHEMATICS FOR QUEENSLAND

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NEIL CRACKNELL

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Please be aware that this publication may contain images of Aboriginal and Torres Strait Islander people who are now deceased. Several variations of Aboriginal and Torres Strait Islander terms and spellings may also appear; no disrespect is intended. Please note that the terms 'Indigenous Australians' and 'Aboriginal and Torres Strait Islander peoples' may be used interchangeably in this publication.

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Specialist Mathematics Units 3 & 4 ISBN 978-1-009-57826-4 Cambridge University Press

Online appendices accessed through the Interactive Textbook or PDF Textbook only

Included in the Interactive and PDF Textbook only

About the lead author and consultants

About the lead author

Michael Evans was a consultant to ACARA on the writing of the Australian Curriculum from which the Queensland syllabus has evolved. He is a consultant with the Australian Mathematical Sciences Institute, and is coordinating author of the ICE-EM 7–10 series also published by Cambridge.

He has also been active in the Australian Mathematics Trust, being involved with the writing of enrichment material and competition questions. He has many years' experience as a Chief Examiner and Chairperson of examination panels.

About the consultants

As at the time this edition was published:

Joel Speranza is currently working with the Toowoomba Catholic Schools Office, delivering Specialist Mathematics hybridly to students from eight different regional schools. He also creates supporting material for the senior mathematics curriculum through the website Maths Video Australia.

Noel Covill is Head of Mathematics at St Joseph's College, Gregory Terrace

Trevor Redmond is Head of Mathematics at Somerville House, South Brisbane

Introduction and overview

Cambridge Senior Mathematics for Queensland Specialist Mathematics Units 3 & 4 provides complete and aligned coverage of the QCAA syllabus to be implemented in Year 12 from 2025. Its four components – the print book, downloadable PDF textbook, online Interactive Textbook (ITB) and Online Teaching Resource (OTS) – contain a huge range of resources, including worked solutions, available to schools in a single package at one convenient price (the OTS is included with class adoptions, conditions apply). There are no extra subscriptions or per-student charges to pay.

New features in the second edition include:

- **Learning intentions** complemented by a Skills Checklist at the end of each chapter that allows you to check your understanding and tick off your achievements,
- **Technology-free and technology-active** short-response and multiple-choice questions are clearly labelled in Chapter review sections and Unit Revision chapters.
- **The problem-solving and modelling task (Appendix A):** There are three appendices at the end of the book. Appendix A is written by consultant Joel Speranza, providing advice on how to complete problem-solving and modelling tasks (PSMTs). This is supported by video resources accessed through QR codes and in the Interactive Textbook.

The Second Edition also features significantly revised and updated material from the first edition, including:

Degree of difficulty classification of questions: in the exercises, questions are classified as simple familiar SF, complex familiar CF, or complex unfamiliar CU questions and are indicated by a strip along the margin. The revision chapters described below also contain model questions for each of these categories.

Three revision chapters of material covered in the course: The first two of these chapters each cover an entire unit, and the last revision chapter contains questions revising the whole book. Each revision chapter is divided into technology-free and technology-active short-response and multiple-choice questions. The revised set of chapter tests provided within the Online Teaching Suite are now also categorised into technology-free and technology-active short-response and multiple-choice questions.

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Calculator guidance: Throughout the book there is guidance for the use of the TI-Nspire CX non-CAS and the Casio fx-CG20AU and fx-CG50AU graphics calculators for the solution of problems. Guidance on the TI-84Plus CE is included in the Interactive Textbook, accessed via icons next to the TI-Nspire boxes. There are also online guides for the general use of each of these calculators.

Assessment: Examination practice questions and various assessment tasks are provided in the revision chapters and the Online Teaching Suite.

Interactive Textbook (ITB)

The Interactive Textbook (ITB) is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

Updated and revised for the new syllabus, the Interactive Textbook includes:

- **Video demonstrations** of all worked examples
- **Quick quizzes** containing auto-marked multiple-choice questions have been thoroughly updated and revised, enabling students to check their understanding.
- A **success criteria** checklist at the end of each chapter with linked questions and examples available for download
- Comprehensive **worked solutions** for all questions are provided in the Interactive Textbook as an option that teacher can choose to enable for their students.
- **Downloadable skillsheets** can be used for homework or in class to focus on a single skill or small set of related skills.
- **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

The Online Teaching Suite (OTS)

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

- A teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.

- The **task manager** allowing to direct students on a custom activity sequence based on their scores in measurable activities
- Quickly create customised tests from a bank of multiple-choice questions using the **test generator**. Tests are auto-marked in the Interactive Textbook or can be printed and used for homework or assessment practice.
- An expanded and revised suite of **chapter tests** and **assignments**
- Editable **curriculum grids** and **teaching programs**.
- A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

A new Exam Generator, available at no extra charge within the Online Teaching Suite, will include a comprehensive bank of QCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic, chapter and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- QCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

Acknowledgements

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1

Trigonometric functions

Chapter contents

Trigonometry is used in many of the topics in this course: vectors, complex numbers, integration, rates of change, differential equations and modelling motion.

This chapter provides an opportunity to revise the trigonometry that you have studied in Specialist Mathematics Units 1 & 2.

In addition, this chapter introduces the three inverse trigonometric functions: arcsine, arccosine and arctangent, which are also written as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$. In Chapter 11, we will see that these three functions have a somewhat surprising role to play in integration.

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1A Trigonometric functions

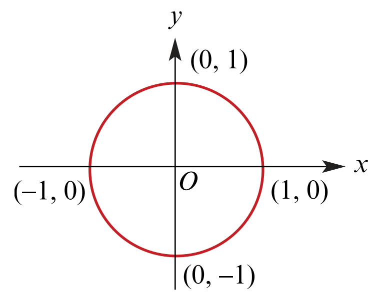
Learning intentions

- To define the sine, cosine and tangent functions in terms of the unit circle.
- To be able to determine exact values and solve simple equations involving these functions.
- To be able to sketch graphs of these functions.

Defining sine, cosine and tangent

The **unit circle** is a circle of radius 1 with centre at the origin. It is the graph of the relation $x^2 + y^2 = 1$.

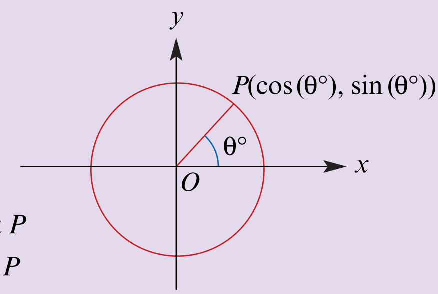
We can define the sine and cosine of any angle by using points on the unit circle.



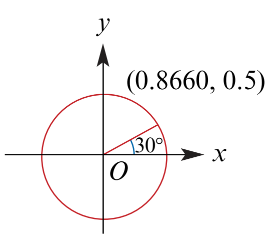
Definition of sine and cosine

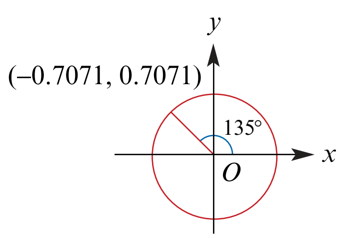
For each angle θ°, there is a point P on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the x-axis.

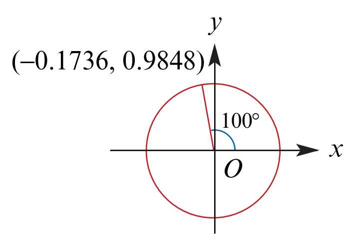
- $\cos(\theta°)$ is defined as the x-coordinate of the point P
- $\sin(\theta°)$ is defined as the y-coordinate of the point P



For example:







$\sin 30° = 0.5$ (exact value) $\qquad \sin 135° = \dfrac{1}{\sqrt{2}} \approx 0.7071 \qquad \sin 100° \approx 0.9848$

$\cos 30° = \dfrac{\sqrt{3}}{2} \approx 0.8660 \qquad \cos 135° = \dfrac{-1}{\sqrt{2}} \approx -0.7071 \qquad \cos 100° \approx -0.1736$

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Definition of tangent

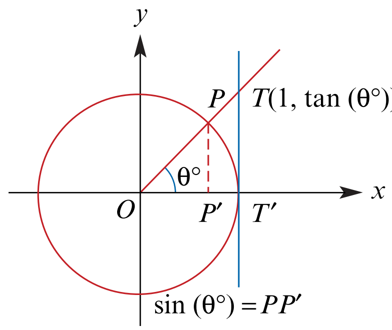
$$\tan(\theta^\circ) = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$$

The value of $\tan(\theta^\circ)$ can be illustrated geometrically through the unit circle.

By considering similar triangles OPP' and OTT', it can be seen that

$$\frac{TT'}{OT'} = \frac{PP'}{OP'}$$

i.e. $$TT' = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)} = \tan(\theta^\circ)$$



The trigonometric ratios

For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle.

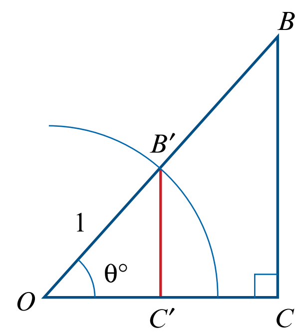
From the diagram:

$$B'C' = \sin(\theta^\circ) \quad \text{and} \quad OC' = \cos(\theta^\circ)$$

The similarity factor is the length OB, giving

$$BC = OB\sin(\theta^\circ) \quad \text{and} \quad OC = OB\cos(\theta^\circ)$$

$$\therefore \quad \frac{BC}{OB} = \sin(\theta^\circ) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta^\circ)$$

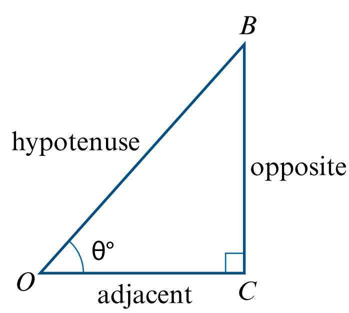


This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta^\circ) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta^\circ) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta^\circ) = \frac{\text{opposite}}{\text{adjacent}}$$



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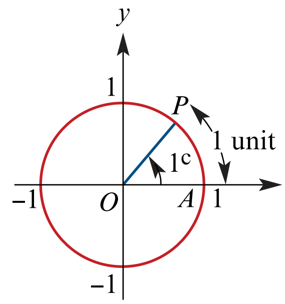
Definition of a radian

In moving around the unit circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Usually the symbol for radians, ^{c}, is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.

The following table displays the conversions of some special angles from degrees to radians.

Angle in degrees	0°	30°	45°	60°	90°	180°	360°
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

Some values for the trigonometric functions are given in the following table.

x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

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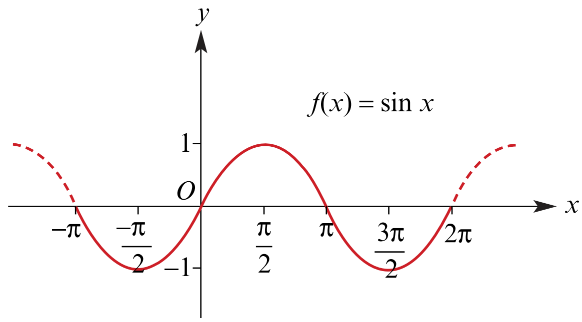
The graphs of sine and cosine

A sketch of the graph of

$$f(x) = \sin x, \quad x \in \mathbb{R}$$

is shown opposite.

As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**. The period is 2π. The amplitude is 1.

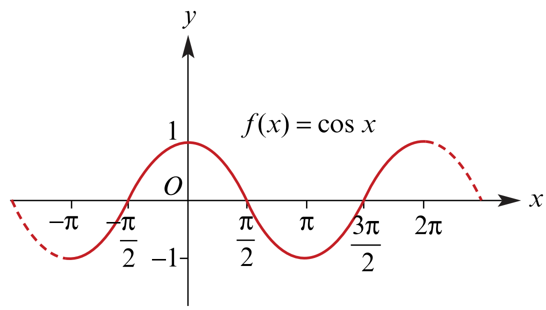


A sketch of the graph of

$$f(x) = \cos x, \quad x \in \mathbb{R}$$

is shown opposite.

The period of the cosine function is 2π. The amplitude is 1.



For the graphs of $y = a\cos(nx)$ and $y = a\sin(nx)$, where $a > 0$ and $n > 0$:

- Period $= \dfrac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a, a]$

Symmetry properties of sine and cosine

The following results may be obtained from the graphs of the functions or from the unit-circle definitions:

$$\begin{aligned}
\sin(\pi - \theta) &= \sin\theta & \cos(\pi - \theta) &= -\cos\theta \\
\sin(\pi + \theta) &= -\sin\theta & \cos(\pi + \theta) &= -\cos\theta \\
\sin(2\pi - \theta) &= -\sin\theta & \cos(2\pi - \theta) &= \cos\theta \\
\sin(-\theta) &= -\sin\theta & \cos(-\theta) &= \cos\theta \\
\sin(\theta + 2n\pi) &= \sin\theta & \cos(\theta + 2n\pi) &= \cos\theta \qquad \text{for } n \in \mathbb{Z} \\
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta
\end{aligned}$$

Example 1

a Convert 135° to radians.

b Convert 1.5^c to degrees, correct to two decimal places.

Solution

a $135° = \dfrac{135 \times \pi^c}{180} = \dfrac{3\pi^c}{4}$

b $1.5^c = \dfrac{1.5 \times 180°}{\pi} = 85.94°$ to two decimal places

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Example 2

Determine the exact value of:

a $\sin 150^\circ$

b $\cos(-585^\circ)$

Solution

a
$$\begin{aligned}\sin 150^\circ &= \sin(180^\circ - 150^\circ)\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\cos(-585^\circ) &= \cos 585^\circ\\ &= \cos(585^\circ - 360^\circ)\\ &= \cos 225^\circ\\ &= -\cos 45^\circ\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

Example 3

Determine the exact value of:

a $\sin\left(\frac{11\pi}{6}\right)$

b $\cos\left(\frac{-45\pi}{6}\right)$

Solution

a
$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{\pi}{6}\right)\\ &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

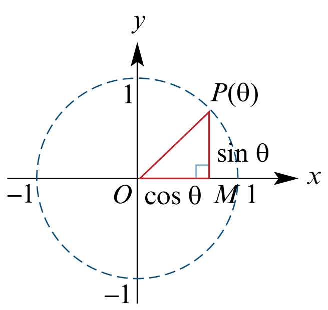
b
$$\begin{aligned}\cos\left(\frac{-45\pi}{6}\right) &= \cos(-7\tfrac{1}{2} \times \pi)\\ &= \cos\left(\frac{\pi}{2}\right)\\ &= 0\end{aligned}$$

The Pythagorean identity

Let $P(\theta)$ be any point on the unit circle. Then Pythagoras' theorem gives

$$MP^2 + OM^2 = OP^2$$

$$\therefore \quad (\sin\theta)^2 + (\cos\theta)^2 = 1$$



Hence, for any value of θ, we have

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

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Example 4

If $\sin(x^\circ) = 0.3$ and $0 < x < 90$, determine:

a $\cos(x^\circ)$ **b** $\tan(x^\circ)$

Solution

a
$$\sin^2(x^\circ) + \cos^2(x^\circ) = 1$$
$$0.09 + \cos^2(x^\circ) = 1$$
$$\cos^2(x^\circ) = 0.91$$
$$\therefore \quad \cos(x^\circ) = \pm\sqrt{0.91}$$

Since $0 < x < 90$, this gives

$$\cos(x^\circ) = \sqrt{0.91} = \sqrt{\frac{91}{100}} = \frac{\sqrt{91}}{10}$$

b
$$\tan(x^\circ) = \frac{\sin(x^\circ)}{\cos(x^\circ)} = \frac{0.3}{\sqrt{0.91}}$$
$$= \frac{3}{\sqrt{91}}$$
$$= \frac{3\sqrt{91}}{91}$$

Solution of equations involving sine and cosine

If a trigonometric equation has a solution, then it will have a corresponding solution in each ‘cycle’ of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one ‘cycle’ of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

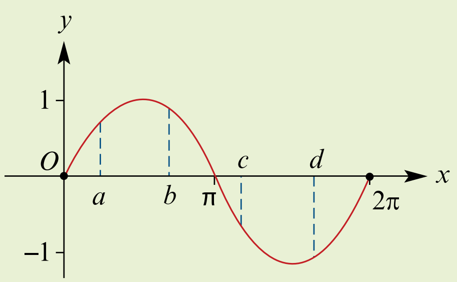
Example 5

The graph of $y = f(x)$ for

$$f(x) = \sin x, \quad x \in [0, 2\pi]$$

is shown.

For each pronumeral marked on the x-axis, determine the other x-value which has the same y-value.



Solution

For $x = a$, the other value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

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Example 6

Solve the equation

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \quad \text{for } x \in [0, 2\pi]$$

Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \le x \le 2\pi \quad &\Leftrightarrow \quad 0 \le 2x \le 4\pi \\ &\Leftrightarrow \quad \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3} \\ &\Leftrightarrow \quad \frac{\pi}{3} \le \theta \le \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin\theta = \frac{1}{2}$ for $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$.

Consider $\sin\theta = \frac{1}{2}$.

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \dots$$

The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for θ.

For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$:

$$\theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12}$$

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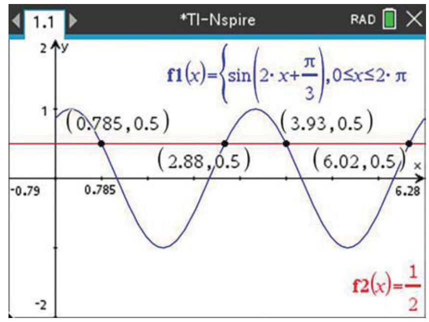
TI-84

Using the TI-Nspire CX non-CAS

- Ensure that the **Graphs** application is in radian mode (menu > **Settings**).
- Plot the graphs of:
  - $f1(x) = \sin\left(2x + \frac{\pi}{3}\right) \mid 0 \le x \le 2\pi$
  - $f2(x) = \frac{1}{2}$
- Use menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.

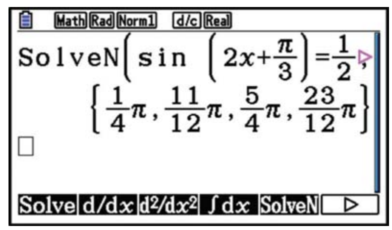


Using the Casio

Method 1: Using the numerical solver

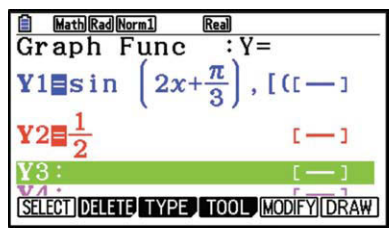
- Press MENU 1 to select **Run-Matrix** mode.
- Ensure that the angle setting is Radians (SHIFT MENU).
- Go to the numerical solver **SolveN** OPTN F4 F5.
- Complete the equation and domain by entering:

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}, x, 0, 2\pi\right)$$



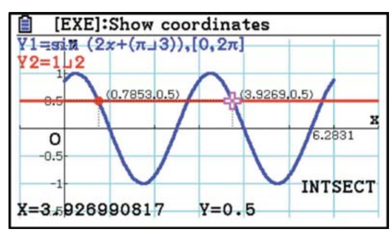
Method 2: Using Graph mode

- Press MENU 5 to select **Graph** mode.
- Enter the two functions:
  - $Y1 = \sin\left(2x + \frac{\pi}{3}\right), [0, 2\pi]$
  - $Y2 = \frac{1}{2}$



- Select **Draw** F6 to view the graph. Adjust the View Window SHIFT F3 if required.
- To determine the intersection points, go to the **G-Solve** menu SHIFT F5 and select **Intersection** F5.

  Use the cursor key ► to determine the next point.



Transformations of the graphs of sine and cosine

The graphs of functions with rules of the form

$$f(x) = a\sin(nx + \varepsilon) + b \qquad \text{and} \qquad f(x) = a\cos(nx + \varepsilon) + b$$

can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.

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Example 7

Sketch the graph of the function

$$h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1, \quad x \in [0, 2\pi]$$

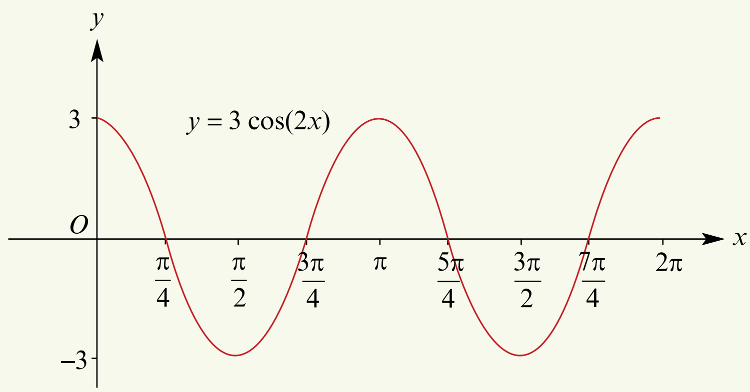
Solution

We can write $h(x) = 3\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 1$.

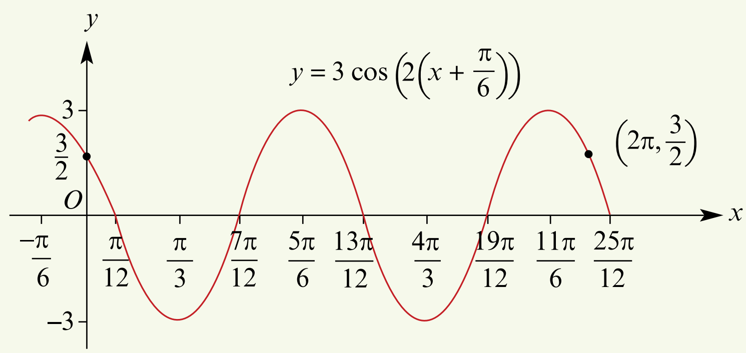
The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

- a dilation of factor $\frac{1}{2}$ from the y-axis
- a dilation of factor 3 from the x-axis
- a translation of $\frac{\pi}{6}$ units in the negative direction of the x-axis
- a translation of 1 unit in the positive direction of the y-axis.

First apply the two dilations to the graph of $y = \cos x$.



Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the x-axis.

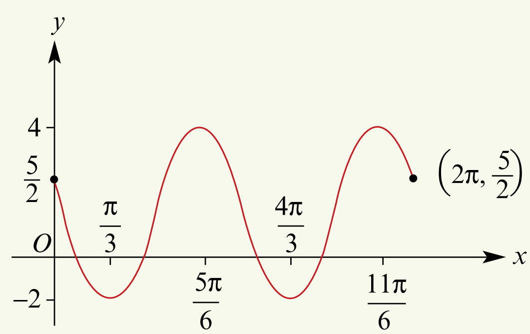


Apply the final translation and restrict the graph to the required domain.

The x-axis intercepts can be found by solving the equation

$3\cos\left(2x + \frac{\pi}{3}\right) + 1 = 0$ for $0 \le x \le 2\pi$.

The approximate values of these are $x = 0.43$ or $x = 1.66$ or $x = 3.57$ or $x = 4.80$



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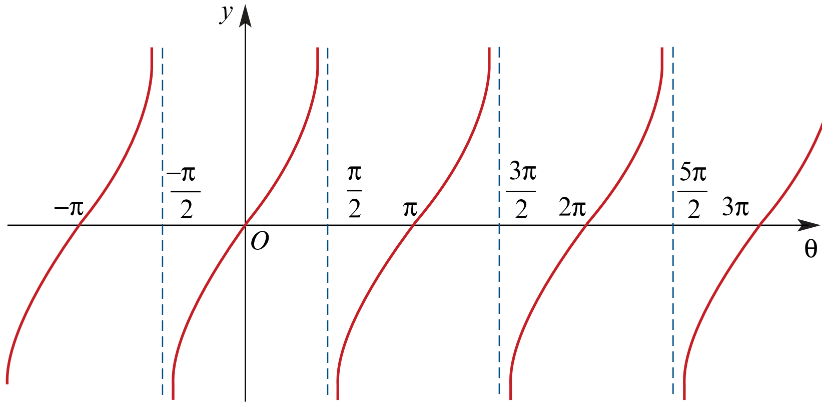
The graph of the tangent function

A sketch of the graph of $y = \tan\theta$ is shown below.



Notes:

- The domain of tan is $\mathbb{R} \setminus \left\{ \frac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}$.
- The range of tan is $\mathbb{R}$.
- The graph repeats itself every π units, i.e. the period of tan is π.
- The vertical asymptotes have equations $\theta = \frac{(2k+1)\pi}{2}$, where $k \in \mathbb{Z}$.

Symmetry properties of tangent

The following results are obtained from the definition of tan:

$$\tan(\pi - \theta) = -\tan\theta \qquad \tan(2\pi - \theta) = -\tan\theta$$

$$\tan(\pi + \theta) = \tan\theta \qquad \tan(-\theta) = -\tan\theta$$

Example 8

Determine the exact value of:

a $\tan 330°$

b $\tan\left(\frac{4\pi}{3}\right)$

Solution

a
$$\begin{aligned} \tan 330° &= \tan(360° - 30°) \\ &= -\tan 30° \\ &= -\frac{1}{\sqrt{3}} \end{aligned}$$

b
$$\begin{aligned} \tan\left(\frac{4\pi}{3}\right) &= \tan\left(\pi + \frac{\pi}{3}\right) \\ &= \tan\left(\frac{\pi}{3}\right) \\ &= \sqrt{3} \end{aligned}$$

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Solution of equations involving tangent

The method here is similar to that used for solving equations involving sin and cos, except that only one solution needs to be found then all other solutions are one period length apart.

Example 9

Solve the following equations:

a $\tan x = -1$ for $x \in [0, 4\pi]$

b $\tan(2x - \pi) = \sqrt{3}$ for $x \in [-\pi, \pi]$

Solution

a $\tan x = -1$

Now $\tan\left(\frac{3\pi}{4}\right) = -1$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{3\pi}{4} + \pi \text{ or } \frac{3\pi}{4} + 2\pi \text{ or } \frac{3\pi}{4} + 3\pi$$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \text{ or } \frac{11\pi}{4} \text{ or } \frac{15\pi}{4}$$

b Let $\theta = 2x - \pi$. Then

$$\begin{aligned} -\pi \le x \le \pi \quad &\Leftrightarrow \quad -2\pi \le 2x \le 2\pi \\ &\Leftrightarrow \quad -3\pi \le 2x - \pi \le \pi \\ &\Leftrightarrow \quad -3\pi \le \theta \le \pi \end{aligned}$$

To solve $\tan(2x - \pi) = \sqrt{3}$, we first solve $\tan\theta = \sqrt{3}$.

$$\theta = \frac{\pi}{3} \text{ or } \frac{\pi}{3} - \pi \text{ or } \frac{\pi}{3} - 2\pi \text{ or } \frac{\pi}{3} - 3\pi$$

$$\therefore \quad \theta = \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3} \text{ or } -\frac{8\pi}{3}$$

$$\therefore \quad 2x - \pi = \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3} \text{ or } -\frac{8\pi}{3}$$

$$\therefore \quad 2x = \frac{4\pi}{3} \text{ or } \frac{\pi}{3} \text{ or } -\frac{2\pi}{3} \text{ or } -\frac{5\pi}{3}$$

$$\therefore \quad x = \frac{2\pi}{3} \text{ or } \frac{\pi}{6} \text{ or } -\frac{\pi}{3} \text{ or } -\frac{5\pi}{6}$$

Skill-sheet

Exercise 1A

Example 1

1 **a** Convert the following angles from degrees to exact values in radians:

i 720° **ii** 540° **iii** −450° **iv** 15° **v** −10° **vi** −315°

b Convert the following angles from radians to degrees:

i $\frac{5\pi}{4}$ **ii** $-\frac{2\pi}{3}$ **iii** $\frac{7\pi}{12}$ **iv** $-\frac{11\pi}{6}$ **v** $\frac{13\pi}{9}$ **vi** $-\frac{11\pi}{12}$

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2 Perform the correct conversion on each of the following angles, giving the answer correct to two decimal places.

a Convert from degrees to radians:

i $7°$ **ii** $-100°$ **iii** $-25°$ **iv** $51°$ **v** $206°$ **vi** $-410°$

b Convert from radians to degrees:

i 1.7^c **ii** -0.87^c **iii** 2.8^c **iv** 0.1^c **v** -3^c **vi** -8.9^c

Example 2

3 Determine the exact value of each of the following:

a $\sin(135°)$ **b** $\cos(-300°)$ **c** $\sin(480°)$

d $\cos(240°)$ **e** $\sin(-225°)$ **f** $\sin(420°)$

Example 3

4 Determine the exact value of each of the following:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\cos\left(-\frac{\pi}{3}\right)$

d $\cos\left(\frac{5\pi}{4}\right)$ **e** $\cos\left(\frac{9\pi}{4}\right)$ **f** $\sin\left(\frac{11\pi}{3}\right)$

g $\cos\left(\frac{31\pi}{6}\right)$ **h** $\cos\left(\frac{29\pi}{6}\right)$ **i** $\sin\left(-\frac{23\pi}{6}\right)$

Example 4

5 If $\sin(x°) = 0.5$ and $90 < x < 180$, determine:

a $\cos(x°)$ **b** $\tan(x°)$

6 If $\cos(x°) = -0.7$ and $180 < x < 270$, determine:

a $\sin(x°)$ **b** $\tan(x°)$

7 If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, determine:

a $\cos x$ **b** $\tan x$

8 If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, determine:

a $\cos x$ **b** $\tan x$

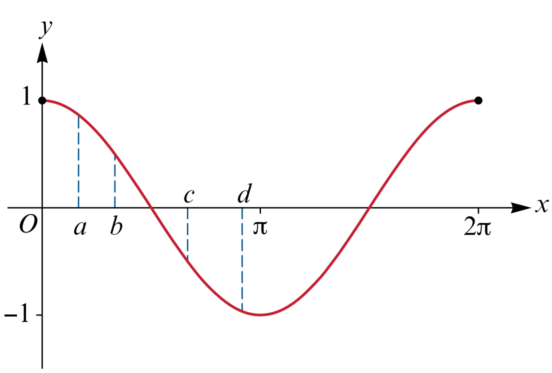
Example 5

9 The graph of $y = f(x)$ for

$$f(x) = \cos x, \quad x \in [0, 2\pi]$$

is shown.

For each pronumeral marked on the x-axis, determine the other x-value which has the same y-value.



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Example 6 **10** Solve each of the following for $x \in [0, 2\pi]$:

a $\sin x = -\dfrac{\sqrt{3}}{2}$

b $\sin(2x) = -\dfrac{\sqrt{3}}{2}$

c $2\cos(2x) = -1$

d $\sin\left(x + \dfrac{\pi}{3}\right) = -\dfrac{1}{2}$

e $2\cos\left(2\left(x + \dfrac{\pi}{3}\right)\right) = -1$

f $2\sin\left(2x + \dfrac{\pi}{3}\right) = -\sqrt{3}$

Example 7 **11** Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin(2x),\ \ x \in [0, 2\pi]$

b $f(x) = \cos\left(x + \dfrac{\pi}{3}\right),\ \ x \in \left[\dfrac{-\pi}{3}, \pi\right]$

c $f(x) = \cos\left(2\left(x + \dfrac{\pi}{3}\right)\right),\ \ x \in [0, \pi]$

d $f(x) = 2\sin(3x) + 1,\ \ x \in [0, \pi]$

e $f(x) = 2\sin\left(x - \dfrac{\pi}{4}\right) + \sqrt{3},\ \ x \in [0, 2\pi]$

Example 8 **12** Determine the exact value of each of the following:

a $\tan\left(\dfrac{5\pi}{4}\right)$ **b** $\tan\left(-\dfrac{2\pi}{3}\right)$ **c** $\tan\left(-\dfrac{29\pi}{6}\right)$ **d** $\tan 240°$

13 If $\tan x = \dfrac{1}{4}$ and $\pi \le x \le \dfrac{3\pi}{2}$, determine the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

14 If $\tan x = -\dfrac{\sqrt{3}}{2}$ and $\dfrac{\pi}{2} \le x \le \pi$, determine the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

Example 9 **15** Solve each of the following for $x \in [0, 2\pi]$:

a $\tan x = -\sqrt{3}$

b $\tan\left(3x - \dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{3}$

c $2\tan\left(\dfrac{x}{2}\right) + 2 = 0$

d $3\tan\left(\dfrac{\pi}{2} + 2x\right) = -3$

16 Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

a $f(x) = \tan(2x)$

b $f(x) = \tan\left(x - \dfrac{\pi}{3}\right)$

c $f(x) = 2\tan\left(2x + \dfrac{\pi}{3}\right)$

d $f(x) = 2\tan\left(2x + \dfrac{\pi}{3}\right) - 2$

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1B The reciprocal trigonometric functions

Learning intentions

- To define the reciprocal trigonometric functions.
- To be able to sketch the graphs of these functions.

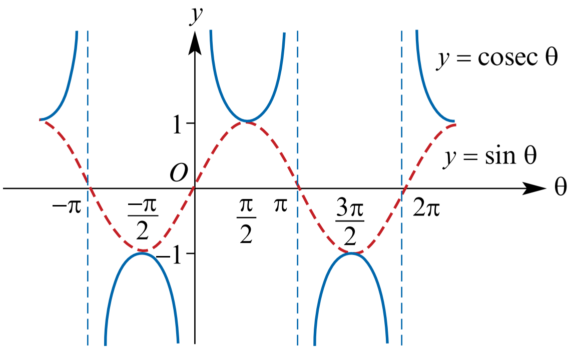
The cosecant function: $y = \operatorname{cosec} \theta$

The cosecant function is defined by

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

provided $\sin \theta \neq 0$.

The graphs of $y = \operatorname{cosec} \theta$ and $y = \sin \theta$ are shown here on the same set of axes.



- **Domain** As $\sin \theta = 0$ when $\theta = n\pi$, $n \in \mathbb{Z}$, the domain of $y = \operatorname{cosec} \theta$ is $\mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$.
- **Range** The range of $y = \sin \theta$ is $[-1, 1]$, so the range of $y = \operatorname{cosec} \theta$ is $\mathbb{R} \setminus (-1, 1)$.
- **Turning points** The graph of $y = \sin \theta$ has turning points at $\theta = \dfrac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$, as does the graph of $y = \operatorname{cosec} \theta$.
- **Asymptotes** The graph of $y = \operatorname{cosec} \theta$ has vertical asymptotes with equations $\theta = n\pi$, for $n \in \mathbb{Z}$.

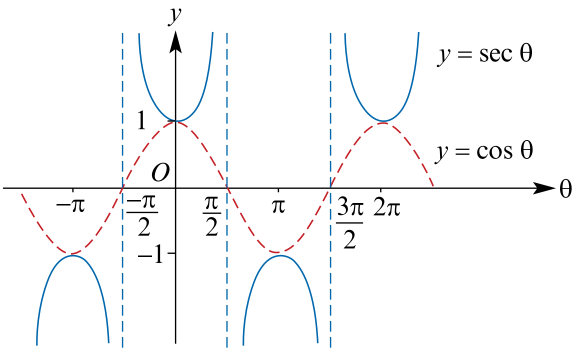
The secant function: $y = \sec \theta$

The secant function is defined by

$$\sec \theta = \frac{1}{\cos \theta}$$

provided $\cos \theta \neq 0$.

The graphs of $y = \sec \theta$ and $y = \cos \theta$ are shown here on the same set of axes.



- **Domain** The domain of $y = \sec \theta$ is $\mathbb{R} \setminus \left\{ \dfrac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$.
- **Range** The range of $y = \sec \theta$ is $\mathbb{R} \setminus (-1, 1)$.
- **Turning points** The graph of $y = \sec \theta$ has turning points at $\theta = n\pi$, for $n \in \mathbb{Z}$.

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- **Asymptotes** The vertical asymptotes have equations $\theta = \frac{(2n+1)\pi}{2}$, for $n \in \mathbb{Z}$.

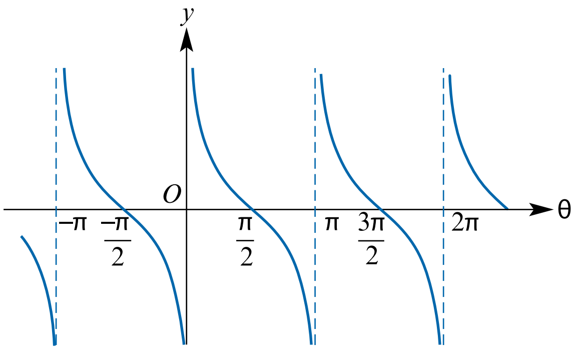
Since the graph of $y = \cos\theta$ is a translation of the graph of $y = \sin\theta$, the graph of $y = \sec\theta$ is a translation of the graph of $y = \operatorname{cosec}\theta$, by $\frac{\pi}{2}$ units in the negative direction of the θ-axis.

The cotangent function: $y = \cot\theta$

The cotangent function is defined by

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

provided $\sin\theta \neq 0$.



Using the complementary properties of sine and cosine, we have

$$\begin{aligned}\cot\theta &= \tan\left(\frac{\pi}{2} - \theta\right)\\ &= -\tan\left(\pi - \left(\frac{\pi}{2} - \theta\right)\right)\\ &= -\tan\left(\theta + \frac{\pi}{2}\right)\end{aligned}$$

Therefore the graph of $y = \cot\theta$, shown above, is obtained from the graph of $y = \tan\theta$ by a translation of $\frac{\pi}{2}$ units in the negative direction of the θ-axis and then a reflection in the θ-axis.

- **Domain** As $\sin\theta = 0$ when $\theta = n\pi$, $n \in \mathbb{Z}$, the domain of $y = \cot\theta$ is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$.
- **Range** The range of $y = \cot\theta$ is $\mathbb{R}$.
- **Asymptotes** The vertical asymptotes have equations $\theta = n\pi$, for $n \in \mathbb{Z}$.

Note: $\cot\theta = \frac{1}{\tan\theta}$ provided $\cos\theta \neq 0$.

Example 10

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}(2x)$

b $y = \sec\left(x + \frac{\pi}{3}\right)$

c $y = \cot\left(x - \frac{\pi}{4}\right)$

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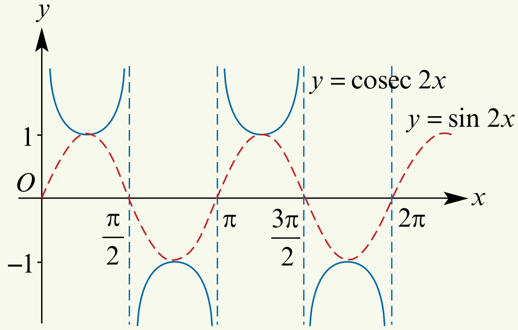
Solution

a The graph of $y = \operatorname{cosec}(2x)$ is obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

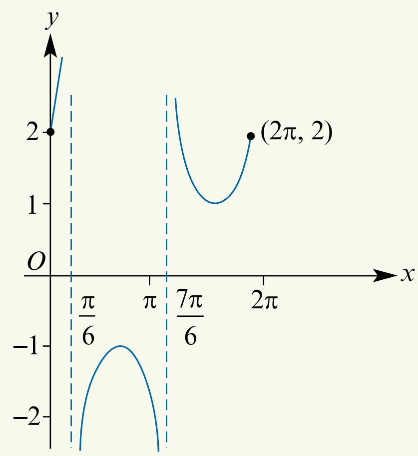
The graph of $y = \sin(2x)$ is also shown.



b The graph of $y = \sec\left(x + \dfrac{\pi}{3}\right)$ is obtained from the graph of $y = \sec x$ by a translation of $\dfrac{\pi}{3}$ units in the negative direction of the x-axis.

The y-axis intercept is $\sec\left(\dfrac{\pi}{3}\right) = 2$.

The asymptotes are $x = \dfrac{\pi}{6}$ and $x = \dfrac{7\pi}{6}$.

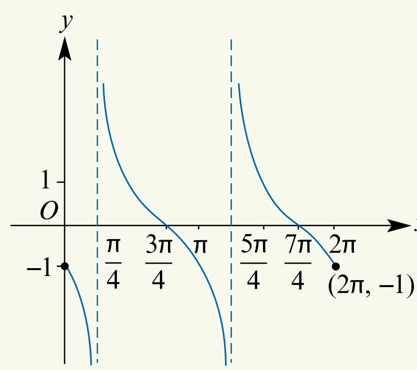


c The graph of $y = \cot\left(x - \dfrac{\pi}{4}\right)$ is obtained from the graph of $y = \cot x$ by a translation of $\dfrac{\pi}{4}$ units in the positive direction of the x-axis.

The y-axis intercept is $\cot\left(-\dfrac{\pi}{4}\right) = -1$.

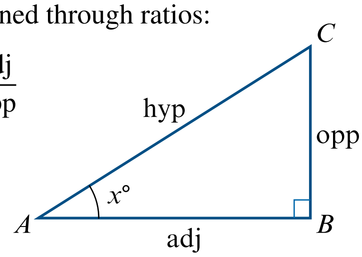
The asymptotes are $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$.

The x-axis intercepts are $\dfrac{3\pi}{4}$ and $\dfrac{7\pi}{4}$.



For right-angled triangles, the reciprocal functions can be defined through ratios:

$$\operatorname{cosec}(x^\circ) = \frac{\text{hyp}}{\text{opp}} \qquad \sec(x^\circ) = \frac{\text{hyp}}{\text{adj}} \qquad \cot(x^\circ) = \frac{\text{adj}}{\text{opp}}$$



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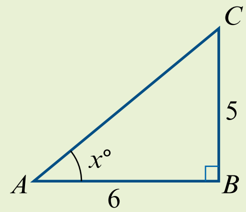
Example 11

In triangle ABC, $\angle ABC = 90°$, $\angle CAB = x°$, $AB = 6$ cm and $BC = 5$ cm. Determine:

a AC

b the trigonometric ratios related to $x°$



Solution

a By Pythagoras' theorem,

$$AC^2 = 5^2 + 6^2 = 61$$

$$\therefore\ AC = \sqrt{61} \text{ cm}$$

b $\sin(x°) = \dfrac{5}{\sqrt{61}}$ $\quad \cos(x°) = \dfrac{6}{\sqrt{61}}$ $\quad \tan(x°) = \dfrac{5}{6}$

$\operatorname{cosec}(x°) = \dfrac{\sqrt{61}}{5}$ $\quad \sec(x°) = \dfrac{\sqrt{61}}{6}$ $\quad \cot(x°) = \dfrac{6}{5}$

Useful properties

The symmetry properties established for sine, cosine and tangent can be used to establish the following results:

$$\sec(\pi - \theta) = -\sec\theta \qquad \operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta \qquad \cot(\pi - \theta) = -\cot\theta$$

$$\sec(\pi + \theta) = -\sec\theta \qquad \operatorname{cosec}(\pi + \theta) = -\operatorname{cosec}\theta \qquad \cot(\pi + \theta) = \cot\theta$$

$$\sec(2\pi - \theta) = \sec\theta \qquad \operatorname{cosec}(2\pi - \theta) = -\operatorname{cosec}\theta \qquad \cot(2\pi - \theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta \qquad \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta \qquad \cot(-\theta) = -\cot\theta$$

The complementary properties are also useful:

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec}\theta \qquad \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

Example 12

Determine the exact value of each of the following:

a $\sec\left(\dfrac{11\pi}{4}\right)$

b $\operatorname{cosec}\left(-\dfrac{23\pi}{4}\right)$

Solution

a
$$\begin{aligned}
\sec\left(\frac{11\pi}{4}\right) &= \sec\left(2\pi + \frac{3\pi}{4}\right) \\
&= \sec\left(\frac{3\pi}{4}\right) \\
&= \frac{1}{\cos(\frac{3\pi}{4})} \\
&= \frac{1}{-\frac{1}{\sqrt{2}}} \\
&= -\sqrt{2}
\end{aligned}$$

b
$$\begin{aligned}
\operatorname{cosec}\left(-\frac{23\pi}{4}\right) &= \operatorname{cosec}\left(-6\pi + \frac{\pi}{4}\right) \\
&= \operatorname{cosec}\left(\frac{\pi}{4}\right) \\
&= \frac{1}{\sin(\frac{\pi}{4})} \\
&= \frac{1}{\frac{1}{\sqrt{2}}} \\
&= \sqrt{2}
\end{aligned}$$

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Two new identities

The Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ holds for all values of θ.

From this identity, we can derive the following two additional identities:

$\tan^2\theta + 1 = \sec^2\theta$ provided $\cos\theta \neq 0$

$\cot^2\theta + 1 = \mathrm{cosec}^2\theta$ provided $\sin\theta \neq 0$

Proof The first identity is obtained by dividing each term in the Pythagorean identity by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \quad \tan^2\theta + 1 = \sec^2\theta$$

The derivation of the second identity is left as an exercise.

Example 13

Simplify the expression

$$\frac{\cos x - \cos^3 x}{\cot x}$$

Solution

$$\begin{aligned}\frac{\cos x - \cos^3 x}{\cot x} &= \frac{\cos x \cdot (1 - \cos^2 x)}{\cot x} \\ &= \cos x \cdot \sin^2 x \cdot \frac{\sin x}{\cos x} \\ &= \sin^3 x\end{aligned}$$

Example 14

If $\tan x = 2$ and $x \in \left[0, \frac{\pi}{2}\right]$, determine:

a $\sec x$ **b** $\cos x$ **c** $\sin x$ **d** $\mathrm{cosec}\, x$

Solution

a $$\begin{aligned}\sec^2 x &= \tan^2 x + 1 \\ &= 4 + 1\end{aligned}$$

$\therefore \ \sec x = \pm\sqrt{5}$

Since $x \in \left(0, \frac{\pi}{2}\right)$, we have $\sec x = \sqrt{5}$.

b $\cos x = \dfrac{1}{\sec x} = \dfrac{\sqrt{5}}{5}$

c $\sin x = \tan x \cdot \cos x = \dfrac{2\sqrt{5}}{5}$

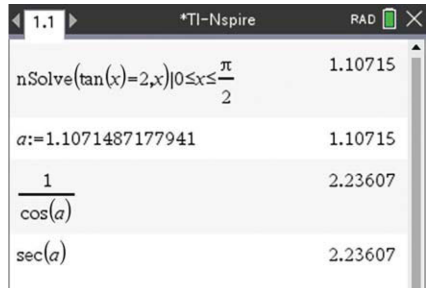
d $\mathrm{cosec}\, x = \dfrac{1}{\sin x} = \dfrac{\sqrt{5}}{2}$

TI-84

Using the TI-Nspire CX non-CAS

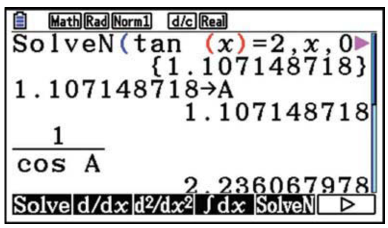
- Use menu > **Algebra** > **Numerical Solve** and complete as shown.
- Assign (ctrl ⁝) or store (ctrl var) the answer as the variable a to obtain the results.

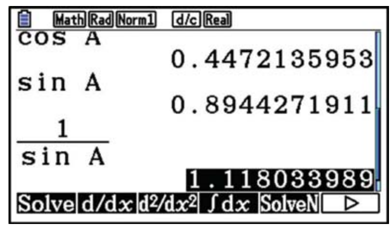
Hint: Use the approximate answers given by the calculator to check your exact answers found by hand.



Using the Casio

- In **Run-Matrix** mode, select the numerical solver **SolveN** OPTN F4 F5.
- Complete the equation and domain by entering: $\tan(x) = 2, x, 0, \frac{\pi}{2})$
- Assign the result to the variable A. Then obtain the required values as shown.





Exercise 1B

SF

Example 10

1 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ **b** $y = \sec\left(x - \frac{\pi}{6}\right)$ **c** $y = \cot\left(x + \frac{\pi}{3}\right)$

2 Sketch the graph of each of the following over the interval $[0, \pi]$:

a $y = \sec(2x)$ **b** $y = \operatorname{cosec}(3x)$ **c** $y = \cot(4x)$

d $y = \operatorname{cosec}\left(2x + \frac{\pi}{2}\right)$ **e** $y = \sec(2x + \pi)$ **f** $y = \cot\left(2x - \frac{\pi}{3}\right)$

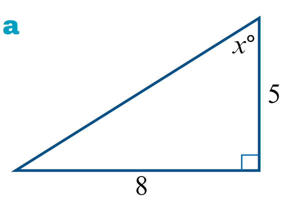
3 Sketch the graph of each of the following over the interval $[-\pi, \pi]$:

a $y = \sec\left(2x - \frac{\pi}{2}\right)$ **b** $y = \operatorname{cosec}\left(2x + \frac{\pi}{3}\right)$ **c** $y = \cot\left(2x - \frac{2\pi}{3}\right)$

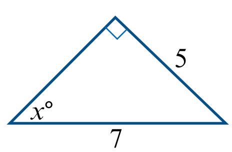
Example 11

4 Determine $\cot(x^\circ)$, $\sec(x^\circ)$ and $\operatorname{cosec}(x^\circ)$ for each of the following triangles:

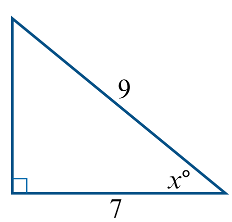
a



b



c



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SF

Example 12

5 Determine the exact value of each of the following:

a $\operatorname{cosec}\left(\frac{\pi}{6}\right)$ **b** $\sec\left(\frac{\pi}{4}\right)$ **c** $\cot\left(-\frac{\pi}{6}\right)$ **d** $\sec\left(-\frac{\pi}{3}\right)$

e $\operatorname{cosec}\left(\frac{3\pi}{4}\right)$ **f** $\cot\left(\frac{9\pi}{4}\right)$ **g** $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ **h** $\sec\left(-\frac{7\pi}{3}\right)$

Example 13

6 Simplify each of the following expressions:

a $\sec^2 x - \tan^2 x$ **b** $\cot^2 x - \operatorname{cosec}^2 x$ **c** $\dfrac{\tan^2 x + 1}{\tan^2 x}$

d $\dfrac{\sin^2 x}{\cos x} + \cos x$ **e** $\sin^4 x - \cos^4 x$ **f** $\tan^3 x + \tan x$

Example 14

7 If $\tan x = -4$ and $x \in \left(-\frac{\pi}{2}, 0\right)$, determine:

a $\sec x$ **b** $\cos x$ **c** $\operatorname{cosec} x$

8 If $\cot x = 3$ and $x \in \left(\pi, \frac{3\pi}{2}\right)$, determine:

a $\operatorname{cosec} x$ **b** $\sin x$ **c** $\sec x$

9 If $\sec x = 10$ and $x \in \left(-\frac{\pi}{2}, 0\right)$, determine:

a $\tan x$ **b** $\sin x$

10 If $\operatorname{cosec} x = -6$ and $x \in \left(\frac{3\pi}{2}, 2\pi\right)$, determine:

a $\cot x$ **b** $\cos x$

11 If $\sin x^\circ = 0.5$ and $90 < x < 180$, determine:

a $\cos x^\circ$ **b** $\cot x^\circ$ **c** $\operatorname{cosec} x^\circ$

12 If $\operatorname{cosec} x^\circ = -3$ and $180 < x < 270$, determine:

a $\sin x^\circ$ **b** $\cos x^\circ$ **c** $\sec x^\circ$

13 If $\cos x^\circ = -0.7$ and $0 < x < 180$, determine:

a $\sin x^\circ$ **b** $\tan x^\circ$ **c** $\cot x^\circ$

14 If $\sec x^\circ = 5$ and $180 < x < 360$, determine:

a $\cos x^\circ$ **b** $\sin x^\circ$ **c** $\cot x^\circ$

CF

15 Simplify each of the following expressions:

a $\sec^2\theta + \operatorname{cosec}^2\theta - \sec^2\theta\operatorname{cosec}^2\theta$ **b** $(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta)$

c $(1 - \cos^2\theta)(1 + \cot^2\theta)$ **d** $\dfrac{\sec^2\theta - \operatorname{cosec}^2\theta}{\tan^2\theta - \cot^2\theta}$

16 Let $x = \sec\theta - \tan\theta$. Prove that $x + \frac{1}{x} = 2\sec\theta$ and also determine a simple expression for $x - \frac{1}{x}$ in terms of θ.

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1C Angle sum and difference identities

Learning intentions

- To be able to use the angle sum and difference identities.
- To be able to use double-angle identities.

The following identities were proved in Specialist Mathematics Units 1 & 2.

Angle sum and difference identities

- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- $\cos(A-B) = \cos A \cos B + \sin A \sin B$
- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Example 15

a Use $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$ to evaluate $\sin\left(\frac{5\pi}{12}\right)$.

b Use $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ to evaluate $\cos\left(\frac{\pi}{12}\right)$.

Solution

a
$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3})\end{aligned}$$

b
$$\begin{aligned}\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3})\end{aligned}$$

Example 16

Suppose $\sin x = 0.2$ and $\cos y = -0.4$, where $x \in \left[0, \frac{\pi}{2}\right]$ and $y \in \left[\pi, \frac{3\pi}{2}\right]$.

a Determine $\cos x$ and $\sin y$.

b Hence, determine $\sin(x+y)$.

Solution

a
$$\begin{aligned}\cos x &= \pm\sqrt{1-0.2^2} && \text{as } \sin x = 0.2 \\ &= \pm\sqrt{0.96} \\ \therefore \cos x &= \sqrt{0.96} && \text{as } x \in \left[0, \frac{\pi}{2}\right] \\ &= \frac{2\sqrt{6}}{5}\end{aligned}$$

$$\begin{aligned}\sin y &= \pm\sqrt{1-(-0.4)^2} && \text{as } \cos y = -0.4 \\ &= \pm\sqrt{0.84} \\ \therefore \sin y &= -\sqrt{0.84} && \text{as } y \in \left[\pi, \frac{3\pi}{2}\right] \\ &= -\frac{\sqrt{21}}{5}\end{aligned}$$

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b Now using the angle sum identity for the sine function,

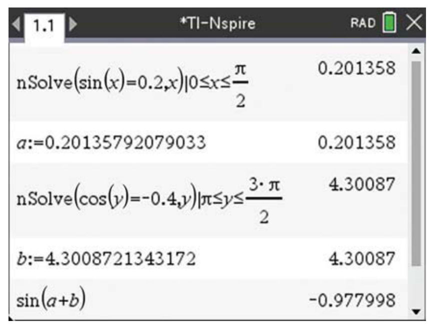
$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ &= 0.2\times(-0.4) + \frac{2\sqrt{6}}{5}\times\left(-\frac{\sqrt{21}}{5}\right)\\ &= -0.08 - \frac{2}{25}\times 3\sqrt{14}\\ &= -\frac{2}{25}(1+3\sqrt{14})\end{aligned}$$

TI-84

Using the TI-Nspire CX non-CAS

- First solve $\sin(x) = 0.2$ for $0 \le x \le \frac{\pi}{2}$.
- Assign the result to a.
- Then solve $\cos(y) = -0.4$ for $\pi \le y \le \frac{3\pi}{2}$.
- Assign the result to b.
- Evaluate $\sin(a + b)$.

Hint: Use this approximate answer to check your exact answer found by hand.

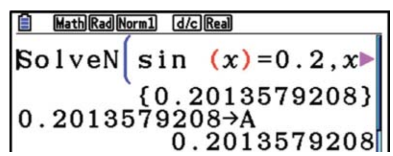


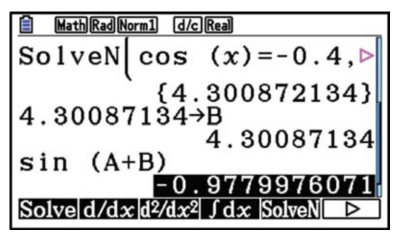
Using the Casio

- In **Run-Matrix** mode, select the numerical solver **SolveN** (OPTN) (F4) (F5) and complete as:

  $\sin(x) = 0.2, x, 0, \frac{\pi}{2})$
- Assign the result to the variable A.
- Select the numerical solver and complete as:

  $\cos(x) = -0.4, x, \pi, \frac{3\pi}{2})$
- Assign the result to the variable B.
- Evaluate $\sin(A + B)$.





Double-angle identities

Double-angle identities

- $\cos(2A) = \cos^2 A - \sin^2 A$
  $= 1 - 2\sin^2 A$
  $= 2\cos^2 A - 1$
- $\sin(2A) = 2\sin A\cos A$

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Proof These identities can be derived from the angle sum identities. For example:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\therefore \quad \cos(A+A) = \cos A \cos A - \sin A \sin A$$

$$\therefore \quad \cos(2A) = \cos^2 A - \sin^2 A$$

The two other expressions for $\cos(2A)$ are obtained using the Pythagorean identity:

$$\begin{aligned} \cos^2 A - \sin^2 A &= (1 - \sin^2 A) - \sin^2 A \\ &= 1 - 2\sin^2 A \end{aligned}$$

and

$$\begin{aligned} \cos^2 A - \sin^2 A &= \cos^2 A - (1 - \cos^2 A) \\ &= 2\cos^2 A - 1 \end{aligned}$$

Example 17

If $\sin\alpha = 0.6$ and $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, determine $\sin(2\alpha)$.

Solution

$$\begin{aligned} \cos\alpha &= \pm\sqrt{1-0.6^2} && \text{since } \sin\alpha = 0.6 \\ &= \pm 0.8 \end{aligned}$$

$$\therefore \quad \cos\alpha = -0.8 \qquad \text{since } \alpha \in \left[\frac{\pi}{2}, \pi\right]$$

Hence

$$\begin{aligned} \sin(2\alpha) &= 2\sin\alpha\cos\alpha \\ &= 2 \times 0.6 \times (-0.8) \\ &= -0.96 \end{aligned}$$

Example 18

If $\cos\alpha = 0.7$ and $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right]$, determine $\sin\left(\frac{\alpha}{2}\right)$.

Solution

We use a double-angle identity:

$$\cos(2x) = 1 - 2\sin^2 x$$

$$\therefore \quad \cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\begin{aligned} 2\sin^2\left(\frac{\alpha}{2}\right) &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{0.15} = \pm\frac{\sqrt{15}}{10}$$

Since $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right]$, we have $\frac{\alpha}{2} \in \left[\frac{3\pi}{4}, \pi\right]$, so $\sin\left(\frac{\alpha}{2}\right)$ is positive. Hence $\sin\left(\frac{\alpha}{2}\right) = \frac{\sqrt{15}}{10}$.

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Skill-sheet

Exercise 1C

SF

Example 15

1 Use the angle sum and difference identities to determine the exact value of each of the following:

a $\sin\left(\frac{\pi}{12}\right)$ **b** $\cos\left(\frac{7\pi}{12}\right)$

2 Use the angle sum and difference identities to expand each of the following:

a $\sin(2x - 5y)$ **b** $\cos(x^2 + y)$

3 Simplify each of the following:

a $\sin(x)\cos(2y) - \cos(x)\sin(2y)$

b $\cos(3x)\cos(2x) + \sin(3x)\sin(2x)$

c $\sin(A + B)\cos(A - B) + \cos(A + B)\sin(A - B)$

d $\cos(y)\cos(-2y) - \sin(y)\sin(-2y)$

4 **a** Expand $\sin(x + 2x)$. **b** Hence express $\sin(3x)$ in terms of $\sin x$.

5 **a** Expand $\cos(x + 2x)$. **b** Hence express $\cos(3x)$ in terms of $\cos x$.

Example 16

6 If $\sin x = 0.6$ and $\tan y = 2.4$, where $x \in \left[\frac{\pi}{2}, \pi\right]$ and $y \in \left[0, \frac{\pi}{2}\right]$, determine the exact value of each of the following:

a $\cos x$ **b** $\sec y$ **c** $\cos y$

d $\sin y$ **e** $\cos(x - y)$ **f** $\sin(x - y)$

7 If $\cos x = -0.7$ and $\sin y = 0.4$, where $x \in \left[\pi, \frac{3\pi}{2}\right]$ and $y \in \left[0, \frac{\pi}{2}\right]$, determine the value of each of the following, correct to two decimal places:

a $\sin x$ **b** $\cos y$ **c** $\sin(x + y)$ **d** $\cos(x + y)$

CF

8 Simplify each of the following:

a $\frac{1}{2}\sin x\cos x$ **b** $\sin^2 x - \cos^2 x$ **c** $\dfrac{\sin^4 x - \cos^4 x}{\cos(2x)}$

d $\dfrac{4\sin^3 x - 2\sin x}{\cos x\cos(2x)}$ **e** $\dfrac{4\sin^2 x - 4\sin^4 x}{\sin(2x)}$

Example 17

9 If $\sin x = -0.8$ and $x \in \left[\pi, \frac{3\pi}{2}\right]$, determine:

a $\sin(2x)$ **b** $\cos(2x)$ **c** $\tan(2x)$

Example 18

10 If $\sin x = -0.75$ and $x \in \left[\pi, \frac{3\pi}{2}\right]$, determine correct to two decimal places:

a $\cos x$ **b** $\sin(\frac{1}{2}x)$

11 If $\cos x = 0.9$ and $x \in \left[0, \frac{\pi}{2}\right]$, determine $\cos(\frac{1}{2}x)$ correct to two decimal places.

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1D The inverse trigonometric functions

Learning intentions

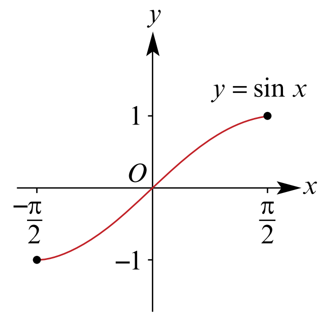
- ▶ To define, use and sketch the inverse trigonometric functions.

By restricting the domains of the trigonometric functions, we can define inverse functions. We restrict the domain so that there is only one x-value for each possible y-value.

The inverse sine function: $y = \sin^{-1} x$

Restricting the sine function

The domain of the restricted sine function is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. There is only one x-value for each possible y-value.



Note: Other intervals (defined through consecutive turning points of the graph) could have been used for the restricted domain, but this is the convention.

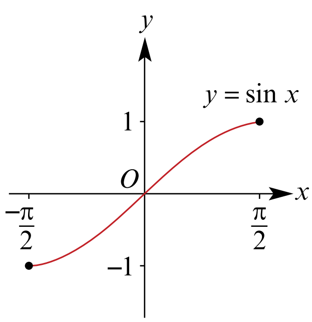
Defining the inverse function

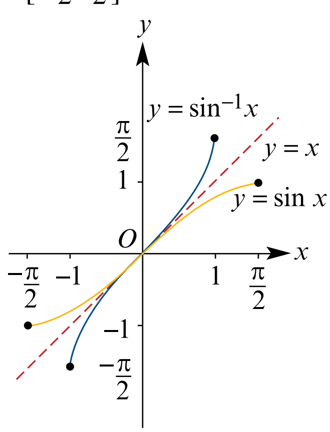
The inverse of the restricted sine function is usually denoted by $\sin^{-1}$ or arcsin.

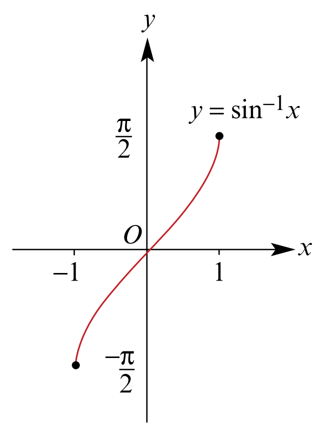
Inverse sine function

$$\sin^{-1} x = y \quad \text{if} \quad \sin y = x, \quad \text{for } x \in [-1, 1] \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Reflect the graph of $y = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in the line $y = x$ to give the graph of $y = \sin^{-1} x$.







- **Domain** Domain of $\sin^{-1}$ = range of restricted sine function = $[-1, 1]$
- **Range** Range of $\sin^{-1}$ = domain of restricted sine function = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- **Inverse relationship**
  - $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$
  - $\sin^{-1}(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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The inverse cosine function: $y = \cos^{-1} x$

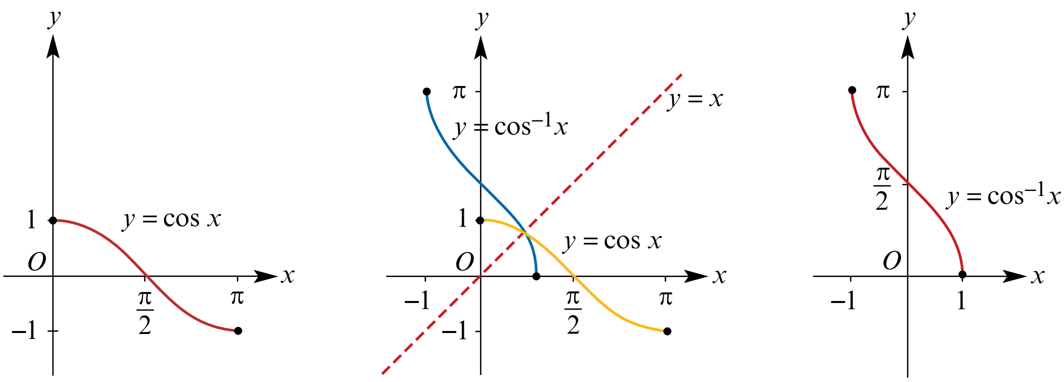
The standard domain for the restricted cosine function is $[0, \pi]$. This ensures that there is only one x-value for each possible y-value.

The inverse of the restricted cosine function is denoted by $\cos^{-1}$ or arccos.

Inverse cosine function

$$\cos^{-1} x = y \quad \text{if} \quad \cos y = x, \quad \text{for } x \in [-1, 1] \text{ and } y \in [0, \pi]$$

The graph of $y = \cos^{-1} x$ is obtained from the graph of $y = \cos x$, $x \in [0, \pi]$, through a reflection in the line $y = x$.



- **Domain** Domain of $\cos^{-1}$ = range of restricted cosine function = $[-1, 1]$
- **Range** Range of $\cos^{-1}$ = domain of restricted cosine function = $[0, \pi]$
- **Inverse relationship**
  - $\cos(\cos^{-1} x) = x$ for all $x \in [-1, 1]$
  - $\cos^{-1}(\cos x) = x$ for all $x \in [0, \pi]$

The inverse tangent function: $y = \tan^{-1} x$

The domain of the restricted tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This ensures that there is only one x-value for each possible y-value.

The inverse of the restricted tangent function is denoted by $\tan^{-1}$ or arctan.

Inverse tangent function

$$\tan^{-1} x = y \quad \text{if} \quad \tan y = x, \quad \text{for } x \in \mathbb{R} \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graph of $y = \tan^{-1} x$ is obtained from the graph of $y = \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, through a reflection in the line $y = x$.

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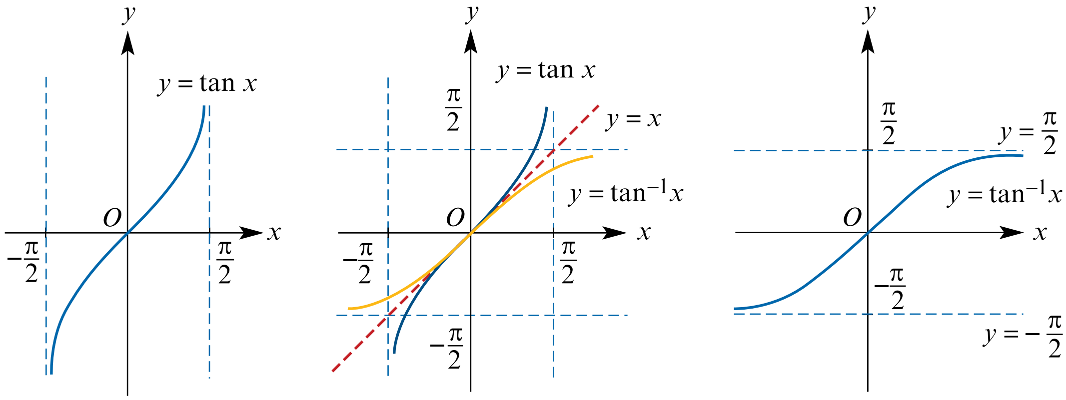


- **Domain** Domain of $\tan^{-1}$ = range of restricted tangent function = $\mathbb{R}$
- **Range** Range of $\tan^{-1}$ = domain of restricted tangent function = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- **Inverse relationship**
  - $\tan(\tan^{-1} x) = x$ for all $x \in \mathbb{R}$
  - $\tan^{-1}(\tan x) = x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 19

Sketch the graph of each of the following functions for the maximal domain:

a $y = \cos^{-1}(2 - 3x)$

b $y = \tan^{-1}(x + 2) + \frac{\pi}{2}$

Solution

a $\cos^{-1}(2 - 3x)$ is defined $\Leftrightarrow \quad -1 \le 2 - 3x \le 1$

$$\Leftrightarrow \quad -3 \le -3x \le -1$$

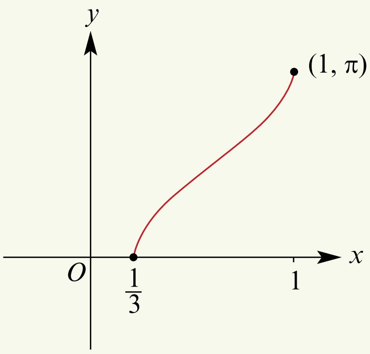
$$\Leftrightarrow \quad \frac{1}{3} \le x \le 1$$

The implied domain is $\left[\frac{1}{3}, 1\right]$.

We can write $y = \cos^{-1}\left(-3\left(x - \frac{2}{3}\right)\right)$.

The graph is obtained from the graph of $y = \cos^{-1} x$ by the following sequence of transformations:

- a dilation of factor $\frac{1}{3}$ from the y-axis
- a reflection in the y-axis
- a translation of $\frac{2}{3}$ units in the positive direction of the x-axis.



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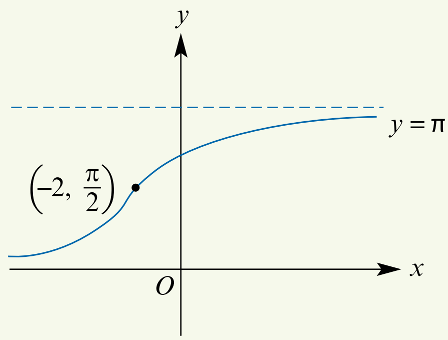
b The domain of $\tan^{-1}$ is $\mathbb{R}$.

The graph of

$$y = \tan^{-1}(x+2) + \frac{\pi}{2}$$

is obtained from the graph of $y = \tan^{-1} x$ by a translation of 2 units in the negative direction of the x-axis and $\frac{\pi}{2}$ units in the positive direction of the y-axis.



Example 20

a Evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

b Simplify:

i $\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

ii $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

iii $\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)$

iv $\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$

Solution

a Evaluating $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equivalent to solving $\sin y = -\frac{\sqrt{3}}{2}$ for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

b **i** Since $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by definition we have

$$\sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$$

ii
$$\begin{aligned}\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) &= \sin^{-1}\left(\sin\left(\pi - \frac{5\pi}{6}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\end{aligned}$$

iii
$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) &= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right)\\ &= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\end{aligned}$$

iv
$$\begin{aligned}\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) &= \sin\left(\frac{\pi}{4}\right)\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

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Example 21

Determine the implied domain and range of:

a $y = \sin^{-1}(2x - 1)$

b $y = 3\cos^{-1}(2 - 2x)$

Solution

a For $\sin^{-1}(2x - 1)$ to be defined:

$$-1 \le 2x - 1 \le 1$$
$$\Leftrightarrow \quad 0 \le 2x \le 2$$
$$\Leftrightarrow \quad 0 \le x \le 1$$

Thus the implied domain is $[0, 1]$.

The range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b For $3\cos^{-1}(2 - 2x)$ to be defined:

$$-1 \le 2 - 2x \le 1$$
$$\Leftrightarrow \quad -3 \le -2x \le -1$$
$$\Leftrightarrow \quad \frac{1}{2} \le x \le \frac{3}{2}$$

Thus the implied domain is $\left[\frac{1}{2}, \frac{3}{2}\right]$.

The range is $[0, 3\pi]$.

Skill-sheet

Exercise 1D

Example 19

1 Sketch the graphs of the following functions, stating clearly the implied domain and the range of each:

a $y = \tan^{-1}(x - 1)$ **b** $y = \cos^{-1}(x + 1)$ **c** $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$

d $y = 2\tan^{-1}(x) + \frac{\pi}{2}$ **e** $y = \cos^{-1}(2x)$ **f** $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$

SF

Example 20a

2 Evaluate each of the following:

a $\arcsin 1$ **b** $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$ **c** $\arcsin 0.5$

d $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **e** $\cos^{-1} 0.5$ **f** $\tan^{-1} 1$

g $\tan^{-1}(-\sqrt{3})$ **h** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **i** $\cos^{-1}(-1)$

Example 20b

3 Simplify:

a $\sin(\cos^{-1} 0.5)$ **b** $\sin^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right)$ **c** $\tan\left(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

d $\cos(\tan^{-1} 1)$ **e** $\tan^{-1}\left(\sin\left(\frac{5\pi}{2}\right)\right)$ **f** $\tan(\cos^{-1} 0.5)$

g $\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)$ **h** $\sin^{-1}\left(\sin\left(-\frac{2\pi}{3}\right)\right)$ **i** $\tan^{-1}\left(\tan\left(\frac{11\pi}{4}\right)\right)$

j $\cos^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$ **k** $\cos^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right)$ **l** $\sin^{-1}\left(\cos\left(-\frac{3\pi}{4}\right)\right)$

CF

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Example 21

SF

4 Given that the domains of sin, cos and tan are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0, \pi]$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively, give the implied domain and range of each of the following:

a $y = \sin^{-1}(2 - x)$
b $y = \sin\left(x + \frac{\pi}{4}\right)$
c $y = \sin^{-1}(2x + 4)$
d $y = \sin\left(3x - \frac{\pi}{3}\right)$
e $y = \cos\left(x - \frac{\pi}{6}\right)$
f $y = \cos^{-1}(x + 1)$
g $y = \cos^{-1}(x^2)$
h $y = \cos\left(2x + \frac{2\pi}{3}\right)$
i $y = \tan^{-1}(x^2)$
j $y = \tan\left(2x - \frac{\pi}{2}\right)$
k $y = \tan^{-1}(2x + 1)$
l $y = \tan(x^2)$

CF

5 Simplify each of the following expressions, in an exact form:

a $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$
b $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$
c $\cos\left(\tan^{-1}\left(\frac{7}{24}\right)\right)$
d $\tan\left(\sin^{-1}\left(\frac{40}{41}\right)\right)$
e $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$
f $\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$
g $\sin(\tan^{-1}(-2))$
h $\cos\left(\sin^{-1}\left(\frac{3}{7}\right)\right)$
i $\sin(\tan^{-1} 0.7)$

6 Let $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}$, where $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$.

a Determine:

i $\cos\alpha$
ii $\cos\beta$

b Use the angle sum and difference identities to show that:

i $\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{5}{13}\right) = \sin^{-1}\left(\frac{16}{65}\right)$
ii $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

7 Given that the domains of sin and cos are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively, explain why each expression cannot be evaluated:

a $\cos(\arcsin(-0.5))$
b $\sin(\cos^{-1}(-0.2))$
c $\cos(\tan^{-1}(-1))$

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1E Solution of equations

Learning intentions

- To solve equations involving other reciprocal trigonometric functions and identities.

We now introduce equations involving the reciprocal trigonometric functions and the use of the double-angle identities.

Example 22

Solve the equation $\sec x = 2$ for $x \in [0, 2\pi]$.

Solution

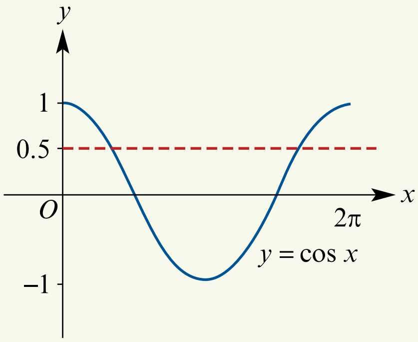
$$\sec x = 2$$

$$\therefore \quad \cos x = \frac{1}{2}$$

We are looking for solutions in $[0, 2\pi]$:

$$x = \frac{\pi}{3} \quad \text{or} \quad x = 2\pi - \frac{\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$



Example 23

Solve the equation $\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = \frac{-2\sqrt{3}}{3}$ for $x \in [0, 2\pi]$.

Solution

$$\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = \frac{-2\sqrt{3}}{3} \implies \sin\left(2x - \frac{\pi}{3}\right) = \frac{-3}{2\sqrt{3}} = \frac{-\sqrt{3}}{2}$$

Let $\theta = 2x - \frac{\pi}{3}$ where $\theta \in \left[-\frac{\pi}{3}, \frac{11\pi}{3}\right]$.

Then $\quad \sin\theta = \frac{-\sqrt{3}}{2}$

$$\therefore \quad \theta = -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad 2x - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad 2x = 0, \frac{5\pi}{3}, 2\pi, \frac{11\pi}{3} \text{ or } 4\pi$$

$$\therefore \quad x = 0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6} \text{ or } 2\pi$$

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General solution of trigonometric equations

We recall the following from Specialist Mathematics Units 1 & 2.

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

Example 24

a Determine all the values of x for which $\cot x = -1$.

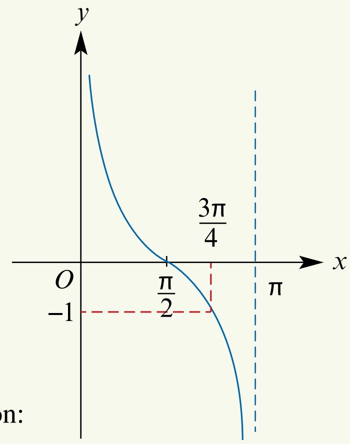
b Determine all the values of x for which $\sec\left(2x - \frac{\pi}{3}\right) = 2$.

Solution

a The period of the function $y = \cot x$ is π.

The solution of $\cot x = -1$ in $[0, \pi]$ is $x = \frac{3\pi}{4}$.

Therefore the solutions of the equation are
$$x = \frac{3\pi}{4} + n\pi \quad \text{where } n \in \mathbb{Z}$$



b First write the equation as
$$\cos\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$$

We now proceed as usual to determine the general solution:
$$2x - \frac{\pi}{3} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$$
$$2x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$$
$$2x - \frac{\pi}{3} = 2n\pi + \frac{\pi}{3} \quad \text{or} \quad 2x - \frac{\pi}{3} = 2n\pi - \frac{\pi}{3}$$
$$2x = 2n\pi + \frac{2\pi}{3} \quad \text{or} \quad 2x = 2n\pi$$
$$\therefore \quad x = n\pi + \frac{\pi}{3} \quad \text{or} \quad x = n\pi \quad \text{where } n \in \mathbb{Z}$$

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Using identities to solve equations

The double-angle identities can be used to help solve trigonometric equations.

Example 25

Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sin(4x) = \sin(2x)$

b $\cos x = \sin\left(\frac{x}{2}\right)$

Solution

a

$$\begin{aligned} \sin(4x) &= \sin(2x) \\ 2\sin(2x)\cos(2x) &= \sin(2x) \\ \sin(2x)(2\cos(2x) - 1) &= 0 \qquad \text{where } 2x \in [0, 4\pi] \end{aligned}$$

Thus $\sin(2x) = 0$ or $2\cos(2x) - 1 = 0$

i.e. $\sin(2x) = 0$ or $\cos(2x) = \frac{1}{2}$

$\therefore \quad 2x = 0, \pi, 2\pi, 3\pi, 4\pi$ or $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ or $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Hence $x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ or 2π.

b

$$\begin{aligned} \cos x &= \sin\left(\frac{x}{2}\right) \\ 1 - 2\sin^2\left(\frac{x}{2}\right) &= \sin\left(\frac{x}{2}\right) \\ 2\sin^2\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) - 1 &= 0 \qquad \text{where } \frac{x}{2} \in [0, \pi] \end{aligned}$$

Let $a = \sin\left(\frac{x}{2}\right)$. Then $a \in [0, 1]$. We have

$$2a^2 + a - 1 = 0$$

$\therefore \quad (2a - 1)(a + 1) = 0$

$\therefore \quad 2a - 1 = 0$ or $a + 1 = 0$

$\therefore \quad a = \frac{1}{2}$ or $a = -1$

Thus $a = \frac{1}{2}$, since $a \in [0, 1]$. We now have

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$$

$\therefore \quad \frac{x}{2} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\therefore \quad x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

Maximum and minimum values

We know that $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$. This can be used to determine the maximum and minimum values of trigonometric functions without using calculus.

For example:

- The function $y = 2\sin x + 3$ has a maximum value of 5 and a minimum value of 1. The maximum value occurs when $\sin x = 1$ and the minimum value occurs when $\sin x = -1$.
- The function $y = \dfrac{1}{2\sin x + 3}$ has a maximum value of 1 and a minimum value of $\dfrac{1}{5}$.

Example 26

Determine the maximum and minimum values of:

a $\sin^2(2x) + 2\sin(2x) + 2$

b $\dfrac{1}{\sin^2(2x) + 2\sin(2x) + 2}$

Solution

a Let $a = \sin(2x)$. Then

$$\begin{aligned}\sin^2(2x) + 2\sin(2x) + 2 &= a^2 + 2a + 2\\ &= (a+1)^2 + 1\\ &= (\sin(2x)+1)^2 + 1\end{aligned}$$

Now $-1 \le \sin(2x) \le 1$.

Therefore the maximum value is 5 and the minimum value is 1.

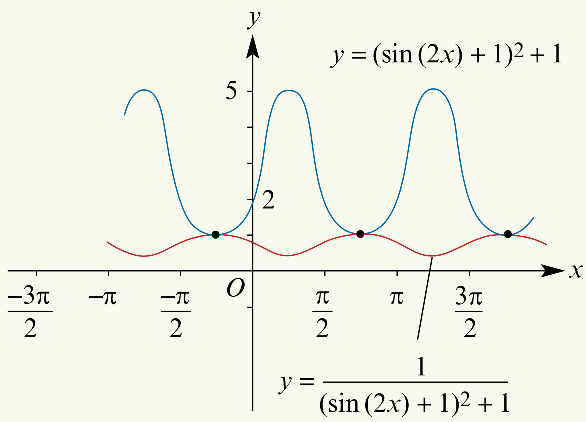
b Note that

$$\sin^2(2x) + 2\sin(2x) + 2 > 0$$

for all x. Thus its reciprocal also has this property.

A local maximum for the original function yields a local minimum for the reciprocal.

A local minimum for the original function yields a local maximum for the reciprocal.



Hence the maximum value is 1 and the minimum value is $\dfrac{1}{5}$.

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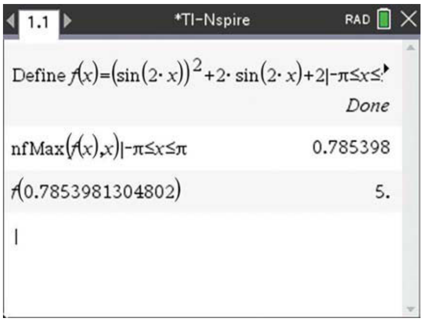
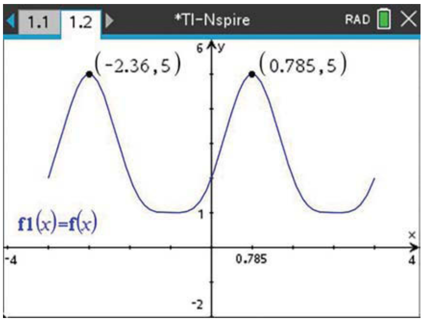
TI-84

Using the TI-Nspire CX non-CAS

- To determine an x-value at which the maximum y-value occurs, use (menu) > **Calculus** > **Numerical Function Maximum**.



- Alternatively, to determine the maximum y-value from the graph, use either (menu) > **Trace** > **Graph Trace** or (menu) > **Analyze Graph** > **Maximum**.



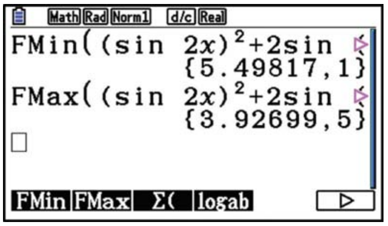
Using the Casio

Method 1: Using Run-Matrix mode

- In **Run-Matrix** mode, go to the **Calculation** menu (OPTN) (F4). Select **FMin** (F6) (F1) or **FMax** (F6) (F2).
- Complete by entering the expression and the domain:

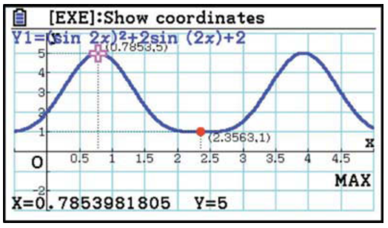
  $(\sin 2x)^2 + 2\sin(2x) + 2, 0, 2\pi)$

- The minimum or maximum value of the expression is the y-coordinate of the given point.



Method 2: Using Graph mode

- Plot the graph of $y = (\sin 2x)^2 + 2\sin(2x) + 2$.
- Adjust the View Window (SHIFT) (F3) if required.
- From the **G-Solve** menu (SHIFT) (F5), select **Minimum** (F3) or **Maximum** (F2).



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Using a graphics calculator to obtain approximate solutions

Many equations involving the trigonometric functions cannot be solved using analytic techniques. A graphics calculator can be used to solve such equations numerically.

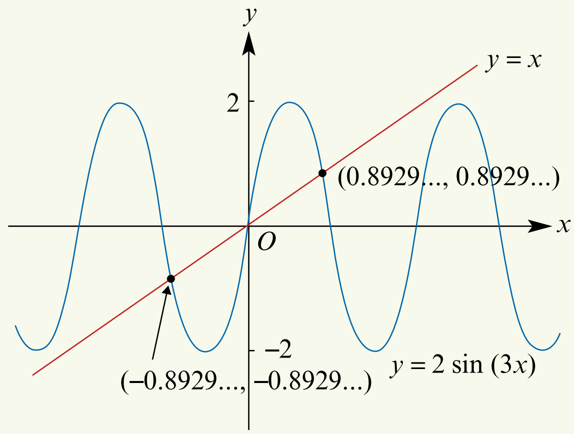
Example 27

Determine the solutions of the equation $2\sin(3x) = x$, correct to three decimal places.

Solution

The graphs of $y = 2\sin(3x)$ and $y = x$ are plotted using a calculator.

The solutions are $x = 0$, $x \approx 0.893$ and $x \approx -0.893$.



Skill-sheet

Exercise 1E

Example 22, 23

1 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\operatorname{cosec} x = -2$ **b** $\operatorname{cosec}\left(x - \frac{\pi}{4}\right) = -2$ **c** $3\sec x = 2\sqrt{3}$

d $\operatorname{cosec}(2x) + 1 = 2$ **e** $\cot x = -\sqrt{3}$ **f** $\cot\left(2x - \frac{\pi}{3}\right) = -1$

2 Solve each of the following equations, giving solutions in the interval $[0, 2\pi]$:

a $\sin x = 0.5$ **b** $\cos x = \frac{-\sqrt{3}}{2}$ **c** $\tan x = \sqrt{3}$

d $\cot x = -1$ **e** $\sec x = -2$ **f** $\operatorname{cosec} x = -\sqrt{2}$

SF

Example 24

3 Determine all the solutions to each of the following equations:

a $\sin x = \frac{1}{\sqrt{2}}$ **b** $\sec x = 1$ **c** $\cot x = \sqrt{3}$

d $\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = 2$ **e** $\operatorname{cosec}\left(3x - \frac{\pi}{3}\right) = \frac{2\sqrt{3}}{3}$ **f** $\sec\left(3x - \frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$

g $\cot\left(2x - \frac{\pi}{6}\right) = \sqrt{3}$ **h** $\cot\left(2x - \frac{\pi}{4}\right) = -1$ **i** $\operatorname{cosec}\left(2x - \frac{\pi}{4}\right) = 1$

CF

4 Solve each of the following in the interval $[-\pi, \pi]$, giving the answers correct to two decimal places:

a $\sec x = 2.5$ **b** $\operatorname{cosec} x = -5$ **c** $\cot x = 0.6$

SF

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Example 25

5 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\cos^2 x - \cos x \sin x = 0$ **b** $\sin(2x) = \sin x$ **c** $\sin(2x) = \cos x$

d $\sin(8x) = \cos(4x)$ **e** $\cos(2x) = \cos x$ **f** $\sec^2 x + \tan x = 1$

g $\tan x(1 + \cot x) = 0$ **h** $\cot x + 3\tan x = 5\,\text{cosec}\,x$ **i** $\sin x + \cos x = 1$

Example 26

6 Determine the maximum and minimum values of each of the following:

a $2 + \sin\theta$ **b** $\dfrac{1}{2 + \sin\theta}$ **c** $\sin^2\theta + 4$

d $\dfrac{1}{\sin^2\theta + 4}$ **e** $\cos^2\theta + 2\cos\theta$ **f** $\cos^2\theta + 2\cos\theta + 6$

Example 27

7 Using a graphics calculator, determine the coordinates of the points of intersection for the graphs of the following pairs of functions. (Give values correct to two decimal places.)

a $y = 2x$ and $y = 3\sin(2x)$ **b** $y = x$ and $y = 2\sin(2x)$

c $y = 3 - x$ and $y = \cos x$ **d** $y = x$ and $y = \tan x$, $x \in [0, 2\pi]$

8 Let $a \in [-1, 1]$ with $a \neq -1$. Consider the equation $\cos x = a$ for $x \in [0, 2\pi]$. If q is one of the solutions, determine the second solution in terms of q.

9 Let $\sin\alpha = a$ where $\alpha \in \left(0, \dfrac{\pi}{2}\right)$. Determine, in terms of α, two values of x in $[0, 2\pi]$ which satisfy each of the following equations:

a $\sin x = -a$ **b** $\cos x = a$

10 Let $\sec\beta = b$ where $\beta \in \left(\dfrac{\pi}{2}, \pi\right)$. Determine, in terms of β, two values of x in $[-\pi, \pi]$ which satisfy each of the following equations:

a $\sec x = -b$ **b** $\text{cosec}\,x = b$

11 Let $\tan\gamma = c$ where $\gamma \in \left(\pi, \dfrac{3\pi}{2}\right)$. Determine, in terms of γ, two values of x in $[0, 2\pi]$ which satisfy each of the following equations:

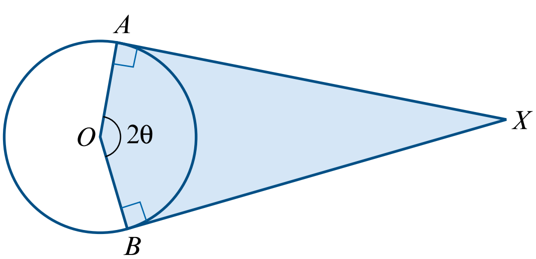
a $\tan x = -c$ **b** $\cot x = c$

12 A curve on a light rail track is an arc of a circle of length 300 m and the straight line joining the two ends of the curve is 270 m long.

a Show that, if the arc subtends an angle of $2\theta°$ at the centre of the circle, then θ is a solution of the equation $\sin\theta° = \dfrac{\pi}{200}\theta°$.

b Solve this equation for θ, correct to two decimal places.

13 Two tangents are drawn from a point so that the area of the shaded region is equal to the area of the remaining region of the circle.



$\angle AOB = 2\theta$

a Show that θ satisfies the equation $\tan\theta = \pi - \theta$.

b Solve for θ, giving the answer correct to three decimal places.

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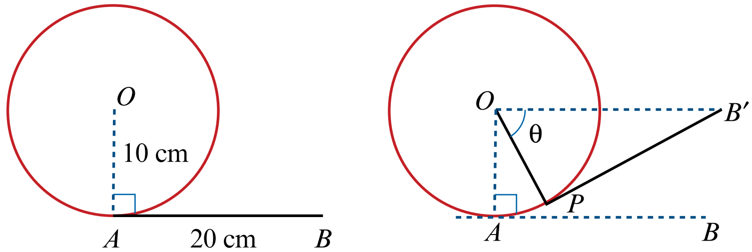
14 Two particles A and B move in a straight line. At time t, their positions relative to a point O are given by

$$x_A = 0.5 \sin t \quad \text{and} \quad x_B = 0.25t^2 + 0.05t$$

Determine the times at which their positions are the same, and give this position. (Distances are measured in centimetres and time in seconds.)

15 A string is wound around a disc and a horizontal length of the string AB is 20 cm long. The radius of the disc is 10 cm. The string is then moved so that the end of the string, B', is moved to a point at the same level as O, the centre of the circle. The line $B'P$ is a tangent to the circle.



a Show that θ satisfies the equation $\frac{\pi}{2} - \theta + \tan\theta = 2$.

b Determine the value of θ, correct to two decimal places, which satisfies this equation.

1F Sums and products of sines and cosines

Learning intentions

- To use the product-to-sum and sum-to-product identities.

In Section 1C, we considered the angle sum and difference identities for sine and cosine. We use them in this section to obtain new identities which allow us to rewrite products of sines and cosines as sums or differences, and vice versa.

Expressing products as sums or differences

Product-to-sum identities

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

Proof We use the angle sum and difference identities for sine and cosine:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad (1)$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B \qquad (2)$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B \qquad (3)$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B \qquad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

Example 28

Express each of the following products as sums or differences:

a $2\sin(3\theta)\cos(\theta)$

b $2\sin 50^\circ \cos 60^\circ$

c $2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right)$

Solution

a Use the third product-to-sum identity:

$$\begin{aligned} 2\sin(3\theta)\cos(\theta) &= \sin(3\theta + \theta) + \sin(3\theta - \theta) \\ &= \sin(4\theta) + \sin(2\theta) \end{aligned}$$

b Use the third product-to-sum identity:

$$\begin{aligned} 2\sin 50^\circ \cos 60^\circ &= \sin 110^\circ + \sin(-10)^\circ \\ &= \sin 110^\circ - \sin 10^\circ \end{aligned}$$

c Use the first product-to-sum identity:

$$\begin{aligned} 2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right) + \cos(2\theta) \\ &= \cos(2\theta) \end{aligned}$$

Expressing sums and differences as products

Sum-to-product identities

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

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Proof Using the first product-to-sum identity, we have

$$\begin{aligned} 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) &= \cos\left(\frac{A+B}{2}-\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}+\frac{A-B}{2}\right) \\ &= \cos B+\cos A \\ &= \cos A+\cos B \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

Example 29

Express each of the following as products:

a $\sin 36° + \sin 10°$

b $\cos 36° + \cos 10°$

Solution

a $\sin 36° + \sin 10° = 2\sin 23° \cos 13°$

b $\cos 36° + \cos 10° = 2\cos 23° \cos 13°$

Example 30

Prove that

$$\frac{\cos(\theta)-\cos(3\theta)}{\sin(3\theta)-\sin(\theta)} = \tan(2\theta)$$

Solution

$$\begin{aligned} \text{LHS} &= \frac{\cos(\theta)-\cos(3\theta)}{\sin(3\theta)-\sin(\theta)} \\ &= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \frac{2\sin(2\theta)\sin(\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS} \end{aligned}$$

Example 31

Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.

Solution

$$\begin{aligned} & \sin(3x)+\sin(11x) = 0 \\ \Leftrightarrow\quad & 2\sin(7x)\cos(4x) = 0 \\ \Leftrightarrow\quad & \sin(7x) = 0 \quad \text{or} \quad \cos(4x) = 0 \\ \Leftrightarrow\quad & 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\ \Leftrightarrow\quad & x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8} \end{aligned}$$

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Exercise 1F

Example 28

SF

1 Express each of the following products as sums or differences:

a $2\sin(4\pi t)\cos(7\pi t)$

b $\sin 50^\circ \cos 10^\circ$

c $3\cos\left(\frac{\pi x}{3}\right)\sin\left(\frac{2\pi x}{3}\right)$

d $2\sin\left(\frac{A+B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$

CF

2 Express $2\sin(4\theta)\sin(\theta)$ as a difference of cosines.

3 Use a product-to-sum identity to derive the expression for $2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$ as a difference of sines.

4 Show that $\cos 75^\circ \cos 15^\circ = \frac{1}{4}$.

Example 29

SF

5 Express each of the following as products:

a $\sin 66^\circ + \sin 34^\circ$

b $\cos 66^\circ + \cos 34^\circ$

c $\sin 66^\circ - \sin 34^\circ$

d $\cos 66^\circ - \cos 34^\circ$

6 Express each of the following as products:

a $\sin(8A) + \sin(2A)$

b $\cos(x) + \cos(4x)$

c $\sin(6x) - \sin(4x)$

d $\cos(5A) - \cos(3A)$

Example 30

CF

7 Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

CU

8 For any three angles α, β and γ, show that

$$\sin(\alpha+\beta)\sin(\alpha-\beta) + \sin(\beta+\gamma)\sin(\beta-\gamma) + \sin(\gamma+\alpha)\sin(\gamma-\alpha) = 0$$

CF

9 Show that $\cos 70^\circ + \sin 40^\circ = \cos 10^\circ$.

10 Show that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

Example 31

11 Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\cos(5x) + \cos(x) = 0$

b $\cos(5x) - \cos(x) = 0$

c $\sin(5x) + \sin(x) = 0$

d $\sin(5x) - \sin(x) = 0$

12 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\cos(2\theta) - \sin(\theta) = 0$

b $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$

c $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$

d $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

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Review

Chapter summary

Reciprocal trigonometric functions

- **Definitions**

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \text{provided } \sin\theta \neq 0$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{provided } \cos\theta \neq 0$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \text{provided } \sin\theta \neq 0$$

- **Symmetry properties**

$$\sec(\pi-\theta) = -\sec\theta \qquad \operatorname{cosec}(\pi-\theta) = \operatorname{cosec}\theta \qquad \cot(\pi-\theta) = -\cot\theta$$

$$\sec(\pi+\theta) = -\sec\theta \qquad \operatorname{cosec}(\pi+\theta) = -\operatorname{cosec}\theta \qquad \cot(\pi+\theta) = \cot\theta$$

$$\sec(2\pi-\theta) = \sec\theta \qquad \operatorname{cosec}(2\pi-\theta) = -\operatorname{cosec}\theta \qquad \cot(2\pi-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta \qquad \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta \qquad \cot(-\theta) = -\cot\theta$$

- **Complementary properties**

$$\sec\left(\frac{\pi}{2}-\theta\right) = \operatorname{cosec}\theta \qquad \operatorname{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$$

$$\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$$

- **Pythagorean identities**

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Angle sum and difference identities

- $\cos(A+B) = \cos A\,\cos B - \sin A\,\sin B$
- $\cos(A-B) = \cos A\,\cos B + \sin A\,\sin B$
- $\sin(A+B) = \sin A\,\cos B + \cos A\,\sin B$
- $\sin(A-B) = \sin A\,\cos B - \cos A\,\sin B$

Double-angle identities

- $\cos(2A) = \cos^2 A - \sin^2 A$

$$= 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1$$

- $\sin(2A) = 2\sin A\,\cos A$

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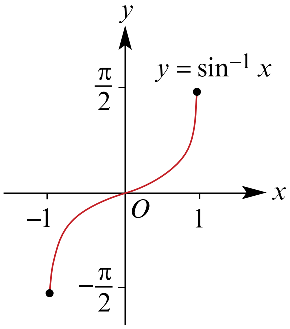
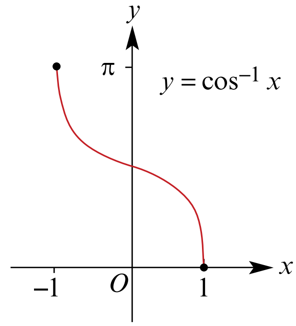
Inverse trigonometric functions

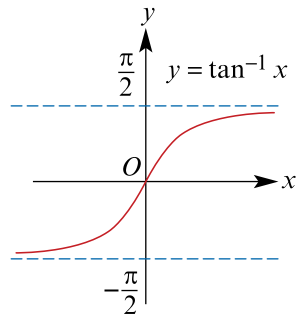
- **Inverse sine (arcsin)**
  $\sin^{-1} x = y$ if $\sin y = x$,
  for $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



- **Inverse cosine (arccos)**
  $\cos^{-1} x = y$ if $\cos y = x$,
  for $x \in [-1, 1]$ and $y \in [0, \pi]$



- **Inverse tangent (arctan)**
  $\tan^{-1} x = y$ if $\tan y = x$,
  for $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Product-to-sum identities

- $2\cos A \cos B = \cos(A - B) + \cos(A + B)$
- $2\sin A \sin B = \cos(A - B) - \cos(A + B)$
- $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

Sum-to-product identities

- $\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$
- $\sin A - \sin B = 2\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

1A **1** I can determine the exact values of trigonometric functions. ☐

See Example 2, Example 3, Example 4, Example 8, Question 3 and Question 4

1A **2** I can solve simple equations involving the trigonometric functions. ☐

See Example 6, Question 11, and Question 16

1A **3** I can sketch the graphs and transformations of the trigonometric functions. ☐

See Example 7, Question 11 and Question 16

1B **4** I can sketch graphs involving reciprocal trigonometric functions. ☐

See Example 10, Question 1, Question 2 and Question 3

1B **5** I can determine exact values of reciprocal trigonometric functions. ☐

See Example 12, Example 14, Question 5 and Question 7

1B **6** I can simplify identities involving reciprocal trigonometric functions. ☐

See Example 13, Question 6 and Question 15

1C **7** I can use the angle sum and difference identities. ☐

See Example 15, Example 16, Question 1, Question 2,Question 3 andQuestion 6

1C **8** I can use the double angle identities. ☐

See Example 17, Example 18, Question 9 and Question 10

1D **9** I can sketch graphs involving the inverse trigonometric functions. ☐

See Example 19 and Question 1

1D **10** I can simplify expressions involving the inverse trigonometric functions. ☐

See Example 20, Question 1, Question 2 and Question 5

1E **11** I can solve harder equations involving trigonometric functions. ☐

See Example 23, Example 25, Question 1, Question 3 and Question 5

1F **12** I can use the product-to-sum identities. ☐

See Example 28, Question 1, Question 2 and Question 3

1F **13** I can use the sum-to-product identities. ☐

See Example 29, Example 30, Example 31, Question 5, Question 7 and Question 11

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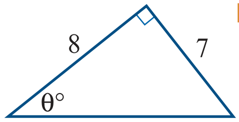
Review

Short-response questions

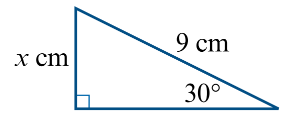
Technology-free short-response questions

SF

1 **a** Determine $\sin \theta°$.



b Determine x.

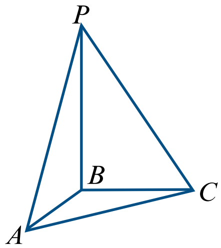


2 **a** Determine the exact value of $\cos 315°$.

b Given that $\tan x° = \frac{3}{4}$ and $180 < x < 270$, determine the exact value of $\cos x°$.

c Determine an angle $A°$ (with $A \neq 330$) such that $\sin A° = \sin 330°$.

3 ABC is a horizontal right-angled triangle with the right angle at B. The point P is 3 cm directly above B. The length of AB is 1 cm and the length of BC is 1 cm. Determine the angle that the triangle ACP makes with the horizontal.



4 Determine all angles θ with $0 \le \theta \le 2\pi$, where:

a $\sin \theta = \frac{1}{2}$ **b** $\cos \theta = \frac{\sqrt{3}}{2}$ **c** $\tan \theta = 1$

5 **a** Solve $2\cos(2x + \pi) - 1 = 0$ for $-\pi \le x \le \pi$.

b Sketch the graph of $y = 2\cos(2x + \pi) - 1$ for $-\pi \le x \le \pi$, clearly labelling the axis intercepts.

c Solve $2\cos(2x + \pi) < 1$ for $-\pi \le x \le \pi$.

6 If θ is an acute angle and $\cos \theta = \frac{4}{5}$, determine:

a $\cos(2\theta)$ **b** $\sin(2\theta)$ **c** $\operatorname{cosec} \theta$ **d** $\cot \theta$

CF

7 Solve each of the following equations for $-\pi < x \le 2\pi$:

a $\sin(2x) = 2\cos x$ **b** $\cos(2x) = \sin x$

c $\cos x - 1 = \cos(2x)$ **d** $\sin^2 x \cos^3 x = \cos x$

e $\sin^2 x - \frac{1}{2}\sin x - \frac{1}{2} = 0$ **f** $2\cos^2 x - 3\cos x + 1 = 0$

8 Solve each of the following equations for $0 \le \theta \le 2\pi$, giving exact answers:

a $2 - \sin\theta = \cos^2\theta + 7\sin^2\theta$ **b** $\sec(2\theta) = 2$

c $\frac{1}{2}(5\cos\theta - 3\sin\theta) = \sin\theta$ **d** $\sec\theta = 2\cos\theta$

SF

9 Determine the exact value of each of the following:

a $\sin\left(\frac{5\pi}{3}\right)$ **b** $\operatorname{cosec}\left(-\frac{5\pi}{3}\right)$ **c** $\sec\left(\frac{7\pi}{3}\right)$

d $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$ **e** $\cot\left(-\frac{3\pi}{4}\right)$ **f** $\cot\left(-\frac{\pi}{6}\right)$

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Review

SF

10 Given that $\tan\alpha = p$, where α is an acute angle, determine each of the following in terms of p:

a $\tan(-\alpha)$ **b** $\tan(\pi - \alpha)$ **c** $\tan\left(\frac{\pi}{2} - \alpha\right)$

d $\tan\left(\frac{3\pi}{2} + \alpha\right)$ **e** $\tan(2\pi - \alpha)$

11 Determine:

a $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **b** $\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ **c** $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

d $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)$ **e** $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ **f** $\cos(\tan^{-1}(-1))$

12 Sketch the graph of each of the following functions, stating the maximal domain and range of each:

a $y = 2\tan^{-1} x$ **b** $y = \sin^{-1}(3 - x)$ **c** $y = 3\cos^{-1}(2x + 1)$

d $y = -\cos^{-1}(2 - x)$ **e** $y = 2\tan^{-1}(1 - x)$

CF

13 Solve the equation $\sin(3x) = \sin(5x)$ for $0 \le x \le \pi$.

CU

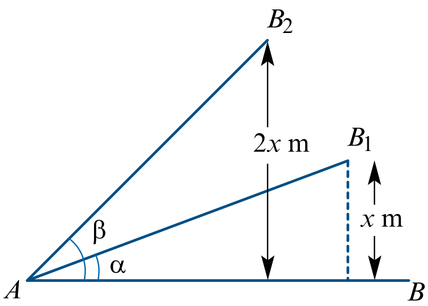
14 Prove the identity $\dfrac{\sin A + \sin B - \sin(A + B)}{\sin A + \sin B + \sin(A + B)} = \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$.

Technology-active short-response questions

CU

15 A horizontal rod is 1 m long. One end is hinged at A, and the other end rests on a support B. The rod can be rotated about A, with the other end taking the two positions B_1 and B_2, which are x m and $2x$ m above the line AB respectively, where $x < 0.5$.

Let $\angle BAB_1 = \alpha$ and $\angle BAB_2 = \beta$.



a Determine each of the following in terms of x:

i $\sin\alpha$ **ii** $\cos\alpha$ **iii** $\tan\alpha$ **iv** $\sin\beta$ **v** $\cos\beta$ **vi** $\tan\beta$

b Using the results of **a**, determine:

i $\sin(\beta - \alpha)$ **ii** $\cos(\beta - \alpha)$ **iii** $\tan(\beta - \alpha)$

iv $\sin(2\alpha)$ **v** $\cos(2\alpha)$ **vi** $\tan(2\alpha)$

c If $x = 0.3$, determine the magnitudes of $\angle B_2AB_1$ and 2α, correct to two decimal places.

16 **a** On the one set of axes, sketch the graphs of the following for $x \in (0, \pi) \cup (\pi, 2\pi)$:

i $y = \operatorname{cosec}(x)$ **ii** $y = \cot(x)$ **iii** $y = \operatorname{cosec}(x) - \cot(x)$

b **i** Show that $\operatorname{cosec} x - \cot x > 0$ for all $x \in (0, \pi)$, and hence that $\operatorname{cosec} x > \cot x$ for all $x \in (0, \pi)$.

ii Show that $\operatorname{cosec} x - \cot x < 0$ for all $x \in (\pi, 2\pi)$, and hence that $\operatorname{cosec} x < \cot x$ for all $x \in (\pi, 2\pi)$.

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c On separate axes, sketch the graph of $y = \cot\left(\frac{x}{2}\right)$ for $x \in (0, 2\pi)$ and the graph of $y = \operatorname{cosec}(x) + \cot(x)$ for $x \in (0, 2\pi) \setminus \{\pi\}$.

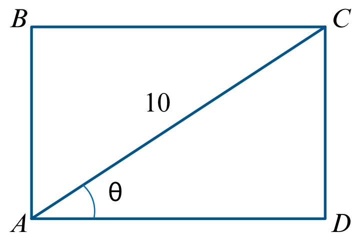
d **i** Prove that $\operatorname{cosec}\theta + \cot\theta = \cot\left(\frac{\theta}{2}\right)$ where $\sin\theta \neq 0$.

ii Use this result to determine $\cot\left(\frac{\pi}{8}\right)$ and $\cot\left(\frac{\pi}{12}\right)$.

iii Use the result $1 + \cot^2\left(\frac{\pi}{8}\right) = \operatorname{cosec}^2\left(\frac{\pi}{8}\right)$ to determine the exact value of $\sin\left(\frac{\pi}{8}\right)$.

e Use the result of **d** to show that $\operatorname{cosec}(\theta) + \operatorname{cosec}(2\theta) + \operatorname{cosec}(4\theta)$ can be expressed as the difference of two cotangents.

17 **a** $ABCD$ is a rectangle with diagonal AC of length 10 units.



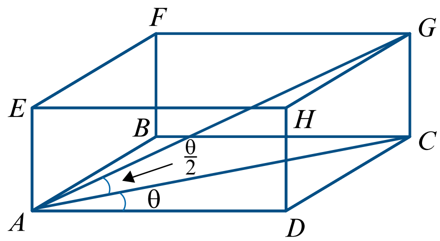
i Determine the area of the rectangle in terms of θ.

ii Sketch the graph of R against θ, where R is the area of the rectangle in square units, for $\theta \in \left(0, \frac{\pi}{2}\right)$.

iii Determine the maximum value of R. (Do not use calculus.)

iv Determine the value of θ for which this maximum occurs.

b $ABCDEFGH$ is a cuboid with $\angle GAC = \frac{\theta}{2}$, $\angle CAD = \theta$ and $AC = 10$.



i Show that the volume, V, of the cuboid is given by

$$V = 1000\cos\theta\,\sin\theta\,\tan\left(\frac{\theta}{2}\right)$$

ii Determine the values of a and b such that $V = a\sin^2\left(\frac{\theta}{2}\right) + b\sin^4\left(\frac{\theta}{2}\right)$.

iii Let $p = \sin^2\left(\frac{\theta}{2}\right)$. Express V as a quadratic in p.

iv Determine the possible values of p for $0 < \theta < \frac{\pi}{2}$.

v Sketch the graphs of V against θ and V against p with the help of a calculator.

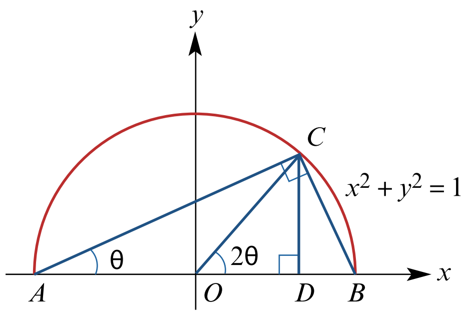
vi Determine the maximum volume of the cuboid and the values of p and θ for which this occurs. (Determine the maximum through the quadratic found in **b iii**.)

c Now assume that the cuboid satisfies $\angle CAD = \theta$, $\angle GAC = \theta$ and $AC = 10$.

i Determine V in terms of θ. **ii** Sketch the graph of V against θ.

iii Discuss the relationship between V and θ using the graph of **c ii**.

18 Points A, B and C lie on a circle with centre O and radius 1 as shown.



a Give reasons why triangle ACD is similar to triangle ABC.

b Give the coordinates of C in terms of trigonometric functions applied to 2θ.

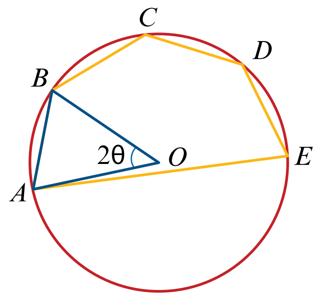
c **i** Determine CA in terms of θ from triangle ABC.

ii Determine CB in terms of θ from triangle ABC.

d Use the results of **b** and **c** to show that $\sin(2\theta) = 2\sin\theta\cos\theta$.

e Use the results of **b** and **c** to show that $\cos(2\theta) = 2\cos^2\theta - 1$.

19 $ABCDE$ is a pentagon inscribed in a circle with $AB = BC = CD = DE = 1$ and $\angle BOA = 2\theta$. The centre of the circle is O. Let $p = AE$.



a Show that $p = \dfrac{\sin(4\theta)}{\sin\theta}$.

b Express p as a function of $\cos\theta$.

Let $x = \cos\theta$.

c **i** If $p = \sqrt{3}$, show that $8x^3 - 4x - \sqrt{3} = 0$.

ii Show that $\dfrac{\sqrt{3}}{2}$ is a solution to the equation and that it is the only real solution.

iii Determine the value of θ for which $p = \sqrt{3}$.

iv Determine the radius of the circle.

d Using a calculator, sketch the graph of p against θ for $\theta \in \left(0, \dfrac{\pi}{4}\right]$.

e If $A = E$, determine the value of θ.

f **i** If $AE = 1$, show that $8x^3 - 4x - 1 = 0$.

ii Hence show that $\dfrac{1}{4}(\sqrt{5} + 1) = \cos\left(\dfrac{\pi}{5}\right)$.

20 **a** **i** Prove that $\tan x + \cot x = 2\,\text{cosec}(2x)$ for $\sin(2x) \neq 0$.

ii Solve the equation $\tan x = \cot x$ for x.

iii On the one set of axes, sketch the graphs of $y = \tan x$, $y = \cot x$ and $y = 2\,\text{cosec}(2x)$ for $x \in (0, 2\pi)$.

b **i** Prove that $\cot(2x) + \tan x = \text{cosec}(2x)$ for $\sin(2x) \neq 0$.

ii Solve the equation $\cot(2x) = \tan x$ for x.

iii On the one set of axes, sketch the graphs of $y = \cot(2x)$, $y = \tan x$ and $y = \text{cosec}(2x)$ for $x \in (0, 2\pi)$.

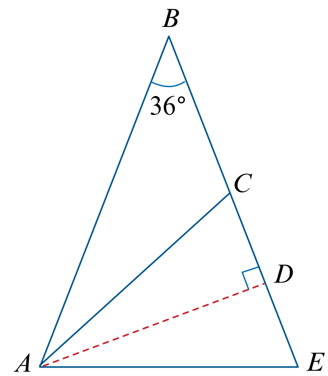
c **i** Prove that $\cot(mx) + \tan(nx) = \dfrac{\cos((m-n)x)}{\sin(mx)\cos(nx)}$, for all $m, n \in \mathbb{Z}$.

ii Hence show that $\cot(6x) + \tan(3x) = \text{cosec}(6x)$.

Review

CU

21 Triangle ABE is isosceles with $AB = BE$, and triangle ACE is isosceles with $AC = AE = 1$.



a **i** Determine the magnitudes of $\angle BAE$, $\angle AEC$ and $\angle ACE$.

ii Hence determine the magnitude of $\angle BAC$.

b Show that $BD = 1 + \sin 18°$.

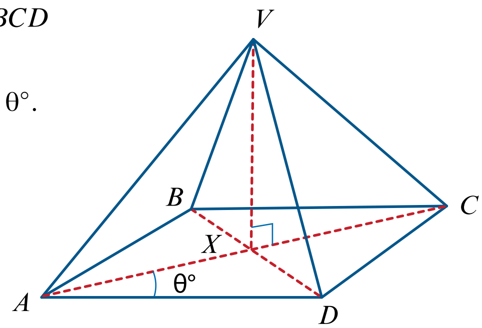
c Use triangle ABD to prove that

$$\cos 36° = \frac{1 + \sin 18°}{1 + 2\sin 18°}$$

d Hence show that $4\sin^2 18° + 2\sin 18° - 1 = 0$.

e Determine $\sin 18°$ in exact form.

22 $VABCD$ is a right pyramid, where the base $ABCD$ is a rectangle with diagonal length $AC = 10$.



a First assume that $\angle CAD = \theta°$ and $\angle VAX = \theta°$.

i Show that the volume, V, of the pyramid is given by

$$V = \frac{500}{3}\sin^2(\theta°)$$

ii Sketch the graph of V against θ for $\theta \in (0, 90)$.

iii Comment on the graph.

b Now assume that $\angle CAD = \theta°$ and $\angle VAX = \dfrac{\theta°}{2}$.

i Show that the volume, V, of the pyramid is given by

$$V = \frac{1000}{3}\sin^2\left(\frac{\theta°}{2}\right)\left(1 - 2\sin^2\left(\frac{\theta°}{2}\right)\right)$$

ii State the maximal domain of the function $V(\theta)$.

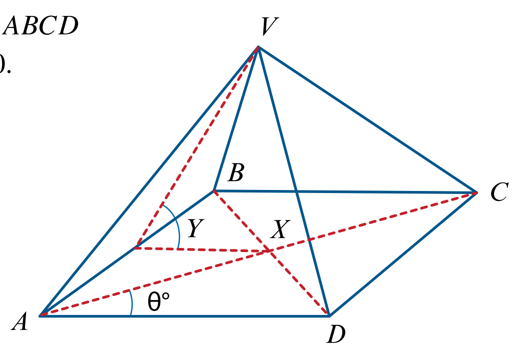
iii Let $a = \sin^2\left(\dfrac{\theta°}{2}\right)$ and write V as a quadratic in a.

iv Hence determine the maximum value of V and the value of θ for which this occurs.

v Sketch the graph of V against θ for the domain established in **b ii**.

23 $VABCD$ is a right pyramid, where the base $ABCD$ is a rectangle with diagonal length $AC = 10$.

Assume that $\angle CAD = \theta°$ and $AY = BY$.



a If $\angle VYX = \theta°$, determine:

i an expression for the volume of the pyramid in terms of θ

ii the maximum volume and the value of θ for which this occurs.

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b If $\angle VYX = \dfrac{\theta°}{2}$:

  i show that $V = \dfrac{500}{3}\cos^2(\theta°)\,(1 - \cos(\theta°))$

  ii state the implied domain for the function.

c Let $a = \cos(\theta°)$. Then $V = \dfrac{500}{3}a^2(1 - a)$. Use a graphics calculator to determine the maximum value of V and the values of a and θ for which this maximum occurs.

Multiple-choice questions

Technology-free multiple-choice questions

1 If $2\cos x° - \sqrt{2} = 0$, then the value of the acute angle $x°$ is

A $30°$ **B** $60°$ **C** $45°$ **D** $25°$

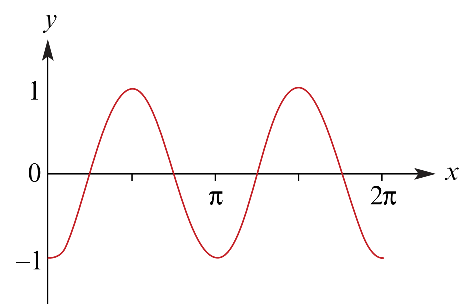
2 The equation of the graph shown is

A $y = \sin\left(2\left(x - \dfrac{\pi}{4}\right)\right)$

B $y = \cos\left(x + \dfrac{\pi}{4}\right)$

C $y = \sin(2x)$

D $y = -2\sin(x)$

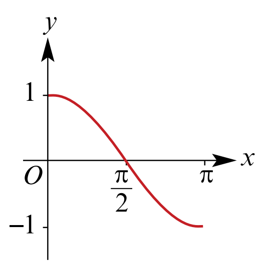


3 The exact value of the expression $\sin\left(\dfrac{2\pi}{3}\right) \times \cos\left(\dfrac{\pi}{4}\right) \times \tan\left(\dfrac{\pi}{6}\right)$ is

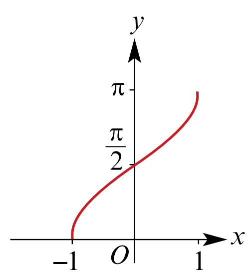
A $\dfrac{1}{\sqrt{2}}$ **B** $\dfrac{1}{\sqrt{3}}$ **C** $\dfrac{\sqrt{2}}{4}$ **D** $\dfrac{\sqrt{3}}{2}$

4 Which of the following is the graph of the function $y = \cos^{-1}(x)$?

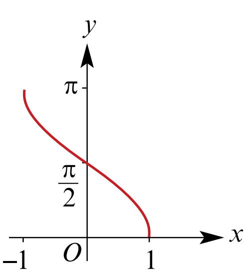
A



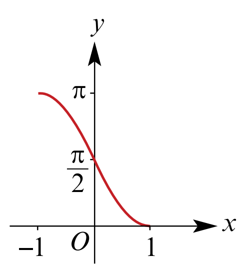
B



C



D



Review

5 If $\cos x = \dfrac{-2}{3}$ and $2\pi < x < 3\pi$, then the exact value of $\sin x$ is

A $2\pi + \dfrac{\sqrt{5}}{3}$ **B** $2\pi - \dfrac{\sqrt{5}}{3}$ **C** $\dfrac{\sqrt{5}}{3}$ **D** $\dfrac{-\sqrt{5}}{3}$

6 Given that $\cos(x) = \dfrac{-1}{10}$ and $x \in \left(\dfrac{\pi}{2}, \pi\right)$, the value of $\cot(x)$ is

A $3\sqrt{11}$ **B** $-3\sqrt{11}$ **C** $\dfrac{\sqrt{11}}{33}$ **D** $\dfrac{-\sqrt{11}}{33}$

7 The graph of the function $y = 2 + \sec(3x)$, for $x \in \left(-\dfrac{\pi}{6}, \dfrac{7\pi}{6}\right)$, has stationary points at

A $x = \dfrac{\pi}{2}$ **B** $x = \dfrac{\pi}{3}, \pi$ **C** $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ **D** $x = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi$

8 If $\sin x = \dfrac{-1}{3}$, then the possible values of $\cos x$ are

A $\dfrac{-2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}$ **B** $\dfrac{-2}{3}, \dfrac{2}{3}$ **C** $\dfrac{-8}{9}, \dfrac{8}{9}$ **D** $\dfrac{-\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$

9 The maximal domain of $y = \cos^{-1}(1 - 5x)$ is given by

A $\left[0, \dfrac{2}{5}\right]$ **B** $\left[\dfrac{1-\pi}{5}, \dfrac{1}{5}\right]$ **C** $[-1, 1]$ **D** $\left[-\dfrac{1}{5}, \dfrac{1}{5}\right]$

10 The number of solutions of $\cos^2(3x) = \dfrac{1}{4}$, given that $0 \le x \le \pi$, is

A 1 **B** 2 **C** 3 **D** 6

Technology-active multiple-choice questions

11 If $\sin A = t$ and $\cos B = t$, where $\dfrac{\pi}{2} < A < \pi$ and $0 < B < \dfrac{\pi}{2}$, then $\cos(B + A)$ is equal to

A $\sqrt{1 - t^2}$ **B** $2t^2 - 1$ **C** $1 - 2t^2$ **D** $-2t\sqrt{1 - t^2}$

12 Correct to two decimal places, the obtuse angle x satisfying $3 \sin x - 1 = 0$ is

A 2.70 **B** 2.80 **C** 0.24 **D** 0.34

13 Suppose θ is an acute angle. If $\cos(2\theta) = 0.3$, then correct to two decimal places, $\sin\theta$ is

A −0.59 **B** 0.59 **C** −0.81 **D** 0.81

14 If $\sec\theta = -\dfrac{3}{2}$ and $\theta \in (\pi, 2\pi)$ then the value of θ correct to two decimal places is

A 2.30 **B** 3.98 **C** 4.21 **D** 4.32

15 If $\operatorname{cosec}\theta = -\dfrac{3}{2}$ and $\theta \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$, then correct to two decimal places, $\cos\theta$ is

A 0.45 **B** −0.45 **C** 0.75 **D** −0.75

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Review

16 Let $m \in \mathbb{R}$ and consider the graphs of $y = \sin x$ and $y = mx$. These graphs intersect n times. The value of n cannot be equal to

A 1 **B** 2 **C** 3 **D** 5

17 Suppose $\sin A = \frac{2}{3}$ and $\cos B = \frac{1}{4}$, where $\frac{\pi}{2} < A < \pi$ and $\frac{3\pi}{2} < B < 2\pi$.

Correct to two decimal places, $\sin(A + B)$ is equal to

A 0.59 **B** 0.69 **C** 0.79 **D** 0.89

18 The minimum value of $\dfrac{1}{\cos(2x) + 2\cos^2(x) + 3}$ correct to two decimal places is

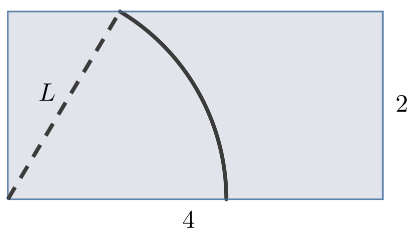
A 0.09 **B** 0.13 **C** 0.17 **D** 0.23

19 Suppose θ is an acute angle in a right-angled triangle. The length of its hypotenuse is 2 and $\cot\theta = \frac{3}{2}$. Correct to two decimal places, the length of its shortest side is

A 1.44 **B** 1.33 **C** 1.22 **D** 1.11

20 A circle arc of radius L is centred at the corner of a 4×2 rectangle. The arc cuts the rectangle into two regions of equal area. Correct to two decimal places, the value of L is

A 1.98 **B** 2.24 **C** 2.33 **D** 2.41



2

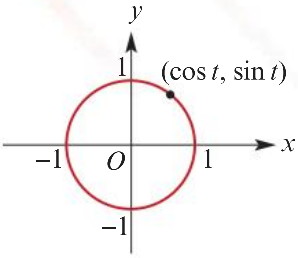
Cartesian and parametric equations

Chapter contents

In this chapter, we first consider the Cartesian equations of three important types of curves in the plane: circles, ellipses and hyperbolas. These curves are called *conic sections*, because they arise as the cross-section of a pair of cones. We will use these curves in our study of vector equations in Chapter 6 and vector calculus in Chapter 7.

We also introduce parametric equations for curves in the plane. For example, the unit circle can be described by the pair of parametric equations

$$x = \cos t \quad \text{and} \quad y = \sin t \quad \text{for } t \in \mathbb{R}$$



Parametric equations will be used in various contexts throughout this book:

- In Chapter 6, they are used to describe lines and curves in three-dimensional space.
- In Chapter 7, they are used in our study of motion along a curve.
- In Chapter 8, they are used to describe the solutions of systems of linear equations.

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2A Circles

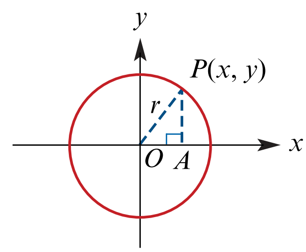
Learning intentions

- To be able to sketch the graphs of circles.
- To be able to determine the equation of a circle.

The set of points r units from the origin is a **circle**. The value of r is called the **radius** of the circle.

If a point with coordinates (x, y) lies on the circle, then Pythagoras' theorem gives

$$x^2 + y^2 = r^2$$



Cartesian equation of a circle

The circle with centre (h, k) and radius r is the graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2$$

Note: This circle is obtained from the circle with equation $x^2 + y^2 = r^2$ by the translation defined by $(x, y) \to (x + h, y + k)$.

Example 1

Sketch the graph of the circle with centre $(-2, 5)$ and radius 2, and state the Cartesian equation for this circle.

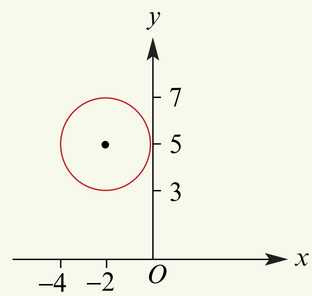
Solution

The equation is

$$(x + 2)^2 + (y - 5)^2 = 4$$

which may also be written as

$$x^2 + y^2 + 4x - 10y + 25 = 0$$



The equation $x^2 + y^2 + 4x - 10y + 25 = 0$ can be restored to the more familiar form by completing the square:

$$\begin{aligned} x^2 + y^2 + 4x - 10y + 25 &= 0 \\ x^2 + 4x + 4 + y^2 - 10y + 25 + 25 &= 29 \\ (x + 2)^2 + (y - 5)^2 &= 4 \end{aligned}$$

This suggests a general form of the equation of a circle:

$$\boxed{x^2 + y^2 + Dx + Ey + F = 0}$$

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Example 2

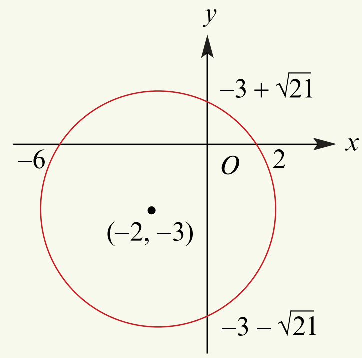
Sketch the graph of $x^2 + y^2 + 4x + 6y - 12 = 0$. State the coordinates of the centre and the radius.

Solution

Complete the square in both x and y:

$$x^2 + y^2 + 4x + 6y - 12 = 0$$
$$x^2 + 4x + 4 + y^2 + 6y + 9 - 12 = 13$$
$$(x+2)^2 + (y+3)^2 = 25$$

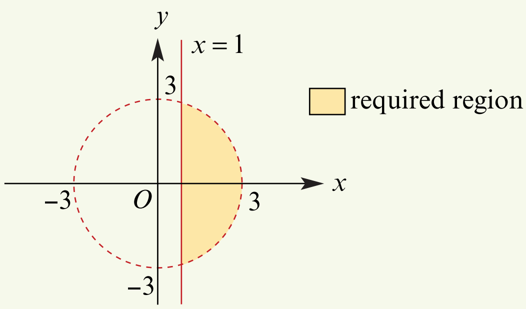
The circle has centre $(-2, -3)$ and radius 5.



Example 3

Sketch a graph of the region of the plane such that $x^2 + y^2 < 9$ and $x \geq 1$.

Solution



Exercise 2A

Example 1 **1** For each of the following, determine the equation of the circle with the given centre and radius:

- **a** centre $(2, 3)$; radius 1
- **b** centre $(-3, 4)$; radius 5
- **c** centre $(0, -5)$; radius 5
- **d** centre $(3, 0)$; radius $\sqrt{2}$

Example 2 **2** Determine the radius and the coordinates of the centre of the circle with equation:

- **a** $x^2 + y^2 + 4x - 6y + 12 = 0$
- **b** $x^2 + y^2 - 2x - 4y + 1 = 0$
- **c** $x^2 + y^2 - 3x = 0$
- **d** $x^2 + y^2 + 4x - 10y + 25 = 0$

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3 Sketch the graph of each of the following:

a $2x^2 + 2y^2 + x + y = 0$ **b** $x^2 + y^2 + 3x - 4y = 6$

c $x^2 + y^2 + 8x - 10y + 16 = 0$ **d** $x^2 + y^2 - 8x - 10y + 16 = 0$

e $2x^2 + 2y^2 - 8x + 5y + 10 = 0$ **f** $3x^2 + 3y^2 + 6x - 9y = 100$

Example 3

4 For each of the following, sketch the graph of the specified region of the plane:

a $x^2 + y^2 \leq 16$ **b** $x^2 + y^2 \geq 9$

c $(x-2)^2 + (y-2)^2 < 4$ **d** $(x-3)^2 + (y+2)^2 > 16$

e $x^2 + y^2 \leq 16$ and $x \leq 2$ **f** $x^2 + y^2 \leq 9$ and $y \geq -1$

5 The points $(8, 4)$ and $(2, 2)$ are the ends of a diameter of a circle. Determine the coordinates of the centre and the radius of the circle.

6 Determine the equation of the circle with centre $(2, -3)$ that touches the x-axis.

CF

7 Determine the equation of the circle that passes through $(3, 1)$, $(8, 2)$ and $(2, 6)$.

8 Consider the circles with equations

$$4x^2 + 4y^2 - 60x - 76y + 536 = 0 \quad \text{and} \quad x^2 + y^2 - 10x - 14y + 49 = 0$$

a Determine the radius and the coordinates of the centre of each circle.

b Determine the coordinates of the points of intersection of the two circles.

9 Determine the coordinates of the points of intersection of the circle with equation $x^2 + y^2 = 25$ and the line with equation:

a $y = x$ **b** $y = 2x$

2B Ellipses and hyperbolas

Learning intentions

▶ To be able to sketch the graphs of ellipses and hyperbolas.

Ellipses and hyperbolas will arise in our study of vector equations in Chapter 6 and vector calculus in Chapter 7. In this section, we sketch the graphs of these curves.

Ellipses

For positive constants a and b, the curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is obtained from the unit circle $x^2 + y^2 = 1$ by applying the following dilations:

- a dilation of factor a from the y-axis, i.e. $(x, y) \to (ax, y)$
- a dilation of factor b from the x-axis, i.e. $(x, y) \to (x, by)$.

The result is the transformation $(x, y) \to (ax, by)$.

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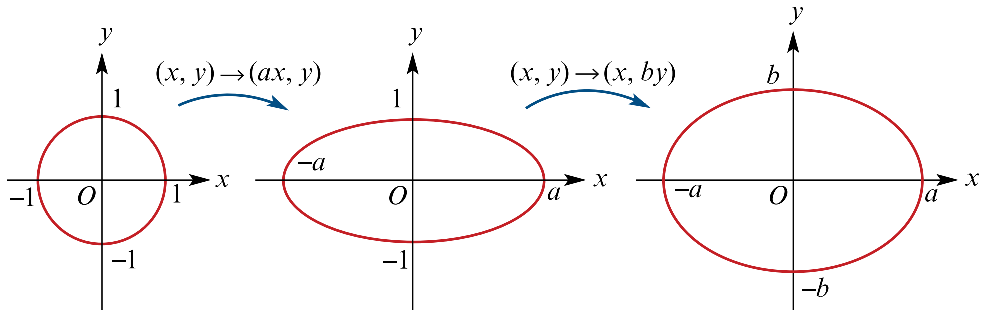


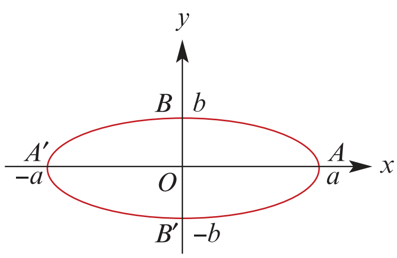
The curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is an ellipse centred at the origin with x-axis intercepts at $(-a, 0)$ and $(a, 0)$ and with y-axis intercepts at $(0, -b)$ and $(0, b)$.

If $a = b$, then the ellipse is a circle centred at the origin with radius a.

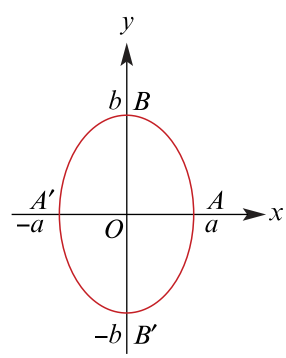
Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a > b$



AA' is the major axis

BB' is the minor axis

Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $b > a$



AA' is the minor axis

BB' is the major axis

Cartesian equation of an ellipse

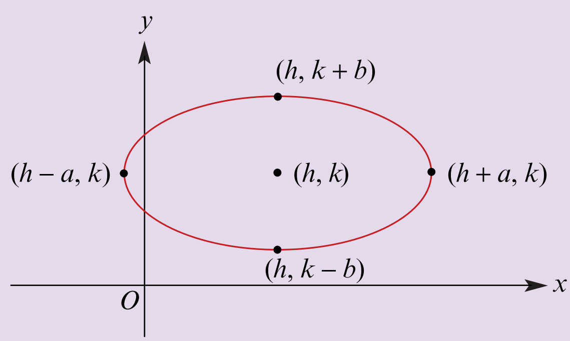
The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse with centre (h, k). It is obtained from the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by the translation $(x, y) \to (x + h, y + k)$.



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Example 4

Sketch the graph of each of the following ellipses. Give the coordinates of the centre and the axis intercepts.

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

b $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

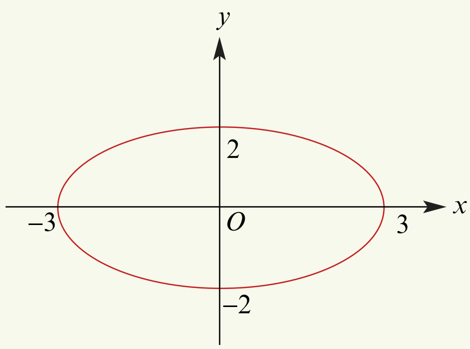
c $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{16} = 1$

d $3x^2 + 24x + y^2 + 36 = 0$

Solution

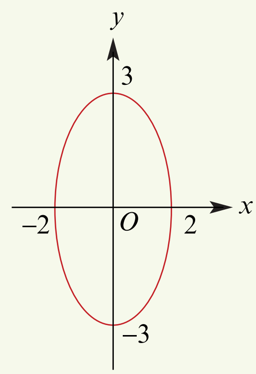
a Centre $(0, 0)$

Axis intercepts $(\pm 3, 0)$ and $(0, \pm 2)$



b Centre $(0, 0)$

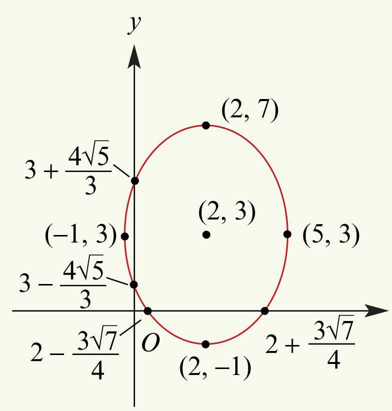
Axis intercepts $(\pm 2, 0)$ and $(0, \pm 3)$



c Centre $(2, 3)$

***y*-axis intercepts**

When $x = 0$:
$$\frac{4}{9} + \frac{(y-3)^2}{16} = 1$$
$$\frac{(y-3)^2}{16} = \frac{5}{9}$$
$$(y-3)^2 = \frac{16 \times 5}{9}$$
$$\therefore \quad y = 3 \pm \frac{4\sqrt{5}}{3}$$



***x*-axis intercepts**

When $y = 0$:
$$\frac{(x-2)^2}{9} + \frac{9}{16} = 1$$
$$\frac{(x-2)^2}{9} = \frac{7}{16}$$
$$(x-2)^2 = \frac{9 \times 7}{16}$$
$$\therefore \quad x = 2 \pm \frac{3\sqrt{7}}{4}$$

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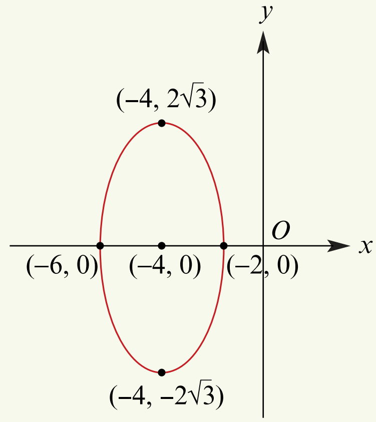
d Completing the square:

$$3x^2 + 24x + y^2 + 36 = 0$$
$$3(x^2 + 8x + 16) + y^2 + 36 - 48 = 0$$
$$3(x+4)^2 + y^2 = 12$$

i.e. $$\frac{(x+4)^2}{4} + \frac{y^2}{12} = 1$$

Centre $(-4, 0)$

Axis intercepts $(-6, 0)$ and $(-2, 0)$



The equation of an ellipse can be written in the form

$$Ax^2 + By^2 + Cx + Ey + F = 0$$

where A and B are positive. If $A = B$, then the graph is a circle.

Hyperbolas

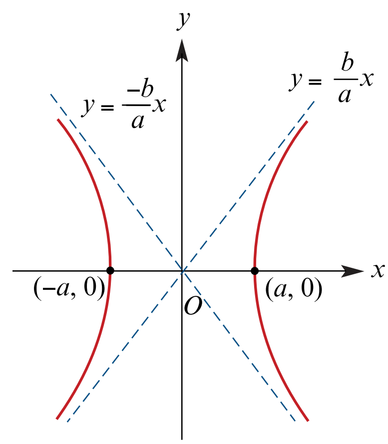
The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is a hyperbola centred at the origin with axis intercepts $(a, 0)$ and $(-a, 0)$.

The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

To see why this should be the case, we rearrange the equation of the hyperbola as follows:



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$
$$\therefore \quad y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

As $x \to \pm\infty$, we have $\frac{a^2}{x^2} \to 0$. This suggest that

$$y^2 \to \frac{b^2x^2}{a^2}$$

i.e. $$y \to \pm\frac{bx}{a}$$

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Cartesian equation of a hyperbola

The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola with centre (h, k). The asymptotes are

$$y - k = \pm\frac{b}{a}(x-h)$$

Note: This hyperbola is obtained from the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the translation defined by $(x, y) \to (x+h, y+k)$.

Example 5

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

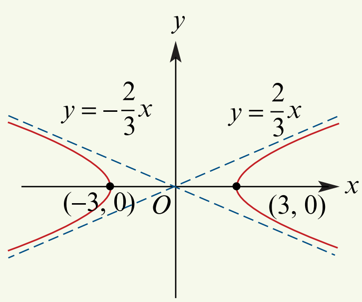
Solution

a Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.



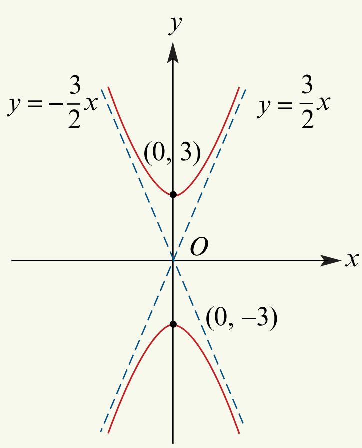
b Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

The y-axis intercepts are $(0, 3)$ and $(0, -3)$.

The centre is $(0, 0)$.



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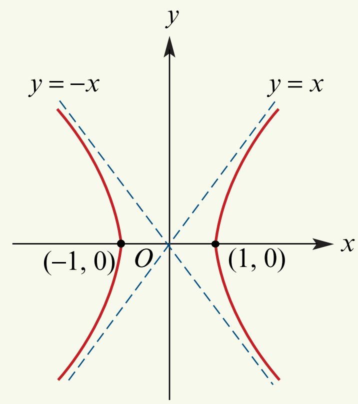
c First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This is called a **rectangular hyperbola**, as its asymptotes are perpendicular.



Now to sketch the graph of

$$(x-1)^2 - (y+2)^2 = 1$$

we apply the translation $(x, y) \to (x+1, y-2)$.

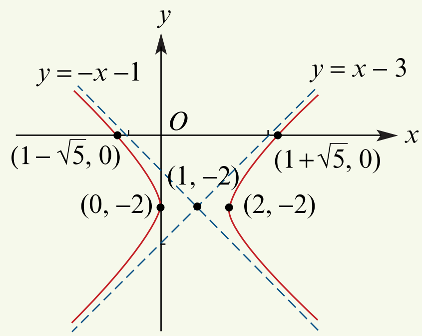
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x-1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

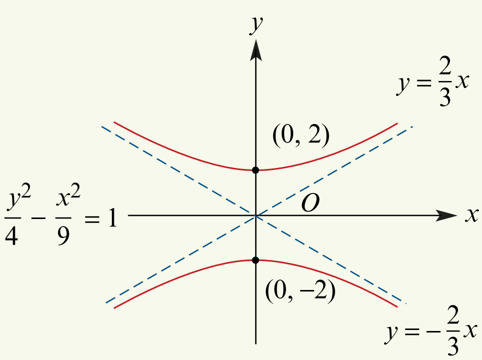
If $x = 0$, then $y = -2$.

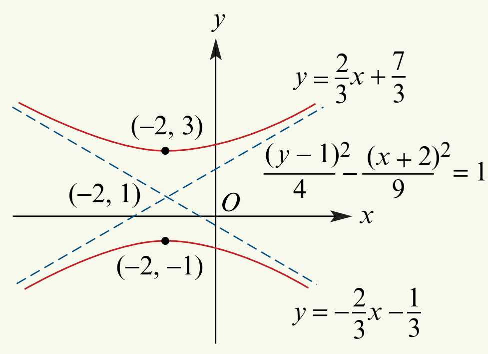
If $y = 0$, then $(x-1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.



d The graph of $\dfrac{(y-1)^2}{4} - \dfrac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ through the translation $(x, y) \to (x-2, y+1)$. Its centre will be $(-2, 1)$.





The axis intercepts are $\left(0, 1 \pm \dfrac{2\sqrt{13}}{3}\right)$.

Note: The hyperbolas $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ and $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ have the same asymptotes; they are called **conjugate hyperbolas**.

Skill-sheet

Exercise 2B

Example 4

1 Sketch the graph of each of the following. Label the axis intercepts and state the coordinates of the centre.

a $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b $25x^2 + 16y^2 = 400$

c $\frac{(x-4)^2}{9} + \frac{(y-1)^2}{16} = 1$

d $x^2 + \frac{(y-2)^2}{9} = 1$

e $9x^2 + 25y^2 - 54x - 100y = 44$

f $9x^2 + 25y^2 = 225$

g $5x^2 + 9y^2 + 20x - 18y - 16 = 0$

h $16x^2 + 25y^2 - 32x + 100y - 284 = 0$

i $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

j $2(x-2)^2 + 4(y-1)^2 = 16$

Example 5

2 Sketch the graph of each of the following. Label the axis intercepts and give the equations of the asymptotes.

a $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b $\frac{y^2}{16} - \frac{x^2}{9} = 1$

c $x^2 - y^2 = 4$

d $2x^2 - y^2 = 4$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$

f $9x^2 - 25y^2 - 90x + 150y = 225$

g $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

h $4x^2 - 8x - y^2 + 2y = 0$

i $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

j $25x^2 - 16y^2 = 400$

3 Determine the coordinates of the points of intersection of $y = \frac{1}{2}x$ with:

a $x^2 - y^2 = 1$

b $\frac{x^2}{4} + y^2 = 1$

4 Show that there is no intersection point of the line $y = x + 5$ with the ellipse $x^2 + \frac{y^2}{4} = 1$.

5 Determine the points of intersection of the curves $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Show that the points of intersection are the vertices of a square.

6 Determine the coordinates of the points of intersection of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and the line with equation $5x = 4y$.

7 On the one set of axes, sketch the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 9$.

8 Consider a hyperbola whose equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a, b > 0$.

a The asymptotes of the hyperbola are $y = 2x + 3$ and $y = -2x + 7$. Determine the values of h and k.

b The hyperbola is also tangent to the y-axis. Determine the values of a and b.

SF

CF

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2C Parametric equations

Learning intentions

- To be able to express parametric equations in Cartesian form and sketch their graphs.

In Chapter 6, we will study motion along a curve. A **parameter** (usually t representing time) will be used to help describe these curves. In this section, we give an introduction to parametric equations of curves in the plane.

The unit circle

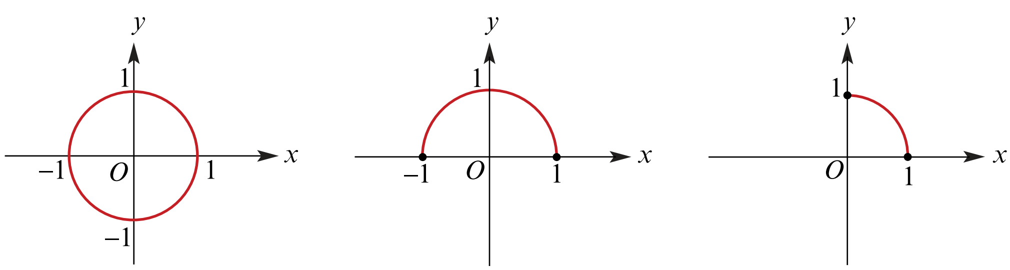
The unit circle can be expressed in Cartesian form as $\{ (x, y) : x^2 + y^2 = 1 \}$. We have seen in Section 1A Chapter 1, Section that the unit circle can also be described by two equations

$$x = \cos t \quad \text{and} \quad y = \sin t \quad \text{for } t \in \mathbb{R}$$

These are **parametric equations** for the unit circle.

We still obtain the entire unit circle if we restrict the values of t to the interval $[0, 2\pi]$. The following three diagrams illustrate the graphs obtained from the parametric equations $x = \cos t$ and $y = \sin t$ for three different sets of values of t.

$t \in [0, 2\pi]$ $\qquad$ $t \in [0, \pi]$ $\qquad$ $t \in \left[0, \frac{\pi}{2}\right]$



Circles

Parametric equations for a circle centred at the origin

The circle with centre the origin and radius a is described by the parametric equations

$$x = a \cos t \quad \text{and} \quad y = a \sin t$$

The entire circle is obtained by taking $t \in [0, 2\pi]$.

Note: To obtain the Cartesian equation, first rearrange the parametric equations as

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{a} = \sin t$$

Square and add these equations to obtain

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \cos^2 t + \sin^2 t = 1$$

This equation can be written as $x^2 + y^2 = a^2$, which is the Cartesian equation of the circle with centre the origin and radius a.

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The domain and range of the circle can be found from the parametric equations:

- **Domain** The range of the function with rule $x = a\cos t$ is $[-a, a]$.
  Hence the domain of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.
- **Range** The range of the function with rule $y = a\sin t$ is $[-a, a]$.
  Hence the range of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.

Example 6

A circle is defined by the parametric equations

$$x = 2 + 3\cos\theta \quad \text{and} \quad y = 1 + 3\sin\theta \quad \text{for } \theta \in [0, 2\pi]$$

Determine the Cartesian equation of the circle, and state the domain and range of this relation.

Solution

Domain

The range of the function with rule $x = 2 + 3\cos\theta$ is $[-1, 5]$. Hence the domain of the corresponding Cartesian relation is $[-1, 5]$.

Range

The range of the function with rule $y = 1 + 3\sin\theta$ is $[-2, 4]$. Hence the range of the corresponding Cartesian relation is $[-2, 4]$.

Cartesian equation

Rewrite the parametric equations as

$$\frac{x-2}{3} = \cos\theta \quad \text{and} \quad \frac{y-1}{3} = \sin\theta$$

Square both sides of each of these equations and add:

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{9} = \cos^2\theta + \sin^2\theta = 1$$

i.e. $\quad (x-2)^2 + (y-1)^2 = 9$

Parametric equations for a circle

The circle with centre (h, k) and radius a is described by the parametric equations

$$x = h + a\cos t \quad \text{and} \quad y = k + a\sin t$$

The entire circle is obtained by taking $t \in [0, 2\pi]$.

Parametric equations in general

A **parametric curve** in the plane is defined by a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**. Each value of t gives a point $\big(f(t), g(t)\big)$ in the plane. The set of all such points will be a curve in the plane.

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Note: Suppose $x = f(t)$ and $y = g(t)$ are parametric equations for a curve C. If we eliminate the parameter t between the two equations, then each point of the curve C lies on the curve represented by the resulting Cartesian equation.

Example 7

A curve is defined parametrically by the equations

$$x = at^2 \quad \text{and} \quad y = 2at \quad \text{for } t \in \mathbb{R}$$

where a is a positive constant. Determine:

a the Cartesian equation of the curve

b the equation of the line passing through the points where $t = 1$ and $t = -2$

c the length of the chord joining the points where $t = 1$ and $t = -2$.

Solution

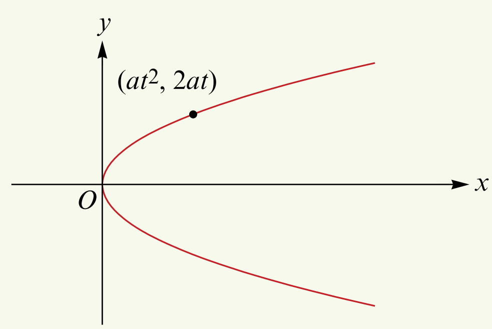
a The second equation gives $t = \dfrac{y}{2a}$.

Substitute this into the first equation:

$$x = at^2 = a\left(\frac{y}{2a}\right)^2$$

$$= a\left(\frac{y^2}{4a^2}\right)$$

$$= \frac{y^2}{4a}$$



This can be written as $y^2 = 4ax$.

b At $t = 1$, $x = a$ and $y = 2a$. This is the point $(a, 2a)$.

At $t = -2$, $x = 4a$ and $y = -4a$. This is the point $(4a, -4a)$.

The gradient of the line is

$$m = \frac{2a - (-4a)}{a - 4a} = \frac{6a}{-3a} = -2$$

Therefore the equation of the line is

$$y - 2a = -2(x - a)$$

which simplifies to $y = -2x + 4a$.

c The chord joining $(a, 2a)$ and $(4a, -4a)$ has length

$$\sqrt{(a - 4a)^2 + (2a - (-4a))^2} = \sqrt{9a^2 + 36a^2}$$

$$= \sqrt{45a^2}$$

$$= 3\sqrt{5}\,a \qquad (\text{since } a > 0)$$

Ellipses

Parametric equations for an ellipse

The ellipse with the Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be described by the parametric equations

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$

The entire ellipse is obtained by taking $t \in [0, 2\pi]$.

Note: We can rearrange these parametric equations as

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

Square and add these equations to obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

The domain and range of the ellipse can be found from the parametric equations:

- **Domain** The range of the function with rule $x = a\cos t$ is $[-a, a]$.
  Hence the domain of the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $[-a, a]$.
- **Range** The range of the function with rule $y = b\sin t$ is $[-b, b]$.
  Hence the range of the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $[-b, b]$.

Example 8

Determine the Cartesian equation of the curve with parametric equations

$$x = 3 + 3\sin t \quad \text{and} \quad y = 2 - 2\cos t \quad \text{for } t \in \mathbb{R}$$

Describe the curve.

Solution

We can rearrange the two equations as

$$\frac{x-3}{3} = \sin t \quad \text{and} \quad \frac{2-y}{2} = \cos t$$

Now square both sides of each equation and add:

$$\frac{(x-3)^2}{9} + \frac{(2-y)^2}{4} = \sin^2 t + \cos^2 t = 1$$

Since $(2-y)^2 = (y-2)^2$, this equation can be written more neatly as

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

This is the equation of an ellipse with centre $(3, 2)$ and axis intercepts at $(3, 0)$ and $(0, 2)$.

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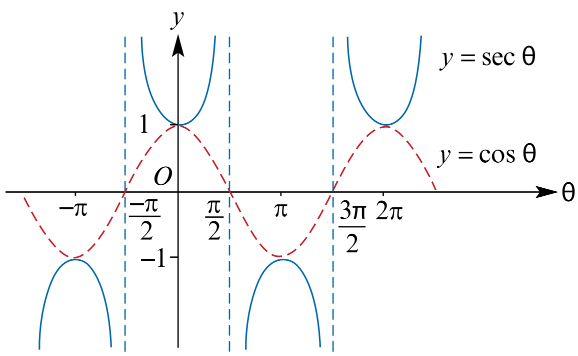
Hyperbolas

In order to give parametric equations for hyperbolas, we will use the **secant function**, which is defined by

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{if } \cos\theta \neq 0$$

The graphs of $y = \sec\theta$ and $y = \cos\theta$ are shown here on the same set of axes. The secant function was revised in Section 1B of Chapter 1.



We will also use one of the alternative forms of the Pythagorean identity from Section 1B of Chapter 1:

$$\tan^2\theta + 1 = \sec^2\theta$$

We will use this identity in the form

$$\sec^2\theta - \tan^2\theta = 1$$

Parametric equations for a hyperbola

The hyperbola with the Cartesian equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ can be described by the parametric equations

$$x = a\sec t \quad \text{and} \quad y = b\tan t \quad \text{for } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Note: We can rearrange these parametric equations as

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

Square and subtract these equations to obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

The domain and range of the hyperbola can be determined from the parametric equations.

- **Domain** There are two cases, giving the left and right branches of the hyperbola:
  - For $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of the function with rule $x = a\sec t$ is $[a, \infty)$.
    The domain $[a, \infty)$ gives the right branch of the hyperbola.
  - For $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, the range of the function with rule $x = a\sec t$ is $(-\infty, a]$.
    The domain $(-\infty, a]$ gives the left branch of the hyperbola.
- **Range** For both sections of the domain, the range of the function with rule $y = b\tan t$ is $\mathbb{R}$.

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Example 9

Determine the Cartesian equation of the curve with parametric equations

$$x = 3\sec t \quad \text{and} \quad y = 4\tan t \quad \text{for } t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Describe the curve.

Solution

Rearrange the two equations:

$$\frac{x}{3} = \sec t \quad \text{and} \quad \frac{y}{4} = \tan t$$

Square both sides of each equation and subtract:

$$\frac{x^2}{9} - \frac{y^2}{16} = \sec^2 t - \tan^2 t = 1$$

The Cartesian equation of the curve is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

The range of the function with rule $x = 3\sec t$ for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is $(-\infty, -3]$. Hence the domain for the graph is $(-\infty, -3]$.

The curve is the left branch of a hyperbola centred at the origin with x-axis intercept at $(-3, 0)$. The equations of the asymptotes are $y = \frac{4x}{3}$ and $y = -\frac{4x}{3}$.

Determining parametric equations for a curve

When converting from a Cartesian equation to a pair of parametric equations, there are many different possible choices.

Example 10

Give parametric equations for each of the following:

a $x^2 + y^2 = 9$

b $\frac{x^2}{16} + \frac{y^2}{4} = 1$

c $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1$

Solution

a One possible solution is $x = 3\cos t$ and $y = 3\sin t$ for $t \in [0, 2\pi]$.

Another solution is $x = -3\cos(2t)$ and $y = 3\sin(2t)$ for $t \in [0, \pi]$.

Yet another solution is $x = 3\sin t$ and $y = 3\cos t$ for $t \in \mathbb{R}$.

b One solution is $x = 4\cos t$ and $y = 2\sin t$.

c One solution is $x = 1 + 3\sec t$ and $y = -1 + 2\tan t$.

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Using a graphics calculator with parametric equations

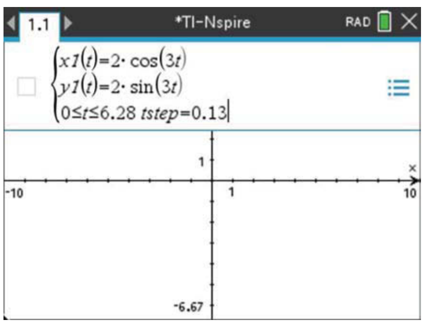
Example 11

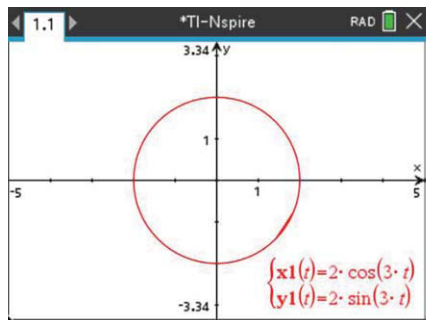
Plot the graph of the parametric curve given by

$$x = 2\cos(3t) \quad \text{and} \quad y = 2\sin(3t)$$

Using the TI-Nspire CX non-CAS

- Open a **Graphs** application (on > **New** > **Add Graphs**).
- Use menu > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.





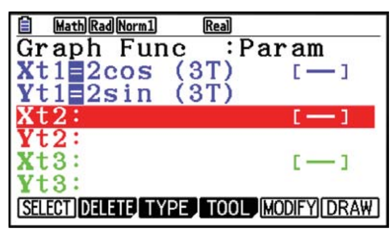
Using the Casio

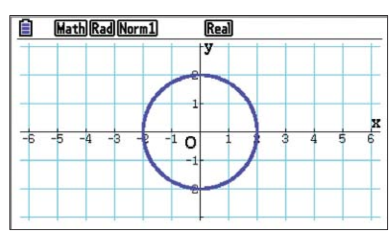
- Press MENU 5 to select **Graph** mode.
- To specify the graph type, go to **Type** F3 and choose **Parametric** F3.
- Enter the rule $x = 2\cos(3t)$ in $Xt1$:

  2 cos (3 X,θ,T) EXE
- Enter the rule $y = 2\sin(3t)$ in $Yt1$:

  2 sin (3 X,θ,T) EXE
- Adjust the View Window:

  SHIFT F3 F1 EXE
- Select **Draw** F6.





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Skill-sheet

Exercise 2C

Example 6

1 Determine the Cartesian equation of the curve with parametric equations $x = 2\cos(3t)$ and $y = 2\sin(3t)$, and determine the domain and range of the corresponding relation.

SF

Example 7

2 A curve is defined parametrically by the equations $x = 4t^2$ and $y = 8t$ for $t \in \mathbb{R}$. Determine:

a the Cartesian equation of the curve

b the equation of the line passing through the points where $t = 1$ and $t = -1$

c the length of the chord joining the points where $t = 1$ and $t = -3$.

Example 8

3 Determine the Cartesian equation of the curve with parametric equations

$$x = 2 + 3\sin t \quad \text{and} \quad y = 3 - 2\cos t \quad \text{for } t \in \mathbb{R}$$

Describe the curve.

Example 9

4 Determine the Cartesian equation of the curve with parametric equations

$$x = 2\sec t \quad \text{and} \quad y = 3\tan t \quad \text{for } t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Describe the curve.

5 Determine the corresponding Cartesian equation for each pair of parametric equations:

a $x = 4\cos(2t)$ and $y = 4\sin(2t)$

b $x = 2\sin(2t)$ and $y = 2\cos(2t)$

c $x = 4\cos t$ and $y = 3\sin t$

d $x = 4\sin t$ and $y = 3\cos t$

e $x = 2\tan(2t)$ and $y = 3\sec(2t)$

f $x = 1 - t$ and $y = t^2 - 4$

g $x = t + 2$ and $y = \dfrac{1}{t}$

h $x = t^2 - 1$ and $y = t^2 + 1$

i $x = t - \dfrac{1}{t}$ and $y = 2\left(t + \dfrac{1}{t}\right)$

6 For each of the following pairs of parametric equations, determine the Cartesian equation of the curve and sketch its graph:

a $x = \sec t,\ y = \tan t,\ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

b $x = 3\cos(2t),\ y = -4\sin(2t)$

c $x = 3 - 3\cos t,\ y = 2 + 2\sin t$

d $x = 3\sin t,\ y = 4\cos t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

e $x = \sec t,\ y = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

f $x = 1 - \sec(2t),\ y = 1 + \tan(2t),\ t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

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SF

7 A circle is defined by the parametric equations

$$x = 2\cos(2t) \quad \text{and} \quad y = -2\sin(2t) \quad \text{for } t \in \mathbb{R}$$

a Determine the coordinates of the point P on the circle where $t = \frac{4\pi}{3}$.

b Determine the equation of the tangent to the circle at P.

Example 10

8 Give parametric equations corresponding to each of the following:

a $x^2 + y^2 = 16$

b $\frac{x^2}{9} - \frac{y^2}{4} = 1$

c $(x-1)^2 + (y+2)^2 = 9$

d $\frac{(x-1)^2}{9} + \frac{(y+3)^2}{4} = 9$

9 A circle has centre $(1, 3)$ and radius 2. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where a, b, c and d are positive constants, state the values of a, b, c and d.

10 An ellipse has x-axis intercepts $(-4, 0)$ and $(4, 0)$ and y-axis intercepts $(0, 3)$ and $(0, -3)$. State a possible pair of parametric equations for this ellipse.

11 The circle with parametric equations $x = 2\cos(2t)$ and $y = 2\sin(2t)$ is dilated by a factor of 3 from the x-axis. For the image curve, state:

a a possible pair of parametric equations

b the Cartesian equation.

CF

12 The ellipse with parametric equations $x = 3 - 2\cos\left(\frac{t}{2}\right)$ and $y = 4 + 3\sin\left(\frac{t}{2}\right)$ is translated 3 units in the negative direction of the x-axis and 2 units in the negative direction of the y-axis. For the image curve, state:

a a possible pair of parametric equations

b the Cartesian equation.

13 Sketch the graph of the curve with parametric equations $x = 2 + 3\sin(2\pi t)$ and $y = 4 + 2\cos(2\pi t)$ for:

a $t \in [0, \frac{1}{4}]$

b $t \in [0, \frac{1}{2}]$

c $t \in [0, \frac{3}{2}]$

For each of these graphs, state the domain and range.

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Review

Chapter summary

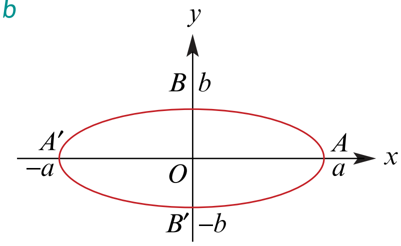
Circles

- The circle with centre at the origin and radius a has Cartesian equation $x^2 + y^2 = a^2$.
- The circle with centre (h, k) and radius a has equation $(x - h)^2 + (y - k)^2 = a^2$.

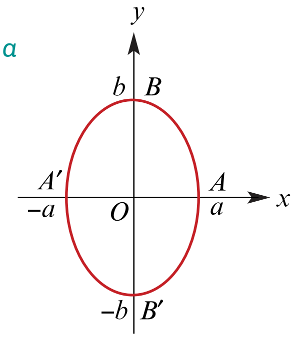
Ellipses

- The curve with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse centred at the origin with axis intercepts $(\pm a, 0)$ and $(0, \pm b)$.

$a > b$



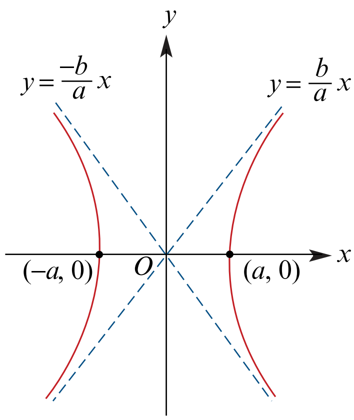
$b > a$



- The curve with equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is an ellipse with centre (h, k).

Hyperbolas

- The curve with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola centred at the origin.
  - The axis intercepts are $(\pm a, 0)$.
  - The asymptotes have equations $y = \pm\frac{b}{a}x$.



- The curve with equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is a hyperbola with centre (h, k). The asymptotes have equations $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.

Parametric equations

- A **parametric curve** in the plane is defined by a pair of functions

  $$x = f(t) \quad \text{and} \quad y = g(t)$$

  where t is called the **parameter** of the curve.

Review

- Parameterisations of familiar curves:

	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = a^2$	$x = a\cos t$ and $y = a\sin t$
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos t$ and $y = b\sin t$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x = a\sec t$ and $y = b\tan t$

Note: To obtain the entire circle or the entire ellipse using these parametric equations, it suffices to take $t \in [0, 2\pi]$.

- Translations of parametric curves: The circle with equation $(x-h)^2 + (y-k)^2 = a^2$ can also be described by the parametric equations $x = h + a\cos t$ and $y = k + a\sin t$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

2A **1** I can sketch the graphs of circles. ☐

See Example 1, Example 2, Example 3, Question 1, Question 2 and Question 3

2A **2** I can determine the equation of a circle. ☐

See Question 6 and Question 7

2B **3** I can sketch the graphs of ellipses. ☐

See Example 4 and Question 1

2B **4** I can sketch the graphs of hyperbolas. ☐

See Example 5 and Question 2

2C **5** I can determine the Cartesian equation corresponding to two parametric equations. ☐

See Example 6, Example 7, Example 8, Question 5 and Question 6

2C **6** I can determine parametric equations corresponding to a Cartesian equation. ☐

See Example 10 and Question 8

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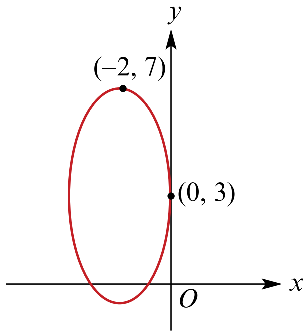
Review

Short-response questions

Technology-free short-response questions

SF

1 Write down the equation of the ellipse shown.



2 Determine the equations of the asymptotes of the hyperbola with rule $x^2 - \dfrac{(y-2)^2}{9} = 15$.

3 A curve is defined by the parametric equations $x = 3\cos(2t) + 4$ and $y = \sin(2t) - 6$. Give the Cartesian equation of the curve.

4 A curve is defined by the parametric equations $x = 2\cos(\pi t)$ and $y = 2\sin(\pi t) + 2$. Give the Cartesian equation of the curve.

5 A circle has centre $(1, 2)$ and radius 3. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where a, b, c and d are positive constants, state the values of a, b, c and d.

6 Determine the centre and radius of the circle with equation $x^2 + 8x + y^2 - 12y + 3 = 0$.

7 Determine the x- and y-axis intercepts of the ellipse with equation $\dfrac{x^2}{81} + \dfrac{y^2}{9} = 1$.

Technology-active short-response questions

CF

8 An ellipse is defined by the rule $\dfrac{x^2}{2} + \dfrac{(y+3)^2}{5} = 1$.

a Determine:

- **i** the domain of the relation
- **ii** the range of the relation
- **iii** the centre of the ellipse.

An ellipse E is given by the rule $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$. The domain of E is $[-1, 3]$ and its range is $[-1, 5]$.

b Determine the values of a, b, h and k.

The line $y = x - 2$ intersects the ellipse E at $A(1, -1)$ and at P.

c Determine the coordinates of the point P.

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A line perpendicular to the line $y = x - 2$ is drawn at P. This line intersects the y-axis at the point Q.

d Determine the coordinates of Q.

e Determine the equation of the circle through A, P and Q.

CF

9 **a** Show that the circle with equation $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the x-axis and the y-axis.

b Show that every circle that touches both the x-axis and the y-axis has an equation of a similar form.

c Hence show that there are exactly two circles that pass through the point $(2, 4)$ and just touch the x-axis and the y-axis, and give their equations.

d For each of these two circles, state the coordinates of the centre and give the radius.

e For each circle, determine the gradient of the line which passes through the centre and the point $(2, 4)$.

f For each circle, determine the equation of the tangent to the circle at the point $(2, 4)$.

CU

10 A circle is defined by the parametric equations $x = a\cos t$ and $y = a\sin t$. Let P be the point with coordinates $(a\cos t, a\sin t)$, with $\sin t \neq 0$.

a Determine the equation of the straight line which passes through the origin and the point P.

b State the coordinates, in terms of t, of the other point of intersection of the circle with the straight line through the origin and P.

c Determine the equation of the tangent to the circle at the point P.

d Determine the coordinates of the points of intersection A and B of the tangent with the x-axis and the y-axis respectively.

e Determine the area of triangle OAB in terms of t if $0 < t < \frac{\pi}{2}$. Determine the value of t for which the area of this triangle is a minimum.

Multiple-choice questions

Technology-free multiple-choice questions

1 A circle has a diameter with endpoints at $(4, -2)$ and $(-2, -2)$. The equation of the circle is

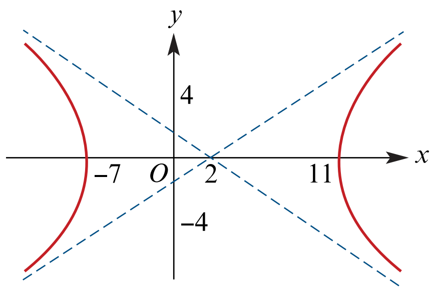
A $(x - 1)^2 + (y - 2)^2 = 3$

B $(x - 1)^2 + (y + 2)^2 = 3$

C $(x + 1)^2 + (y - 2)^2 = 6$

D $(x - 1)^2 + (y + 2)^2 = 9$

2 The equation of the graph shown is



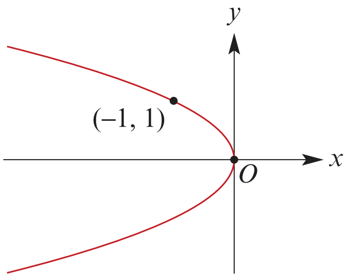
A $\dfrac{(x+2)^2}{27} - \dfrac{y^2}{108} = 1$

B $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{34} = 1$

C $\dfrac{(x+2)^2}{81} - \dfrac{y^2}{324} = 1$

D $\dfrac{(x-2)^2}{81} - \dfrac{y^2}{324} = 1$

3 Which of the following pairs of parametric equations describes the parabola shown?



A $x = t, \quad y = t^2$

B $x = t, \quad y = \sqrt{t}$

C $x = t^2, \quad y = t$

D $x = -t^2, \quad y = t$

4 If the line $x = k$ is a tangent to the circle with equation $(x-1)^2 + (y+2)^2 = 1$, then k is equal to

A 0 or 2 **B** 1 or −2

C 1 or 3 **D** −1 or −3

5 The curve with equation $x^2 - 2x = y^2$ is

A an ellipse with centre $(1, 0)$ **B** a hyperbola with centre $(1, 0)$

C a circle with centre $(1, 0)$ **D** an ellipse with centre $(-1, 0)$

6 A curve is defined parametrically by the equations $x = 2\cos(t)$ and $y = 2\cos(2t)$. The Cartesian equation of the curve is

A $y = 2 + x^2$ **B** $y = x^2 - 2$

C $y = 2x$ **D** $y = x$

7 A curve is defined parametrically by the equations $x = 2\sec t$ and $y = 3\tan t$. The point on the curve where $t = -\dfrac{\pi}{3}$ is

A $(4, 3\sqrt{3})$ **B** $(4, -3\sqrt{3})$

C $(4\sqrt{3}, -4)$ **D** $(-4, -3\sqrt{3})$

8 The asymptotes of a hyperbola are $y = 3x - 2$ and $y = -3x + 4$. The equation of the hyperbola could be

A $\dfrac{(y+1)^2}{9} - \dfrac{(x-1)^2}{3} = 1$ **B** $\dfrac{(y-1)^2}{9} - \dfrac{(x+1)^2}{3} = 1$

C $\dfrac{(x-1)^2}{9} - (y-1)^2 = 1$ **D** $\dfrac{(y-1)^2}{9} - (x-1)^2 = 1$

Review

Technology-active multiple-choice questions

9 The ellipse given by $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$ has two axial intercepts. Accurate to 2 decimal places, the distance between these intercepts is equal to

A 3.87 **B** 3.74
C 3.61 **D** 3.45

10 Consider the pair of parametric equations $x = 3 + 2\cos(2t)$ and $y = -1 + 2\sin(2t)$ defined on the interval $t \in \left[0, \dfrac{\pi}{4}\right]$. The length of the curve that this defines is equal to

A 2π units **B** π units
C $\dfrac{\pi}{2}$ units **D** $\dfrac{\pi}{4}$ units

11 A curve is parameterised by the equations $x = t^2$ and $y = t + 1$ where $t \in \mathbb{R}$. The graph of $y = x - 11$ intersects the curves at points A and B. Accurate to 2 decimal places, length AB is

A 9.80 **B** 9.85
C 9.90 **D** 9.95

12 The graph of $x^2 + y^2 = 2ax$ will be a circle with

A centre $(-a, 0)$ and radius a **B** centre $(a, 0)$ and radius a
C centre $(-a, 0)$ and radius $2a$ **D** centre $(a, 0)$ and radius $\sqrt{2a}$

13 The graph of $x^2 + 4y^2 + 8y = c$ will be an ellipse provided

A $c < 8$ **B** $c > -8$
C $c > -4$ **D** $c < 4$

14 A hyperbola is parameterised by the equations $x = 1 + \sec t$ and $y = 3\tan t$. The asymptotes of this hyperbola are

A $y = 3x - 3$ and $y = -3x + 3$ **B** $y = 3x + 3$ and $y = -3x - 3$
C $y = 3x - 1$ and $y = -3x + 1$ **D** $y = 3x + 1$ and $y = -3x - 1$

15 A ellipse is parameterised by the equations $x = a + b\cos t$ and $y = c + d\sin t$. For the ellipse to pass through the origin we require that

A $a^2d^2 - b^2c^2 = b^2d^2$ **B** $a^2d^2 + b^2c^2 = b^2d^2$
C $a^2b^2 - c^2d^2 = b^2d^2$ **D** $a^2b^2 + c^2d^2 = b^2d^2$

16 The equation of an ellipse is $(x-2)^2 + 4y^2 = 1$. If $y = mx$ is tangent to the ellipse, then

A $m = \pm\dfrac{\sqrt{2}}{6}$ **B** $m = \pm\dfrac{\sqrt{10}}{10}$
C $m = \pm\dfrac{\sqrt{2}}{4}$ **D** $m = \pm\dfrac{\sqrt{3}}{6}$

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3

Further complex numbers

Chapter contents

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as 'impossible', they arose in calculations to determine real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano's formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

This chapter covers Unit 3 Topic 1: Further complex numbers. It also contains the proof by induction of de Moivre's theorem and proving multi-angle trigonometric identities from Unit 3 Topic 2: Mathematical induction and trigonometric proofs.

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3A Building the complex numbers

Learning intentions

- ▶ To be able to identify the real and imaginary parts of a complex number.
- ▶ To be able to add, subtract and multiply complex numbers.

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can determine square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)}\\ &= \sqrt{4} \times \sqrt{-1}\\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b, but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers. Note that every real number is a complex number.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$$

The letter often used to denote a complex number is z.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\mathrm{Re}(z) = a \quad \text{and} \quad \mathrm{Im}(z) = b$$

where $\mathrm{Re}(z)$ is called the **real part** of z and $\mathrm{Im}(z)$ is called the **imaginary part** of z.

Note: Both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers.

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Example 1

Let $z = 4 - 5i$. Determine:

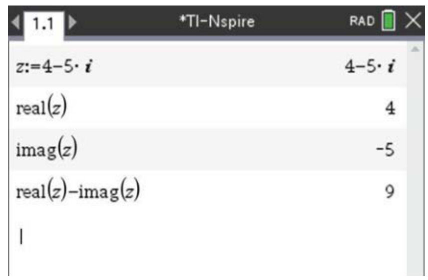
a $\text{Re}(z)$ **b** $\text{Im}(z)$ **c** $\text{Re}(z) - \text{Im}(z)$

Solution

a $\text{Re}(z) = 4$ **b** $\text{Im}(z) = -5$ **c** $\text{Re}(z) - \text{Im}(z) = 4 - (-5) = 9$

Using the TI-Nspire CX non-CAS

- Assign the complex number z, as shown in the first line. Use (π▸) to access i.
- To determine the real part, use (menu) > **Number** > **Complex Number Tools** > **Real Part**, or just type real(.
- For the imaginary part, use (menu) > **Number** > **Complex Number Tools** > **Imaginary Part**.



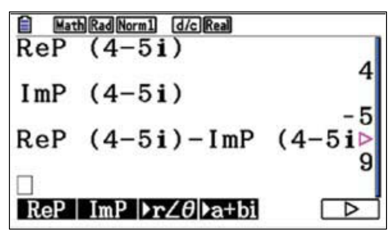
Hint: You do not need to be in complex mode. If you use i in the input, then it will display in the same format.

Using the Casio

Calculations with complex numbers can be performed in **Run-Matrix** mode using the **Complex numbers** menu (OPTN) (F3).

To determine the real and imaginary parts of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- For the real part, select **ReP** (F6) (F1) and enter the complex number as shown. (For the symbol i, select **i** (F6) (F1).)
- For the imaginary part, use **ImP** (F6) (F2).



Hint: Copy and paste the previous entry lines to determine the difference.

Example 2

a Represent $\sqrt{-5}$ as an imaginary number. **b** Simplify $2\sqrt{-9} + 4i$.

Solution

a
$$\begin{aligned}\sqrt{-5} &= \sqrt{5 \times (-1)}\\ &= \sqrt{5} \times \sqrt{-1}\\ &= i\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}2\sqrt{-9} + 4i &= 2\sqrt{9 \times (-1)} + 4i\\ &= 2 \times 3 \times i + 4i\\ &= 6i + 4i\\ &= 10i\end{aligned}$$

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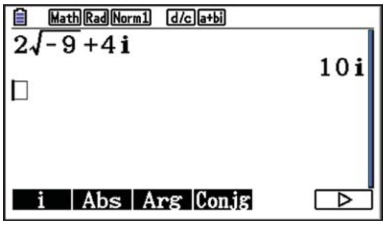
Using the TI-Nspire CX non-CAS

Enter the expression and press enter.

Using the Casio

- In **Run-Matrix** mode, ensure that the Complex Mode setting is $a + bi$. (To change this setting, go to the set-up screen SHIFT MENU.)
- Enter the expression as shown and press EXE.



Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 3

Solve the equation $(2a - 3) + 2bi = 5 + 6i$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Solution

If $(2a - 3) + 2bi = 5 + 6i$, then

$$2a - 3 = 5 \quad \text{and} \quad 2b = 6$$

$$\therefore \quad a = 4 \quad \text{and} \quad b = 3$$

Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$.

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The following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

Multiplication by a scalar

If $z = a + bi$ and $k \in \mathbb{R}$, then $kz = k(a + bi) = ka + kbi$.

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

It is easy to check that $k(z_1 + z_2) = kz_1 + kz_2$, for all $k \in \mathbb{R}$.

Example 4

Let $z_1 = 2 - 3i$ and $z_2 = 1 + 4i$. Simplify:

a $z_1 + z_2$ **b** $z_1 - z_2$ **c** $3z_1 - 2z_2$

Solution

a $z_1 + z_2$
$= (2 - 3i) + (1 + 4i)$
$= 3 + i$

b $z_1 - z_2$
$= (2 - 3i) - (1 + 4i)$
$= 1 - 7i$

c $3z_1 - 2z_2$
$= 3(2 - 3i) - 2(1 + 4i)$
$= 4 - 17i$

TI-84

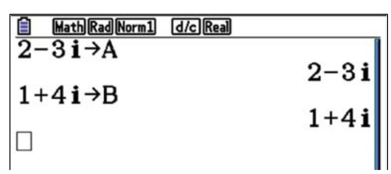
Using the TI-Nspire CX non-CAS

Enter the expressions as shown.

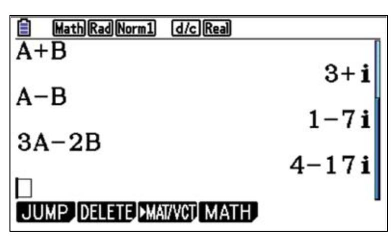
Using the Casio

- In **Run-Matrix** mode, store the two complex numbers as A and B.

Hint: For the symbol i, you can press SHIFT 0.



- Enter the expressions as shown.



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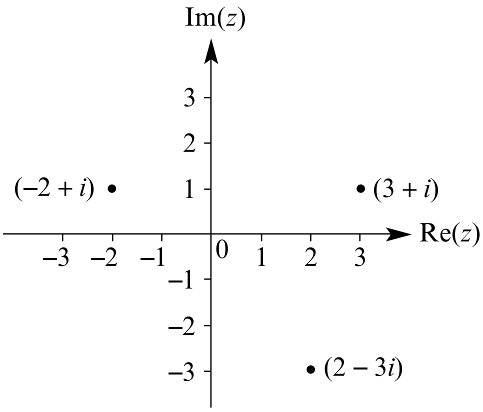
Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. In a vector sense, a complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\text{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.



Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.

A complex number written as $a + bi$ is said to be in **Cartesian form**.

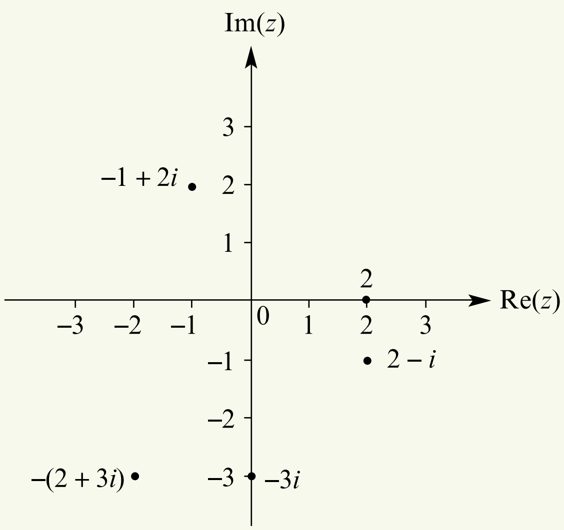
Example 5

Represent the following complex numbers as points on an Argand diagram:

a 2 **b** $-3i$ **c** $2 - i$

d $-(2 + 3i)$ **e** $-1 + 2i$

Solution



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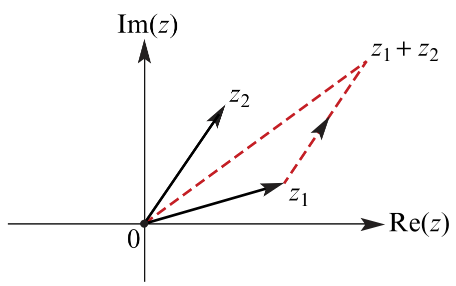
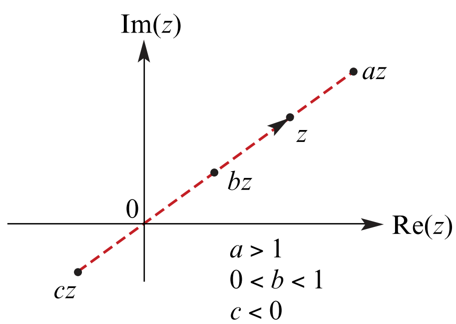
Geometric representation of the basic operations on complex numbers

Addition of complex numbers is analogous to addition of vectors. The sum of two complex numbers corresponds to the sum of their position vectors.

Multiplication of a complex number by a scalar corresponds to the multiplication of its position vector by the scalar.





The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

Example 6

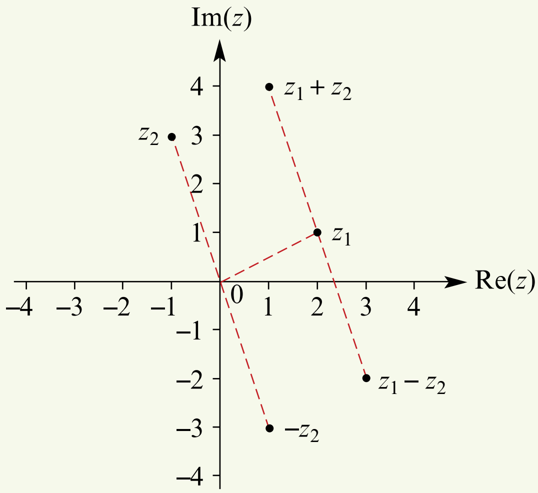
Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

Solution

$$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$$

$$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$$



Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad (\text{since } i^2 = -1) \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$. The following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z \times 1 = z$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 7

Simplify:

a $(2 + 3i)(1 - 5i)$ **b** $3i(5 - 2i)$ **c** i^3

Solution

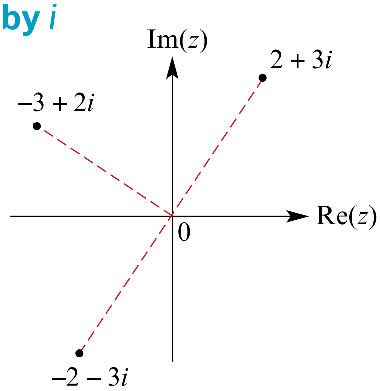
a $$\begin{aligned} (2 + 3i)(1 - 5i) &= 2 - 10i + 3i - 15i^2 \\ &= 2 - 10i + 3i + 15 \\ &= 17 - 7i \end{aligned}$$

b $$\begin{aligned} 3i(5 - 2i) &= 15i - 6i^2 \\ &= 15i + 6 \\ &= 6 + 15i \end{aligned}$$

c $$\begin{aligned} i^3 &= i \times i^2 \\ &= -i \end{aligned}$$

Geometric significance of multiplication by *i*

When the complex number $2 + 3i$ is multiplied by -1, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.



When the complex number $2 + 3i$ is multiplied by i, we obtain

$$\begin{aligned} i(2 + 3i) &= 2i + 3i^2 \\ &= 2i - 3 \\ &= -3 + 2i \end{aligned}$$

The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

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Powers of i

Successive multiplication by i gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \ldots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Exercise 3A

SF

Example 1

1 Let $z = 6 - 7i$. Determine:

a $\mathrm{Re}(z)$ **b** $\mathrm{Im}(z)$ **c** $\mathrm{Re}(z) - \mathrm{Im}(z)$

Example 2

2 Simplify each of the following:

a $\sqrt{-25}$ **b** $\sqrt{-27}$ **c** $2i - 7i$

d $5\sqrt{-16} - 7i$ **e** $\sqrt{-8} + \sqrt{-18}$ **f** $i\sqrt{-12}$

g $i(2 + i)$ **h** $\mathrm{Im}(2\sqrt{-4})$ **i** $\mathrm{Re}(5\sqrt{-49})$

Example 3

3 Solve the following equations for real values x and y:

a $x + yi = 5$ **b** $x + yi = 2i$

c $x = yi$ **d** $x + yi = (2 + 3i) + 7(1 - i)$

e $2x + 3 + 8i = -1 + (2 - 3y)i$ **f** $x + yi = (2y + 1) + (x - 7)i$

Example 4

4 Let $z_1 = 2 - i$, $z_2 = 3 + 2i$ and $z_3 = -1 + 3i$. Determine:

a $z_1 + z_2$ **b** $z_1 + z_2 + z_3$ **c** $2z_1 - z_3$

d $3 - z_3$ **e** $4i - z_2 + z_1$ **f** $\mathrm{Re}(z_1)$

g $\mathrm{Im}(z_2)$ **h** $\mathrm{Im}(z_3 - z_2)$ **i** $\mathrm{Re}(z_2) - i\,\mathrm{Im}(z_2)$

Example 5

5 Represent each of the following complex numbers on an Argand diagram:

a $-4i$ **b** -3 **c** $2(1 + i)$

d $3 - i$ **e** $-(3 + 2i)$ **f** $-2 + 3i$

Example 6

6 Let $z_1 = 1 + 2i$ and $z_2 = 2 - i$.

a Represent the following complex numbers on an Argand diagram:

i z_1 **ii** z_2 **iii** $2z_1 + z_2$ **iv** $z_1 - z_2$

b Verify that parts **iii** and **iv** correspond to vector addition and subtraction.

Example 7

7 Simplify each of the following:

a $(5 - i)(2 + i)$ **b** $(4 + 7i)(3 + 5i)$ **c** $(2 + 3i)(2 - 3i)$

d $(1 + 3i)^2$ **e** $(2 - i)^2$ **f** $(1 + i)^3$

g i^4 **h** $i^{11}(6 + 5i)$ **i** i^{70}

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SF

8 Solve each of the following equations for real values x and y:

a $2x + (y + 4)i = (3 + 2i)(2 - i)$

b $(x + yi)(3 + 2i) = -16 + 11i$

c $(x + 2i)^2 = 5 - 12i$

d $(x + yi)^2 = -18i$

e $i(2x - 3yi) = 6(1 + i)$

9 **a** Represent each of the following complex numbers on an Argand diagram:

i $1 + i$ **ii** $(1 + i)^2$ **iii** $(1 + i)^3$ **iv** $(1 + i)^4$

b Describe any geometric pattern observed in the position of these complex numbers.

10 Let $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$. Let P, Q and R be the points defined on an Argand diagram by z_1, z_2 and $z_2 - z_1$ respectively.

a Show that $\overrightarrow{PQ} = \overrightarrow{OR}$.

b Hence determine QP.

3B Modulus, conjugate and division

Learning intentions

- To be able to determine the modulus and the complex conjugate of a complex number.
- To be able to divide complex numbers.

The modulus of a complex number

Definition of the modulus

For $z = a + bi$, the **modulus** of z is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2}$$

This is the distance of the complex number from the origin.

For example, if $z_1 = 3 + 4i$ and $z_2 = -3 + 4i$, then

$$|z_1| = \sqrt{3^2 + 4^2} = 5 \quad \text{and} \quad |z_2| = \sqrt{(-3)^2 + 4^2} = 5$$

Both z_1 and z_2 are a distance of 5 units from the origin.

Properties of the modulus

- $|z_1 z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)
- $|z_1 + z_2| \le |z_1| + |z_2|$ (triangle inequality)

These results will be proved in Exercise 3B.

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The conjugate of a complex number

Definition of the complex conjugate

For $z = a + bi$, the **complex conjugate** of z is denoted by $\overline{z}$ and is defined by

$$\overline{z} = a - bi$$

Properties of the complex conjugate

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- $\overline{kz} = k\overline{z}$, for $k \in \mathbb{R}$
- $z\overline{z} = |z|^2$
- $z + \overline{z} = 2\,\mathrm{Re}(z)$

Proof The first three results will be proved in Exercise 3B. To prove the remaining two results, consider a complex number $z = a + bi$. Then $\overline{z} = a - bi$ and therefore

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned} \qquad \begin{aligned} z + \overline{z} &= (a + bi) + (a - bi) \\ &= 2a \\ &= 2\,\mathrm{Re}(z) \end{aligned}$$

It follows from these two results that if $z \in \mathbb{C}$, then $z\overline{z}$ and $z + \overline{z}$ are real numbers. We can prove a partial converse to this property of the complex conjugate:

Let $z, w \in \mathbb{C} \setminus \mathbb{R}$ such that zw and $z + w$ are real numbers. Then $w = \overline{z}$.

Proof Write $z = a + bi$ and $w = c + di$, where $b, d \neq 0$. Then

$$\begin{aligned} z + w &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \end{aligned}$$

Since $z + w$ is real, we have $b + d = 0$. Therefore $d = -b$ and so

$$\begin{aligned} zw &= (a + bi)(c - bi) \\ &= (ac + b^2) + (bc - ab)i \end{aligned}$$

Since zw is real, we have $bc - ab = b(c - a) = 0$. As $b \neq 0$, this implies that $c = a$. We have shown that $w = a - bi = \overline{z}$.

Example 8

Determine the complex conjugate and modulus of each of the following:

a $z = 3 - 2i$ **b** $z = 2$ **c** $z = 3i$

Solution

a If $z = 3 - 2i$, then $\overline{z} = 3 + 2i$ and $|z| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$.

b If $z = 2$, then $\overline{z} = 2$ and $|z| = 2$.

c If $z = 3i$, then $\overline{z} = -3i$ and $|z| = 3$.

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Using the TI-Nspire CX non-CAS

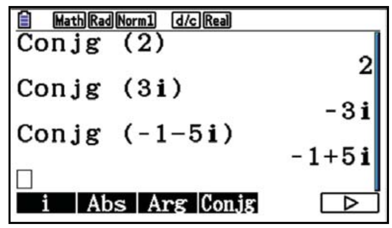
To determine the complex conjugate, use menu > **Number** > **Complex Number Tools** > **Complex Conjugate**, or just type conj(.

Note: Use $\pi\blacktriangleright$ to access i.

Using the Casio

To determine the conjugate of a complex number:

- In **Run-Matrix** mode, go to the **Complex numbers** menu OPTN F3.
- Select **Congj** F4 and enter the complex number as shown.



Division of complex numbers

We begin with some familiar algebra that will motivate the definition:

$$\frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

We can see that

$$(a+bi) \times \frac{a-bi}{a^2+b^2} = 1$$

Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a+bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a-bi}{a^2+b^2} = \frac{\overline{z}}{|z|^2}$$

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1\overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}$$

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Multiply the numerator and denominator by the conjugate of z_2:

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$
$$= \frac{(a+bi)(c-di)}{c^2+d^2}$$

We complete the division by simplifying the last expression. This process is demonstrated in the next example.

Example 9

a Write each of the following in the form $a + bi$, where $a, b \in \mathbb{R}$:

i $\dfrac{1}{3-2i}$

ii $\dfrac{4+i}{3-2i}$

b Simplify $\dfrac{(1+2i)^2}{i(1+3i)}$.

Solution

a i
$$\frac{1}{3-2i} = \frac{1}{3-2i} \times \frac{3+2i}{3+2i}$$
$$= \frac{3+2i}{3^2-(2i)^2}$$
$$= \frac{3+2i}{13}$$
$$= \frac{3}{13} + \frac{2}{13}i$$

ii
$$\frac{4+i}{3-2i} = \frac{4+i}{3-2i} \times \frac{3+2i}{3+2i}$$
$$= \frac{(4+i)(3+2i)}{3^2+2^2}$$
$$= \frac{12+8i+3i-2}{13}$$
$$= \frac{10}{13} + \frac{11}{13}i$$

b
$$\frac{(1+2i)^2}{i(1+3i)} = \frac{1+4i-4}{-3+i}$$
$$= \frac{-3+4i}{-3+i} \times \frac{-3-i}{-3-i}$$
$$= \frac{9+3i-12i+4}{(-3)^2-i^2}$$
$$= \frac{13-9i}{10}$$
$$= \frac{13}{10} - \frac{9}{10}i$$

Note: There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

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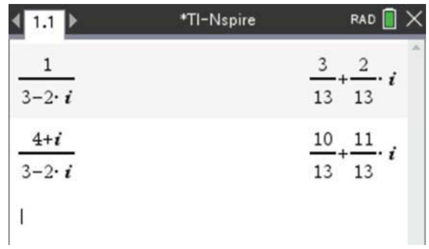
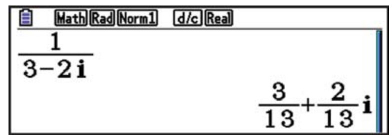
Using the TI-Nspire CX non-CAS

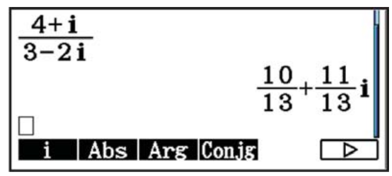
Complete as shown.



Using the Casio

- In **Run-Matrix** mode, go to the **Complex numbers** menu (OPTN) (F3).
- Enter the expressions as shown.





Exercise 3B

SF

Example 8 **1** Determine the complex conjugate and modulus of each of the following complex numbers:

a $\sqrt{3}$ **b** $8i$ **c** $4 - 3i$

d $-(1 + 2i)$ **e** $4 + 2i$ **f** $-3 - 2i$

Example 9 **2** Simplify each of the following, giving your answer in the form $a + bi$:

a $\dfrac{2 + 3i}{3 - 2i}$ **b** $\dfrac{i}{-1 + 3i}$ **c** $\dfrac{-4 - 3i}{i}$

d $\dfrac{3 + 7i}{1 + 2i}$ **e** $\dfrac{\sqrt{3} + i}{-1 - i}$ **f** $\dfrac{17}{4 - i}$

3 Let $z = a + bi$ and $w = c + di$. Show that:

a $\overline{z + w} = \overline{z} + \overline{w}$ **b** $\overline{zw} = \overline{z}\,\overline{w}$ **c** $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

d $|zw| = |z|\,|w|$ **e** $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$

4 Let $z = 2 - i$. Simplify the following:

a $z(z + 1)$ **b** $\overline{z} + 4$ **c** $\overline{z - 2i}$

d $\dfrac{z - 1}{z + 1}$ **e** $(z - i)^2$ **f** $(z + 1 + 2i)^2$

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5 For $z = a + bi$, write each of the following in terms of a and b: SF

a $z\bar{z}$

b $\dfrac{z}{|z|^2}$

c $z + \bar{z}$

d $z - \bar{z}$

e $\dfrac{z}{\bar{z}}$

f $\dfrac{\bar{z}}{z}$

6 Let $z, w \in \mathbb{C}$. If $|z| = |w| = 2$ and $|z + w| = 3$, then determine the value of $\left|\dfrac{1}{z} + \dfrac{1}{w}\right|$. CF

7 Determine all complex numbers z for which $\bar{z} = z^2$.

8 Prove that $|z_1 + z_2| \le |z_1| + |z_2|$ for all $z_1, z_2 \in \mathbb{C}$. CU

3C The polar form of a complex number

Learning intentions

- ▶ To be able to determine the polar form of a complex number.

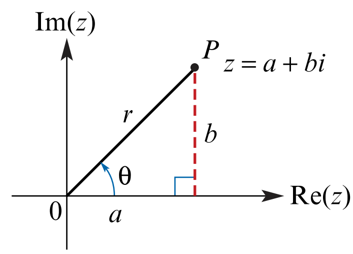
In the preceding sections, we have expressed complex numbers in Cartesian form. Another way of expressing complex numbers is by using polar form.

Each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

Polar form

The diagram shows the point P corresponding to the complex number $z = a + bi$. We see that $a = r\cos\theta$ and $b = r\sin\theta$, and so we can write

$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + (r\sin\theta)\,i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$



This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r\operatorname{cis}\theta$$

- The distance $r = \sqrt{a^2 + b^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ, measured anticlockwise from the horizontal axis, is called the **argument** of z and is denoted by $\arg z$.

Polar form for complex numbers is also called **modulus–argument form**.

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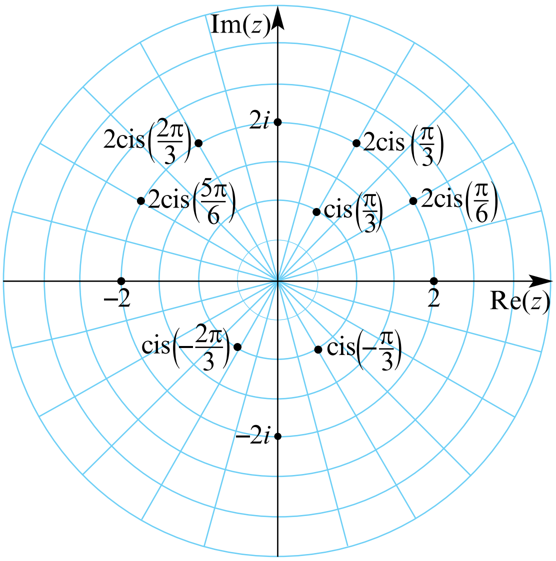
This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15°$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r\operatorname{cis}\theta = r\operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

Principal value of the argument

For a non-zero complex number z, the argument of z that belongs to the interval $(-\pi, \pi]$ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$. That is,
$-\pi < \operatorname{Arg} z \leq \pi$

Example 10

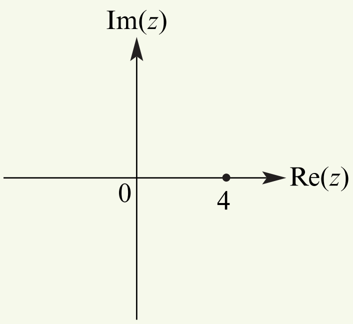
Determine the modulus and principal argument of each of the following complex numbers:

a 4 **b** $-2i$

c $1 + i$ **d** $4 - 3i$

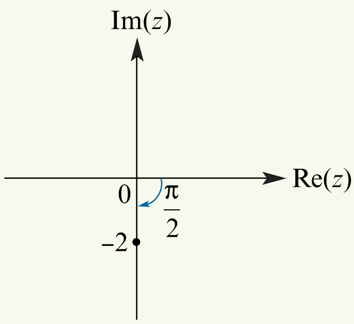
Solution

a



$|4| = 4, \quad \operatorname{Arg}(4) = 0$

b



$|-2i| = 2, \quad \operatorname{Arg}(-2i) = -\frac{\pi}{2}$

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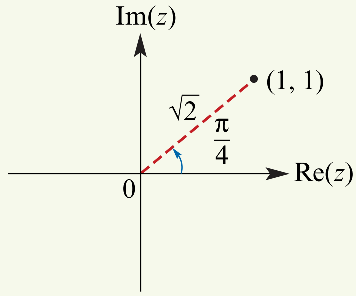
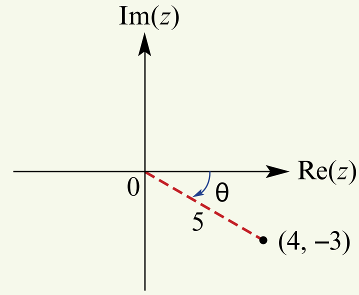
c



$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\operatorname{Arg}(1 + i) = \frac{\pi}{4}$$

d



$$|4 - 3i| = \sqrt{4^2 + (-3)^2} = 5$$

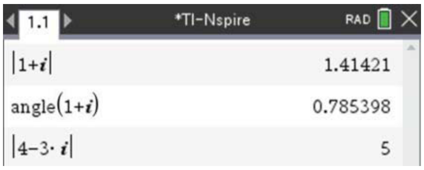
$$\operatorname{Arg}(4 - 3i) = -\tan^{-1}\left(\frac{3}{4}\right) \approx -0.64 \text{ rad}$$

TI-84

Using the TI-Nspire CX non-CAS

- To determine the modulus of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Magnitude**.
  Alternatively, use |□| from the 2D-template palette or type abs(.
- To determine the principal value of the argument, use (menu) > **Number** > **Complex Number Tools** > **Polar Angle**.

Note: Use (π▸) to access i.



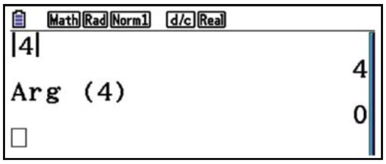
Using the Casio

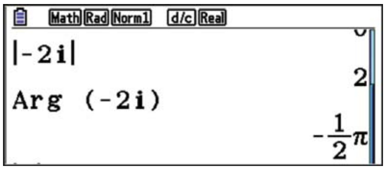
To determine the modulus of a complex number:

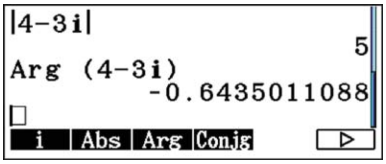
- In **Run-Matrix** mode, go to the **Complex numbers** menu (OPTN) (F3).
- Select **Abs** (F2) and enter the complex number as shown.

To determine the principal argument of a complex number:

- In **Run-Matrix** mode, go to the **Complex numbers** menu (OPTN) (F3).
- Select **Arg** (F3) and enter the complex number as shown.







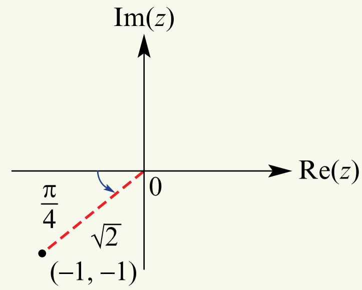
Example 11

Determine the argument of $-1-i$ in the interval $[0, 2\pi]$.

Solution

Choosing the angle in the interval $[0, 2\pi]$ gives

$$\arg(-1-i) = \frac{5\pi}{4}$$



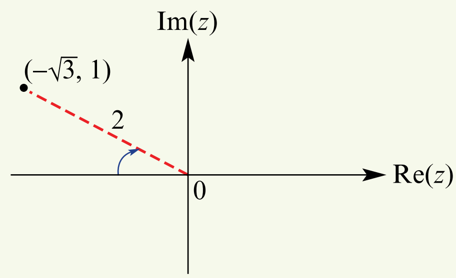
Example 12

Express $-\sqrt{3}+i$ in the form $r \operatorname{cis} \theta$, where $\theta = \operatorname{Arg}(-\sqrt{3}+i)$.

Solution

$$r = \left|-\sqrt{3}+i\right|$$
$$= \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\theta = \operatorname{Arg}(-\sqrt{3}+i) = \frac{5\pi}{6}$$



Therefore $-\sqrt{3}+i = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$

Example 13

Express $2\operatorname{cis}\left(\frac{-3\pi}{4}\right)$ in the form $a+bi$.

Solution

$$\begin{aligned} a &= r\cos\theta \\ &= 2\cos\left(\frac{-3\pi}{4}\right) \\ &= -2\cos\left(\frac{\pi}{4}\right) \\ &= -2 \times \frac{1}{\sqrt{2}} \\ &= -\sqrt{2} \end{aligned} \qquad \begin{aligned} b &= r\sin\theta \\ &= 2\sin\left(\frac{-3\pi}{4}\right) \\ &= -2\sin\left(\frac{\pi}{4}\right) \\ &= -2 \times \frac{1}{\sqrt{2}} \\ &= -\sqrt{2} \end{aligned}$$

Therefore $2\operatorname{cis}\left(\frac{-3\pi}{4}\right) = -\sqrt{2}-\sqrt{2}i$

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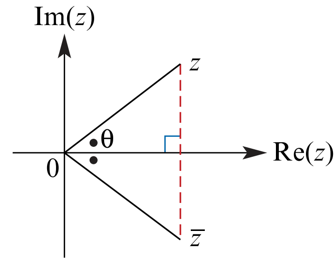
Complex conjugate in polar form

It is easy to show that the complex conjugate, $\bar{z}$, is a reflection of the point z in the horizontal axis.

Therefore, if $z = r \operatorname{cis} \theta$, then $\bar{z} = r \operatorname{cis}(-\theta)$.



Exercise 3C

Example 10

1 Find the modulus and principal argument of each of the following complex numbers:

a -3 **b** $5i$ **c** $i - 1$
d $\sqrt{3} + i$ **e** $2 - 2\sqrt{3}i$ **f** $(2 - 2\sqrt{3}i)^2$

2 Find the principal argument of each of the following, correct to two decimal places:

a $5 + 12i$ **b** $-8 + 15i$ **c** $-4 - 3i$
d $1 - \sqrt{2}i$ **e** $\sqrt{2} + \sqrt{3}i$ **f** $-(3 + 7i)$

Example 11

3 Determine the argument of each of the following in the interval stated:

a $1 - \sqrt{3}i$ in $[0, 2\pi]$ **b** $-7i$ in $[0, 2\pi]$
c $-3 + \sqrt{3}i$ in $[0, 2\pi]$ **d** $\sqrt{2} + \sqrt{2}i$ in $[0, 2\pi]$
e $\sqrt{3} + i$ in $[-2\pi, 0]$ **f** $2i$ in $[-2\pi, 0]$

4 Convert each of the following arguments into principal arguments:

a $\dfrac{5\pi}{4}$ **b** $\dfrac{17\pi}{6}$ **c** $\dfrac{-15\pi}{8}$ **d** $\dfrac{-5\pi}{2}$

Example 12

5 Convert each of the following complex numbers from Cartesian form $a + bi$ into the form $r \operatorname{cis} \theta$, where $\theta = \operatorname{Arg}(a + bi)$:

a $-1 - i$ **b** $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ **c** $\sqrt{3} - \sqrt{3}i$
d $\dfrac{1}{\sqrt{3}} + \dfrac{1}{3}i$ **e** $\sqrt{6} - \sqrt{2}i$ **f** $-2\sqrt{3} + 2i$

Example 13

6 Convert each of the following complex numbers into the form $a + bi$:

a $2 \operatorname{cis}\left(\dfrac{3\pi}{4}\right)$ **b** $5 \operatorname{cis}\left(\dfrac{-\pi}{3}\right)$ **c** $2\sqrt{2} \operatorname{cis}\left(\dfrac{\pi}{4}\right)$
d $3 \operatorname{cis}\left(\dfrac{-5\pi}{6}\right)$ **e** $6 \operatorname{cis}\left(\dfrac{\pi}{2}\right)$ **f** $4 \operatorname{cis} \pi$

7 Let $z = \operatorname{cis} \theta$. Show that:

a $|z| = 1$ **b** $\dfrac{1}{z} = \operatorname{cis}(-\theta)$

8 Determine the complex conjugate of each of the following:

a $2 \operatorname{cis}\left(\dfrac{3\pi}{4}\right)$ **b** $7 \operatorname{cis}\left(\dfrac{-2\pi}{3}\right)$ **c** $-3 \operatorname{cis}\left(\dfrac{2\pi}{3}\right)$ **d** $5 \operatorname{cis}\left(\dfrac{-\pi}{4}\right)$

SF

3D Operations in polar form

Learning intentions

- To be able to multiply and divide complex numbers in polar form.

Addition and subtraction

There is no simple way to add or subtract complex numbers in the form $r \operatorname{cis} \theta$. Complex numbers need to be expressed in the form $a + bi$ before these operations can be carried out.

Example 14

Simplify $2 \operatorname{cis}\left(\frac{\pi}{3}\right) + 3 \operatorname{cis}\left(\frac{2\pi}{3}\right)$.

Solution

First convert to Cartesian form:

$$\begin{aligned} 2 \operatorname{cis}\left(\frac{\pi}{3}\right) &= 2\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right) \\ &= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \\ &= 1 + \sqrt{3} i \end{aligned}$$

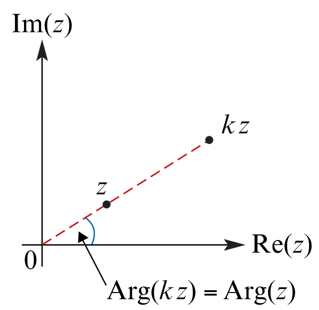
$$\begin{aligned} 3 \operatorname{cis}\left(\frac{2\pi}{3}\right) &= 3\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right) \\ &= 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \\ &= -\frac{3}{2} + \frac{3\sqrt{3}}{2} i \end{aligned}$$

Now we have

$$\begin{aligned} 2 \operatorname{cis}\left(\frac{\pi}{3}\right) + 3 \operatorname{cis}\left(\frac{2\pi}{3}\right) &= (1 + \sqrt{3} i) + \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2} i\right) \\ &= -\frac{1}{2} + \frac{5\sqrt{3}}{2} i \end{aligned}$$

Multiplication by a scalar

Positive scalar If $k \in \mathbb{R}^+$, then $\operatorname{Arg}(kz) = \operatorname{Arg}(z)$.



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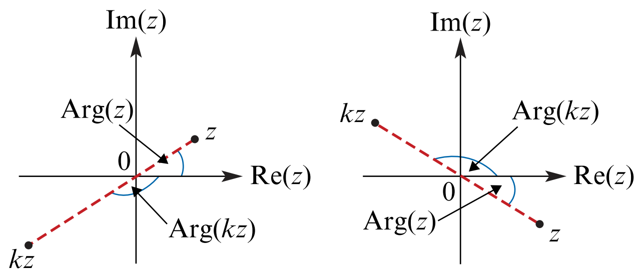
Negative scalar If $k \in \mathbb{R}^-$, then

$$\operatorname{Arg}(kz) = \begin{cases} \operatorname{Arg}(z) - \pi, & 0 < \operatorname{Arg}(z) \le \pi \\ \operatorname{Arg}(z) + \pi, & -\pi < \operatorname{Arg}(z) \le 0 \end{cases}$$



Multiplication of complex numbers

Multiplication in polar form

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{(multiply the moduli and add the angles)}$$

Proof We have

$$\begin{aligned} z_1 z_2 &= r_1 \operatorname{cis} \theta_1 \times r_2 \operatorname{cis} \theta_2 \\ &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &= r_1 r_2 \big((\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)\big) \end{aligned}$$

Now use the angle sum identities from Chapter 1:

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$$
$$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$$

Hence
$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \end{aligned}$$

Here are two useful properties of the modulus and the principal argument with regard to multiplication of complex numbers:

- $|z_1 z_2| = |z_1|\,|z_2|$
- $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0$, 1 or -1

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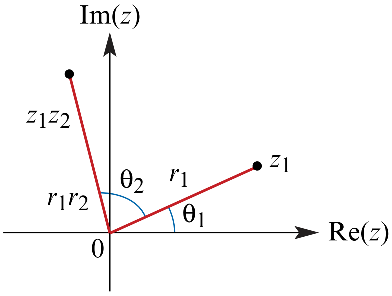
Geometric interpretation of multiplication

We have seen that:

- The modulus of the product of two complex numbers is the product of their moduli.
- The argument of the product of two complex numbers is the sum of their arguments.



Geometrically, the effect of multiplying a complex number z_1 by the complex number $z_2 = r_2 \operatorname{cis} \theta_2$ is to produce an enlargement of Oz_1, where O is the origin, by a factor of r_2 and an anticlockwise turn through an angle θ_2 about the origin.

If $r_2 = 1$, then only the turning effect will take place.

Let $z = \operatorname{cis} \theta$. Multiplication by z^2 is, in effect, the same as a multiplication by z followed by another multiplication by z. The effect is a turn of θ followed by another turn of θ. The end result is an anticlockwise turn of 2θ. This is also shown by determining z^2:

$$\begin{aligned} z^2 = z \times z = \operatorname{cis} \theta \times \operatorname{cis} \theta &= \operatorname{cis}(\theta + \theta) \qquad \text{(using the multiplication rule)} \\ &= \operatorname{cis}(2\theta) \end{aligned}$$

Division of complex numbers

Division in polar form

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$ with $r_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{(divide the moduli and subtract the angles)}$$

Proof We have already seen in Exercise 3C that $\dfrac{1}{\operatorname{cis} \theta_2} = \operatorname{cis}(-\theta_2)$.

We can now use the rule for multiplication in polar form to obtain

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis} \theta_1 \operatorname{cis}(-\theta_2) = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Here are three useful properties of the modulus and the principal argument with regard to division of complex numbers:

- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
- $\operatorname{Arg}\left(\dfrac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1
- $\operatorname{Arg}\left(\dfrac{1}{z}\right) = -\operatorname{Arg}(z)$, provided z is not a negative real number

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Example 15

Simplify:

a $2\operatorname{cis}\left(\frac{\pi}{3}\right) \times \sqrt{3}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

b $\dfrac{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}{4\operatorname{cis}\left(\frac{\pi}{5}\right)}$

Solution

a
$$\begin{aligned} 2\operatorname{cis}\left(\frac{\pi}{3}\right) \times \sqrt{3}\operatorname{cis}\left(\frac{3\pi}{4}\right) &= 2\sqrt{3}\operatorname{cis}\left(\frac{\pi}{3} + \frac{3\pi}{4}\right) \\ &= 2\sqrt{3}\operatorname{cis}\left(\frac{13\pi}{12}\right) \\ &= 2\sqrt{3}\operatorname{cis}\left(-\frac{11\pi}{12}\right) \end{aligned}$$

b
$$\begin{aligned} \frac{2\operatorname{cis}\left(\frac{2\pi}{3}\right)}{4\operatorname{cis}\left(\frac{\pi}{5}\right)} &= \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3} - \frac{\pi}{5}\right) \\ &= \frac{1}{2}\operatorname{cis}\left(\frac{7\pi}{15}\right) \end{aligned}$$

Note: A solution giving the principal value of the argument, that is, the argument in the range $(-\pi, \pi]$, is preferred unless otherwise stated.

Skill-sheet

Exercise 3D

SF

Example 14

1 Simplify $4\operatorname{cis}\left(\frac{\pi}{6}\right) + 6\operatorname{cis}\left(\frac{2\pi}{3}\right)$.

Example 15

2 Simplify each of the following:

a $4\operatorname{cis}\left(\frac{2\pi}{3}\right) \times 3\operatorname{cis}\left(\frac{3\pi}{4}\right)$

b $\dfrac{\sqrt{2}\operatorname{cis}\left(\frac{\pi}{2}\right)}{\sqrt{8}\operatorname{cis}\left(\frac{5\pi}{6}\right)}$

c $\frac{1}{2}\operatorname{cis}\left(\frac{-2\pi}{5}\right) \times \frac{7}{3}\operatorname{cis}\left(\frac{\pi}{3}\right)$

d $\dfrac{4\operatorname{cis}\left(\frac{-\pi}{4}\right)}{\frac{1}{2}\operatorname{cis}\left(\frac{7\pi}{10}\right)}$

e $\dfrac{4\operatorname{cis}\left(\frac{2\pi}{3}\right)}{32\operatorname{cis}\left(\frac{-\pi}{3}\right)}$

3 For each of the following, determine $\operatorname{Arg}(z_1z_2)$ and $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ and comment on their relationship:

a $z_1 = \operatorname{cis}\left(\frac{\pi}{4}\right)$ and $z_2 = \operatorname{cis}\left(\frac{\pi}{3}\right)$

b $z_1 = \operatorname{cis}\left(\frac{-2\pi}{3}\right)$ and $z_2 = \operatorname{cis}\left(\frac{-3\pi}{4}\right)$

c $z_1 = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ and $z_2 = \operatorname{cis}\left(\frac{\pi}{2}\right)$

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CF

4 Show that if $-\frac{\pi}{2} < \text{Arg}(z_1) < \frac{\pi}{2}$ and $-\frac{\pi}{2} < \text{Arg}(z_2) < \frac{\pi}{2}$, then

$$\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) \quad \text{and} \quad \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$$

5 For $z = 1 + i$, determine:

a $\text{Arg}\, z$ **b** $\text{Arg}(-z)$ **c** $\text{Arg}\left(\frac{1}{z}\right)$

6 **a** Express each of the following in modulus–argument form, where $0 < \theta < \frac{\pi}{2}$:

i $1 + i\tan\theta$ **ii** $1 + i\cot\theta$ **iii** $\frac{1}{\sin\theta} + \frac{1}{\cos\theta}i$

b Hence simplify each of the following:

i $(1 + i\tan\theta)^2$ **ii** $(1 + i\cot\theta)^{-3}$ **iii** $\frac{1}{\sin\theta} - \frac{1}{\cos\theta}i$

7 Let $u = \sqrt{2} + \sqrt{2}i$ and $v = \sqrt{3} + i$.

a Evaluate uv in Cartesian form.

b Determine the polar forms of u and v.

c Hence, evaluate uv by multiplying in polar form.

d Hence, determine the exact values of $\cos\left(\frac{5\pi}{6}\right)$ and $\sin\left(\frac{5\pi}{6}\right)$.

3E De Moivre's theorem

Learning intentions

- To be able to use de Moivre's theorem to determine powers of complex numbers.

De Moivre's theorem allows us to readily simplify expressions of the form z^n when z is expressed in polar form.

De Moivre's theorem

$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$, where $n \in \mathbb{Z}$

Proof For each natural number n, let $P(n)$ be the proposition:

$$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$$

Step 1 First consider $P(1)$. Note that

$$(r\,\text{cis}\,\theta)^1 = r\,\text{cis}\,\theta = r^1\,\text{cis}(1 \times \theta).$$

Therefore $P(1)$ is true.

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Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$(r\operatorname{cis}\theta)^k = r^k\operatorname{cis}(k\theta)$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$(r\operatorname{cis}\theta)^{k+1} = r^{k+1}\operatorname{cis}((k+1)\theta)$$

We have

$$\begin{aligned}\text{LHS of } P(k+1) &= (r\operatorname{cis}\theta)^{k+1}\\ &= (r\operatorname{cis}\theta)^k(r\operatorname{cis}\theta)\\ &= (r^k\operatorname{cis}(k\theta))(r\operatorname{cis}\theta) \qquad \text{(using } P(k))\\ &= r^{k+1}\operatorname{cis}(k\theta+\theta)\\ &= r^{k+1}\operatorname{cis}((k+1)\theta)\\ &= \text{RHS of } P(k+1)\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

Finally, to obtain the result for negative integers, again let $z = \operatorname{cis}\theta$. Then

$$z^{-1} = \frac{1}{z} = \bar{z} = \operatorname{cis}(-\theta)$$

For $k \in \mathbb{N}$, we have

$$z^{-k} = (z^{-1})^k = (\operatorname{cis}(-\theta))^k = \operatorname{cis}(-k\theta)$$

using the result for positive integers.

Example 16

Simplify:

a $\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9$

b $\dfrac{\operatorname{cis}\left(\frac{7\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^7}$

Solution

a
$$\begin{aligned}\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9 &= \operatorname{cis}\left(9\times\frac{\pi}{3}\right)\\ &= \operatorname{cis}(3\pi)\\ &= \operatorname{cis}\pi\\ &= \cos\pi + i\sin\pi\\ &= -1\end{aligned}$$

b
$$\begin{aligned}\frac{\operatorname{cis}\left(\frac{7\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^7} &= \operatorname{cis}\left(\frac{7\pi}{4}\right)\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^{-7}\\ &= \operatorname{cis}\left(\frac{7\pi}{4}\right)\operatorname{cis}\left(\frac{-7\pi}{3}\right)\\ &= \operatorname{cis}\left(\frac{7\pi}{4}-\frac{7\pi}{3}\right)\\ &= \operatorname{cis}\left(\frac{-7\pi}{12}\right)\end{aligned}$$

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Example 17

Simplify $\dfrac{(1+i)^3}{(1-\sqrt{3}i)^5}$.

Solution

First convert the numerator and denominator into polar form:

$$1+i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$1-\sqrt{3}i = 2\operatorname{cis}\left(\frac{-\pi}{3}\right)$$

Therefore

$$\begin{aligned}
\frac{(1+i)^3}{(1-\sqrt{3}i)^5} &= \frac{\left(\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^3}{\left(2\operatorname{cis}\left(\frac{-\pi}{3}\right)\right)^5} \\
&= \frac{2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{32\operatorname{cis}\left(\frac{-5\pi}{3}\right)} && \text{(by De Moivre's theorem)} \\
&= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{3\pi}{4} - \left(\frac{-5\pi}{3}\right)\right) \\
&= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{29\pi}{12}\right) \\
&= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{5\pi}{12}\right)
\end{aligned}$$

Example 18

a Expand $(\cos\theta + i\sin\theta)^3$.

b Hence, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Solution

a Expanding gives

$$\begin{aligned}
(\cos\theta + i\sin\theta)^3 &= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3 \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

b By de Moivre's theorem, we also know that $(\operatorname{cis}\theta)^3 = \operatorname{cis} 3\theta$. Therefore

$$\begin{aligned}(\operatorname{cis}\theta)^3 &= \operatorname{cis} 3\theta \\ &= \cos 3\theta + i\sin 3\theta.\end{aligned}$$

If we equate the real parts of these two expressions we see that

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta,\end{aligned}$$

as required.

Exercise 3E

SF

Example 16

1 Simplify each of the following:

a $2\operatorname{cis}\left(\frac{5\pi}{6}\right) \times \left(\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{8}\right)\right)^4$

b $\dfrac{1}{\left(\frac{3}{2}\operatorname{cis}\left(\frac{5\pi}{8}\right)\right)^3}$

c $\left(\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^8 \times \left(\sqrt{3}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^6$

d $\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{2}\right)\right)^{-5}$

e $\left(2\operatorname{cis}\left(\frac{3\pi}{2}\right) \times 3\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^3$

f $\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{8}\right)\right)^{-6} \times \left(4\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^2$

g $\dfrac{\left(6\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{\left(\frac{1}{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^{-5}}$

Example 17

2 Simplify each of the following, giving your answer in polar form $r\operatorname{cis}\theta$, with $r > 0$ and $\theta \in (-\pi, \pi]$:

a $(1+\sqrt{3}i)^6$

b $(1-i)^{-5}$

c $i(\sqrt{3}-i)^7$

d $(-3+\sqrt{3}i)^{-3}$

e $\dfrac{(1+\sqrt{3}i)^3}{i(1-i)^5}$

f $\dfrac{(-1+\sqrt{3}i)^4(-\sqrt{2}-\sqrt{2}i)^3}{\sqrt{3}-3i}$

g $(-1+i)^5\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^3$

h $\dfrac{\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{(1-\sqrt{3}i)^2}$

i $\left((1-i)\operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^7$

CF

3 **a** Show that $\sin\theta + i\cos\theta = \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$.

b Simplify each of the following:

i $(\sin\theta + i\cos\theta)^7$

ii $(\sin\theta + i\cos\theta)(\cos\theta + i\sin\theta)$

iii $(\sin\theta + i\cos\theta)^{-4}$

iv $(\sin\theta + i\cos\theta)(\sin\varphi + i\cos\varphi)$

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4 **a** Show that $\cos\theta - i\sin\theta = \text{cis}(-\theta)$.

b Simplify each of the following:

i $(\cos\theta - i\sin\theta)^5$

ii $(\cos\theta - i\sin\theta)^{-3}$

iii $(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)$

iv $(\cos\theta - i\sin\theta)(\sin\theta + i\cos\theta)$

5 **a** Show that $\sin\theta - i\cos\theta = \text{cis}\left(\theta - \dfrac{\pi}{2}\right)$.

b Simplify each of the following:

i $(\sin\theta - i\cos\theta)^6$

ii $(\sin\theta - i\cos\theta)^{-2}$

iii $(\sin\theta - i\cos\theta)^2(\cos\theta - i\sin\theta)$

iv $\dfrac{\sin\theta - i\cos\theta}{\cos\theta + i\sin\theta}$

Example 18

6 Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

7 We will now use De Moivre's theorem to determine expressions for $\cos 4\theta$ and $\sin 4\theta$.

a Expand $(\cos\theta + i\sin\theta)^4$ using the binomial theorem.

b By De Moivre's theorem, we know that $(\text{cis}\,\theta)^4 = \text{cis}\,4\theta$. Use this result and the result of **a** to show that:

i $\cos 4\theta = 1 - 8\cos^2\theta + 8\cos^4\theta$

ii $\sin 4\theta = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$

3F Solving quadratic equations over the complex numbers

Learning intentions

▶ To be able to use complex numbers to solve quadratic equations with negative discriminants.

Factorisation of quadratics

Quadratic polynomials with a negative discriminant cannot be factorised over the real numbers. The introduction of complex numbers enables us to factorise such quadratics.

Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

Example 19

Factorise:

a $z^2 + 16$

b $2z^2 + 6$

Solution

a
$$\begin{aligned} z^2 + 16 &= z^2 - 16i^2 \\ &= (z + 4i)(z - 4i) \end{aligned}$$

b
$$\begin{aligned} 2z^2 + 6 &= 2(z^2 + 3) \\ &= 2(z^2 - 3i^2) \\ &= 2(z + \sqrt{3}i)(z - \sqrt{3}i) \end{aligned}$$

Note: The discriminant of $z^2 + 16$ is $\Delta = 0 - 4 \times 16 = -64$.
The discriminant of $2z^2 + 6$ is $\Delta = 0 - 4 \times 2 \times 6 = -48$.

Example 20

Factorise:

a $z^2 + z + 3$

b $2z^2 - z + 1$

c $2z^2 - 2(3 - i)z + 4 - 3i$

Solution

a Let $P(z) = z^2 + z + 3$. Then, by completing the square, we have

$$\begin{aligned} P(z) &= \left(z^2 + z + \frac{1}{4}\right) + 3 - \frac{1}{4} \\ &= \left(z + \frac{1}{2}\right)^2 + \frac{11}{4} \\ &= \left(z + \frac{1}{2}\right)^2 - \frac{11}{4}i^2 \\ &= \left(z + \frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{11}}{2}i\right) \end{aligned}$$

b Let $P(z) = 2z^2 - z + 1$. Then

$$\begin{aligned} P(z) &= 2\left(z^2 - \frac{1}{2}z + \frac{1}{2}\right) \\ &= 2\left(\left(z^2 - \frac{1}{2}z + \frac{1}{16}\right) + \frac{1}{2} - \frac{1}{16}\right) \\ &= 2\left(\left(z - \frac{1}{4}\right)^2 + \frac{7}{16}\right) \\ &= 2\left(\left(z - \frac{1}{4}\right)^2 - \frac{7}{16}i^2\right) \\ &= 2\left(z - \frac{1}{4} + \frac{\sqrt{7}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{7}}{4}i\right) \end{aligned}$$

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c Let $P(z) = 2z^2 - 2(3 - i)z + 4 - 3i$. Then

$$\begin{aligned}
P(z) &= 2\left(z^2 - (3 - i)z + \frac{4 - 3i}{2}\right) \\
&= 2\left(z^2 - (3 - i)z + \left(\frac{3 - i}{2}\right)^2 + \frac{4 - 3i}{2} - \left(\frac{3 - i}{2}\right)^2\right) \\
&= 2\left(z - \frac{3 - i}{2}\right)^2 + 4 - 3i - \frac{(3 - i)^2}{2} \\
&= 2\left(z - \frac{3 - i}{2}\right)^2 + \frac{8 - 6i - 9 + 6i + 1}{2} \\
&= 2\left(z - \frac{3 - i}{2}\right)^2
\end{aligned}$$

Solving quadratic equations

In the previous example, we used the method of completing the square to factorise quadratic expressions. This method can also be used to solve quadratic equations.

Alternatively, a quadratic equation of the form $az^2 + bz + c = 0$ can be solved by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is obtained by completing the square in the expression $az^2 + bz + c$.

Example 21

Solve each of the following equations for z:

a $z^2 + z + 3 = 0$

b $2z^2 - z + 1 = 0$

c $z^2 = 2z - 5$

d $2z^2 - 2(3 - i)z + 4 - 3i = 0$

Solution

a From Example 20a:

$$z^2 + z + 3 = \left(z - \left(-\frac{1}{2} - \frac{\sqrt{11}}{2}i\right)\right)\left(z - \left(-\frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\right)$$

Hence $z^2 + z + 3 = 0$ has solutions

$$z = -\frac{1}{2} - \frac{\sqrt{11}}{2}i \quad \text{and} \quad z = -\frac{1}{2} + \frac{\sqrt{11}}{2}i$$

b From Example 20b:

$$2z^2 - z + 1 = 2\left(z - \left(\frac{1}{4} - \frac{\sqrt{7}}{4}i\right)\right)\left(z - \left(\frac{1}{4} + \frac{\sqrt{7}}{4}i\right)\right)$$

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Hence $2z^2 - z + 1 = 0$ has solutions

$$z = \frac{1}{4} - \frac{\sqrt{7}}{4}i \quad \text{and} \quad z = \frac{1}{4} + \frac{\sqrt{7}}{4}i$$

c Rearrange the expression to give $z^2 - 2z + 5 = 0$. Now use the quadratic formula:

$$\begin{aligned} z &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \end{aligned}$$

The solutions are $1 + 2i$ and $1 - 2i$.

d From Example 20c, we have

$$2z^2 - 2(3 - i)z + 4 - 3i = 2\left(z - \frac{3 - i}{2}\right)^2$$

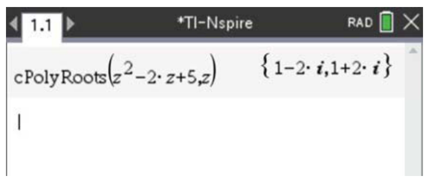
Hence $2z^2 - 2(3 - i)z + 4 - 3i = 0$ has solution $z = \dfrac{3 - i}{2}$.

Note: In parts **a**, **b** and **c** of this example, the two solutions are conjugates of each other. We explore this further in the next section.

TI-84

Using the TI-Nspire CX non-CAS

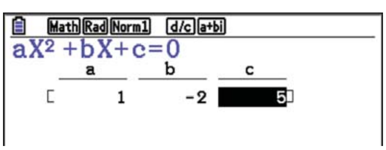
To determine the zeroes of a polynomial over the complex numbers, use (menu) > **Algebra** > **Polynomial Tools** > **Complex Roots of a Polynomial**.

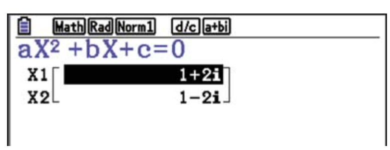


Using the Casio

To solve the quadratic equation $z^2 - 2z + 5 = 0$ over the complex numbers:

- Select **Equation** mode ((MENU) (ALPHA) (X,θ,T)).
- Ensure that the Complex Mode setting is $a + bi$ ((SHIFT) (MENU)).
- Select **Polynomial** (F2); choose degree 2 (F1).
- Enter the coefficients of the equation in the table as shown; select **Solve** (F1).





We can see that any quadratic polynomial can be factorised into linear factors over the complex numbers. In the next section, we determine that any higher degree polynomial can also be factorised into linear factors over the complex numbers.

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Skill-sheet

Exercise 3F

Example 19, 20

1 Factorise each of the following into linear factors over $\mathbb{C}$:

a $z^2 + 16$
b $z^2 + 5$
c $z^2 + 2z + 5$
d $z^2 - 3z + 4$
e $2z^2 - 8z + 9$
f $3z^2 + 6z + 4$
g $3z^2 + 2z + 2$
h $2z^2 - z + 3$

Example 21

2 Solve each of the following equations over $\mathbb{C}$:

a $x^2 + 25 = 0$
b $x^2 + 8 = 0$
c $x^2 - 4x + 5 = 0$
d $3x^2 + 7x + 5 = 0$
e $x^2 = 2x - 3$
f $5x^2 + 1 = 3x$
g $z^2 + (1 + 2i)z + (-1 + i) = 0$
h $z^2 + z + (1 - i) = 0$

Hint: Show that $-3 + 4i = (1 + 2i)^2$.

3G Solving polynomial equations over the complex numbers

Learning intentions

- To be able to determine all complex solutions of a polynomial equation.

You have studied polynomials over the real numbers in Mathematical Methods. We now extend this study to polynomials over the complex numbers.

For $n \in \mathbb{N} \cup \{0\}$, a polynomial of degree n is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where the coefficients a_i are complex numbers and $a_n \neq 0$.

When we divide the polynomial $P(z)$ by the polynomial $D(z)$ we obtain two polynomials, $Q(z)$ the **quotient** and $R(z)$ the **remainder**, such that

$$P(z) = D(z)Q(z) + R(z)$$

and either $R(z) = 0$ or $R(z)$ has degree less than $D(z)$.

If $R(z) = 0$, then $D(z)$ is a **factor** of $P(z)$.

The remainder theorem and the factor theorem are true for polynomials over $\mathbb{C}$.

Remainder theorem

Let $\alpha \in \mathbb{C}$. When a polynomial $P(z)$ is divided by $z - \alpha$, the remainder is $P(\alpha)$.

Proof Dividing the polynomial $P(z)$ by $z - \alpha$, we can write

$$P(z) = (z - \alpha)Q(z) + R$$

where $Q(z)$ is the quotient and R is the remainder, with $R \in \mathbb{C}$. Therefore

$$P(\alpha) = (\alpha - \alpha)Q(\alpha) + R = R$$

and so the remainder is $R = P(\alpha)$.

Factor theorem

Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.

Proof This theorem follows straight from the remainder theorem, since $z - \alpha$ is a factor of $P(z)$ if and only if the remainder is zero when $P(z)$ is divided by $z - \alpha$.

Example 22

Factorise $P(z) = z^3 + z^2 + 4$.

Solution

Use the factor theorem to determine the first factor:

$$P(-1) = -1 + 1 + 4 \neq 0$$

$$P(-2) = -8 + 4 + 4 = 0$$

Therefore $z + 2$ is a factor. By division, we obtain

$$P(z) = (z + 2)(z^2 - z + 2)$$

We can factorise $z^2 - z + 2$ by completing the square:

$$\begin{aligned} z^2 - z + 2 &= \left(z^2 - z + \frac{1}{4}\right) + 2 - \frac{1}{4} \\ &= \left(z - \frac{1}{2}\right)^2 - \frac{7}{4}i^2 \\ &= \left(z - \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{7}}{2}i\right) \end{aligned}$$

Hence $\quad P(z) = (z + 2)\left(z - \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{7}}{2}i\right)$

Example 23

Factorise $z^3 - iz^2 - 4z + 4i$.

Solution

Factorise by grouping:

$$\begin{aligned} z^3 - iz^2 - 4z + 4i &= z^2(z - i) - 4(z - i) \\ &= (z - i)(z^2 - 4) \\ &= (z - i)(z - 2)(z + 2) \end{aligned}$$

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The conjugate root theorem

In every example thus far, whenever a polynomial has had real coefficients, the complex solutions have occured in **conjugate pairs**. One can prove this always occurs.

Conjugate root theorem

Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of the equation $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Proof We will prove the theorem for quadratics, as it gives the idea of the general proof.

Let $P(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Assume that α is a solution of the equation $P(z) = 0$. Then $P(\alpha) = 0$. That is,

$$a\alpha^2 + b\alpha + c = 0$$

Take the conjugate of both sides of this equation and use properties of conjugates:

$$\overline{a\alpha^2 + b\alpha + c} = \overline{0}$$

$$\overline{a\alpha^2} + \overline{b\alpha} + \overline{c} = 0$$

$$a(\overline{\alpha^2}) + b\overline{\alpha} + c = 0 \qquad \text{since } a, b \text{ and } c \text{ are real numbers}$$

$$a(\overline{\alpha})^2 + b\overline{\alpha} + c = 0$$

Hence $P(\overline{\alpha}) = 0$. That is, $\overline{\alpha}$ is a solution of the equation $P(z) = 0$.

Factorisation of cubic polynomials

Over the complex numbers, every cubic polynomial has three linear factors.

If the coefficients of the cubic are real, then at least one factor must be real (as complex factors occur in pairs). The usual method of solution, already demonstrated in Example 22, is to first determine the real linear factor using the factor theorem and then complete the square on the resulting quadratic factor. The cubic polynomial can also be factorised if one complex root is given, as shown in the next example.

Example 24

Let $P(z) = z^3 - 3z^2 + 5z - 3$.

a Use the factor theorem to show that $z - 1 + \sqrt{2}i$ is a factor of $P(z)$.

b Determine the other linear factors of $P(z)$.

Solution

a To show that $z - (1 - \sqrt{2}i)$ is a factor, we must check that $P(1 - \sqrt{2}i) = 0$.

We have

$$P(1 - \sqrt{2}i) = (1 - \sqrt{2}i)^3 - 3(1 - \sqrt{2}i)^2 + 5(1 - \sqrt{2}i) - 3 = 0$$

Therefore $z - (1 - \sqrt{2}i)$ is a factor of $P(z)$.

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b Since the coefficients of $P(z)$ are real, the complex linear factors occur in conjugate pairs, so $z - (1 + \sqrt{2}i)$ is also a factor.

To determine the third linear factor, first multiply the two complex factors together:

$$\begin{aligned}
&\big(z - (1 - \sqrt{2}i)\big)\big(z - (1 + \sqrt{2}i)\big) \\
&= z^2 - (1 - \sqrt{2}i)z - (1 + \sqrt{2}i)z + (1 - \sqrt{2}i)(1 + \sqrt{2}i) \\
&= z^2 - (1 - \sqrt{2}i + 1 + \sqrt{2}i)z + 1 + 2 \\
&= z^2 - 2z + 3
\end{aligned}$$

Therefore, by inspection, the linear factors of $P(z) = z^3 - 3z^2 + 5z - 3$ are

$$z - 1 + \sqrt{2}i, \quad z - 1 - \sqrt{2}i \quad \text{and} \quad z - 1$$

Factorisation of higher degree polynomials

Polynomials of the form $z^4 - a^4$ and $z^6 - a^6$ are considered in the following two examples.

Example 25

Factorise $z^4 - 16$ over $\mathbb{C}$.

Solution

$$\begin{aligned}
z^4 - 16 &= (z^2 + 4)(z^2 - 4) && \text{difference of two squares} \\
&= (z + 2i)(z - 2i)(z + 2)(z - 2)
\end{aligned}$$

Example 26

Factorise $z^6 - 1$ over $\mathbb{C}$.

Solution

First note that $z^6 - 1 = (z^3 + 1)(z^3 - 1)$.

Now we factorise $z^3 + 1$ and $z^3 - 1$.

We have

$$\begin{aligned}
z^3 + 1 &= (z + 1)(z^2 - z + 1) \\
&= (z + 1)\left(\left(z^2 - z + \frac{1}{4}\right) + 1 - \frac{1}{4}\right) \\
&= (z + 1)\left(\left(z - \frac{1}{2}\right)^2 - \frac{3}{4}i^2\right) \\
&= (z + 1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)
\end{aligned}$$

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Similarly, we determine

$$\begin{aligned} z^3 - 1 &= (z-1)(z^2+z+1) \\ &= (z-1)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \end{aligned}$$

Therefore

$$\begin{aligned} z^6 - 1 &= (z^3+1)(z^3-1) \\ &= (z+1)(z-1)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \end{aligned}$$

The fundamental theorem of algebra

The following important theorem has been attributed to Gauss (1799).

Fundamental theorem of algebra

Every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ of degree n, where $n \geq 1$ and the coefficients a_i are complex numbers, has at least one linear factor in the complex number system.

Given any polynomial $P(z)$ of degree $n \geq 1$, the theorem tells us that we can factorise $P(z)$ as

$$P(z) = (z - \alpha_1)Q(z)$$

for some $\alpha_1 \in \mathbb{C}$ and some polynomial $Q(z)$ of degree $n - 1$.

By applying the fundamental theorem of algebra repeatedly, it can be shown that:

A polynomial of degree n can be factorised into n linear factors in $\mathbb{C}$:

i.e. $P(z) = a_n(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)\cdots(z-\alpha_n)$, where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{C}$

A polynomial equation can be solved by first rearranging it into the form $P(z) = 0$, where $P(z)$ is a polynomial, and then factorising $P(z)$ and extracting a solution from each factor.

If $P(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$, then the solutions of $P(z) = 0$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$.

The solutions of the equation $P(z) = 0$ are also referred to as the **zeroes** or the **roots** of the polynomial $P(z)$.

Example 27

Solve each of the following equations over $\mathbb{C}$:

a $z^3 + 3z^2 + 7z + 5 = 0$

b $z^3 - iz^2 - 4z + 4i = 0$

Solution

a Let $P(z) = z^3 + 3z^2 + 7z + 5$.

Then $P(-1) = 0$, so $z + 1$ is a factor, by the factor theorem.

$$\begin{aligned} P(z) &= (z+1)(z^2 + 2z + 5) \\ &= (z+1)(z^2 + 2z + 1 + 4) \\ &= (z+1)((z+1)^2 - (2i)^2) \\ &= (z+1)(z+1-2i)(z+1+2i) \end{aligned}$$

The solutions of the equation $P(z) = 0$ are $z = -1$, $z = -1 + 2i$ and $z = -1 - 2i$.

b From Example 23:

$$z^3 - iz^2 - 4z + 4i = 0$$
$$(z-i)(z-2)(z+2) = 0$$
$$\therefore \quad z = i, \quad z = 2 \text{ or } z = -2$$

Exercise 3G

Example 22, 23

1 Factorise each of the following polynomials into linear factors over $\mathbb{C}$:

a $z^3 - 4z^2 - 4z - 5$ **b** $z^3 - z^2 - z + 10$ **c** $3z^3 - 13z^2 + 5z - 4$

d $2z^3 + 3z^2 - 4z + 15$ **e** $z^3 - (2 - i)z^2 + z - 2 + i$

Example 24

2 Let $P(z) = z^3 + 4z^2 - 10z + 12$.

a Use the factor theorem to show that $z - 1 - i$ is a linear factor of $P(z)$.

b Write down another complex linear factor of $P(z)$.

c Hence determine all the linear factors of $P(z)$ over $\mathbb{C}$.

3 Let $P(z) = 2z^3 + 9z^2 + 14z + 5$.

a Use the factor theorem to show that $z + 2 - i$ is a linear factor of $P(z)$.

b Write down another complex linear factor of $P(z)$.

c Hence determine all the linear factors of $P(z)$ over $\mathbb{C}$.

4 Let $P(z) = z^4 + 8z^2 + 16z + 20$.

a Use the factor theorem to show that $z - 1 + 3i$ is a linear factor of $P(z)$.

b Write down another complex linear factor of $P(z)$.

c Hence determine all the linear factors of $P(z)$ over $\mathbb{C}$.

SF

Example 25, 26

5 Factorise each of the following into linear factors over $\mathbb{C}$:

a $z^4 - 81$ **b** $z^6 - 64$

6 For each of the following, factorise the first expression into linear factors over $\mathbb{C}$, given that the second expression is one of the linear factors:

a $z^3 + (1 - i)z^2 + (1 - i)z - i, \quad z - i$ **b** $z^3 - (2 - i)z^2 - (1 + 2i)z - i, \quad z + i$

c $z^3 - (2 + 2i)z^2 - (3 - 4i)z + 6i, \quad z - 2i$ **d** $2z^3 + (1 - 2i)z^2 - (5 + i)z + 5i, \quad z - i$

CF

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SF

7 For each of the following, determine the value of p given that:

a $z + 2$ is a factor of $z^3 + 3z^2 + pz + 12$

b $z - i$ is a factor of $z^3 + pz^2 + z - 4$

c $z + 1 - i$ is a factor of $2z^3 + z^2 - 2z + p$

Example 27

8 Solve each of the following equations over $\mathbb{C}$:

a $x^3 + x^2 - 6x - 18 = 0$

b $x^3 - 6x^2 + 11x - 30 = 0$

c $2x^3 + 3x^2 = 11x^2 - 6x - 16$

d $x^4 + x^2 = 2x^3 + 36$

CF

9 Let $z^2 + az + b = 0$, where $a, b \in \mathbb{R}$. Determine a and b if one of the solutions is:

a $2i$

b $3 + 2i$

c $-1 + 3i$

10 **a** $1 + 3i$ is a solution of the equation $3z^3 - 7z^2 + 32z - 10 = 0$. Find the other solutions.

b $-2 - i$ is a solution of the equation $z^4 - 5z^2 + 4z + 30 = 0$. Find the other solutions.

11 For a cubic polynomial $P(x)$ with real coefficients, $P(2 + i) = 0$, $P(1) = 0$ and $P(0) = 10$. Express $P(x)$ in the form $P(x) = ax^3 + bx^2 + cx + d$ and solve the equation $P(x) = 0$.

12 If $z = 1 + i$ is a zero of the polynomial $z^3 + az^2 + bz + 10 - 6i$, determine the constants a and b, given that they are real.

13 The polynomial $P(z) = 2z^3 + az^2 + bz + 5$, where a and b are real numbers, has $2 - i$ as one of its zeroes.

a Determine a quadratic factor of $P(z)$, and hence calculate the real constants a and b.

b Determine the solutions to the equation $P(z) = 0$.

14 For the polynomial $P(z) = az^4 + az^2 - 2z + d$, where a and d are real numbers:

a Evaluate $P(1 + i)$.

b Given that $P(1 + i) = 0$, determine the values of a and d.

c Show that $P(z)$ can be written as the product of two quadratic factors with real coefficients, and hence solve the equation $P(z) = 0$.

15 The solutions of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Determine the complex numbers p and q.

SF

16 Given that $1 - i$ is a solution of $z^3 - 4z^2 + 6z - 4 = 0$, determine the other two solutions.

17 Solve each of the following for z:

a $z^2 - (6 + 2i)z + (8 + 6i) = 0$ **b** $z^3 - 2iz^2 - 6z + 12i = 0$ **c** $z^3 - z^2 + 6z - 6 = 0$

d $z^3 - z^2 + 2z - 8 = 0$ **e** $6z^2 - 3\sqrt{2}\,z + 6 = 0$ **f** $z^3 + 2z^2 + 9z = 0$

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3H Roots of complex numbers

Learning intentions

- To be able to use de Moivre's theorem to determine the roots of a complex number.

Equations of the form $z^n = a$, where $a \in \mathbb{C}$, are often solved by using de Moivre's theorem.

Write both z and a in polar form, as $z = r \operatorname{cis} \theta$ and $a = q \operatorname{cis} \varphi$.

Then $z^n = a$ becomes

$$(r \operatorname{cis} \theta)^n = q \operatorname{cis} \varphi$$

$$\therefore \quad r^n \operatorname{cis}(n\theta) = q \operatorname{cis} \varphi \qquad \text{(using de Moivre's theorem)}$$

Compare modulus and argument:

$$\begin{aligned} r^n &= q & \operatorname{cis}(n\theta) &= \operatorname{cis} \varphi & \\ r &= \sqrt[n]{q} & n\theta &= \varphi + 2k\pi & \text{where } k \in \mathbb{Z} \\ & & \theta &= \frac{1}{n}(\varphi + 2k\pi) & \text{where } k \in \mathbb{Z} \end{aligned}$$

This will provide all the solutions of the equation.

Example 28

Solve $z^3 = 1$.

Solution

Let $z = r \operatorname{cis} \theta$. Then

$$(r \operatorname{cis} \theta)^3 = 1 \operatorname{cis} 0$$

$$\therefore \quad r^3 \operatorname{cis}(3\theta) = 1 \operatorname{cis} 0$$

$$\therefore \quad r^3 = 1 \quad \text{and} \quad 3\theta = 0 + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\therefore \quad r = 1 \quad \text{and} \quad \theta = \frac{2k\pi}{3} \quad \text{where } k \in \mathbb{Z}$$

Hence the solutions are of the form $z = \operatorname{cis}\left(\frac{2k\pi}{3}\right)$, where $k \in \mathbb{Z}$.

We start determining solutions.

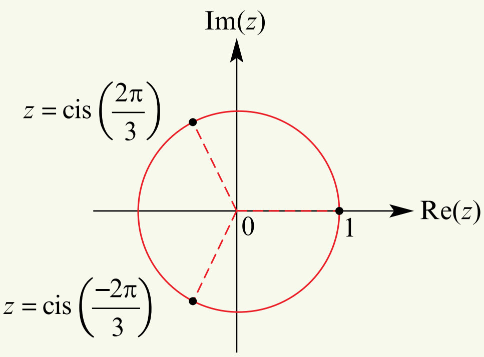
For $k = 0$: $\quad z = \operatorname{cis} 0 = 1$

For $k = 1$: $\quad z = \operatorname{cis}\left(\frac{2\pi}{3}\right)$

For $k = 2$: $\quad z = \operatorname{cis}\left(\frac{4\pi}{3}\right) = \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

For $k = 3$: $\quad z = \operatorname{cis}(2\pi) = 1$

The solutions begin to repeat.



The three solutions are 1, $\operatorname{cis}\left(\frac{2\pi}{3}\right)$ and $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

The solutions are shown to lie on the unit circle at intervals of $\frac{2\pi}{3}$ around the circle.

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Note: An equation of the form $z^3 = a$, where $a \in \mathbb{R}$, has three solutions. Since $a \in \mathbb{R}$, two of the solutions will be conjugate to each other and the third must be a real number.

In Example 28, we found the three cube roots of the number 1:

$$1, \quad w = \operatorname{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{and} \quad w^2 = \operatorname{cis}\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

More generally:

Solutions of $z^n = 1$

For $n \in \mathbb{N}$, the solutions of the equation $z^n = 1$ are called the ***n*th roots of unity**.

- The solutions of $z^n = 1$ lie on the unit circle.
- There are n solutions and they are equally spaced around the circle at intervals of $\frac{2\pi}{n}$. This observation can be used to determine all solutions, since $z = 1$ is one solution.

Example 29

Solve $z^2 = 1 + i$.

Solution

Let $z = r \operatorname{cis} \theta$. Note that $1 + i = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$.

$$(r \operatorname{cis} \theta)^2 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$\therefore \quad r^2 \operatorname{cis}(2\theta) = 2^{\frac{1}{2}} \operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$\therefore \quad r = 2^{\frac{1}{4}} \quad \text{and} \quad 2\theta = \frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

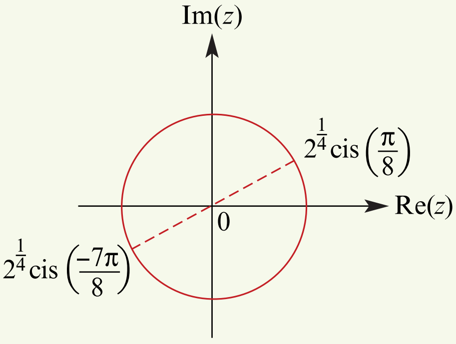
$$\therefore \quad r = 2^{\frac{1}{4}} \quad \text{and} \quad \theta = \frac{\pi}{8} + k\pi \quad \text{where } k \in \mathbb{Z}$$

Hence $z = 2^{\frac{1}{4}} \operatorname{cis}\left(\frac{\pi}{8} + k\pi\right)$, where $k \in \mathbb{Z}$.

For $k = 0$: $\quad z = 2^{\frac{1}{4}} \operatorname{cis}\left(\frac{\pi}{8}\right)$

For $k = 1$: $\quad z = 2^{\frac{1}{4}} \operatorname{cis}\left(\frac{9\pi}{8}\right)$

$$= 2^{\frac{1}{4}} \operatorname{cis}\left(\frac{-7\pi}{8}\right)$$



Note: If z_1 is a solution of $z^2 = a$, where $a \in \mathbb{C}$, then the other solution is $z_2 = -z_1$.

In Example 29, we found the two square roots of the complex number $1 + i$. More generally:

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Solutions of $z^n = a$

For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, the solutions of the equation $z^n = a$ are called the ***n*th roots** of a.

- The solutions of $z^n = a$ lie on a circle with centre the origin and radius $|a|^{\frac{1}{n}}$.
- There are n solutions and they are equally spaced around the circle at intervals of $\dfrac{2\pi}{n}$. This observation can be used to determine all solutions if one is known.

The following example shows an alternative method for solving equations of the form $z^2 = a$, where $a \in \mathbb{C}$.

Example 30

Solve $z^2 = 5 + 12i$ using $z = a + bi$, where $a, b \in \mathbb{R}$. Hence factorise $z^2 - 5 - 12i$.

Solution

Let $z = a + bi$. Then
$$\begin{aligned} z^2 &= (a+bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ &= (a^2 - b^2) + 2abi \end{aligned}$$

So $z^2 = 5 + 12i$ becomes

$$(a^2 - b^2) + 2abi = 5 + 12i$$

Equating coefficients:

$$a^2 - b^2 = 5 \quad \text{and} \quad 2ab = 12$$

$$a^2 - \left(\frac{6}{a}\right)^2 = 5 \qquad\qquad b = \frac{6}{a}$$

$$\begin{aligned} a^2 - \frac{36}{a^2} &= 5 \\ a^4 - 36 &= 5a^2 \\ a^4 - 5a^2 - 36 &= 0 \\ (a^2 - 9)(a^2 + 4) &= 0 \\ a^2 - 9 &= 0 \\ (a+3)(a-3) &= 0 \\ \therefore \qquad a &= -3 \text{ or } a = 3 \end{aligned}$$

When $a = -3$, $b = -2$ and when $a = 3$, $b = 2$.

So the solutions to the equation $z^2 = 5 + 12i$ are $z = -3 - 2i$ and $z = 3 + 2i$.

Hence $z^2 - 5 - 12i = (z + 3 + 2i)(z - 3 - 2i)$.

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Skill-sheet

Exercise 3H

SF

1 For each of the following, solve the equation over $\mathbb{C}$ and show the solutions on an Argand diagram:

Example 28, 29

a $z^2 + 1 = 0$ **b** $z^3 = 27i$ **c** $z^2 = 1 + \sqrt{3}i$

d $z^2 = 1 - \sqrt{3}i$ **e** $z^3 = i$ **f** $z^3 + i = 0$

2 Determine all the cube roots of the following complex numbers:

a $4\sqrt{2} - 4\sqrt{2}i$ **b** $-4\sqrt{2} + 4\sqrt{2}i$ **c** $-4\sqrt{3} - 4i$

d $4\sqrt{3} - 4i$ **e** $-125i$ **f** $-1 + i$

Example 30

CF

3 Let $z = a + bi$ such that $z^2 = 3 + 4i$, where $a, b \in \mathbb{R}$.

a Determine equations in terms of a and b by equating real and imaginary parts.

b Determine the values of a and b and hence determine the square roots of $3 + 4i$.

4 Using the method of Question **3**, determine the square roots of each of the following:

a $-15 - 8i$ **b** $24 + 7i$

c $-3 + 4i$ **d** $-7 + 24i$

5 Determine the solutions of the equation $z^4 - 2z^2 + 4 = 0$ in polar form.

6 Determine the solutions of the equation $z^2 - i = 0$ in Cartesian form. Hence factorise $z^2 - i$.

7 Determine the solutions of the equation $z^8 + 1 = 0$ in polar form. Hence factorise $z^8 + 1$.

8 **a** Determine the square roots of $1 + i$ by using:

i Cartesian methods **ii** de Moivre's theorem.

b Hence determine exact values of $\cos\left(\dfrac{\pi}{8}\right)$ and $\sin\left(\dfrac{\pi}{8}\right)$.

CU

9 **a** Use de Moivre's theorem to solve $z^4 = -64$.

b Hence, determine real numbers a and b for which $z^4 + 64 = (z^2 + az + b)(z^2 - az + b)$.

10 Let $w = \text{cis}\left(\dfrac{2\pi}{5}\right)$.

a Show that $1, w, w^2, w^3$ and w^4 are all solutions of $z^5 = 1$.

b For any $z \in \mathbb{C}$, show that $z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4)$. **Hint:** Expand the right-hand side.

c Hence, show that the sum of the roots of $z^5 = 1$ is equal to zero.

d Hence, simplify each of the following:

i $w + w^2 + w^3 + w^4$

ii $(1 + w + w^2 + w^3)^5$

iii $\dfrac{1 + w^4}{w + w^2 + w^3}$

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3I Sketching subsets of the complex plane

Learning intentions

- To be able to sketch regions of the complex plane including circles, lines and rays.

Particular sets of points of the complex plane can be described by placing restrictions on z. For example:

- $\{ z : \text{Re}(z) = 6 \}$ is the straight line parallel to the imaginary axis with each point on the line having real part 6.
- $\{ z : \text{Im}(z) = 2\,\text{Re}(z) \}$ is the straight line through the origin with gradient 2.

The set of all points which satisfy a given condition is called the **locus** of the condition (plural loci). When sketching a locus, a solid line is used for a boundary which is included in the locus, and a dashed line is used for a boundary which is not included.

Example 31

On an Argand diagram, sketch the subset S of the complex plane, where

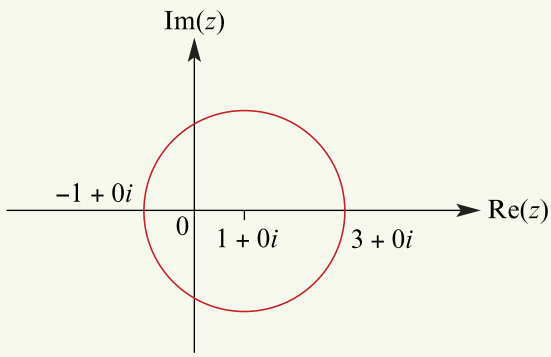
$$S = \{ z : |z - 1| = 2 \}$$

Solution

Method 1: Using algebra

Let $z = x + yi$. Then

$$\begin{aligned} |z - 1| &= 2 \\ |x + yi - 1| &= 2 \\ |(x - 1) + yi| &= 2 \\ \sqrt{(x-1)^2 + y^2} &= 2 \\ \therefore \quad (x-1)^2 + y^2 &= 4 \end{aligned}$$



This demonstrates that S is represented by the circle with centre $1 + 0i$ and radius 2.

Method 2: Using geometry

If z_1 and z_2 are complex numbers, then $|z_1 - z_2|$ is the distance between the points on the complex plane corresponding to z_1 and z_2.

Hence $\{ z : |z - 1| = 2 \}$ is the set of all points that are distance 2 from $1 + 0i$. That is, the set S is represented by the circle with centre $1 + 0i$ and radius 2.

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Example 32

On an Argand diagram, sketch the subset S of the complex plane, where

$$S = \{ z : |z - 2| = |z - (1 + i)| \}$$

Solution

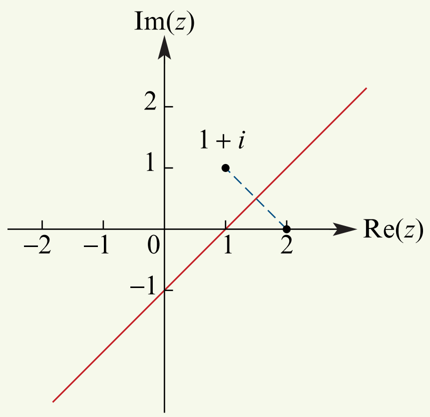
Method 1: Using algebra

Let $z = x + yi$. Then

$$|z - 2| = |z - (1 + i)|$$
$$|x + yi - 2| = |x + yi - (1 + i)|$$
$$|x - 2 + yi| = |x - 1 + (y - 1)i|$$
$$\therefore \quad \sqrt{(x - 2)^2 + y^2} = \sqrt{(x - 1)^2 + (y - 1)^2}$$

Squaring both sides of the equation and expanding:

$$x^2 - 4x + 4 + y^2 = x^2 - 2x + 1 + y^2 - 2y + 1$$
$$-4x + 4 = -2x - 2y + 2$$
$$\therefore \quad y = x - 1$$



Method 2: Using geometry

The set S consists of all points in the complex plane that are equidistant from 2 and $1 + i$.

In the Cartesian plane, this set corresponds to the perpendicular bisector of the line segment joining $(2, 0)$ and $(1, 1)$. The midpoint of the line segment is $(\frac{3}{2}, \frac{1}{2})$, and the gradient of the line segment is -1.

Therefore the equation of the perpendicular bisector is

$$y - \tfrac{1}{2} = 1(x - \tfrac{3}{2})$$

which simplifies to $y = x - 1$.

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Example 33

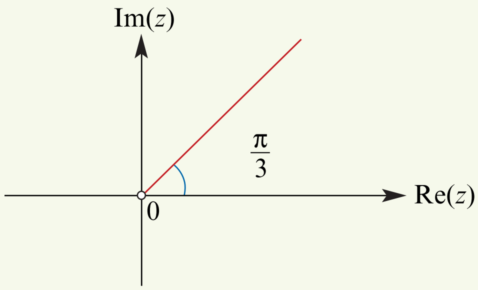
Sketch the subset of the complex plane defined by each of the following conditions:

a $\text{Arg}(z) = \frac{\pi}{3}$ **b** $\text{Arg}(z+3) = -\frac{\pi}{3}$ **c** $\text{Arg}(z) \le \frac{\pi}{3}$

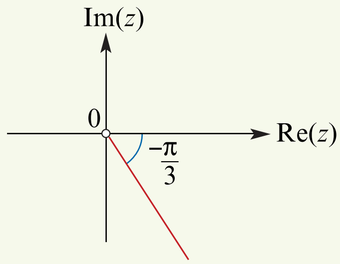
Solution

a $\text{Arg}(z) = \frac{\pi}{3}$ defines a ray or a half line.

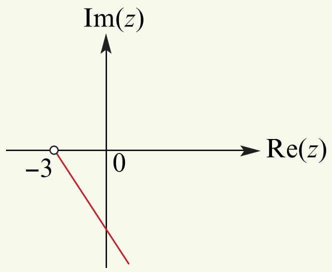
Note: The origin is not included.



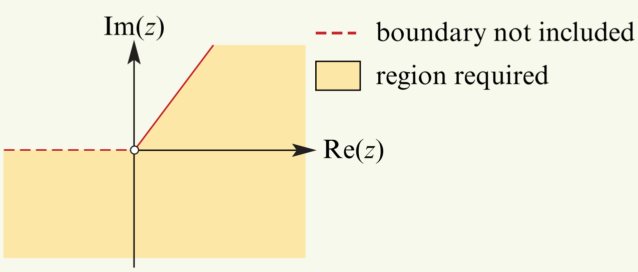
b First draw the graph of $\text{Arg}(z) = -\frac{\pi}{3}$.



The graph of $\text{Arg}(z+3) = -\frac{\pi}{3}$ is obtained by a translation of 3 units to the left.



c Since $-\pi < \text{Arg}(z) \le \pi$ in general, the condition $\text{Arg}(z) \le \frac{\pi}{3}$ implies $-\pi < \text{Arg}(z) \le \frac{\pi}{3}$.



Example 34

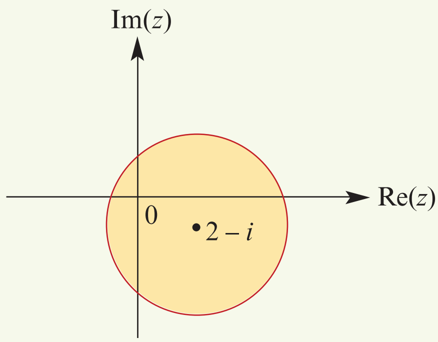
Sketch the region corresponding to each of the following:

a $|z - (2 - i)| \le 3$

b $\{ z : 2 < |z| \le 4 \} \cap \left\{ z : \frac{\pi}{6} < \operatorname{Arg}(z) \le \frac{\pi}{3} \right\}$

Solution

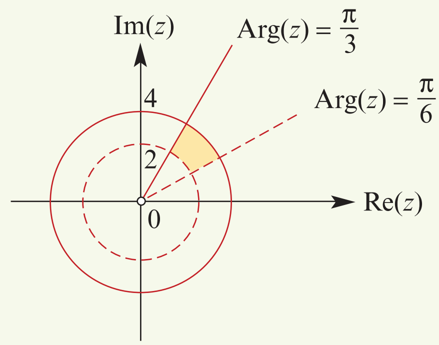
a



Region required

The condition defines a disc of radius 3 and centre $2 - i$. The Cartesian relation is $\{ (x, y) : (x - 2)^2 + (y + 1)^2 \le 3^2 \}$.

b



Region required

Example 35

Describe the locus defined by $|z + 3| = 2|z - i|$.

Solution

Let $z = x + yi$. Then

$$|z + 3| = 2|z - i|$$

$$|(x + 3) + yi| = 2|x + (y - 1)i|$$

$$\therefore \quad \sqrt{(x + 3)^2 + y^2} = 2\sqrt{x^2 + (y - 1)^2}$$

Squaring both sides gives

$$x^2 + 6x + 9 + y^2 = 4(x^2 + y^2 - 2y + 1)$$

$$0 = 3x^2 + 3y^2 - 6x - 8y - 5$$

$$5 = 3(x^2 - 2x) + 3\left(y^2 - \frac{8}{3}y\right)$$

$$\frac{5}{3} = (x^2 - 2x + 1) + \left(y^2 - \frac{8}{3}y + \frac{16}{9}\right) - \frac{25}{9}$$

$$\therefore \quad \frac{40}{9} = (x - 1)^2 + \left(y - \frac{4}{3}\right)^2$$

The locus is the circle with centre $1 + \frac{4}{3}i$ and radius $\frac{2\sqrt{10}}{3}$.

Note: For $a, b \in \mathbb{C}$ and $k \in \mathbb{R}^+ \setminus \{1\}$, the equation $|z - a| = k|z - b|$ defines a circle.

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Skill-sheet

Exercise 3I

Example 31

1 Illustrate each of the following on an Argand diagram:

a $2\operatorname{Im}(z) = \operatorname{Re}(z)$ **b** $\operatorname{Im}(z) + \operatorname{Re}(z) = 1$ **c** $|z - 2| = 3$

d $|z - i| = 4$ **e** $|z - (1 + \sqrt{3}i)| = 2$ **f** $|z - (1 - i)| = 6$

2 Sketch $\{ z : z = i\bar{z} \}$ in the complex plane.

Example 32

3 Describe the subset of the complex plane defined by $\{ z : |z - 1| = |z + 1| \}$.

Example 33

4 Sketch the subset of the complex plane defined by each of the following conditions:

a $\operatorname{Arg}(z) = \dfrac{\pi}{4}$ **b** $\operatorname{Arg}(z - 2) = -\dfrac{\pi}{4}$ **c** $\operatorname{Arg}(z) \le \dfrac{\pi}{4}$

Example 34

5 Sketch each of the following regions of the complex plane:

a $\{ z : |z - 1| \le 2 \}$ **b** $\{ z : 2 \le |z| \le 3 \} \cap \left\{ z : \dfrac{\pi}{4} < \operatorname{Arg}(z) \le \dfrac{3\pi}{4} \right\}$

Example 35

6 Sketch each of the following:

a $\{ z : |z + 2i| = 2|z - i| \}$ **b** $\{ z : \operatorname{Im}(z) = -2 \}$ **c** $\{ z : z + \bar{z} = 5 \}$

d $\{ z : z\bar{z} = 5 \}$ **e** $\left\{ z : \operatorname{Arg}(z - i) = \dfrac{\pi}{3} \right\}$

7 Sketch each of the following:

a $\{ z : |z - i| > 1 \}$ **b** $\{ z : |z + i| \le 2 \}$ **c** $\{ z : \operatorname{Re}(z) \ge 0 \}$

d $\{ z : 2\operatorname{Re}(z) + \operatorname{Im}(z) \le 0 \}$ **e** $\{ z : \operatorname{Re}(z) > 2 \text{ and } \operatorname{Im}(z) \ge 1 \}$

8 On an Argand diagram, sketch the set $S = \{ z : \operatorname{Re}(z) \le 1 \} \cap \{ z : 0 \le \operatorname{Im}(z) \le 3 \}$.

9 Sketch the region of the complex plane for which $\operatorname{Re}(z) \ge 0$ and $|z + 2i| \le 1$.

10 Sketch the locus defined by $|z - 2 + 3i| \le 2$.

SF

11 On the Argand plane, sketch the curve defined by each of the following equations:

a $\left| \dfrac{z - 2}{z} \right| = 1$ **b** $\left| \dfrac{z - 1 - i}{z} \right| = 1$

12 If the real part of $\dfrac{z + 1}{z - 1}$ is zero, determine the locus of z in the complex plane.

13 On an Argand diagram with origin O, the point P represents z and Q represents $\dfrac{1}{\bar{z}}$. Prove that O, P and Q are collinear and determine the ratio $OP : OQ$ in terms of $|z|$.

CF

14 Determine the locus of points described by each of the following conditions:

a $|z - (1 + i)| = 1$ **b** $|z - 2| = |z + 2i|$ **c** $\operatorname{Arg}(z - 1) = \dfrac{\pi}{2}$ **d** $\operatorname{Arg}(z + i) = \dfrac{\pi}{4}$

SF

15 Let $w = 2z$. Describe the locus of w if z describes a circle with centre $1 + 2i$ and radius 3.

16 **a** Determine the solutions of the equation $z^2 + 2z + 4 = 0$.

b Show that the solutions satisfy:

i $|z| = 2$ **ii** $|z - 1| = \sqrt{7}$ **iii** $z + \bar{z} = -2$

c On a single diagram, sketch the loci defined by the equations in **b**.

CF

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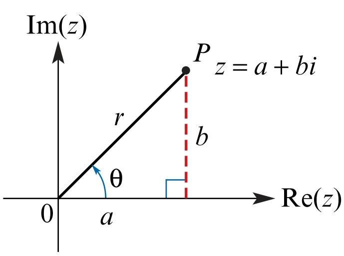
Chapter summary

- The imaginary number i has the property $i^2 = -1$.
- The set of complex numbers is $\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\operatorname{Re}(z) = a$
  - the **imaginary part** of z is $\operatorname{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z, denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The **argument** of z is an angle measured anticlockwise about the origin from the positive direction of the real axis to the line joining the origin to z.
- The **principal value** of the argument, denoted by $\operatorname{Arg} z$, is the angle in the interval $(-\pi, \pi]$.
- The complex number $z = a + bi$ can be expressed in **polar form** as

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= r\operatorname{cis}\theta \end{aligned}$$

  where $r = |z| = \sqrt{a^2 + b^2}$, $\cos\theta = \dfrac{a}{r}$, $\sin\theta = \dfrac{b}{r}$.
  This is also called **modulus–argument form**.



- The **complex conjugate** of z, denoted by $\bar{z}$, is the reflection of z in the real axis. If $z = a + bi$, then $\bar{z} = a - bi$. If $z = r\operatorname{cis}\theta$, then $\bar{z} = r\operatorname{cis}(-\theta)$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

- Multiplication and division in polar form:
  Let $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$. Then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

- **De Moivre's theorem** $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$
- **Conjugate root theorem** If a polynomial has real coefficients, then the complex roots occur in conjugate pairs.
- **Fundamental theorem of algebra** Every non-constant polynomial with complex coefficients has at least one linear factor in the complex number system.
- A polynomial of degree n can be factorised over $\mathbb{C}$ into a product of n linear factors.
- If z_1 is a solution of $z^2 = a$, where $a \in \mathbb{C}$, then the other solution is $z_2 = -z_1$.
- The solutions of $z^n = a$, where $a \in \mathbb{C}$, lie on the circle centred at the origin with radius $|a|^{\frac{1}{n}}$. The solutions are equally spaced around the circle at intervals of $\dfrac{2\pi}{n}$.
- The distance between z_1 and z_2 in the complex plane is $|z_1 - z_2|$.
  For example, the set $\{\, z : |z - (1 + i)| = 2 \,\}$ is a circle with centre $1 + i$ and radius 2.

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

3A **1** I can identify the real and imaginary parts of a complex number. ☐

See Example 1 and Question 1

3A **2** I can add and subtract complex numbers. ☐

See Example 4 and Question 4

3A **3** I can represent complex numbers on an Argand diagram. ☐

See Example 5, Example 6, Question 5 and Question 6

3A **4** I can multiply complex numbers. ☐

See Example 7, Question 7 and Question 8

3B **5** I can determine the complex conjugate and modulus of a complex number. ☐

See Example 8, Question 1 and Question 3

3C **6** I can convert between the Cartesian and polar forms of a complex number. ☐

See Example 12, Example 13, Question 5 and Question 6

3D **7** I can multiply and divide complex numbers in polar form. ☐

See Example 15 and Question 2

3E **8** I can use de Moivre's theorem to simplify powers of complex numbers. ☐

See Example 16, Example 17, Question 1 and Question 2

3F **9** I can factorise and solve quadratics over the complex numbers. ☐

See Example 19, Example 20, Example 21, Question 1 and Question 2

3G **10** I can factorise polynomials over the complex numbers. ☐

See Example 22, Example 23, Example 24, Question 1, Question 2 and Question 3

3G **11** I can solve polynomial equations over the complex numbers. ☐

See Example 27, Question 8, Question 9 and Question 10

3H **12** I can determine the roots of a complex number using de Moivre's theorem. ☐

See Example 28, Example 29 and Question 1

3I **13** I can sketch subsets of the complex plane including circles, lines and rays. ☐

See Example 31, Example 32, Example 33, Question 1, Question 3 and Question 4

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Review

3F **14** I can multiply and divide complex numbers in polar form. ☐

See Example 21, Example 22, Example 23 and Question 3

3G **15** I can use de Moivre's theorem to simplify powers of complex numbers. ☐

See Example 24, Example 25, Question 1, Question 2 and Question 3

3H **16** I can sketch subsets of the complex plane including circles, lines and rays. ☐

See Example 27, Example 29, Question 2, Question 4 and Question 5

Short-response questions

Technology-free short-response questions

SF

1 Express each of the following in the form $a + bi$, where $a, b \in \mathbb{R}$:

a $3 + 2i + 5 - 7i$ **b** i^3 **c** $(3 - 2i)(5 + 7i)$

d $(3 - 2i)(3 + 2i)$ **e** $\dfrac{2}{3 - 2i}$ **f** $\dfrac{5 - i}{2 + i}$

g $\dfrac{3i}{2 + i}$ **h** $(1 - 3i)^2$ **i** $\dfrac{(5 + 2i)^2}{3 - i}$

2 Solve each of the following equations for z:

a $(z - 2)^2 + 9 = 0$ **b** $\dfrac{z - 2i}{z + (3 - 2i)} = 2$ **c** $z^2 + 6z + 12 = 0$

d $z^4 + 81 = 0$ **e** $z^3 - 27 = 0$ **f** $8z^3 + 27 = 0$

3 **a** Show that $2 - i$ is a solution of the equation $z^3 - 2z^2 - 3z + 10 = 0$. Hence solve the equation for z.

b Show that $3 - 2i$ is a solution of the equation $x^3 - 5x^2 + 7x + 13 = 0$. Hence solve the equation for $x \in \mathbb{C}$.

c Show that $1 + i$ is a solution of the equation $z^3 - 4z^2 + 6z - 4 = 0$. Hence determine the other solutions of this equation.

4 Express each of the following polynomials as a product of linear factors:

a $2x^2 + 3x + 2$ **b** $x^3 - x^2 + x - 1$ **c** $x^3 + 2x^2 - 4x - 8$

CF

5 If $(a + bi)^2 = 3 - 4i$, determine the possible values of a and b, where $a, b \in \mathbb{R}$.

SF

6 Pair each of the transformations given on the left with the appropriate operation on the complex numbers given on the right:

a reflection in the real axis **i** multiply by -1

b rotation anticlockwise by 90° about O **ii** multiply by i

c rotation through 180° about O **iii** multiply by $-i$

d rotation anticlockwise about O through 270° **iv** take the conjugate

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Review

CF

7 If $(a + bi)^2 = -24 - 10i$, determine the possible values of a and b, where $a, b \in \mathbb{R}$.

8 Determine the values of a and b if $f(z) = z^2 + az + b$ and $f(-1 - 2i) = 0$, where $a, b \in \mathbb{R}$.

9 Express $\dfrac{1}{1 + \sqrt{3}i}$ in the form $r \operatorname{cis} \theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

10 On an Argand diagram with origin O, the point P represents $3 + i$. The point Q represents $a + bi$, where both a and b are positive. If the triangle OPQ is equilateral, determine a and b.

SF

11 Let $z = 1 - i$. Determine:

a $2\bar{z}$ **b** $\dfrac{1}{z}$

c $|z^7|$ **d** $\operatorname{Arg}(z^7)$

12 Let $w = 1 + i$ and $z = 1 - \sqrt{3}i$.

a Write down:

i $|w|$ **ii** $|z|$ **iii** $\operatorname{Arg} w$ **iv** $\operatorname{Arg} z$

b Hence write down $\left|\dfrac{w}{z}\right|$ and $\operatorname{Arg}(wz)$.

13 Express $\sqrt{3} + i$ in polar form. Hence determine $(\sqrt{3} + i)^7$ and express in Cartesian form.

CF

14 Consider the equation $z^4 - 2z^3 + 11z^2 - 18z + 18 = 0$. Determine all real values of r for which $z = ri$ is a solution of the equation. Hence determine all the solutions of the equation.

15 Express $(1 - i)^9$ in Cartesian form.

16 Consider the polynomial $P(z) = z^3 + (2 + i)z^2 + (2 + 2i)z + 4$. Determine the real numbers k such that ki is a zero of $P(z)$. Hence, or otherwise, determine the three zeroes of $P(z)$.

SF

17 **a** Determine the three linear factors of $z^3 - 2z + 4$.

b What is the remainder when $z^3 - 2z + 4$ is divided by $z - 3$?

CF

18 If a and b are complex numbers such that $\operatorname{Im}(a) = 2$, $\operatorname{Re}(b) = -1$ and $a + b = -ab$, determine a and b.

SF

19 **a** Express $S = \{ z : |z - (1 + i)| \leq 1 \}$ in Cartesian form.

b Sketch S on an Argand diagram.

20 Describe $\{ z : |z + i| = |z - i| \}$.

CF

21 Let $S = \left\{ z : z = 2 \operatorname{cis} \theta,\ 0 \leq \theta \leq \dfrac{\pi}{2} \right\}$. Sketch:

a S **b** $T = \{ w : w = z^2,\ z \in S \}$ **c** $U = \left\{ v : v = \dfrac{2}{z},\ z \in S \right\}$

22 Determine the centre of the circle which passes through the points $-2i$, 1 and $2 - i$.

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Review

23 On an Argand diagram, points A and B represent $a = 5 + 2i$ and $b = 8 + 6i$.

CF

a Determine $i(a - b)$ and show that it can be represented by a vector perpendicular to $\overrightarrow{AB}$ and of the same length as $\overrightarrow{AB}$.

b Hence determine complex numbers c and d, represented by C and D, such that $ABCD$ is a square.

24 Solve each of the following for $z \in \mathbb{C}$:

SF

a $z^3 = -8$

b $z^2 = 2 + 2\sqrt{3}i$

25 **a** Factorise $x^6 - 1$ over $\mathbb{R}$.

CF

b Factorise $x^6 - 1$ over $\mathbb{C}$.

c Determine all the sixth roots of unity. (That is, solve $x^6 = 1$ for $x \in \mathbb{C}$.)

26 Let z be a complex number with a non-zero imaginary part. Simplify:

a $\left|\dfrac{\bar{z}}{z}\right|$

b $\dfrac{i(\text{Re}(z) - z)}{\text{Im}(z)}$

c $\text{Arg}\, z + \text{Arg}\left(\dfrac{1}{z}\right)$

27 If $\text{Arg}\, z = \dfrac{\pi}{4}$ and $\text{Arg}(z - 3) = \dfrac{\pi}{2}$, determine $\text{Arg}(z - 6i)$.

28 **a** If $\text{Arg}(z + 2) = \dfrac{\pi}{2}$ and $\text{Arg}(z) = \dfrac{2\pi}{3}$, determine z.

b If $\text{Arg}(z - 3) = -\dfrac{3\pi}{4}$ and $\text{Arg}(z + 3) = -\dfrac{\pi}{2}$, determine z.

29 A complex number z satisfies the inequality $\left|z + 2 - 2\sqrt{3}i\right| \le 2$.

a Sketch the corresponding region representing all possible values of z.

b **i** Determine the least possible value of $|z|$.

ii Determine the greatest possible value of $\text{Arg}\, z$.

Technology-active short-response questions

30 Let $z = 4\,\text{cis}\left(\dfrac{5\pi}{6}\right)$ and $w = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)$.

CU

a Determine $|z^7|$ and $\text{Arg}(z^7)$.

b Show z^7 on an Argand diagram.

c Express $\dfrac{z}{w}$ in the form $r\,\text{cis}\,\theta$.

d Express z and w in Cartesian form, and hence express $\dfrac{z}{w}$ in Cartesian form.

e Use the results of **d** to determine an exact value for $\tan\left(\dfrac{7\pi}{12}\right)$ in the form $a + \sqrt{b}$, where a and b are rational.

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31 Let $v = 2 + i$ and $P(z) = z^3 - 7z^2 + 17z - 15$.

a Show by substitution that $P(2 + i) = 0$.

b Determine the other two solutions of the equation $P(z) = 0$.

c Let $\boldsymbol{i}$ be the unit vector in the positive $\text{Re}(z)$-direction and let $\boldsymbol{j}$ be the unit vector in the positive $\text{Im}(z)$-direction.
Let A be the point on the Argand diagram corresponding to $v = 2 + i$.
Let B be the point on the Argand diagram corresponding to $1 - 2i$.
Show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$.

32 **a** Determine the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$, writing your solutions in Cartesian form.

b **i** Plot the two solutions from **a** on an Argand diagram.

ii Determine the equation of the circle, with centre the origin, which passes through these two points.

iii Determine the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

iv Let $Q(z) = (z^2 + 4)(z^2 - 2\sqrt{3}z + 4)$. Determine the polynomial $P(z)$ such that $Q(z)P(z) = z^6 + 64$ and explain the significance of the result.

33 **a** Express $-4\sqrt{3} - 4i$ in exact polar form.

b Determine the cube roots of $-4\sqrt{3} - 4i$.

c Carefully plot the three cube roots of $-4\sqrt{3} - 4i$ on an Argand diagram.

d **i** Show that the cubic equation $z^3 - 3\sqrt{3}iz^2 - 9z + 3\sqrt{3}i = -4\sqrt{3} - 4i$ can be written in the form $(z - w)^3 = -4\sqrt{3} - 4i$, where w is a complex number.

ii Hence determine the solutions of the equation $z^3 - 3\sqrt{3}iz^2 - 9z + (3\sqrt{3} + 4)i + 4\sqrt{3} = 0$, in exact Cartesian form.

34 The points X, Y and Z correspond to the numbers $4\sqrt{3} + 2i$, $5\sqrt{3} + i$ and $6\sqrt{3} + 4i$.

a Determine the vector $\overrightarrow{XY}$ and the vector $\overrightarrow{XZ}$.

b Let z_1 and z_2 be the complex numbers corresponding to the vectors $\overrightarrow{XY}$ and $\overrightarrow{XZ}$. Determine z_3 such that $z_2 = z_3 z_1$.

c By writing z_3 in modulus–argument form, show that XYZ is half an equilateral triangle XWZ and give the complex number to which W corresponds.

d The triangle XYZ is rotated through an angle of $\frac{\pi}{3}$ anticlockwise about Y. Determine the new position of X.

35 **a** Sketch the region T in the complex plane which is obtained by reflecting

$$S = \{ z : \text{Re}(z) \le 2 \} \cap \{ z : \text{Im}(z) < 2 \} \cap \left\{ z : \frac{\pi}{6} < \text{Arg}(z) < \frac{\pi}{3} \right\}$$

in the line defined by $|z + i| = |z - 1|$.

b Describe the region T by using set notation in a similar way to that used in **a** to describe S.

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36 Consider the equation $x^2 + 4x - 1 + k(x^2 + 2x + 1) = 0$. Determine the set of real values k, where $k \neq -1$, for which the two solutions of the equation are:

a real and distinct **b** real and equal

c complex with positive real part and non-zero imaginary part.

37 **a** If $z = \cos\theta + i\sin\theta$, prove that $\dfrac{1-z}{1+z} = -i\tan\left(\dfrac{\theta}{2}\right)$.

b On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers 0, 1, z, $1+z$ and $1-z$ respectively. Show these points on a diagram.

c Prove that the magnitude of $\angle POQ$ is $\dfrac{\pi}{2}$. Determine, in terms of θ, the ratio $\dfrac{|OP|}{|OQ|}$.

38 A regular hexagon $LMNPQR$ has its centre at the origin O and its vertex L at the point $z = 4$.

a Indicate in a diagram the region in the hexagon in which the inequalities $|z| \geq 2$ and $\dfrac{-\pi}{3} \leq \operatorname{Arg} z \leq \dfrac{\pi}{3}$ are satisfied.

b Determine, in the form $|z - c| = a$, the equation of the circle through O, M and R.

c Determine the complex numbers corresponding to the points N and Q.

d The hexagon is rotated clockwise about the origin by 45°. Express in the form $r \operatorname{cis}\theta$ the complex numbers corresponding to the new positions of N and Q.

39 **a** A complex number $z = a + bi$ is such that $|z| = 1$. Show that $\dfrac{1}{z} = \bar{z}$.

b Let $z_1 = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ and $z_2 = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$. If $z_3 = \dfrac{1}{z_1} + \dfrac{1}{z_2}$. Determine z_3 in polar form.

c On a diagram, show the points z_1, z_2, z_3 and $z_4 = \dfrac{1}{z_3}$.

40 **a** Let $P(z) = z^3 + 3pz + q$. It is known that $P(z) = (z-k)^2(z-a)$.

i Show that $p = -k^2$. **ii** Determine q in terms of k.

iii Show that $4p^3 + q^2 = 0$.

b Let $h(z) = z^3 - 6iz + 4 - 4i$. It is known that $h(z) = (z-b)^2(z-c)$. Determine the values of b and c.

41 **a** Let z be a complex number with $|z| = 6$. Let A be the point representing z. Let B be the point representing $(1+i)z$.

i Determine $|(1+i)z|$.

ii Determine $|(1+i)z - z|$.

iii Prove that OAB is an isosceles right-angled triangle.

b Let z_1 and z_2 be non-zero complex numbers satisfying $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. If $z_1 = \alpha z_2$:

i Show that $\alpha = 1 + i$ or $\alpha = 1 - i$.

ii For each of these values of α, describe the geometric nature of the triangle whose vertices are the origin and the points representing z_1 and z_2.

CU

42 **a** Let $z = -12 + 5i$. Determine:

i $|z|$

ii $\operatorname{Arg}(z)$ correct to two decimal places in degrees

b Let $w^2 = -12 + 5i$ and $\alpha = \operatorname{Arg}(w^2)$.

i Write $\cos\alpha$ and $\sin\alpha$ in exact form.

ii Using the result $r^2(\cos(2\theta) + i\sin(2\theta)) = |w^2|(\cos\alpha + i\sin\alpha)$, write r, $\cos(2\theta)$ and $\sin(2\theta)$ in exact form, where $w = r\operatorname{cis}\theta$.

iii Use the result of **ii** to determine $\sin\theta$ and $\cos\theta$.

iv Determine the two values of w.

c Use a Cartesian method to determine w.

d Determine the square roots of $12 + 5i$ and comment on their relationship with the square roots of $-12 + 5i$.

43 **a** Determine the locus defined by $2z\bar{z} + 3z + 3\bar{z} - 10 = 0$.

b Determine the locus defined by $2z\bar{z} + (3 + i)z + (3 - i)\bar{z} - 10 = 0$.

c Determine the locus defined by $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$, where α, β and γ are real.

d Determine the locus defined by $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$, where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

44 **a** Expand $(\cos\theta + i\sin\theta)^5$.

b By De Moivre's theorem, we know that $(\operatorname{cis}\theta)^5 = \operatorname{cis}5\theta$. Use this result and the result of **a** to show that:

i $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

ii $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$ if $\sin\theta \neq 0$

45 **a** If $\bar{z}$ denotes the complex conjugate of the number $z = x + yi$, determine the Cartesian equation of the line given by $(1 + i)z + (1 - i)\bar{z} = -2$.
Sketch on an Argand diagram the set $\left\{z : (1 + i)z + (1 - i)\bar{z} = -2,\ \operatorname{Arg} z \le \dfrac{\pi}{2}\right\}$.

b Let $S = \left\{z : \left|z - (2\sqrt{2} + 2\sqrt{2}i)\right| \le 2\right\}$.

i Sketch S on an Argand diagram.

ii If z belongs to S, determine the maximum and minimum values of $|z|$.

iii If z belongs to S, determine the maximum and minimum values of $\operatorname{Arg}(z)$.

46 Let $u = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ and $v = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$

a Evaluate $\dfrac{u}{v}$ in Cartesian form.

b Determine the polar forms of u and v.

c Hence, evaluate $\dfrac{u}{v}$ by dividing in polar form.

d Hence, determine the exact values of $\cos\left(\dfrac{\pi}{12}\right)$ and $\sin\left(\dfrac{\pi}{12}\right)$.

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Multiple-choice questions

Technology-free multiple-choice questions

1 If $z_1 = 5\operatorname{cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2\operatorname{cis}\left(\frac{3\pi}{4}\right)$, then $z_1 z_2$ is equal to

A $7\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **B** $7\operatorname{cis}\left(\frac{13\pi}{12}\right)$ **C** $10\operatorname{cis}\left(\frac{\pi}{4}\right)$ **D** $10\operatorname{cis}\left(\frac{-11\pi}{12}\right)$

2 If $(x + yi)^2 = -32i$ for real values of x and y, then

A $x = 4, y = 4$ **B** $x = -4, y = 4$
C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$

3 If $u = 1 - i$, then $\frac{1}{3 - u}$ is equal to

A $\frac{2}{5} - \frac{1}{5}i$ **B** $\frac{2}{3} + \frac{1}{3}i$ **C** $\frac{2}{5} + \frac{1}{5}i$ **D** $\frac{2}{3} - \frac{1}{3}i$

4 Factorising $z^2 + 6z + 10$ over $\mathbb{C}$ gives

A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$
C $(z + 3 - i)(z + 3 + i)$ **D** $(z + 3 + i)(z - 3 + i)$

5 The solutions of the equation $z^3 + 8i = 0$ are

A $\sqrt{3} - i,\ -2i,\ 2i$ **B** $\sqrt{3} - i,\ -\sqrt{3} - i,\ 2i$
C $-\sqrt{3} - i,\ -2,\ -2i$ **D** $-\sqrt{3} - i,\ \sqrt{3} - i,\ -2i$

6 $\frac{\sqrt{6}}{2}(1 + i)$ can be expressed in polar form as

A $\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **B** $\sqrt{3}\operatorname{cis}\left(-\frac{7\pi}{4}\right)$ **C** $-\sqrt{3}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **D** $-\sqrt{3}\operatorname{cis}\left(-\frac{7\pi}{4}\right)$

7 If $z = 1 + i$ is one solution of an equation of the form $z^4 = a$, where $a \in \mathbb{C}$, then the other solutions are

A $-1,\ 1,\ 0$ **B** $-1,\ 1,\ 1 - i$ **C** $-1 + i,\ -1 - i$ **D** $-1 + i, -1 - i, 1 - i$

8 The square roots of $-2 - 2\sqrt{3}i$ in polar form are

A $2\operatorname{cis}\left(-\frac{2\pi}{3}\right),\ 2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **B** $2\operatorname{cis}\left(-\frac{\pi}{3}\right),\ 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$
C $4\operatorname{cis}\left(-\frac{2\pi}{3}\right),\ 4\operatorname{cis}\left(\frac{\pi}{3}\right)$ **D** $4\operatorname{cis}\left(-\frac{\pi}{3}\right),\ 4\operatorname{cis}\left(\frac{2\pi}{3}\right)$

9 The zeroes of the polynomial $2x^2 + 6x + 7$ are α and β. The value of $|\alpha - \beta|$ is

A $\sqrt{5}$ **B** $2\sqrt{5}$ **C** $3\sqrt{5}$ **D** $4\sqrt{5}$

10 The points z_1, z_2, z_3 and z_4 in the complex plane are the vertices of a parallelogram (taken in order) if and only if

A $z_1 + z_2 + z_3 + z_4 = 0$ **B** $z_1 + z_2 = z_3 + z_4$
C $z_1 + z_4 = z_2 + z_3$ **D** $z_1 + z_3 = z_2 + z_4$

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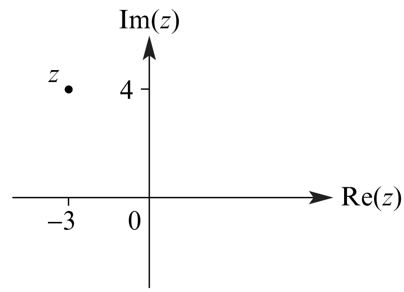
Technology-active multiple-choice questions

11 The complex number z shown in the diagram is best represented by

A $5\operatorname{cis}(2.12)$
B $25\operatorname{cis}(2.12)$
C $5\operatorname{cis}(2.21)$
D $25\operatorname{cis}(2.12)$



12 The expression $(1+i)(1-i)(2+2i)(2-2i)(3+3i)(3-3i)$ is equal to

A 576 **B** 288 **C** 144 **D** 72

13 The points U and V correspond to the complex numbers $u = 1 - i$ and $v = 4 + 5i$. The line UV intersects the positive direction of the real axis at an acute angle of approximately

A 63.4° **B** 61.2° **C** 60.9° **D** 59.1°

14 A region in the complex plane is defined by the equations $|z| < 5$ and $\dfrac{\pi}{6} < \operatorname{Arg} z < \dfrac{5\pi}{6}$. The area of this region is closest to

A 22.4 **B** 24.6 **C** 26.2 **D** 28.6

15 The ray $\operatorname{Arg}(z) = \dfrac{\pi}{4}$ cuts the region defined by $|z - 6i| < 6$ into two sub-regions. The smaller of these has an area that is approximately equal to

A 11.4 **B** 11.2 **C** 10.8 **D** 10.3

16 A set of points is defined by $|z - 1 - i)| = |z + 1 + i|$. The shortest distance from a point in this set to the point $z = 3i$ is closest to

A 2.1 **B** 2.3 **C** 2.5 **D** 2.7

17 Consider all complex numbers z and w satisfying $|z| = 1$ and $|w - 3 - 3i| = 7$. The minimum value of $|z - w|$ is approximately

A 1.66 **B** 1.84 **C** 1.76 **D** 1.69

18 The three solutions of $z^3 = 7 + \sqrt{15}i$ define the vertices of an equilateral triangle. The area of this triangle is approximately equal to

A 7.1 **B** 6.3 **C** 5.2 **D** 4.8

19 Let $a > 0$ be a real number. The four solutions of $z^4 = a$ define the vertices of a square. If the side length of the square is $\sqrt{3}$, then a is equal to

A 2.25 **B** 1.96 **C** 1.69 **D** 1.44

20 If $w = \sqrt{2} + i$, the largest positive integer n for which w^n lies in $\{z \in \mathbb{C} : |z| \le 20\}$ is

A 4 **B** 5 **C** 6 **D** 7

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4

Mathematical induction

Chapter contents

In this chapter, we introduce an important technique for proving statements that are true for each natural number n.

For example, you may remember from your study of arithmetic sequences that the sum of the first n odd numbers is n^2. This result involves a statement, $P(n)$, about each natural number n.

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

We can prove results like this using **mathematical induction**.

In the first section of this chapter, we revise concepts of proof from Specialist Mathematics Units 1 & 2. We consider alternative proofs of some of the results that will be proved later in the chapter using mathematical induction.

The first section also provides an opportunity to revise both divisibility and partial sums.

This chapter together with Chapter 1 covers Unit 3 Topic 2: Mathematical induction and trigonometric proofs. For the proof by induction of de Moivre's theorem and associated exercises see Chapter 3.

4A Revision of proof techniques

Learning intentions

- To revise the concept of **divisibility** for integers.
- To revise basic concepts of **proof** including:
  - conditional statements
  - equivalent statements
  - proof by contradiction
  - counterexamples.

We start by revising the fundamental ideas of proof introduced in Specialist Mathematics Units 1 & 2. In this section we will not use mathematical induction.

Divisibility of integers

- The set of **natural numbers** is $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.
- The set of **integers** is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Let a and b be integers. Then we say that a is **divisible** by b if there exists an integer k such that $a = bk$. In this case, we also say that b is a **divisor** of a.

For example, the integer 12 is divisible by 3, since $12 = 3 \times 4$.

Example 1

Let $n \in \mathbb{N}$. Prove that $n^3 - n$ is divisible by 3.

Solution

Note that $n^3 - n = n(n-1)(n+1)$.

When the natural number n is divided by 3, the remainder must be 0, 1 or 2. Therefore n can be written in the form $3k$, $3k+1$ or $3k+2$, for some integer k.

Case 1: $n = 3k$. Then
$$n^3 - n = n(n-1)(n+1) = 3k(3k-1)(3k+1)$$

Case 2: $n = 3k+1$. Then
$$\begin{aligned} n^3 - n &= n(n-1)(n+1) \\ &= (3k+1)(3k)(3k+2) \\ &= 3k(3k+1)(3k+2) \end{aligned}$$

Case 3: $n = 3k+2$. Then
$$\begin{aligned} n^3 - n &= n(n-1)(n+1) \\ &= (3k+2)(3k+1)(3k+3) \\ &= 3(k+1)(3k+1)(3k+2) \end{aligned}$$

In all three cases, we see that $n^3 - n$ is divisible by 3.

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Partial sums of sequences

In Mathematical Methods Units 1 & 2, you have used the formula for the sum of the first n terms of an arithmetic sequence

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{n}{2}\big(2a + (n - 1)d\big)$$

and the formula for the sum of the first n terms of a geometric sequence

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \qquad (\text{if } r \neq 1)$$

In Section 4B, we use mathematical induction to establish these and other similar results.

The next example illustrates an alternative technique called 'telescopic cancelling'. This technique can be used to find the partial sums of some sequences.

Example 2

Prove that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Solution

We observe that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

We use this result to expand each term of the series:

$$\begin{aligned}
\text{LHS} &= \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \qquad \text{(by cancelling)} \\
&= \frac{n}{n+1} = \text{RHS}
\end{aligned}$$

Proof by contradiction

The basic outline of a **proof by contradiction** is:

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

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Example 3

Suppose x satisfies $5^x = 2$. Use proof by contradiction to show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} 5^x = 2 \quad &\Rightarrow \quad 5^{\frac{m}{n}} = 2 \\ &\Rightarrow \quad \left(5^{\frac{m}{n}}\right)^n = 2^n \qquad \text{(raise both sides to the power } n\text{)} \\ &\Rightarrow \quad 5^m = 2^n \end{aligned}$$

The left-hand side of this equation is odd and the right-hand side is even. This gives a contradiction, and so x is not rational.

Implication and equivalence

Conditional statements

Consider the following sentence about a quadrilateral $ABCD$:

Statement	If	$ABCD$ is a square	then	$ABCD$ has equal diagonals.

This is called a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as

$$P \Rightarrow Q$$

which is read 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

The converse of a conditional statement

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

The converse of a true statement may not be true. For example:

Statement	If $ABCD$ is a square, then $ABCD$ has equal diagonals.	(true)
Converse	If $ABCD$ has equal diagonals, then $ABCD$ is a square.	(false)

In this case, we know that the converse statement is false, since any rectangle has diagonals of equal length.

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Equivalent statements

Now consider the following two statements about a quadrilateral $ABCD$:

P: $ABCD$ is a rectangle

Q: $ABCD$ has equal diagonals that bisect each other

Both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements and we write $P \Leftrightarrow Q$.

We can also say that P is true **if and only if** Q is true. So in the above example, we can say that a quadrilateral is a rectangle if and only if it has equal diagonals that bisect each other.

To prove that two statements P and Q are equivalent, you have to prove two things:

$P \Rightarrow Q$ and $Q \Rightarrow P$

Example 4

Let $n \in \mathbb{Z}$. Prove that n is divisible by 3 if and only if n^2 is divisible by 3.

Solution

(⇒) Assume that n is divisible by 3. We want to show that n^2 is divisible by 3.

Since n is divisible by 3, there exists an integer k such that $n = 3k$. Therefore $n^2 = (3k)^2 = 3(3k^2)$. Hence n^2 is divisible by 3.

(⇐) Assume that n^2 is divisible by 3. We want to show that n is divisible by 3.

When the integer n is divided by 3, the remainder must be 0, 1 or 2. Therefore n can be written in the form $3k$, $3k + 1$ or $3k + 2$, for some integer k.

Case 1: $n = 3k$. This is the case where n is divisible by 3.

Case 2: $n = 3k + 1$. Then $n^2 = 9k^2 + 6k + 1$, which leaves remainder 1 when divided by 3. This contradicts our assumption that n^2 is divisible by 3. So this case cannot occur.

Case 3: $n = 3k + 2$. Then $n^2 = 9k^2 + 12k + 4$, which leaves remainder 1 when divided by 3. This contradicts our assumption that n^2 is divisible by 3. So this case cannot occur.

Hence n must be divisible by 3.

Quantification and counterexamples

A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**. For example:

Statement	For all real numbers x, the number $x^2 - x$ is positive.

To disprove a universal statement, we simply need to give one example where it does not hold. Such an example is called a **counterexample**.

Example 5

Disprove the statement: For all real numbers x, the number $x^2 - x$ is positive.

Solution

When $x = 0$, we obtain $x^2 - x = 0$, which is not positive.

The negation of a universal statement is an **existence statement**; it claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.

Negation	There exists a real number x such that $x^2 - x$ is not positive.

When we disprove a universal statement, we are finding an example to show that its negation is true.

Notation

The words 'for all' can be abbreviated using the *turned A* symbol, $\forall$. The words 'there exists' can be abbreviated using the *turned E* symbol, $\exists$. For example:

- 'For all natural numbers n, we have $2n \geq n + 1$' can be written as $(\forall n \in \mathbb{N})\ 2n \geq n + 1$.
- 'There exists an integer m such that $m^2 = 25$' can be written as $(\exists m \in \mathbb{Z})\ m^2 = 25$.

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

Exercise 4A

SF

Example 1 **1** Let $n \in \mathbb{N}$. Prove that $n^2 - n$ is even.
Hint: Consider the cases when n is odd and n is even.

2 Assume that m is divisible by 5 and n is divisible by 11. Prove that:

a mn is divisible by 55 **b** m^2n is divisible by 275.

CF

3 Assume that m and n are perfect cubes. Show that mn is a perfect cube.

4 Let m and n be integers. Prove that $(2m + n)^2 - (2m - n)^2$ is divisible by 8.

5 Suppose that n is an odd integer. Prove that $n^2 + 8n + 3$ is even.

6 Let $n \in \mathbb{Z}$. Prove that $3n^2 + 7n + 11$ is odd.
Hint: Consider the cases when n is odd and n is even.

Example 2 **7** **a** By simplifying the right-hand side, show that

$$\frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

b Use part **a** to prove that

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \text{for all } n \in \mathbb{N}$$

8 a Show that

$$k(k+1) = \frac{1}{3}\Big(k(k+1)(k+2) - (k-1)k(k+1)\Big)$$

b Use part a to prove that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} \quad \text{for all } n \in \mathbb{N}$$

9 a Show that

$$(2k-1)(2k+1) = \frac{1}{6}\Big((2k-1)(2k+1)(2k+3) - (2k-3)(2k-1)(2k+1)\Big)$$

b Use part a to prove that

$$1 \cdot 3 + 3 \cdot 5 + \cdots + (2n-1)(2n+1) = \frac{(4n^2-1)(2n+3)+3}{6} \quad \text{for all } n \in \mathbb{N}$$

Example 3 10 Use proof by contradiction to show that $\log_2 7$ is irrational.

Example 4 11 Let $n \in \mathbb{Z}$. Prove that n is divisible by 5 if and only if n^2 is divisible by 5.

12 Let n be a positive integer. Prove that if n^3 is divisible by 3, then n is divisible by 3.

13 Prove by contradiction that $\sqrt[3]{3}$ is irrational.
Hint: Use the result from Question 12.

Example 5 14 Provide a counterexample to show that $n^2 + n + 1$ is not always a prime number, where n is a positive integer.

15 Let $m, n \in \mathbb{Z}$. Consider the statement: If m is even and n is odd, then mn is even.

a Write down the converse of this statement.

b Show that the converse is not true.

16 Prove that, for any two positive integers that are not divisible by 3, the difference between their squares is divisible by 3.

17 Show that the sum of three consecutive positive integers is a divisor of the sum of the cubes of these three integers.

18 Let $k \in \mathbb{N}$. Prove that the product of k consecutive positive integers is divisible by $k!$.
Hint: Consider the binomial coefficient ${}^{n+k}C_k$.

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4B Mathematical induction

Learning intentions

- To understand the **principle of mathematical induction**.
- To use mathematical induction to prove results involving:
  - divisibility
  - partial sums of sequences.

In Example 2 from the previous section, we considered the result

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

This result involves an infinite sequence of propositions, one for each natural number:

$$P(1): \quad \frac{1}{1 \times 2} = \frac{1}{1+1}$$

$$P(2): \quad \frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{2}{2+1}$$

$$P(3): \quad \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} = \frac{3}{3+1}$$

$$\vdots$$

We proved that the proposition $P(n)$ is true for every natural number n. In this section, we give an alternative proof of this result using mathematical induction.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n.

We can prove that $P(n)$ is true for every natural number n as follows:

a Show that $P(1)$ is true.

b Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \quad \Rightarrow \quad P(2) \text{ is true} \quad \Rightarrow \quad P(3) \text{ is true} \quad \Rightarrow \quad \cdots$$

Let's see how mathematical induction is used in practice.

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Using induction for partial sums

Mathematical induction is useful for proving many results about partial sums.

Example 6

Prove using mathematical induction that

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Step 1 $P(1)$ is the proposition $\frac{1}{1\times 2}=\frac{1}{1+1}$, that is, $\frac{1}{2}=\frac{1}{2}$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

We have

$$\begin{aligned}\text{LHS of } P(k+1) &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \qquad \text{(using } P(k))\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

It follows by the principle of mathematical induction that $P(n)$ is true for every natural number n.

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In the next example, we use induction to establish the formula for the sum of the first n odd numbers.

Example 7

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Step 1 $P(1)$ is the proposition $1 = 1^2$, that is, $1 = 1$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

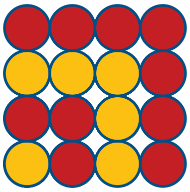
We have

$$\begin{aligned} \text{LHS of } P(k+1) &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) && \text{(using } P(k)\text{)} \\ &= (k + 1)^2 \\ &= \text{RHS of } P(k + 1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

While mathematical induction is good for proving that formulas are true, it rarely indicates why they should be true in the first place. The formula $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ can be discovered in the diagram shown on the right.



Using induction for divisibility results

We now use mathematical induction to prove results about divisibility. You should compare the next example with Example 1 from the previous section.

Example 8

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition: $n^3 - n$ is divisible by 3.

Step 1 $P(1)$ is the proposition $1^3 - 1 = 0$ is divisible by 3. Clearly, $P(1)$ is true.

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Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$k^3 - k = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, we have to prove that the number $(k+1)^3 - (k+1)$ is divisible by 3. We have

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= k^3 - k + 3k^2 + 3k\\ &= 3m + 3k^2 + 3k \qquad \text{(using } P(k))\\ &= 3(m + k^2 + k)\end{aligned}$$

Therefore $(k+1)^3 - (k+1)$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Example 9

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. So $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3. We have

$$\begin{aligned}7^{k+1} - 4 &= 7 \times 7^k - 4\\ &= 7(3m + 4) - 4 \qquad \text{(using } P(k))\\ &= 21m + 28 - 4\\ &= 21m + 24\\ &= 3(7m + 8)\end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

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Summation notation

Suppose that m and n are integers with $m < n$. Then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

This notation, which is called **summation notation** or **sigma notation**, is very convenient for concisely representing sums. The notation uses the symbol Σ, which is the uppercase Greek letter *sigma*.

The notation $\sum_{i=m}^{n} a_i$ is read: 'the sum of the numbers a_i from i equals m to i equals n'.

The expression $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is called the **expanded form** of $\sum_{i=m}^{n} a_i$.

Example 10

Write $\sum_{i=1}^{5} 2^i$ in expanded form and evaluate.

Solution

$$\begin{aligned}\sum_{i=1}^{5} 2^i &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 2 + 4 + 8 + 16 + 32 \\ &= 62\end{aligned}$$

You may prefer to use this notation in induction proofs involving partial sums. The next example uses this notation to give the sum of the cubes of the first n odd numbers.

Example 11

Prove using the principle of mathematical induction that

$$\sum_{r=1}^{n} (2r-1)^3 = n^2(2n^2 - 1)$$

Solution

For each natural number n, let $P(n)$ be the proposition:

$$\sum_{r=1}^{n} (2r-1)^3 = n^2(2n^2 - 1)$$

Step 1 First consider $P(1)$:

$$\text{LHS of } P(1) = \sum_{r=1}^{1} (2r-1)^3 = (2 \cdot 1 - 1)^3 = 1$$

$$\text{RHS of } P(1) = 1^2(2 \cdot 1^2 - 1) = 1$$

Therefore $P(1)$ is true.

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Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$\sum_{r=1}^{k}(2r-1)^3 = k^2(2k^2-1)$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$\sum_{r=1}^{k+1}(2r-1)^3 = (k+1)^2(2(k+1)^2-1)$$

We have

$$\begin{aligned}\text{LHS of } P(k+1) &= \sum_{r=1}^{k+1}(2r-1)^3 \\ &= \sum_{r=1}^{k}(2r-1)^3 + (2(k+1)-1)^3 \\ &= k^2(2k^2-1) + (2k+1)^3 \qquad \text{(using } P(k)\text{)} \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1\end{aligned}$$

$$\begin{aligned}\text{RHS of } P(k+1) &= (k+1)^2(2(k+1)^2-1) \\ &= 2(k+1)^4 - (k+1)^2 \\ &= 2k^4 + 8k^3 + 11k^2 + 6k + 1\end{aligned}$$

Therefore $P(k+1)$ is true. Hence we have shown that $P(k)$ implies $P(k+1)$, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

Exercise 4B

Example 6

1 Use the principle of mathematical induction to prove that

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all $n \in \mathbb{N}$.

2 Prove each of the following using mathematical induction:

Example 7

a $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

b $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

c $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

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3 Prove each of the following using mathematical induction:

a $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$, where $x \neq 1$

b $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

c $\dfrac{1}{1 \times 5} + \dfrac{1}{5 \times 9} + \cdots + \dfrac{1}{(4n-3)(4n+1)} = \dfrac{n}{4n+1}$

Example 8

4 Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

5 Prove by mathematical induction that $n(n+1)(n+2)$ is divisible by 3 for all $n \in \mathbb{N}$.

6 Prove by mathematical induction that $n^5 - n$ is divisible by 5 for all $n \in \mathbb{N}$.

7 **a** Expand $(2k-1)^5$ and $(2k+1)^5$.

b Prove by induction that $n^5 - n$ is divisible by 240 for each odd positive integer n.

Example 9

8 Prove each of the following divisibility statements by mathematical induction:

a $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$

b $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$

c $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$

d $5^n + 6 \times 7^n + 1$ is divisible by 4 for all $n \in \mathbb{N}$

9 Prove by induction that 3^n is odd for every $n \in \mathbb{N}$.

CU

10 The Fibonacci sequence is defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$.

a Determine f_n for $n = 1, 2, \ldots, 10$.

b Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

c Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

d Try to determine a general formula for the expression from part **c**.

e Confirm that your formula works using mathematical induction.

f Using induction, prove that every third Fibonacci number, f_{3n}, is even.

CF

11 Prove that $4^n + 5^n$ is divisible by 9 for every odd natural number n.

12 Prove each of the following statements by induction:

a $(2n+1)^2 - 1$ is divisible by 8 for $n = 1, 2, 3, \ldots$

b $8^n - (-6)^n$ is divisible by 7 for $n = 1, 2, 3, \ldots$

c $5^{2n} - 13^n$ is divisible by 6 for $n = 1, 2, 3, \ldots$

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SF

13 Use mathematical induction to prove the following results:

a $a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \dfrac{n}{2}\big(2a + (n - 1)d\big)$

b $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$, where $r \neq 1$

CF

14 **a** Prove by induction that $3^n - (-2)^n$ is divisible by 5 for all $n \in \mathbb{N}$.

b Prove by induction that $4^n - (-3)^n$ is divisible by 7 for all $n \in \mathbb{N}$.

c State a similar divisibility result for an odd number of your choice. Prove your result by induction.

CU

15 Use the product rule and proof by induction to show that $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for each positive integer n.

SF

16 Prove each of the following using mathematical induction:

a $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \dfrac{1}{6}n(n + 1)(2n + 7)$

b $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n + 1) = n(n + 1)^2$

Example 10

17 Write each of the following in expanded form and evaluate:

a $\displaystyle\sum_{i=1}^{4} i^3$ **b** $\displaystyle\sum_{k=1}^{5} k^3$ **c** $\displaystyle\sum_{i=1}^{5} (-1)^i\, i$ **d** $\displaystyle\frac{1}{5}\sum_{i=1}^{5} i$

e $\displaystyle\sum_{i=1}^{6} i$ **f** $\displaystyle\sum_{k=1}^{4} (k - 1)^2$ **g** $\displaystyle\frac{1}{3}\sum_{i=1}^{4} (i - 2)^2$ **h** $\displaystyle\sum_{i=1}^{6} i^2$

18 Write each of the following in expanded form:

a $\displaystyle\sum_{i=1}^{n} r^i$ **b** $\displaystyle\sum_{i=0}^{n} r^i \cdot 2^{5-i}$ **c** $\displaystyle\sum_{i=0}^{n} (2r)^i \cdot 3^{6-i}$ **d** $\displaystyle\sum_{i=0}^{n} (r - r_i)^i$

Example 11

CF

19 Prove each of the following using mathematical induction:

a $\displaystyle\sum_{r=1}^{n} r(r + 4) = \frac{n(n + 1)(2n + 13)}{6}$

b $\displaystyle\sum_{r=1}^{n} (2r + 1) \cdot 2^{r-1} = 1 + (2n - 1) \cdot 2^n$

20 Prove that $3^{3n+1} + 9 \times 2^{n+3}$ is divisible by 25, for each natural number n.

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Chapter summary

Basic concepts of proof

- A **conditional statement** has the form: If P is true, then Q is true.
  This can be abbreviated as $P \Rightarrow Q$, which is read 'P implies Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**. The symbol for this is $\forall$.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.The symbol for this is $\exists$.
- A **counterexample** can be used to demonstrate that a universal statement is false.

Proof by mathematical induction

Mathematical induction is used to prove that a statement is true for all natural numbers. The basic outline of a proof by mathematical induction is:

0. Define the proposition $P(n)$ for $n \in \mathbb{N}$.
1. Show that $P(1)$ is true.
2. Assume that $P(k)$ is true for some $k \in \mathbb{N}$.
3. Show that $P(k + 1)$ is true.
4. Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

4A **1** I can directly prove simple mathematical statements. ☐

See Example 1, Question 1, Question 2, Question 3 and Question 4

4A **2** I can determine partial sums using a telescoping series. ☐

See Example 2, Question 7, Question 8 and Question 9

4A **3** I can give proofs by contradiction. ☐

See Example 3 and Question 10

4A **4** I can show that two mathematical statements are equivalent. ☐

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Review

See Example 4 and Question 11

4A **5** I can use counterexamples to disprove mathematical statements. ☐

See Example 5 and Question 14

4B **6** I can use mathematical induction to prove results related to sums. ☐

See Example 6, Example 7, Question 1 and Question 2

4B **7** I can use mathematical induction to prove divisibility results. ☐

See Example 8, Example 9, Question 4 and Question 8

Short-response questions

1 Using mathematical induction, prove each of the following for $n \in \mathbb{N}$: SF

a $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n - 1)}{2}$

b $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$

c $3^{-1} + 3^{-2} + \cdots + 3^{-n} = \dfrac{3^n - 1}{2 \times 3^n}$

d $\dfrac{1}{2^1} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots + \dfrac{n}{2^n} = 2 - \dfrac{n + 2}{2^n}$

2 Using mathematical induction, prove each of the following for $n \in \mathbb{N}$: CF

a $\displaystyle\sum_{r=1}^{n} (r^2 - 1) = \frac{1}{6} n(n - 1)(2n + 5)$

b $\displaystyle\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n + 1}$

c $\displaystyle\sum_{r=1}^{n} r2^r = 2 + (n - 1)2^{n+1}$

d $\displaystyle\sum_{r=2}^{n} \frac{1}{r^2 - 1} = \frac{(n - 1)(3n + 2)}{4n(n + 1)}$

3 Prove each of the following divisibility results for every natural number n:

a $5^n + 3$ is divisible by 4

b $3^{2n} + 7$ is divisible by 8

c $4^n + 6n - 1$ is divisible by 3

d $7^{2n-1} + 5$ is divisible by 12

e $10^n + 18n - 1$ is divisible by 27

f $5^n - 4n - 1$ is divisible by 16

g $10^n + 7^n - 5$ is divisible by 6

h $(2n + 1)7^n - 1$ is divisible by 4

4 Prove by induction that $n^2 + 5n + 6$ is even for all $n \in \mathbb{N}$. SF

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5 Prove by induction that the sum of the cubes of any three consecutive positive integers is always divisible by 9.

6 **a** Prove that $n^2 + n + 2$ is divisible by 2 for all $n \in \mathbb{N}$.

b Hence, prove by induction that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

7 **a** Use a trigonometric identity to show that

$$2\sin(A)\cos(kA) = \sin((k+1)A) - \sin((k-1)A)$$

b Use part **a** and 'telescopic cancelling' to prove that

$$2\sin(A)\big(\cos(A) + \cos(3A) + \cdots + \cos((2n-1)A)\big) = \sin(2nA)$$

for all $n \in \mathbb{N}$.

c Use mathematical induction to give an alternative proof of the result from part **b**.

8 **a** Use a trigonometric identity to show that

$$2\sin(A)\sin(kA) = \cos((k-1)A) - \cos((k+1)A)$$

b Use part **a** and 'telescopic cancelling' to prove that

$$\sin(A) + \sin(3A) + \sin(5A) + \cdots + \sin((2n-1)A) = \sin^2(nA)\operatorname{cosec}(A)$$

for all $n \in \mathbb{N}$.

c Use mathematical induction to give an alternative proof of the result from part **b**.

9 **a** Show that

$$2\sin(A)\cos(2kA) = \sin((2k+1)A) - \sin((2k-1)A)$$

b Use part **a** and 'telescopic cancelling' to prove that

$$\cos(2A) + \cos(4A) + \cdots + \cos(2nA) = \frac{\sin(nA)\cos((n+1)A)}{\sin(A)}$$

for all $n \in \mathbb{N}$.

c Prove this result by induction.

10 Prove using mathematical induction that

$$\sum_{r=1}^{n} \sin(2rA) = \frac{\cos(A) - \cos((2n+1)A)}{2\sin(A)}$$

for every natural number n.

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Review

Multiple-choice questions

1 Mathematical induction can be used to prove that

$$2^{n+1} \geq n^2$$

for all $n \in \mathbb{N}$. To prove the base case, we note that

A $2^1 \geq 1^2$ **B** $1^2 \geq 2^2$ **C** $2^2 \geq 2^2$ **D** $2^2 \geq 1^2$

2 Let $P(n)$ be the statement that

$$\sum_{j=1}^{n} (2j - 1) = n^2, n \in \mathbb{N}.$$

What is the statement $P(1)$?

A $(-1)^2 = 1^2$ **B** $0 = 0^2$

C $1 = 1^2$ **D** $1 + 3 = 2^2$

3 Mathematical induction can be used to prove that $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$. To prove the inductive step, we would have to show that

A $7^{k+1} - 3^{k+1} = 4m$ for some $m \in \mathbb{N}$ **B** $7^k - 3^k = 4m$ for some $m \in \mathbb{N}$

C $7^{k+1} - 3^{k+1} = \dfrac{m}{4}$ for some $m \in \mathbb{N}$ **D** $7^k - 3^k = \dfrac{m}{4}$ for some $m \in \mathbb{N}$

4 Mathematical induction can be used to prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

To prove the inductive step, we would have to show that

A $\sum_{i=1}^{k} \dfrac{1}{i(i+1)} = \dfrac{k}{k+1}$ **B** $\sum_{i=1}^{k+1} \dfrac{1}{i(i+1)} = \dfrac{k}{k+1}$

C $\sum_{i=1}^{k} \dfrac{1}{i(i+1)} = \dfrac{k+1}{k+2}$ **D** $\sum_{i=1}^{k+1} \dfrac{1}{i(i+1)} = \dfrac{k+1}{k+2}$

5 Mathematical induction can be used to prove that $7^{2n} + 16n - 1$ is divisible by some integer d for all $n \in \mathbb{N}$. The value of d could be

A 64 **B** 75 **C** 49 **D** 12

6 Mathematical induction can be used to prove that $16 \times 4^n - 3n - 7$ is divisible by some integer d for all $n \in \mathbb{N}$. The value of d could be

A 2 **B** 6 **C** 9 **D** 12

7 Mathematical induction can be used to prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all $n \in \mathbb{N}$. Before proving the inductive step, we first assume that

A $2^{k+3} + 3^{2k+3} = 7m$ for some $m \in \mathbb{Z}$ **B** $2^{k+2} + 3^{2k+1} = 7m$ for some $m \in \mathbb{Z}$

C $2^{k+2} + 3^{2k+1} = \dfrac{m}{7}$ for some $m \in \mathbb{Z}$ **D** $2^{k+3} + 3^{2k+3} = \dfrac{m}{7}$ for some $m \in \mathbb{Z}$

5

Vectors in two and three dimensions

Chapter contents

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length	30 cm is the length of the page of a particular book
time	10 s is the time for one athlete to run 100 m

More is required to describe displacement, velocity or force. The direction must be recorded as well as the magnitude.

displacement	30 km in the direction north
velocity	60 km/h in the direction south-east

A quantity that has both a magnitude and a direction is called a **vector**.

Chapter 5 together with Chapter 6 covers Unit 3 Topic 3: Vectors in two and three dimensions.

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5A Introduction to vectors

Learning intentions

- To be able to represent vectors and work with operations on vectors.

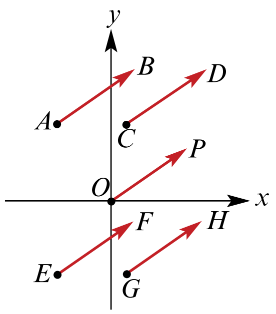
A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

Directed line segments

The five directed line segments shown all have the same length and direction, and so they are equivalent.



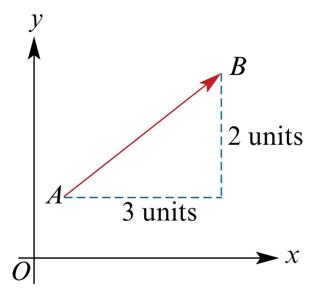
A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.

Column vectors

An alternative way to represent a vector is as a column of numbers. The column of numbers corresponds to a set of equivalent directed line segments.



For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across to the right and 2 up.

Vector notation

A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter, such as $\boldsymbol{v}$. We can write $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.

Magnitude of vectors

The magnitude of vector $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$. Likewise, the magnitude of vector $\boldsymbol{v}$ is denoted by $|\boldsymbol{v}|$. The magnitude of a vector is represented by the length of a directed line segment corresponding to the vector.

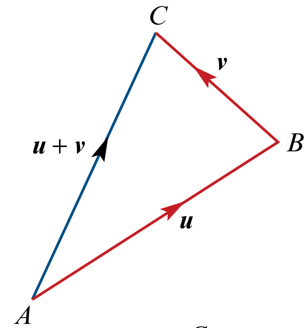
For $\overrightarrow{AB}$ in the diagram above, we have $|\overrightarrow{AB}| = \sqrt{3^2 + 2^2} = \sqrt{13}$ using Pythagoras' theorem.

In general, if $\overrightarrow{AB}$ is represented by the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then its magnitude is given by

$$|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$$

Addition of vectors

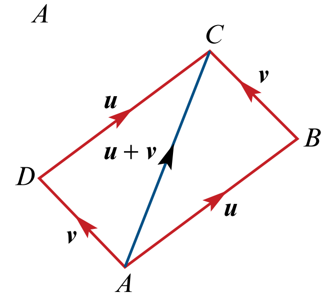
Adding vectors geometrically



Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment representing $\boldsymbol{v}$ from B to C.

The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

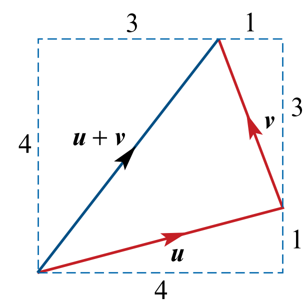


The same result is achieved if the order is reversed. This is represented in the diagram on the right:

$$\begin{aligned}\boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u}\end{aligned}$$

Hence addition of vectors is commutative.

Adding column vectors



Two vectors can be added using column-vector notation.

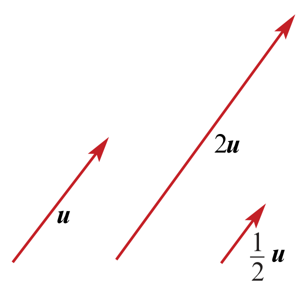
For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$

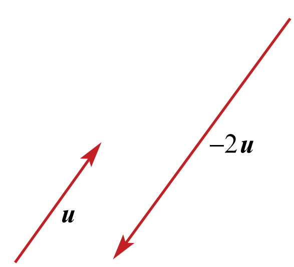


We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.

When a vector is multiplied by -2, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by -1, the vector's direction is reversed and the length remains the same.



If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$. The directed line segment $-\overrightarrow{AB}$ goes from B to A.

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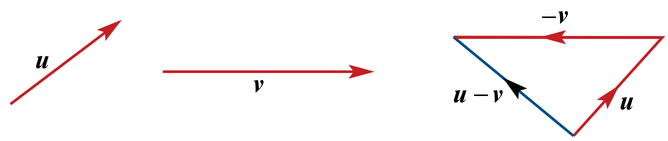
Zero vector

The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction. The magnitude of the zero vector is 0. Note that $0\boldsymbol{a} = \mathbf{0}$ and $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$.

In two dimensions, the zero vector can be written as $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Subtraction of vectors

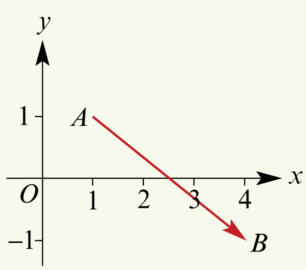
To determine $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.



Example 1

Draw a directed line segment representing the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and state the magnitude of this vector.

Solution



The magnitude is

$$\sqrt{3^2 + (-2)^2} = \sqrt{13}$$

Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 across to the right and 2 down'.

Note: Here the segment starts at $(1, 1)$ and goes to $(4, -1)$. It can start at any point.

Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from $(2, 6)$ to $(3, 1)$.

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, determine a and b.

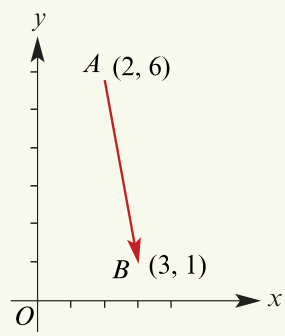
Solution

From the diagram:

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\therefore \quad \boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Hence $a = 1$ and $b = -5$.



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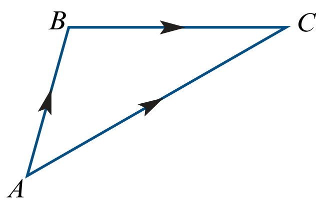
Polygons of vectors

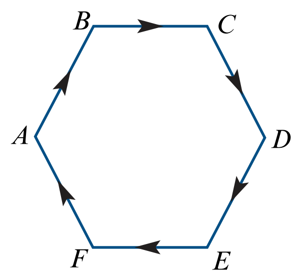
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$

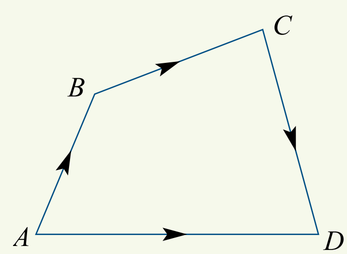


Example 3

Illustrate the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, where A, B, C and D are points in the plane.

Solution

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$$



Parallel vectors

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.

Position vectors

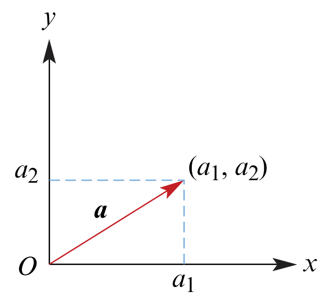
We can use a point O, the origin, as a starting point for a vector to indicate the position of a point A in space relative to O.

For a point A, the **position vector** is $\overrightarrow{OA}$.

The two-dimensional vector

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

is associated with the point (a_1, a_2). The vector $\boldsymbol{a}$ can be represented by the directed line segment from the origin to the point (a_1, a_2).



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Properties of the basic operations on vectors

The following properties can be established from our definitions of the basic operations on vectors. (In fact, these properties are used to generalise the concept of a vector, but this is beyond the scope of the course.)

commutative law for vector addition	$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$	
associative law for vector addition	$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$	
zero vector	$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$	
additive inverse	$\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$	
distributive laws	$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$	for $m \in \mathbb{R}$
	$(\ell + m)\boldsymbol{a} = \ell\boldsymbol{a} + m\boldsymbol{a}$	for $\ell, m \in \mathbb{R}$
compatibility of multiplication	$(\ell m)\boldsymbol{a} = \ell(m\boldsymbol{a})$	for $\ell, m \in \mathbb{R}$
identity of scalar multiplication	$1\boldsymbol{a} = \boldsymbol{a}$	

Thus, many of the ordinary rules of algebra apply to vectors.

Example 4

Simplify the following vector expression:

$$2(\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}) + \frac{3}{2}(\boldsymbol{b} - 4\boldsymbol{c})$$

Solution

$$\begin{aligned} 2(\boldsymbol{a} - \boldsymbol{b} + 3\boldsymbol{c}) + \frac{3}{2}(\boldsymbol{b} - 4\boldsymbol{c}) &= 2\boldsymbol{a} - 2\boldsymbol{b} + 6\boldsymbol{c} + \frac{3}{2}\boldsymbol{b} - 6\boldsymbol{c} \\ &= 2\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} \end{aligned}$$

Vectors in three dimensions

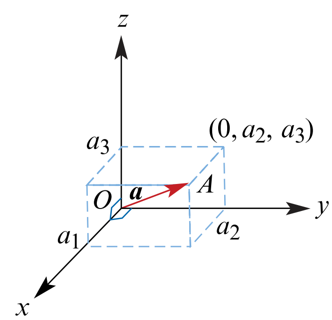
The definition of a vector is, of course, also valid in three dimensions. The properties which hold in two dimensions also hold in three dimensions.

For vectors in three dimensions, we use a third axis, denoted by z. The third axis is at right angles to the other two axes. The x-axis is drawn at an angle to indicate a direction out of the page towards you.

Vectors in three dimensions can also be written using column-vector notation:

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The vector $\boldsymbol{a}$ can be represented by the directed line segment from the origin to the point $A(a_1, a_2, a_3)$.



Example 5

Let $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Determine:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{a} - 2\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **d** $|\boldsymbol{a}|$

Solution

a $\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$

b $\boldsymbol{a} - 2\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 3 \end{bmatrix}$

c $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$

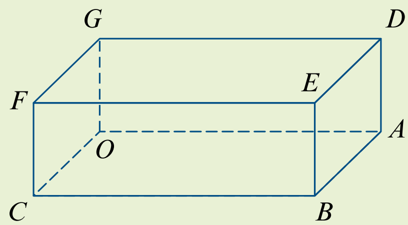
d $|\boldsymbol{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

Example 6

$OABCDEFG$ is a cuboid as shown.

Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{g} = \overrightarrow{OG}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

Determine the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{g}$ and $\boldsymbol{c}$:



a $\overrightarrow{OB}$ **b** $\overrightarrow{OF}$ **c** $\overrightarrow{GD}$ **d** $\overrightarrow{GB}$ **e** $\overrightarrow{FA}$

Solution

a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \boldsymbol{a} + \boldsymbol{c}$ (as $\overrightarrow{AB} = \overrightarrow{OC}$)

b $\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF}$

$= \boldsymbol{c} + \boldsymbol{g}$ (as $\overrightarrow{CF} = \overrightarrow{OG}$)

c $\overrightarrow{GD} = \overrightarrow{OA}$

$= \boldsymbol{a}$

d $\overrightarrow{GB} = \overrightarrow{GO} + \overrightarrow{OA} + \overrightarrow{AB}$

$= -\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{c}$

e $\overrightarrow{FA} = \overrightarrow{FG} + \overrightarrow{GO} + \overrightarrow{OA}$

$= -\boldsymbol{c} - \boldsymbol{g} + \boldsymbol{a}$

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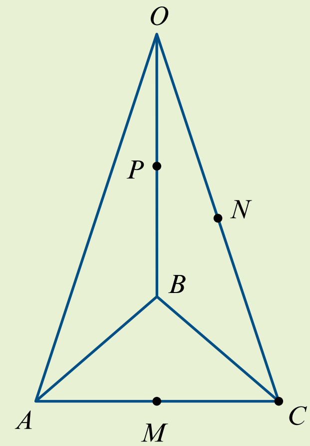
Example 7

$OABC$ is a tetrahedron,
M is the midpoint of AC,
N is the midpoint of OC,
P is the midpoint of OB.

Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

Determine in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

a $\overrightarrow{AC}$ **b** $\overrightarrow{OM}$ **c** $\overrightarrow{CN}$ **d** $\overrightarrow{MN}$ **e** $\overrightarrow{MP}$



Solution

a $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= -\boldsymbol{a} + \boldsymbol{c}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$
$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c})$
$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$

c $\overrightarrow{CN} = \frac{1}{2}\overrightarrow{CO}$
$= \frac{1}{2}(-\boldsymbol{c})$
$= -\frac{1}{2}\boldsymbol{c}$

d $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}) + \frac{1}{2}\boldsymbol{c}$
$= -\frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{c}$
$= -\frac{1}{2}\boldsymbol{a}$

(So MN is parallel to AO.)

e $\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP}$
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}) + \frac{1}{2}\boldsymbol{b}$
$= \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a} - \boldsymbol{c})$

Note: In this example, we found the position vector of the midpoint M by basic principles. We can also use the following general formula.

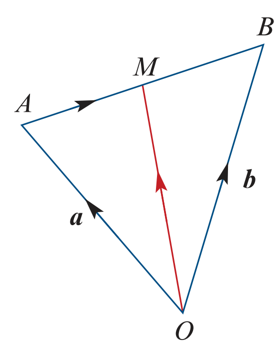
Midpoint of a line segment

Let M be the midpoint of a line segment AB, where points A and B have position vectors $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. Then

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$$

The position vector of the midpoint M is

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB}\\ &= \boldsymbol{a} + \tfrac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\\ &= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$



If M is the midpoint of line segment AB, then

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

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Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then

$$m\boldsymbol{a} - p\boldsymbol{a} = q\boldsymbol{b} - n\boldsymbol{b}$$

$$\therefore \quad (m-p)\boldsymbol{a} = (q-n)\boldsymbol{b}$$

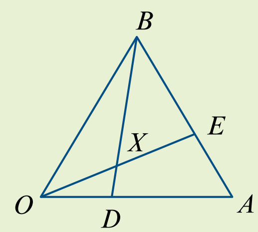
If $m \neq p$ or $n \neq q$, we could therefore write

$$\boldsymbol{a} = \frac{q-n}{m-p}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} = \frac{m-p}{q-n}\boldsymbol{a}$$

But this is not possible, as $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel.
Therefore $m = p$ and $n = q$.

Example 8

Points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O.



The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and the point E is such that $\overrightarrow{AE} = \ell\overrightarrow{AB}$. The line segments BD and OE intersect at X.

Assume that $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$.

a Express $\overrightarrow{XB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and k.

b Express $\overrightarrow{OX}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and ℓ.

c Express $\overrightarrow{XB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and ℓ.

d Determine k and ℓ.

Solution

a
$$\begin{aligned}\overrightarrow{XB} &= \frac{4}{5}\overrightarrow{DB}\\ &= \frac{4}{5}(-\overrightarrow{OD} + \overrightarrow{OB})\\ &= \frac{4}{5}(-k\overrightarrow{OA} + \overrightarrow{OB})\\ &= \frac{4}{5}(-k\boldsymbol{a} + \boldsymbol{b})\\ &= -\frac{4k}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{OX} &= \frac{2}{5}\overrightarrow{OE}\\ &= \frac{2}{5}(\overrightarrow{OA} + \overrightarrow{AE})\\ &= \frac{2}{5}(\overrightarrow{OA} + \ell\overrightarrow{AB})\\ &= \frac{2}{5}(\boldsymbol{a} + \ell(\boldsymbol{b} - \boldsymbol{a}))\\ &= \frac{2}{5}(1-\ell)\boldsymbol{a} + \frac{2\ell}{5}\boldsymbol{b}\end{aligned}$$

c
$$\begin{aligned}\overrightarrow{XB} &= \overrightarrow{XO} + \overrightarrow{OB}\\ &= -\overrightarrow{OX} + \overrightarrow{OB}\\ &= -\frac{2}{5}(1-\ell)\boldsymbol{a} - \frac{2\ell}{5}\boldsymbol{b} + \boldsymbol{b}\\ &= \frac{2}{5}(\ell - 1)\boldsymbol{a} + \left(1 - \frac{2\ell}{5}\right)\boldsymbol{b}\end{aligned}$$

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d As $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel, the vector $\overrightarrow{XB}$ has a unique representation in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. From parts **a** and **c**, we have

$$-\frac{4k}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b} = \frac{2}{5}(\ell - 1)\boldsymbol{a} + \left(1 - \frac{2\ell}{5}\right)\boldsymbol{b}$$

Hence

$$-\frac{4k}{5} = \frac{2}{5}(\ell - 1) \quad (1) \qquad \text{and} \qquad \frac{4}{5} = 1 - \frac{2\ell}{5} \quad (2)$$

From equation (2), we have

$$\frac{2\ell}{5} = \frac{1}{5}$$

$$\therefore \quad \ell = \frac{1}{2}$$

Substitute in (1):

$$-\frac{4k}{5} = \frac{2}{5}\left(\frac{1}{2} - 1\right)$$

$$\therefore \quad k = \frac{1}{4}$$

Exercise 5A

Example 1

1 Draw a directed line segment representing the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and state the magnitude of this vector.

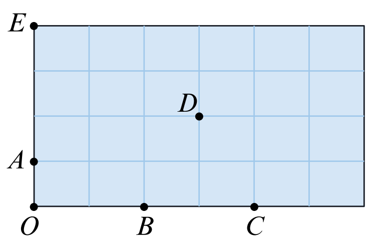
Example 2

2 The vector $\boldsymbol{u}$ is defined by the directed line segment from $(-2, 4)$ to $(1, 6)$.
If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, determine a and b.

Example 3

3 Draw distinct points O, A, B, C, D and E and illustrate the vector sum $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

4 In the diagram, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.



a Determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OC}$ **ii** $\overrightarrow{OE}$ **iii** $\overrightarrow{OD}$
iv $\overrightarrow{DC}$ **v** $\overrightarrow{DE}$

b If $|\boldsymbol{a}| = 1$ and $|\boldsymbol{b}| = 2$, determine:

i $|\overrightarrow{OC}|$ **ii** $|\overrightarrow{OE}|$ **iii** $|\overrightarrow{OD}|$

5 If the vector $\boldsymbol{a}$ has magnitude 3, determine the magnitude of:

a $2\boldsymbol{a}$ **b** $\frac{3}{2}\boldsymbol{a}$ **c** $-\frac{1}{2}\boldsymbol{a}$

Example 4

6 Simplify each of the following vector expressions:

a $3(\boldsymbol{a} - \boldsymbol{b} - 2\boldsymbol{c}) + \frac{5}{2}(3\boldsymbol{a} + \boldsymbol{b} - 6\boldsymbol{c})$

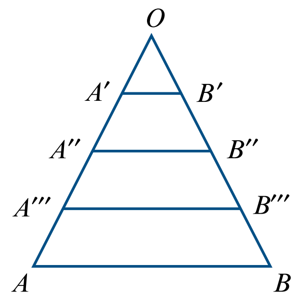
b $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}) + \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a}) + \frac{1}{2}(\boldsymbol{c} + \boldsymbol{a} - \boldsymbol{b})$

SF

7 $OA' = A'A'' = A''A''' = A'''A$
$OB' = B'B'' = B''B''' = B'''B$
If $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$, determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

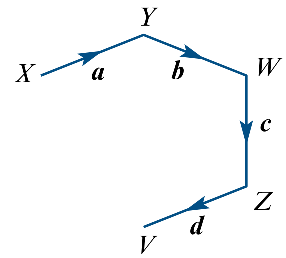
a **i** $\overrightarrow{OA'}$ **ii** $\overrightarrow{OB'}$ **iii** $\overrightarrow{A'B'}$ **iv** $\overrightarrow{AB}$

b **i** $\overrightarrow{OA''}$ **ii** $\overrightarrow{OB''}$ **iii** $\overrightarrow{A''B''}$



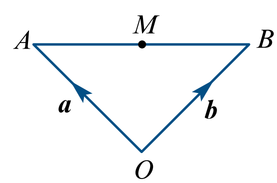
8 Determine in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$:

a $\overrightarrow{XW}$ **b** $\overrightarrow{VX}$ **c** $\overrightarrow{ZY}$



9 The position vectors of two points A and B are $\boldsymbol{a}$ and $\boldsymbol{b}$. The point M is the midpoint of AB. Determine:

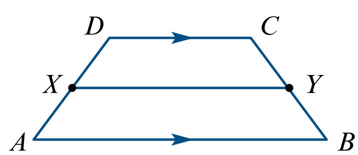
a $\overrightarrow{AB}$ **b** $\overrightarrow{AM}$ **c** $\overrightarrow{OM}$



10 $ABCD$ is a trapezium with AB parallel to DC. X and Y are the midpoints of AD and BC respectively.

a Express $\overrightarrow{XY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{DC}$.

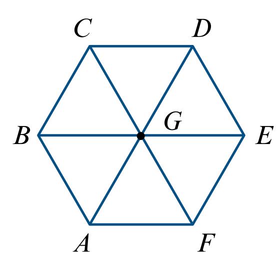
b Show that XY is parallel to AB.



11 $ABCDEF$ is a regular hexagon with centre G. The position vectors of A, B and C, relative to an origin O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a Express $\overrightarrow{OG}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

b Express $\overrightarrow{CD}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.



Example 5

12 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. Determine:

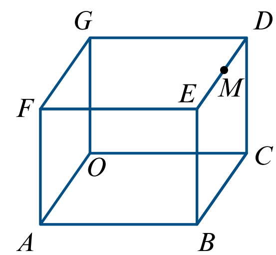
a $\boldsymbol{a} + \boldsymbol{c}$ **b** $2\boldsymbol{b} + \boldsymbol{c}$ **c** $-\boldsymbol{a} + 2\boldsymbol{b} - 2\boldsymbol{c}$

d $3\boldsymbol{c} - 4\boldsymbol{a}$ **e** $|\boldsymbol{a}|$ **f** $|\boldsymbol{c}|$

13 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Determine the values of x and y for which:

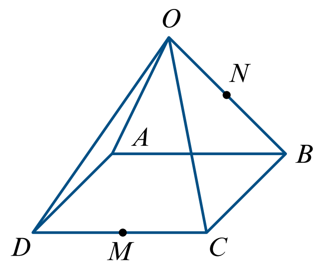
a $x\boldsymbol{a} = (y-1)\boldsymbol{b}$ **b** $(2-x)\boldsymbol{a} = 3\boldsymbol{a} + (7-3y)\boldsymbol{b}$

c $(5+2x)(\boldsymbol{a}+\boldsymbol{b}) = y(3\boldsymbol{a}+2\boldsymbol{b})$



Example 6 **14** For the cuboid shown, let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{g} = \overrightarrow{OG}$.
Let M be the midpoint of ED.
Determine each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{g}$:

a $\overrightarrow{EF}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{EM}$ **d** $\overrightarrow{OM}$ **e** $\overrightarrow{AM}$



Example 7 **15** $OABCD$ is a right square pyramid.
Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{d} = \overrightarrow{OD}$.

a **i** Determine $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
ii Determine $\overrightarrow{DC}$ in terms of $\boldsymbol{c}$ and $\boldsymbol{d}$.
iii Use the fact that $\overrightarrow{AB} = \overrightarrow{DC}$ to determine a relationship between $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

b **i** Determine $\overrightarrow{BC}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.
ii Let M be the midpoint of DC and N the midpoint of OB. Determine $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

16 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors that are not parallel.

a If $k\boldsymbol{a} + \ell\boldsymbol{b} = 3\boldsymbol{a} + (1 - \ell)\boldsymbol{b}$, determine the values of k and ℓ.

b If $2(\ell - 1)\boldsymbol{a} + \left(1 - \frac{\ell}{5}\right)\boldsymbol{b} = -\frac{4k}{5}\boldsymbol{a} + 3\boldsymbol{b}$, determine the values of k and ℓ.

Example 6 **17** Points P, Q and R have position vectors $2\boldsymbol{a} - \boldsymbol{b}$, $3\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} + 4\boldsymbol{b}$ respectively, relative to an origin O, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors. The point S is on the line OP with $\overrightarrow{OS} = k\overrightarrow{OP}$ and $\overrightarrow{RS} = m\overrightarrow{RQ}$.

a Express $\overrightarrow{OS}$ in terms of:

i k, $\boldsymbol{a}$ and $\boldsymbol{b}$ **ii** m, $\boldsymbol{a}$ and $\boldsymbol{b}$

b Hence evaluate k and m.

18 The position vectors of points A and B, relative to an origin O, are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors. The point P is such that $\overrightarrow{OP} = 4\overrightarrow{OB}$. The midpoint of AB is the point Q. The point R is such that $\overrightarrow{OR} = \frac{8}{5}\overrightarrow{OQ}$.

a Determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OQ}$ **ii** $\overrightarrow{OR}$ **iii** $\overrightarrow{AR}$ **iv** $\overrightarrow{RP}$

b Show that R lies on AP and state the ratio $AR : RP$.

c Given that the point S is such that $\overrightarrow{OS} = \lambda\overrightarrow{OQ}$, determine the value of λ such that PS is parallel to BA.

19 Suppose that the point X lies between A and B on the line AB, with $\overrightarrow{AX} = k\overrightarrow{AB}$.

a Determine $\frac{AX}{AB}$ in terms of k. **b** Show that $0 < k < 1$.

c Determine $\frac{AX}{XB}$ in terms of k. **d** Let $m = \frac{AX}{XB}$. Express k in terms of m.

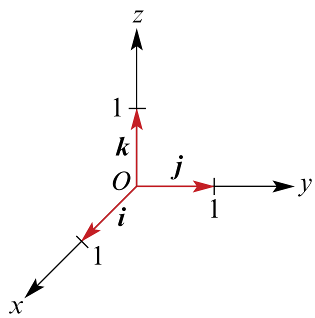
5B Resolution of a vector into rectangular components

Learning intentions

- To be able to work with vectors written in component form.

A **unit vector** is a vector of magnitude 1. For a non-zero vector $\boldsymbol{a}$, the unit vector with the same direction as $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$



- The unit vector in the positive direction of the x-axis is $\hat{\boldsymbol{i}}$.
- The unit vector in the positive direction of the y-axis is $\hat{\boldsymbol{j}}$.
- The unit vector in the positive direction of the z-axis is $\hat{\boldsymbol{k}}$.

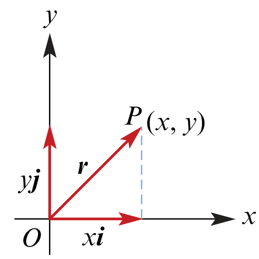
In two dimensions: $\hat{\boldsymbol{i}} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\hat{\boldsymbol{j}} = \begin{bmatrix}0\\1\end{bmatrix}$.

In three dimensions: $\hat{\boldsymbol{i}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\hat{\boldsymbol{j}} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\hat{\boldsymbol{k}} = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Every vector in two or three dimensions can be expressed uniquely as a linear combination of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$:

e.g. $$\boldsymbol{r} = \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = \begin{bmatrix}r_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\r_2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\r_3\end{bmatrix} = r_1\hat{\boldsymbol{i}} + r_2\hat{\boldsymbol{j}} + r_3\hat{\boldsymbol{k}}$$

Two dimensions

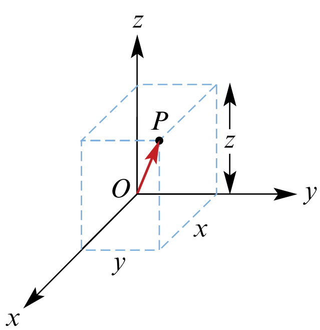


For the point $P(x, y)$:

$$\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$$

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$$

Three dimensions



For the point $P(x, y, z)$:

$$\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$$

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

Basic operations in component form

Let $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$.

Then $$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\hat{\boldsymbol{i}} + (a_2 + b_2)\hat{\boldsymbol{j}} + (a_3 + b_3)\hat{\boldsymbol{k}}$$

$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\hat{\boldsymbol{i}} + (a_2 - b_2)\hat{\boldsymbol{j}} + (a_3 - b_3)\hat{\boldsymbol{k}}$$

and $$m\boldsymbol{a} = ma_1\hat{\boldsymbol{i}} + ma_2\hat{\boldsymbol{j}} + ma_3\hat{\boldsymbol{k}} \quad \text{for a scalar } m$$

Equivalence

If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Magnitude

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 9

a Using the vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, give the vectors:

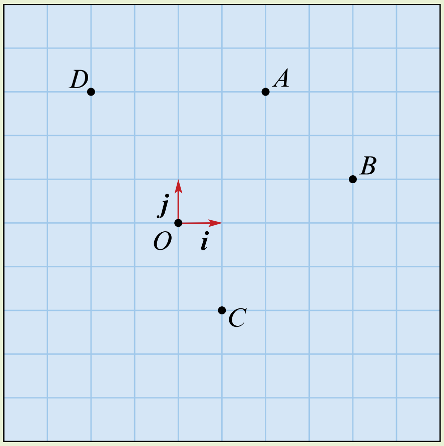
i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{OD}$

b Using the vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, give the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$

c Determine the magnitudes of the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$



Solution

a **i** $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **ii** $\overrightarrow{OB} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ **iii** $\overrightarrow{OC} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ **iv** $\overrightarrow{OD} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

b **i**
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\\ &= 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\end{aligned}$$

ii
$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= -4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\\ &= -3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\end{aligned}$$

c **i**
$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{2^2 + (-2)^2}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

ii
$$\begin{aligned}|\overrightarrow{BC}| &= \sqrt{(-3)^2 + (-3)^2}\\ &= \sqrt{18}\\ &= 3\sqrt{2}\end{aligned}$$

Example 10

Let $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. Determine:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{a} - 2\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **d** $|\boldsymbol{a}|$

Solution

a
$$\begin{aligned}\boldsymbol{a} + \boldsymbol{b} &= (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) + (3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}})\\ &= 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}\end{aligned}$$

b
$$\begin{aligned}\boldsymbol{a} - 2\boldsymbol{b} &= (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) - 2(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}})\\ &= -5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}\end{aligned}$$

c
$$\begin{aligned}\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} &= (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) + (3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}) + (2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})\\ &= 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}\end{aligned}$$

d $|\boldsymbol{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

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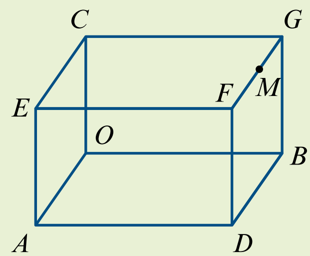
Example 11

A cuboid is labelled as shown.

$\overrightarrow{OA} = 3\hat{\boldsymbol{i}}$, $\overrightarrow{OB} = 5\hat{\boldsymbol{j}}$, $\overrightarrow{OC} = 4\hat{\boldsymbol{k}}$



a Determine in terms of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$:

i $\overrightarrow{DB}$ **ii** $\overrightarrow{OD}$ **iii** $\overrightarrow{DF}$ **iv** $\overrightarrow{OF}$

b Determine $|\overrightarrow{OF}|$.

c If M is the midpoint of FG, determine:

i $\overrightarrow{OM}$ **ii** $|\overrightarrow{OM}|$

Solution

a **i** $\overrightarrow{DB} = \overrightarrow{AO}$
$= -\overrightarrow{OA}$
$= -3\hat{\boldsymbol{i}}$

ii $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$
$= 5\hat{\boldsymbol{j}} + \overrightarrow{OA}$
$= 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{i}}$
$= 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

iii $\overrightarrow{DF} = \overrightarrow{OC}$
$= 4\hat{\boldsymbol{k}}$

iv $\overrightarrow{OF} = \overrightarrow{OD} + \overrightarrow{DF}$
$= 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

b $|\overrightarrow{OF}| = \sqrt{9 + 25 + 16}$
$= \sqrt{50}$
$= 5\sqrt{2}$

c **i** $\overrightarrow{OM} = \overrightarrow{OD} + \overrightarrow{DF} + \overrightarrow{FM}$
$= 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + \frac{1}{2}(-\overrightarrow{GF})$
$= 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + \frac{1}{2}(-3\hat{\boldsymbol{i}})$
$= \frac{3}{2}\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

ii $|\overrightarrow{OM}| = \sqrt{\frac{9}{4} + 25 + 16}$
$= \frac{1}{2}\sqrt{9 + 100 + 64}$
$= \frac{1}{2}\sqrt{173}$

Example 12

If $\boldsymbol{a} = x\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 8\hat{\boldsymbol{i}} + 2y\hat{\boldsymbol{j}}$ such that $\boldsymbol{a} + \boldsymbol{b} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, determine the values of x and y.

Solution

$\boldsymbol{a} + \boldsymbol{b} = (x + 8)\hat{\boldsymbol{i}} + (2y + 3)\hat{\boldsymbol{j}} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

$\therefore \quad x + 8 = -2 \quad$ and $\quad 2y + 3 = 4$

i.e. $\quad x = -10 \quad$ and $\quad y = \frac{1}{2}$

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Example 13

Let $A = (2, -3)$, $B = (1, 4)$ and $C = (-1, -3)$. The origin is O. Determine:

a **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$

b F such that $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA}$

c G such that $\overrightarrow{AG} = 3\overrightarrow{BC}$

Solution

a **i** $\overrightarrow{OA} = 2\boldsymbol{i} - 3\boldsymbol{j}$

ii $$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{i} + 4\boldsymbol{j}\\ &= -\boldsymbol{i} + 7\boldsymbol{j}\end{aligned}$$

iii $$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= -\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{i} - 3\boldsymbol{j}\\ &= -2\boldsymbol{i} - 7\boldsymbol{j}\end{aligned}$$

b $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}(2\boldsymbol{i} - 3\boldsymbol{j}) = \boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$

Hence $F = (1, -\frac{3}{2})$

c $\overrightarrow{AG} = 3\overrightarrow{BC} = 3(-2\boldsymbol{i} - 7\boldsymbol{j}) = -6\boldsymbol{i} - 21\boldsymbol{j}$

Therefore

$$\begin{aligned}\overrightarrow{OG} &= \overrightarrow{OA} + \overrightarrow{AG}\\ &= 2\boldsymbol{i} - 3\boldsymbol{j} - 6\boldsymbol{i} - 21\boldsymbol{j}\\ &= -4\boldsymbol{i} - 24\boldsymbol{j}\end{aligned}$$

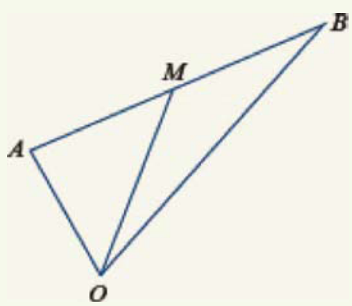
Hence $G = (-4, -24)$

Example 14

Let $A = (2, -4, 5)$ and $B = (5, 1, 7)$. Determine M, the midpoint of AB.

Solution

We have $\overrightarrow{OA} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\overrightarrow{OB} = 5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$.



Thus $$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k} + 5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}\\ &= 3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}\end{aligned}$$

and so $\overrightarrow{AM} = \dfrac{1}{2}(3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k})$

Now $$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \overrightarrow{AM}\\ &= 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k} + \frac{3}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + \boldsymbol{k}\\ &= \frac{7}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + 6\boldsymbol{k}\end{aligned}$$

Hence $M = \left(\dfrac{7}{2}, -\dfrac{3}{2}, 6\right)$

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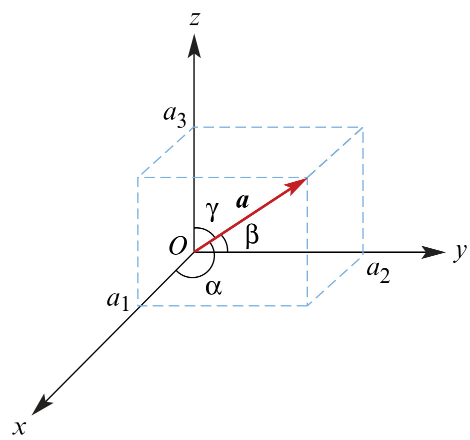
Angle made by a vector with an axis

The *direction* of a vector can be given by the angles which the vector makes with the $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ directions.



If the vector $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ makes angles α, β and γ with the positive directions of the x-, y- and z-axes respectively, then

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \quad \cos\beta = \frac{a_2}{|\boldsymbol{a}|}, \quad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

The derivation of these results is left as an exercise.

Example 15

Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$.

For each of these vectors, determine:

a its magnitude

b the angle the vector makes with the y-axis.

Solution

a $|\boldsymbol{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

$|\boldsymbol{b}| = \sqrt{1^2 + 4^2 + (-3)^2} = \sqrt{26}$

b The angle that $\boldsymbol{a}$ makes with the y-axis is

$$\cos^{-1}\left(\frac{-1}{\sqrt{5}}\right) \approx 116.57°$$

The angle that $\boldsymbol{b}$ makes with the y-axis is

$$\cos^{-1}\left(\frac{4}{\sqrt{26}}\right) \approx 38.33°$$

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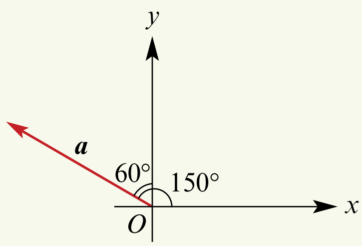
Example 16

A position vector in two dimensions has magnitude 5 and its direction, measured anticlockwise from the x-axis, is 150°. Express this vector in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.

Solution

Let $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}}$.



The vector $\boldsymbol{a}$ makes an angle of 150° with the x-axis and an angle of 60° with the y-axis.

Therefore

$$\cos 150^\circ = \frac{a_1}{|\boldsymbol{a}|} \quad \text{and} \quad \cos 60^\circ = \frac{a_2}{|\boldsymbol{a}|}$$

Since $|\boldsymbol{a}| = 5$, this gives

$$a_1 = |\boldsymbol{a}|\cos 150^\circ = \frac{-5\sqrt{3}}{2}$$

$$a_2 = |\boldsymbol{a}|\cos 60^\circ = \frac{5}{2}$$

$$\therefore \quad \boldsymbol{a} = \frac{-5\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{5}{2}\hat{\boldsymbol{j}}$$

Example 17

Let $\hat{\boldsymbol{i}}$ be a unit vector in the east direction and let $\hat{\boldsymbol{j}}$ be a unit vector in the north direction, with units in kilometres.

a Show that the unit vector in the direction N60°W is $-\frac{\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}}$.

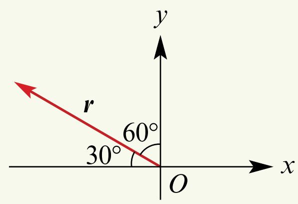
b If a car drives 3 km in the direction N60°W, determine the position vector of the car with respect to its starting point.

c The car then drives 6.5 km due north. Determine:

- **i** the position vector of the car
- **ii** the distance of the car from the starting point
- **iii** the bearing of the car from the starting point.

Solution

a Let $\boldsymbol{r}$ denote the unit vector in the direction N60°W.



Then $\quad \boldsymbol{r} = -\cos 30^\circ\hat{\boldsymbol{i}} + \cos 60^\circ\hat{\boldsymbol{j}}$

$$= -\frac{\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}}$$

Note: $|\boldsymbol{r}| = 1$

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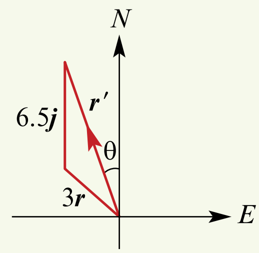
b The position vector is

$$3\boldsymbol{r} = 3\left(-\frac{\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{1}{2}\hat{\boldsymbol{j}}\right) = -\frac{3\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{3}{2}\hat{\boldsymbol{j}}$$

c Let $\boldsymbol{r}'$ denote the new position vector.



i $\boldsymbol{r}' = 3\boldsymbol{r} + 6.5\hat{\boldsymbol{j}}$

$$= -\frac{3\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{3}{2}\hat{\boldsymbol{j}} + \frac{13}{2}\hat{\boldsymbol{j}}$$

$$= -\frac{3\sqrt{3}}{2}\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$$

ii $|\boldsymbol{r}'| = \sqrt{\dfrac{9 \times 3}{4} + 64}$

$$= \sqrt{\frac{27 + 256}{4}}$$

$$= \frac{1}{2}\sqrt{283}$$

iii Since $\boldsymbol{r}' = -\dfrac{3\sqrt{3}}{2}\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$, we have

$$\tan\theta^\circ = \frac{3\sqrt{3}}{16}$$

$$\therefore \quad \theta^\circ = \tan^{-1}\left(\frac{3\sqrt{3}}{16}\right) \approx 18^\circ$$

The bearing is 342°, correct to the nearest degree.

Notation for vectors

The unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ provide a neat way to represent vectors in three dimensions. However, it is also common to use column or row vectors. For example, the position vector of the point $P(1, -3, 4)$ can be written as

$$\overrightarrow{OP} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} \quad \text{or} \quad \overrightarrow{OP} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} \quad \text{or} \quad \overrightarrow{OP} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}$$

You should be able to work with these notations.

Skill-sheet

Exercise 5B

Example 9

SF

1 **a** Give each of the following vectors in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$:

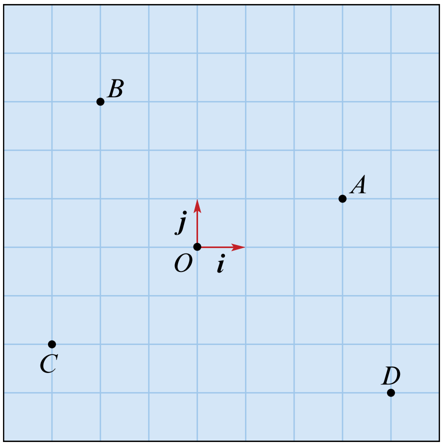
i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{OD}$

b Determine each of the following:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{CD}$ **iii** $\overrightarrow{DA}$

c Determine the magnitude of each of the following:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{DA}$



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SF

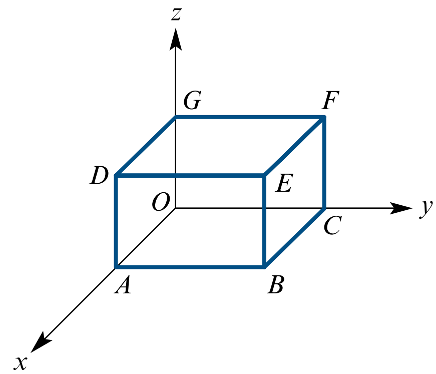
Example 10

2 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 4\hat{\boldsymbol{k}}$. Determine:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $2\boldsymbol{a} + \boldsymbol{c}$ **c** $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}$ **d** $\boldsymbol{c} - 4\boldsymbol{a}$ **e** $|\boldsymbol{b}|$ **f** $|\boldsymbol{c}|$

Example 11

3 $OABCDEFG$ is a cuboid set on Cartesian axes with $\overrightarrow{OA} = 5\hat{\boldsymbol{i}}$, $\overrightarrow{OC} = 2\hat{\boldsymbol{j}}$ and $\overrightarrow{OG} = 3\hat{\boldsymbol{k}}$.



a Determine:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{CF}$ **iii** $\overrightarrow{AB}$
iv $\overrightarrow{OD}$ **v** $\overrightarrow{OE}$ **vi** $\overrightarrow{GE}$
vii $\overrightarrow{EC}$ **viii** $\overrightarrow{DB}$ **ix** $\overrightarrow{DC}$
x $\overrightarrow{BG}$ **xi** $\overrightarrow{GB}$ **xii** $\overrightarrow{FA}$

b Evaluate:

i $|\overrightarrow{OD}|$ **ii** $|\overrightarrow{OE}|$ **iii** $|\overrightarrow{GE}|$

c Let M be the midpoint of CB. Determine:

i $\overrightarrow{CM}$ **ii** $\overrightarrow{OM}$ **iii** $\overrightarrow{DM}$

d Let N be the point on FG such that $\overrightarrow{FN} = 2\overrightarrow{NG}$. Determine:

i $\overrightarrow{FN}$ **ii** $\overrightarrow{GN}$ **iii** $\overrightarrow{ON}$ **iv** $\overrightarrow{NA}$ **v** $\overrightarrow{NM}$

e Evaluate:

i $|\overrightarrow{NM}|$ **ii** $|\overrightarrow{DM}|$ **iii** $|\overrightarrow{AN}|$

Example 12

4 Determine the values of x and y if:

a $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\boldsymbol{b} = x\hat{\boldsymbol{i}} + 3y\hat{\boldsymbol{j}}$, $\boldsymbol{a} + \boldsymbol{b} = 7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

b $\boldsymbol{a} = x\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 5y\hat{\boldsymbol{j}}$, $\boldsymbol{a} - \boldsymbol{b} = 6\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

c $\boldsymbol{a} = 6\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, $\boldsymbol{b} = x\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, $\boldsymbol{a} + 2\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

Example 13

5 Let $A = (-2, 4)$, $B = (1, 6)$ and $C = (-1, -6)$. Let O be the origin. Determine:

a **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$

b F such that $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA}$

c G such that $\overrightarrow{AG} = 3\overrightarrow{BC}$

Example 14

6 Let $A = (1, -6, 7)$ and $B = (5, -1, 9)$. Determine M, the midpoint of AB.

7 Points A, B, C and D have position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$, $\boldsymbol{c} = 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{d} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ respectively.

a Determine:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DA}$

b Evaluate:

i $|\overrightarrow{AC}|$ **ii** $|\overrightarrow{BD}|$

c Determine the two parallel vectors in **a**.

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8 Points A and B are defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ respectively. The point M is on the line segment AB such that $AM : MB = 4 : 1$.

a Determine:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{AM}$ **iii** $\overrightarrow{OM}$

b Determine the coordinates of M.

9 $A = (2, 1)$, $B = (1, -3)$, $C = (-5, 2)$, $D = (3, 5)$ and O is the origin.

a Determine:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$ **iv** $\overrightarrow{BD}$

b Show that $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are parallel.

c What can be said about the points A, B and D?

10 Let $A = (1, 4, -4)$, $B = (2, 3, 1)$, $C = (0, -1, 4)$ and $D = (4, 5, 6)$.

a Determine:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{AC}$ **iii** $\overrightarrow{BD}$ **iv** $\overrightarrow{CD}$

b Show that $\overrightarrow{OB}$ and $\overrightarrow{CD}$ are parallel.

11 Let $A = (1, 4, -2)$, $B = (3, 3, 0)$, $C = (2, 5, 3)$ and $D = (0, 6, 1)$.

a Determine:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DA}$

b Describe the quadrilateral $ABCD$.

12 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Determine:

a D such that $\overrightarrow{AB} = \overrightarrow{CD}$

b E such that $\overrightarrow{AE} = -\overrightarrow{BC}$

c G such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

13 $ABCD$ is a parallelogram, where $A = (2, 1)$, $B = (-5, 4)$, $C = (1, 7)$ and $D = (x, y)$.

a Determine:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{AD}$ (in terms of x and y)

b Hence determine the coordinates of D.

14 **a** Let $A = (1, 4, 3)$ and $B = (2, -1, 5)$. Use a vector method to determine the coordinates of M, the midpoint of the line segment AB.

b Use a similar method to determine M, the midpoint of XY, where X and Y have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

15 Let $A = (5, 4, 1)$ and $B = (3, 1, -4)$. Determine M on line segment AB such that $AM = 4MB$.

16 Let $A = (4, -3)$ and $B = (7, 1)$. Determine N such that $\overrightarrow{AN} = 3\overrightarrow{BN}$.

17 Determine the point P on the line $x - 6y = 11$ such that $\overrightarrow{OP}$ is parallel to the vector $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

CF SF CF

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SF

18 The points A, B, C and D have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. Show that, if $ABCD$ is a parallelogram, then $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$.

19 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = 4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$.

a Determine:

i $\frac{1}{2}\boldsymbol{a}$ **ii** $\boldsymbol{b} - \boldsymbol{c}$ **iii** $3\boldsymbol{b} - \boldsymbol{a} - 2\boldsymbol{c}$

b Determine values for k and ℓ such that $k\boldsymbol{a} + \ell\boldsymbol{b} = \boldsymbol{c}$.

20 Let $\boldsymbol{a} = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 8\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$.

a Determine:

i $2\boldsymbol{a} - \boldsymbol{b}$ **ii** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **iii** $0.5\boldsymbol{a} + 0.4\boldsymbol{b}$

b Determine values for k and ℓ such that $k\boldsymbol{a} + \ell\boldsymbol{b} = \boldsymbol{c}$.

Example 15

21 Let $\boldsymbol{a} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$.

a Determine:

i $|\boldsymbol{a}|$ **ii** $|\boldsymbol{b}|$ **iii** $|\boldsymbol{a} + 2\boldsymbol{b}|$ **iv** $|\boldsymbol{c} - \boldsymbol{d}|$

b Determine, correct to two decimal places, the angle which each of the following vectors makes with the positive direction of the x-axis:

i $\boldsymbol{a}$ **ii** $\boldsymbol{a} + 2\boldsymbol{b}$ **iii** $\boldsymbol{c} - \boldsymbol{d}$

Example 16

22 The table gives the magnitudes of vectors in two dimensions and the angle they each make with the x-axis (measured anticlockwise).
Express each of the vectors in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, correct to two decimal places.

	Magnitude	Angle
a	10	110°
b	8.5	250°
c	6	40°
d	5	300°

23 The following table gives the magnitudes of vectors in three dimensions and the angles they each make with the x-, y- and z-axes, correct to two decimal places. Express each of the vectors in terms of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, correct to two decimal places.

	Magnitude	Angle with x-axis	Angle with y-axis	Angle with z-axis
a	10	130°	80°	41.75°
b	8	50°	54.52°	120°
c	7	28.93°	110°	110°
d	12	121.43°	35.5°	75.2°

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24 Show that if a vector in three dimensions makes angles α, β and γ with the x-, y- and z-axes respectively, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

25 Points A, B and C have position vectors $\boldsymbol{a} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ respectively. Let M be the midpoint of BC.

a Show that $\triangle ABC$ is isosceles.
b Determine $\overrightarrow{OM}$.
c Determine $\overrightarrow{AM}$.
d Determine the area of $\triangle ABC$.

26 $OABCV$ is a square-based right pyramid with V the vertex. The base diagonals OB and AC intersect at the point M. If $\overrightarrow{OA} = 5\hat{\boldsymbol{i}}$, $\overrightarrow{OC} = 5\hat{\boldsymbol{j}}$ and $\overrightarrow{MV} = 3\hat{\boldsymbol{k}}$, determine each of the following:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OM}$ **c** $\overrightarrow{OV}$ **d** $\overrightarrow{BV}$ **e** $|\overrightarrow{OV}|$

27 Points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Let M and N be the midpoints of OA and OB respectively, where O is the origin.

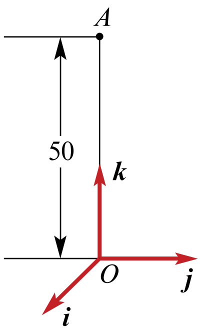
a Show that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$.
b Hence describe the geometric relationships between line segments MN and AB.

Example 17

28 Let $\hat{\boldsymbol{i}}$ be the unit vector in the east direction and let $\hat{\boldsymbol{j}}$ be the unit vector in the north direction, with units in kilometres. A runner sets off on a bearing of 120°.

a Determine a unit vector in this direction.
b The runner covers 3 km. Determine the position of the runner with respect to her starting point.
c The runner now turns and runs for 5 km in a northerly direction. Determine the position of the runner with respect to her original starting point.
d Determine the distance of the runner from her starting point.

29 A hang-glider jumps from a 50 m cliff.



a Give the position vector of point A with respect to O.
b After a short period of time, the hang-glider has position B given by $\overrightarrow{OB} = -80\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}} + 40\hat{\boldsymbol{k}}$ metres.
  i Determine the vector $\overrightarrow{AB}$.
  ii Determine the magnitude of $\overrightarrow{AB}$.
c The hang-glider then moves 600 m in the $\hat{\boldsymbol{j}}$-direction and 60 m in the $\hat{\boldsymbol{k}}$-direction. Give the new position vector of the hang-glider.

30 A light plane takes off (from a point which will be considered as the origin) so that its position after a short period of time is given by $\boldsymbol{r}_1 = 1.5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 0.9\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}$ is a unit vector in the east direction, $\hat{\boldsymbol{j}}$ is a unit vector in the north direction and measurements are in kilometres.

a Determine the distance of the plane from the origin.
b The position of a second plane at the same time is given by $\boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 0.8\hat{\boldsymbol{k}}$.

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i Determine $\boldsymbol{r}_1 - \boldsymbol{r}_2$.

ii Determine the distance between the two aircraft.

c Give a unit vector which would describe the direction in which the first plane must fly to pass over the origin at a height of 900 m.

31 Jan starts at a point O and walks on level ground 200 metres in a north-westerly direction to P. She then walks 50 metres due north to Q, which is at the bottom of a building. Jan then climbs to T, the top of the building, which is 30 metres vertically above Q. Let $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ be unit vectors in the east, north and vertically upwards directions respectively. Express each of the following in terms of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{OQ}$ **d** $\overrightarrow{QT}$ **e** $\overrightarrow{OT}$

32 A ship leaves a port and sails north-east for 100 km to a point P. Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be the unit vectors in the east and north directions respectively, with units in kilometres.

a Determine the position vector of point P.

b If B is the point on the shore with position vector $\overrightarrow{OB} = 100\hat{\boldsymbol{i}}$, determine:

i $\overrightarrow{BP}$

ii the bearing of P from B.

33 Let $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = m\boldsymbol{a} + (1 - m)\boldsymbol{b}$.

a Determine $\boldsymbol{c}$ in terms of m.

b Hence determine p if $\boldsymbol{c} = 7\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + p\hat{\boldsymbol{k}}$.

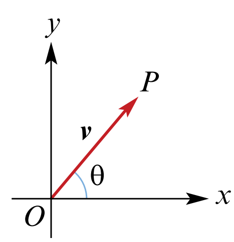
5C Polar form of a vector

Learning intentions

- ▶ To be able to work with vectors written in polar form.

Polar form in two dimensions

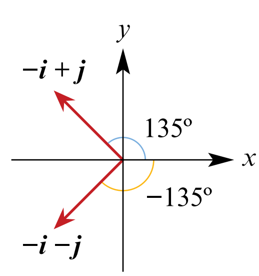
Each vector $\boldsymbol{v} = \overrightarrow{OP}$ in the plane can be written in polar form as $\boldsymbol{v} = [r, \theta]$.



- The number r is the magnitude of $\boldsymbol{v}$.
- The angle θ is measured from the positive x-direction to the vector $\boldsymbol{v}$. The range is $-180° < \theta \le 180°$, where positive angles are formed by moving towards the positive y-direction (i.e. anticlockwise).

For example:

- The vector $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ can be written as $[\sqrt{2}, 135°]$.
- The vector $-\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ can be written as $[\sqrt{2}, -135°]$

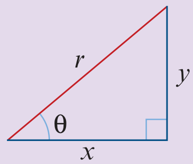


- **Converting from polar form to component form**
  Given $\boldsymbol{v} = [r, \theta]$, we can write $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, where

  $$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- **Converting from component form to polar form**
  Given $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, we can write $\boldsymbol{v} = [r, \theta]$, where

  $$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$



Warning: When converting to polar form, you need to take care to choose the angle θ in the correct quadrant.

Example 18

Write each of the following vectors in component form:

a $[3, 30°]$ **b** $[4, -60°]$ **c** $[2, 50°]$

Solution

a $x = 3\cos 30°$

$y = 3\sin 30°$

$\therefore [3, 30°] = \dfrac{3\sqrt{3}}{2}\hat{\boldsymbol{i}} + \dfrac{3}{2}\hat{\boldsymbol{j}}$

b $x = 4\cos(-60°)$

$y = 4\sin(-60°)$

$\therefore [4, -60°] = 2\hat{\boldsymbol{i}} - 2\sqrt{3}\hat{\boldsymbol{j}}$

c $x = 2\cos 50°$

$y = 2\sin 50°$

$\therefore [2, 50°] \approx 1.29\hat{\boldsymbol{i}} + 1.53\hat{\boldsymbol{j}}$

Example 19

Write each of the following vectors in polar form:

a $-4\sqrt{3}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ **b** $-4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

Solution

a Here $x = -4\sqrt{3}$ and $y = 4$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{48 + 16} \\ &= 8 \end{aligned}$$

$$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}}$$

Note that, since $x < 0$ and $y > 0$, we have $90° < \theta < 180°$. So $\theta = 150°$.

Hence $-4\sqrt{3}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} = [8, 150°]$.

b Here $x = -4$ and $y = -4$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{16 + 16} \\ &= 4\sqrt{2} \end{aligned}$$

$$\tan\theta = \frac{y}{x} = 1$$

Note that, since $x < 0$ and $y < 0$, we have $-180° < \theta < -90°$. So $\theta = -135°$.

Hence $-4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} = [4\sqrt{2}, -135°]$.

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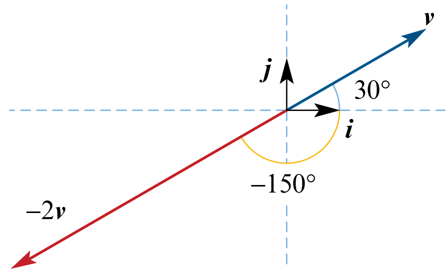
Basic operations on vectors in polar form

We can easily perform scalar multiplication for vectors written in polar form. For example, if $\boldsymbol{v} = [3, 30°]$, then

$$\begin{aligned} -2\boldsymbol{v} &= [2 \times 3, (30 - 180)°] \\ &= [6, -150°] \end{aligned}$$

Here the magnitude is doubled and the direction is reversed.



The simplest method for determining the sum of vectors written in polar form is to start by converting them into component form.

Example 20

Determine the sum of the vectors $\boldsymbol{u} = [10, 30°]$ and $\boldsymbol{v} = [12, 90°]$.

Solution

We have $\quad \boldsymbol{u} = 10\cos 30°\, \hat{\boldsymbol{i}} + 10\sin 30°\, \hat{\boldsymbol{j}}$

$\qquad\qquad = 5\sqrt{3}\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

and $\quad \boldsymbol{v} = 12\hat{\boldsymbol{j}}$

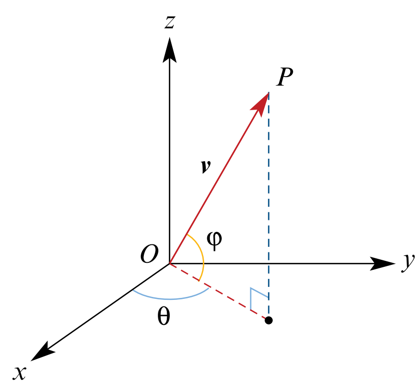
Hence $\quad \boldsymbol{u} + \boldsymbol{v} = 5\sqrt{3}\hat{\boldsymbol{i}} + 17\hat{\boldsymbol{j}}$

Note: If required, we can convert $\boldsymbol{u} + \boldsymbol{v}$ into polar form $[r, \theta]$ using

$$r = \sqrt{(5\sqrt{3})^2 + 17^2} = 2\sqrt{91} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{17}{5\sqrt{3}}\right) \approx 63.004°$$

Polar form in three dimensions

Each vector $\boldsymbol{v} = \overrightarrow{OP}$ in three-dimensional space can be written in polar form as $\boldsymbol{v} = [r, \theta, \varphi]$.



- The number r is the magnitude of $\boldsymbol{v}$.
- The angle θ is measured from the positive x-direction to the projection of $\boldsymbol{v}$ onto the x–y plane.

  The range is $-180° < \theta \le 180°$, where positive angles are formed by moving towards the positive y-direction.
- The angle φ is measured from the projection of $\boldsymbol{v}$ onto the x–y plane to the vector $\boldsymbol{v}$.

  The range is $-90° \le \varphi \le 90°$, where positive angles are formed by moving towards the positive z-direction.

Note: Here φ is called the **altitude angle**.

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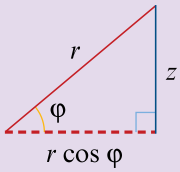
■ **Converting from polar form to component form**

Given $\boldsymbol{v} = [r, \theta, \varphi]$, we can write $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, where

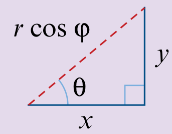
$$x = r\cos\theta\,\cos\varphi, \quad y = r\sin\theta\,\cos\varphi \quad \text{and} \quad z = r\sin\varphi$$



■ **Converting from component form to polar form**

Given $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, we can write $\boldsymbol{v} = [r, \theta, \varphi]$, where

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x} \quad \text{and} \quad \sin\varphi = \frac{z}{r}$$



Warning: When converting to polar form, you need to take care to choose the angle θ in the correct quadrant of the x–y plane.

Example 21

a Express the vector $[12, 45°, 60°]$ in component form.

b Express the vector $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ in polar form.

Solution

a Let $\boldsymbol{a} = [12, 45°, 60°]$.

Then

$$\begin{aligned} x &= r\cos\theta\,\cos\varphi \\ &= 12\cos 45°\cos 60° \\ &= 3\sqrt{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta\,\cos\varphi \\ &= 12\sin 45°\cos 60° \\ &= 3\sqrt{2} \end{aligned} \qquad \begin{aligned} z &= r\sin\varphi \\ &= 12\sin 60° \\ &= 6\sqrt{3} \end{aligned}$$

Hence

$$\boldsymbol{a} = 3\sqrt{2}\hat{\boldsymbol{i}} + 3\sqrt{2}\hat{\boldsymbol{j}} + 6\sqrt{3}\hat{\boldsymbol{k}}$$

b Let $\boldsymbol{b} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.

Then

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{1 + 4 + 4} = 3 \end{aligned} \qquad \tan\theta = \frac{y}{x} = -2 \qquad \sin\varphi = \frac{z}{r} = \frac{2}{3}$$

Note that, since $x > 0$ and $y < 0$, we must have $-90° < \theta < 0°$. Hence

$$\begin{aligned} \boldsymbol{b} &= [3, \tan^{-1}(-2), \sin^{-1}(\tfrac{2}{3})] \\ &\approx [3, -63.43°, 41.81°] \end{aligned}$$

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Exercise 5C

SF

1 For each of the following, draw a sketch to illustrate the vector and write the vector in component form:

a $[2, 30°]$ **b** $[4, -30°]$ **c** $[2, 135°]$ **d** $[4, -120°]$ **e** $[4, 150°]$

Example 18

2 Write each of the following vectors in component form:

a $[8, 60°]$ **b** $[10, 30°]$ **c** $[6, 150°]$ **d** $[8, -45°]$ **e** $[12, -150°]$

Example 19

3 Write each of the following vectors in polar form:

a $4\hat{i} - 4\hat{j}$ **b** $-3\hat{i} + 3\hat{j}$ **c** $-2\sqrt{3}\hat{i} + 2\hat{j}$ **d** $3\hat{i} + 4\hat{j}$ **e** $-5\hat{i} - 12\hat{j}$

4 Write each of the following vectors in polar form:

a $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ **d** $\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ **e** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

5 Write each vector in component form, giving values correct to two decimal places:

a $[7, 40°]$ **b** $[10, 35°]$ **c** $[11, 155°]$ **d** $[9, -35°]$ **e** $[12, -125°]$

6 Express each of the following vectors in polar form:

a $2\boldsymbol{u}$, where $\boldsymbol{u} = [10, 60°]$ **b** $-\frac{1}{2}\boldsymbol{v}$, where $\boldsymbol{v} = [2, 15°]$ **c** $-3\boldsymbol{w}$, where $\boldsymbol{w} = [6, -20°]$

Example 20

7 Determine each vector sum in component form:

a $[8, 30°] + [12, 60°]$ **b** $[10, 45°] + [10, -90°]$ **c** $[4, 20°] + [6, 80°]$

Example 21a

8 Write each of the following three-dimensional vectors in component form:

a $[12, 30°, 60°]$ **b** $[30, 45°, 45°]$ **c** $[24, 135°, -45°]$
d $[60, -120°, 30°]$ **e** $[200, 90°, -60°]$ **f** $[200, 90°, -90°]$

9 Write each of the following three-dimensional vectors in polar form:

a $\begin{bmatrix} \sqrt{3} \\ 1 \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} \sqrt{3} \\ 2 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example 21b

10 Write each of the following three-dimensional vectors in polar form:

a $4\hat{i} - 4\hat{j} + \hat{k}$ **b** $-3\hat{i} + 3\hat{j} - \hat{k}$ **c** $-2\sqrt{3}\hat{i} + 2\hat{j} - 2\hat{k}$
d $3\hat{i} + 4\hat{j} - 4\hat{k}$ **e** $-5\hat{i} - 12\hat{j} + 2\hat{k}$ **f** $-6\hat{j} + 4\hat{k}$

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5D Scalar product of vectors

Learning intentions

- ► To be able to work with the scalar product.

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors in three dimensions $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The scalar product of two vectors in two dimensions is defined similarly.

Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

The scalar product is often called the **dot product**.

Example 22

Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$. determine:

a $\boldsymbol{a} \cdot \boldsymbol{b}$ **b** $\boldsymbol{a} \cdot \boldsymbol{a}$

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \times (-2) + (-2) \times 3 + 3 \times 4 = 4$

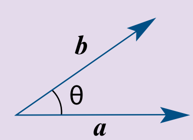
b $\boldsymbol{a} \cdot \boldsymbol{a} = 1^2 + (-2)^2 + 3^2 = 14$

Geometric description of the scalar product

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

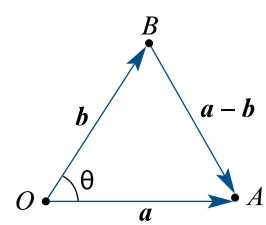
$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



Proof Let $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$. The cosine rule in $\triangle OAB$ gives

$$\begin{aligned}
|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta &= |\boldsymbol{a} - \boldsymbol{b}|^2 \\
(a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\
2(a_1b_1 + a_2b_2 + a_3b_3) &= 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta \\
a_1b_1 + a_2b_2 + a_3b_3 &= |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta \\
\therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} &= |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta
\end{aligned}$$



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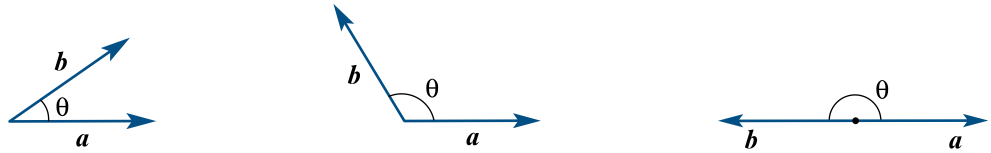
Note: When two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are placed so that their initial points coincide, the angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$ is chosen as shown in the diagrams. Note that $0° \le \theta \le 180°$.



Example 23

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 30° \\ &= 20 \times \frac{\sqrt{3}}{2} \\ &= 10\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 150° \\ &= 20 \times \frac{-\sqrt{3}}{2} \\ &= -10\sqrt{3}\end{aligned}$$

Properties of the scalar product

The following properties can be established from the definition of the scalar product:

commutative law for scalar product	$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
distributive law	$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
compatibility with scalar multiplication	$k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
scalar product with zero	$\boldsymbol{a} \cdot \boldsymbol{0} = 0$

Several further properties follow from the geometric description of the scalar product:

- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$

- For the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, we have $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$ and $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$.

Example 24

a Simplify $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c})$.

b Expand the following:

i $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$ **ii** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$

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Solution

a $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{b} \cdot (\boldsymbol{a} - \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{c}$

$= \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$

b **i** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b})$

$= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b}$

$= \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b}$

ii $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})$

$= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b}$

$= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b}$

Example 25

Solve the equation $(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (3\hat{\boldsymbol{i}} - x\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 4$ for x.

Solution

$$\begin{aligned}(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (3\hat{\boldsymbol{i}} - x\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) &= 4 \\ 3 - x - 2 &= 4 \\ 1 - x &= 4 \\ \therefore \quad x &= -3\end{aligned}$$

Determining the magnitude of the angle between two vectors

The angle between two vectors can be found by using the two forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Therefore

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Example 26

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}, \quad \boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} \quad \text{and} \quad \boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$$

Determine the magnitude of $\angle ABC$, correct to one decimal place.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

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We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = 1 - 6 + 2 = -3$$

$$|\overrightarrow{BA}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$|\overrightarrow{BC}| = \sqrt{1 + 9 + 4} = \sqrt{14}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-3}{\sqrt{6}\sqrt{14}}$$

Hence $\angle ABC = 109.1°$, correct to one decimal place.

(Alternatively, we can write $\angle ABC = 1.9^c$, correct to one decimal place.)

Exercise 5D

SF

Example 22

1 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. Determine:

- **a** $\boldsymbol{a} \cdot \boldsymbol{a}$
- **b** $\boldsymbol{b} \cdot \boldsymbol{b}$
- **c** $\boldsymbol{c} \cdot \boldsymbol{c}$
- **d** $\boldsymbol{a} \cdot \boldsymbol{b}$
- **e** $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$
- **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$
- **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$. Determine:

- **a** $\boldsymbol{a} \cdot \boldsymbol{a}$
- **b** $\boldsymbol{b} \cdot \boldsymbol{b}$
- **c** $\boldsymbol{a} \cdot \boldsymbol{b}$
- **d** $\boldsymbol{a} \cdot \boldsymbol{c}$
- **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

Example 23

3 **a** If $|\boldsymbol{a}| = 6$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 6$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $120°$, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

Example 24

4 Expand and simplify:

- **a** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$
- **b** $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$
- **c** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$
- **d** $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

Example 25

5 Solve each of the following equations:

- **a** $(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) \cdot (5\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = -6$
- **b** $(x\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (-4\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}) = 10$
- **c** $(x\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{k}}) \cdot (-2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}) = x$
- **d** $x(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + x\hat{\boldsymbol{k}}) = 6$

Example 26

6 If A and B are points defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ respectively, determine:

- **a** $\overrightarrow{AB}$
- **b** $|\overrightarrow{AB}|$
- **c** the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

7 Let C and D be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, determine $|\overrightarrow{CD}|$.

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8 $OABC$ is a rhombus with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$. CF

a Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

b Determine $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

c Prove that the diagonals of a rhombus intersect at right angles.

9 From the following list, determine three pairs of perpendicular vectors: SF

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{b} = -4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, \quad \boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k},$$
$$\boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{e} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{f} = -\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$$

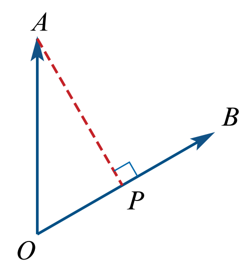
10 The four vertices of a regular tetrahedron have the following position vectors: CF

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}, \quad \boldsymbol{d} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

a Show that all the vertices are the same distance from the origin.

b Show that the angle between any two of these vectors is the same. Determine this angle in degrees correct to two decimal places.

11 Points A and B are defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.
Let P be the point on OB such that AP is perpendicular to OB.
Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant q.



a Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to determine the value of q.

c Determine the coordinates of the point P.

12 If $x\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + y\hat{\boldsymbol{k}}$ is perpendicular to vectors $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, determine x and y.

13 Determine the angle, in radians, between each of the following pairs of vectors, correct to three significant figures:

a $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

b $-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $-2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

c $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$

d $7\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ and $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

14 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

For Questions 15–18, determine the angles in degrees correct to two decimal places.

15 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ respectively. Let M be the midpoint of AB. Determine: CF

a $\overrightarrow{OM}$ **b** $\angle AOM$ **c** $\angle BMO$

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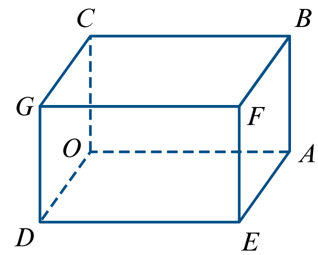
16 $OABCDEFG$ is a cuboid, set on axes at O, such that $\overrightarrow{OD} = \hat{\boldsymbol{i}}$, $\overrightarrow{OA} = 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OC} = 2\hat{\boldsymbol{k}}$. Determine:



a **i** $\overrightarrow{GB}$ **ii** $\overrightarrow{GE}$

b $\angle BGE$

c the angle between diagonals $\overrightarrow{CE}$ and $\overrightarrow{GA}$

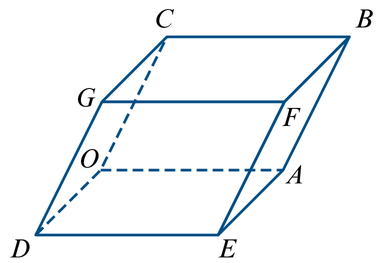
17 Let A, B and C be the points defined by the position vectors $4\hat{\boldsymbol{i}}$, $5\hat{\boldsymbol{j}}$ and $-2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{k}}$ respectively. Let M and N be the midpoints of AB and AC respectively. Determine:

a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$ **b** $\angle MON$ **c** $\angle MOC$

18 A parallelepiped is an oblique prism that has a parallelogram cross-section. It has three pairs of parallel and congruent faces.

$OABCDEFG$ is a parallelepiped with $\overrightarrow{OA} = 3\hat{\boldsymbol{j}}$, $\overrightarrow{OC} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\overrightarrow{OD} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

Show that the diagonals DB and CE bisect each other, and determine the acute angle between them.



5E Vector projections

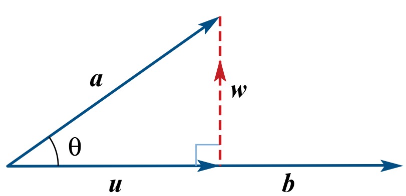
Learning intentions

- To be able to work with vector projections

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that

$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$



where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.

For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{b} &= 0 \\ (\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} &= 0 \\ \boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) &= 0 \end{aligned}$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

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Vector resolute

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\,\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}$.

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**.

Example 27

Let $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Determine the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 - 2 = -4$

$\boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 + 4 = 6$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} = -\frac{4}{6}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = -\frac{2}{3}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$$

Example 28

Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$. Determine the scalar resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = -2 - 3 = -5$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{-5}{\sqrt{10}} = -\frac{\sqrt{10}}{2}$$

b $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -5$

$|\boldsymbol{a}| = \sqrt{4 + 4 + 1} = 3$

The scalar resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{|\boldsymbol{a}|} = -\frac{5}{3}$$

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Example 29

Resolve $\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ into rectangular components, one of which is parallel to $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

Solution

Let $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\boldsymbol{b}$.

We have

$$\boldsymbol{a}\cdot\boldsymbol{b} = 2 - 6 + 1 = -3$$
$$\boldsymbol{b}\cdot\boldsymbol{b} = 4 + 4 + 1 = 9$$

Therefore the vector resolute is

$$\frac{-3}{9}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = -\frac{1}{3}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

The perpendicular component is

$$\begin{aligned}\boldsymbol{a} - \left(-\frac{1}{3}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})\right) &= (\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) + \frac{1}{3}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \\ &= \frac{5}{3}\hat{\boldsymbol{i}} + \frac{7}{3}\hat{\boldsymbol{j}} - \frac{4}{3}\hat{\boldsymbol{k}} \\ &= \frac{1}{3}(5\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}})\end{aligned}$$

Hence we can write

$$\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} = -\frac{1}{3}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) + \frac{1}{3}(5\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}})$$

Check: As a check, we verify that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(5\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}})\cdot(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 10 - 14 + 4 = 0$, as expected.

Skill-sheet

Exercise 5E

SF

1 Points A and B are defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.

- **a** Determine $\hat{\boldsymbol{a}}$.
- **b** Determine $\hat{\boldsymbol{b}}$.
- **c** Determine $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.

2 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

- **a** Determine:
  - **i** $\hat{\boldsymbol{a}}$
  - **ii** $\hat{\boldsymbol{b}}$
- **b** Determine the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

CF

3 Points A and B are defined by the position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{k}}$.

- **a** Determine:
  - **i** $\hat{\boldsymbol{a}}$
  - **ii** $\hat{\boldsymbol{b}}$
- **b** Determine the unit vector which bisects $\angle AOB$.

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Example 27

4 For each pair of vectors, determine the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:

a $\boldsymbol{a} = \hat{\imath} + 3\hat{\jmath}$ and $\boldsymbol{b} = \hat{\imath} - 4\hat{\jmath} + \hat{k}$

b $\boldsymbol{a} = \hat{\imath} - 3\hat{k}$ and $\boldsymbol{b} = \hat{\imath} - 4\hat{\jmath} + \hat{k}$

c $\boldsymbol{a} = 4\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\boldsymbol{b} = 4\hat{\imath} - \hat{k}$

Example 28

5 For each of the following pairs of vectors, determine the scalar resolute of the first vector in the direction of the second vector:

a $\boldsymbol{a} = 2\hat{\imath} + \hat{\jmath}$ and $\boldsymbol{b} = \hat{\imath}$

b $\boldsymbol{a} = 3\hat{\imath} + \hat{\jmath} - 3\hat{k}$ and $\boldsymbol{c} = \hat{\imath} - 2\hat{\jmath}$

c $\boldsymbol{b} = 2\hat{\jmath} + \hat{k}$ and $\boldsymbol{a} = 2\hat{\imath} + \sqrt{3}\hat{\jmath}$

d $\boldsymbol{b} = \hat{\imath} - \sqrt{5}\hat{\jmath}$ and $\boldsymbol{c} = -\hat{\imath} + 4\hat{\jmath}$

Example 29

6 For each of the following pairs of vectors, determine the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:

a $\boldsymbol{a} = 2\hat{\imath} + \hat{\jmath} + \hat{k},\ \boldsymbol{b} = 5\hat{\imath} - \hat{k}$

b $\boldsymbol{a} = 3\hat{\imath} + \hat{\jmath},\ \boldsymbol{b} = \hat{\imath} + \hat{k}$

c $\boldsymbol{a} = -\hat{\imath} + \hat{\jmath} + \hat{k},\ \boldsymbol{b} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$

7 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\boldsymbol{b} = \hat{\jmath} + \hat{k}$ respectively. Determine:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b a unit vector in the direction of the component of $\boldsymbol{a}$ perpendicular to OB

8 Let A and B be the points defined by the position vectors $\boldsymbol{a} = 4\hat{\imath} + \hat{\jmath}$ and $\boldsymbol{b} = \hat{\imath} - \hat{\jmath} - \hat{k}$ respectively. Determine:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$

c the shortest distance from A to line OB

SF

9 Points A, B and C have position vectors $\boldsymbol{a} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\boldsymbol{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\boldsymbol{c} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$. Determine:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to line AC

d the area of triangle ABC

10 **a** Verify that vectors $\boldsymbol{a} = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$ and $\boldsymbol{b} = 5\hat{\imath} + \hat{\jmath} + \hat{k}$ are perpendicular to each other.

b If $\boldsymbol{c} = 2\hat{\imath} - \hat{k}$, determine:

i $\boldsymbol{d}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$

ii $\boldsymbol{e}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$.

c Determine $\boldsymbol{f}$ such that $\boldsymbol{c} = \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$.

d Hence show that $\boldsymbol{f}$ is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

CF

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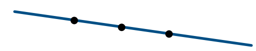
5F Collinearity

Learning intentions

- To be able to use vectors to establish when points are collinear.

We are already familiar with midpoints of line segments. We know that, if C is the midpoint of AB, then we can write $\overrightarrow{OC} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$. We now consider the more general situation where C is on the line AB.

Three or more points are said to be **collinear** if they all lie on a single line.



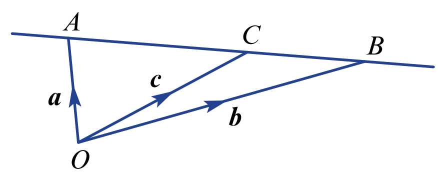
Three distinct points A, B and C are collinear if and only if there exists a non-zero real number m such that $\overrightarrow{AC} = m\overrightarrow{AB}$ (that is, if and only if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel).

A property of collinearity

Let points A, B and C have position vectors $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Then

$$\overrightarrow{AC} = m\overrightarrow{AB} \quad \text{if and only if} \quad \boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b}$$

Proof If $\overrightarrow{AC} = m\overrightarrow{AB}$, then we have



$$\begin{aligned}\boldsymbol{c} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \overrightarrow{OA} + m\overrightarrow{AB}\\ &= \boldsymbol{a} + m(\boldsymbol{b}-\boldsymbol{a})\\ &= \boldsymbol{a} + m\boldsymbol{b} - m\boldsymbol{a} = (1-m)\boldsymbol{a} + m\boldsymbol{b}\end{aligned}$$

Similarly, we can show that if $\boldsymbol{c} = (1-m)\boldsymbol{a} + m\boldsymbol{b}$, then $\overrightarrow{AC} = m\overrightarrow{AB}$.

Note: It follows from this result that if distinct points A, B and C are collinear, then we can write $\overrightarrow{OC} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}$, where $\lambda + \mu = 1$. If C is between A and B, then $0 < \mu < 1$.

Example 30

For distinct points A and B, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. Express $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where C is:

a the midpoint of AB

b the point of trisection of AB nearer to A

c the point C such that $\overrightarrow{AC} = -2\overrightarrow{AB}$.

Solution

a $\overrightarrow{AC} = \dfrac{1}{2}\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \boldsymbol{a} + \frac{1}{2}\overrightarrow{AB}\\ &= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})\\ &= \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\end{aligned}$$

b $\overrightarrow{AC} = \dfrac{1}{3}\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \boldsymbol{a} + \frac{1}{3}\overrightarrow{AB}\\ &= \boldsymbol{a} + \frac{1}{3}(\boldsymbol{b}-\boldsymbol{a})\\ &= \frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}\end{aligned}$$

c $\overrightarrow{AC} = -2\overrightarrow{AB}$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC}\\ &= \boldsymbol{a} - 2\overrightarrow{AB}\\ &= \boldsymbol{a} - 2(\boldsymbol{b}-\boldsymbol{a})\\ &= 3\boldsymbol{a} - 2\boldsymbol{b}\end{aligned}$$

Note: Alternatively, we could have used the previous result in this example.

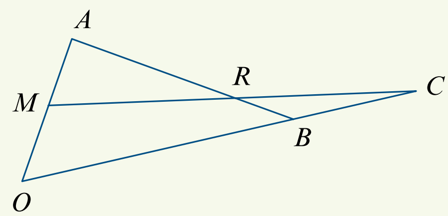
Example 31

Consider a triangle OAB. Let M be the midpoint of OA, let C be the point such that $\overrightarrow{OC} = \frac{4}{3}\overrightarrow{OB}$ and let R be the point of intersection of lines AB and MC.

a Determine $\overrightarrow{OR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.

b Hence determine $AR : RB$.

Solution



a We have $\overrightarrow{OM} = \frac{1}{2}\boldsymbol{a}$ and $\overrightarrow{OC} = \frac{4}{3}\boldsymbol{b}$.

Since M, R and C are collinear, there exists $m \in \mathbb{R}$ with

$$\begin{aligned}\overrightarrow{MR} &= m\overrightarrow{MC}\\ &= m(\overrightarrow{MO} + \overrightarrow{OC})\\ &= m\left(-\frac{1}{2}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}\right)\end{aligned}$$

Thus
$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OM} + \overrightarrow{MR}\\ &= \frac{1}{2}\boldsymbol{a} + m\left(-\frac{1}{2}\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}\right)\\ &= \frac{1-m}{2}\boldsymbol{a} + \frac{4m}{3}\boldsymbol{b}\end{aligned}$$

Since A, R and B are collinear, there exists $n \in \mathbb{R}$ with

$$\begin{aligned}\overrightarrow{AR} &= n\overrightarrow{AB}\\ &= n(\overrightarrow{AO} + \overrightarrow{OB})\\ &= n(-\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$

Thus
$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OA} + \overrightarrow{AR}\\ &= \boldsymbol{a} + n(-\boldsymbol{a} + \boldsymbol{b})\\ &= (1-n)\boldsymbol{a} + n\boldsymbol{b}\end{aligned}$$

We have now shown that

$$\overrightarrow{OR} = \frac{1-m}{2}\boldsymbol{a} + \frac{4m}{3}\boldsymbol{b} = (1-n)\boldsymbol{a} + n\boldsymbol{b}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel, it follows that

$$\frac{1-m}{2} = 1 - n \quad \text{and} \quad \frac{4m}{3} = n$$

This gives $m = \frac{3}{5}$ and $n = \frac{4}{5}$. Hence $\overrightarrow{OR} = \frac{1}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b}$.

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b From part **a**, we have

$$\begin{aligned}\overrightarrow{AR} &= \overrightarrow{AO} + \overrightarrow{OR}\\ &= -\boldsymbol{a} + \frac{1}{5}\boldsymbol{a} + \frac{4}{5}\boldsymbol{b}\\ &= \frac{4}{5}(\boldsymbol{b} - \boldsymbol{a})\\ &= \frac{4}{5}\overrightarrow{AB}\end{aligned}$$

Hence $AR : RB = 4 : 1$.

Exercise 5F

Example 30

1 Points A, B and R are collinear, with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Express $\overrightarrow{OR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, where R is the point:

a of trisection of AB nearer to B

b between A and B such that $AR : RB = 3 : 2$.

2 Let $\overrightarrow{OA} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. Determine $\overrightarrow{OR}$, where R is:

a the midpoint of line segment AB

b the point such that $\overrightarrow{AR} = \frac{4}{3}\overrightarrow{AB}$

c the point such that $\overrightarrow{AR} = -\frac{1}{3}\overrightarrow{AB}$.

3 The position vectors of points P, Q and R are $\boldsymbol{a}$, $3\boldsymbol{a} - 4\boldsymbol{b}$ and $4\boldsymbol{a} - 6\boldsymbol{b}$ respectively.

a Show that P, Q and R are collinear.

b Determine $PQ : QR$.

4 In triangle OAB, $\overrightarrow{OA} = a\hat{\boldsymbol{i}}$ and $\overrightarrow{OB} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. Let C be the midpoint of AB.

a Determine $\overrightarrow{OC}$.

b Deduce, by vector method, the relationship between x, y and a if the vector $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$.

5 In parallelogram $OAUB$, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Let $\overrightarrow{OM} = \frac{1}{5}\boldsymbol{a}$ and $MP : PB = 1 : 5$, where P is on the line segment MB.

a Prove that P is on the diagonal OU.

b Hence determine $OP : PU$.

6 $OABC$ is a square with $\overrightarrow{OA} = -4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OC} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.

a Determine $\overrightarrow{OB}$.

b Given that D is the point on AB such that $\overrightarrow{BD} = \frac{1}{3}\overrightarrow{BA}$, determine $\overrightarrow{OD}$.

c Given that OD intersects AC at E and that $\overrightarrow{OE} = (1 - \lambda)\overrightarrow{OA} + \lambda\overrightarrow{OC}$, determine λ.

CF

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CF

7 In triangle OAB, $\overrightarrow{OA} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{OB} = \hat{i} + 2\hat{j} - 2\hat{k}$.

a Use the scalar product to show that $\angle AOB$ is an obtuse angle.

b Determine $\overrightarrow{OP}$, where P is:

i the midpoint of AB

ii the point on AB such that OP is perpendicular to AB

iii the point where the bisector of $\angle AOB$ intersects AB.

5G Applications of vectors

Learning intentions

- To be able to apply vectors to the study of displacement and velocity.
- To be able to apply vectors to the study of relative and resultant velocities.

We refer the reader to Chapter 7, Vector calculus, for further discussion on displacement and velocity. We will not discuss forces in this section but refer the reader to Chapter 14, Modelling motion.

For the remainder of this chapter, we will be working with vector and scalar quantities:

Applications of vectors: displacement and velocity

- A **vector quantity** has both magnitude and direction. In this and subsequent chapters we will introduce the vector quantities displacement, velocity and force.
- A **scalar quantity** has only magnitude. We will use the scalar quantities distance, time, speed and mass.

Displacement

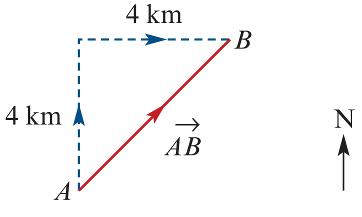
We have been describing points in the plane using position vectors. Points A and B have position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

If an object moves from point A to point B, then the **displacement** of the object is the change in position of the object; it is described by the vector $\overrightarrow{AB}$.

For example, suppose that a person walks 4 km north and then 4 km east.

The person's displacement is $4\sqrt{2}$ km north-east.

Note: The total distance that the person has walked is 8 km, which is not equal to the magnitude of the displacement vector.



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Example 32

A particle moves from point $A(2, 2, 1)$ to point $B(-1, 3, 2)$. Express the displacement vector of the particle in component form.

Solution

We have $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.

The displacement vector is

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -(2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) + -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}\\ &= -3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\end{aligned}$$

Velocity

Velocity is the rate of change of position with respect to time.

Velocity is a vector quantity; it has magnitude and direction. The units of velocity which will be used in this chapter are metres per second (ms^{-1}) and kilometres per hour (km/h).

Some examples of velocity vectors are:

- 80 km/h in the direction north
- 10 km/h on a bearing of 080°
- $3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}$ ms^{-1}

The first two vectors have magnitudes 80 km/h and 10 km/h respectively. The third vector has magnitude $|3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}| = \sqrt{3^2 + 4^2 + 12^2} = 13$ ms^{-1}. The magnitude of velocity is called **speed**.

Motion with constant velocity

In this chapter, we only deal with constant velocity (that is, the velocity does not change over a particular time interval). Consider the following two examples:

- If a car travels for 2 hours with a constant velocity of 80 km/h north, then its displacement is $2 \times 80 = 160$ km north.
- If a particle starts at the origin and moves with a velocity of $3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}$ ms^{-1} for 2 seconds, then its position is $2(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}) = 6\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + 24\hat{\boldsymbol{k}}$ m.

If an object moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by

$$\boldsymbol{s} = t\boldsymbol{v}$$

Note: Here $\boldsymbol{s}$ and $\boldsymbol{v}$ are vector quantities and t is a scalar quantity. So this is an example of scalar multiplication.

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Example 33

A particle starts at the point A with position vector $\overrightarrow{OA} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, where the unit is metres. The particle begins moving with a constant velocity of $2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ ms^{-1}. Determine the position vector of the particle after:

a 5 seconds **b** t seconds.

Solution

a Let P be the point that the particle reaches after 5 seconds. Then

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + 5(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})\\ &= \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + 10\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}} + 15\hat{\boldsymbol{k}}\\ &= 11\hat{\boldsymbol{i}} + 23\hat{\boldsymbol{j}} + 17\hat{\boldsymbol{k}}\end{aligned}$$

b Let Q be the point that the particle reaches after t seconds. Then

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OA} + t(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})\\ &= \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + 2t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}} + 3t\hat{\boldsymbol{k}}\\ &= (1 + 2t)\hat{\boldsymbol{i}} + (3 + 4t)\hat{\boldsymbol{j}} + (2 + 3t)\hat{\boldsymbol{k}}\end{aligned}$$

Direction of motion

The velocity vector is in the direction of motion. We often use the unit vector of the velocity vector to describe the direction of motion.

For example, if $\boldsymbol{v} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \sqrt{11}\hat{\boldsymbol{k}}$, then the unit vector $\hat{\boldsymbol{v}} = \frac{1}{6}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \sqrt{11}\hat{\boldsymbol{k}})$ is in the direction of motion.

Example 34

Particle A starts moving from point O with a constant velocity of $\boldsymbol{v}_A = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \sqrt{11}\hat{\boldsymbol{k}}$ ms^{-1}. Three seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 8 ms^{-1}. When and where will B catch up to A?

Solution

At time t seconds, particle A is at the point with position vector

$$\overrightarrow{OP_A} = t(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \sqrt{11}\hat{\boldsymbol{k}})$$

At time t seconds, for $t \geq 3$, particle B has been moving for $t - 3$ seconds and is at the point with position vector

$$\overrightarrow{OP_B} = \frac{4(t-3)}{3}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \sqrt{11}\hat{\boldsymbol{k}})$$

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The two particles are at the same point when

$$\frac{4(t-3)}{3} = t$$

$$4(t-3) = 3t$$

$$t = 12$$

$$\therefore \quad t = 12$$

Particle B catches up to particle A at time $t = 12$ seconds.

At this time, both particles have position vector $12(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \sqrt{11}\hat{\boldsymbol{k}})$.

Example 35

A particle starts from O with a constant velocity of $\boldsymbol{v}_1 = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ ms^{-1}. At the same time, a second particle starts moving with constant velocity from point B, where $\overrightarrow{OB} = 25\hat{\boldsymbol{j}}$.

Given that the two particles meet and their paths are at right angles, determine:

a the position vector of the point where they meet

b the velocity of the second particle.

Solution

a Assume that the particles meet at the point P at time t seconds. Since their paths are at right angles, we have

$$\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$$

At time t seconds, the position vector of the first particle is

$$\overrightarrow{OP} = t(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) = 3t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}$$

Therefore

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{BO} + \overrightarrow{OP}\\ &= -25\hat{\boldsymbol{j}} + (3t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}})\\ &= 3t\hat{\boldsymbol{i}} + (4t-25)\hat{\boldsymbol{j}}\end{aligned}$$

Since $\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$, we obtain

$$\begin{aligned}(3t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}) \cdot (3t\hat{\boldsymbol{i}} + (4t-25)\hat{\boldsymbol{j}}) &= 0\\ 9t^2 + 4t(4t-25) &= 0\\ 25t^2 - 100t &= 0\\ \therefore \quad t(t-4) &= 0\end{aligned}$$

The particles do not meet at time 0 s, so they meet at time $t = 4$ s.

The position vector of the point where they meet is

$$\overrightarrow{OP} = 4(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) = 12\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}}$$

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b Let $\boldsymbol{v}$ m/s be the velocity of the second particle. We use the formula $\boldsymbol{s} = t\boldsymbol{v}$.

At time $t = 4$, the displacement of the second particle is $\overrightarrow{BP}$. Therefore

$$\begin{aligned} \overrightarrow{BP} &= 4\boldsymbol{v} \\ \overrightarrow{BO} + \overrightarrow{OP} &= 4\boldsymbol{v} \\ -25\hat{\boldsymbol{j}} + (12\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}}) &= 4\boldsymbol{v} \\ 12\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}} &= 4\boldsymbol{v} \end{aligned}$$

Hence the velocity of the second particle is $\boldsymbol{v} = 3\hat{\boldsymbol{i}} - \frac{9}{4}\hat{\boldsymbol{j}}$ ms^{-1}

Resultant velocity

If two or more velocity vectors are added, then the sum is called a **resultant velocity**.

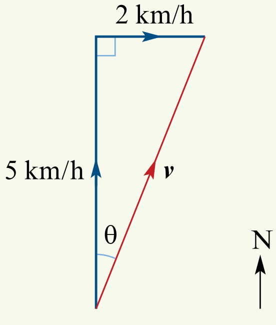
Example 36

A river is flowing north at 5 km/h. Mila can swim at 2 km/h in still water. She dives in from the west bank of the river and swims towards the opposite bank.

a In which direction does she travel? **b** What is her actual speed?

Solution

The swimmer's actual velocity, $\boldsymbol{v}$, is the vector sum of her velocity relative to the water (2 km/h east) and the water's velocity (5 km/h north).



a From the diagram, we have

$$\tan\theta = \frac{2}{5}$$

$$\therefore \quad \theta \approx 21.8^\circ$$

She is travelling on a bearing of 022°.

b Her actual speed is

$$|\boldsymbol{v}| = \sqrt{2^2 + 5^2} \approx 5.39 \text{ km/h}$$

Relative velocity

In the previous example, the velocity of the water is given *relative to the bank* and the velocity of the swimmer is given *relative to the water*. The velocity of the swimmer *relative to the bank* is found by taking the vector sum. That is:

Velocity of swimmer relative to bank	=	Velocity of swimmer relative to water	+	Velocity of water relative to bank

The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.

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Consider another example: A train is travelling north at 60 km/h, and a passenger walks at 3 km/h along the corridor towards the back of the train.

Velocity of passenger relative to Earth	=	Velocity of passenger relative to train	+	Velocity of train relative to Earth

The passenger is moving with a velocity of 57 km/h north relative to Earth.

In general, if an object A is in motion relative to another object B, then we can determine the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

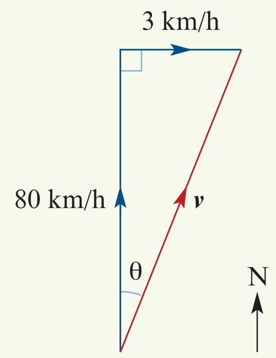
Velocities measured relative to Earth are often called **true velocities** or **actual velocities**.

Example 37

A train is moving with a constant velocity of 80 km/h north. A passenger walks straight across a carriage from the west side to the east side at 3 km/h. What is the true velocity of the passenger?

Solution

The passenger's true velocity, $\boldsymbol{v}$, is the vector sum of his velocity relative to the train (3 km/h east) and the train's velocity (80 km/h north).



Speed: $|\boldsymbol{v}| = \sqrt{80^2 + 3^2}$

$= \sqrt{6409}$

≈ 80.06 km/h

Direction: $\tan\theta = \dfrac{3}{80}$

$\therefore \quad \theta \approx 2.15^\circ$

The passenger's true velocity is 80.06 km/h on a bearing of 002°.

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Example 38

Car A is moving with a velocity of 50 km/h due north, while car B is moving with a velocity of 120 km/h due west. What is the velocity of car A relative to car B?

Solution

Let $\boldsymbol{v}_A$ be the velocity of car A, and let $\boldsymbol{v}_B$ be the velocity of car B.

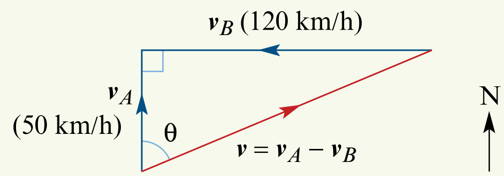
The velocity of car A relative to car B is given by $\boldsymbol{v} = \boldsymbol{v}_A - \boldsymbol{v}_B$.

Speed: $|\boldsymbol{v}| = \sqrt{50^2 + 120^2}$

$= 130$ km/h

Direction: $\theta = \tan^{-1}\left(\dfrac{12}{5}\right)$

$\approx 67.38°$



The velocity of car A relative to car B is 130 km/h on a bearing of 067°.

Wind effect on flight paths

The **airspeed** of an aircraft is its speed relative to air. In the next example, we see how the wind affects the actual velocity of an aircraft.

Example 39

A light aircraft has an airspeed of 250 km/h. The pilot sets a course due north. If the wind is blowing from the north-west at 80 km/h, what is the true speed and direction of the aircraft?

Solution

We can use the cosine rule to determine the true speed:

$$|\boldsymbol{v}| = \sqrt{250^2 + 80^2 - 2 \times 250 \times 80 \cos 45°}$$

$$= 201.5334\ldots \text{ km/h}$$

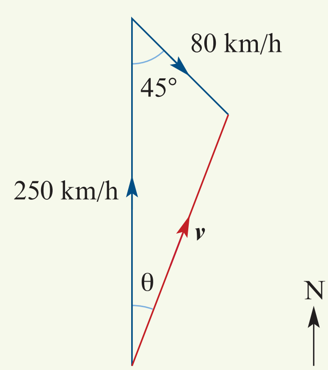
We can now use the sine rule to determine the angle θ:

$$\frac{80}{\sin\theta} = \frac{|\boldsymbol{v}|}{\sin 45°}$$

$$\sin\theta = \frac{80 \sin 45°}{|\boldsymbol{v}|}$$

$$= 0.2806\ldots$$

$$\therefore \quad \theta \approx 16.30°$$



The aircraft is flying at 201.53 km/h on a bearing of 016°.

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To fly an aircraft in a given direction, the pilot must compensate for the effect of the wind.

Example 40

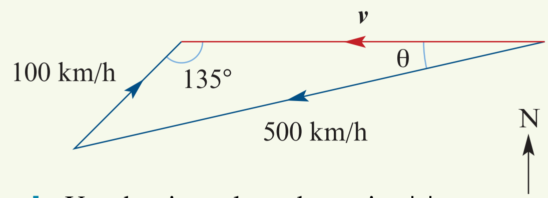
An aeroplane is scheduled to travel from a point P to a point Q, which is 1000 km due west of P. The aeroplane's airspeed is 500 km/h and the wind is blowing from the south-west at 100 km/h.

a In which direction should the pilot set the course?

b How long will the flight take?

Solution

We want to ensure that the plane's true velocity, $\boldsymbol{v}$, is due west.



a Use the sine rule to determine θ:

$$\frac{500}{\sin 135^\circ} = \frac{100}{\sin\theta}$$

$$\sin\theta = \frac{100\sin 135^\circ}{500}$$

$$= 0.1414\ldots$$

$$\therefore \quad \theta = (8.130\ldots)^\circ$$

The pilot should head on a bearing of 262°.

b Use the sine rule to determine $|\boldsymbol{v}|$:

$$\frac{500}{\sin 135^\circ} = \frac{|\boldsymbol{v}|}{\sin(36.869\ldots)^\circ}$$

$$\therefore \quad |\boldsymbol{v}| = \frac{500\sin(36.869\ldots)^\circ}{\sin 135^\circ}$$

$$= 424.264\ldots$$

$$\approx 424.26 \text{ km/h}$$

The plane's speed relative to the ground is approximately 424 km/h. The flight will take approximately 2.4 hours.

Exercise 5G

Example 32

SF

1 For each of the following, determine the displacement vector in component form for a particle that moves from point A to point B:

a $A(3, 7)$, $B(2, -4)$ **b** $A(-2, 4)$, $B(3, -2)$

c $A(3, 1)$, $B(4, 6)$ **d** $A(3, 7, 2)$, $B(3, -4, 6)$

e $A(-2, -7, -4)$, $B(2, -7, 8)$ **f** $A(5, -6, 9)$, $B(11, 5, -4)$

2 Give the corresponding speed for each of the following velocity vectors:

a $5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s **b** $3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ m/s

c $-\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s **d** $-2\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ m/s

e $5\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ m/s **f** $-7\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ m/s

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In each of the following questions, the unit of distance is metres.

SF

Example 33

3 A particle starts from the point A with position vector $\overrightarrow{OA} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and moves with a constant velocity of $5\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ ms^{-1}. Determine the position vector of the particle after:

a 5 seconds **b** t seconds.

4 An object takes 5 seconds to move with constant velocity from point A to point B, where $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = -15\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$. Determine the velocity of the object.

5 A particle starts from the point B with position vector $\overrightarrow{OB} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and moves with a constant velocity of $7\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ m/s.

a Determine the position vector of the particle after:

i 4 seconds **ii** t seconds.

b Determine the particle's distance from the origin after:

i 4 seconds **ii** t seconds.

CF

6 Let O be the origin and let A and B be the points with $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = -5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. A particle moves with constant velocity from A to B in 10 seconds. Determine:

a the velocity of the particle **b** the speed of the particle.

Example 34

7 Particle A starts moving from point O with a constant velocity of $\boldsymbol{v}_A = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ m/s. Two seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 6 m/s. When and where will B catch up to A?

Example 35

8 A particle starts from O with a constant velocity of $\boldsymbol{v}_1 = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ m/s. At the same time, a second particle starts moving with constant velocity from point B, where $\overrightarrow{OB} = 20\hat{\boldsymbol{j}}$. Given that the two particles meet and their paths are at right angles, determine:

a the position vector of the point where they meet

b the velocity of the second particle.

9 Points A and B have position vectors $\overrightarrow{OA} = 10\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = 20\hat{\boldsymbol{i}}$. A particle starts moving from point A with a constant velocity of $\boldsymbol{v}_1 = 2\hat{\boldsymbol{i}}$ m/s. At the same time, a second particle starts moving from point B with constant velocity. Given that the two particles meet and their paths are at right angles, determine:

a the position vector of the point where they meet

b the velocity of the second particle.

Example 36

10 A river is flowing south at 4 km/h. Max can swim at 3 km/h in still water. He dives in from the west bank of the river and swims towards the opposite bank.

a In which direction does he travel? **b** What is his actual speed?

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Example 37

11 A train is moving due north at 100 km/h. A passenger walks straight across a carriage from the east side to the west side at 4 km/h. What is the true velocity of the passenger?

12 Cars A and B are driving along a straight level road that runs east–west.

a If car A has a velocity of 100 km/h west and car B has a velocity of 80 km/h west, what is the velocity of car A relative to car B?

b If car A has a velocity of 100 km/h west and car B has a velocity of 80 km/h east, what is the velocity of car A relative to car B?

13 A cricketer is on a moving walkway which runs from south to north at 2 m/s. He bowls his fastest delivery, which is 45 m/s, again in a direction north. What is the velocity of the ball (relative to Earth)?

14 A ship is moving in a straight line at 15 m/s. A bird flies horizontally from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

15 Car A is travelling north at 60 km/h along a straight level road. Car B is on the same road travelling north at 40 km/h. Determine:

a the velocity of car A relative to car B **b** the velocity of car B relative to car A.

16 A plane is heading due north, its airspeed is 240 km/h and there is an 80 km/h wind blowing from west to east. What is the velocity of the plane relative to Earth?

Example 38

17 Car A is moving with a velocity of 60 km/h due north, while car B is moving with a velocity of 80 km/h due west. What is the velocity of car A relative to car B?

18 A glider P is travelling due north at 60 km/h, and another glider Q is travelling north-west at 40 km/h. Determine the velocity of P relative to Q.

19 Two particles, A and B, are moving with constant velocities of $\boldsymbol{v}_A = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ m/s and $\boldsymbol{v}_B = 5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ m/s respectively.

a Determine the velocity of B relative to A.

b Determine the magnitude of this relative velocity.

20 A ship is moving in a straight line at 15 m/s. A bird flies at an angle of 18° to the horizontal from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

Example 39

21 A light aircraft has an airspeed of 240 km/h. The pilot sets a course due north. The wind is blowing from the north-east at 70 km/h. What is the true speed and direction of the aircraft?

Example 40

22 An aeroplane with an airspeed of 200 km/h is flying to an airport south-west of its present position. There is a wind blowing at 70 km/h from the east.

a Determine the course that the pilot must set.

b Determine the speed of the aeroplane relative to the ground.

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5H Geometric proofs

Learning intentions

- To be able to complete geometric proofs using vector techniques.

In this section we use vectors to prove geometric results in two and three dimensions. The following properties of vectors will be useful:

Parallel vectors

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$.
- Given points P, A and B, with $\overrightarrow{PA} = \boldsymbol{a}$ and $\overrightarrow{PB} = \boldsymbol{b}$, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then P, A and B lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A, B and C are collinear.

Scalar product

- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear combinations of non-parallel vectors

- For two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are not parallel, if $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$, then $m = p$ and $n = q$.

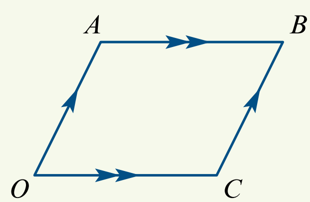
Vector proofs in two-dimensional geometry

Example 41

Prove that the diagonals of a rhombus are perpendicular.

Solution

$OABC$ is a rhombus.



Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$. The diagonals of the rhombus are OB and AC.

$$\begin{aligned}\text{Now}\quad \overrightarrow{OB} &= \overrightarrow{OC} + \overrightarrow{CB} \\ &= \overrightarrow{OC} + \overrightarrow{OA} \\ &= \boldsymbol{c} + \boldsymbol{a}\end{aligned}$$

$$\begin{aligned}\text{and}\quad \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= -\boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

Consider the scalar product of $\overrightarrow{OB}$ and $\overrightarrow{AC}$:

$$\begin{aligned}\overrightarrow{OB} \cdot \overrightarrow{AC} &= (\boldsymbol{c} + \boldsymbol{a}) \cdot (\boldsymbol{c} - \boldsymbol{a}) \\ &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{a} \\ &= |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2\end{aligned}$$

A rhombus has all sides of equal length, and therefore $|\boldsymbol{c}| = |\boldsymbol{a}|$. Hence $\overrightarrow{OB} \cdot \overrightarrow{AC} = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 = 0$. This implies that AC is perpendicular to OB.

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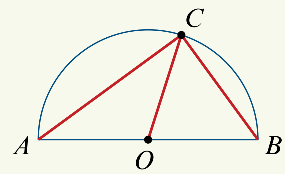
Example 42

Prove that the angle subtended by a diameter in a circle is a right angle.

Solution

Let O be the centre of the circle and let AB be a diameter.



Then $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = r$, where r is the radius.

Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Then $\overrightarrow{OB} = -\boldsymbol{a}$.

We have $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$.

$$\begin{aligned}\text{Thus } \overrightarrow{AC} \cdot \overrightarrow{BC} &= (-\boldsymbol{a} + \boldsymbol{c}) \cdot (\boldsymbol{a} + \boldsymbol{c}) \\ &= -\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c} \\ &= -|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2\end{aligned}$$

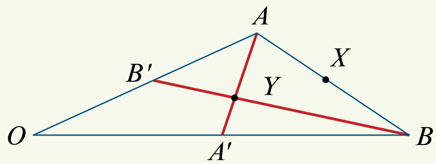
But $|\boldsymbol{a}| = |\boldsymbol{c}|$ and therefore $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$. Hence $AC \perp BC$.

Example 43

Prove that the medians of a triangle are concurrent.

Solution

Consider triangle OAB. Let A', B' and X be the midpoints of OB, OA and AB respectively.



Let Y be the point of intersection of the medians AA' and BB'.

Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.

We start by showing that $AY : YA' = BY : YB' = 2 : 1$.

We have $\overrightarrow{AY} = \lambda\overrightarrow{AA'}$ and $\overrightarrow{BY} = \mu\overrightarrow{BB'}$, for some $\lambda, \mu \in \mathbb{R}$.

$$\text{Now} \quad \begin{aligned}\overrightarrow{AA'} &= \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB} \\ &= -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\overrightarrow{BB'} &= \overrightarrow{BO} + \frac{1}{2}\overrightarrow{OA} \\ &= -\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}\end{aligned}$$

$$\therefore \quad \overrightarrow{AY} = \lambda\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right) \qquad \therefore \quad \overrightarrow{BY} = \mu\left(-\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}\right)$$

But $\overrightarrow{BY}$ can also be obtained as follows:

$$\begin{aligned}\overrightarrow{BY} &= \overrightarrow{BA} + \overrightarrow{AY} \\ &= \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AY} \\ &= -\boldsymbol{b} + \boldsymbol{a} + \lambda\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)\end{aligned}$$

$$\therefore \quad -\mu\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{a} = (1 - \lambda)\boldsymbol{a} + \left(\frac{\lambda}{2} - 1\right)\boldsymbol{b}$$

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Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel, we now have

$$\frac{\mu}{2} = 1 - \lambda \quad (1) \qquad \text{and} \qquad -\mu = \frac{\lambda}{2} - 1 \quad (2)$$

Multiply (1) by 2 and add to (2):

$$0 = 2 - 2\lambda + \frac{\lambda}{2} - 1$$

$$1 = \frac{3\lambda}{2}$$

$$\therefore \quad \lambda = \frac{2}{3}$$

Substitute in (1) to determine $\mu = \frac{2}{3}$. We have shown that $AY : YA' = BY : YB' = 2 : 1$.

Now, by symmetry, the point of intersection of the medians AA' and OX must also divide AA' in the ratio $2 : 1$, and therefore must be Y.

Hence the three medians are concurrent at Y.

Note: The point where the three medians intersect is called the **centroid** of the triangle.

Vector proofs in three-dimensional geometry

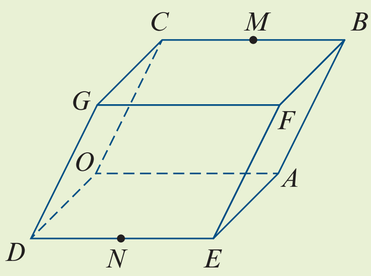
Example 44

Consider a parallelepiped $OABCDEFG$ as shown.

a Prove that the diagonals OF and CE bisect each other.

b Let M be the midpoint of CB, and let N be the midpoint of DE.

Prove that the midpoint of MN is the point where the diagonals OF and CE intersect.



Solution

Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{d} = \overrightarrow{OD}$.

a Let X be the midpoint of OF. Then

$$\begin{aligned}\overrightarrow{OX} &= \tfrac{1}{2}\overrightarrow{OF} \\ &= \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BF}) \\ &= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})\end{aligned}$$

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Let Y be the midpoint of CE. Then

$$\begin{aligned}\overrightarrow{OY} &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OE}) \\ &= \tfrac{1}{2}(\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{AE}) \\ &= \tfrac{1}{2}(\boldsymbol{c} + \boldsymbol{a} + \boldsymbol{d}) \\ &= \overrightarrow{OX}\end{aligned}$$

Therefore $X = Y$, and so the diagonals OF and CE bisect each other.

b We have

$$\overrightarrow{OM} = \overrightarrow{OC} + \tfrac{1}{2}\overrightarrow{CB} = \boldsymbol{c} + \tfrac{1}{2}\boldsymbol{a}$$

$$\overrightarrow{ON} = \overrightarrow{OD} + \tfrac{1}{2}\overrightarrow{DE} = \boldsymbol{d} + \tfrac{1}{2}\boldsymbol{a}$$

Let Z be the midpoint of MN. Then

$$\begin{aligned}\overrightarrow{OZ} &= \tfrac{1}{2}(\overrightarrow{OM} + \overrightarrow{ON}) \\ &= \tfrac{1}{2}(\boldsymbol{c} + \tfrac{1}{2}\boldsymbol{a} + \boldsymbol{d} + \tfrac{1}{2}\boldsymbol{a}) \\ &= \tfrac{1}{2}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})\end{aligned}$$

Therefore $Z = X$, where X is the point of intersection of OF and CE found in part **a**. Hence X is the midpoint of MN.

Exercise 5H

Vector proofs in two-dimensional geometry

SF

1 Prove that the diagonals of a parallelogram bisect each other.

2 Prove that if the midpoints of the sides of a rectangle are joined, then a rhombus is formed.

3 Prove that if the midpoints of the sides of a square are joined, then another square is formed.

4 Prove that the median to the base of an isosceles triangle is perpendicular to the base.

5 Prove that if the diagonals of a parallelogram are of equal length, then the parallelogram is a rectangle.

6 Prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

7 Prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.

8 Prove that if the midpoints of the sides of a quadrilateral are joined, then a parallelogram is formed.

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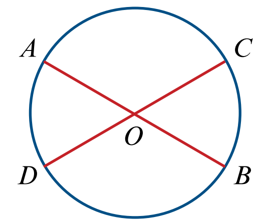
9 $ABCD$ is a parallelogram, M is the midpoint of AB and P is the point of trisection of MD nearer to M. Prove that A, P and C are collinear and that P is a point of trisection of AC.

10 $ABCD$ is a parallelogram with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. The point P lies on AD and is such that $AP : PD = 1 : 2$ and the point Q lies on BD and is such that $BQ : QD = 2 : 1$. Show that PQ is parallel to AC.

11 AB and CD are diameters of a circle with centre O. Prove that $ACBD$ is a rectangle.



12 In triangle AOB, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and M is the midpoint of AB.

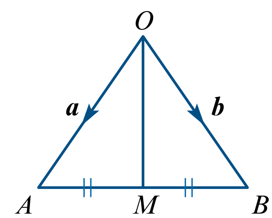
a Determine:

i $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

ii $\overrightarrow{OM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

b Determine $\overrightarrow{AM} \cdot \overrightarrow{AM} + \overrightarrow{OM} \cdot \overrightarrow{OM}$.

c Hence prove that $OA^2 + OB^2 = 2OM^2 + 2AM^2$.



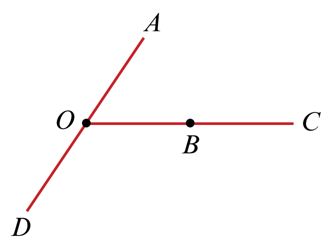
13 In the figure, O is the midpoint of AD and B is the midpoint of OC. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.

Let P be the point such that $\overrightarrow{OP} = \dfrac{1}{3}(\boldsymbol{a} + 4\boldsymbol{b})$.

a Prove that A, P and C are collinear.

b Prove that D, B and P are collinear.

c Determine $DB : BP$.



14 In triangle AOB, $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point P is on AB such that the length of AP is twice the length of BP. The point Q is such that $\overrightarrow{OQ} = 3\overrightarrow{OP}$.

a Determine each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OP}$ **ii** $\overrightarrow{OQ}$ **iii** $\overrightarrow{AQ}$

b Hence show that $\overrightarrow{AQ}$ is parallel to $\overrightarrow{OB}$.

15 $ORST$ is a parallelogram, U is the midpoint of RS and V is the midpoint of ST. Relative to the origin O, the position vectors of points R, S, T, U and V are $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.

b Express $\boldsymbol{u}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{s}$, and express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

c Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

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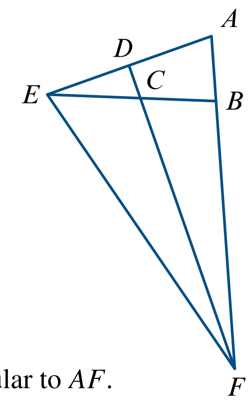
CF

16 The points A, B, C, D and E shown in the diagram have position vectors

$$\boldsymbol{a} = \hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}} \qquad \boldsymbol{b} = 2\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} \qquad \boldsymbol{c} = -\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$$
$$\boldsymbol{d} = -2\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} \qquad \boldsymbol{e} = -4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$$

respectively. The lines AB and DC intersect at F as shown.



a Show that E lies on the lines DA and BC.

b Determine $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

c Determine the position vector of the point F.

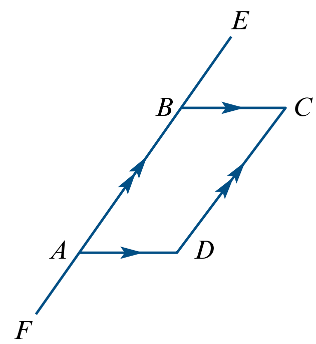
d Show that FD is perpendicular to EA and that EB is perpendicular to AF.

e Determine the position vector of the centre of the circle through E, D, B and F.

17 Coplanar points A, B, C, D and E have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ and $\boldsymbol{e}$ respectively, relative to an origin O. The point A is the midpoint of OB and the point E divides AC in the ratio $1 : 2$. If $\boldsymbol{e} = \frac{1}{3}\boldsymbol{d}$, show that $OCDB$ is a parallelogram.

18 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O. The point P divides the line segment OA in the ratio $1 : 3$ and the point R divides the line segment AB in the ratio $1 : 2$. Given that $PRBQ$ is a parallelogram, determine the position of Q.

19 $ABCD$ is a parallelogram, AB is extended to E and BA is extended to F such that $BE = AF = BC$. Line segments EC and FD are extended to meet at X.



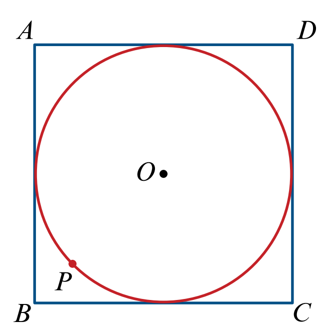
a Prove that the lines EX and FX meet at right angles.

b If $\overrightarrow{EX} = \lambda\overrightarrow{EC}$, $\overrightarrow{FX} = \mu\overrightarrow{FD}$ and $|\overrightarrow{AB}| = k|\overrightarrow{BC}|$, determine the values of λ and μ in terms of k.

c Determine the values of λ and μ if $ABCD$ is a rhombus.

d If $|\overrightarrow{EX}| = |\overrightarrow{FX}|$, prove that $ABCD$ is a rectangle.

20 In the figure, the circle has centre O and radius r. The circle is inscribed in a square $ABCD$, and P is any point on the circle.



a Show that $\overrightarrow{AP} \cdot \overrightarrow{AP} = 3r^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OA}$.

b Hence determine $AP^2 + BP^2 + CP^2 + DP^2$ in terms of r.

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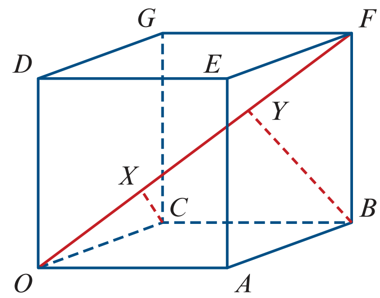
Vector proofs in three-dimensional geometry

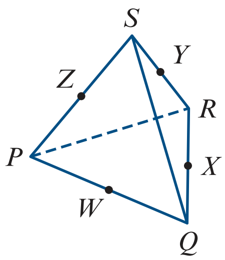
21 A 'space diagonal' of a polyhedron is a line segment connecting two vertices that are not on the same face. Prove that the space diagonals of a rectangular prism are of equal length and bisect each other.

22 Consider a rectangular prism $OABCDEFG$ as shown. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\boldsymbol{d} = \overrightarrow{OD}$. Let $a = |\boldsymbol{a}|$, $c = |\boldsymbol{c}|$ and $d = |\boldsymbol{d}|$.



- **a** Let X be the point on diagonal OF such that CX is perpendicular to OF. Determine the position vector of X in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.
- **b** Let Y be the point on diagonal OF such that BY is perpendicular to OF. Determine the position vector of Y in terms of $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.
- **c** If $a = c = d = 1$, determine:
  - **i** the position vectors of X and Y
  - **ii** the magnitude of $\angle CXA$
  - **iii** the magnitude of $\angle BYG$

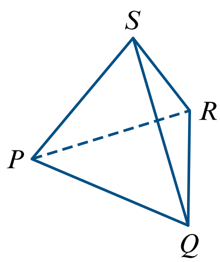
23 Let P, Q, R and S be four points in space that do not lie in the same plane. Let W, X, Y and Z be the midpoints of PQ, QR, RS and SP respectively. Relative to an origin O, denote the position vectors of points $P, Q, R, \ldots, Y, Z$ by $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \ldots, \boldsymbol{y}, \boldsymbol{z}$ respectively. Prove that $WXYZ$ is a parallelogram.



24 A tetrahedron is a polyhedron with four triangular faces. In a regular tetrahedron, each face is an equilateral triangle.

Prove that, for a regular tetrahedron, the line segments joining the midpoints of opposite edges have a common midpoint.

Note: For a tetrahedron $PQRS$, the edges PQ and RS are opposite, the edges PR and QS are opposite, and the edges PS and QR are opposite.



25 Point C is a vertex of the regular tetrahedron $OABC$. Point G is the centroid of triangle OAB. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

- **a** Determine $\overrightarrow{OG}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- **b** Prove that $\overrightarrow{CG}$ is perpendicular to $\overrightarrow{OG}$.

26 Prove that opposite edges of a regular tetrahedron are perpendicular.

27 Let $OABC$ be a tetrahedron. Assume that edge OA is perpendicular to edge BC, and that edge OB is perpendicular to edge AC. Prove that edge OC is perpendicular to edge AB.

CU

28 Let $OABC$ be a tetrahedron such that opposite edges are perpendicular. Show that

$$OA^2 + BC^2 = OB^2 + AC^2 = OC^2 + AB^2$$

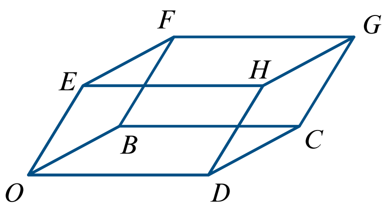
29 A regular tetrahedron $VABC$ has edges of length 4 cm.

a Let T be the point on VC such that AT is perpendicular to VC. Determine the value of λ such that $\overrightarrow{VT} = \lambda\overrightarrow{VC}$.

b Prove that BT is perpendicular to VC.

c Determine the magnitude of $\angle ATB$.

30 $OBCDEFGH$ is a parallelepiped. Let $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{d} = \overrightarrow{OD}$ and $\boldsymbol{e} = \overrightarrow{OE}$.



a Express each of the vectors $\overrightarrow{OG}$, $\overrightarrow{DF}$, $\overrightarrow{BH}$ and $\overrightarrow{CE}$ in terms of $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{e}$.

b Determine $|\overrightarrow{OG}|^2$, $|\overrightarrow{DF}|^2$, $|\overrightarrow{BH}|^2$ and $|\overrightarrow{CE}|^2$ in terms of $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{e}$.

c Show that $|\overrightarrow{OG}|^2 + |\overrightarrow{DF}|^2 + |\overrightarrow{BH}|^2 + |\overrightarrow{CE}|^2 = 4(|\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2)$.

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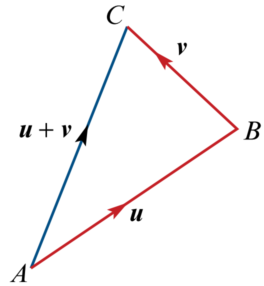
Chapter summary

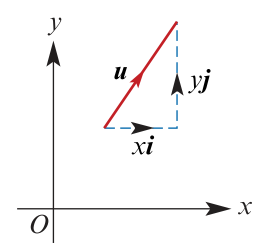
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors

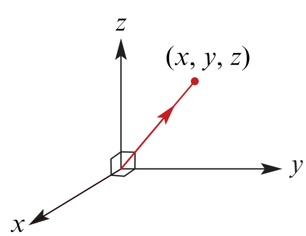


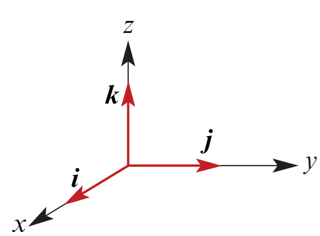
- **Addition**
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ is obtained geometrically as shown.
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
- **Scalar multiplication**
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- **Subtraction** $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Component form



- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, where
  - $\hat{\boldsymbol{i}}$ is the unit vector in the positive direction of the x-axis
  - $\hat{\boldsymbol{j}}$ is the unit vector in the positive direction of the y-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- In three dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are unit vectors as shown.
- If $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.





- If the vector $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ makes angles α, β and γ with the positive directions of the x-, y- and z-axes respectively, then

$$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, \qquad \cos\beta = \frac{a_2}{|\boldsymbol{a}|} \qquad \text{and} \qquad \cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$$

- The **unit vector** in the direction of vector $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

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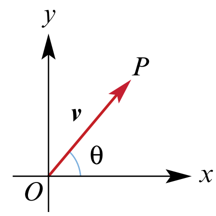
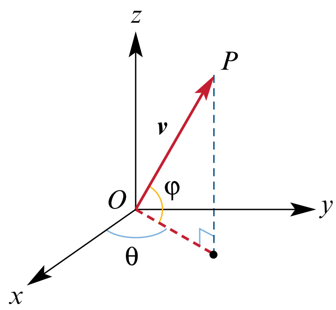
Polar form in two dimensions



- In two dimensions, each vector $\boldsymbol{v} = \overrightarrow{OP}$ can be written in polar form as $\boldsymbol{v} = [r, \theta]$.
  - The number r is the magnitude of $\boldsymbol{v}$.
  - The angle θ is measured from the positive x-direction to the vector $\boldsymbol{v}$. The range is $-180° < \theta \le 180°$, where positive angles are formed by moving towards the positive y-direction (i.e. anticlockwise).

Polar form in three dimensions



- In three dimensions, each vector $\boldsymbol{v} = \overrightarrow{OP}$ can be written in polar form as $\boldsymbol{v} = [r, \theta, \varphi]$.
  - The number r is the magnitude of $\boldsymbol{v}$.
  - The angle θ is measured from the positive x-direction to the projection of $\boldsymbol{v}$ onto the x–y plane. The range is $-180° < \theta \le 180°$, where positive angles are formed by moving towards the positive y-direction.
  - The angle φ is measured from the projection of $\boldsymbol{v}$ onto the x–y plane to the vector $\boldsymbol{v}$. The range is $-90° \le \varphi \le 90°$, where positive angles are formed by moving towards the positive z-direction.
- **Polar form to component form** If $\boldsymbol{v} = [r, \theta, \varphi]$, then $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, where

$$x = r\cos\theta\cos\varphi, \quad y = r\sin\theta\cos\varphi \quad \text{and} \quad z = r\sin\varphi$$

- **Component form to polar form** If $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$, then $\boldsymbol{v} = [r, \theta, \varphi]$, where

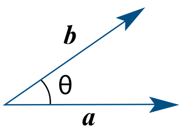
$$r = \sqrt{x^2 + y^2 + z^2}, \quad \tan\theta = \frac{y}{x} \quad \text{and} \quad \sin\varphi = \frac{z}{r}$$

- When converting to polar form, take care to choose the angle θ in the correct quadrant of the x–y plane.

Scalar product and vector projections

- The **scalar product** of vectors $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$



- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

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Displacement and velocity

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B, then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- **Motion with constant velocity** If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

Relative velocity

- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.
- If an object A is in motion relative to another object B, we can determine the velocity of A using a vector sum:

$$\boxed{\text{Velocity of } A \text{ relative to Earth}} = \boxed{\text{Velocity of } A \text{ relative to } B} + \boxed{\text{Velocity of } B \text{ relative to Earth}}$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

5A **1** I can draw a directed line segment corresponding to a vector and determine its magnitude.

See Example 1, Example 2 and Questions 1 and 2

5A **2** I can illustrate a vector sum.

See Example 3 and Question 3

5A **3** I can apply the rules of vector algebra.

See Example 4 and Question 6

5A **4** I can apply the rules of vector algebra.

See Example 4, Example 5 and Questions 6 and 12

5A **5** I can apply the rules of vector algebra to determine vectors in three dimensional space.

See Example 6, Example 7, Example 8 and Questions 14, 15 and 17

5B **6** I can work with vectors written in component form.

See Example 9, Example 10, Example 11, Example 12, Example 13, Example 14 and Questions 1, 2, 3, 4, 5 and 6

5B **7** I can determine the angle between two vectors.

See Example 15, Example 16, Example 17 and Questions 21, 22 and 28

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5C **8** I can work with vectors written in polar form. ☐

See Example 18, Example 19, Example 20, Example 21 and Questions 2, 3, 7, 8 and 10

5D **9** I can work with the properties of the scalar product of vectors. ☐

See Example 22, Example 23, Example 24, Example 25 Example 26 and Questions 1, 3, 4, 5 and 6

5E **10** I can determine the vector and scalar resolute of one vector in the direction of a second. ☐

See Example 27, Example 28, Example 29 and Questions 4, 5 and 6

5F **11** I can solve problems involving collinearity. ☐

See Example 30, Example 31 and Question 1

5G **12** Given that a particle moves from point A to point B, both with given coordinates, I can determine the displacement vector in component form. ☐

See Example 32 and Question 1

5G **13** Given that a particle starts from a given point and moves in a straight line with constant velocity, I can determine the position of the particle at a given time. ☐

See Example 33 and Question 3

5G **14** I can investigate whether two particles each travelling with constant velocity meet, if given sufficient information. ☐

See Example 34, Example 35 and Questions 7 and 8

5G **15** I can determine the resultant velocity of a particle by adding two velocity vectors. ☐

See Example 36 and Questions 10

5G **16** I can determine the relative velocity of one particle with respect to another if both particles have constant velocity. ☐

See Example 37, Example 38 and Questions 11 and 17

5G **17** I can model the motion of an aeroplane using resultant velocity. ☐

See Example 39, Example 40 and Questions 21 and 22

5H **18** I can use vectors to complete geometric proofs ☐

See Example 41, Example 42, Example 43, Example 44

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Review

Short-response questions

Technology-free short-response questions

SF

1 $ABCD$ is a parallelogram, where A, B and C have position vectors $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $4\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}}$ respectively. Determine:

a $\overrightarrow{AD}$ **b** the cosine of $\angle BAD$

2 Points A, B and C are defined by position vectors $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ respectively. Point M is on the line segment AB such that $|\overrightarrow{AM}| = |\overrightarrow{AC}|$.

a Determine:

i $\overrightarrow{AM}$ **ii** the position vector of N, the midpoint of CM

b Hence show that $\overrightarrow{AN} \perp \overrightarrow{CM}$.

3 Let $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + x\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = y\hat{\boldsymbol{i}} + z\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$. Determine:

a x such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other

b y and z such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are mutually perpendicular

4 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and let $\boldsymbol{b}$ be a vector such that the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\hat{\boldsymbol{b}}$.

a Determine the cosine of the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Determine $|\boldsymbol{b}|$ if the vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is $2\hat{\boldsymbol{a}}$.

5 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.

a Determine $\boldsymbol{c}$, the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

b Determine $\boldsymbol{d}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$.

c Hence show that $|\boldsymbol{a}|\,|\boldsymbol{d}| = |\boldsymbol{c}|^2$.

6 Points A and B have position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Point C has position vector $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + (1 + 3t)\hat{\boldsymbol{j}} + (-1 + 2t)\hat{\boldsymbol{k}}$.

a Determine in terms of t:

i $\overrightarrow{CA}$ **ii** $\overrightarrow{CB}$

b Determine the values of t for which $\angle BCA = 90°$.

7 $OABC$ is a parallelogram, where A and C have position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ respectively.

a Determine:

i $|\boldsymbol{a} - \boldsymbol{c}|$ **ii** $|\boldsymbol{a} + \boldsymbol{c}|$ **iii** $(\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{a} + \boldsymbol{c})$

b Hence determine the acute angle between the diagonals of the parallelogram.

8 Write the vector $\boldsymbol{v} = [20, -60°, 30°]$ in component form.

9 Write the vector $\boldsymbol{v} = 5\sqrt{3}\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 10\sqrt{3}\hat{\boldsymbol{k}}$ in polar form.

Review

SF

10 $OABC$ is a trapezium with $\overrightarrow{OC} = 2\overrightarrow{AB}$. If $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\overrightarrow{OC} = 6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, determine:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BC}$ **c** the cosine of $\angle BAC$.

CF

11 The position vectors of A and B, relative to an origin O, are $6\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $3\hat{\boldsymbol{i}} + p\hat{\boldsymbol{j}}$.

a Express $\overrightarrow{AO} \cdot \overrightarrow{AB}$ in terms of p.

b Determine the value of p for which $\overrightarrow{AO}$ is perpendicular to $\overrightarrow{AB}$.

c Determine the cosine of $\angle OAB$ when $p = 6$.

12 Points A, B and C have position vectors $\boldsymbol{p} + \boldsymbol{q}$, $3\boldsymbol{p} - 2\boldsymbol{q}$ and $6\boldsymbol{p} + m\boldsymbol{q}$ respectively, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are non-zero, non-parallel vectors. Determine the value of m such that the points A, B and C are collinear.

13 If $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$, $\boldsymbol{s} = \hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ and $\boldsymbol{t} = -2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, Determine the values of λ and μ such that the vector $\boldsymbol{r} + \lambda\boldsymbol{s} + \mu\boldsymbol{t}$ is parallel to the x-axis.

14 Show that the points $A(4, 3, 0)$, $B(5, 2, 3)$, $C(4, -1, 3)$ and $D(2, 1, -3)$ form a trapezium and state the ratio of the parallel sides.

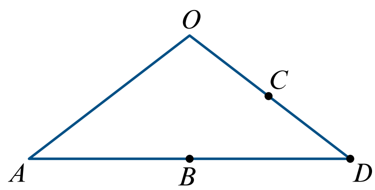
15 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, show that $\boldsymbol{a} + \boldsymbol{b}$ is perpendicular to $\boldsymbol{b}$ and determine the cosine of the angle between the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$.

16 O, A and B are the points with coordinates $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a Let C be the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$. Determine the coordinates of C.

b Let D be the point $(1, 24)$. If $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, determine the values of h and k.

17 Relative to O, the position vectors of A, B and C are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Points B and C are the midpoints of AD and OD respectively.



a Determine $\overrightarrow{OD}$ and $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

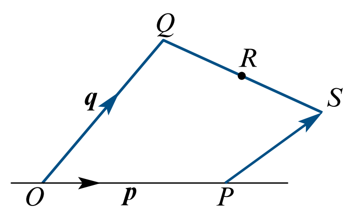
b Determine $\boldsymbol{b}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

c Point E on the extension of OA is such that $\overrightarrow{OE} = 4\overrightarrow{AE}$. If $\overrightarrow{CB} = k\overrightarrow{AE}$, determine the value of k.

18 $\overrightarrow{OP} = \boldsymbol{p}$ $\quad$ $\overrightarrow{OQ} = \boldsymbol{q}$

$\overrightarrow{OR} = \frac{1}{3}\boldsymbol{p} + k\boldsymbol{q}$ $\quad$ $\overrightarrow{OS} = h\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$

Given that R is the midpoint of QS, determine h and k.



19 ABC is a right-angled triangle with the right angle at B. If $\overrightarrow{AC} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\overrightarrow{AB}$ is parallel to $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, determine $\overrightarrow{AB}$.

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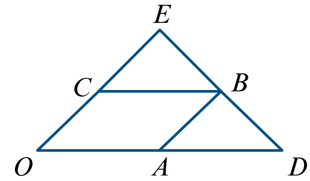
CF

20 In this diagram, $OABC$ is a parallelogram with $\overrightarrow{OA} = 2\overrightarrow{AD}$. Let $\boldsymbol{a} = \overrightarrow{AD}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express $\overrightarrow{DB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b Use a vector method to prove that $\overrightarrow{OE} = 3\overrightarrow{OC}$.

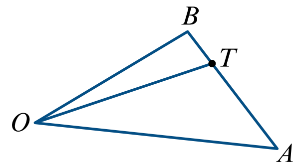
21 For a quadrilateral $OABC$, let D be the point of trisection of OC nearer O and let E be the point of trisection of AB nearer A. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Determine:

i $\overrightarrow{OD}$ **ii** $\overrightarrow{OE}$ **iii** $\overrightarrow{DE}$

b Hence prove that $3\overrightarrow{DE} = 2\overrightarrow{OA} + \overrightarrow{CB}$.

22 In triangle OAB, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and T is a point on AB such that $AT = 3TB$.



a Determine $\overrightarrow{OT}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b If M is a point such that $\overrightarrow{OM} = \lambda\overrightarrow{OT}$, where $\lambda > 1$, determine:

i $\overrightarrow{BM}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and λ **ii** λ, if $\overrightarrow{BM}$ is parallel to $\overrightarrow{OA}$.

Technology-active short-response questions

CU

23 A spider builds a web in a garden. Relative to an origin O, the position vectors of the ends A and B of a strand of the web are $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.

a **i** Determine $\overrightarrow{AB}$. **ii** Determine the length of the strand.

b A small insect is at point C, where $\overrightarrow{OC} = 2.5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 1.5\hat{\boldsymbol{k}}$. Unluckily, it flies in a straight line and hits the strand of web between A and B. Let Q be the point at which the insect hits the strand, where $\overrightarrow{AQ} = \lambda\overrightarrow{AB}$.

i Determine $\overrightarrow{CQ}$ in terms of λ.

ii If the insect hits the strand at right angles, determine the value of λ and the vector $\overrightarrow{OQ}$.

c Another strand MN of the web has endpoints M and N with position vectors $\overrightarrow{OM} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\overrightarrow{ON} = 6\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{k}}$. The spider decides to continue AB to join MN. Determine the position vector of the point of contact.

24 The position vectors of points A and B are $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

a **i** Determine $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$. **ii** Determine $\overrightarrow{AB}$.

b Let X be the midpoint of line segment AB.

i Determine $\overrightarrow{OX}$. **ii** Show that $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{AB}$.

c Determine the position vector of a point C such that $OACB$ is a parallelogram.

d Show that the diagonal OC is perpendicular to the diagonal AB by considering the scalar product $\overrightarrow{OC} \cdot \overrightarrow{AB}$.

e **i** Determine a vector of magnitude $\sqrt{195}$ that is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

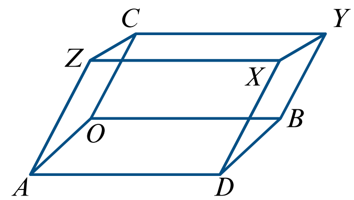
ii Show that this vector is also perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

iii Comment on the relationship between the vector found in **e i** and the parallelogram $OACB$.

25 Points A, B and C have position vectors

$$\overrightarrow{OA} = 5\hat{\boldsymbol{i}}, \quad \overrightarrow{OB} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}, \quad \text{and} \quad \overrightarrow{OC} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$$

The parallelepiped has OA, OB and OC as three edges and remaining vertices X, Y, Z and D as shown in the diagram.



CU

a Write down the position vectors of X, Y, Z and D in terms of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ and calculate the lengths of OD and OY.

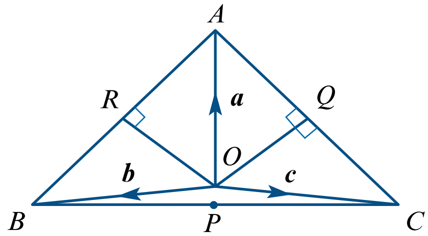
b Calculate the size of angle OZY.

c The point P divides CZ in the ratio $\lambda : 1$. That is, $CP : PZ = \lambda : 1$.

  i Give the position vector of P.

  ii Determine λ if $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CZ}$.

26 ABC is a triangle as shown in the diagram. The points P, Q and R are the midpoints of the sides BC, CA and AB respectively. Point O is the point of intersection of the perpendicular bisectors of CA and AB.
Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$	**ii** $\overrightarrow{BC}$	**iii** $\overrightarrow{CA}$
iv $\overrightarrow{OP}$	**v** $\overrightarrow{OQ}$	**vi** $\overrightarrow{OR}$

b Prove that OP is perpendicular to BC.

c Hence prove that the perpendicular bisectors of the sides of a triangle are concurrent.

d Prove that $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$.

27 The position vectors of two points B and C, relative to an origin O, are denoted by $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a In terms of $\boldsymbol{b}$ and $\boldsymbol{c}$, determine the position vector of L, the point on BC between B and C such that $BL : LC = 2 : 1$.

b Let $\boldsymbol{a}$ be the position vector of a point A such that O is the midpoint of AL. Prove that $3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c} = \boldsymbol{0}$.

c Let M be the point on CA between C and A such that $CM : MA = 3 : 2$.

  i Prove that B, O and M are collinear.

  ii Determine the ratio $BO : OM$.

d Let N be the point on AB such that C, O and N are collinear. Determine the ratio $AN : NB$.

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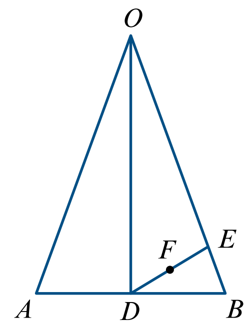
28 OAB is an isosceles triangle with $OA = OB$.
Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$.



a Let D be the midpoint of AB and let E be a point on OB.
Determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

- **i** $\overrightarrow{OD}$
- **ii** $\overrightarrow{DE}$ if $\overrightarrow{OE} = \lambda\overrightarrow{OB}$

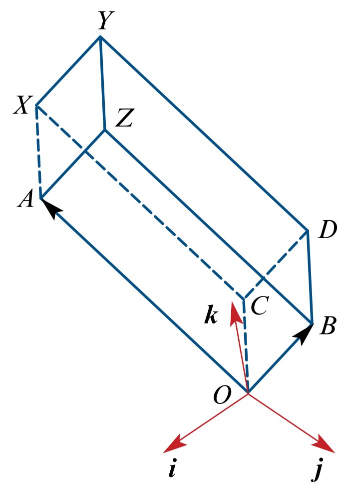
b If DE is perpendicular to OB, show that

$$\lambda = \frac{1}{2}\frac{(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b})}{\boldsymbol{b}\cdot\boldsymbol{b}}$$

c Now assume that DE is perpendicular to OB and that $\lambda = \dfrac{5}{6}$.

- **i** Show that $\cos\theta = \dfrac{2}{3}$, where θ is the magnitude of $\angle AOB$.
- **ii** Let F be the midpoint of DE. Show that OF is perpendicular to AE.

29 A cuboid is positioned on level ground so that it rests on one of its vertices, O. Vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are on the ground.



$$\overrightarrow{OA} = 3\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$$

$$\overrightarrow{OB} = 2\hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

$$\overrightarrow{OC} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

a **i** Determine $\overrightarrow{OA}\cdot\overrightarrow{OB}$ in terms of a.
ii Determine a.

b **i** Use the fact that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OC}$ to write an equation relating x and y.
ii Determine the values of x and y.

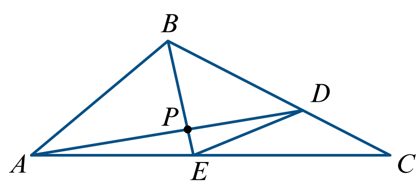
c Determine the position vectors:
i $\overrightarrow{OD}$ **ii** $\overrightarrow{OX}$ **iii** $\overrightarrow{OY}$

d State the height of points X and Y above the ground.

30 In the diagram, D is a point on BC with $\dfrac{BD}{DC} = 3$ and E is a point on AC with $\dfrac{AE}{EC} = \dfrac{3}{2}$.
Let P be the point of intersection of AD and BE. Let $\boldsymbol{a} = \overrightarrow{BA}$ and $\boldsymbol{c} = \overrightarrow{BC}$.

a Determine:

- **i** $\overrightarrow{BD}$ in terms of $\boldsymbol{c}$
- **ii** $\overrightarrow{BE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$
- **iii** $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$



b Let $\overrightarrow{BP} = \mu\overrightarrow{BE}$ and $\overrightarrow{AP} = \lambda\overrightarrow{AD}$.
Determine λ and μ.

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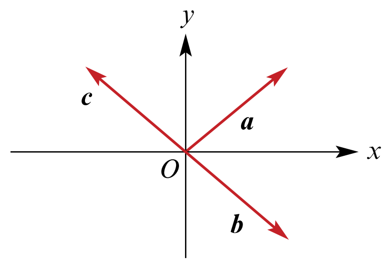
Review

CU

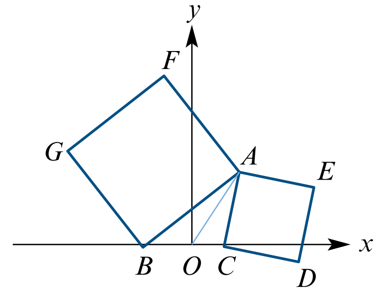
31 **a** Let $\boldsymbol{a} = p\hat{\boldsymbol{i}} + q\hat{\boldsymbol{j}}$. The vector $\boldsymbol{b}$ is obtained by rotating $\boldsymbol{a}$ clockwise through 90° about the origin. The vector $\boldsymbol{c}$ is obtained by rotating $\boldsymbol{a}$ anticlockwise through 90° about the origin. Determine $\boldsymbol{b}$ and $\boldsymbol{c}$ in terms of p, q, $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.



b In the diagram, $ABGF$ and $AEDC$ are squares with $OB = OC = 1$. Let $\overrightarrow{OA} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

i Determine $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of x, y, $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.

ii Use the results of **a** to determine $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of x, y, $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.



c **i** Prove that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{EF}$.

ii Prove that $|\overrightarrow{EF}| = 2|\overrightarrow{OA}|$.

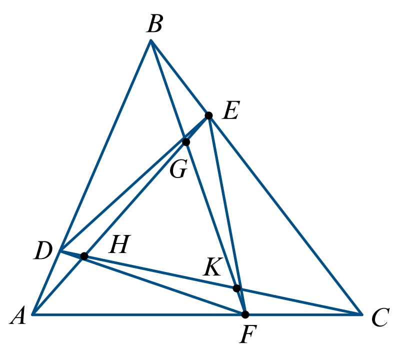
32 Triangle ABC is equilateral and $AD = BE = CF$.

a Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be unit vectors in the directions of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively. Let $\overrightarrow{AB} = m\boldsymbol{u}$ and $\overrightarrow{AD} = n\boldsymbol{u}$.

i Determine $\overrightarrow{BC}$, $\overrightarrow{BE}$, $\overrightarrow{CA}$ and $\overrightarrow{CF}$.

ii Determine $|\overrightarrow{AE}|$ and $|\overrightarrow{FB}|$ in terms of m and n.

b Show that $\overrightarrow{AE} \cdot \overrightarrow{FB} = \dfrac{1}{2}(m^2 - mn + n^2)$.

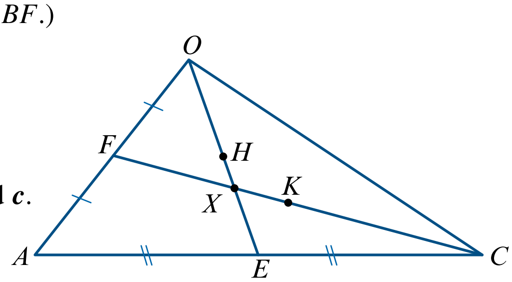


c Show that triangle GHK is equilateral.
(G is the point of intersection of BF and AE.
H is the point of intersection of AE and CD.
K is the point of intersection of CD and BF.)

33 AOC is a triangle. The medians CF and OE intersect at X.
Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Determine $\overrightarrow{CF}$ and $\overrightarrow{OE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b **i** If $\overrightarrow{OE}$ is perpendicular to $\overrightarrow{AC}$, prove that $\triangle OAC$ is isosceles.



ii If furthermore $\overrightarrow{CF}$ is perpendicular to $\overrightarrow{OA}$, determine the magnitude of angle AOC, and hence prove that $\triangle AOC$ is equilateral.

c Let H and K be the midpoints of OE and CF respectively.

i Show that $\overrightarrow{HK} = \lambda\boldsymbol{c}$ and $\overrightarrow{FE} = \mu\boldsymbol{c}$, for some $\lambda, \mu \in \mathbb{R} \setminus \{0\}$.

ii Give reasons why $\triangle HXK$ is similar to $\triangle EXF$. (Vector method not required.)

iii Hence prove that $OX : XE = 2 : 1$.

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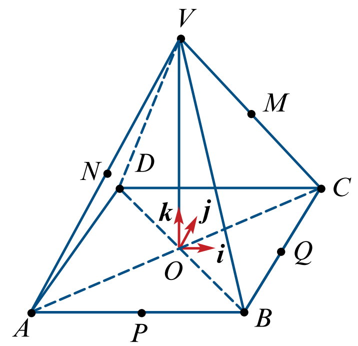
34 $VABCD$ is a square-based pyramid:



- The origin O is the centre of the base.
- The unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are in the directions of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{OV}$ respectively with 1 unit = 1 cm.
- $AB = BC = CD = DA = 4$ cm
- $OV = 2h$ cm, where h is a positive real number.
- P, Q, M and N are the midpoints of AB, BC, VC and VA respectively.

a Determine the position vectors of A, B, C and D relative to O.

b Determine vectors $\overrightarrow{PM}$ and $\overrightarrow{QN}$ in terms of h.

c Determine the position vector $\overrightarrow{OX}$, where X is the point of intersection of QN and PM.

d If OX is perpendicular to VB:

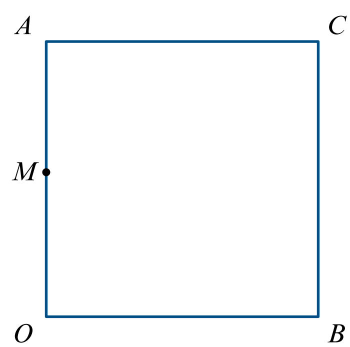
i Determine the value of h.

ii Determine the acute angle between PM and QN, correct to the nearest degree.

e **i** Prove that $NMQP$ is a rectangle.

ii Determine h if $NMQP$ is a square.

35 $OACB$ is a square with $\overrightarrow{OA} = a\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = a\hat{\boldsymbol{i}}$. Point M is the midpoint of OA.



a Determine in terms of a:

i $\overrightarrow{OM}$ **ii** $\overrightarrow{MC}$

b P is a point on MC such that $\overrightarrow{MP} = \lambda\overrightarrow{MC}$. Determine $\overrightarrow{MP}$, $\overrightarrow{BP}$ and $\overrightarrow{OP}$ in terms of λ and a.

c If BP is perpendicular to MC:

i Determine the values of λ, $|\overrightarrow{BP}|$, $|\overrightarrow{OP}|$ and $|\overrightarrow{OB}|$

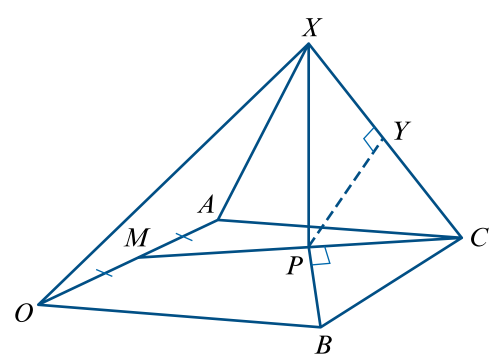
ii Evaluate $\cos\theta$, where $\theta = \angle PBO$.

d If $|\overrightarrow{OP}| = |\overrightarrow{OB}|$, determine the possible values of λ and illustrate these two cases carefully.

e In the diagram:

- $\overrightarrow{OA} = a\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = a\hat{\boldsymbol{i}}$
- M is the midpoint of OA
- BP is perpendicular to MC
- $\overrightarrow{PX} = a\hat{\boldsymbol{k}}$
- Y is a point on XC such that PY is perpendicular to XC.

Determine $\overrightarrow{OY}$.



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Review

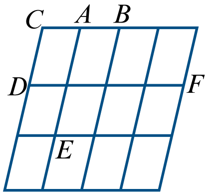
Multiple-choice questions

Technology-free multiple-choice questions

1 If $\overrightarrow{OX} = \boldsymbol{a} + 2\boldsymbol{b}$ and $\overrightarrow{XY} = \boldsymbol{a} - \boldsymbol{b}$, then $\overrightarrow{OY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to

A $\boldsymbol{b}$ **B** $3\boldsymbol{b}$ **C** $2\boldsymbol{a} + \boldsymbol{b}$ **D** $2\boldsymbol{a} + 3\boldsymbol{b}$

2 The grid shown is made up of identical parallelograms. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{c} = \overrightarrow{CD}$. Then the vector $\overrightarrow{EF}$ is equal to



A $\boldsymbol{a} + 3\boldsymbol{c}$ **B** $-3\boldsymbol{a} + \boldsymbol{c}$
C $-3\boldsymbol{a} - \boldsymbol{c}$ **D** $3\boldsymbol{a} - \boldsymbol{c}$

3 $ABCD$ is a parallelogram with $\overrightarrow{AB} = \boldsymbol{u}$ and $\overrightarrow{BC} = \boldsymbol{v}$. If M is the midpoint of AB, then the vector $\overrightarrow{DM}$ expressed in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to

A $\frac{1}{2}\boldsymbol{u} + \boldsymbol{v}$ **B** $\frac{1}{2}\boldsymbol{u} - \boldsymbol{v}$ **C** $\boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$ **D** $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$

4 If $A = (3, 6)$ and $B = (11, 1)$, then the vector $\overrightarrow{AB}$ in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ is equal to

A $3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ **B** $8\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ **C** $8\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ **D** $14\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$

5 Let OAB be a triangle such that $\overrightarrow{AO} \cdot \overrightarrow{AB} = \overrightarrow{BO} \cdot \overrightarrow{BA}$ and $|\overrightarrow{AB}| \neq |\overrightarrow{OB}|$. Then triangle OAB must be

A scalene **B** equilateral **C** isosceles **D** right-angled

6 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors such that $x(\boldsymbol{a} + \boldsymbol{b}) = 2y\boldsymbol{a} + (y + 3)\boldsymbol{b}$, then the values of x and y are

A $x = -6,\ y = -3$ **B** $x = -2,\ y = -1$
C $x = 2,\ y = 1$ **D** $x = 6,\ y = 3$

7 If A and B are points defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ respectively, then $|\overrightarrow{AB}|$ is equal to

A 29 **B** $\sqrt{29}$ **C** 11 **D** $\sqrt{21}$

8 Let $\boldsymbol{x} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{y} = -5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. The scalar resolute of $\boldsymbol{x}$ in the direction of $\boldsymbol{y}$ is

A $\dfrac{21}{\sqrt{27}}$ **B** $\dfrac{-13\sqrt{23}}{23}$ **C** $\dfrac{-13\sqrt{29}}{29}$ **D** $\dfrac{-13\sqrt{27}}{27}$

9 Let $ABCD$ be a rectangle such that $|\overrightarrow{BC}| = 3|\overrightarrow{AB}|$. If $\overrightarrow{AB} = \boldsymbol{a}$, then $|\overrightarrow{AC}|$ in terms of $|\boldsymbol{a}|$ is equal to

A $2|\boldsymbol{a}|$ **B** $\sqrt{10}\,|\boldsymbol{a}|$ **C** $4|\boldsymbol{a}|$ **D** $10|\boldsymbol{a}|$

10 If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are mutually perpendicular unit vectors then $|\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}|$ is equal to

A 1 **B** $\sqrt{2}$ **C** $\sqrt{3}$ **D** 3

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Technology-active multiple-choice questions

11 The angle between the vector $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \sqrt{2}\hat{\boldsymbol{k}}$ and $5\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ is approximately

A $0.72°$ **B** $0.77°$ **C** $43.85°$ **D** $46.15°$

12 The angle between vectors $\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $a\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ is $\frac{\pi}{4}$. The possible values of a are

A $\pm\sqrt{2}$ **B** ± 2 **C** $\pm\sqrt{1.8}$ **D** ± 1.8

13 Points $A(a, 2, 1)$, $B(-1, 0, 3)$ and $C(3, 1, -1)$ have the property that $\angle ABC$ is a right angle.
The value of a is

A $-\frac{7}{2}$ **B** $-\frac{3}{2}$ **C** $\frac{1}{2}$ **D** -2

14 A unit vector $\boldsymbol{a}$ makes an angle of $\frac{\pi}{3}$ with $\hat{\boldsymbol{i}}$ and an angle $\frac{\pi}{4}$ with $\hat{\boldsymbol{j}}$. The angle θ that $\boldsymbol{a}$ makes with $\hat{\boldsymbol{k}}$ is such that $\frac{\pi}{2} < \theta < \pi$. The value of θ is

A $\frac{5\pi}{6}$ **B** $\frac{3\pi}{4}$ **C** $\frac{2\pi}{3}$ **D** $\frac{11\pi}{12}$

15 If $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$ and $|2\boldsymbol{a} - \boldsymbol{b}| = 5$ then $|2\boldsymbol{a} + \boldsymbol{b}|$ is equal to

A 12 **B** 11 **C** 7 **D** 5

16 Let $\boldsymbol{a} = (2 + m)\hat{\boldsymbol{i}} - 3m\hat{\boldsymbol{j}} + (8 + 2m)\hat{\boldsymbol{k}}$. If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ then the value of m is

A $-\frac{2}{3}$ **B** 0 **C** $-\frac{1}{3}$ **D** $\frac{4}{3}$

17 If $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$ and $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 5$ and $|\boldsymbol{c}| = 6$ then the acute angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ to the nearest degree has magnitude

A $32°$ **B** $34°$ **C** $56°$ **D** $86°$

18 Let $\boldsymbol{a} = m\hat{\boldsymbol{i}} + 2m\hat{\boldsymbol{j}} + n\hat{\boldsymbol{k}}$ where m and n are positive real numbers and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$. If $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ and $|\boldsymbol{a}| = 16$, the values of m and n correct to two decimal places are

A $n = 1.15$ and $m = 3.07$ **B** $n = 9.56$ and $m = 5.74$
C $n = 9.10$ and $b = 4.02$ **D** $n = 8.2$ and $m = 4.3$

19 The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are such that $|\boldsymbol{a}| = 7$, $|\boldsymbol{b}| = 6$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 11$. The value of $|\boldsymbol{a} + \boldsymbol{b}|$ is

A 13 **B** 2 **C** $\sqrt{107}$ **D** $\sqrt{\frac{42}{11}}$

20 The coordinates of a point A in 3-dimensional space are $(7, 1, 13)$. The altitude angle of $\overrightarrow{OA}$ in radians correct to two decimal places is

A 0.90 **B** 1.71 **C** 1.51 **D** 1.07

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6 Vector and Cartesian equations

Chapter contents

In this chapter, we continue our study of vectors. We use them to describe curves, lines, planes and spheres in three dimensions.

We know that a line in two-dimensional space can be simply described by a Cartesian equation of the form $ax + by = c$. We will now study how to describe lines in three-dimensional space using both Cartesian and vector equations.

Chapter 6 covers Unit 3 Topic 3 Subtopic: Vector and Cartesian equations.

6A Vector functions

Learning intentions

- To be able to describe curves using vector functions.

Describing a particle's path using a vector function

Consider the vector $\boldsymbol{r} = (3+t)\hat{\boldsymbol{i}} + (1-2t)\hat{\boldsymbol{j}}$, where $t \in \mathbb{R}$.

Then $\boldsymbol{r}$ represents a family of vectors defined by different values of t.

If the variable t represents time, then $\boldsymbol{r}$ is a vector function of time. We write

$$\boldsymbol{r}(t) = (3+t)\hat{\boldsymbol{i}} + (1-2t)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$$

Further, if $\boldsymbol{r}(t)$ represents the position of a particle with respect to time, then the endpoints of the vectors $\boldsymbol{r}(t)$ will trace out the path of the particle in the Cartesian plane.

A table of values for a range of values of t is given below. These position vectors can be represented in the Cartesian plane as shown in Figure A.

t	-3	-2	-1	0	1	2	3
$\boldsymbol{r}(t)$	$7\hat{\boldsymbol{j}}$	$\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$	$2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$	$3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$	$4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$	$5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$	$6\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$

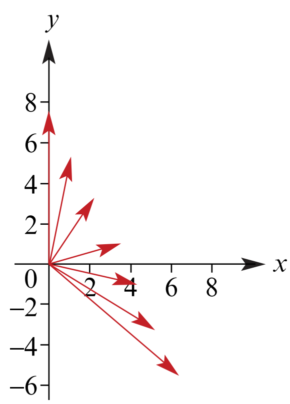

Figure A

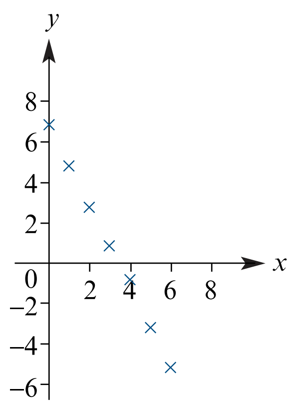

Figure B

The graph of the position vectors (Figure A) is not helpful. But when only the endpoints are plotted (Figure B), the pattern of the path is more obvious. We can determine the Cartesian equation for the path as follows.

Let (x, y) be the point on the path at time t.

Then $\boldsymbol{r}(t) = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and therefore

$$x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} = (3+t)\hat{\boldsymbol{i}} + (1-2t)\hat{\boldsymbol{j}}$$

This implies that

$$x = 3 + t \quad (1) \qquad \text{and} \qquad y = 1 - 2t \quad (2)$$

Now we eliminate the parameter t from the equations.

From (1), we have $t = x - 3$. Substituting in (2) gives $y = 1 - 2(x-3) = 7 - 2x$.

The particle's path is the straight line with equation $y = 7 - 2x$.

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Describing curves in the plane using vector functions

Now consider the Cartesian equation $y = x^2$. The graph can also be described by a vector function using a parameter t, which does not necessarily represent time.

Define the vector function $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$.

Using similar reasoning as before, if $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} = t\,\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$, then $x = t$ and $y = t^2$, so eliminating t yields $y = x^2$.

This representation is not unique. For instance, $\boldsymbol{r}(t) = t^3\hat{\boldsymbol{i}} + t^6\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$, also represents the graph with Cartesian equation $y = x^2$. Note that if these two vector functions are used to describe the motion of particles, then the paths are the same, but the particles are at different locations at a given time (with the exception of $t = 0$ and $t = 1$).

Also note that $\boldsymbol{r}(t) = t^2\hat{\boldsymbol{i}} + t^4\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$, only represents the equation $y = x^2$ for $x \ge 0$.

In the rest of this section, we consider graphs defined by vector functions, but without relating them to the motion of a particle. We view a vector function as a mapping from a subset of the real numbers into the set of all two-dimensional vectors.

Example 1

Determine the Cartesian equation for the graph represented by each vector function:

a $\boldsymbol{r}(t) = (2 - t)\hat{\boldsymbol{i}} + (3 + t^2)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

b $\boldsymbol{r}(t) = (1 - \cos t)\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

Solution

a Let (x, y) be any point on the curve.

Then $\quad x = 2 - t \qquad (1)$

and $\quad y = 3 + t^2 \qquad (2)$

Equation (1) gives $t = 2 - x$.

Substitute in (2):

$$y = 3 + (2 - x)^2$$

$$\therefore \quad y = x^2 - 4x + 7, \quad x \in \mathbb{R}$$

b Let (x, y) be any point on the curve.

Then $\quad x = 1 - \cos t \qquad (3)$

and $\quad y = \sin t \qquad (4)$

From (3): $\quad \cos t = 1 - x$.

From (4):

$$\begin{aligned} y^2 = \sin^2 t &= 1 - \cos^2 t \\ &= 1 - (1 - x)^2 \\ &= -x^2 + 2x \end{aligned}$$

The Cartesian equation is $y^2 = -x^2 + 2x$.

For a vector function $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$:

- The **domain** of the Cartesian relation is given by the range of the function $x(t)$.
- The **range** of the Cartesian relation is given by the range of the function $y(t)$.

In Example 1b, the domain of the corresponding Cartesian relation is the range of the function $x(t) = 1 - \cos t$, which is $[0, 2]$. The range of the Cartesian relation is the range of the function $y(t) = \sin t$, which is $[-1, 1]$.

Note that the Cartesian equation $y^2 = -x^2 + 2x$ can be written as $(x - 1)^2 + y^2 = 1$; it is the circle with centre $(1, 0)$ and radius 1.

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Example 2

Determine the Cartesian equation of each of the following. State the domain and range and sketch the graph of each of the relations.

a $\boldsymbol{r}(t) = \cos^2(t)\,\hat{\boldsymbol{i}} + \sin^2(t)\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$ **b** $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + (1-t)\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

Solution

a Let (x, y) be any point on the curve defined by $\boldsymbol{r}(t) = \cos^2(t)\,\hat{\boldsymbol{i}} + \sin^2(t)\,\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$. Then

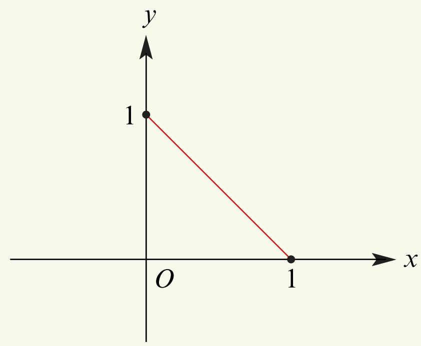
$$x = \cos^2(t) \quad \text{and} \quad y = \sin^2(t)$$

Therefore

$$y = \sin^2(t) = 1 - \cos^2(t) = 1 - x$$

Hence $y = 1 - x$.

Note that $0 \le \cos^2(t) \le 1$ and $0 \le \sin^2(t) \le 1$, for all $t \in \mathbb{R}$. The domain of the relation is $[0, 1]$ and the range is $[0, 1]$.

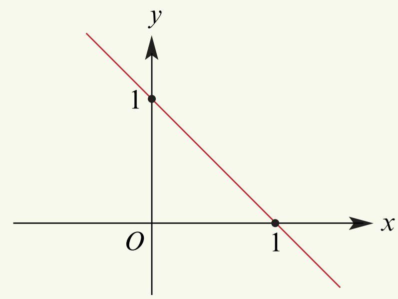


b Let (x, y) be any point on the curve defined by $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + (1-t)\,\hat{\boldsymbol{j}}$, $t \in \mathbb{R}$. Then

$$x = t \quad \text{and} \quad y = 1 - t$$

Hence $y = 1 - x$.

The domain is $\mathbb{R}$ and the range is $\mathbb{R}$.



Example 3

For each of the following, state the Cartesian equation, the domain and range of the corresponding Cartesian relation and sketch the graph:

a $\boldsymbol{r}(\lambda) = (1 - 2\cos(\lambda))\,\hat{\boldsymbol{i}} + 3\sin(\lambda)\,\hat{\boldsymbol{j}}$ **b** $\boldsymbol{r}(\lambda) = 2\sec(\lambda)\,\hat{\boldsymbol{i}} + \tan(\lambda)\,\hat{\boldsymbol{j}}$

Solution

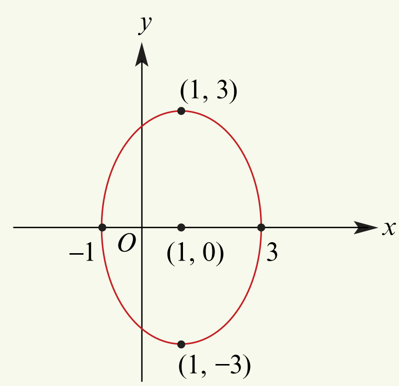
a Let $x = 1 - 2\cos(\lambda)$ and $y = 3\sin(\lambda)$. Then

$$\frac{x-1}{-2} = \cos(\lambda) \quad \text{and} \quad \frac{y}{3} = \sin(\lambda)$$

Squaring each and adding yields

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = \cos^2(\lambda) + \sin^2(\lambda) = 1$$

The graph is an ellipse with centre $(1, 0)$. The domain of the relation is $[-1, 3]$ and the range is $[-3, 3]$.



Note: The entire ellipse is obtained by taking $\lambda \in [0, 2\pi]$.

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b $\boldsymbol{r}(\lambda) = 2\sec(\lambda)\,\hat{\boldsymbol{i}} + \tan(\lambda)\,\hat{\boldsymbol{j}}$, for $\lambda \in \mathbb{R} \setminus \left\{ \dfrac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$

Let (x, y) be any point on the curve. Then

$$x = 2\sec(\lambda) \quad \text{and} \quad y = \tan(\lambda)$$

$$\therefore \quad x^2 = 4\sec^2(\lambda) \quad \text{and} \quad y^2 = \tan^2(\lambda)$$

$$\therefore \quad \frac{x^2}{4} = \sec^2(\lambda) \quad \text{and} \quad y^2 = \tan^2(\lambda)$$

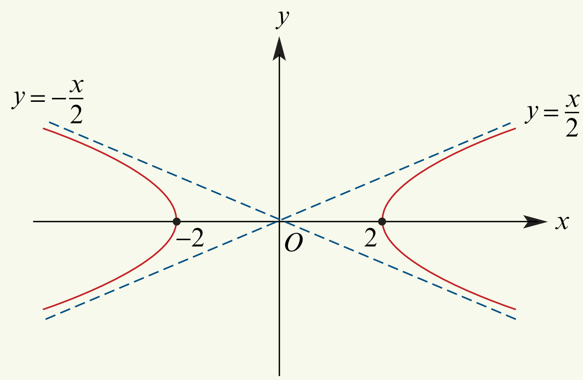
But $\sec^2(\lambda) - \tan^2(\lambda) = 1$ and therefore

$$\frac{x^2}{4} - y^2 = 1$$

The domain of the relation is the range of $x(\lambda) = 2\sec(\lambda)$, which is $(-\infty, -2] \cup [2, \infty)$. The range of the relation is the range of $y(\lambda) = \tan(\lambda)$, which is $\mathbb{R}$.

The graph is a hyperbola centred at the origin. The asymptotes have equations $y = \pm\dfrac{x}{2}$.

Note: The graph is produced for $\lambda \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.



Exercise 6A

Example 1, 2

1 For each of the following vector functions, determine the corresponding Cartesian equation, and state the domain and range of the Cartesian relation:

a $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + 2t\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

b $\boldsymbol{r}(t) = 2\hat{\boldsymbol{i}} + 5t\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

c $\boldsymbol{r}(t) = -t\,\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

d $\boldsymbol{r}(t) = (2-t)\hat{\boldsymbol{i}} + (t+7)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

e $\boldsymbol{r}(t) = t^2\hat{\boldsymbol{i}} + (2-3t)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

f $\boldsymbol{r}(t) = (t-3)\hat{\boldsymbol{i}} + (t^3+1)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

g $\boldsymbol{r}(t) = (2t+1)\hat{\boldsymbol{i}} + 3^t\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

h $\boldsymbol{r}(t) = \left(t - \dfrac{\pi}{2}\right)\hat{\boldsymbol{i}} + \cos(2t)\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

i $\boldsymbol{r}(t) = \dfrac{1}{t+4}\,\hat{\boldsymbol{i}} + (t^2+1)\hat{\boldsymbol{j}}, \quad t \neq -4$

j $\boldsymbol{r}(t) = \dfrac{1}{t}\,\hat{\boldsymbol{i}} + \dfrac{1}{t+1}\,\hat{\boldsymbol{j}}, \quad t \neq 0, -1$

Example 3

2 For each of the following vector functions, determine the corresponding Cartesian relation, state the domain and range of the relation and sketch the graph:

a $\boldsymbol{r}(t) = 2\cos(t)\,\hat{\boldsymbol{i}} + 3\sin(t)\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

b $\boldsymbol{r}(t) = 2\cos^2(t)\,\hat{\boldsymbol{i}} + 3\sin^2(t)\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

c $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + 3t^2\hat{\boldsymbol{j}}, \quad t \geq 0$

d $\boldsymbol{r}(t) = t^3\hat{\boldsymbol{i}} + 3t^2\hat{\boldsymbol{j}}, \quad t \geq 0$

e $\boldsymbol{r}(\lambda) = \cos(\lambda)\,\hat{\boldsymbol{i}} + \sin(\lambda)\,\hat{\boldsymbol{j}}, \quad \lambda \in \left[0, \dfrac{\pi}{2}\right]$

SF

f $\boldsymbol{r}(\lambda) = 3\sec(\lambda)\,\hat{\boldsymbol{i}} + 2\tan(\lambda)\,\hat{\boldsymbol{j}}, \quad \lambda \in \left(0, \frac{\pi}{2}\right)$

g $\boldsymbol{r}(t) = 4\cos(2t)\,\hat{\boldsymbol{i}} + 4\sin(2t)\,\hat{\boldsymbol{j}}, \quad t \in \left[0, \frac{\pi}{2}\right]$

h $\boldsymbol{r}(\lambda) = 3\sec^2(\lambda)\,\hat{\boldsymbol{i}} + 2\tan^2(\lambda)\,\hat{\boldsymbol{j}}, \quad \lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

i $\boldsymbol{r}(t) = (3 - t)\hat{\boldsymbol{i}} + (5t^2 + 6t)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$

SF

3 Determine a vector function which corresponds to each of the following. Note that the answers given are not the only possible answers.

a $y = 3 - 2x$ **b** $x^2 + y^2 = 4$ **c** $(x - 1)^2 + y^2 = 4$
d $x^2 - y^2 = 4$ **e** $y = (x - 3)^2 + 2(x - 3)$ **f** $2x^2 + 3y^2 = 12$

CF

4 A circle of radius 5 has its centre at the point C with position vector $2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ relative to the origin O. A general point P on the circle has position $\boldsymbol{r}$ relative to O. The angle between $\hat{\boldsymbol{i}}$ and $\overrightarrow{CP}$, measured anticlockwise from $\hat{\boldsymbol{i}}$ to $\overrightarrow{CP}$, is denoted by θ.

a Give the vector function for P. **b** Give the Cartesian equation for P.

6B Position vectors as a function of time

Learning intentions

- ▶ To be able to describe motion using vector functions.

Consider a particle travelling at a constant speed along a circular path with radius length 1 unit and centre O. The path is represented in **Cartesian form** as

$$\{\,(x, y) : x^2 + y^2 = 1\,\}$$

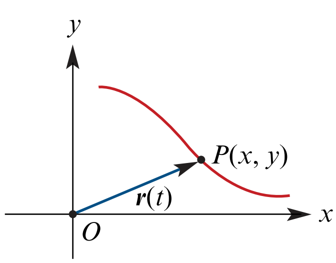
If the particle starts at the point $(1, 0)$ and travels anticlockwise, taking 2π units of time to complete one circle, then its path is represented in **parametric form** as

$$\{\,(x, y) : x = \cos t \text{ and } y = \sin t, \text{ for } t \geq 0\,\}$$

This is expressed in vector form as

$$\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}$$

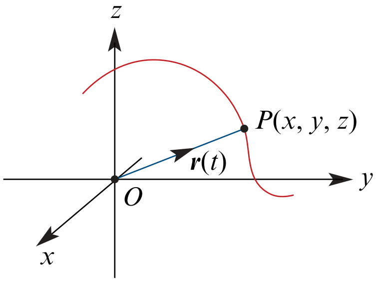
where $\boldsymbol{r}(t)$ is the position vector of the particle at time t.



The graph of a vector function is the set of points determined by the function $\boldsymbol{r}(t)$ as t varies.

In two dimensions, the x- and y-axes are used.

In three dimensions, three mutually perpendicular axes are used. We consider the x- and y-axes as in the horizontal plane and the z-axis as vertical and through the point of intersection of the x- and y-axes.



Information from the vector function

The vector function gives much more information about the motion of the particle than the Cartesian equation of its path.

For example, the vector function $\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}$, $t \geq 0$, indicates that:

- At time $t = 0$, the particle has position vector $\boldsymbol{r}(0) = \hat{\boldsymbol{i}}$. That is, the particle starts at $(1, 0)$.
- The particle moves with constant speed on the curve with equation $x^2 + y^2 = 1$.
- The particle moves in an anticlockwise direction.
- The particle moves around the circle with a period of 2π, i.e. it takes 2π units of time to complete one circle.

The vector function $\boldsymbol{r}(t) = \cos(2\pi t)\,\hat{\boldsymbol{i}} + \sin(2\pi t)\,\hat{\boldsymbol{j}}$ also describes a particle moving anticlockwise around the circle with equation $x^2 + y^2 = 1$, but this time the period is 1 unit of time.

The vector function $\boldsymbol{r}(t) = -\cos(2\pi t)\,\hat{\boldsymbol{i}} + \sin(2\pi t)\,\hat{\boldsymbol{j}}$ again describes a particle moving around the unit circle, but the particle starts at $(-1, 0)$ and moves clockwise.

Example 4

Sketch the path of a particle where the position at time t is given by

$$\boldsymbol{r}(t) = 2t\,\hat{\boldsymbol{i}} + t^2\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

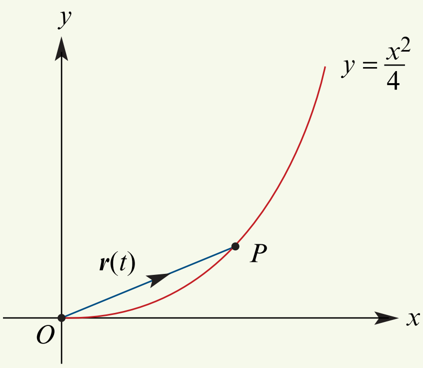
Solution

Now $x = 2t$ and $y = t^2$.

This implies $t = \dfrac{x}{2}$ and so $y = \left(\dfrac{x}{2}\right)^2$.

The Cartesian form is $y = \dfrac{x^2}{4}$, for $x \geq 0$.

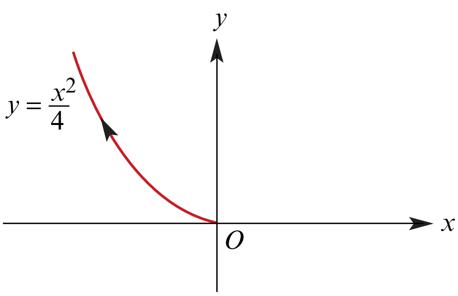
Since $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\boldsymbol{r}(1) = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, it can be seen that the particle starts at the origin and moves along the parabola $y = \dfrac{x^2}{4}$ with $x \geq 0$.



Notes:

- The equation $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + \frac{1}{4}t^2\,\hat{\boldsymbol{j}}$, $t \geq 0$, gives the same Cartesian path, but the rate at which the particle moves along the path is different.
- If $\boldsymbol{r}(t) = -t\,\hat{\boldsymbol{i}} + \frac{1}{4}t^2\,\hat{\boldsymbol{j}}$, $t \geq 0$, then again the Cartesian equation is $y = \dfrac{x^2}{4}$, but $x \leq 0$.

  Hence the motion is along the curve shown and in the direction indicated.



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- **Motion in two dimensions**
  When a particle moves along a curve in a plane, its position is specified by a vector function of the form

  $$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$$

- **Motion in three dimensions**
  When a particle moves along a curve in three-dimensional space, its position is specified by a vector function of the form

  $$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$$

Example 5

An object moves along a path where the position vector is given by

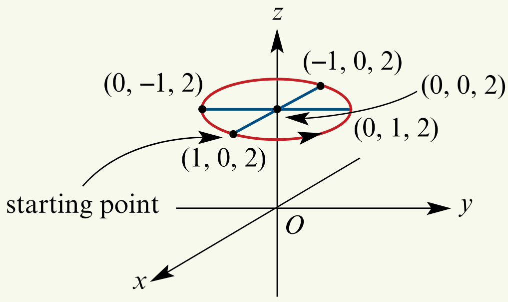
$$\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}, \quad t \geq 0$$

Describe the motion of the object.

Solution

Being unfamiliar with the graphs of relations in three dimensions, it is probably best to determine a number of position vectors (points) and try to visualise joining the dots.

t	$\boldsymbol{r}(t)$	Point
0	$\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}$	$(1, 0, 2)$
$\frac{\pi}{2}$	$\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$	$(0, 1, 2)$
π	$-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}$	$(-1, 0, 2)$
$\frac{3\pi}{2}$	$-\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$	$(0, -1, 2)$
2π	$\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}$	$(1, 0, 2)$



The object is moving along a circular path, with centre $(0, 0, 2)$ and radius length 1, starting at $(1, 0, 2)$ and moving anticlockwise when viewed from above, always at a distance of 2 above the x–y plane (horizontal plane).

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Example 6

The motion of two particles is given by the vector functions $\boldsymbol{r}_1(t) = (2t - 3)\hat{\boldsymbol{i}} + (t^2 + 10)\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(t) = (t + 2)\hat{\boldsymbol{i}} + 7t\hat{\boldsymbol{j}}$, where $t \geq 0$. Determine:

a the point at which the particles collide

b the points at which the two paths cross

c the distance between the particles when $t = 1$.

Solution

a The two particles collide when they share the same position at the same time:

$$\boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$$

$$(2t - 3)\hat{\boldsymbol{i}} + (t^2 + 10)\hat{\boldsymbol{j}} = (t + 2)\hat{\boldsymbol{i}} + 7t\hat{\boldsymbol{j}}$$

Therefore

$$2t - 3 = t + 2 \quad (1) \qquad \text{and} \qquad t^2 + 10 = 7t \quad (2)$$

From (1), we have $t = 5$.

Check in (2): $t^2 + 10 = 35 = 7t$.

The particles are at the same point when $t = 5$, i.e. they collide at the point $(7, 35)$.

b At the points where the paths cross, the two paths share common points which may occur at different times for each particle. Therefore we need to distinguish between the two time variables:

$$\boldsymbol{r}_1(t) = (2t - 3)\hat{\boldsymbol{i}} + (t^2 + 10)\hat{\boldsymbol{j}}$$

$$\boldsymbol{r}_2(s) = (s + 2)\hat{\boldsymbol{i}} + 7s\hat{\boldsymbol{j}}$$

When the paths cross:

$$2t - 3 = s + 2 \qquad (3)$$

$$t^2 + 10 = 7s \qquad (4)$$

We now solve these equations simultaneously.

Equation (3) becomes $s = 2t - 5$.

Substitute in (4):

$$t^2 + 10 = 7(2t - 5)$$

$$t^2 - 14t + 45 = 0$$

$$(t - 9)(t - 5) = 0$$

$$\therefore \quad t = 5 \text{ or } t = 9$$

The corresponding values for s are 5 and 13.

These values can be substituted back into the vector equations to obtain the points at which the paths cross, i.e. $(7, 35)$ and $(15, 91)$.

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c When $t = 1$: $\boldsymbol{r}_1(1) = -\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}$

$\boldsymbol{r}_2(1) = 3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$

The vector representing the displacement between the two particles after 1 second is

$$\boldsymbol{r}_1(1) - \boldsymbol{r}_2(1) = -4\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$$

The distance between the two particles is $\sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ units.

Exercise 6B

Example 4

CF

1 The path of a particle with respect to an origin is described as a function of time, t, by the vector equation $\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}$, $t \geq 0$.

a Determine the Cartesian equation of the path.

b Sketch the path of the particle.

c Determine the times at which the particle crosses the y-axis.

2 Repeat Question 1 for the paths described by the following vector functions:

a $\boldsymbol{r}(t) = (t^2 - 9)\hat{\boldsymbol{i}} + 8t\hat{\boldsymbol{j}}, \quad t \geq 0$

b $\boldsymbol{r}(t) = (t+1)\hat{\boldsymbol{i}} + \dfrac{1}{t+2}\hat{\boldsymbol{j}}, \quad t > -2$

c $\boldsymbol{r}(t) = \dfrac{t-1}{t+1}\hat{\boldsymbol{i}} + \dfrac{2}{t+1}\hat{\boldsymbol{j}}, \quad t > -1$

Example 6

3 The paths of two particles with respect to time t are described by the vector equations $\boldsymbol{r}_1(t) = (3t-5)\hat{\boldsymbol{i}} + (8-t^2)\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(t) = (3-t)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$, where $t \geq 0$. Determine:

a the point at which the two particles collide

b the points at which the two paths cross

c the distance between the two particles when $t = 3$.

4 Repeat Question 3 for the paths described by the vector equations $\boldsymbol{r}_1(t) = (2t^2+4)\hat{\boldsymbol{i}} + (t-2)\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(t) = 9t\,\hat{\boldsymbol{i}} + 3(t-1)\hat{\boldsymbol{j}}$, where $t \geq 0$.

5 The positions of two particles at time t, for $t \geq 0$, are given by

$$\boldsymbol{r}_1(t) = (2t+4)\hat{\boldsymbol{i}} + (t-5)\hat{\boldsymbol{j}} + (t+6)\hat{\boldsymbol{k}}$$
$$\boldsymbol{r}_2(t) = (3t-5)\hat{\boldsymbol{i}} + (2t-14)\hat{\boldsymbol{j}} + (2t-3)\hat{\boldsymbol{k}}$$

a Determine the time at which the positions of the two particles coincide.

b Determine the position vector of the particles at this time.

6 Repeat Question 5 for each of the following:

a $\boldsymbol{r}_1(t) = \cos(2\pi t)\,\hat{\boldsymbol{i}} + \sin(2\pi t)\,\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ and $\boldsymbol{r}_2(t) = -2t\,\hat{\boldsymbol{i}} + (2t-1)\hat{\boldsymbol{j}} + (2t+4)\hat{\boldsymbol{k}}$

b $\boldsymbol{r}_1(t) = t^2\hat{\boldsymbol{i}} + (t^2+1)\hat{\boldsymbol{j}} + (2t-1)\hat{\boldsymbol{k}}$ and $\boldsymbol{r}_2(t) = (3t-2)\hat{\boldsymbol{i}} + (2t+1)\hat{\boldsymbol{j}} + (3t-3)\hat{\boldsymbol{k}}$

c $\boldsymbol{r}_1(t) = (3t-2)\hat{\boldsymbol{i}} + (t^2-2t-2)\hat{\boldsymbol{j}} + (3t-3)\hat{\boldsymbol{k}}$ and $\boldsymbol{r}_2(t) = t^2\hat{\boldsymbol{i}} + (2t^2-5t)\hat{\boldsymbol{j}} + (2t-1)\hat{\boldsymbol{k}}$

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7 The path of a particle defined as a function of time t is given by the vector equation $\boldsymbol{r}(t) = (1+t)\hat{\boldsymbol{i}} + (3t+2)\hat{\boldsymbol{j}}$. Determine:

a the distance of the particle from the origin when $t = 3$

b the times at which the distance of the particle from the origin is 1 unit.

8 Let $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ be the vector equation representing the motion of a particle with respect to time t, where $t \geq 0$. Determine:

a the position, A, of the particle when $t = 3$

b the distance of the particle from the origin when $t = 3$

c the position, B, of the particle when $t = 4$

d the displacement of the particle in the fourth second in vector form.

9 Let $\boldsymbol{r}(t) = (t+1)\hat{\boldsymbol{i}} + (3-t)\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$ be the vector equation representing the motion of a particle with respect to time t, where $t \geq 0$. Determine:

a the position of the particle when $t = 2$

b the distance of the particle from the point $(4, -1, 1)$ when $t = 2$.

10 Let $\boldsymbol{r}(t) = at^2\hat{\boldsymbol{i}} + (b-t)\hat{\boldsymbol{j}}$ be the vector equation representing the motion of a particle with respect to time t. When $t = 3$, the position of the particle is $(6, 4)$. Determine a and b.

11 A particle travels in a path such that the position vector, $\boldsymbol{r}(t)$, at time t is given by $\boldsymbol{r}(t) = 3\cos(t)\,\hat{\boldsymbol{i}} + 2\sin(t)\,\hat{\boldsymbol{j}}$, $t \geq 0$.

a Express this vector function as a Cartesian relation.

b Determine the initial position of the particle.

c The positive y-axis points north and the positive x-axis points east. Determine, correct to two decimal places, the bearing of the point P, the position of the particle at $t = \dfrac{3\pi}{4}$, from:

i the origin **ii** the initial position.

12 An object moves so that the position vector at time t is given by $\boldsymbol{r}(t) = e^t\,\hat{\boldsymbol{i}} + e^{-t}\hat{\boldsymbol{j}}$, $t \geq 0$.

a Express this vector function as a Cartesian relation.

b Determine the initial position of the object.

c Sketch the graph of the path travelled by the object, indicating the direction of motion.

13 An object is moving so that its position, $\boldsymbol{r}$, at time t is given by $\boldsymbol{r}(t) = (e^t + e^{-t})\hat{\boldsymbol{i}} + (e^t - e^{-t})\hat{\boldsymbol{j}}$, $t \geq 0$.

a Determine the initial position of the object.

b Determine the position at $t = \ln 2$.

c Determine the Cartesian equation of the path.

Hint: Square the x and y components and determine their difference.

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14 An object is projected so that its position, $\boldsymbol{r}$, at time t is given by
$\boldsymbol{r}(t) = 100t\,\hat{\boldsymbol{i}} + (100\sqrt{3}t - 5t^2)\hat{\boldsymbol{j}}$, for $0 \le t \le 20\sqrt{3}$.

a Determine the initial and final positions of the object.

b Determine the Cartesian form of the path.

c Sketch the graph of the path, indicating the direction of motion.

15 Two particles A and B have position vectors $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$ respectively at time t, given by $\boldsymbol{r}_A(t) = 6t^2\hat{\boldsymbol{i}} + (2t^3 - 18t)\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_B(t) = (13t - 6)\hat{\boldsymbol{i}} + (3t^2 - 27)\hat{\boldsymbol{j}}$, where $t \ge 0$. Determine where and when the particles collide.

Example 5 **16** The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = 3\cos t\,\hat{\boldsymbol{i}} + 3\sin t\,\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $t \ge 0$. Describe the motion of the particle.

17 The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}} + t\hat{\boldsymbol{k}}$, $t \ge 0$. Describe the motion of the particle.

18 The motion of a particle is described by the vector equation
$\boldsymbol{r}(t) = (1 - 2\cos(2t))\,\hat{\boldsymbol{i}} + (3 - 5\sin(2t))\,\hat{\boldsymbol{j}}$, for $t \ge 0$. Determine:

a the Cartesian equation of the path

b the position at:

i $t = 0$ **ii** $t = \dfrac{\pi}{4}$ **iii** $t = \dfrac{\pi}{2}$

c the time taken by the particle to return to its initial position

d the direction of motion along the curve.

19 For each of the following vector equations:

i Determine the Cartesian equation of the body's path

ii sketch the path

iii describe the motion of the body.

a $\boldsymbol{r}(t) = \cos^2(3\pi t)\,\hat{\boldsymbol{i}} + 2\cos^2(3\pi t)\,\hat{\boldsymbol{j}}$, $t \ge 0$

b $\boldsymbol{r}(t) = \cos(2\pi t)\,\hat{\boldsymbol{i}} + \cos(4\pi t)\,\hat{\boldsymbol{j}}$, $t \ge 0$

c $\boldsymbol{r}(t) = e^t\,\hat{\boldsymbol{i}} + e^{-2t}\hat{\boldsymbol{j}}$, $t \ge 0$

20 Particle A moves along a parabolic path with its position at time t given by

$$\boldsymbol{r}_A(t) = 2t\,\hat{\boldsymbol{i}} + (4t^2 - 8t)\hat{\boldsymbol{j}}, \quad t \ge 0$$

Particle B moves along a straight-line path with its position at time t given by

$$\boldsymbol{r}_B(t) = \frac{6t - 7}{2}\,\hat{\boldsymbol{i}} + (12t - 21)\hat{\boldsymbol{j}}, \quad t \ge 0$$

a Determine the Cartesian equations of the paths of A and B.

b Determine the coordinates of the points of intersection of the paths of A and B.

c Determine when and where particles A and B collide.

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6C Vector equations of lines

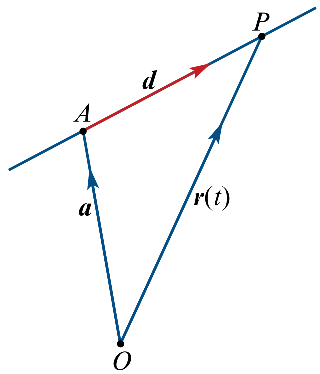
Learning intentions

- To be able to use vector, parametric and Cartesian equations of straight lines.
- To be able to calculate the distance from a point to a line.

Vector equation of a line given by a point and a direction

A line ℓ in two- or three-dimensional space may be described using two vectors:

- the position vector $\boldsymbol{a}$ of a point A on the line
- a vector $\boldsymbol{d}$ parallel to the line.



We can describe the line as

$$\ell = \{\, P : \overrightarrow{OP} = \boldsymbol{a} + t\boldsymbol{d} \text{ for some } t \in \mathbb{R} \,\}$$

Usually we omit the set notation. We write $\boldsymbol{r}(t)$ for the position vector of a point P on the line, and therefore

$$\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$$

This is a **vector equation** of the line ℓ.

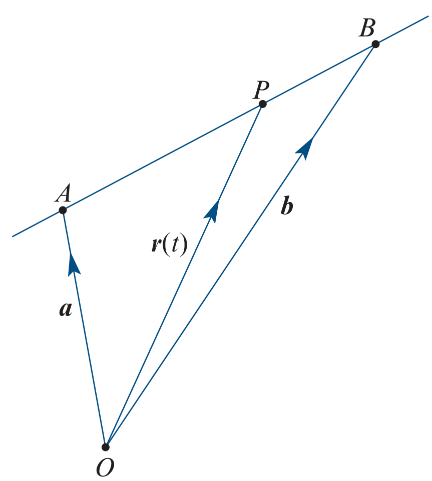
As the value of t varies over the real numbers, the position vector $\boldsymbol{r}(t)$ varies over all the points on the line ℓ. We sometimes express this idea by saying that t is a **parameter** and that $\boldsymbol{r}(t)$ is a **parameterisation** of the line ℓ.

If it is understood that t is the parameter, then we may write $\boldsymbol{r}$ instead of $\boldsymbol{r}(t)$.

Note: There is no unique vector equation of a given line. We can choose any point A as the 'starting point' on the line and any vector $\boldsymbol{d}$ parallel to the line and any multiple of the direction vector can be used.

Vector equation of a line given by two points

If the position vectors $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$ of two points on a line ℓ are known, then the line may be described by



$$\begin{aligned} \boldsymbol{r}(t) &= \overrightarrow{OA} + t\overrightarrow{AB} \\ &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}), \quad t \in \mathbb{R} \end{aligned}$$

This is also a vector equation of the line ℓ.

This vector equation can be rewritten as

$$\boldsymbol{r}(t) = (1 - t)\boldsymbol{a} + t\boldsymbol{b}, \quad t \in \mathbb{R}$$

In Section 5F, we derived this expression for the position vector of a point collinear with A and B.

Note: As already noted above, there is no unique vector equation of a given line. Here we can choose any two distinct points A and B on the line.

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Example 7

Verify that the point $P(-7, 4, -14)$ lies on the line represented by the vector equation

$$\boldsymbol{r}(t) = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}), \quad t \in \mathbb{R}$$

Solution

The point $P(-7, 4, -14)$ has position vector $-7\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 14\hat{\boldsymbol{k}}$.

By equating coefficients of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$, we can see that the point P lies on the line if there exists $t \in \mathbb{R}$ such that

$$5 + 2t = -7$$

$$-2 - t = 4$$

$$4 + 3t = -14$$

A solution for each of these equations is $t = -6$. Hence P lies on the line.

Example 8

Determine a vector equation of the line AB, where the points A and B have position vectors

$$\overrightarrow{OA} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} \quad \text{and} \quad \overrightarrow{OB} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$$

Solution

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the position vectors of points A and B respectively. Then a vector equation of the line is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \\ &= \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t((2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) - (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})) \\ &= \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}), \quad t \in \mathbb{R}\end{aligned}$$

Note: This can also be written as $\boldsymbol{r} = (1 + t)\hat{\boldsymbol{i}} + (1 - 2t)\hat{\boldsymbol{j}} + (-2 + t)\hat{\boldsymbol{k}}$, $t \in \mathbb{R}$.

Example 9

Determine a vector equation for each of the following lines:

a the line through $A(1, 2)$ that is parallel to $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

b the line passing through the points $A(3, -5, 4)$ and $B(-4, 3, 10)$

Solution

a Point A has position vector $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$. So a vector equation of the line is

$$\boldsymbol{r}(t) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}), \quad t \in \mathbb{R}$$

b The points A and B have position vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$ respectively. So a vector equation of the line is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \\ &= 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t((-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}) - (3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}})) \\ &= 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t(-7\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}), \quad t \in \mathbb{R}\end{aligned}$$

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Cartesian equation of a line in two dimensions

From a vector equation to the Cartesian equation

- For example, start with the vector equation

$$\boldsymbol{r} = \hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}), \quad t \in \mathbb{R}$$

- Rearrange this equation as

$$\boldsymbol{r} = (1 + t)\hat{\boldsymbol{i}} + (5 + 2t)\hat{\boldsymbol{j}}$$

Let $P(x, y)$ be the point on the line with position vector $\boldsymbol{r}$, so that $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$. Then, we have

$$x = 1 + t \quad \text{and} \quad y = 5 + 2t$$

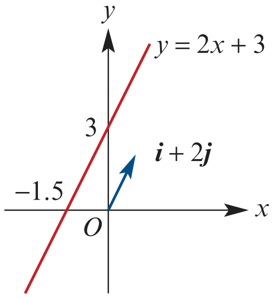
These are parametric equations for the line.

- Now eliminate t to determine y in terms of x.
We have $t = x - 1$, so $y = 5 + 2(x - 1) = 2x + 3$.
The Cartesian equation of the line is $y = 2x + 3$.

From the Cartesian equation to a vector equation

For example, start with the Cartesian equation $y = 2x + 3$.
A point on the line is $(0, 3)$, with position vector $3\hat{\boldsymbol{j}}$. The line has gradient 2, so a vector parallel to the line is $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$.
Therefore a vector equation of the line is

$$\boldsymbol{r} = 3\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}), \quad t \in \mathbb{R}$$



Note: For a line with equation $y = mx + c$, you can choose the point $(0, c)$ on the line and the vector $\hat{\boldsymbol{i}} + m\hat{\boldsymbol{j}}$ parallel to the line.

Cartesian form for a line in three dimensions

From a vector equation to Cartesian form

- For example, the line through the point $(5, -2, 4)$ that is parallel to the vector $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ can be described by the vector equation

$$\boldsymbol{r} = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}), \quad t \in \mathbb{R}$$

- Let $P(x, y, z)$ be the point on the line with position vector $\boldsymbol{r}$. Then we can write the vector equation as

$$x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}} = (5 + 2t)\hat{\boldsymbol{i}} + (-2 - t)\hat{\boldsymbol{j}} + (4 + 3t)\hat{\boldsymbol{k}}$$

The corresponding parametric equations are

$$x = 5 + 2t, \quad y = -2 - t \quad \text{and} \quad z = 4 + 3t$$

- Solving each of these equations for t and then equating gives

$$\frac{x - 5}{2} = \frac{y + 2}{-1} = \frac{z - 4}{3} = t$$

This is in **Cartesian form**. You cannot describe a line in three dimensions using a single linear Cartesian equation.

From Cartesian form to a vector equation To convert from Cartesian form to a vector equation, we can perform these steps in the reverse order. See Example 11.

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We have seen that a straight line can be described by a vector equation, by parametric equations or in Cartesian form.

Lines in three dimensions

A line in three-dimensional space can be described in the following three ways, where $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ is the position vector of a point A on the line, and $\boldsymbol{d} = d_1\hat{\boldsymbol{i}} + d_2\hat{\boldsymbol{j}} + d_3\hat{\boldsymbol{k}}$ is a vector parallel to the line.

Vector equation	Parametric equations	Cartesian form
$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$	$x = a_1 + d_1t$ $y = a_2 + d_2t$ $z = a_3 + d_3t$	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Parallel and perpendicular lines

For two lines $\ell_1 : \boldsymbol{r}_1 = \boldsymbol{a}_1 + t\boldsymbol{d}_1, t \in \mathbb{R}$, and $\ell_2 : \boldsymbol{r}_2 = \boldsymbol{a}_2 + s\boldsymbol{d}_2, s \in \mathbb{R}$:

- The lines ℓ_1 and ℓ_2 are parallel if and only if $\boldsymbol{d}_1$ is parallel to $\boldsymbol{d}_2$.
- The lines ℓ_1 and ℓ_2 are perpendicular if and only if $\boldsymbol{d}_1$ is perpendicular to $\boldsymbol{d}_2$.

Example 10

Let ℓ be the line with vector equation

$$\boldsymbol{r} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}), \quad t \in \mathbb{R}$$

a Determine a vector equation of the line through $A(1, 3, 2)$ that is parallel to the line ℓ.

b Determine a vector equation of the line through $A(1, 3, 2)$ that is perpendicular to the line ℓ and parallel to the x–y plane.

Solution

a The position vector of A is $\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, and a vector parallel to ℓ is $-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$.

Therefore a vector equation of the line through A parallel to ℓ is

$$\boldsymbol{r} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + s(-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}), \quad s \in \mathbb{R}$$

b If a vector is parallel to the x–y plane, then its $\hat{\boldsymbol{k}}$-component is zero. So we want to determine a vector $\boldsymbol{d} = d_1\hat{\boldsymbol{i}} + d_2\hat{\boldsymbol{j}}$ that is perpendicular to $-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$.

Therefore we require

$$(d_1\hat{\boldsymbol{i}} + d_2\hat{\boldsymbol{j}}) \cdot (-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}) = 0$$

i.e. $$-d_1 - 3d_2 = 0$$

We see that we can choose $d_1 = 3$ and $d_2 = -1$. So $\boldsymbol{d} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

Hence a vector equation of the required line is

$$\boldsymbol{r} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + s(3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}), \quad s \in \mathbb{R}$$

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Distance from a point to a line

We can use the scalar product to determine the distance from a point to a line.

Example 11

a Determine a vector equation for the line whose Cartesian equation is

$$1 - x = \frac{2 - y}{3}, z = 2$$

b Determine the distance to the line from the point $A(1, 3, 2)$.

Solution

Let $t = 1 - x = \dfrac{2 - y}{3}$ and $z = 2$ so that

$x = 1 - t$

$y = 2 - 3t$

$z = 2$

Therefore,

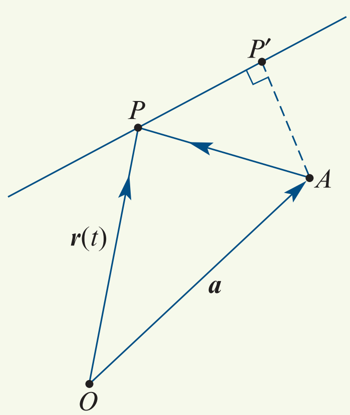
$$\boldsymbol{r}(t) = (1 - t)\hat{\boldsymbol{i}} + (2 - 3t)\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

The equation of the line can be written as

$$\boldsymbol{r}(t) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + t(-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}), \quad t \in \mathbb{R}$$

So the vector $\boldsymbol{d} = -\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ is parallel to the line.

a The required distance is $|\overrightarrow{AP'}|$, where P' is the point on the line such that AP' is perpendicular to the line.



For any point P on the line with $\overrightarrow{OP} = \boldsymbol{r}(t)$, we have

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{AO} + \overrightarrow{OP} \\ &= -(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) + (1 - t)\hat{\boldsymbol{i}} + (2 - 3t)\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} \\ &= -t\hat{\boldsymbol{i}} + (-1 - 3t)\hat{\boldsymbol{j}}\end{aligned}$$

$$\begin{aligned}\therefore \quad \overrightarrow{AP} \cdot \boldsymbol{d} &= (-t\hat{\boldsymbol{i}} + (-1 - 3t)\hat{\boldsymbol{j}}) \cdot (-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}) \\ &= t - 3(-1 - 3t) \\ &= 10t + 3\end{aligned}$$

If $\overrightarrow{AP'} \cdot \boldsymbol{d} = 0$, then $t = -\dfrac{3}{10}$ and so $\overrightarrow{AP'} = \dfrac{3}{10}\hat{\boldsymbol{i}} - \dfrac{1}{10}\hat{\boldsymbol{j}}$.

The distance from the point A to the line is $|\overrightarrow{AP'}| = \dfrac{\sqrt{10}}{10}$.

Describing line segments

We can use a vector equation to describe a line segment by restricting the values of the parameter. Consider a vector equation $\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{d}$ with parameter t.

- As the value of t varies over $\mathbb{R}$, the position vector $\boldsymbol{r}(t)$ varies over all the points on a line.
- If the value of t only varies over an interval $[p, q]$, then the position vector $\boldsymbol{r}(t)$ only varies over the points on the line segment between $\boldsymbol{r}(p)$ and $\boldsymbol{r}(q)$.

Example 12

Points A and B have position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}}$ respectively.

a Show that the vector equation $\boldsymbol{r}(t) = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$, $t \in \mathbb{R}$, represents the line through A and B.

b Determine the set of values of t which, together with this vector equation, describes the line segment AB.

c Determine the set of values of t which, together with this vector equation, describes the line segment AC, where $C(4, 8, -9)$ is a point on the line AB.

Solution

a An equation of the line AB is

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})\\ &= \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}), \quad t \in \mathbb{R}\end{aligned}$$

b Taking $t = 0$ gives $\boldsymbol{r}(0) = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} = \boldsymbol{a}$.

To determine the value of t which gives $\boldsymbol{b}$, consider

$$\begin{aligned}\boldsymbol{r}(t) &= \boldsymbol{b}\\ \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) &= 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}}\\ (1 + t)\hat{\boldsymbol{i}} + 4(t - 1)\hat{\boldsymbol{j}} - 3t\hat{\boldsymbol{k}} &= 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}}\end{aligned}$$

Therefore $t = 1$.

So the line segment AB is described by

$$\boldsymbol{r}(t) = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}), \quad t \in [0, 1]$$

c To determine the value of t which gives $\overrightarrow{OC}$, consider

$$\begin{aligned}\boldsymbol{r}(t) &= \overrightarrow{OC}\\ \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) &= 4\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} - 9\hat{\boldsymbol{k}}\\ (1 + t)\hat{\boldsymbol{i}} + 4(t - 1)\hat{\boldsymbol{j}} - 3t\hat{\boldsymbol{k}} &= 4\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} - 9\hat{\boldsymbol{k}}\end{aligned}$$

Therefore $t = 3$.

So the line segment AC is described by

$$\boldsymbol{r}(t) = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}), \quad t \in [0, 3]$$

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Exercise 6C

SF

Example 7

1 For each of the following, determine whether the point lies on the line:

a $(4, 2, 1)$, $\quad \boldsymbol{r}(t) = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

b $(3, -3, -4)$, $\quad \boldsymbol{r}(t) = 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

c $(3, -1, -1)$, $\quad \boldsymbol{r}(t) = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + t(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

Example 8

2 For each of the following, determine a vector equation of the line through the points A and B:

a $\overrightarrow{OA} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\quad \overrightarrow{OB} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

b $\overrightarrow{OA} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}}$, $\quad \overrightarrow{OB} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

c $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\quad \overrightarrow{OB} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

d $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $\quad \overrightarrow{OB} = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

Example 9

3 For each of the following, determine a vector equation of the line that passes through the points A and B:

a $A(3, 1)$, $\quad B(-2, 2)$

b $A(-1, 5)$, $\quad B(2, -1)$

c $A(1, 2, 3)$, $\quad B(2, 0, -1)$

d $A(1, -4, 0)$, $\quad B(2, 3, 1)$

4 Convert each vector equation found in Question 3 into:

i parametric equations

ii Cartesian form.

5 Consider the line with equation $2x + 3y = 12$.

a Show that the point $(3, 2)$ lies on the line and that the vector $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ is parallel to the line. Hence give a vector equation for the line.

b Show that the point $(0, 4)$ lies on the line and that the vector $-9\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ is parallel to the line. Hence give another vector equation for the line.

c Show that the point (6,0) lies on the line. Using this information and the fact that (0,4) also lies on the line, give yet another vector equation for the line.

Example 10

6 Determine a vector equation of the line through the point $A(2, 1, 0)$ that is:

a parallel to the line $\boldsymbol{r} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}),\ t \in \mathbb{R}$

b perpendicular to the line $\boldsymbol{r} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}),\ t \in \mathbb{R}$, and parallel to the x–y plane.

7 Determine a vector equation of the line through the origin that is:

a parallel to the vector $2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

b perpendicular to the line $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}),\ t \in \mathbb{R}$, and in the y–z plane.

8 **a** Determine a vector equation of the line AB, where points A and B are defined by the position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ respectively.

b Determine which of the following points are on this line:

i $(5, 0)$ $\quad$ **ii** $(0, 7)$ $\quad$ **iii** $(8, -3)$

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CF

9 The line ℓ is given by the vector equation $\boldsymbol{r} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$, $t \in \mathbb{R}$.

a Determine a vector equation of the line which passes through the point $(0, 1, 1)$ and is parallel to the line ℓ.

b Verify that the two equations do not represent the same line ℓ.

c The point $(2, m, n)$ lies on the line ℓ. Determine the values of m and n.

10 **a** Let $\boldsymbol{v} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. Determine a vector that is perpendicular to the vector $\boldsymbol{v}$ and has the same magnitude as $\boldsymbol{v}$.

b Points A and B are given by the position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ respectively. Determine a vector equation of the line which passes through B and is perpendicular to $\overrightarrow{BA}$.

c Determine the x- and y-axis intercepts of this line.

11 Determine parametric equations and Cartesian equations for each line:

a $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

b $\boldsymbol{r} = 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

Example 11

12 **a** Determine a vector equation for the line whose Cartesian equation is

$$\frac{x-1}{2} = \frac{2-y}{3} = \frac{z}{5}$$

b Determine the distance to the line from the point $A(-1, 3, 1)$.

13 For each of the following, determine the distance from the point to the line:

a $(0, 0, 0)$, $\boldsymbol{r} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

b $(1, 10, -2)$, $\boldsymbol{r} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

c $(1, 2, 3)$, $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

d $(1, 1, 4)$, $\boldsymbol{r} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$

Example 12

14 Points A, B and C are defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ respectively.

a Show that the vector equation $\boldsymbol{r} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$, $t \in \mathbb{R}$, represents the line through the points A and B.

b Show that the point C is also on this line.

c Determine the set of values of t which, together with the vector equation, describe the line segment BC.

15 Determine the coordinates of the closest point to $(2, 1, 3)$ on the line given by the equation $\boldsymbol{r} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, $t \in \mathbb{R}$.

16 Determine a vector equation to represent the line through the point $(-2, 2, 1)$ that is parallel to the x-axis.

17 Determine the distance from the origin to the line that passes through the point $(3, 1, 5)$ and is parallel to the vector $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

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CF

18 For each of the following, give the coordinates of the endpoints of the line segment described by the vector equation:

a $\boldsymbol{r} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}),\ t \in [1, 3]$

b $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}),\ t \in [-1, 2]$

19 Let ℓ be the line with vector equation $\boldsymbol{r}(t) = (3-t)\hat{\boldsymbol{i}} + (3-t)\hat{\boldsymbol{j}} + t\hat{\boldsymbol{k}}$.

a Determine the point on ℓ closest to the origin. (**Hint:** determine the point P' on ℓ such that OP' is perpendicular to ℓ.)

b Let P be a point on ℓ with position vector $\boldsymbol{r}(t)$. Show that $|\overrightarrow{OP}|^2 = 3t^2 - 12t + 18$.

c For which value of t is the quadratic function $f(t) = 3t^2 - 12t + 18$ minimised?

d Use parts **b** and **c** to determine the point on ℓ closest to the origin by another method.

20 A line is given by the vector equation $\boldsymbol{r} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}),\ t \in \mathbb{R}$.

a Determine the vector $\overrightarrow{OB}$ in terms of t, where B is a point on the line.

b Determine $|\overrightarrow{OB}|$ in terms of t.

c Hence determine the minimum value of $|\overrightarrow{OB}|$. That is, determine the shortest distance from the origin to a point on the line.

d Let A be the point $(1, 3, 2)$. Determine the shortest distance from A to a point on the line.

6D Intersection of lines and skew lines

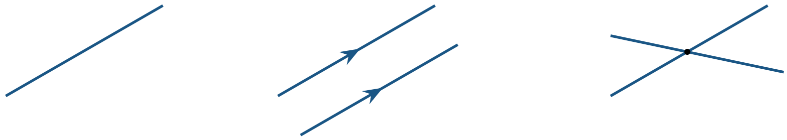
Learning intentions

- To be able to determine if a pair of lines in three dimensions are parallel, intersect, coincide or are skew and to determine the angle between two lines.
- To be able to determine the coordinates of a point of intersection of lines.

Lines in two-dimensional space

From Mathematical Methods Units 1 & 2, you know that there are three possibilities for a pair of lines in two-dimensional space:

- the lines coincide
- the lines are parallel and distinct
- the lines intersect at a point.



For example, the two lines

$$\ell_1 : \boldsymbol{r}_1(\lambda) = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}),\ \lambda \in \mathbb{R} \quad \text{and} \quad \ell_2 : \boldsymbol{r}_2(\mu) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \mu(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}),\ \mu \in \mathbb{R}$$

are parallel, since the direction vectors $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ are parallel. To check whether two parallel lines coincide, we choose a point on one line and check whether it also lies on the other line. For example, the point with position vector $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ lies on line ℓ_2. This point lies on ℓ_1 if there is a value of λ such that $2 + \lambda = 2$ and $2 - \lambda = 3$.

No such λ exists, so the lines ℓ_1 and ℓ_2 are parallel and distinct.

Note: When we are considering a pair of lines, we should use different parameters for the two vector equations. (Here we used λ and μ.)

Example 13

Determine the position vector of the point of intersection of the lines

$$\boldsymbol{r}_1(\lambda) = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}),\ \lambda \in \mathbb{R} \quad \text{and} \quad \boldsymbol{r}_2(\mu) = -\hat{\boldsymbol{j}} + \mu(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}),\ \mu \in \mathbb{R}$$

Solution

At the point of intersection, we have $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$ and so

$$2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}) = -\hat{\boldsymbol{j}} + \mu(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$$

$$\therefore \quad (2 + \lambda)\hat{\boldsymbol{i}} + (2 - \lambda)\hat{\boldsymbol{j}} = 3\mu\hat{\boldsymbol{i}} + (-1 + 2\mu)\hat{\boldsymbol{j}}$$

Equate coefficients of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$:

$$2 + \lambda = 3\mu \qquad (1)$$

$$2 - \lambda = -1 + 2\mu \qquad (2)$$

Solve simultaneously by adding (1) and (2):

$$4 = -1 + 5\mu$$

Hence $\mu = 1$ and so $\lambda = 1$.

Substituting $\lambda = 1$ into the equation $\boldsymbol{r}_1(\lambda) = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$ gives $\boldsymbol{r}_1(1) = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

The point of intersection has position vector $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

Lines in three-dimensional space

There are four possibilities for a pair of lines in three-dimensional space: the lines may coincide, they may be parallel and distinct, they may intersect at a point and they may also be **skew**.

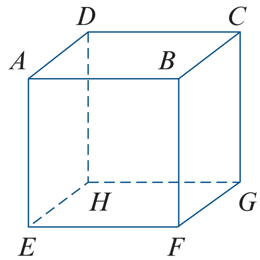
Skew lines

Two lines are **skew lines** if they do not intersect and are not parallel.

Two lines are skew if and only if they do not lie in the same plane.

For example, consider the cube $ABCDEFGH$ as shown.

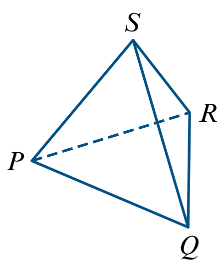
Lines AB and FG are skew. We can see that AB and FG do not lie in the same plane.



As another example, consider the tetrahedron $SPQR$ as shown.

We can see three pairs of skew lines:

- lines SP and RQ are skew
- lines SR and PQ are skew
- lines SQ and PR are skew.



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Coincident and parallel lines

We can determine whether two lines in three dimensions are coincident or parallel by similar methods as in two dimensions.

Consider two lines $\ell_1 : \boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\ell_2 : \boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$.

- Lines ℓ_1 and ℓ_2 are parallel if and only if the direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are parallel (i.e. $\boldsymbol{d}_1 = m\boldsymbol{d}_2$ for some real number m).
- If lines ℓ_1 and ℓ_2 are parallel, then we can check whether they coincide by checking whether a point on ℓ_1 (such as the point with position vector $\boldsymbol{a}_1$) also lies on ℓ_2.

Intersecting lines

Two lines $\ell_1 : \boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\ell_2 : \boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$ have a point in common if there exist values of λ and μ such that $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$.

Example 14

Determine the point of intersection of the lines

$$\boldsymbol{r}_1(\lambda) = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \quad \text{and} \quad \boldsymbol{r}_2(\mu) = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}} + \mu(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$$

Solution

At the point of intersection, we have $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$ and so

$$5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}} + \mu(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$$

Equate coefficients of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$:

$$5 + 2\lambda = -3 + \mu \qquad (1)$$
$$2 + \lambda = 4 - \mu \qquad (2)$$
$$\lambda = 6 - 2\mu \qquad (3)$$

From (1) and (2), we have

$$7 + 3\lambda = 1$$
$$\therefore \quad \lambda = -2$$

Substitute in (1) to determine $\mu = 4$.

Now we must check that these values also satisfy equation (3):

$$\text{RHS} = 6 - 2 \times 4 = -2 = \text{LHS}$$

Hence the lines intersect where $\lambda = -2$ and $\mu = 4$.

The point of intersection has the position vector

$$\begin{aligned}\boldsymbol{r}_1(-2) &= 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \\ &= \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}\end{aligned}$$

Hence the lines intersect at the point $(1, 0, -2)$.

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Example 15

Show that the following two lines are skew lines:

$$\boldsymbol{r}_1(\lambda) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{k}} + \lambda(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}), \qquad \lambda \in \mathbb{R}$$
$$\boldsymbol{r}_2(\mu) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \mu(4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}), \quad \mu \in \mathbb{R}$$

Solution

We first note that the lines are not parallel, since $\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} \neq m(4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$, for all $m \in \mathbb{R}$.

We now show that the lines do not meet. If they did meet, then equating coefficients of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ would give

$$1 + \lambda = 2 + 4\mu \qquad (1)$$
$$3\lambda = 3 - \mu \qquad (2)$$
$$1 + 4\lambda = \mu \qquad (3)$$

From (1) and (2), we have $\lambda = 1$ and $\mu = 0$. But this is not consistent with equation (3). So there are no values of λ and μ such that $\boldsymbol{r}_1(\lambda) = \boldsymbol{r}_2(\mu)$.

The two lines are skew, as they are not parallel and do not intersect.

Concurrence of three lines

A point of **concurrence** is where three or more lines meet.

Example 16

Determine the point of concurrence of the following three lines:

$$\ell_1: \quad \boldsymbol{r}_1(t) = -2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}), \qquad t \in \mathbb{R}$$
$$\ell_2: \quad \boldsymbol{r}_2(s) = \hat{\boldsymbol{j}} + s(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}), \qquad s \in \mathbb{R}$$
$$\ell_3: \quad \boldsymbol{r}_3(u) = 8\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + u(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}), \quad u \in \mathbb{R}$$

Solution

The point of intersection of lines ℓ_1 and ℓ_2 can be found from the values of t and s such that $\boldsymbol{r}_1(t) = \boldsymbol{r}_2(s)$. Equating coefficients of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$, we obtain

$$-2 + t = s \qquad (1)$$
$$1 + t = 1 + 2s \qquad (2)$$

Solving simultaneously gives $s = 2$ and $t = 4$. Taking $t = 4$ gives $\boldsymbol{r}_1(4) = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$.

Thus lines ℓ_1 and ℓ_2 intersect at the point $(2, 5)$.

For this to be a point of concurrence, the point must also lie on ℓ_3. We must determine a value of u such that

$$2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} = 8\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + u(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$$

We see that $u = 2$ gives the result. The three lines are concurrent at the point $(2, 5)$.

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Angle between two lines

In Section 5D, we used the scalar product to determine the angle between two vectors.

If two lines have vector equations $\boldsymbol{r}_1(\lambda) = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r}_2(\mu) = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$, then they are in the directions of vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ respectively. The angle θ between the two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ can be found using the scalar product:

$$\cos\theta = \frac{\boldsymbol{d}_1 \cdot \boldsymbol{d}_2}{|\boldsymbol{d}_1|\,|\boldsymbol{d}_2|}$$

The angle between the two lines is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.

This applies to any pair of lines, whether parallel, intersecting or skew.

The two lines are perpendicular if and only if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.

Example 17

Determine the acute angle between the following two straight lines:

$$\boldsymbol{r}_1(\lambda) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$$
$$\boldsymbol{r}_2(\mu) = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + \mu(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}})$$

Solution

The vectors $\boldsymbol{d}_1 = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{d}_2 = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ give the directions of the two lines.

We have $|\boldsymbol{d}_1| = \sqrt{38}$, $|\boldsymbol{d}_2| = \sqrt{38}$ and $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = -11$.

Let θ be the angle between $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$. Then

$$\cos\theta = \frac{\boldsymbol{d}_1 \cdot \boldsymbol{d}_2}{|\boldsymbol{d}_1|\,|\boldsymbol{d}_2|} = -\frac{11}{38}$$

The acute angle between the lines is 73.17°, correct to two decimal places.

Exercise 6D

Example 13 **1** Determine the position vector of the point of intersection of the lines with equations

$$\boldsymbol{r}_1(\lambda) = 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \lambda(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$$
$$\boldsymbol{r}_2(\mu) = -2\hat{\boldsymbol{j}} + \mu(4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$$

Example 14 **2** Determine the coordinates of the point of intersection of the lines with equations

$$\boldsymbol{r}_1(\lambda) = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + \lambda(-2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$$
$$\boldsymbol{r}_2(\mu) = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}} + \mu(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$$

Example 15 **3** Show that the following two lines are skew lines:

$$\boldsymbol{r}_1(\lambda) = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + \lambda(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$$
$$\boldsymbol{r}_2(\mu) = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + \mu(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})$$

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SF

4 For each pair of lines, answer the following questions:

i Are the lines parallel?

ii Are the lines perpendicular?

iii Do the lines coincide?

iv If they intersect at a point, what is the point of intersection?

a $\boldsymbol{r}_1(t) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$
$\boldsymbol{r}_2(s) = -\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + s(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$

b $\boldsymbol{r}_1(t) = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$
$\boldsymbol{r}_2(s) = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + s(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$

c $\boldsymbol{r}_1(t) = 5\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}} + t(-2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}})$
$\boldsymbol{r}_2(s) = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + s(4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}})$

d $\boldsymbol{r}_1(t) = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$
$\boldsymbol{r}_2(s) = 7\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + s(-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$

e $\boldsymbol{r}_1(t) = 5\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$

f $\boldsymbol{r}_1(t) = 7\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}} + t(3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

g $\boldsymbol{r}_1(t) = 6\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

h $\boldsymbol{r}_1(t) = 4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = -\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}} + s(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

i $\boldsymbol{r}_1(t) = -3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + t(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}})$

j $\boldsymbol{r}_1(t) = 7\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}} + s(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

Example 16

5 For each of the following, determine the point of concurrence (if it exists) of the lines:

a $\boldsymbol{r}_1(t) = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + s(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$
$\boldsymbol{r}_3(u) = -\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + u(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

b $\boldsymbol{r}_1(t) = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = 25\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$
$\boldsymbol{r}_3(u) = 5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + u(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

c $\boldsymbol{r}_1(t) = 5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = 10\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
$\boldsymbol{r}_3(u) = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + u(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

d $\boldsymbol{r}_1(t) = -5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}})$
$\boldsymbol{r}_2(s) = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
$\boldsymbol{r}_3(u) = 5\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} + u(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

Example 17

6 Determine the acute angle between each of the following pairs of lines:

a $\boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$
$\boldsymbol{r}_2 = 5\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + s(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}})$

b $\boldsymbol{r}_1 = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$
$\boldsymbol{r}_2 = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} - s(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}})$

7 The lines ℓ_1 and ℓ_2 are given by the equations

$$\ell_1: \quad \boldsymbol{r}_1 = \hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$$
$$\ell_2: \quad \boldsymbol{r}_2 = 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}})$$

a Determine the acute angle between the lines.

b Show that these lines are skew lines.

8 The lines ℓ_1 and ℓ_2 are given by the equations

$$\ell_1: \quad \boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$$
$$\ell_2: \quad \boldsymbol{r}_2 = 4\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

a Determine the coordinates of the point of intersection of the lines.

b Determine the cosine of the angle between the lines.

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9 Three lines are represented by vector equations as follows:

$$\ell_1: \quad \boldsymbol{r}_1 = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}} + t_1(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}), \qquad t_1 \in \mathbb{R}$$

$$\ell_2: \quad \boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t_2(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}), \qquad t_2 \in \mathbb{R}$$

$$\ell_3: \quad \boldsymbol{r}_3 = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t_3(\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}), \qquad t_3 \in \mathbb{R}$$

For each pair of lines, determine whether they intersect or not. If they intersect, then determine their point of intersection.

6E Vector product

Learning intentions

- To be able to calculate and use the vector product.

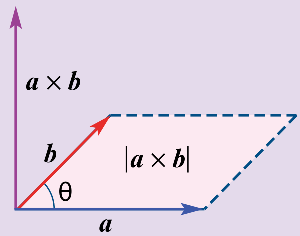
The vector product is an operation that takes two vectors and produces another vector.

Geometric definition of the vector product

Definition of the vector product

The **vector product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \times \boldsymbol{b}$.

- The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to $|\boldsymbol{a}|\,|\boldsymbol{b}| \sin \theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$, in the sense of the right-hand rule explained below.



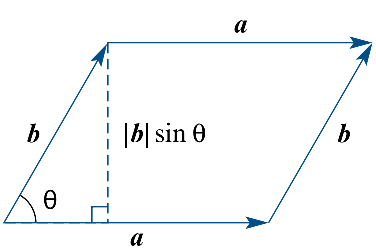
Note: The vector product is often called the **cross product**.

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$

By definition, we have

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin \theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

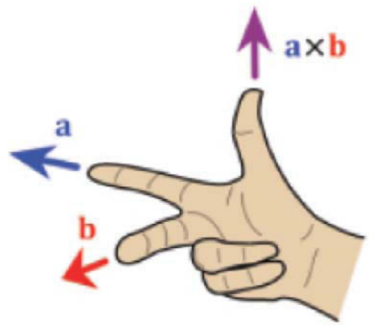


From the diagram on the right, we see that $|\boldsymbol{a} \times \boldsymbol{b}|$ is the area of the parallelogram 'spanned' by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

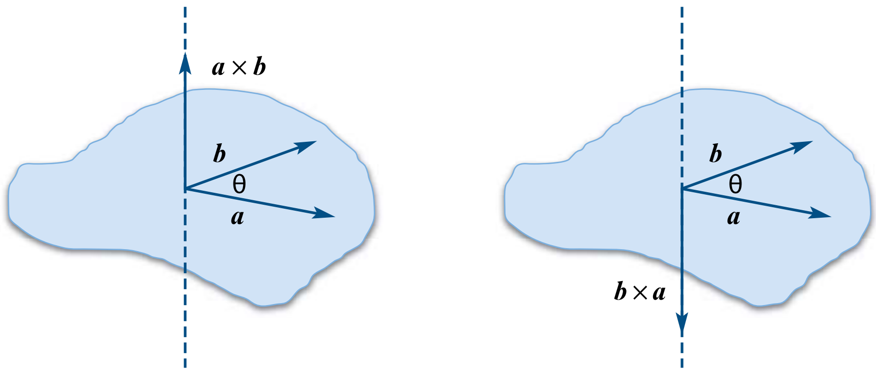
The direction of $\boldsymbol{a} \times \boldsymbol{b}$ using the right-hand rule

To determine the direction of the vector $\boldsymbol{a} \times \boldsymbol{b}$ using your right hand:

- Point your index finger along the vector $\boldsymbol{a}$.
- Point your middle finger along the vector $\boldsymbol{b}$.
- Keep your thumb at right angles to both $\boldsymbol{a}$ and $\boldsymbol{b}$, as in the picture. The direction of your thumb gives the direction of the vector $\boldsymbol{a} \times \boldsymbol{b}$.



The following two diagrams show $\boldsymbol{a} \times \boldsymbol{b}$ and $\boldsymbol{b} \times \boldsymbol{a}$.



The vector $\boldsymbol{b} \times \boldsymbol{a}$ has the same magnitude as $\boldsymbol{a} \times \boldsymbol{b}$, but the opposite direction. We can see that

$$\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b})$$

Thus the vector product is *not* commutative.

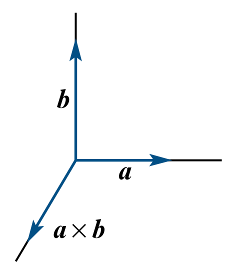
Note: The vector product is also *not* associative: in general, we have $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

Vector product of parallel vectors

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors, then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, since $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 0° = 0$.

Conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors such that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

Vector product of perpendicular vectors

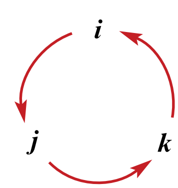


If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors, then

$$\begin{aligned} |\boldsymbol{a} \times \boldsymbol{b}| &= |\boldsymbol{a}|\,|\boldsymbol{b}| \sin 90° \\ &= |\boldsymbol{a}|\,|\boldsymbol{b}| \end{aligned}$$

The three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ form a right-handed system of mutually perpendicular vectors, as shown in the diagram.

Vector product in component form



Using the previous observations about the vector product of parallel and perpendicular vectors:

- $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{i}} = \boldsymbol{0}$
- $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{j}} = \boldsymbol{0}$
- $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \boldsymbol{0}$
- $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}$
- $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{i}}$
- $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}$
- $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{i}} = -\hat{\boldsymbol{k}}$
- $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{j}} = -\hat{\boldsymbol{i}}$
- $\hat{\boldsymbol{i}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{j}}$

The diagram on the right may help you to follow the pattern among these vector products.

The vector product distributes over addition. That is:

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$$

These facts can be used to establish the following result.

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Vector product in component form

If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$, then

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2b_3 - a_3b_2)\hat{\boldsymbol{i}} - (a_1b_3 - a_3b_1)\hat{\boldsymbol{j}} + (a_1b_2 - a_2b_1)\hat{\boldsymbol{k}}$$

Note: In Specialist Mathematics Units 1 & 2, you may have seen how to determine the determinant of a 3×3 matrix. This gives a way of evaluating the vector product as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\hat{\boldsymbol{k}}$$

$$= (a_2b_3 - a_3b_2)\hat{\boldsymbol{i}} - (a_1b_3 - a_3b_1)\hat{\boldsymbol{j}} + (a_1b_2 - a_2b_1)\hat{\boldsymbol{k}}$$

To obtain the $\hat{\boldsymbol{i}}$-component, we 'delete' the $\hat{\boldsymbol{i}}$-row and the $\hat{\boldsymbol{i}}$-column of the 3×3 matrix. Likewise for the $\hat{\boldsymbol{j}}$- and $\hat{\boldsymbol{k}}$-components.
(Here we are using $|\mathbf{A}|$ to denote the determinant of a square matrix $\mathbf{A}$.)

Example 18

Determine the vector product of $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, and hence determine a unit vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution

The vector product can be evaluated as follows:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 3 & 3 & 8 \\ 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 8 \\ -3 & 2 \end{vmatrix}\hat{\boldsymbol{i}} - \begin{vmatrix} 3 & 8 \\ 1 & 2 \end{vmatrix}\hat{\boldsymbol{j}} + \begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix}\hat{\boldsymbol{k}}$$

$$= (3 \times 2 - 8 \times (-3))\hat{\boldsymbol{i}} - (3 \times 2 - 8 \times 1)\hat{\boldsymbol{j}} + (3 \times (-3) - 3 \times 1)\hat{\boldsymbol{k}}$$

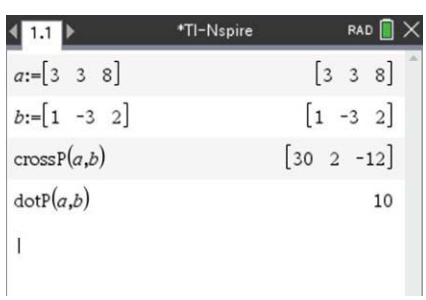
$$= 30\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 12\hat{\boldsymbol{k}}$$

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is $\sqrt{30^2 + 2^2 + 12^2} = 2\sqrt{262}$.

Hence a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\dfrac{1}{2\sqrt{262}}(30\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 12\hat{\boldsymbol{k}})$.

Using the TI-Nspire CX non-CAS

- Assign the vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ as shown.
- Determine the vector product using menu > **Matrix & Vector** > **Vector** > **Cross Product**.
  The vector product is $30\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 12\hat{\boldsymbol{k}}$.
- The scalar product of two vectors can also be found, using menu > **Matrix & Vector** > **Vector** > **Dot Product**.



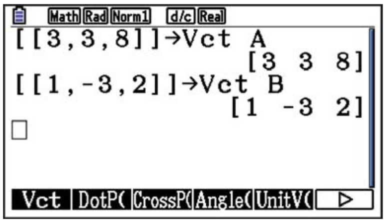
Note: You can enter the vectors directly into the vector commands if preferred.

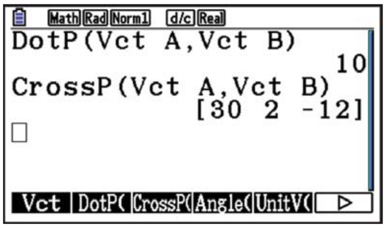
Using the Casio

- Press (MENU) (1) to select **Run-Matrix** mode.
- Store $3\hat{i} + 3\hat{j} + 8\hat{k}$ as Vector A by entering:

  [[3, 3, 8]] → Vct A

  Hint: To obtain 'Vct', go to the **Vector** menu (OPTN) (F2) (F6) (F6) and select **Vct** (F1).
- Store $\hat{i} - 3\hat{j} + 2\hat{k}$ as Vector B similarly.
- Determine the scalar product by going to the **Vector** menu (OPTN) (F2) (F6) (F6) and using **DotP(** (F2).
- Determine the vector product by going to the **Vector** menu (OPTN) (F2) (F6) (F6) and using **CrossP(** (F3).





Example 19

a Simplify:

i $\boldsymbol{a} \times (\boldsymbol{a} - \boldsymbol{b})$ **ii** $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a}$

b Given that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \times \boldsymbol{a}$, with $\boldsymbol{a} \neq \boldsymbol{0}$, show that $\boldsymbol{b} = -\boldsymbol{c}$ or $\boldsymbol{a} = k(\boldsymbol{b} + \boldsymbol{c})$ for some $k \in \mathbb{R}$.

Solution

a **i** Since the vector product distributes over addition, we have

$$\begin{aligned}\boldsymbol{a} \times (\boldsymbol{a} - \boldsymbol{b}) &= \boldsymbol{a} \times (\boldsymbol{a} + (-\boldsymbol{b}))\\ &= \boldsymbol{a} \times \boldsymbol{a} + \boldsymbol{a} \times (-\boldsymbol{b})\\ &= \boldsymbol{0} + \boldsymbol{a} \times (-\boldsymbol{b})\\ &= -(\boldsymbol{a} \times \boldsymbol{b})\\ &= \boldsymbol{b} \times \boldsymbol{a}\end{aligned}$$

ii Since $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$, we have $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$.

b By assumption, we have

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{c} \times \boldsymbol{a}\\ \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{c} \times \boldsymbol{a} &= \boldsymbol{0}\\ \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c} &= \boldsymbol{0}\\ \therefore \quad \boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{0}\end{aligned}$$

Since $\boldsymbol{a} \neq \boldsymbol{0}$, it follows that either $\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$ or the vectors $\boldsymbol{a}$ and $\boldsymbol{b} + \boldsymbol{c}$ are parallel. Hence we must have $\boldsymbol{b} = -\boldsymbol{c}$ or $\boldsymbol{a} = k(\boldsymbol{b} + \boldsymbol{c})$ for some $k \in \mathbb{R}$.

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Skill-sheet

Exercise 6E

SF

Example 18

1 Use the vector product to determine a vector perpendicular to the two given vectors:

a $\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

b $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

c $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}}$

d $2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $2\hat{\boldsymbol{j}}$

e $2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ and $-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$

f $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $-\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

g $-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}}$

h $-2\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}}$ and $2\hat{\boldsymbol{j}}$

Example 19

2 Simplify:

a $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{b}$

b $(\boldsymbol{a} + \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b})$

c $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b})$

d $(\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})) \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot ((\boldsymbol{b} + \boldsymbol{c}) \times \boldsymbol{a})$

f $((\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}))$

3 Determine a vector of magnitude 5 that is perpendicular to $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.

4 Determine a vector perpendicular to $\boldsymbol{a} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$.

5 A parallelogram $OABC$ has one vertex at the origin O and two other vertices at the points $A(0, 1, 3)$ and $B(0, 2, 5)$. Determine the area of $OABC$.

6 Determine the area of the triangle PQR with vertices $P(1, 5, -2)$, $Q(0, 0, 0)$ and $R(3, 5, 1)$.

CF

7 The three vertices of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Show that the area of the triangle is $\frac{1}{2}|(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{c} \times \boldsymbol{a})|$.

8 Let $\boldsymbol{v}$ be a vector parallel to a line ℓ, and let $\boldsymbol{u}$ be a vector from any point on the line to a point P not on the line. Show that the distance from the point P to the line ℓ is $\dfrac{|\boldsymbol{u} \times \boldsymbol{v}|}{|\boldsymbol{v}|}$.

9 In this question, we verify that the component form of the vector product has some of its desired properties. Consider vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as follows:

$$\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$$
$$\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$$
$$\boldsymbol{c} = (a_2b_3 - a_3b_2)\hat{\boldsymbol{i}} - (a_1b_3 - a_3b_1)\hat{\boldsymbol{j}} + (a_1b_2 - a_2b_1)\hat{\boldsymbol{k}}$$

a Verify that $\boldsymbol{c}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Verify that if we swap $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\boldsymbol{c}$ becomes $-\boldsymbol{c}$.

c From Section 5D we know that

$$a_1b_1 + a_2b_2 + a_3b_3 = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Using this result and the Pythagorean identity, verify that $|\boldsymbol{c}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$.

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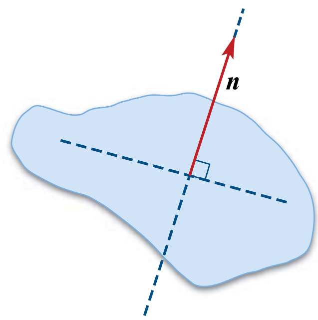
6F Vector equations of planes

Learning intentions

- To be able to determine the vector and Cartesian equations of a plane.

Normal vectors to planes

At each point on a smooth surface there is a line perpendicular to the surface. For a plane, these perpendiculars all point in the same direction.



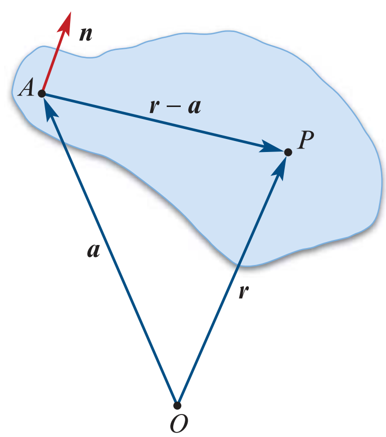
A vector that is perpendicular to a plane is called a **normal** to the plane.

Note: There is not a unique normal vector for a given plane. If the vector $\boldsymbol{n}$ is normal to the plane, then so are the vectors $k\boldsymbol{n}$ and $-k\boldsymbol{n}$, for all $k \in \mathbb{R}^+$.

Equations of planes

A plane Π in three-dimensional space may be described using two vectors:

- the position vector $\boldsymbol{a}$ of a point A on the plane
- a vector $\boldsymbol{n}$ that is normal to the plane.



Let $\boldsymbol{r}$ be the position vector of any other point P on the plane. Then the vector $\overrightarrow{AP} = \boldsymbol{r} - \boldsymbol{a}$ lies in the plane, and is therefore perpendicular to $\boldsymbol{n}$. Hence

$$(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = 0$$

This can be written as

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

This is a **vector equation** of the plane.

If we write the position vector of the point P as $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ and write the normal vector as $\boldsymbol{n} = n_1\hat{\boldsymbol{i}} + n_2\hat{\boldsymbol{j}} + n_3\hat{\boldsymbol{k}}$, then we obtain a **Cartesian equation** of the plane:

$$n_1x + n_2y + n_3z = \boldsymbol{a} \cdot \boldsymbol{n}$$

This is often written as

$$n_1x + n_2y + n_3z = k$$

where $k = \boldsymbol{a} \cdot \boldsymbol{n}$.

Note: We can also give parametric equations for a plane. This appears in section 6I.

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Planes in three dimensions

A plane in three-dimensional space can be described as follows, where $\boldsymbol{a}$ is the position vector of a point A on the plane, the vector $\boldsymbol{n} = n_1\hat{\boldsymbol{i}} + n_2\hat{\boldsymbol{j}} + n_3\hat{\boldsymbol{k}}$ is normal to the plane, and $k = \boldsymbol{a} \cdot \boldsymbol{n}$.

Vector equation	Cartesian equation
$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$	$n_1x + n_2y + n_3z = k$

We now look at examples of determining equations of planes given different information:

- a point on the plane and a normal vector to the plane
- three points on the plane that are not collinear
- two lines in the plane that intersect at a point.

Determining the plane defined by a point and a normal vector

The following example illustrates two methods for determining an equation of a plane.

Example 20

For a plane Π, the vector $-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ is normal to the plane and the point A with position vector $-3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ is on the plane. Determine a vector equation and a Cartesian equation of the plane.

Solution

Method 1: Determining a vector equation first

Using the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$, a vector equation is

$$\boldsymbol{r} \cdot (-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = (-3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}) \cdot (-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$$

i.e. $\quad \boldsymbol{r} \cdot (-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = 5$

For a Cartesian equation, write $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$. Then

$$(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}) \cdot (-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = 5$$

i.e. $\quad -x + 5y - 3z = 5$

Method 2: Determining a Cartesian equation first

The vector $\boldsymbol{n} = -\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ is normal to the plane, so a Cartesian equation is

$$-x + 5y - 3z = k$$

for some $k \in \mathbb{R}$. Since the point $A(-3, 4, 6)$ is on the plane, we have

$$-(-3) + 5(4) - 3(6) = k$$

Therefore $k = 5$, and a Cartesian equation is $-x + 5y - 3z = 5$.

Hence a vector equation is $\boldsymbol{r} \cdot (-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = 5$.

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Determining the plane defined by three points

Three points determine a plane provided they are not collinear.

Example 21

Consider the plane containing the points $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$.

a Determine a Cartesian equation of the plane.

b Determine the axis intercepts of the plane, and hence sketch a graph of the plane.

Solution

a $\overrightarrow{AB} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}}$ and $\overrightarrow{AC} = -2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

The vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ is $-\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.

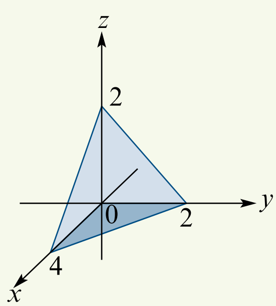
Therefore the vector $\boldsymbol{n} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ is normal to the plane.

Using the point A and the normal $\boldsymbol{n}$, we can use either of the two methods to determine the Cartesian equation $-x - 2y - 2z = -4$.

b We can write the Cartesian equation of the plane more neatly as $x + 2y + 2z = 4$.

x-axis intercept: Let $y = z = 0$. Then $x = 4$.
y-axis intercept: Let $x = z = 0$. Then $2y = 4$, so $y = 2$.
z-axis intercept: Let $x = y = 0$. Then $2z = 4$, so $z = 2$.

The axis intercepts of the plane are $(4, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.



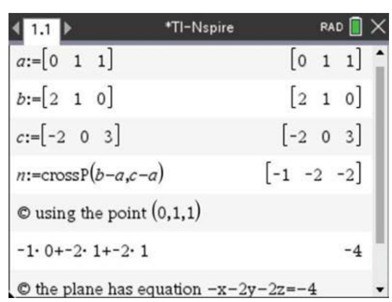
TI-84

Using the TI-Nspire CX non-CAS

Consider the plane containing $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$.

Determining a Cartesian equation of the plane

- Assign the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as shown.
- Determine a normal vector $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ using menu > **Matrix & Vector** > **Vector** > **Cross Product**. Hence $\boldsymbol{n} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.
- Determine the value of k by substituting the point A into the Cartesian equation $n_1 x + n_2 y + n_3 z = k$. Hence $k = -4$.
- The plane has equation $-x - 2y - 2z = -4$.



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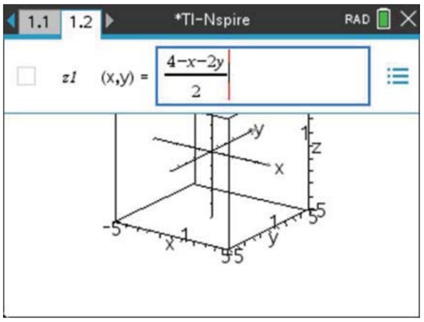
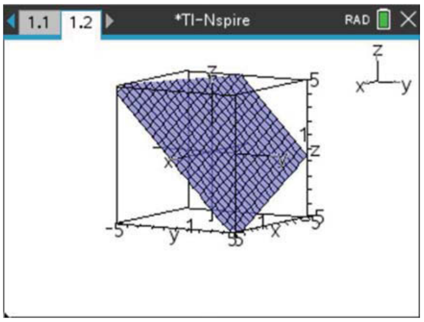
Plotting a 3D graph of the plane

- Transpose the Cartesian equation of the plane to make z the subject:

$$z = \frac{4 - x - 2y}{2}$$



- In a **Graphs** application, use (menu) > **View** > **3D Graphing**. Enter the expression for z in $z1(x, y)$ as shown.



- To rotate the view of the plane, use (menu) > **Actions** > **Rotate** (or press (r)) and then use the arrow keys.
- Use (menu) to change other attributes as desired.

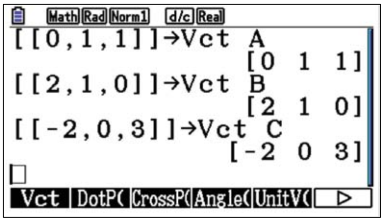
Using the Casio

Consider the plane containing $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$.

Determining a Cartesian equation of the plane

An equation for the plane can be determined in **Run-Matrix** mode by using commands from the **Vector** menu (OPTN) (F2) (F6) (F6).

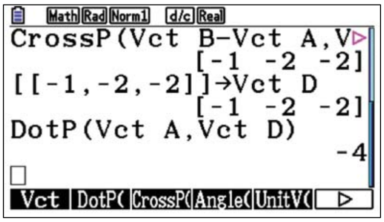
- Store the three vectors A, B and C as shown.



- Determine a normal vector $\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ by entering:

  CrossP(Vct B − Vct A, Vct C − Vct A)

  Hence $\boldsymbol{n} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.



- Store the normal vector $\boldsymbol{n}$ as Vector D.
- Determine the value of $k = \boldsymbol{a} \cdot \boldsymbol{n}$ by entering:

  DotP(Vct A, Vct D)

  Hence $k = -4$.
- The plane has equation $-x - 2y - 2z = -4$.

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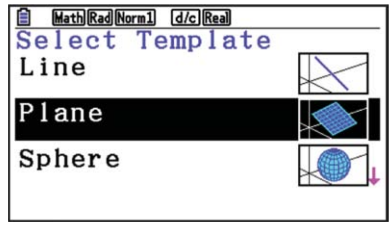
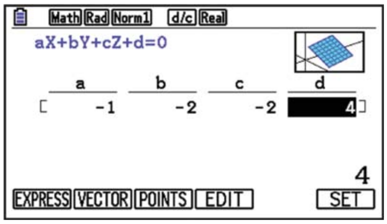
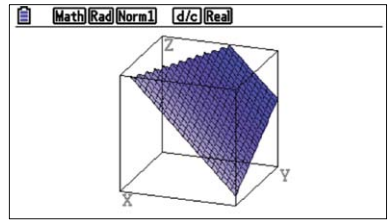
Plotting a 3D graph of the plane

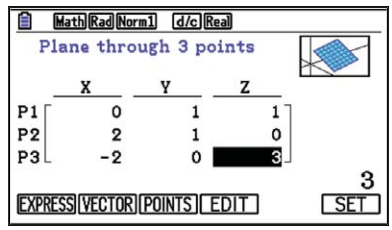
- Select **3D Graph** mode (MENU) (ALPHA) ( )).
- Go to **Type** (F3); select **Plane** by pressing ▼ (EXE).
- Select the **Express** template (F1).
- Enter the coefficients of the Cartesian equation $-x - 2y - 2z + 4 = 0$ as shown below.
- Select **Set** (F6), then **Draw** (F6).







- Alternatively, you can plot the graph by selecting the **Points** template (F3) and entering the coordinates of the three points $A(0, 1, 1)$, $B(2, 1, 0)$ and $C(-2, 0, 3)$ into the table as shown.



Determining the plane defined by two intersecting lines

Two lines that intersect at a single point can be used to determine a plane.

Example 22

Determine a vector equation and a Cartesian equation of the plane containing the lines

$$\boldsymbol{r}_1 = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \lambda(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$$

$$\boldsymbol{r}_2 = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}} + \mu(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$$

Note: From Example 14, we know that these lines intersect at the point $(1, 0, -2)$.

Solution

We know that $\boldsymbol{a} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ is the position vector of a point on the plane.

We want to determine a normal vector. It must be perpendicular to both $\boldsymbol{d}_1 = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{d}_2 = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, so we can choose

$$\boldsymbol{n} = \boldsymbol{d}_1 \times \boldsymbol{d}_2 = -\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$$

Hence a vector equation of the plane is

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

i.e. $\quad \boldsymbol{r} \cdot (-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = 5$

The corresponding Cartesian equation is $-x + 5y - 3z = 5$.

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Skill-sheet

Exercise 6F

Example 20

1 In each of the following, a vector $\boldsymbol{n}$ normal to the plane and a point A on the plane are given. Determine a vector equation and a Cartesian equation of each plane.

a $\boldsymbol{n} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $A(1, -2, 4)$ **b** $\boldsymbol{n} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$, $A(3, 1, 0)$

c $\boldsymbol{n} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $A(2, -3, -5)$ **d** $\boldsymbol{n} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $A(1, -2, 3)$

Example 21

2 Points $A = (2, 1, -1)$, $B = (1, 3, 1)$ and $C = (3, -2, 2)$ lie in a plane. Determine a unit vector normal to this plane and determine a vector equation of this plane.

Example 22

3 Determine a vector equation and a Cartesian equation of the plane containing the lines $\boldsymbol{r}_1 = \hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + \lambda(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ and $\boldsymbol{r}_2 = -3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \mu(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$.

4 The point $A = (-3, 1, 1)$ and the line ℓ lie in the same plane. The line ℓ is defined by the equation $\boldsymbol{r} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$, $t \in \mathbb{R}$.

a Determine a vector normal to this plane.

b Determine a vector equation of the line through A that is normal to this plane.

5 Points $A = (1, 1, 3)$, $B = (1, 5, -2)$ and $C = (0, 3, -1)$ lie in a plane. Determine a unit vector normal to this plane and determine a vector equation of this plane.

6 A plane is defined by the vector equation $\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = 7$. Show that each of the following is the position vector of a point on this plane:

a $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ **b** $3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ **c** $-\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ **d** $2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

7 A plane is defined by the vector equation $\boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 10$. Show that each of the following is a point on this plane:

a $(2, 2, -2)$ **b** $(1, 5, -2)$ **c** $(3, 4, 3)$ **d** $(2, 0, -4)$

8 Determine x in each of the following:

a The point $(1, x, 2)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}) = 5$.

b The point $(2, -1, 0)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}) = x$.

c The point $(1, -3, 2)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} + x\hat{\boldsymbol{k}}) = 8$.

d The point $(x, 1, -2)$ lies on the plane given by the equation $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 5$.

9 Determine a Cartesian equation of the plane containing the three points $A(0, 3, 4)$, $B(1, 2, 0)$ and $C(-1, 6, 4)$.

10 Determine a Cartesian equation of the plane that is at right angles to the line given by $x = 4 + t,\ y = 1 - 2t,\ z = 8t$ and goes through the point $P(3, 2, 1)$.

11 Determine a Cartesian equation of the plane that is parallel to the plane with equation $5x - 3y + 2z = 6$ and goes through the point $P(4, -1, 2)$.

12 Determine a Cartesian equation of the plane that contains the intersecting lines given by $x = 4 + t_1,\ y = 2t_1,\ z = 1 - 3t_1$ and $x = 4 - 3t_2,\ y = 3t_2,\ z = 1 + 2t_2$.

13 Determine a Cartesian equation of the plane that is at right angles to the plane with equation $3x + 2y - z = 4$ and goes through the points $P(1, 2, 4)$ and $Q(-1, 3, 2)$.

6G Distances, angles and intersections

Learning intentions

- ▶ To be able to determine the distance of a point to a plane and the distance between two parallel planes.
- ▶ To be able to determine the angle between two planes and the coordinates of a point of intersection of a line with a plane.

Distance from a point to a plane

The distance from a point P to a plane Π is given by $d = |\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}|$

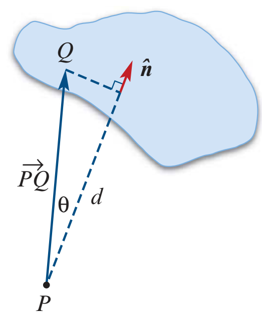
where $\hat{\boldsymbol{n}}$ is a unit vector normal to the plane and Q is any point on the plane.

Proof For the situation shown in the diagram, we can see that the distance from P to the plane is

$$d = |\overrightarrow{PQ}| \cos\theta$$

where θ is the angle between $\overrightarrow{PQ}$ and $\hat{\boldsymbol{n}}$. Therefore

$$d = |\overrightarrow{PQ}|\,|\hat{\boldsymbol{n}}| \cos\theta = \overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$$



The other situation is where the unit normal $\hat{\boldsymbol{n}}$ points in the opposite direction. In this case, we will obtain $d = -\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$.

Hence, in general, the distance is the absolute value of $\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}$.

Example 23

Determine the distance from the point $P(1, -4, -3)$ to the plane with equation $2x - 3y + 6z = -1$.

Solution

A normal vector to the plane is $\boldsymbol{n} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$. So a unit vector normal to the plane is $\hat{\boldsymbol{n}} = \frac{1}{7}(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}})$. Let $Q(x, y, z)$ be any point on the plane. Note that this implies $2x - 3y + 6z = -1$.

We want to determine the projection of $\overrightarrow{PQ}$ onto $\hat{\boldsymbol{n}}$. We have

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x-1)\hat{\boldsymbol{i}} + (y+4)\hat{\boldsymbol{j}} + (z+3)\hat{\boldsymbol{k}}$$

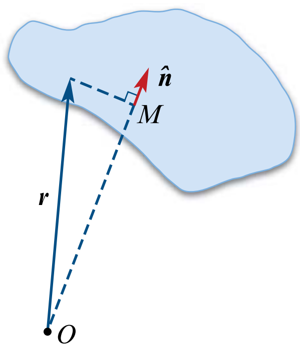
Therefore

$$\begin{aligned}\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}} &= \frac{1}{7}(2(x-1) - 3(y+4) + 6(z+3)) \\ &= \frac{1}{7}(2x - 3y + 6z - 2 - 12 + 18) \\ &= \frac{1}{7}(-1 + 4) = \frac{3}{7} \qquad \text{(since } 2x - 3y + 6z = -1\text{)}\end{aligned}$$

The distance from the point P to the plane is $\frac{3}{7}$.

Note: Alternatively, we could have chosen Q to be a specific point on the plane, such as $(1, 1, 0)$. This would give $\overrightarrow{PQ} = 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and therefore $\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}} = \frac{3}{7}$.

Distance of a plane from the origin



A plane that does not pass through the origin is described by a vector equation of the form $\boldsymbol{r} \cdot \boldsymbol{n} = k$, where $k \neq 0$.

The point M on the plane that is closest to the origin has a position vector of the form $\overrightarrow{OM} = m\hat{\boldsymbol{n}}$, where $|m|$ is the distance of the plane from the origin.

If $\boldsymbol{n}$ points towards the plane from the origin, then $m > 0$, and if $\boldsymbol{n}$ points away from the plane, then $m < 0$. So we can say that m is the 'signed distance' of the plane from the origin (relative to the normal vector $\boldsymbol{n}$).

Since the point M lies on the plane, we know that $(m\hat{\boldsymbol{n}}) \cdot \boldsymbol{n} = k$. But $(m\hat{\boldsymbol{n}}) \cdot \boldsymbol{n} = m(\hat{\boldsymbol{n}} \cdot \boldsymbol{n}) = m|\boldsymbol{n}|$. So we have $m|\boldsymbol{n}| = k$ and therefore $m = \dfrac{k}{|\boldsymbol{n}|}$.

> For a plane with vector equation $\boldsymbol{r} \cdot \boldsymbol{n} = k$, where $k \neq 0$, the signed distance of the plane from the origin (relative to the normal vector $\boldsymbol{n}$) is given by $\dfrac{k}{|\boldsymbol{n}|}$.

Distance between two parallel planes

To determine the distance between parallel planes Π_1 and Π_2, we can choose any point P on Π_1 and then determine the distance from the point P to the plane Π_2.

In the following example, we use an alternative method.

Example 24

Consider the parallel planes given by the equations

$$\Pi_1 : 2x - y + 2z = 5 \quad \text{and} \quad \Pi_2 : 2x - y + 2z = -2$$

a Determine the distance of each plane from the origin.

b Determine the distance between the two planes.

Solution

The vector $\boldsymbol{n} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ is normal to both planes, with $|\boldsymbol{n}| = 3$.

a Relative to $\boldsymbol{n}$, the signed distance of plane Π_1 from the origin is $\dfrac{5}{|\boldsymbol{n}|} = \dfrac{5}{3}$.

So the distance of plane Π_1 from the origin is $\dfrac{5}{3}$.

Relative to $\boldsymbol{n}$, the signed distance of plane Π_2 from the origin is $\dfrac{-2}{|\boldsymbol{n}|} = -\dfrac{2}{3}$.

So the distance of plane Π_2 from the origin is $\dfrac{2}{3}$.

b Relative to the normal vector $\boldsymbol{n}$, the two planes are on different sides of the origin. So the distance between them is $\dfrac{5}{3} + \dfrac{2}{3} = \dfrac{7}{3}$.

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In the following example we determine the distance between two parallel planes by using a normal vector common to both planes.

Example 25

Let $A(3, 1, -1)$, $B(2, 1, 2)$ and $C(3, 2, 4)$ be three points on a plane Π_1.

a Determine a vector and a Cartesian equation of the plane Π_1.

b A second plane Π_2 is parallel to Π_1 and contains the point $(6, 5, 1)$. Determine the Cartesian equation of Π_2.

c Determine the shortest distance between Π_1 and Π_2.

Solution

a $\boldsymbol{u} = \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$

$\boldsymbol{v} = \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$

$\boldsymbol{a} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the plane and $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to the plane.

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ -1 & 0 & 3 \\ 0 & 1 & 5 \end{vmatrix} = -3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$$

Therefore the vector equation of the plane Π_1 is:

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{n} &= \boldsymbol{a} \cdot \boldsymbol{n} \\ &= (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (-3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \end{aligned}$$

That is, $\boldsymbol{r} \cdot \boldsymbol{n} = -3$

Therefore Cartesian equation of the plane Π_1 is $3x - 5y + z = 3$

b For the plane, Π_2, $\boldsymbol{a} = 6\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{n} = -3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

The vector equation is:

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{n} &= \boldsymbol{a} \cdot \boldsymbol{n} \\ &= (6\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \cdot (-3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 6 \end{aligned}$$

The Cartesian equation is $3x - 5y + z = -6$

c Let $\overrightarrow{OA}$ be the position vector of the point $(3, 1, -1)$ in Π_1 and let $\overrightarrow{OA'}$ be the position vector of the point $(6, 5, 1)$ in Π_2.

We consider the scalar resolute of $\overrightarrow{AA'}$ in the direction of $\boldsymbol{n}$

$$\overrightarrow{AA'} = 6\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} - (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

The unit vector in the direction of $\boldsymbol{n}$ is

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{35}}(-3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

$$\begin{aligned} \overrightarrow{AA'} \cdot \hat{\boldsymbol{n}} &= (3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{n}} \\ &= \frac{1}{\sqrt{35}}(-9 + 20 - 2) \\ &= \frac{9}{\sqrt{35}} \end{aligned}$$

The distance between the two planes is $\dfrac{9}{\sqrt{35}}$

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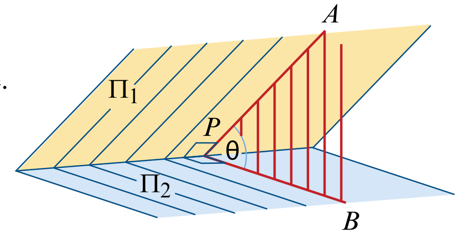
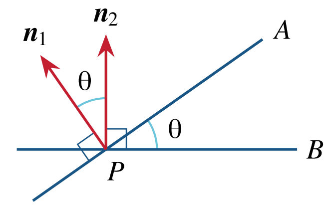
Intersections and angles

Intersection of two planes

Two planes that are not parallel will intersect in a line.



Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between planes Π_1 and Π_2.



To determine the angle between the two planes, we first determine the angle θ between two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ that are normal to the two planes. The angle between the planes is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.

- Two planes are parallel if and only if the two normal vectors are parallel.
- Two planes are perpendicular if and only if the two normal vectors are perpendicular.

Example 26

Let Π_1 and Π_2 be the planes represented by the vector equations

$$\Pi_1 : \boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = 6 \quad \text{and} \quad \Pi_2 : \boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 4$$

a Determine the angle between the planes.

b Determine a vector equation of the line of intersection of the planes.

Solution

a We first determine the angle between normals to the planes.

A normal to plane Π_1 is $\boldsymbol{n}_1 = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$, and a normal to plane Π_2 is $\boldsymbol{n}_2 = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

Let θ be the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Then

$$\begin{aligned} \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 &= |\boldsymbol{n}_1|\,|\boldsymbol{n}_2| \cos\theta \\ (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) \cdot (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) &= \sqrt{11}\sqrt{6}\cos\theta \\ \therefore \qquad -2 &= \sqrt{66}\cos\theta \end{aligned}$$

Hence $\theta \approx 104.25°$. The acute angle between the planes is $180° - 104.25° = 75.75°$, correct to two decimal places.

b Consider Cartesian equations for the two planes:

$$x + y - 3z = 6 \qquad (1)$$
$$2x - y + z = 4 \qquad (2)$$

Add (1) and (2):

$$3x - 2z = 10 \qquad (3)$$

Let $x = \lambda$. Then $z = \dfrac{3\lambda - 10}{2}$, from (3), and $y = \dfrac{7\lambda - 18}{2}$, from (2).

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This gives us parametric equations for the line of intersection:

$$x = \lambda, \quad y = \frac{7\lambda - 18}{2}, \quad z = \frac{3\lambda - 10}{2}$$

These convert to the vector equation

$$\boldsymbol{r} = -9\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}} + \lambda\left(\hat{\boldsymbol{i}} + \frac{7}{2}\hat{\boldsymbol{j}} + \frac{3}{2}\hat{\boldsymbol{k}}\right), \quad \lambda \in \mathbb{R}$$

Note: Alternatively, we can use the parametric equations to determine a point $A(0, -9, -5)$ on the line. A vector $\boldsymbol{d}$ parallel to the line must be perpendicular to the two normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. Hence we can choose $\boldsymbol{d} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$.

Intersection of a line and a plane

A line and a plane that are not parallel intersect at a point. The angle between a line and a plane is equal to $90° - \theta$, where θ is the angle between the line and a normal to the plane.

Example 27

Consider the line represented by the equation $\boldsymbol{r} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ and the plane represented by the equation $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 2$.

a Determine the point of intersection of the line and the plane.

b Determine the angle between the line and the plane.

Solution

a To determine the point of intersection, we want to determine the value of t for which

$$\boldsymbol{r} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

represents a point on the plane. That is,

$$\begin{aligned} (3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})) \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) &= 2 \\ (3 + t) + (-1 + 2t) + 2(-1 - t) &= 2 \\ \therefore \qquad t &= 2 \end{aligned}$$

The point of intersection has position vector

$$\boldsymbol{r} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + 2(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$$

The point of intersection is $(5, 3, -3)$.

b We first determine the angle between the line and the normal to the plane.

The vector $\boldsymbol{d} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ is parallel to the line, and the vector $\boldsymbol{n} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ is normal to the plane. Let θ be the angle between $\boldsymbol{d}$ and $\boldsymbol{n}$. Then

$$\begin{aligned} \boldsymbol{d} \cdot \boldsymbol{n} &= |\boldsymbol{d}|\,|\boldsymbol{n}| \cos\theta \\ (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) &= \sqrt{6}\sqrt{6}\cos\theta \\ \therefore \qquad 1 &= 6\cos\theta \end{aligned}$$

So $\theta = 80.4°$, correct to one decimal place. Hence the angle between the line and the plane is $90° - 80.4° = 9.6°$, correct to one decimal place.

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Exercise 6G

Example 23

1 Determine the distance from the point $(1, 3, 2)$ to each of the following planes:

a $\boldsymbol{r} \cdot (7\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}) = 9$ **b** $6x + 6y + 3z = 8$

SF

Example 24, 25

2 Determine the distance between the following pair of parallel planes:

$$\Pi_1: \quad x + 2y - 2z = 4$$
$$\Pi_2: \quad x + 2y - 2z = 12$$

CF

Example 26

3 Let Π_1 and Π_2 be the planes represented by the vector equations

$$\Pi_1: \quad \boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 8$$
$$\Pi_2: \quad \boldsymbol{r} \cdot (\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 6$$

a Determine the angle between the planes.

b Determine a vector equation of the line of intersection of the planes.

Example 27

4 Consider the line represented by the equation $\boldsymbol{r} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$ and the plane represented by the equation $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 4$.

a Determine the point of intersection of the line and the plane.

b Determine the angle between the line and the plane.

5 Let $A = (2, 0, -1)$, $B = (1, -3, 1)$, $C = (0, -1, 2)$ and $D = (3, -2, 2)$.

a Determine a vector normal to the plane containing points A, B and C.

b Determine a vector normal to the plane containing points B, C and D.

c Use the two normal vectors to determine the angle between these two planes.

6 In each of the following, a pair of vector equations is given that represent a line and a plane respectively. Determine the point of intersection of the line and the plane and determine the angle between the line and the plane, correct to two decimal places.

a $\boldsymbol{r} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$
$\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 7$

b $\boldsymbol{r} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$
$\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 7$

c $\boldsymbol{r} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}} + t(3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$
$\boldsymbol{r} \cdot (-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 4$

d $\boldsymbol{r} = -\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$
$\boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = -10$

7 The vector $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ is normal to a plane Π which contains the point $A(5, 4, -1)$.

a Determine a vector equation of the plane.

b Determine the distance of the plane from the origin.

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8 **a** Determine the distance from the origin to the plane $\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}) = 7$.

b Determine the vector projection of $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ in the direction of $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.

c Determine the magnitude of this vector projection.

d Hence determine the distance from the point $(1, 1, -1)$ to the given plane.

9 Using the method of Question 8, determine the distance from the point $(2, -1, 3)$ to the plane given by the equation $\boldsymbol{r} \cdot (-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = -3$.

10 **a** Determine the point of intersection of the line $\boldsymbol{r} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}})$ and the plane $\boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 11$.

b Determine the acute angle between the line and the plane, correct to one decimal place.

11 Points A, B and C have position vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ respectively.

a Determine a Cartesian equation of the plane containing A, B and C.

b Determine the area of triangle ABC.

c Determine the position vector of the foot of the perpendicular from the origin O to the plane ABC.

12 Let Π_1 and Π_2 be the planes represented by the vector equations

$$\Pi_1 : \quad \boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}) = 3$$
$$\Pi_2 : \quad \boldsymbol{r} \cdot (8\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 1$$

a Determine the angle between the planes.

b Determine a vector equation of the line of intersection of the planes.

13 Let $A = (0, 2, -1)$, $B = (1, 1, 1)$, $C = (-1, 0, 2)$ and $D = (2, -2, 2)$.

a Determine a vector normal to the plane containing points A, B and C.

b Determine a vector normal to the plane containing points B, C and D.

c Use the two normal vectors to determine the angle between these two planes.

14 **a** Determine a vector which is perpendicular to the two lines given by

$$\boldsymbol{r}_1 = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t_1(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}), \qquad t_1 \in \mathbb{R}$$
$$\boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t_2(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}), \quad t_2 \in \mathbb{R}$$

b Determine a vector equation of the line which is normal to the plane containing these two lines and which passes through their point of intersection.

CF

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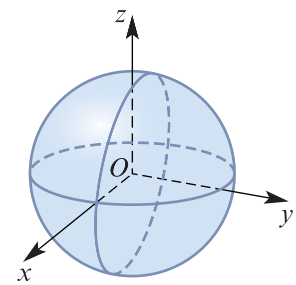
6H Equations of spheres

Learning intentions

- To be able to work with Cartesian and vector equations of spheres.

The unit sphere is the set of points $P(x, y, z)$ that are 1 unit from the origin. i.e. $|OP| = 1$. Let $P(x, y, z)$ be a point on the unit sphere.



Then $|\overrightarrow{OP}| = 1$ and therefore

$$x^2 + y^2 + z^2 = 1$$

Conversely, any point $P(x, y, z)$ which satisfies $x^2 + y^2 + z^2 = 1$ lies on the unit sphere.

Similarly, we can obtain the general Cartesian equation of a sphere.

Cartesian equation of a sphere

The sphere with centre $C(h, k, \ell)$ and radius a has Cartesian equation

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = a^2$$

We note that this again depends on the idea of a set of points which are equidistant from a given point. The vector equation of a sphere is also derived from this observation.

Vector equation of a sphere

The sphere with centre C and radius a has vector equation

$$|\boldsymbol{r} - \overrightarrow{OC}| = a$$

A point P lies on the sphere if and only if its position vector $\boldsymbol{r}$ satisfies this condition.

Example 28

For the sphere with centre $(1, -2, 3)$ and radius 6, determine:

a the Cartesian equation **b** the vector equation.

Solution

a $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 36$ **b** $|\boldsymbol{r} - (\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})| = 6$

Example 29

Determine the points of intersection of the line $\boldsymbol{r} = t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$ and the sphere $x^2 + y^2 + z^2 = 9$.

Solution

The line $\boldsymbol{r} = t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$ is described by the parametric equations

$$x = 2t, \quad y = t, \quad z = -2t$$

Substituting in the equation of the sphere gives $4t^2 + t^2 + 4t^2 = 9$

Therefore $t = \pm 1$. The points of intersection are $(2, 1, -2)$ and $(-2, -1, 2)$.

Exercise 6H

SF

Example 28

1 For the sphere with centre $(-1, 3, 2)$ and radius 2, determine:

a the Cartesian equation

b the vector equation.

2 For the sphere with centre $(-1, -3, 1)$ and radius 4, determine:

a the Cartesian equation

b the vector equation.

Example 29

3 Determine the points of intersection of the line $x = 2t,\ y = 3t,\ z = -2t,\ t > 0$ and the sphere $x^2 + y^2 + z^2 = 16$.

4 Determine the points of intersection of the line $\boldsymbol{r} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}),\ t \in \mathbb{R}$, and the sphere $x^2 + y^2 + z^2 = 36$.

5 Determine the intersection of the line $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}),\ t \in \mathbb{R}$, and the sphere $(x-2)^2 + (y-3)^2 + (z-4)^2 = 36$.

6 For each of the following, give the coordinates of the centre and the radius of the circle formed by the given plane cutting the sphere $x^2 + y^2 + z^2 = 36$:

a $z = 3$ **b** $x = 3$ **c** $y = x$

7 The equation of a sphere is

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$$

Determine the coordinates of the centre and the radius of the sphere.

8 Determine the Cartesian equation of each of the following spheres:

a centre $(1, 0, -1)$ and radius 4

b centre $(1, -3, 2)$ and passes through the origin

c centre $(3, -2, 4)$ and passes through $(7, 2, 3)$

d centre the origin and passes through $(1, 2, 2)$

9 The equation of a sphere is $(x-2)^2 + (y-3)^2 + (z-4)^2 = 29$.

a Determine the intercepts with each of the axes.

b Determine a vector equation of the line which passes through the centre of the sphere and the point $X(4, 0, 0)$.

c Determine a Cartesian equation of the plane which contains the point $X(4, 0, 0)$ and is perpendicular to the radius joining the centre of the sphere to X.

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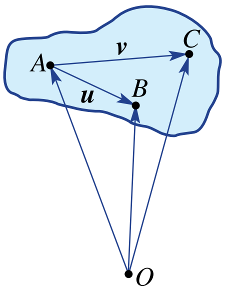
6I Parametric equations of planes(Optional)

In Section 6F we discussed the vector equation of a plane

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

where $\boldsymbol{a}$ is the position vector of a point A in the plane, $\boldsymbol{r}$ is the position vector of any other point in the plane and $\boldsymbol{n}$ is a vector normal to the plane. In this section we develop what is called the **parametric vector equation** of a plane.



Let A, B and C be points in the plane which do not all lie on the same straight line. Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

Let R be an arbitrary point in the plane. Then we can write $\overrightarrow{AR}$ as a linear combination of the non-parallel vectors $\boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{c} - \boldsymbol{a}$,

$$\overrightarrow{AR} = s(\boldsymbol{b} - \boldsymbol{a}) + t(\boldsymbol{c} - \boldsymbol{a})$$

Hence, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$$= \boldsymbol{a} + s(\boldsymbol{b} - \boldsymbol{a}) + t(\boldsymbol{c} - \boldsymbol{a})$$

We let $\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{v} = \boldsymbol{c} - \boldsymbol{a}$ and $\overrightarrow{OR} = \boldsymbol{r}$

and we can write:

$$\boldsymbol{r}(s, t) = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}, \qquad s, t \in \mathbb{R}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are any two non-parallel vectors in the plane.

The parameters s and t vary over all real numbers, and so the vector $\boldsymbol{r}(s, t)$ varies over all position vectors of points on the plane.

If we write the vectors $\boldsymbol{r}, \boldsymbol{a}, \boldsymbol{u}, \boldsymbol{v}$ in component form, then we have

$$\begin{aligned} \boldsymbol{r}(s, t) &= x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}} \\ &= a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}} + s(u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}) + t(v_1\hat{\boldsymbol{i}} + v_2\hat{\boldsymbol{j}} + v_3\hat{\boldsymbol{k}}) \end{aligned}$$

Note: We verify that, for any s and t, the vector $\boldsymbol{r}(s, t)$ satisfies the vector equation $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ of the plane.
Let $\boldsymbol{n}$ be a vector normal to the plane. Then using the parametric vector equation we have

$$\begin{aligned} \boldsymbol{r} \cdot \boldsymbol{n} &= (\boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}) \cdot \boldsymbol{n} \\ &= \boldsymbol{a} \cdot \boldsymbol{n} + s\boldsymbol{u} \cdot \boldsymbol{n} + t\boldsymbol{v} \cdot \boldsymbol{n} \\ &= \boldsymbol{a} \cdot \boldsymbol{n} \text{ (since } \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n} = 0) \end{aligned}$$

That is,

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$$

Therefore the arbitrary position vector satisfies the vector equation of a plane.

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Example 30

Determine a parametric vector equation of the plane passing through the points $A(2, 0, 1)$, $B(1, -2, 1)$ and $C(2, 1, -1)$.

Solution

The position vectors of A, B and C are $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$ where:

$\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$

$\boldsymbol{b} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

$\boldsymbol{c} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}.$

The vectors $\boldsymbol{u} = \overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ and $\boldsymbol{v} = \overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$ are vectors in the plane and
$\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{a} = -\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

$\boldsymbol{v} = \boldsymbol{c} - \boldsymbol{a} = \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}.$

Therefore a parametric vector equation of the plane is:

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}\\ &= 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}} + s(-\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}) + t(\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})\\ &= (2 - s)\hat{\boldsymbol{i}} + (-2s + t)\hat{\boldsymbol{j}} + (1 - 2t)\hat{\boldsymbol{k}}\end{aligned}$$

Example 31

a The point $(31, 18, -9)$ lies on the plane with a parametric vector equation

$$\boldsymbol{r} = (2 + 3s + t)\hat{\boldsymbol{i}} + (1 + s + t)\hat{\boldsymbol{j}} + (3 - 2s)\hat{\boldsymbol{k}}.$$

Determine the corresponding values of s and t.

b Show that the point $(3, 18, 15)$ does not lie on the plane with a parametric vector equation
$\boldsymbol{r} = (2 + 3s + t)\hat{\boldsymbol{i}} + (1 + s + t)\hat{\boldsymbol{j}} + (3 - 2s)\hat{\boldsymbol{k}}.$

Solution

a We have the equations:

$$\begin{aligned}2 + 3s + t &= 31 \quad \ldots(1)\\ 1 + s + t &= 18 \quad \ldots(2)\\ 3 - 2s &= -9 \quad \ldots(3)\end{aligned}$$

From equation (3), $s = 6$.
Sustituting in equation (2),
$1 + 6 + t = 18 \Rightarrow t = 11$

b The equations to consider are

$$\begin{aligned}2 + 3s + t &= 3 \quad \ldots(1)\\ 1 + s + t &= 18 \quad \ldots(2)\\ 3 - 2s &= 15 \quad \ldots(3)\end{aligned}$$

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From (3), $s = -6$.

Substitute in (2)

$1 - 6 + t = 18 \Rightarrow t = 23$.

Check in (1).

LHS $= 2 - 18 + 23 = 7 \neq$RHS.

Parametric equations of the plane

In the above we wrote the parametric vector equations in component form as

$$x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}} + s(u_1\hat{\boldsymbol{i}} + u_2\hat{\boldsymbol{j}} + u_3\hat{\boldsymbol{k}}) + t(v_1\hat{\boldsymbol{i}} + v_2\hat{\boldsymbol{j}} + v_3\hat{\boldsymbol{k}})$$

$$= (a_1 + su_1 + tv_1)\hat{\boldsymbol{i}} + (a_2 + su_2 + tv_2)\hat{\boldsymbol{j}} + (a_3 + su_3 + tv_3)\hat{\boldsymbol{j}}$$

Writing the Cartesian coordinate variables x, y and z in terms of the parameters s and t gives a set of equations for the plane known as **parametric equations**

We can obtain parametric equations for a plane from this by equating the components.

$$x = a_1 + su_1 + tv_1$$

$$y = a_2 + su_2 + tv_2$$

$$z = a_3 + su_3 + tv_3$$

Example 32

The following parametric equations describe a plane.

$x = 1 - t \ldots (1)$

$y = s - t \ldots (2)$

$z = 1 - s \ldots (3)$

Determine the Cartesian equation of the plane.

Solution

From (1), $t = 1 - x$

From (3), $s = 1 - z$

Substitute for t and s in (2).

$y = 1 - z - (1 - x)$

The Cartesian equation of the plane is

$x - y - z = 0$.

It is a plane containing the origin.

Example 33

A plane, Π_1, is described by the parametric equations:

$$x = 3 + s - 2t$$
$$y = 1 + s - t$$
$$z = -1 + 2s + t$$

a Determine a parametric vector equation of the plane, Π_1. That is, in the form, $\boldsymbol{r}(s,t) = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}$.

b Determine a vector equation of the plane. That is, in the form, $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$.

c Determine a Cartesian equation of Π_1.

d A second plane Π_2 is parallel to Π_1 and contains the point $(6, 5, 1)$. Determine the Cartesian equation of Π_2.

e Determine the shortest distance between Π_1 and Π_2.

Solution

a $\boldsymbol{r}(s,t) = (3 + s - 2t)\hat{\boldsymbol{i}} + (1 + s - t)\hat{\boldsymbol{j}} + (-1 + 2s + t)\hat{\boldsymbol{k}}$

$\quad = (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) + s(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) + t(-2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

b Two vectors in the plane are $\boldsymbol{u} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{v} = -2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

The vector $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to the plane.

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ 1 & 1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = (3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$$

The position vector of a point A on the plane is

$\boldsymbol{a} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

Therefore the vector equation of the plane is:

$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

$\quad = (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

That is,

$\boldsymbol{r} \cdot \boldsymbol{n} = 3$

c $\boldsymbol{r} \cdot \boldsymbol{n} = (x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}) \cdot (3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 3$

Therefore Cartesian equation of the plane is

$3x - 5y + z = 3$

d For the plane, Π_2, $\boldsymbol{a} = 6\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{n} = 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

The vector equation is:

$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

$\quad = (6\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \cdot (3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = -6$

The Cartesian equation is $3x - 5y + z = -6$

Note: The second Cartesian equation can be found by letting $3x - 5y + z = m$ and substituting in $x = 6, y = 5, z = 1$ to determine that $m = -6$.

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e Method 1

From the discussion in **Section 5E** we see that the distance between the two planes is $\dfrac{3}{\sqrt{35}} + \dfrac{6}{\sqrt{35}} = \dfrac{9}{\sqrt{35}}$

Method 2

Let $\overrightarrow{OA}$ be the position vector of the point $(3, 1, -1)$ in Π_1 and let $\overrightarrow{OA'}$ be the position vector of the point $(6, 5, 1)$ in Π_2.

We consider the scalar resolute of $\overrightarrow{AA'}$ in the direction of $\boldsymbol{n}$

$$\overrightarrow{AA'} = 6\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} - (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

The unit vector in the direction of $\boldsymbol{n}$ is

$$\hat{\boldsymbol{n}} = \frac{1}{\sqrt{35}}(3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$$

$$\begin{aligned}\overrightarrow{AA'} \cdot \hat{\boldsymbol{n}} &= (3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{n}} \\ &= \frac{1}{\sqrt{35}}(9 - 20 + 2) \\ &= -\frac{9}{\sqrt{35}}\end{aligned}$$

The distance is again found to be $\dfrac{9}{\sqrt{35}}$.

Example 34

The Cartesian equation of a plane is $2x + 3y - z = 4$. Determine

a a vector equation of the plane in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$.

b a parametric vector equation of the plane in the form $\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}$.

c the parametric equations derived directly from the parametric vector equation of the plane derived in **b.**

Solution

a We first determine a point on the plane and determine its position vector.

Let $x = 1$ and $y = 1$. Then substituting in the equation $2x + 3y - z = 4$ we have $z = 1$.

We choose the point $A(1, 1, 1)$.

We have $\overrightarrow{OA} = \boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

Note: We could have chosen any point on the plane. For example, $\boldsymbol{a} = 2\hat{\boldsymbol{i}}$ since $(2, 0, 0)$ is on the plane.

We know that a vector normal to the plane is $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

A vector equation of the plane is

$$\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \cdot (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$$

That is,

$$\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 4$$

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b Method 1

We determine the position vectors of three points A, B and C in the plane.

Using the Cartesian equation of the plane, $2x + 3y - z = 4$.

When $y = 0$ and $z = 0$, $x = 2$.

When $x = 0$ and $z = 0$, $y = \frac{4}{3}$.

When $x = 0$ and $y = 0$, $z = -4$.

The position vectors of the points A, B and C are $\boldsymbol{a} = 2\hat{\boldsymbol{i}}$, $\boldsymbol{b} = \frac{4}{3}\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -4\hat{\boldsymbol{k}}$ respectively.

Hence we have vectors in the plane

$$\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{a} = -2\hat{\boldsymbol{i}} + \frac{4}{3}\hat{\boldsymbol{j}}$$

$$\boldsymbol{v} = \boldsymbol{c} - \boldsymbol{a} = -2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{k}}$$

Hence, a parametric vector equation of the plane is

$$\boldsymbol{r}(s, t) = 2\hat{\boldsymbol{i}} + s\left(-2\hat{\boldsymbol{i}} + \frac{4}{3}\hat{\boldsymbol{j}}\right) + t(-2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{k}})$$

Method 2

As in **a** we choose the vector $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

Further to this we need two vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, in the plane.

These vectors will be perpendicular to $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

With this observation consider:

$$(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + m_1\hat{\boldsymbol{k}}) = 0, \qquad m_1 \in \mathbb{R}$$

which implies $m_1 = 5$.

The vector $\boldsymbol{u} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ is in the plane.

Similarly you can show that the vector $\boldsymbol{v} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}$ is in the plane.

So a parametric vector equation for this plane is

$$\boldsymbol{r}(s, t) = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}) + t(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}})$$

Note: Our choice of considering $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + m_1\hat{\boldsymbol{k}}$ and similarly $\boldsymbol{v} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + m_2\hat{\boldsymbol{k}}$ is arbitrary.

c From **b Method 2**

$$\boldsymbol{r}(s, t) = (1 + s + t)\hat{\boldsymbol{i}} + (1 + s + 2t)\hat{\boldsymbol{j}} + (1 + 5s + 8t)\hat{\boldsymbol{k}}$$

Hence the parametric equations are,

$x = 1 + s + t$

$y = 1 + s + 2t$

$z = 1 + 5s + 8t$

Note: The parametric equations are far from unique. This is demonstrated in **b.**

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Skill-sheet

Exercise 6I

Example 30

1 Write parametric vector equations of the planes which the following given points.

a $A(1, 1, 1)$, $B(1, -1, 1)$, $C(1, 1, -1)$

b $A(2, 0, -1)$, $B(1, -1, 2)$, $C(3, 1, -1)$

c $A(-1, 1, 0)$, $B(2, -1, 2)$, $C(0, 1, -1)$

d $A(1, 0, 1)$, $B(-1, -2, 3)$, $C(3, 1, -1)$

Example 31

2 **a** The point $(13, 11, -2)$ lies on the plane with a parametric vector equation

$$\boldsymbol{r} = (2 + 2s + t)\hat{\boldsymbol{i}} + (1 + s + 2t)\hat{\boldsymbol{j}} + (3 - 2s + t)\hat{\boldsymbol{k}}.$$

Determine the corresponding values of s and t.

b Show that the point $(3, 18, 15)$ does not lie on the plane with a parametric vector equation
$\boldsymbol{r} = (2 + 3s + t)\hat{\boldsymbol{i}} + (1 + s + t)\hat{\boldsymbol{j}} + (3 - 2s)\hat{\boldsymbol{k}}$

3 **a** If $m\hat{\boldsymbol{i}}$ is the position vector of a point on the plane with parametric vector equation

$$\boldsymbol{r} = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) + s(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}) + t(2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}), \qquad s, t \in \mathbb{R}$$

determine the value of m.

b If $m\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ is the position vector of a point on the plane with parametric vector equation

$$\boldsymbol{r} = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) + t(3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}), \qquad s, t \in \mathbb{R}$$

determine the value of m.

c If $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ is the position vector of a point on the plane with parametric vector equation

$$\boldsymbol{r} = m\hat{\boldsymbol{j}} + s(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) + t(4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}}), \qquad s, t \in \mathbb{R}$$

determine the value of m.

4 Use the equation of the plane

$$\boldsymbol{r}(s, t) = \boldsymbol{a} + s(\boldsymbol{b} - \boldsymbol{a}) + t(\boldsymbol{c} - \boldsymbol{a})$$

where $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are position vectors of points A, B and C on the plane to show that there exist real numbers ℓ, m and n such that:

$$\boldsymbol{r} = \ell\boldsymbol{a} + m\boldsymbol{b} + n\boldsymbol{c}, \text{ where } \ell + m + n = 1$$

Example 32

5 The parametric equations of a plane are:

$$x = 1 + 2s - 2t$$
$$y = 1 + t$$
$$z = -3s + t$$

Determine the Cartesian equation of the plane.

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Example 33

6 A plane, Π_1, is described by the parametric equations:

$$x = 1 + s + 2t$$
$$y = 2 - s - t$$
$$z = 1 + 2s + t$$

a Determine a parametric vector equation of the plane, Π_1. That is, in the form, $\boldsymbol{r}(s,t) = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}$.

b Determine the vector equation of the plane. That is, in the form, $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$.

c Determine a Cartesian equation of Π_1.

d A second plane Π_2 parallel to Π_1 contains the point $(1, 2, -1)$. Determine the Cartesian equation of Π_2.

e Determine the shortest distance between Π_1 and Π_2.

Example 33

7 The Cartesian equation of a plane is $x - 2y + z = 4$. Determine

a a vector equation of the plane in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

b a parametric vector equation of the plane in the form $\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}$

c parametric equations derived directly from the parametric vector equation of the plane derived in **b.**

8 For each of the parametric vector equations below determine

i a Cartesian equation

ii a vector equation

a $\boldsymbol{r} = (4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) + s(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) + t(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$

b $\boldsymbol{r} = (\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}) + s(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) + t(-\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$

c $\boldsymbol{r} = (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) + s(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}) + t(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

9 A line ℓ has a vector equation $\boldsymbol{r}(\lambda) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \lambda(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$. A plane, Π_1, is described by the parametric vector equation $\boldsymbol{r}(s,t) = 4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}) + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

a Determine the position vector of the point A where the line ℓ intersects the plane Π_1.

b Determine the angle, to the nearest whole degree, between the line ℓ, and the plane Π_1.

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Review

Chapter summary

Vector functions

- The motion of a particle in three dimensions can be specified by a **vector function** of the form

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$$

where $\boldsymbol{r}(t)$ is the position vector of the particle at time t.

- The set of all points $(x(t), y(t), z(t))$ forms a curve in three-dimensional space; this curve is the path of the particle. We say that $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$ is a **vector equation** of the curve (with parameter t).

Lines

A line in three dimensions can be described as follows, where $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ is the position vector of a point A on the line, and $\boldsymbol{d} = d_1\hat{\boldsymbol{i}} + d_2\hat{\boldsymbol{j}} + d_3\hat{\boldsymbol{k}}$ is parallel to the line.

Vector equation	Parametric equations	Cartesian form
$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}, \quad t \in \mathbb{R}$	$x = a_1 + d_1 t$ $y = a_2 + d_2 t$ $z = a_3 + d_3 t$	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Planes

A plane in three dimensions can be described as follows, where $\boldsymbol{a}$ is the position vector of a point A on the plane, the vector $\boldsymbol{n} = n_1\hat{\boldsymbol{i}} + n_2\hat{\boldsymbol{j}} + n_3\hat{\boldsymbol{k}}$ is normal to the plane, and $k = \boldsymbol{a} \cdot \boldsymbol{n}$.

Vector equation	Cartesian equation
$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$	$n_1 x + n_2 y + n_3 z = k$

Spheres

A sphere in three dimensions can be described as follows, where $\boldsymbol{c} = c_1\hat{\boldsymbol{i}} + c_2\hat{\boldsymbol{j}} + c_3\hat{\boldsymbol{k}}$ is the position vector of the centre C, and a is the radius.

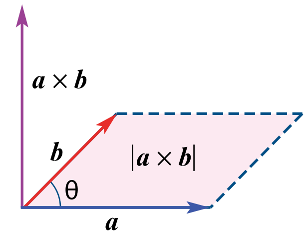
Vector equation	Cartesian equation
$\lvert\boldsymbol{r} - \boldsymbol{c}\rvert = a$	$(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2$

Vector product

- If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}}$, then

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\hat{\boldsymbol{i}} - (a_1 b_3 - a_3 b_1)\hat{\boldsymbol{j}} + (a_1 b_2 - a_2 b_1)\hat{\boldsymbol{k}}$$

- The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to $|\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (provided $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors and not parallel).



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Distances and angles

- **Distance from a point to a line** The distance from a point P to a line ℓ is given by $|\overrightarrow{PQ}|$, where Q is the point on the line such that PQ is perpendicular to the line.
- **Distance from a point to a plane** The distance from a point P to a plane Π is given by $|\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}|$, where $\hat{\boldsymbol{n}}$ is a unit vector normal to the plane and Q is any point on the plane.
- **Angle between two lines** First determine the angle θ between two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ that are parallel to the two lines. The angle between the lines is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.
- **Angle between two planes** First determine the angle θ between two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ that are normal to the two planes. The angle between the planes is θ or $180° - \theta$, whichever is in the interval $[0°, 90°]$.
- **Angle between a line and a plane** The angle between a line ℓ and a plane Π is $90° - \theta$, where θ is the acute angle between the line and a normal to the plane.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

6A **1** I can determine a Cartesian equation for a curve given the vector function for this curve. ☐

See Example 1, Example 2 and Question 1

6A **2** I can determine a Cartesian equation and the corresponding domain and range for a curve given the vector function for this curve. ☐

See Example 3 and Question 2

6B **3** I can determine the path of a particle given a function for its position at time t. ☐

See Example 4, Example 5 and Questions 1

6B **4** I can determine if two moving particles collide or their paths cross given functions for their position at time t. ☐

See Example 6 and Question 3

6C **5** I can determine if a point lies on a line described by is vector equation. ☐

See Example 7 and Question 1

6C **6** I can determine the vector equation of a line given sufficient information. ☐

See Example 8, Example 9, Example 10 and Questions 2, 3 and 6

6C **7** I can determine the distance of a point from a line. ☐

See Example 11 and Question 12

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- 6C **8** I can describe line segments with vector equations. ☐
  See Example 12 and Question 13
- 6D **9** I can determine the position vector of the point of intersection of two lines. ☐
  See Example 13, Example 14 and Question 3
- 6D **10** I can determine if two lines are skew lines. ☐
  See Example 15 and Questions 1 and 2
- 6D **11** I can determine if three lines are concurrent. ☐
  See Example 16 and Question 5
- 6D **12** I can determine the angle between two lines. ☐
  See Example 17 and Question 6
- 6E **13** I can determine the vector product of two vectors. ☐
  See Example 18 and Question 1
- 6E **14** I can simplify expressions involving vector products. ☐
  See Example 19 and Question 2
- 6F **15** I can determine the vector and Cartesian equation of a plane given sufficient information. ☐
  See Example 20, Example 21, Example 21 and Questions 1, 2 and 3
- 6G **16** I can determine the distance of a point from a plane. ☐
  See Example 23 and Question 1
- 6G **17** I can determine the distance between two parallel planes. ☐
  See Example 24, Example 25 and Question 2
- 6G **18** I can determine the angle between two planes and the vector equation of the line of intersection of two planes. ☐
  See Example 26 and Question 3
- 6G **19** I can determine the point of intersection of a line with a plane and determine the angle between a line and a plane. ☐
  See Example 27 and Question 4
- 6G **20** I can determine the Cartesian and vector equations of a sphere. ☐
  See Example 28 and Question 1
- 6G **21** I can determine the points of intersection of a line with a sphere. ☐
  See Example 29 and Question 3

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Short-response questions

Technology-free short-response questions

1 The position of a particle at time t is given by

$$\boldsymbol{r}(t) = (3t+1)\hat{\boldsymbol{i}} + (t^2 - t - 1)\hat{\boldsymbol{j}} + (4 - t^2)\hat{\boldsymbol{k}}, \quad t \geq 0$$

a What is the particle's position vector at time $t = 2$?

b Determine the time at which the particle's position vector is $4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$.

2 The motion of two particles with respect to time t is given by the vector functions $\boldsymbol{r}_1(t) = (4t-5)\hat{\boldsymbol{i}} + (t^2+12)\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(t) = (3t-2)\hat{\boldsymbol{i}} + 7t\hat{\boldsymbol{j}}$, where $t \geq 0$. Determine:

a the point at which the particles collide

b the points at which the two paths cross

c the distance between the particles at time $t = 1$.

3 Determine Cartesian equations of the curves given by the following vector equations:

a $\boldsymbol{r}(t) = 3\cos(2t)\,\hat{\boldsymbol{i}} - 4\sin(2t)\,\hat{\boldsymbol{j}}$, for $0 \leq t \leq 2\pi$

b $\boldsymbol{r}(t) = 2t\,\hat{\boldsymbol{i}} + 3t^2\hat{\boldsymbol{j}}$, for $t \geq 3$

c $\boldsymbol{r}(t) = 3\cos(2t)\,\hat{\boldsymbol{i}} - 3\cos(t)\,\hat{\boldsymbol{j}}$, for $0 \leq t \leq 2\pi$

d $\boldsymbol{r}(t) = (5 - \sin(3t))\,\hat{\boldsymbol{i}} - 4\cos(3t)\,\hat{\boldsymbol{j}}$, for $0 \leq t \leq 2\pi$

4 Determine the domain and range of each Cartesian relation found in Question 3.

5 Determine the position vector of the point of intersection of the lines $\boldsymbol{r}_1 = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + \lambda(3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$ and $\boldsymbol{r}_2 = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}} + \mu(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}})$.

6 Show that the lines $\boldsymbol{r}_1 = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \lambda(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}})$ and $\boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \mu(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ do not intersect.

7 Determine a Cartesian equation of the plane through the point $(1, 2, 3)$ with normal vector $4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$.

8 Determine a vector equation of the line parallel to the x-axis that contains the point $(-2, 2, 1)$.

9 Determine the coordinates of the nearest point to $(2, 1, 3)$ on the line $\boldsymbol{r} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$.

10 Determine the distance from the origin to the line passing through the point $(3, 1, 5)$ parallel to the vector $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

11 Determine the coordinates of the point of intersection of the line $\boldsymbol{r} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$, $t \in \mathbb{R}$, and the plane $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}) = 13$.

12 Determine a vector that is perpendicular to the vectors $8\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

Review SF CF SF CF SF

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Review

13 The line ℓ passes through the points $A(-1, -3, -3)$ and $B(5, 0, 6)$. Determine a vector equation of the line ℓ. Determine the point P on the line ℓ such that OP is perpendicular to the line, where O is the origin. CF

14 Show that the lines $\boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + \lambda(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ and $\boldsymbol{r}_2 = \hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}} + \mu(\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}})$ are skew lines. Determine the cosine of the angle between the lines.

15 Determine a Cartesian equation of the plane that contains the points $P(1, -2, 0)$, $Q(3, 1, 4)$ and $R(0, -1, 2)$. SF

16 Determine a Cartesian equation of the plane that contains the points $P(1, -2, 1)$, $Q(-2, 5, 0)$ and $R(-4, 3, 2)$.

17 Determine an equation of the plane through $A(-1, 2, 0)$, $B(3, 1, 1)$ and $C(1, 0, 3)$ in:

a vector form **b** Cartesian form.

18 **a** Determine a Cartesian equation of the plane passing through the origin O and the points $A(1, 1, 1)$ and $B(0, 1, 2)$. CF

b Determine the area of triangle OAB.

c Show that the point $C(-2, 2, 6)$ lies on the plane and determine the point of intersection of the lines OB and AC.

19 The origin O and the point $A(2, -1, -1)$ are two vertices of an equilateral triangle OAB in the plane $x + y + z = 0$. Determine the coordinates of the vertex B.

20 Show that the four points $(1, 0, 0)$, $(2, 1, 0)$, $(3, 2, 1)$ and $(4, 3, 2)$ are coplanar. SF

21 For vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ such that $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$, show that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c} \times \boldsymbol{a}$. CF

Technology-active short-response questions

22 Two lines are represented by vector equations $\boldsymbol{r}_1 = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t_1(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, $t_1 \in \mathbb{R}$, and $\boldsymbol{r}_2 = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t_2(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, $t_2 \in \mathbb{R}$. CF

a Show that these lines intersect and determine their point of intersection, P.

b The vector equation $\boldsymbol{r}_3 = t_3(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, $t_3 \in \mathbb{R}$, represents a line through the origin. Determine the distance from the point of intersection P to this line.

23 The points A, B and C have position vectors $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, $\overrightarrow{OB} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ and $\overrightarrow{OC} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}}$. The plane Π_1 has vector equation $\boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 6$.

a Show that the point C is on the plane Π_1.

b Show that the point B is the reflection in the plane Π_1 of the point A.

c Determine the length of the projection of $\overrightarrow{AC}$ onto the plane Π_1.

The plane Π_2 has Cartesian equation $12x - 4y + 3z = k$, where k is a positive constant.

d Determine the acute angle between planes Π_1 and Π_2.

e Given that the distance from the point C to the plane Π_2 is 3, determine the value of k.

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Review

24 Consider the two lines given by CU

$$\ell_1: \quad \boldsymbol{r}_1 = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}), \qquad t \in \mathbb{R}$$
$$\ell_2: \quad \boldsymbol{r}_2 = 16\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + s(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}), \qquad s \in \mathbb{R}$$

a Show that ℓ_1 and ℓ_2 are skew lines.

b Verify that both ℓ_1 and ℓ_2 are perpendicular to the vector $\boldsymbol{n} = 5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

c The point $A(3, 2, 1)$ lies on line ℓ_1. Write down a vector equation of the line ℓ_3 through A in the direction of $\boldsymbol{n}$.

d Determine the point of intersection, B, of the lines ℓ_2 and ℓ_3, and determine the length of the line segment AB.

25 The point O is the origin and the points A, B, C and D have position vectors CF

$$\overrightarrow{OA} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}, \quad \overrightarrow{OB} = 6\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}, \quad \overrightarrow{OC} = 9\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}, \quad \overrightarrow{OD} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

Prove that:

a the triangle OAB is isosceles

b the point D lies in the plane OAB

c the line CD is perpendicular to the plane OAB

d the line AC is inclined at an angle of 60° to the plane OAB.

26 A Cartesian equation of the plane Π_1 is $y + z = 0$ and a vector equation of the line ℓ is $\boldsymbol{r} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})$, where $t \in \mathbb{R}$. Determine: CU

a the position vector of the point of intersection of the line ℓ and the plane Π_1

b the length of the perpendicular from the origin to the line ℓ

c a Cartesian equation of the plane Π_2 which contains the line ℓ and the origin

d the acute angle between the planes Π_1 and Π_2, correct to one decimal place.

27 $ABCD$ is a parallelogram where A, B, C have position vectors $\overrightarrow{OA} = (\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$, $\overrightarrow{OB} = (2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$ and $\overrightarrow{OC} = (3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ respectively. CF

a Determine the vector which represents D.

b Determine a vector equation for the line BC.

c If E is a point on the line BC such that AE is perpendicular to BC then determine the coordinates of E.

28 Two lines are represented by the vector equations $\boldsymbol{r}_1 = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ and $\boldsymbol{r}_2 = \lambda\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}} + s(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$.

a Determine the value of λ for which the lines intersect.

b For this value of λ, determine a vector perpendicular to the plane which contains these two lines.

c Determine a Cartesian equation for this plane.

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29 The position vectors of two points $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ lie in plane Π_1 which passes through the origin O.

- **a** Determine the equation of plane Π_1.
- **b** Point A has position vector $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and lies in a plane Π_2 which is parallel to plane Π_1. Determine the equation of plane Π_2.
- **c** Determine the distance between the two planes.

30 A sphere is defined by the equation $|\boldsymbol{r} - (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})| = 7$.

- **a** Determine the coordinates of the points where the sphere intersects the x-axis.
- **b** Show that the point A with position vector $\overrightarrow{OA} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}}$ is a point on the sphere.
- **c** A is a point on the line $\boldsymbol{r} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$. Determine the other point on the line which also lies on the sphere.

31 The points A, B, and C have position vectors with respect to a point O of $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ respectively.

- **a** Determine the equation of the plane defined by points A, B, and C .
- **b** Let $\boldsymbol{v} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + (1 - \lambda - \mu)\boldsymbol{c}$. Determine an expression for $\boldsymbol{v}$ in terms of λ and μ.
- **c** Show that $\boldsymbol{v}$ lies on the plane found in part **a** for all values of λ and μ.

32 The equations of two planes are given as $x + 2z = 3$ and $y - z = 2$.

- **a** For $z = t$, solve the above equations to determine expressions for x and y in terms of t.
- **b** The expressions for x, y and z in terms of t represent the parametric representation of the line of intersection of the two planes. Write down the vector equation of this line.
- **c** A third plane has the equation $3x + y - 4z = 2$. Determine the point of intersection of the line found in part **b** and this plane.
- **d** Comment on the concurrence of the 3 planes.

33 The Cartesian form of a line in three dimensions uses two linear equations in x, y and z. In this question, we show that a line in three dimensions can also be described by a single quadratic equation in x, y and z.

- **a** Let $a, b \in \mathbb{R}$. Show that $a = b = 0$ if and only if $a^2 + b^2 = 0$.
- **b** Show that the z-axis is described by the Cartesian equation $x^2 + y^2 = 0$.
- **c** Show that the line given by the Cartesian equations

$$x - 3 = 2(y - 4) = z + 1$$

is also given by the single quadratic equation

$$(x - 2y + 5)^2 + (x - z - 4)^2 = 0$$

- **d** Write a single quadratic equation for the line with Cartesian form

$$\frac{x}{2} = y - 3 = 4z + 5$$

- **e** Write a single quadratic equation for the line with vector equation

$$\boldsymbol{r}(t) = (2 + t)\hat{\boldsymbol{i}} + (3 - t)\hat{\boldsymbol{j}} + 5t\hat{\boldsymbol{k}}, \quad t \in \mathbb{R}$$

34 An aeroplane takes off from an airport. With respect to a given frame of reference, its position at time t seconds after take-off is given by the vector function

$$\boldsymbol{r}(t) = (5 - 3t)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}} + t\hat{\boldsymbol{k}}, \quad t \geq 0$$

a Determine the position vector of the aeroplane at take-off.

b Determine:

i the position vector of the aeroplane at time t_1 and at time t_2

ii the aeroplane's displacement vector between times t_1 and t_2, where $t_1 < t_2$.

c Hence show that the aeroplane is travelling along a straight line, and state a vector in the direction of the flight path.

d A road on the ground is described by the vector function $\boldsymbol{r}_1(s) = s\hat{\boldsymbol{i}}$, $s \leq 0$.

i Determine the magnitude of the acute angle between the flight path and the road, correct to two decimal places.

ii Determine the shortest distance from the aeroplane to the road at time 6 seconds after take-off, correct to two decimal places.

35 The vector function $\boldsymbol{r}_1(t) = (2 - t)\hat{\boldsymbol{i}} + (2t + 1)\hat{\boldsymbol{j}}$ represents the position of a particle at time t, measured in seconds.

a Determine a Cartesian equation that describes the path of the particle. (Assume $t \geq 0$.)

b **i** Rearrange the vector function in the form $\boldsymbol{r}_1(t) = \boldsymbol{a} + t\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vector constants.

ii Describe the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ geometrically with respect to the path of the particle.

c The vector function $\boldsymbol{r}_2(t) = \boldsymbol{c} + t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$ represents the position of a second particle at time t. Both particles start moving at the same time ($t = 0$) and they collide after 5 seconds.

i Determine $\boldsymbol{c}$.

ii Determine the distance between the two starting points.

36 The paths of two aeroplanes in an aerial display are simultaneously defined by the vector functions

$$\boldsymbol{r}_1(t) = (16 - 3t)\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}} + (3 + 2t)\hat{\boldsymbol{k}}$$
$$\boldsymbol{r}_2(t) = (3 + 2t)\hat{\boldsymbol{i}} + (1 + t)\hat{\boldsymbol{j}} + (11 - t)\hat{\boldsymbol{k}}$$

where t represents time in minutes ($t \geq 0$). Determine:

a the position of the first aeroplane after 1 minute

b the unit vector in the direction of motion of each of the two aeroplanes

c the acute angle between their flight paths, correct to two decimal places

d the point at which the two flight paths cross

e the vector $\boldsymbol{r}_2(t) - \boldsymbol{r}_1(t)$ representing the displacement between the two aeroplanes at time t minutes

f the shortest distance between the two aeroplanes during their flights.

Review

CU

37 A regular tetrahedron has vertices

$$A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1), \quad D = (1, 1, 1)$$

a Show that every edge of the tetrahedron has the same length. What is it?

b Show that every face of the tetrahedron is an equilateral triangle with the same area. What is it?

c Determine the equations of the planes ABC and BCD. What is the angle between them?

d Show that the angle between any two faces of the tetrahedron is $\cos^{-1}\left(\frac{1}{3}\right)$.

38 A regular octahedron has vertices

$$A = (1, 0, 0), \quad B = (0, 1, 0), \quad C = (0, 0, 1),$$
$$A' = (-1, 0, 0), \quad B' = (0, -1, 0), \quad C' = (0, 0, -1)$$

and its edges are $AB, AC, AB', AC', BC, BA', BC', CA', CB', A'B', A'C'$ and $B'C'$.

a Show that every edge of the octahedron has the same length. What is it?

b Show that every face of the octahedron is an equilateral triangle with the same area. What is it?

c Show that opposite faces of the octahedron lie in parallel planes.

d Determine the equations of the planes ABC and $A'BC$. What is the angle between them?

e Show that obtuse the angle between any two adjacent faces of the octahedron is $\cos^{-1}\left(-\frac{1}{3}\right)$.

Note: It follows from Questions 37 and 38 that the angle between faces in a regular tetrahedron and the angle between faces in a regular octahedron sum to 180°. In fact, the regular tetrahedron and octahedron in these questions fit together and extend in a pattern to form a tessellation of three-dimensional space!

Multiple-choice questions

Technology-free multiple-choice questions

1 The vector function $\boldsymbol{r}(t) = (2t+1)\hat{\boldsymbol{i}} + (4t-1)\hat{\boldsymbol{j}} + (3-2t)\hat{\boldsymbol{k}}$ gives the position of a particle at time t, where $t \geq 0$. The position of the particle at time $t = 1$ is

A $(1, 1, 1)$ **B** $(3, 3, 1)$ **C** $(1, -1, 1)$ **D** $(3, -3, 1)$

2 The vector function $\boldsymbol{r}(t) = (t^2-5t+6)\hat{\boldsymbol{i}} + (t^2-3t-4)\hat{\boldsymbol{j}} + (4-2t)\hat{\boldsymbol{k}}$ gives the position of a particle at time t, where $t \geq 0$. The particle has position $(2, -6, 2)$ at time

A $t = 1$ **B** $t = 2$ **C** $t = 3$ **D** $t = 4$

3 The three vertices of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. The area of the triangle is equal to

A $\boldsymbol{a} \times \boldsymbol{b}$ **B** $\frac{1}{2}|\boldsymbol{b} \times \boldsymbol{c}|$
C $\frac{1}{2}|(\boldsymbol{a}-\boldsymbol{b}) \times (\boldsymbol{b}-\boldsymbol{c})|$ **D** $(\boldsymbol{b}-\boldsymbol{c}) \times (\boldsymbol{a}-\boldsymbol{b})$

4 A plane has equation $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}) = 5$. The distance from the origin to this plane is

A 5 **B** $\dfrac{5}{4}$ **C** $\dfrac{5}{6}$ **D** $\dfrac{5}{\sqrt{6}}$

5 Which of the following is the Cartesian equation of the plane that has axis intercepts at $x = 1$, $y = 2$ and $z = 3$?

A $x + 2y + 3z = 0$ **B** $x + 2y + 3z = 1$
C $x + 2y + 3z = 6$ **D** $6x + 3y + 2z = 6$

6 The line given by $\boldsymbol{r} = 5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + \lambda(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$, $\lambda \in \mathbb{R}$, intersects the plane given by $\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = -6$ at the point with position vector

A $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ **B** $5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ **C** $3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ **D** $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

7 The distance from the point $P(1, 5)$ to the line that passes through $(0, 0)$ and $(1, 1)$ is

A 1 **B** $\sqrt{2}$ **C** 2 **D** $2\sqrt{2}$

8 Which of the following vector equations does *not* describe the line passing through the points $(2, 0, 1)$ and $(3, 3, 3)$?

A $\boldsymbol{r} = (2+t)\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}} + (1+2t)\hat{\boldsymbol{k}}$ **B** $\boldsymbol{r} = (3-t)\hat{\boldsymbol{i}} + (3-3t)\hat{\boldsymbol{j}} + (3-2t)\hat{\boldsymbol{k}}$
C $\boldsymbol{r} = (2+3t)\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}} + (1+3t)\hat{\boldsymbol{k}}$ **D** $\boldsymbol{r} = (1+t)\hat{\boldsymbol{i}} + (-3+3t)\hat{\boldsymbol{j}} + (-1+2t)\hat{\boldsymbol{k}}$

9 Which of the following equations describes the plane that contains the points $(0, 0, 1)$, $(1, 1, 1)$ and $(2, 0, 0)$?

A $x - y + 2z = 2$ **B** $x - y + z = 1$
C $x + 3y - z = 4$ **D** $x - y = 0$

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10 Which of the following vectors is parallel to the line of intersection of the planes $2x + y + z = 6$ and $x + z = 0$?

A $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ **B** $\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$
C $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ **D** $-3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

11 The distance from the origin to the plane $x + 2y + 2z = 5$ is

A $\frac{1}{5}$ **B** $\frac{1}{3}$ **C** 1 **D** $\frac{5}{3}$

12 The triangle with vertices $(0, 0, 1)$, $(1, 1, 1)$ and $(2, 3, 2)$ has area

A $\frac{1}{5}$ **B** $\frac{\sqrt{2}}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** 1

13 The origin, O, is a point on a sphere with centre $(6, 8, 0)$. The equation of the sphere is

A $(x + 6)^2 + (y + 8)^2 = 14$ **B** $(x - 6)^2 + (y - 8)^2 = 100$
C $(x - 6)^2 + (y - 8)^2 + z^2 = 100$ **D** $(x - 6)^2 + (y - 8)^2 + z^2 = 10$

14 The line with equation $\boldsymbol{r} = t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}})$ intersects the sphere $|\boldsymbol{r} - \hat{\boldsymbol{i}}| = 5$ for

A $t = -3$ or $t = 4$ **B** $t = -4$ or $t = 3$
C $t = -5$ or $t = 1$ **D** $t = -1$ or $t = 5$

Technology-active multiple-choice questions

15 The line $\boldsymbol{r} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ passes through the plane $5x - y - 4z = 2$ at the point with coordinates

A $(1, 1, 2)$ **B** $(3, -3, 4)$ **C** $(3, -3, -4)$ **D** $(3, 3, 4)$

16 If $\boldsymbol{r} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, $t \in [0, 2]$ defines one diameter of a sphere then an equation representing the sphere could be

A $|\boldsymbol{r} - (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}| = \sqrt{6}$ **B** $|\boldsymbol{r} - (2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{k}}| = \sqrt{6}$
C $|\boldsymbol{r} - (\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}| = 2$ **D** $|\boldsymbol{r} - (3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}| = \sqrt{46}$

17 The acute angle between the planes $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}) = 2$ and $\boldsymbol{r} \cdot (3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}) = 5$ is closest to

A 12° **B** 32° **C** 78° **D** 82°

18 The acute angle between the line $\boldsymbol{r} = (2\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}) + \lambda(3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ and the plane $\boldsymbol{r} \cdot (5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}) = 2$ is closest to

A 7° **B** 22° **C** 68° **D** 83°

19 A parallelogram has vertices $A(1, 0, 2)$, $B(3, 3, 3)$, $C(7, 5, 8)$ and $D(5, 2, 7)$. The area of the parallelogram correct to one decimal place is

A 14.2 **B** 15.6 **C** 16.4 **D** 17.2

20 The angle between the planes $x - y + 3z = 1$ and $3x + y - z = 2$ correct to one decimal place is.

A 87.1° **B** 95.2° **C** 98.6° **D** 83.7°

7

Vector calculus

Chapter contents

In Mathematical Methods Units 3 & 4, you have studied motion in a straight line, and used the vector quantities of position, velocity and acceleration to describe the motion. In this chapter, we consider motion in two dimensions.

We have already seen in Chapter 6 that the motion of a particle in two dimensions can be represented by a vector function of the form

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$$

where $\boldsymbol{r}(t)$ is the position vector of the particle at time t.

We have used vector functions to determine Cartesian equations for the paths of particles, and to determine whether or not two particles will collide.

In this chapter, we will see how to differentiate and antidifferentiate vector functions. This means that we will be able to determine the velocity and acceleration of a particle moving along a curve in two dimensions.

This chapter covers Unit 3 Topic 4: Vector calculus.

7A Summary of differentiation and anti-differentiation

Learning intentions

- To revise basic differentiation and anti-differentiation techniques.

Differentiation

The derivative of a function f is denoted by f' and is defined by

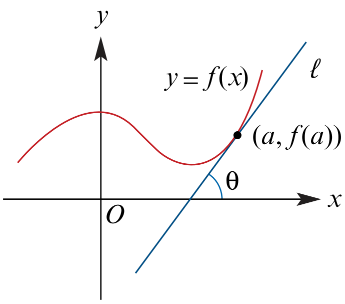
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative f' is also known as the gradient function.

If $(a, f(a))$ is a point on the graph of $y = f(x)$, then the gradient of the graph at that point is $f'(a)$.

If the line ℓ is the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ and ℓ makes an angle of θ with the positive direction of the x-axis, as shown, then

$$f'(a) = \text{gradient of } \ell = \tan\theta$$



Review of differentiation

Here we summarise basic derivatives and rules for differentiation covered in Mathematical Methods Units 3 & 4.

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^a	ax^{a-1}	where $a \in \mathbb{R} \setminus \{0\}$
e^x	e^x	
$\ln x$	$\frac{1}{x}$	for $x > 0$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	

Chain rule

- If $q(x) = f(g(x))$, then

  $$q'(x) = f'(g(x))\, g'(x)$$

- If $y = f(u)$ and $u = g(x)$, then

  $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Product rule

- If $f(x) = u(x) \cdot v(x)$, then

  $$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

- If $y = uv$, then

  $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

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Quotient rule

- If $f(x) = \dfrac{u(x)}{v(x)}$, then
$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$
- If $y = \dfrac{u}{v}$, then
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 1

Differentiate each of the following with respect to x:

a $\sqrt{x}\sin x$ **b** $\dfrac{x^2}{\sin x}$ **c** $\cos(x^2 + 1)$

Solution

a Let $f(x) = \sqrt{x}\sin x$.

Applying the product rule:
$$\begin{aligned} f'(x) &= x^{\frac{1}{2}} \cos x + \frac{1}{2}x^{-\frac{1}{2}} \sin x \\ &= \sqrt{x}\cos x + \frac{\sqrt{x}\sin x}{2x} \end{aligned}$$

b Let $f(x) = \dfrac{x^2}{\sin x}$.

Applying the quotient rule:
$$f'(x) = \frac{2x\sin x - x^2 \cos x}{\sin^2 x}$$

c Let $y = \cos(x^2 + 1)$ and let $u = x^2 + 1$.
Then $y = \cos u$, so by the chain rule:
$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\sin u \cdot 2x \\ &= -2x\sin(x^2 + 1) \end{aligned}$$

The derivative of $\tan(ax + b)$

- If $f(x) = \tan x$, then $f'(x) = \sec^2 x$.
- If $f(x) = \tan(ax + b)$, then $f'(x) = a\sec^2(ax + b)$.

Proof Let $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. Then the quotient rule yields
$$\begin{aligned} f'(x) &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

The second result is obtained from the first result by using the chain rule.

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Example 2

Differentiate each of the following with respect to x:

a $\tan(5x^2 + 3)$ **b** $\tan^3 x$ **c** $\sec^2(3x)$

Solution

a Let $f(x) = \tan(5x^2 + 3)$.
By the chain rule with $g(x) = 5x^2 + 3$, we have

$$\begin{aligned} f'(x) &= \sec^2(5x^2 + 3) \cdot 10x \\ &= 10x \sec^2(5x^2 + 3) \end{aligned}$$

b Let $f(x) = \tan^3 x = (\tan x)^3$.
By the chain rule with $g(x) = \tan x$, we have

$$\begin{aligned} f'(x) &= 3(\tan x)^2 \cdot \sec^2 x \\ &= 3 \tan^2 x \sec^2 x \end{aligned}$$

c Let
$$\begin{aligned} y &= \sec^2(3x) \\ &= \tan^2(3x) + 1 \qquad \text{(using the Pythagorean identity)} \\ &= (\tan(3x))^2 + 1 \end{aligned}$$

Let $u = \tan(3x)$. Then $y = u^2 + 1$ and the chain rule gives

$$\begin{aligned} \frac{dy}{dx} &= 2u \cdot 3 \sec^2(3x) \\ &= 6 \tan(3x) \sec^2(3x) \end{aligned}$$

Second derivatives

In Mathematical Methods Units 3 & 4, you will study the second derivative and its important role in graph sketching. In this chapter, we use the second derivative to describe acceleration.

The second derivative of a function is just the derivative of the derivative. For example, consider the function f with rule $f(x) = 2x^3 - 4x^2$.

- The derivative has rule $f'(x) = 6x^2 - 8x$.
- The second derivative has rule $f''(x) = 12x - 8$.

In Leibniz notation, the second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Example 3

Determine the second derivative of each of the following with respect to x:

a $f(x) = 6x^4 - 4x^3 + 4x$ **b** $y = e^x \sin x$

Solution

a
$$\begin{aligned} f(x) &= 6x^4 - 4x^3 + 4x \\ f'(x) &= 24x^3 - 12x^2 + 4 \\ f''(x) &= 72x^2 - 24x \end{aligned}$$

b
$$\begin{aligned} y &= e^x \sin x \\ \frac{dy}{dx} &= e^x \cos x + e^x \sin x \qquad \text{(by the product rule)} \\ \frac{d^2y}{dx^2} &= -e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x \\ &= 2e^x \cos x \end{aligned}$$

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Anti-differentiation

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of determining a function from its derivative is called **anti-differentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

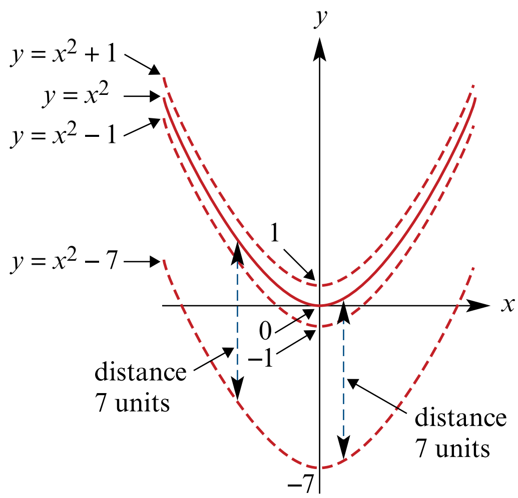
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then it can be proved that they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



Notation

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x \, dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\begin{aligned}\int 2x \, dx &= \{ f(x) : f'(x) = 2x \} \\ &= \{ x^2 + c : c \in \mathbb{R} \}\end{aligned}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

In general:

> If $F'(x) = f(x)$, then $\int f(x) \, dx = F(x) + c$, where c is an arbitrary real number.

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Basic antiderivatives

The following antiderivatives are covered in Mathematical Methods Units 3 & 4.

$f(x)$	$\int f(x)\,dx$	
x^n	$\dfrac{x^{n+1}}{n+1} + c$	where $n \neq -1$
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + c$	where $n \neq -1$
x^{-1}	$\ln x + c$	for $x > 0$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln(ax+b) + c$	for $ax+b > 0$
e^{ax+b}	$\dfrac{1}{a}e^{ax+b} + c$	
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$	
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$	

Example 4

Antidifferentiate each of the following:

a $\sin\left(3x - \dfrac{\pi}{4}\right)$ **b** e^{3x+4} **c** $6x^3 - \dfrac{2}{x^2}$

Solution

a $\sin\left(3x - \dfrac{\pi}{4}\right)$ is of the form $\sin(ax+b)$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\therefore \quad \int \sin\left(3x - \frac{\pi}{4}\right)dx = -\frac{1}{3}\cos\left(3x - \frac{\pi}{4}\right) + c$$

b e^{3x+4} is of the form e^{ax+b}

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

$$\therefore \quad \int e^{3x+4}\,dx = \frac{1}{3}e^{3x+4} + c$$

c
$$\begin{aligned}\int 6x^3 - \frac{2}{x^2}\,dx &= \int 6x^3 - 2x^{-2}\,dx \\ &= \frac{6x^4}{4} + 2x^{-1} + c \\ &= \frac{3}{2}x^4 + \frac{2}{x} + c\end{aligned}$$

Given extra information, we can determine a unique antiderivative. This is demonstrated in the next example.

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Example 5

a Given that $f'(x) = 3x^2 + 1$ and $f(0) = 1$, determine $f(x)$.

b Given that $f''(x) = -9.8$ with $f(0) = 0$ and $f'(0) = 3$, determine $f(x)$.

Solution

a $\int 3x^2 + 1 \, dx = x^3 + x + c$

Thus $f(x) = x^3 + x + c$ for some real number c.

Since $f(0) = 1$, we must have $c = 1$.

Hence $f(x) = x^3 + x + 1$.

b We are given

$$f''(x) = -9.8$$
$$\therefore \quad f'(x) = -9.8x + c_1$$

Since $f'(0) = 3$, we have $c_1 = 3$ and so

$$f'(x) = -9.8x + 3$$
$$\therefore \quad f(x) = -4.9x^2 + 3x + c_2$$

Since $f(0) = 0$, we have $c_2 = 0$ and so

$$f(x) = -4.9x^2 + 3x$$

Skill-sheet

Exercise 7A

SF

Example 1

1 Determine the derivative of each of the following with respect to x:

a $x^5 \sin x$ **b** $\sqrt{x} \cos x$ **c** $e^x \cos x$ **d** $x^3 e^x$ **e** $\sin x \cos x$

Example 2

2 Determine the derivative of each of the following with respect to x:

a $e^x \tan x$ **b** $x^4 \tan x$ **c** $\tan x \ln x$ **d** $\sin x \tan x$ **e** $\sqrt{x} \tan x$

3 Determine the derivative of each of the following using the quotient rule:

a $\dfrac{x}{\ln x}$ **b** $\dfrac{\sqrt{x}}{\tan x}$ **c** $\dfrac{e^x}{\tan x}$ **d** $\dfrac{\tan x}{\ln x}$

e $\dfrac{\sin x}{x^2}$ **f** $\dfrac{\tan x}{\cos x}$ **g** $\dfrac{\cos x}{e^x}$ **h** $\dfrac{\cos x}{\sin x}$ $(= \cot x)$

4 Determine the derivative of each of the following using the chain rule:

a $\tan(x^2 + 1)$ **b** $\sin^2 x$ **c** $e^{\tan x}$ **d** $\tan^5 x$

e $\sin(\sqrt{x})$ **f** $\sqrt{\tan x}$ **g** $\cos\left(\dfrac{1}{x}\right)$ **h** $\sec^2 x$

i $\tan\left(\dfrac{x}{4}\right)$ **j** $\cot x$ **Hint:** Use $\cot x = \tan\left(\dfrac{\pi}{2} - x\right)$.

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SF

5 Use appropriate techniques to determine the derivative of each of the following:

a $\tan(kx),\ k \in \mathbb{R}$ **b** $e^{\tan(2x)}$ **c** $\tan^2(3x)$ **d** $\ln(x)\, e^{\sin x}$

e $\sin^3(x^2)$ **f** $\dfrac{e^{3x+1}}{\cos x}$ **g** $e^{3x}\tan(2x)$ **h** $\sqrt{x}\,\tan(\sqrt{x})$

i $\dfrac{\tan^2 x}{(x+1)^3}$ **j** $\sec^2(5x^2)$

6 Determine $\dfrac{dy}{dx}$ for each of the following:

a $y = (x-1)^5$ **b** $y = \ln(4x)$ **c** $y = e^x \tan(3x)$ **d** $y = e^{\cos x}$

e $y = \cos^3(4x)$ **f** $y = (\sin x + 1)^4$ **g** $y = \sin(2x)\cos x$ **h** $y = \dfrac{x^2+1}{x}$

i $y = \dfrac{x^3}{\sin x}$ **j** $y = \dfrac{1}{x \ln x}$

Example 3

7 Determine the second derivative of each of the following:

a $(2x+5)^8$ **b** $\sin(2x)$ **c** $\cos\left(\dfrac{x}{3}\right)$ **d** $\tan x$

e e^{-4x} **f** $\ln(6x)$ **g** $\ln(\sin x)$ **h** $\tan(1-3x)$

i $\sec\left(\dfrac{x}{3}\right)$ **j** $\operatorname{cosec}\left(\dfrac{x}{4}\right)$

Example 4

8 Antidifferentiate each of the following:

a $\sin\left(2x + \dfrac{\pi}{4}\right)$ **b** $\cos(\pi x)$ **c** $\sin\left(\dfrac{2\pi x}{3}\right)$

d e^{3x+1} **e** $e^{5(x+4)}$ **f** $\dfrac{3}{2x^2}$

g $6x^3 - 2x^2 + 4x + 1$ **h** $\cos\left(\dfrac{3x}{2}\right)$

9 Antidifferentiate each of the following:

a $(3x+2)^5$ **b** $\dfrac{1}{3x-2},\quad x > \frac{2}{3}$ **c** $\sqrt{3x+2}$ **d** $\dfrac{1}{(3x+2)^2}$

e $(5x-1)^{\frac{1}{3}}$ **f** $\dfrac{3x+1}{x+1},\quad x > -1$ **g** $\dfrac{2x+1}{x+3},\quad x > -3$

Example 5a

10 **a** Given that $f'(x) = x^2$ and $f(0) = 6$, determine $f(x)$.

b Given that $f'(x) = \sin x$ and $f(\pi) = 4$, determine $f(x)$.

c Given that $f'(x) = x^2 + 2x - 3$ and $f(1) = 4$, determine $f(x)$.

Example 5b

11 **a** Given that $f''(x) = -9.8$ with $f(0) = 20$ and $f'(0) = 0$, determine $f(x)$.

b Given that $f''(x) = 2x$ with $f(1) = 5$ and $f'(1) = 5$, determine $f(x)$.

c Given that $f''(x) = \cos x$ with $f(0) = 3$ and $f'(0) = 0$, determine $f(x)$.

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7B Position vectors as a function of time

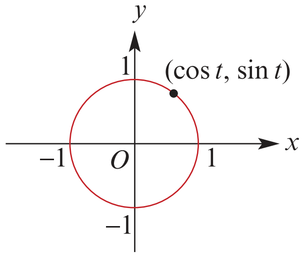
Learning intentions

- To be able to describe paths using position vectors.

In this section, we revise the idea of position vectors as a function of time from Section 6B. We will use the Cartesian and parametric equations of circles, ellipses and hyperbolas; these were introduced in Chapter 2.

Consider a particle travelling at a constant speed along a circular path with radius length 1 unit and centre O.

Suppose that the particle starts at the point $(1, 0)$ and travels anticlockwise, taking 2π units of time to complete one circle.



We have seen that we can represent the path in the following three ways.

Cartesian form: $x^2 + y^2 = 1$

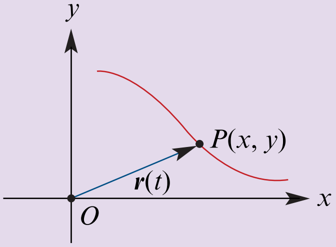
Parametric form: $x = \cos t$ and $y = \sin t$, $\quad t \geq 0$

Vector form: $\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}, \quad t \geq 0$

The vector function gives the position vector of the particle, $\boldsymbol{r}(t)$, at time t.

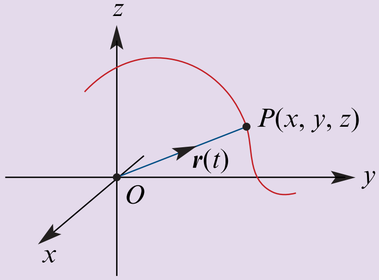
- **Motion in two dimensions**
  When a particle moves along a curve in a plane, its position is specified by a vector function of the form

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$$



- **Motion in three dimensions**
  When a particle moves along a curve in three-dimensional space, its position is specified by a vector function of the form

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$$



Information from the vector function

The vector function gives much more information about the motion of the particle than the Cartesian equation of its path. In the next example, we use the vector function to determine:

- the starting point of the particle's motion
- the direction of motion
- the period of motion.

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Example 6

Each of the following vector functions describes the motion of a particle by giving its position vector, $\boldsymbol{r}(t)$, at time t. For each vector function:

i Determine the Cartesian relation that represents the path of the particle. Determine its domain and range, and sketch its graph.

ii Give the starting point of the particle's motion, the direction of motion and the period of motion (if applicable).

a $\boldsymbol{r}(t) = -5\cos\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{i}} + 5\sin\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{j}}, \quad t \geq 0$

b $\boldsymbol{r}(t) = -5\cos\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{i}} + 12\sin\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{j}}, \quad t \geq 0$

c $\boldsymbol{r}(t) = 4\tan\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{i}} - 3\sec\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{j}}, \quad t \in (1, 3)$

Solution

a $\boldsymbol{r}(t) = -5\cos\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{i}} + 5\sin\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{j}}, \quad t \geq 0$

i Let (x, y) be any point on the path. Then

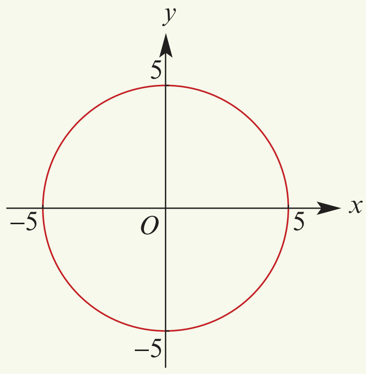
$$x = -5\cos\left(\frac{\pi t}{2}\right) \quad \text{and} \quad y = 5\sin\left(\frac{\pi t}{2}\right)$$

$$\therefore \quad -\frac{x}{5} = \cos\left(\frac{\pi t}{2}\right) \quad \text{and} \quad \frac{y}{5} = \sin\left(\frac{\pi t}{2}\right)$$

Squaring and adding gives

$$\left(-\frac{x}{5}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

i.e. $\quad x^2 + y^2 = 25$



The path is a circle of radius 5 with centre at the origin.

The domain is $[-5, 5]$ and the range is $[-5, 5]$.

ii Determine the position vectors at $t = 0$ and $t = 1$:

$$\boldsymbol{r}(0) = -5\cos(0)\hat{\boldsymbol{i}} + 5\sin(0)\hat{\boldsymbol{j}} = -5\hat{\boldsymbol{i}}$$

$$\boldsymbol{r}(1) = -5\cos\left(\frac{\pi}{2}\right)\hat{\boldsymbol{i}} + 5\sin\left(\frac{\pi}{2}\right)\hat{\boldsymbol{j}} = 5\hat{\boldsymbol{j}}$$

The particle starts at $(-5, 0)$ and moves clockwise.

The period is $2\pi \div \frac{\pi}{2} = 4$.

b $\boldsymbol{r}(t) = -5\cos\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{i}} + 12\sin\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{j}}, \quad t \ge 0$

i Let (x, y) be any point on the path. Then

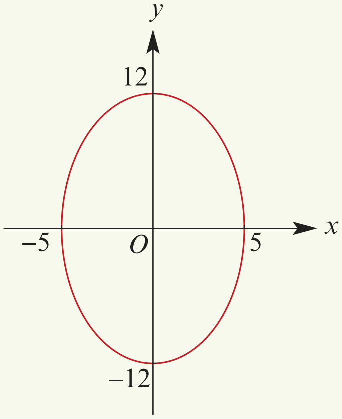
$$x = -5\cos\left(\frac{\pi t}{2}\right) \quad \text{and} \quad y = 12\sin\left(\frac{\pi t}{2}\right)$$

$$\therefore \quad -\frac{x}{5} = \cos\left(\frac{\pi t}{2}\right) \quad \text{and} \quad \frac{y}{12} = \sin\left(\frac{\pi t}{2}\right)$$

Squaring and adding gives

$$\left(-\frac{x}{5}\right)^2 + \left(\frac{y}{12}\right)^2 = 1$$

i.e. $$\frac{x^2}{25} + \frac{y^2}{144} = 1$$



The path is an ellipse centred at the origin with axis intercepts at $(\pm 5, 0)$ and $(0, \pm 12)$. The domain is $[-5, 5]$ and the range is $[-12, 12]$.

ii Determine the position vectors at $t = 0$ and $t = 1$:

$$\boldsymbol{r}(0) = -5\cos(0)\,\hat{\boldsymbol{i}} + 12\sin(0)\,\hat{\boldsymbol{j}} = -5\hat{\boldsymbol{i}}$$

$$\boldsymbol{r}(1) = -5\cos\left(\frac{\pi}{2}\right)\hat{\boldsymbol{i}} + 12\sin\left(\frac{\pi}{2}\right)\hat{\boldsymbol{j}} = 12\hat{\boldsymbol{j}}$$

The particle starts at $(-5, 0)$ and moves clockwise. The period is $2\pi \div \frac{\pi}{2} = 4$.

c $\boldsymbol{r}(t) = 4\tan\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{i}} - 3\sec\left(\frac{\pi t}{2}\right)\hat{\boldsymbol{j}}, \quad t \in (1, 3)$

i Let (x, y) be any point on the path. Then

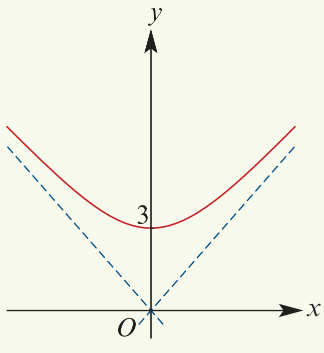
$$x = 4\tan\left(\frac{\pi t}{2}\right) \quad \text{and} \quad y = -3\sec\left(\frac{\pi t}{2}\right)$$

$$\therefore \quad \frac{x}{4} = \tan\left(\frac{\pi t}{2}\right) \quad \text{and} \quad -\frac{y}{3} = \sec\left(\frac{\pi t}{2}\right)$$

Squaring and subtracting gives

$$\left(-\frac{y}{3}\right)^2 - \left(\frac{x}{4}\right)^2 = 1$$

i.e. $$\frac{y^2}{9} - \frac{x^2}{16} = 1$$



This is the equation of a hyperbola centred at the origin with y-axis intercepts at $(0, \pm 3)$.

Since $t \in (1, 3)$, the domain is $\mathbb{R}$ and the range is $[3, \infty)$. So the path is only the upper branch of the hyperbola.

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ii Determine the position vectors at $t = \frac{3}{2}$, $t = 2$ and $t = \frac{5}{2}$:

$$\boldsymbol{r}\left(\frac{3}{2}\right) = 4\tan\left(\frac{3\pi}{4}\right)\hat{\boldsymbol{i}} - 3\sec\left(\frac{3\pi}{4}\right)\hat{\boldsymbol{j}} = -4\hat{\boldsymbol{i}} + 3\sqrt{2}\hat{\boldsymbol{j}}$$

$$\boldsymbol{r}(2) = 4\tan(\pi)\hat{\boldsymbol{i}} - 3\sec(\pi)\hat{\boldsymbol{j}} = 3\hat{\boldsymbol{j}}$$

$$\boldsymbol{r}\left(\frac{5}{2}\right) = 4\tan\left(\frac{5\pi}{4}\right)\hat{\boldsymbol{i}} - 3\sec\left(\frac{5\pi}{4}\right)\hat{\boldsymbol{j}} = 4\hat{\boldsymbol{i}} + 3\sqrt{2}\hat{\boldsymbol{j}}$$

There is no starting point or period in this case. The particle traverses the upper branch of the hyperbola from left to right.

Exercise 7B

For further questions, see Exercises 2C, 6A and 6B

Example 6

SF

1 Each of the following vector functions describes the motion of a particle by giving its position vector, $\boldsymbol{r}(t)$, at time t. For each vector function:

i Determine the Cartesian relation that represents the path of the particle. Determine its domain and range, and sketch its graph.

ii Give the starting point of the particle's motion, the direction of motion and the period of motion (if applicable).

a $\boldsymbol{r}(t) = 3\cos\left(\frac{\pi t}{3}\right)\hat{\boldsymbol{i}} - 3\sin\left(\frac{\pi t}{3}\right)\hat{\boldsymbol{j}}, \quad t \geq 0$

b $\boldsymbol{r}(t) = -4\cos\left(\frac{\pi t}{3}\right)\hat{\boldsymbol{i}} - 3\sin\left(\frac{\pi t}{3}\right)\hat{\boldsymbol{j}}, \quad t \geq 0$

c $\boldsymbol{r}(t) = -5\tan(t)\hat{\boldsymbol{i}} + 12\sec(t)\hat{\boldsymbol{j}}, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

d $\boldsymbol{r}(t) = (2 - 3\cos t)\hat{\boldsymbol{i}} - (4 + 3\sin t)\hat{\boldsymbol{j}}, \quad t \geq 0$

e $\boldsymbol{r}(t) = (2 - 5\cos t)\hat{\boldsymbol{i}} - (4 + 12\sin t)\hat{\boldsymbol{j}}, \quad t \geq 0$

f $\boldsymbol{r}(t) = (2 - 4\sec t)\hat{\boldsymbol{i}} - (4 + 12\tan t)\hat{\boldsymbol{j}}, \quad t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

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7C Vector calculus

Learning intentions

- To be able to differentiate and antidifferentiate vector functions.

Consider the curve defined by a vector function $\boldsymbol{r}(t)$.

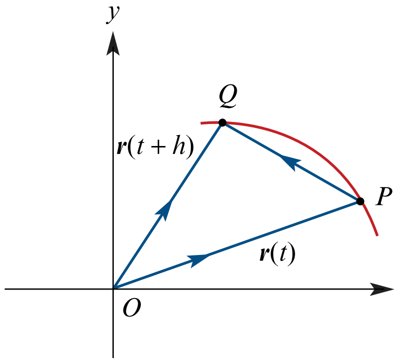
Let P and Q be points on the curve with position vectors $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+h)$ respectively.

Then $\overrightarrow{PQ} = \boldsymbol{r}(t+h) - \boldsymbol{r}(t)$.

It follows that

$$\frac{1}{h}(\boldsymbol{r}(t+h) - \boldsymbol{r}(t))$$

is a vector parallel to $\overrightarrow{PQ}$.



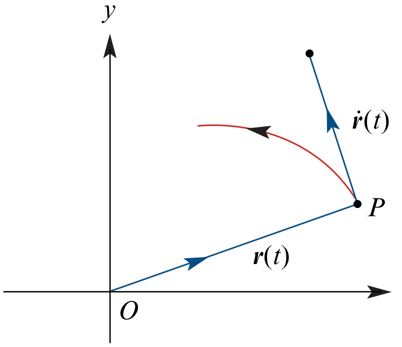
As $h \to 0$, the point Q approaches P along the curve.

The derivative of $\boldsymbol{r}$ with respect to t is denoted by $\dot{\boldsymbol{r}}$ and is defined by

$$\dot{\boldsymbol{r}}(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

provided that this limit exists.

The vector $\dot{\boldsymbol{r}}(t)$ points along the tangent to the curve at P, in the direction of increasing t.



Note: The derivative of a vector function $\boldsymbol{r}(t)$ is also denoted by $\dfrac{d\boldsymbol{r}}{dt}$ or $\boldsymbol{r}'(t)$.

Derivative of a vector function

Let $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$. If both $x(t)$ and $y(t)$ are differentiable, then

$$\dot{\boldsymbol{r}}(t) = \frac{dx}{dt}\hat{\boldsymbol{i}} + \frac{dy}{dt}\hat{\boldsymbol{j}} = \dot{x}(t)\hat{\boldsymbol{i}} + \dot{y}(t)\hat{\boldsymbol{j}}$$

Proof By the definition, we have

$$\begin{aligned}
\dot{\boldsymbol{r}}(t) &= \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} \\
&= \lim_{h \to 0} \frac{(x(t+h)\hat{\boldsymbol{i}} + y(t+h)\hat{\boldsymbol{j}}) - (x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}})}{h} \\
&= \lim_{h \to 0} \frac{x(t+h)\hat{\boldsymbol{i}} - x(t)\hat{\boldsymbol{i}}}{h} + \lim_{h \to 0} \frac{y(t+h)\hat{\boldsymbol{j}} - y(t)\hat{\boldsymbol{j}}}{h} \\
&= \left(\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}\right)\hat{\boldsymbol{i}} + \left(\lim_{h \to 0} \frac{y(t+h) - y(t)}{h}\right)\hat{\boldsymbol{j}}
\end{aligned}$$

$$\therefore \quad \dot{\boldsymbol{r}}(t) = \frac{dx}{dt}\hat{\boldsymbol{i}} + \frac{dy}{dt}\hat{\boldsymbol{j}}$$

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The second derivative of $\boldsymbol{r}(t)$ is

$$\ddot{\boldsymbol{r}}(t) = \frac{d^2x}{dt^2}\hat{\boldsymbol{i}} + \frac{d^2y}{dt^2}\hat{\boldsymbol{j}} = \ddot{x}(t)\hat{\boldsymbol{i}} + \ddot{y}(t)\hat{\boldsymbol{j}}$$

This can be extended to three-dimensional vector functions:

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$$

$$\dot{\boldsymbol{r}}(t) = \frac{dx}{dt}\hat{\boldsymbol{i}} + \frac{dy}{dt}\hat{\boldsymbol{j}} + \frac{dz}{dt}\hat{\boldsymbol{k}}$$

$$\ddot{\boldsymbol{r}}(t) = \frac{d^2x}{dt^2}\hat{\boldsymbol{i}} + \frac{d^2y}{dt^2}\hat{\boldsymbol{j}} + \frac{d^2z}{dt^2}\hat{\boldsymbol{k}}$$

Example 7

Determine $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ if $\boldsymbol{r}(t) = 20t\,\hat{\boldsymbol{i}} + (15t - 5t^2)\hat{\boldsymbol{j}}$.

Solution

$$\boldsymbol{r}(t) = 20t\,\hat{\boldsymbol{i}} + (15t - 5t^2)\hat{\boldsymbol{j}}$$

$$\dot{\boldsymbol{r}}(t) = 20\hat{\boldsymbol{i}} + (15 - 10t)\hat{\boldsymbol{j}}$$

$$\ddot{\boldsymbol{r}}(t) = -10\hat{\boldsymbol{j}}$$

Example 8

Determine $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ if $\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} - \sin t\,\hat{\boldsymbol{j}} + 5t\hat{\boldsymbol{k}}$.

Solution

$$\boldsymbol{r}(t) = \cos t\,\hat{\boldsymbol{i}} - \sin t\,\hat{\boldsymbol{j}} + 5t\hat{\boldsymbol{k}}$$

$$\dot{\boldsymbol{r}}(t) = -\sin t\,\hat{\boldsymbol{i}} - \cos t\,\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$$

$$\ddot{\boldsymbol{r}}(t) = -\cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}$$

Example 9

If $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + ((t-1)^3 + 1)\hat{\boldsymbol{j}}$, determine $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$, where $\boldsymbol{r}(\alpha) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

Solution

$$\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + ((t-1)^3 + 1)\hat{\boldsymbol{j}}$$

$$\dot{\boldsymbol{r}}(t) = \hat{\boldsymbol{i}} + 3(t-1)^2\hat{\boldsymbol{j}}$$

$$\ddot{\boldsymbol{r}}(t) = 6(t-1)\hat{\boldsymbol{j}}$$

We have

$$\boldsymbol{r}(\alpha) = \alpha\hat{\boldsymbol{i}} + ((\alpha-1)^3 + 1)\hat{\boldsymbol{j}} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$$

Therefore $\alpha = 1$, and $\dot{\boldsymbol{r}}(1) = \hat{\boldsymbol{i}}$ and $\ddot{\boldsymbol{r}}(1) = \boldsymbol{0}$.

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Example 10

If $\boldsymbol{r}(t) = e^t \hat{\boldsymbol{i}} + ((e^t - 1)^3 + 1)\hat{\boldsymbol{j}}$, determine $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$, where $\boldsymbol{r}(\alpha) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

Solution

$$\boldsymbol{r}(t) = e^t \hat{\boldsymbol{i}} + ((e^t - 1)^3 + 1)\hat{\boldsymbol{j}}$$
$$\dot{\boldsymbol{r}}(t) = e^t \hat{\boldsymbol{i}} + 3e^t(e^t - 1)^2\hat{\boldsymbol{j}}$$
$$\ddot{\boldsymbol{r}}(t) = e^t \hat{\boldsymbol{i}} + (6e^{2t}(e^t - 1) + 3e^t(e^t - 1)^2)\hat{\boldsymbol{j}}$$

We have

$$\boldsymbol{r}(\alpha) = e^\alpha \hat{\boldsymbol{i}} + ((e^\alpha - 1)^3 + 1)\hat{\boldsymbol{j}} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$$

Therefore $\alpha = 0$, and $\dot{\boldsymbol{r}}(0) = \hat{\boldsymbol{i}}$ and $\ddot{\boldsymbol{r}}(0) = \hat{\boldsymbol{i}}$.

Example 11

A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos t\,\hat{\boldsymbol{i}} + 3\sin t\,\hat{\boldsymbol{j}}$.

a Determine:

i $\dot{\boldsymbol{r}}(t)$ **ii** $\ddot{\boldsymbol{r}}(t)$

b Determine the gradient of the curve at the point (x, y), where $x = 2\cos t$ and $y = 3\sin t$.

Solution

a **i** $\dot{\boldsymbol{r}}(t) = -2\sin t\,\hat{\boldsymbol{i}} + 3\cos t\,\hat{\boldsymbol{j}}$

ii $\ddot{\boldsymbol{r}}(t) = -2\cos t\,\hat{\boldsymbol{i}} - 3\sin t\,\hat{\boldsymbol{j}}$

b Using the chain rule, we can write

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$$

We have

$$\frac{dx}{dt} = -2\sin t \quad \text{and} \quad \frac{dy}{dt} = 3\cos t$$

Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\,\frac{dt}{dx}\\ &= 3\cos t \cdot \frac{1}{-2\sin t}\\ &= -\frac{3}{2}\cot t\end{aligned}$$

Note that the gradient is undefined when $\sin t = 0$.

Note: Part **b** of this example involves **related rates**, which are considered in more detail in Section 13G.

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Example 12

A curve is described by the vector equation $\boldsymbol{r}(t) = \sec(t)\,\hat{\boldsymbol{i}} + \tan(t)\,\hat{\boldsymbol{j}}$, with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a Determine the gradient of the curve at the point (x, y), where $x = \sec(t)$ and $y = \tan(t)$.

b Determine the gradient of the curve where $t = \frac{\pi}{4}$.

Solution

a $x = \sec(t) = \dfrac{1}{\cos(t)} = (\cos t)^{-1}$ and $y = \tan(t)$

$$\frac{dx}{dt} = -(\cos t)^{-2}(-\sin t)$$

$$= \frac{\sin(t)}{\cos^2(t)}$$

$$= \tan(t)\sec(t)$$

$$\frac{dy}{dt} = \sec^2(t)$$

Hence

$$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$$

$$= \sec^2(t) \cdot \frac{1}{\tan(t)\sec(t)}$$

$$= \sec(t)\cot(t)$$

$$= \frac{1}{\sin(t)}$$

b When $t = \frac{\pi}{4}$, $\dfrac{dy}{dx} = \dfrac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$

We have the following results for differentiating vector functions.

Properties of the derivative of a vector function

- $\dfrac{d}{dt}(\boldsymbol{c}) = \boldsymbol{0}$, where $\boldsymbol{c}$ is a constant vector
- $\dfrac{d}{dt}(k\boldsymbol{r}(t)) = k\,\dfrac{d}{dt}(\boldsymbol{r}(t))$, where k is a real number
- $\dfrac{d}{dt}(\boldsymbol{r}_1(t) + \boldsymbol{r}_2(t)) = \dfrac{d}{dt}(\boldsymbol{r}_1(t)) + \dfrac{d}{dt}(\boldsymbol{r}_2(t))$
- $\dfrac{d}{dt}(f(t)\,\boldsymbol{r}(t)) = f(t)\,\dfrac{d}{dt}(\boldsymbol{r}(t)) + \dfrac{d}{dt}(f(t))\,\boldsymbol{r}(t)$, where f is a real-valued function

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Anti-differentiation

Consider $\int \boldsymbol{r}(t)\,dt = \int x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}\,dt$

$$= \left(\int x(t)\,dt\right)\hat{\boldsymbol{i}} + \left(\int y(t)\,dt\right)\hat{\boldsymbol{j}} + \left(\int z(t)\,dt\right)\hat{\boldsymbol{k}}$$

$$= X(t)\hat{\boldsymbol{i}} + Y(t)\hat{\boldsymbol{j}} + Z(t)\hat{\boldsymbol{k}} + \boldsymbol{c}$$

where $\frac{dX}{dt} = x(t)$, $\frac{dY}{dt} = y(t)$, $\frac{dZ}{dt} = z(t)$ and $\boldsymbol{c}$ is a constant vector. Note that $\frac{d\boldsymbol{c}}{dt} = \boldsymbol{0}$.

Example 13

Given that $\ddot{\boldsymbol{r}}(t) = 10\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{k}}$, determine:

a $\dot{\boldsymbol{r}}(t)$ if $\dot{\boldsymbol{r}}(0) = 30\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$

b $\boldsymbol{r}(t)$ if also $\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

Solution

a $\dot{\boldsymbol{r}}(t) = 10t\,\hat{\boldsymbol{i}} - 12t\hat{\boldsymbol{k}} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector

$\dot{\boldsymbol{r}}(0) = 30\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$

Thus $\boldsymbol{c}_1 = 30\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$

and $\dot{\boldsymbol{r}}(t) = 10t\,\hat{\boldsymbol{i}} - 12t\hat{\boldsymbol{k}} + 30\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}}$

$= (10t + 30)\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + (10 - 12t)\hat{\boldsymbol{k}}$

b $\boldsymbol{r}(t) = (5t^2 + 30t)\hat{\boldsymbol{i}} - 20t\hat{\boldsymbol{j}} + (10t - 6t^2)\hat{\boldsymbol{k}} + \boldsymbol{c}_2$, where $\boldsymbol{c}_2$ is a constant vector

$\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

Thus $\boldsymbol{c}_2 = 2\hat{\boldsymbol{k}}$

and $\boldsymbol{r}(t) = (5t^2 + 30t)\hat{\boldsymbol{i}} - 20t\hat{\boldsymbol{j}} + (10t - 6t^2 + 2)\hat{\boldsymbol{k}}$

Example 14

Given $\ddot{\boldsymbol{r}}(t) = -9.8\hat{\boldsymbol{j}}$ with $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\dot{\boldsymbol{r}}(0) = 30\hat{\boldsymbol{i}} + 40\hat{\boldsymbol{j}}$, determine $\boldsymbol{r}(t)$.

Solution

$\ddot{\boldsymbol{r}}(t) = -9.8\hat{\boldsymbol{j}}$

$\therefore\quad \dot{\boldsymbol{r}}(t) = \left(\int 0\,dt\right)\hat{\boldsymbol{i}} + \left(\int -9.8\,dt\right)\hat{\boldsymbol{j}}$

$= -9.8t\hat{\boldsymbol{j}} + \boldsymbol{c}_1$

But $\dot{\boldsymbol{r}}(0) = 30\hat{\boldsymbol{i}} + 40\hat{\boldsymbol{j}}$, giving $\boldsymbol{c}_1 = 30\hat{\boldsymbol{i}} + 40\hat{\boldsymbol{j}}$.

$\therefore\quad \dot{\boldsymbol{r}}(t) = 30\hat{\boldsymbol{i}} + (40 - 9.8t)\hat{\boldsymbol{j}}$

Thus $\boldsymbol{r}(t) = \left(\int 30\,dt\right)\hat{\boldsymbol{i}} + \left(\int 40 - 9.8t\,dt\right)\hat{\boldsymbol{j}}$

$= 30t\,\hat{\boldsymbol{i}} + (40t - 4.9t^2)\hat{\boldsymbol{j}} + \boldsymbol{c}_2$

Now $\boldsymbol{r}(0) = \boldsymbol{0}$ and therefore $\boldsymbol{c}_2 = \boldsymbol{0}$.

Hence $\boldsymbol{r}(t) = 30t\,\hat{\boldsymbol{i}} + (40t - 4.9t^2)\hat{\boldsymbol{j}}$.

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Skill-sheet

Exercise 7C

SF

Example 7, 8

1 Determine $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ for each of the following:

a $\boldsymbol{r}(t) = e^t\,\hat{\boldsymbol{i}} + e^{-t}\,\hat{\boldsymbol{j}}$

b $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + t^2\,\hat{\boldsymbol{j}}$

c $\boldsymbol{r}(t) = \frac{1}{2}t\,\hat{\boldsymbol{i}} + t^2\,\hat{\boldsymbol{j}}$

d $\boldsymbol{r}(t) = 16t\,\hat{\boldsymbol{i}} - 4(4t-1)^2\,\hat{\boldsymbol{j}}$

e $\boldsymbol{r}(t) = \sin(t)\,\hat{\boldsymbol{i}} + \cos(t)\,\hat{\boldsymbol{j}}$

f $\boldsymbol{r}(t) = (3+2t)\hat{\boldsymbol{i}} + 5t\hat{\boldsymbol{j}}$

g $\boldsymbol{r}(t) = 100t\,\hat{\boldsymbol{i}} + (100\sqrt{3}t - 4.9t^2)\hat{\boldsymbol{j}}$

h $\boldsymbol{r}(t) = \tan(t)\,\hat{\boldsymbol{i}} + \cos^2(t)\,\hat{\boldsymbol{j}}$

Example 9, 10

2 Sketch graphs for each of the following, for $t \geq 0$, and determine $\boldsymbol{r}(t_0)$, $\dot{\boldsymbol{r}}(t_0)$ and $\ddot{\boldsymbol{r}}(t_0)$ for the given t_0:

a $\boldsymbol{r}(t) = e^t\,\hat{\boldsymbol{i}} + e^{-t}\,\hat{\boldsymbol{j}}, \quad t_0 = 0$

b $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + t^2\,\hat{\boldsymbol{j}}, \quad t_0 = 1$

c $\boldsymbol{r}(t) = \sin(t)\,\hat{\boldsymbol{i}} + \cos(t)\,\hat{\boldsymbol{j}}, \quad t_0 = \dfrac{\pi}{6}$

d $\boldsymbol{r}(t) = 16t\,\hat{\boldsymbol{i}} - 4(4t-1)^2\,\hat{\boldsymbol{j}}, \quad t_0 = 1$

e $\boldsymbol{r}(t) = \dfrac{1}{t+1}\,\hat{\boldsymbol{i}} + (t+1)^2\,\hat{\boldsymbol{j}}, \quad t_0 = 1$

Example 11, 12

3 Determine the gradient at the point on the curve determined by the given value of t for each of the following:

a $\boldsymbol{r}(t) = \cos(t)\,\hat{\boldsymbol{i}} + \sin(t)\,\hat{\boldsymbol{j}}, \quad t = \dfrac{\pi}{4}$

b $\boldsymbol{r}(t) = \sin(t)\,\hat{\boldsymbol{i}} + \cos(t)\,\hat{\boldsymbol{j}}, \quad t = \dfrac{\pi}{2}$

c $\boldsymbol{r}(t) = e^t\,\hat{\boldsymbol{i}} + e^{-2t}\,\hat{\boldsymbol{j}}, \quad t = 1$

d $\boldsymbol{r}(t) = 2t^2\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}, \quad t = 2$

e $\boldsymbol{r}(t) = (t+2)\hat{\boldsymbol{i}} + (t^2 - 2t)\hat{\boldsymbol{j}}, \quad t = 3$

f $\boldsymbol{r}(t) = \cos(\pi t)\,\hat{\boldsymbol{i}} + \cos(2\pi t)\,\hat{\boldsymbol{j}}, \quad t = \dfrac{1}{4}$

Example 13, 14

4 Determine $\boldsymbol{r}(t)$ for each of the following:

a $\dot{\boldsymbol{r}}(t) = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, where $\boldsymbol{r}(0) = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

b $\dot{\boldsymbol{r}}(t) = 2t\,\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3t^2\hat{\boldsymbol{k}}$, where $\boldsymbol{r}(0) = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

c $\dot{\boldsymbol{r}}(t) = e^{2t}\hat{\boldsymbol{i}} + 2e^{0.5t}\hat{\boldsymbol{j}}$, where $\boldsymbol{r}(0) = \frac{1}{2}\hat{\boldsymbol{i}}$

d $\ddot{\boldsymbol{r}}(t) = \hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$, where $\dot{\boldsymbol{r}}(0) = \hat{\boldsymbol{i}}$ and $\boldsymbol{r}(0) = \boldsymbol{0}$

e $\ddot{\boldsymbol{r}}(t) = \sin(2t)\,\hat{\boldsymbol{i}} - \cos(\frac{1}{2}t)\,\hat{\boldsymbol{j}}$, where $\dot{\boldsymbol{r}}(0) = -\frac{1}{2}\hat{\boldsymbol{i}}$ and $\boldsymbol{r}(0) = 4\hat{\boldsymbol{j}}$

CF

5 The position of a particle at time t is given by $\boldsymbol{r}(t) = \sin(t)\,\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}} + \cos(t)\,\hat{\boldsymbol{k}}$, where $t \geq 0$. Prove that $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are always perpendicular.

6 The position of a particle at time t is given by $\boldsymbol{r}(t) = 2t\,\hat{\boldsymbol{i}} + 16t^2(3-t)\hat{\boldsymbol{j}}$, where $t \geq 0$. Determine:

a when $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are perpendicular

b the pairs of perpendicular vectors $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$.

7 A particle has position $\boldsymbol{r}(t)$ at time t determined by $\boldsymbol{r}(t) = at\,\hat{\boldsymbol{i}} + \dfrac{a^2t^2}{4}\hat{\boldsymbol{j}}$, $a > 0$ and $t \geq 0$.

a Sketch the graph of the path of the particle.

b Determine when the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ is 45°.

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CF

8 A particle has position $\boldsymbol{r}(t)$ at time t specified by $\boldsymbol{r}(t) = 2t\,\hat{\boldsymbol{i}} + (t^2 - 4)\hat{\boldsymbol{j}}$, where $t \geq 0$.

a Sketch the graph of the path of the particle.

b Determine the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ at $t = 1$.

c Determine when the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ is 30°.

9 Given $\boldsymbol{r} = 3t\,\hat{\boldsymbol{i}} + \frac{1}{3}t^3\hat{\boldsymbol{j}} + t^3\hat{\boldsymbol{k}}$, determine:

a $\dot{\boldsymbol{r}}$ **b** $|\dot{\boldsymbol{r}}|$ **c** $\ddot{\boldsymbol{r}}$

d $|\ddot{\boldsymbol{r}}|$ **e** t when $|\ddot{\boldsymbol{r}}| = 16$

10 Given that $\boldsymbol{r} = (V\cos\alpha)t\,\hat{\boldsymbol{i}} + ((V\sin\alpha)t - \frac{1}{2}gt^2)\,\hat{\boldsymbol{j}}$ specifies the position of an object at time $t \geq 0$, determine:

a $\dot{\boldsymbol{r}}$ **b** $\ddot{\boldsymbol{r}}$

c when $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular

d the position of the object when $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular.

7D Velocity and acceleration for motion along a curve

Learning intentions

- To be able to determine the velocity and acceleration of a particle moving along a curve.

Consider a particle moving along a curve in the plane, with position vector at time t given by

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}}$$

We can determine the particle's velocity and acceleration at time t as follows.

Velocity

Velocity is the rate of change of position.

Therefore $\boldsymbol{v}(t)$, the velocity at time t, is given by

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \dot{x}(t)\hat{\boldsymbol{i}} + \dot{y}(t)\hat{\boldsymbol{j}}$$

The velocity vector gives the direction of motion at time t.

Acceleration

Acceleration is the rate of change of velocity.

Therefore $\boldsymbol{a}(t)$, the acceleration at time t, is given by

$$\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) = \ddot{x}(t)\hat{\boldsymbol{i}} + \ddot{y}(t)\hat{\boldsymbol{j}}$$

Speed

Speed is the magnitude of velocity. At time t, the speed is $|\dot{\boldsymbol{r}}(t)|$.

Distance between two points on the curve

The (shortest) distance between two points on the curve is found using $|\boldsymbol{r}(t_1) - \boldsymbol{r}(t_0)|$.

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Example 15

The position of an object is $\boldsymbol{r}(t)$ metres at time t seconds, where $\boldsymbol{r}(t) = e^t\,\hat{\boldsymbol{i}} + \frac{2}{9}e^{2t}\hat{\boldsymbol{j}}$, $t \geq 0$.

Determine at time t:

a the velocity vector **b** the acceleration vector **c** the speed.

Solution

a $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = e^t\,\hat{\boldsymbol{i}} + \frac{4}{9}e^{2t}\hat{\boldsymbol{j}}$

b $\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) = e^t\,\hat{\boldsymbol{i}} + \frac{8}{9}e^{2t}\hat{\boldsymbol{j}}$

c Speed $= |\boldsymbol{v}(t)| = \sqrt{(e^t)^2 + (\frac{4}{9}e^{2t})^2} = \sqrt{e^{2t} + \frac{16}{81}e^{4t}}$ m/s

Example 16

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = (2t - t^2)\hat{\boldsymbol{i}} + (t^2 - 3t)\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$, where $t \geq 0$. Determine:

a the velocity of the particle at time t

b the speed of the particle at time t

c the minimum speed of the particle.

Solution

a $\dot{\boldsymbol{r}}(t) = (2 - 2t)\hat{\boldsymbol{i}} + (2t - 3)\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

b
$$\begin{aligned}\text{Speed} = |\dot{\boldsymbol{r}}(t)| &= \sqrt{4 - 8t + 4t^2 + 4t^2 - 12t + 9 + 4}\\ &= \sqrt{8t^2 - 20t + 17}\end{aligned}$$

c Minimum speed occurs when $8t^2 - 20t + 17$ is a minimum.

$$\begin{aligned}8t^2 - 20t + 17 &= 8\left(t^2 - \frac{5t}{2} + \frac{17}{8}\right)\\ &= 8\left(t^2 - \frac{5t}{2} + \frac{25}{16} + \frac{17}{8} - \frac{25}{16}\right)\\ &= 8\left(\left(t - \frac{5}{4}\right)^2 + \frac{9}{16}\right)\\ &= 8\left(t - \frac{5}{4}\right)^2 + \frac{9}{2}\end{aligned}$$

Hence the minimum speed is $\sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$.

(This occurs when $t = \frac{5}{4}$.)

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Example 17

The position of a projectile at time t is given by $\boldsymbol{r}(t) = 40t\,\hat{\boldsymbol{i}} + (50t - 5t^2)\hat{\boldsymbol{j}}$, for $t \geq 0$, where $\hat{\boldsymbol{i}}$ is a unit vector in a horizontal direction and $\hat{\boldsymbol{j}}$ is a unit vector vertically up. The projectile is fired from a point on the ground. Determine:

a the time taken to reach the ground again

b the speed at which the projectile hits the ground

c the maximum height of the projectile

d the initial speed of the projectile.

Solution

a The projectile is at ground level when the $\hat{\boldsymbol{j}}$-component of $\boldsymbol{r}$ is zero:

$$50t - 5t^2 = 0$$
$$5t(10 - t) = 0$$
$$\therefore \quad t = 0 \text{ or } t = 10$$

The projectile reaches the ground again at $t = 10$.

b $\dot{\boldsymbol{r}}(t) = 40\hat{\boldsymbol{i}} + (50 - 10t)\hat{\boldsymbol{j}}$

The velocity of the projectile when it hits the ground is

$$\dot{\boldsymbol{r}}(10) = 40\hat{\boldsymbol{i}} - 50\hat{\boldsymbol{j}}$$

Therefore the speed is

$$|\dot{\boldsymbol{r}}(100)| = \sqrt{40^2 + 50^2}$$
$$= 10\sqrt{41}$$

The projectile hits the ground with speed $10\sqrt{41}$.

c The projectile reaches its maximum height when the $\hat{\boldsymbol{j}}$-component of $\dot{\boldsymbol{r}}$ is zero:

$$50 - 10t = 0$$
$$\therefore \quad t = 5$$

The maximum height is $50 \times 5 - 5 \times 5^2 = 125$.

d The initial velocity is

$$\dot{\boldsymbol{r}}(0) = 40\hat{\boldsymbol{i}} + 50\hat{\boldsymbol{j}}$$

So the initial speed is

$$|\dot{\boldsymbol{r}}(0)| = \sqrt{40^2 + 50^2}$$
$$= 10\sqrt{41}$$

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Example 18

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 2\sin(2t)\,\hat{\boldsymbol{i}} + \cos(2t)\,\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$, where $t \geq 0$. Determine:

a the velocity at time t

b the speed of the particle at time t

c the maximum speed

d the minimum speed.

Solution

a $\dot{\boldsymbol{r}}(t) = 4\cos(2t)\,\hat{\boldsymbol{i}} - 2\sin(2t)\,\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

b
$$\begin{aligned}\text{Speed} = |\dot{\boldsymbol{r}}(t)| &= \sqrt{16\cos^2(2t) + 4\sin^2(2t) + 4}\\ &= \sqrt{12\cos^2(2t) + 8}\end{aligned}$$

c Maximum speed $= \sqrt{20} = 2\sqrt{5}$, when $\cos(2t) = \pm 1$

d Minimum speed $= \sqrt{8} = 2\sqrt{2}$, when $\cos(2t) = 0$

Example 19

The position vectors, at time $t \geq 0$, of particles A and B are given by

$$\boldsymbol{r}_A(t) = (t^3 - 9t + 8)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$$
$$\boldsymbol{r}_B(t) = (2 - t^2)\hat{\boldsymbol{i}} + (3t - 2)\hat{\boldsymbol{j}}$$

Prove that A and B collide while travelling at the same speed but at right angles to each other.

Solution

When the particles collide, they must be at the same position at the same time:

$$(t^3 - 9t + 8)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}} = (2 - t^2)\hat{\boldsymbol{i}} + (3t - 2)\hat{\boldsymbol{j}}$$

Thus $\quad t^3 - 9t + 8 = 2 - t^2 \quad (1)$

and $\quad t^2 = 3t - 2 \quad (2)$

From (1): $\quad t^3 + t^2 - 9t + 6 = 0 \quad (3)$

From (2): $\quad t^2 - 3t + 2 = 0 \quad (4)$

Equation (4) is simpler to solve:

$$(t - 2)(t - 1) = 0$$

$\therefore \quad t = 2$ or $t = 1$

Now check in (3):

$t = 1 \qquad \text{LHS} = 1 + 1 - 9 + 6 = -1 \neq \text{RHS}$

$t = 2 \qquad \text{LHS} = 8 + 4 - 18 + 6 = 0 = \text{RHS}$

The particles collide when $t = 2$.

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Now consider the speeds when $t = 2$.

$$\dot{\boldsymbol{r}}_A(t) = (3t^2 - 9)\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}} \qquad \dot{\boldsymbol{r}}_B(t) = -2t\,\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$$

$$\therefore \quad \dot{\boldsymbol{r}}_A(2) = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} \qquad \dot{\boldsymbol{r}}_B(2) = -4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$$

The speed of particle A is $\sqrt{3^2 + 4^2} = 5$.

The speed of particle B is $\sqrt{(-4)^2 + 3^2} = 5$.

The speeds of the particles are equal at the time of collision.

Consider the scalar product of the velocity vectors for A and B at time $t = 2$.

$$\begin{aligned}\dot{\boldsymbol{r}}_A(2) \cdot \dot{\boldsymbol{r}}_B(2) &= (3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) \cdot (-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \\ &= -12 + 12 \\ &= 0\end{aligned}$$

Hence the velocities are perpendicular at $t = 2$.

The particles are travelling at right angles at the time of collision.

Skill-sheet

Exercise 7D

All distances are measured in metres and time in seconds.

Example 15

1 The position of a particle at time t is given by $\boldsymbol{r}(t) = t^2\hat{\boldsymbol{i}} - (1 + 2t)\hat{\boldsymbol{j}}$, for $t \geq 0$. Determine: SF

- **a** the velocity at time t
- **b** the acceleration at time t
- **c** the average velocity for the first 2 seconds, i.e. $\dfrac{\boldsymbol{r}(2) - \boldsymbol{r}(0)}{2}$.

2 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$, where $g = 9.8$. Determine:

- **a** the velocity at time t if $\dot{\boldsymbol{r}}(0) = 2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$
- **b** the position at time t if $\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$.

Example 16

3 The velocity of a particle at time t is given by $\dot{\boldsymbol{r}}(t) = 3\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}} + (1 - 4t)\hat{\boldsymbol{k}}$, for $t \geq 0$. CF

- **a** Determine the acceleration of the particle at time t.
- **b** Determine the position of the particle at time t if initially the particle is at $\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.
- **c** Determine an expression for the speed at time t.
- **d** **i** Determine the time at which the minimum speed occurs.
  - **ii** Determine this minimum speed.

4 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = 10\hat{\boldsymbol{i}} - g\hat{\boldsymbol{k}}$, where $g = 9.8$. Determine: SF

- **a** the velocity of the particle at time t, given that $\dot{\boldsymbol{r}}(0) = 20\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + 40\hat{\boldsymbol{k}}$
- **b** the position of the particle at time t, given that $\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}} + 0\hat{\boldsymbol{k}}$.

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5 The position of an object at time t is given by $\boldsymbol{r}(t) = 5\cos(1 + t^2)\,\hat{\boldsymbol{i}} + 5\sin(1 + t^2)\,\hat{\boldsymbol{j}}$. Determine the speed of the object at time t.

6 The position of a particle, $\boldsymbol{r}(t)$, at time t seconds is given by $\boldsymbol{r}(t) = 2t\,\hat{\boldsymbol{i}} + (t^2 - 4)\hat{\boldsymbol{j}}$. Determine the magnitude of the angle between the velocity and acceleration vectors at $t = 1$.

7 The position vector of a particle is given by $\boldsymbol{r}(t) = 12\sqrt{t}\,\hat{\boldsymbol{i}} + t^{\frac{3}{2}}\hat{\boldsymbol{j}}$, for $t \geq 0$. Determine the minimum speed of the particle and its position when it has this speed.

Example 17

8 The position, $\boldsymbol{r}(t)$, of a projectile at time t is given by

$$\boldsymbol{r}(t) = 40t\,\hat{\boldsymbol{i}} + (30t - 4.9t^2)\hat{\boldsymbol{j}}, \quad t \geq 0$$

If the projectile is initially at ground level, determine:

a the time taken to return to the ground
b the speed at which the object hits the ground
c the maximum height reached
d the initial speed of the object
e the initial angle of projection from the horizontal.

9 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -3(\sin(3t)\,\hat{\boldsymbol{i}} + \cos(3t)\,\hat{\boldsymbol{j}})$.

a Determine the position vector $\boldsymbol{r}(t)$, given that $\dot{\boldsymbol{r}}(0) = \hat{\boldsymbol{i}}$ and $\boldsymbol{r}(0) = -3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.
b Show that the path of the particle is circular and state the position of its centre.
c Show that the acceleration is always perpendicular to the velocity.

Example 18

10 The position vector of a particle at time t is $\boldsymbol{r}(t) = 2\cos(t)\,\hat{\boldsymbol{i}} + 4\sin(t)\,\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$. Determine the maximum and minimum speeds of the particle.

11 The velocity vector of a particle at time t seconds is given by

$$\boldsymbol{v}(t) = (2t + 1)^2\hat{\boldsymbol{i}} + \frac{1}{\sqrt{2t + 1}}\,\hat{\boldsymbol{j}}$$

a Determine the magnitude and direction of the acceleration after 1 second.
b Determine the position vector at time t seconds if the particle is initially at O.

Example 19

12 Particles A and B move in the x–y plane with constant velocities.

- $\dot{\boldsymbol{r}}_A(t) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_A(2) = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$
- $\dot{\boldsymbol{r}}_B(t) = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_B(3) = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

Prove that the particles collide, determining:

a the time of collision
b the position vector of the point of collision.

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13 A body moves horizontally along a straight line in a direction Nα°W with a constant speed of 20 m/s. Let $\hat{\boldsymbol{i}}$ be a horizontal unit vector due east and let $\hat{\boldsymbol{j}}$ be a horizontal unit vector due north. Given that $\tan \alpha° = \frac{4}{3}$, determine:

CF

a the velocity of the body at time t

b the position of the body after 5 seconds, relative to its initial position.

14 The position vector of a particle at time t is

SF

$$\boldsymbol{r}(t) = 4\sin(2t)\,\hat{\boldsymbol{i}} + 4\cos(2t)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

a Determine the velocity at time t.

b Determine the speed at time t.

c Determine the acceleration in terms of $\boldsymbol{r}$.

15 The velocity of a particle is given by

$$\dot{\boldsymbol{r}}(t) = (2t - 5)\hat{\boldsymbol{i}}, \quad t \geq 0$$

Initially, the position of the particle relative to an origin O is $-2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$.

a Determine the position of the particle at time t.

b Determine the position of the particle when it is instantaneously at rest.

c Determine the Cartesian equation of the path followed by the particle.

16 A particle has path defined by

$$\boldsymbol{r}(t) = 6\sec(t)\,\hat{\boldsymbol{i}} + 4\tan(t)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

a Determine the Cartesian equation of the path.

b Determine the particle's velocity at time t.

17 A particle moves such that its position vector, $\boldsymbol{r}(t)$, at time t is given by

CF

$$\boldsymbol{r}(t) = 4\cos(t)\,\hat{\boldsymbol{i}} + 3\sin(t)\,\hat{\boldsymbol{j}}, \quad 0 \leq t \leq 2\pi$$

a Determine the Cartesian equation of the path of the particle and sketch the path.

b **i** Determine when the velocity of the particle is perpendicular to its position vector.

ii Determine the position vector of the particle at each of these times.

c **i** Determine the speed of the particle at time t.

ii Write the speed in terms of $\cos^2 t$.

iii State the maximum and minimum speeds of the particle.

7E Motion in a straight line

Learning intentions

▶ To be able to use vector functions to describe motion in a straight line.

You have studied motion in a straight line in Mathematical Methods Units 3 & 4, and you will study it further in Chapter 14. In this section, we consider motion in a straight line from the perspective of vector calculus.

Example 20

A particle moves along a straight line such that its position vector, $\boldsymbol{r}(t)$ cm, at time t seconds is given by $\boldsymbol{r}(t) = (3t - t^3)\hat{\boldsymbol{i}}$, for $t \geq 0$. Determine:

a its initial position

b its position when $t = 2$

c its initial velocity

d its velocity when $t = 2$

e its speed when $t = 2$

f when and where the velocity is zero.

Solution

a The initial position is $\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}} = \boldsymbol{0}$ cm.

b When $t = 2$, the position is $\boldsymbol{r}(2) = (3 \times 2 - 2^3)\hat{\boldsymbol{i}} = -2\hat{\boldsymbol{i}}$ cm.

c $\dot{\boldsymbol{r}}(t) = (3 - 3t^2)\hat{\boldsymbol{i}}$

The initial velocity is $\dot{\boldsymbol{r}}(0) = 3\hat{\boldsymbol{i}}$ cm s^{-1}.

d When $t = 2$, the velocity is $\dot{\boldsymbol{r}}(2) = (3 - 3 \times 2^2)\hat{\boldsymbol{i}} = -9\hat{\boldsymbol{i}}$ cm s^{-1}.

e When $t = 2$, the speed is 9 cm s^{-1}.(Speed is the magnitude of velocity.)

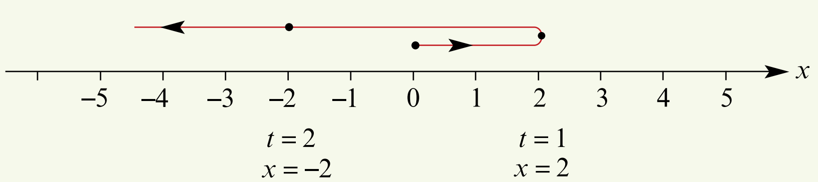
f The velocity is zero when $\dot{\boldsymbol{r}}(t) = \boldsymbol{0}$:

$$\begin{aligned} (3 - 3t^2)\hat{\boldsymbol{i}} &= \boldsymbol{0} \\ 3 - 3t^2 &= 0 \\ 1 - t^2 &= 0 \\ \therefore \quad t = 1 \text{ or } t &= -1 \end{aligned}$$

But $t \geq 0$ and so $t = 1$.

The velocity is zero at time $t = 1$; the position is $\boldsymbol{r}(1) = (3 \times 1 - 1^3)\hat{\boldsymbol{i}} = 2\hat{\boldsymbol{i}}$ cm.

Note: The motion of the particle can now be shown on a number line, where $\hat{\boldsymbol{i}}$ is the unit vector in the positive x-direction.



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Example 21

A cricket ball is projected vertically upwards from ground level with an initial speed of 15 m/s. Take the origin O to be the point of projection and take $\hat{\boldsymbol{j}}$ to be the unit vector vertically up, where the unit of distance is metres.

Relative to this frame of reference, let $\boldsymbol{r}(t)$ m be the position of the ball at time t seconds. Then $\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$, where g m/s^2 is the magnitude of acceleration due to gravity ($g \approx 9.8$). Determine:

a an expression for $\dot{\boldsymbol{r}}(t)$

b an expression for $\boldsymbol{r}(t)$

c the maximum height reached by the ball

d when the ball returns to ground level.

Solution

a We are given $\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$.

antidifferentiate to determine

$$\dot{\boldsymbol{r}}(t) = -gt\hat{\boldsymbol{j}} + \boldsymbol{c}_1, \quad \text{where } \boldsymbol{c}_1 \text{ is a constant vector}$$

Since $\dot{\boldsymbol{r}}(0) = 15\hat{\boldsymbol{j}}$, we have $\boldsymbol{c}_1 = 15\hat{\boldsymbol{j}}$ and therefore

$$\dot{\boldsymbol{r}}(t) = (15 - gt)\hat{\boldsymbol{j}}$$

b antidifferentiate again to determine

$$\boldsymbol{r}(t) = (15t - \tfrac{1}{2}gt^2)\hat{\boldsymbol{j}} + \boldsymbol{c}_2, \quad \text{where } \boldsymbol{c}_2 \text{ is a constant vector}$$

Since $\boldsymbol{r}(0) = \boldsymbol{0}$, we have $\boldsymbol{c}_2 = \boldsymbol{0}$ and therefore

$$\boldsymbol{r}(t) = (15t - \tfrac{1}{2}gt^2)\hat{\boldsymbol{j}}$$

c The maximum height is reached when $\dot{\boldsymbol{r}}(t) = \boldsymbol{0}$:

$$15 - gt = 0$$

$$\therefore \quad t = \frac{15}{g}$$

The position at this time is

$$\boldsymbol{r}\left(\frac{15}{g}\right) = \left(15 \times \frac{15}{g} - \frac{1}{2}g \times \left(\frac{15}{g}\right)^2\right)\hat{\boldsymbol{j}} = \frac{225}{2g}\hat{\boldsymbol{j}}$$

So the maximum height is $\dfrac{225}{2g} \approx 11.48$ m.

d The ball returns to ground level when $\boldsymbol{r}(t) = \boldsymbol{0}$:

$$15t - \tfrac{1}{2}gt^2 = 0$$

$$t(15 - \tfrac{1}{2}gt) = 0$$

$$\therefore \quad t = 0 \text{ or } \tfrac{1}{2}gt = 15$$

So the ball returns to ground level at time $t = \dfrac{30}{g} \approx 3.06$ seconds.

Note: The time to return to ground level is twice the time to reach the maximum height.

Exercise 7E

Example 20

1 A particle moves along a straight line such that its position vector, $\boldsymbol{r}(t)$ cm, at time t seconds is given by $\boldsymbol{r}(t) = (3t - t^2)\hat{\boldsymbol{i}}$, for $t \geq 0$.

a Determine $\boldsymbol{r}(0)$, $\boldsymbol{r}(1)$, $\boldsymbol{r}(2)$, $\boldsymbol{r}(3)$ and $\boldsymbol{r}(4)$. Illustrate the motion on a number line.

b Determine the displacement of the particle in the fifth second. That is, determine $\boldsymbol{r}(5) - \boldsymbol{r}(4)$.

c Determine the average velocity in the first 4 seconds. That is, determine $\dfrac{\boldsymbol{r}(4) - \boldsymbol{r}(0)}{4}$.

d Determine $\dot{\boldsymbol{r}}(t)$.

e Determine the velocity of the particle when $t = 2.5$.

f Determine when and where the particle changes direction.

g Determine the distance travelled in the first 4 seconds.

2 An object moves in a straight line such that its position vector, $\boldsymbol{r}(t)$ cm, at time t seconds is given by $\boldsymbol{r}(t) = (-3t^2 + 10t + 8)\hat{\boldsymbol{i}}$, $t \geq 0$.

a Determine the velocity, $\dot{\boldsymbol{r}}(t)$, at time t seconds.

b Determine the acceleration, $\ddot{\boldsymbol{r}}(t)$, at time t seconds.

c Represent the motion of the object on a number line for $0 \leq t \leq 6$.

d Determine the displacement of the object in the third second.

e Determine the distance travelled in the first 3 seconds.

3 A particle moves along a straight line such that its position vector, $\boldsymbol{r}(t)$ cm, at time t seconds is given by $\boldsymbol{r}(t) = (t^3 - 9t^2 + 24t)\hat{\boldsymbol{i}}$, $t \geq 0$.

a Determine the values of t for which the velocity is instantaneously zero.

b Determine the acceleration when $t = 5$.

c Determine the average velocity of the particle during the first 2 seconds.

Example 21

4 A golf ball is projected vertically upwards from a height of 10 m with an initial speed of 20 m/s. Take the origin O to be the point at ground level below the point of projection and take $\hat{\boldsymbol{j}}$ to be the unit vector vertically up. Let $\boldsymbol{r}(t)$ m be the position of the ball at time t seconds. Then $\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$, where $g \approx 9.8$. Determine:

a an expression for $\dot{\boldsymbol{r}}(t)$

b an expression for $\boldsymbol{r}(t)$

c the maximum height reached by the ball

d when the ball reaches ground level.

5 Particles A and B are each moving along straight-line paths such that their positions at time t are given by the vector functions

$$\boldsymbol{r}_A(t) = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$$
$$\boldsymbol{r}_B(t) = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + t(6\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}})$$

The unit of distance is metres and the unit of time is seconds.

a Determine the speeds of the two particles at time t.

b Determine an expression for the distance between the two particles at time t.

c Determine the time and position at which the two particles collide.

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6 Repeat Question 5 for the following pair of vector functions:

$$\boldsymbol{r}_A(t) = -5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 9\hat{\boldsymbol{k}} + t(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$$
$$\boldsymbol{r}_B(t) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}})$$

SF

7 The velocity, $\boldsymbol{v}(t)$ m/s, of a particle at time t seconds is given by

$$\boldsymbol{v}(t) = \cos(2t)\,(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}), \quad t \geq 0$$

The particle starts at the origin.

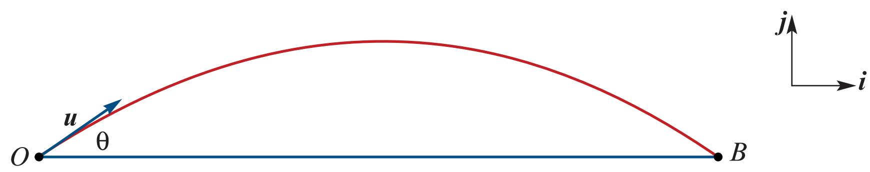
a Determine the initial speed of the particle.
b Determine the position of the particle at time t seconds.
c Determine the acceleration of the particle at time t seconds.
d Determine the position of the particle when its velocity is zero.
e Determine the magnitude of the acceleration of the particle when its velocity is zero.
f Determine the position vector of the centre of motion of the particle.

7F Projectile motion

Learning intentions

- To be able to use vector calculus in the solution of projectile motion questions.

Suppose that a particle is projected at an angle of $\theta°$ to the horizontal with initial velocity $\boldsymbol{u}$.



Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the horizontal and vertical directions as shown, and let $\boldsymbol{r}(t)$ be the position vector of the particle at time t. Then we can write

$$\dot{\boldsymbol{r}}(0) = \boldsymbol{u} = u\cos\theta\,\hat{\boldsymbol{i}} + u\sin\theta\,\hat{\boldsymbol{j}}$$

where u is the magnitude of the initial velocity $\boldsymbol{u}$.

We will assume that the only force acting on the particle is gravity. So we have

$$\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$$

where g is the acceleration due to gravity. Integrating with respect to t gives

$$\dot{\boldsymbol{r}}(t) = -gt\hat{\boldsymbol{j}} + \boldsymbol{c}$$

We see that $\boldsymbol{c} = \dot{\boldsymbol{r}}(0) = u\cos\theta\,\hat{\boldsymbol{i}} + u\sin\theta\,\hat{\boldsymbol{j}}$. So we obtain

$$\dot{\boldsymbol{r}}(t) = u\cos\theta\,\hat{\boldsymbol{i}} + (u\sin\theta - gt)\hat{\boldsymbol{j}}$$

If we assume that $\boldsymbol{r}(0) = \boldsymbol{0}$, then integrating again with respect to t gives

$$\boldsymbol{r}(t) = ut\cos\theta\,\hat{\boldsymbol{i}} + \left(ut\sin\theta - \frac{gt^2}{2}\right)\hat{\boldsymbol{j}}$$

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Equations of projectile motion

For an object projected from the origin with initial velocity $\dot{\boldsymbol{r}}(0) = u\cos\theta\,\hat{\boldsymbol{i}} + u\sin\theta\,\hat{\boldsymbol{j}}$:

Acceleration $\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$

Velocity $\dot{\boldsymbol{r}}(t) = u\cos\theta\,\hat{\boldsymbol{i}} + (u\sin\theta - gt)\hat{\boldsymbol{j}}$

Position $\boldsymbol{r}(t) = ut\cos\theta\,\hat{\boldsymbol{i}} + \left(ut\sin\theta - \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

Note: Close to the Earth's surface, we can take $g \approx 9.8$ m/s^2.

Cartesian equation of the projectile's path

We can write the position function, $\boldsymbol{r}(t)$, as parametric equations:

$$x = ut\cos\theta \quad (1) \qquad y = ut\sin\theta - \frac{gt^2}{2} \quad (2)$$

Solve equation (1) for t and substitute into equation (2):

$$y = u\left(\frac{x}{u\cos\theta}\right)\sin\theta - \frac{g}{2}\left(\frac{x}{u\cos\theta}\right)^2$$

Hence the Cartesian equation of the projectile's path is

$$y = x\tan\theta - \frac{gx^2}{2u^2}\sec^2\theta$$

Maximum height of the projectile

The maximum height is reached when the $\hat{\boldsymbol{j}}$-component of the velocity, $\dot{\boldsymbol{r}}(t)$, is zero. This implies that $u\sin\theta - gt = 0$, and so

$$t = \frac{u\sin\theta}{g}$$

Therefore the position vector of the particle at its maximum height is

$$\begin{aligned}
\boldsymbol{r}(t) &= ut\cos\theta\,\hat{\boldsymbol{i}} + \left(ut\sin\theta - \frac{gt^2}{2}\right)\hat{\boldsymbol{j}} \\
&= \frac{u^2\sin\theta\,\cos\theta}{g}\,\hat{\boldsymbol{i}} + \left(\frac{u^2\sin^2\theta}{g} - \frac{gu^2\sin^2\theta}{2g^2}\right)\hat{\boldsymbol{j}} \\
&= \frac{u^2}{2g}(\sin(2\theta)\,\hat{\boldsymbol{i}} + \sin^2\theta\,\hat{\boldsymbol{j}})
\end{aligned}$$

Hence the maximum height is $\dfrac{u^2}{2g}\sin^2\theta$.

Range of the projectile

If the projectile returns to the same horizontal level as the point of projection, then the total horizontal distance travelled (the projectile's range) is twice the horizontal distance travelled to reach the maximum height.

Hence the range of the projectile is $\dfrac{u^2}{g}\sin(2\theta)$.

The maximum range is obtained when $\theta = 45°$.

Example 22

A particle is projected from a point on horizontal ground with speed 50 m/s at an angle of 30° to the horizontal. Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the horizontal (x) and vertical (y) directions respectively. Neglecting air resistance, determine:

a the initial velocity vector

b the velocity vector at time t seconds

c the position vector at time t seconds

d the Cartesian equation of the path.

Solution

a The initial velocity is

$$\begin{aligned}\boldsymbol{u} &= 50\cos 30^\circ\,\hat{\boldsymbol{i}} + 50\sin 30^\circ\,\hat{\boldsymbol{j}} \\ &= 25\sqrt{3}\,\hat{\boldsymbol{i}} + 25\hat{\boldsymbol{j}}\end{aligned}$$

b We have

$$\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$$

$$\therefore\ \dot{\boldsymbol{r}}(t) = -gt\hat{\boldsymbol{j}} + \boldsymbol{c}_1$$

We see that $\boldsymbol{c}_1 = \dot{\boldsymbol{r}}(0) = 25\sqrt{3}\,\hat{\boldsymbol{i}} + 25\hat{\boldsymbol{j}}$. Hence

$$\dot{\boldsymbol{r}}(t) = 25\sqrt{3}\,\hat{\boldsymbol{i}} + (25 - gt)\hat{\boldsymbol{j}}$$

c We have

$$\dot{\boldsymbol{r}}(t) = 25\sqrt{3}\,\hat{\boldsymbol{i}} + (25 - gt)\hat{\boldsymbol{j}}$$

$$\therefore\ \boldsymbol{r}(t) = 25\sqrt{3}t\,\hat{\boldsymbol{i}} + (25t - \tfrac{1}{2}gt^2)\hat{\boldsymbol{j}} + \boldsymbol{c}_2$$

But $\boldsymbol{r}(0) = \boldsymbol{0}$ implies that $\boldsymbol{c}_2 = \boldsymbol{0}$. Hence

$$\boldsymbol{r}(t) = 25\sqrt{3}t\,\hat{\boldsymbol{i}} + (25t - \tfrac{1}{2}gt^2)\hat{\boldsymbol{j}}$$

d From part **c**, we can write

$$x = 25\sqrt{3}t \quad \text{and} \quad y = 25t - \tfrac{1}{2}gt^2$$

Eliminating t gives

$$y = \frac{25x}{25\sqrt{3}} - \frac{1}{2}g\left(\frac{x}{25\sqrt{3}}\right)^2$$

$$\therefore\ y = \frac{\sqrt{3}x}{3} - \frac{gx^2}{3750}$$

Exercise 7F

Skill-sheet

Example 22

1 A particle is projected from a point on horizontal ground with speed 98 m/s at an angle of 30° to the horizontal. Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the horizontal (x) and vertical (y) directions respectively. Neglecting air resistance, determine:

a the initial velocity vector

b the velocity vector at time t seconds

c the position vector at time t seconds

d the Cartesian equation of the path.

CF

2 A particle is projected from a height of 50 m at an angle of 30° to the horizontal, with an initial speed of $10g$ m/s. After t seconds, the particle is at a height of y m above ground level and at a horizontal distance of x m from the point of projection.

a Express y in terms of x.

b Hence determine, in terms of g, the particle's horizontal distance from the point of projection when it is at a height of 25 m above ground level.

3 A ball is thrown horizontally at 15 m/s from the window of a tall building, and it hits the ground after 2.5 seconds. Determine the height above ground level of the point from which the ball was thrown.

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4 An object slides down an inclined plane, which slopes downwards at an angle of 20° to the horizontal. The object reaches a speed of 40 m/s at the end of the slide, which is 15 m above the ground. Take the end of the slide as the origin, and let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the forwards and upwards directions respectively.

a Let $\boldsymbol{r}(t)$ be the position of the object at time t seconds after leaving the end of the slide. Using $\ddot{\boldsymbol{r}}(t) = -g\hat{\boldsymbol{j}}$, determine $\boldsymbol{r}(t)$.

Hence determine, correct to one decimal place:

b the horizontal distance, in metres, travelled by the object to reach the ground after it leaves the end of the slide

c the angle upwards from the horizontal, in degrees, at which it hits the ground.

5 A stone is thrown to hit a small target, which is at a distance of 14 m horizontally from the point of projection and 5.5 m above ground level. The stone is thrown from a height of 2 m above the horizontal ground with a speed of 42 m/s. Determine, in degrees correct to one decimal place, the angle from the horizontal at which the stone should be thrown to hit the target. (Assume that there is no air resistance.)

6 An object is launched upwards at an angle of α from the horizontal with an initial speed of u m/s. The only force acting is gravity.

a Express $\dot{\boldsymbol{r}}$, the velocity of the object (in m/s) at time t s, in terms of t, u, g and α.

b At time T s, the object is moving in a direction perpendicular to that of projection. Determine the relationship between T, u, g and α.

7 A ball is thrown from a point O at ground level with an initial speed of 50 m/s. There is a wall of height 20 m at a distance of 100 m from point O. Determine the smallest possible angle of projection for the ball to pass over the wall. (Consider the ball as a particle.)

8 A ball is thrown at an angle of $\arctan\left(\frac{12}{5}\right)$ to the horizontal. If the ball hits a building 10 m away at a height of 14 m, determine its initial speed.

9 A ball is projected from ground level over a wall of height 5 m. The point of projection is 20 m from the base of the wall. The initial speed of the ball is 40 m/s at an angle of 30° to the horizontal. Assume that air resistance is negligible.

a How far above the top of the wall does the ball pass? (Give your answer in metres correct to one decimal place.)

b What is the speed of the ball as it passes over the top of the wall? (Give your answer in m/s correct to one decimal place.)

10 Particle X is projected from the origin with an initial velocity of $16\hat{\boldsymbol{i}} + 30\hat{\boldsymbol{j}}$ m/s. At the same time, particle Y is projected from a point 60 m to the right of the origin and 25 m higher with an initial velocity of $-8\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$ m/s.

a Determine the position vector of particle X at time t seconds.

b Determine the time and the point at which the two particles collide.

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11 A particle is projected with initial velocity $\boldsymbol{u}$ from a point O and experiences a constant downwards acceleration equal to $-g\hat{\boldsymbol{j}}$.

a Write an expression for the velocity, $\boldsymbol{v}$, of the particle at time t.

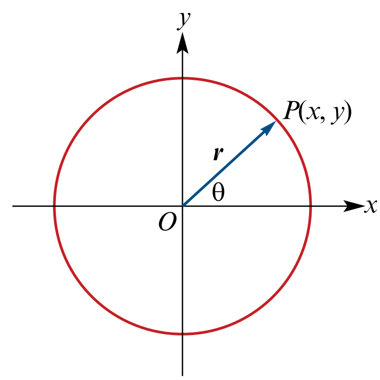
b Write an expression for the position, $\boldsymbol{r}$, of the particle at time t.

c Prove that, at the time t when the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular, we have $|\boldsymbol{u}|^2 + |\boldsymbol{v}|^2 = \dfrac{4|\boldsymbol{r}|^2}{t^2}$ and $|\boldsymbol{r}| = \dfrac{1}{2}gt^2$.

7G Circular motion

Learning intentions

▶ To be able to use vector calculus to solve circular motion problems.



Suppose that a particle P is moving around the circle centred at the origin with radius r. The position of the particle is given by

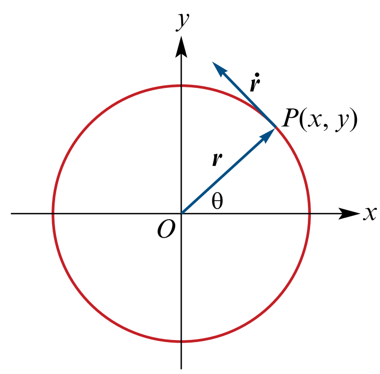
$$\boldsymbol{r} = r\cos\theta\,\hat{\boldsymbol{i}} + r\sin\theta\,\hat{\boldsymbol{j}}$$

where $\theta = f(t)$. That is, we consider the angle, θ, to be a function of time, t.

We can determine the velocity of the particle by first using the chain rule to obtain

$$\frac{d}{dt}(\cos\theta) = -\sin\theta \cdot \frac{d\theta}{dt} = -\dot{\theta}\sin\theta$$

$$\frac{d}{dt}(\sin\theta) = \cos\theta \cdot \frac{d\theta}{dt} = \dot{\theta}\cos\theta$$



Hence

$$\begin{aligned}\dot{\boldsymbol{r}} &= -r\dot{\theta}\sin\theta\,\hat{\boldsymbol{i}} + r\dot{\theta}\cos\theta\,\hat{\boldsymbol{j}}\\ &= r\dot{\theta}(-\sin\theta\,\hat{\boldsymbol{i}} + \cos\theta\,\hat{\boldsymbol{j}})\end{aligned}$$

It can be seen that $-\sin\theta\,\hat{\boldsymbol{i}} + \cos\theta\,\hat{\boldsymbol{j}}$ is a unit vector, and so $r|\dot{\theta}|$ is the magnitude of $\dot{\boldsymbol{r}}$.

We also observe that $\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = 0$. Therefore the velocity vector is perpendicular to the position vector.

Angular velocity

The **angular velocity** of the particle, denoted by ω, is the rate of change of the angle θ with respect to time:

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

Note: The standard unit for angular velocity is radians per second.

We have seen that the magnitude of the velocity is $|\dot{\boldsymbol{r}}| = r|\dot{\theta}|$. So we can now write

$v = r|\omega|$, where v is the speed of the particle.

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Uniform circular motion

Throughout the rest of this section, we will consider circular motion where the angular velocity ω is constant. This is called **uniform circular motion**.

If the particle starts at an angle of $\theta = 0$ at time $t = 0$, then we have $\theta = \omega t$ and the equations of motion are

$$\boldsymbol{r} = r(\cos(\omega t)\,\hat{\boldsymbol{i}} + \sin(\omega t)\,\hat{\boldsymbol{j}})$$
$$\dot{\boldsymbol{r}} = r\omega(-\sin(\omega t)\,\hat{\boldsymbol{i}} + \cos(\omega t)\,\hat{\boldsymbol{j}})$$
$$\ddot{\boldsymbol{r}} = r\omega^2(-\cos(\omega t)\,\hat{\boldsymbol{i}} - \sin(\omega t)\,\hat{\boldsymbol{j}})$$

We see that the acceleration is directed along a radius towards the centre of the circle.

Uniform circular motion

For a particle moving around a circle of radius r with constant angular velocity $\omega > 0$:

- **Speed** The speed of the particle is $v = r\omega$.
- **Period** The time to complete one revolution is $T = \dfrac{2\pi}{\omega}$.

Example 23

A particle is moving around a circle of radius 3 m with a constant speed of 2 m/s. Given that $\theta = 0$ at time $t = 0$, determine:

a the angular velocity of the particle

b the position of the particle at time $t = \pi$ seconds

c the velocity of the particle at time $t = \pi$ seconds

d the acceleration of the particle at time $t = \pi$ seconds.

Solution

a We are given that $v = 2$ m/s and $r = 3$ m.

Therefore the angular velocity is $\omega = \dfrac{v}{r} = \dfrac{2}{3}$ radians per second.

b When $t = \pi$, the particle is at an angle of $\theta = \omega t = \dfrac{2\pi}{3}$.

So
$$\begin{aligned}\boldsymbol{r} &= r(\cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}})\\ &= 3\left(\cos\left(\frac{2\pi}{3}\right)\hat{\boldsymbol{i}} + \sin\left(\frac{2\pi}{3}\right)\hat{\boldsymbol{j}}\right)\\ &= -\frac{3}{2}\hat{\boldsymbol{i}} + \frac{3\sqrt{3}}{2}\hat{\boldsymbol{j}}\end{aligned}$$

c
$$\begin{aligned}\dot{\boldsymbol{r}} &= r\omega(-\sin\theta\,\hat{\boldsymbol{i}} + \cos\theta\,\hat{\boldsymbol{j}})\\ &= 3\times\frac{2}{3}\left(-\sin\left(\frac{2\pi}{3}\right)\hat{\boldsymbol{i}} + \cos\left(\frac{2\pi}{3}\right)\hat{\boldsymbol{j}}\right)\\ &= -\sqrt{3}\,\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\ \text{ m/s}\end{aligned}$$

d
$$\begin{aligned}\ddot{\boldsymbol{r}} &= r\omega^2(-\cos\theta\,\hat{\boldsymbol{i}} - \sin\theta\,\hat{\boldsymbol{j}})\\ &= 3\times\left(\frac{2}{3}\right)^2\left(-\cos\left(\frac{2\pi}{3}\right)\hat{\boldsymbol{i}} - \sin\left(\frac{2\pi}{3}\right)\hat{\boldsymbol{j}}\right)\\ &= \frac{2}{3}\,\hat{\boldsymbol{i}} - \frac{2\sqrt{3}}{3}\,\hat{\boldsymbol{j}}\ \text{ m/s}^2\end{aligned}$$

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Example 24

A particle moves at a constant speed of 8 m/s around a circle with a radius of 4 m. Assume that $\theta = 0$ when $t = 0$.

a Determine the position of the particle, relative to the centre of the circle, at time t seconds.

b Determine the velocity of the particle at time t seconds.

c Determine the acceleration of the particle at time t seconds.

Solution

The angular velocity is $\omega = \dfrac{v}{r} = 2$ radians per second.

At time t seconds, the angle is $\theta = 2t$.

a $\boldsymbol{r} = 4\cos(2t)\hat{\boldsymbol{i}} + 4\sin(2t)\hat{\boldsymbol{j}}$

b $\dot{\boldsymbol{r}} = -8\sin(2t)\hat{\boldsymbol{i}} + 8\cos(2t)\hat{\boldsymbol{j}}$

c $\ddot{\boldsymbol{r}} = -16\cos(2t)\hat{\boldsymbol{i}} - 16\sin(2t)\hat{\boldsymbol{j}}$

Skill-sheet

Exercise 7G

Example 23

1 A particle is moving around a circle of radius 2.5 m with a constant speed of 5 m/s. Given that $\theta = 0$ at time $t = 0$, determine: SF

a the angular velocity of the particle

b the position of the particle at time $t = \pi$ seconds

c the velocity of the particle at time $t = \pi$ seconds

d the acceleration of the particle at time $t = \pi$ seconds.

2 A particle is moving around a circle with a constant speed of 2 m/s. The radius of the circle is 2 m. Given that $\theta = 0$ at time $t = 0$, determine:

a the angular velocity of the particle

b the position of the particle at time $t = \frac{\pi}{2}$ seconds

c the velocity of the particle at time $t = \frac{\pi}{2}$ seconds

d the acceleration of the particle at time $t = \frac{\pi}{2}$ seconds.

3 An electric fan is spinning at 350 revolutions per minute. The fan's diameter is 20 cm.

a Determine the angular velocity of a point at the end of a fan blade.

b Determine the speed of a point at the end of a fan blade.

Example 24

4 A car is driving around a circular track with a radius of 25 m at a constant speed of 10 m/s. Assume that $\theta = 0$ when $t = 0$.

a Determine the position of the car, relative to the centre of the circle, at time t seconds.

b Determine the velocity of the car at time t seconds.

c Determine the acceleration of the car at time t seconds.

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SF

5 A tyre of radius 25 cm is rotating at a constant rate such that a point on its rim has speed $\frac{25}{3}$ m/s. Determine the angular velocity of a point on its rim, in radians per second.

6 A particle moves in a circle of radius 2 m and completes 1.5 revolutions per second. Determine:

a the angular velocity
b the speed
c the magnitude of the acceleration
d the period of the motion.

CF

7 A particle moves such that its position vector, $\boldsymbol{r}(t)$, at time t is given by

$$\boldsymbol{r}(t) = 3\sin(4\pi t)\,\hat{\boldsymbol{i}} - 3\cos(4\pi t)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

All distances are in metres and time is in seconds.

a Determine the speed of the particle in m/s.
b Determine the magnitude of the acceleration in m/s^2.
c Determine the position, velocity and acceleration vectors at time $t = \frac{1}{2}$ second.
d Determine the angular velocity of the particle in radians per second.

8 A bicycle wheel of radius 0.35 m completes 120 revolutions per minute.

a Determine the angular velocity of the wheel (in radians per second).
b Determine the speed of a point on the rim of the wheel (in metres per second).
c How far will the bicycle travel in 720 revolutions of the wheel? (Assume that both wheels of the bicycle have the same radius.)

SF

9 A cyclist is travelling at a constant speed of 12 m/s. The radius of each bicycle wheel is 0.35 m. Determine the angular velocity of each wheel.

CF

10 A particle moves so that its position vector at time t is given by

$$\boldsymbol{r} = 4\cos(t^2)\,\hat{\boldsymbol{i}} + 4\sin(t^2)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

a Show that the particle moves in a circle.
b Determine $\dot{\boldsymbol{r}}$ and hence show that the particle does not move with constant speed.
c Determine $\ddot{\boldsymbol{r}}$ and express your answer in the form $\ddot{\boldsymbol{r}} = f(t)\boldsymbol{r} + g(t)\dot{\boldsymbol{r}}$.

11 A particle moves such that its position vector, $\boldsymbol{r}(t)$, is given by

$$\boldsymbol{r}(t) = (4 + 3\cos(4\pi t))\,\hat{\boldsymbol{i}} + (2 - 3\sin(4\pi t))\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

All distances are in metres and time is in seconds.

a Determine the speed of the particle in m/s.
b Determine the magnitude of the acceleration in m/s^2.
c Determine the position, velocity and acceleration vectors at time $t = 1$ second.
d Determine the angular velocity of the particle relative to the centre of motion.
e Determine the Cartesian equation of the particle's path.

Chapter summary

- We state the following results for motion in three dimensions. The statements for two dimensions are analogous.
  - The position of a particle at time t can be described by a vector function:
    $$\boldsymbol{r}(t) = f(t)\hat{\boldsymbol{i}} + g(t)\hat{\boldsymbol{j}} + h(t)\hat{\boldsymbol{k}}$$
  - The velocity of the particle at time t is
    $$\dot{\boldsymbol{r}}(t) = f'(t)\hat{\boldsymbol{i}} + g'(t)\hat{\boldsymbol{j}} + h'(t)\hat{\boldsymbol{k}}$$
  - The acceleration of the particle at time t is
    $$\ddot{\boldsymbol{r}}(t) = f''(t)\hat{\boldsymbol{i}} + g''(t)\hat{\boldsymbol{j}} + h''(t)\hat{\boldsymbol{k}}$$
- The velocity vector $\dot{\boldsymbol{r}}(t)$ has the direction of the motion of the particle at time t.
- Speed is the magnitude of velocity. At time t, the speed is $|\dot{\boldsymbol{r}}(t)|$.
- The distance between the points on the path corresponding to $t = t_0$ and $t = t_1$ is given by $|\boldsymbol{r}(t_1) - \boldsymbol{r}(t_0)|$.
- Circular motion
  - For a particle moving around the origin in the x–y plane, let θ be the angle that its position vector makes with the positive direction of the x-axis. The **angular velocity** of the particle, ω, is the rate of change of θ with respect to time.
  - For a particle moving around a circle of radius r with constant angular velocity $\omega > 0$:
    - the speed of the particle is $v = r\omega$
    - the period of the motion is $T = \dfrac{2\pi}{\omega}$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

7A **1** I have reviewed differentiation and anti-differentiation techniques. ☐

See Example 1, Example 2, Example 3, Example 4, Example 5 and Questions 1, 2, 7, 8, 10, 11

7B **2** I can describe paths using position vectors. ☐

See Example 6 and Question 1

7C **3** I am able to differentiate vector functions. ☐

See Example 7, Example 8, Example 9, Example 10, Example 11, Example 12 and Questions 1, 2 and 3

7C **4** I am able to antidifferentiate vector functions. ☐

See Example 13, Example 14 and Question 4

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Review

7D **5** I am able to determine the velocity and acceleration of an object given its position vector at time t. ☐

See Example 15, Example 16, Example 17, Example 18, Example 19 and Questions 1, 3, 8, 10 and 12

7E **6** I am able to describe motion in a straight line using vector functions. ☐

See Example 20, Example 21 and Questions 1 and 4

7F **7** I am able to describe projectile motion with vector functions. ☐

See Example 22 and Question 1

7G **8** I am able to describe circular motion with vector functions. ☐

See Example 23, Example 24 and Questions 1 and 4

Short-response questions

Technology-free short-response questions

SF

1 The position, $\boldsymbol{r}(t)$ metres, of a particle moving in a plane is given by $\boldsymbol{r}(t) = 2t\,\hat{\boldsymbol{i}} + (t^2 - 4)\hat{\boldsymbol{j}}$ at time t seconds.

a Determine the velocity and acceleration when $t = 2$.

b Determine the Cartesian equation of the path.

2 Determine the velocity and acceleration vectors of the position vectors:

a $\boldsymbol{r} = 2t^2\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}$

b $\boldsymbol{r} = 4\sin t\,\hat{\boldsymbol{i}} + 4\cos t\,\hat{\boldsymbol{j}} + t^2\hat{\boldsymbol{k}}$

3 At time t, a particle has coordinates $(6t, t^2 + 4)$. Determine the unit vector along the tangent to the path when $t = 4$.

4 The position vector of a particle is given by $\boldsymbol{r}(t) = 10\sin(2t)\,\hat{\boldsymbol{i}} + 5\cos(2t)\,\hat{\boldsymbol{j}}$.

a Determine its position vector when $t = \frac{\pi}{6}$.

b Determine the cosine of the angle between its directions of motion at $t = 0$ and $t = \frac{\pi}{6}$.

5 Determine the unit tangent vector of the curve $\boldsymbol{r} = (\cos t + t\sin t)\hat{\boldsymbol{i}} + (\sin t - t\cos t)\hat{\boldsymbol{j}}$, $t > 0$.

CF

6 A particle moves on a curve with equation $\boldsymbol{r} = 5(\cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}})$. Determine:

a the velocity at time t

b the speed at time t

c the acceleration at time t

d $\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}$, and comment.

7 Particles A and B move with velocities $\boldsymbol{V}_A = \cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}$ and $\boldsymbol{V}_B = \sin t\,\hat{\boldsymbol{i}} + \cos t\,\hat{\boldsymbol{j}}$ respectively. At time $t = 0$, the position vectors of A and B are $\boldsymbol{r}_A = \hat{\boldsymbol{i}}$ and $\boldsymbol{r}_B = \hat{\boldsymbol{j}}$. Prove that the particles collide, finding the time of collision.

8 The position vector of a particle at any time t is given by $\boldsymbol{r} = (1 + \sin t)\hat{\boldsymbol{i}} + (1 - \cos t)\hat{\boldsymbol{j}}$. CF

a Show that the magnitudes of the velocity and acceleration are constants.

b Determine the Cartesian equation of the path described by the particle.

c Determine the first instant that the position is perpendicular to the velocity.

9 The velocities of two particles A and B are given by $\boldsymbol{V}_A = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{V}_B = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. The initial position vector of particle A is $\boldsymbol{r}_A = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$. If the particles collide after 3 seconds, determine the initial position vector of particle B.

10 A particle starts from point $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and travels with a velocity given by $t\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, at time t seconds from the start. A second particle travels in the same plane and its position vector is given by $\boldsymbol{r} = (s - 4)\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, at time s seconds after it started.

a Determine an expression for the position of the first particle.

b Determine the point at which their paths cross.

c If the particles actually collide, determine the time between the two starting times.

11 A particle travels with constant acceleration, given by $\ddot{\boldsymbol{r}}(t) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$. Two seconds after starting, the particle passes through the point $\hat{\boldsymbol{i}}$, travelling at a velocity of $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$. Determine: SF

a an expression for the velocity of the particle at time t

b an expression for its position

c the initial position and velocity of the particle.

12 Two particles travel with constant acceleration given by $\ddot{\boldsymbol{r}}_1(t) = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\ddot{\boldsymbol{r}}_2(t) = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$. The initial velocity of the second particle is $-4\hat{\boldsymbol{i}}$ and that of the first particle is $k\hat{\boldsymbol{j}}$. CF

a Determine an expression for:

  i the velocity of the second particle

  ii the velocity of the first particle.

b At one instant both particles have the same velocity. Determine:

  i the time elapsed before that instant

  ii the value of k

  iii the common velocity.

13 The position of an object is given by $\boldsymbol{r}(t) = e^t\hat{\boldsymbol{i}} + 4e^{2t}\hat{\boldsymbol{j}}$, $t \geq 0$.

a Show that the path of the object is the graph of $y = 4x^2$ for $x \geq 1$.

b Determine:

  i the velocity vector at time t

  ii the initial velocity

  iii the time at which the velocity is parallel to the vector $\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$.

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Review

14 The velocity of a particle is given by $\dot{\boldsymbol{r}}(t) = (t-3)\hat{\boldsymbol{j}}$, $t \geq 0$.

CF

a Show that the path of this particle is linear.

b Initially, the position of the particle is $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

i Determine the Cartesian equation of the path followed by the particle.

ii Determine the point at which the particle is momentarily at rest.

15 A particle is moving such that its position vector, $\boldsymbol{r}(t)$, is given by

$$\boldsymbol{r}(t) = 3\sin(2\pi t)\,\hat{\boldsymbol{i}} + 3\cos(2\pi t)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

All distances are in metres and time is in seconds.

a Determine the speed of the particle in m/s.

b Determine the magnitude of the acceleration in m/s^2.

c Determine the position, velocity and acceleration vectors at time $t = 1$ second.

d Determine the angular velocity of the particle in radians per second.

16 A particle is projected with an initial speed of 20 m/s from a point at ground level. The angle of projection is 60° to the horizontal. Determine the value of a such that the time taken to return to ground level is $\dfrac{a}{g}$ seconds.

17 A wheel of radius 0.5 m is rotating at a constant rate of 360 revolutions per minute. For a point on the circumference of the wheel, determine:

a its angular velocity (in radians per second)

b its speed (in metres per second).

18 A particle moves in a circle such that its position vector at time t is given by

$$\boldsymbol{r}(t) = a\cos(nt)\,\hat{\boldsymbol{i}} + a\sin(nt)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

where a and n are positive constants with units of a being metres and units of t being seconds. Given that the circle has a radius of length 2 metres and the particle completes 80 revolutions per minute, determine:

a the constants a and n

b the particle's speed in m/s

c the magnitude of the acceleration in m/s^2.

19 A particle is projected from a point O. Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the horizontal (x) and vertical (y) directions respectively. At time $t = 1$, the position vector of the particle is $a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}$. Determine an expression for the speed of projection in terms of a, b and g.

Technology-active short-response questions

20 A particle is moving with a velocity given by $\boldsymbol{v}(t) = (4-t)\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 3t\boldsymbol{k}$. The original position of the particle is at $(8, 0, 20)$.

CF

a Determine an expression for the position of the particle at time t.

b Determine an expression for the speed of the particle at time t.

c Determine the time t when the speed is a minimum and determine the position of the particle at this time.

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21 A particle is projected with a velocity of 20 ms^{-1} at an angle of elevation θ.
Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ be unit vectors in the horizontal (x) and vertical (y) directions respectively. The particle is projected from the origin and it hits the ground 10 m from the origin. The acceleration on the particle is given as $\boldsymbol{a}(t) = -g\hat{\boldsymbol{j}}$.

a Determine an expression for the velocity of the particle at time t.

b Determine an expression for the position of the particle at time t.

c Determine two expressions for t_1, the time when the particle hits the ground.

d Use these two expressions found in part **c** to find the two possible values for θ.

22 Two particles start simultaneously from different locations and their positions at time t seconds are given by
$\boldsymbol{r}_1 = (2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}) + t(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$ and
$\boldsymbol{r}_2 = (-8\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}) + t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$.

a Determine a vector $\boldsymbol{x}$ which defines the displacement of the particles from each other at time t.

b Determine t when the particles are closest to each other.

c Determine the distance between them at this time.

23 The position of a particle at time t sec is given by
$\boldsymbol{r} = (3 - 2t)\hat{\boldsymbol{i}} + (3 - t)\hat{\boldsymbol{j}} + (9 - t)\hat{\boldsymbol{k}}$

a Determine the velocity vector of the particle.

b Determine t when the velocity vector is perpendicular to the position vector.

The position of a second particle at time s sec is given by
$\boldsymbol{r} = (s - 3)\hat{\boldsymbol{i}} + (2s - 6)\hat{\boldsymbol{j}} + (\lambda + s)\hat{\boldsymbol{k}}$

c Determine the value of λ which will cause the particles to collide.

d Determine the location of the collision in that case.

24 The position of a particle is given by
$\boldsymbol{r} = \sin t\hat{\boldsymbol{i}} + \sin 2t\hat{\boldsymbol{j}},\ t \geq 0$.

a Determine the velocity vector of the particle.

b State the solutions for $\cos t = 0$ and $\cos 2t = 0$ and hence show that this particle will never be at rest.

c Determine the value of t for which the position vector and the velocity vector are at right angles for the first time, $t > 0$.

25 A child is sitting still in some long grass watching a bee. The bee flies at a constant speed in a straight line from its beehive to a flower and reaches the flower 3 seconds later. The position vector of the beehive relative to the child is $10\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ and the position vector of the flower relative to the child is $7\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$, where all the distances are measured in metres.

a If B is the position of the beehive and F the position of the flower, determine $\overrightarrow{BF}$.

b Determine the distance BF.

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c Determine the speed of the bee.

d Determine the velocity of the bee.

e Determine the time when the bee is closest to the child and its distance from the child at this time.

26 Two particles P and Q are moving in a horizontal plane. The particles are moving with velocities $9\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ m/s and $5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s respectively.

a Determine the speeds of the particles.

b At time $t = 4$, particles P and Q have position vectors $\boldsymbol{r}_P(4) = 96\hat{\boldsymbol{i}} + 44\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_Q(4) = 100\hat{\boldsymbol{i}} + 96\hat{\boldsymbol{j}}$. (Distances are measured in metres.)

i Determine the position vectors of P and Q at time $t = 0$.

ii Determine the vector $\overrightarrow{PQ}$ at time t.

c Determine the time at which P and Q are nearest to each other and the magnitude of $\overrightarrow{PQ}$ at this instant.

27 Two particles A and B move in the plane. The velocity of A is $(-3\hat{\boldsymbol{i}} + 29\hat{\boldsymbol{j}})$ m/s while that of B is $v(\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}})$ m/s, where v is a constant. (All distances are measured in metres.)

a Determine the vector $\overrightarrow{AB}$ at time t seconds, given that $\overrightarrow{AB} = -56\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ when $t = 0$.

b Determine the value of v such that the particles collide.

c If $v = 3$:

i Determine $\overrightarrow{AB}$. **ii** Determine the time when the particles are closest.

28 Initially, a motor boat is at a point J at the end of a jetty and a police boat is at a point P. The position vector of P relative to J is $400\hat{\boldsymbol{i}} - 600\hat{\boldsymbol{j}}$. The motor boat leaves the point J and travels with constant velocity $6\hat{\boldsymbol{i}}$. At the same time, the police boat leaves its position at P and travels with constant velocity $u(8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}})$, where u is a real number. All distances are measured in metres and all times are measured in seconds.

a If the police boat meets the motor boat after t seconds, determine:

i the value of t **ii** the value of u

iii the speed of the police boat **iv** the position of the point where they meet.

b Determine the time at which the police boat was closest to J and its distance from J at this time.

29 A particle A is at rest on a smooth horizontal table at a point with position vector $-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, relative to an origin O. Point B is on the table such that $\overrightarrow{OB} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$. (All distances are measured in metres and time in seconds.) At time $t = 0$, the particle is projected along the table with velocity $(6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$ m/s.

a Determine:

i $\overrightarrow{OA}$ at time t

ii $\overrightarrow{BA}$ at time t.

b Determine the time when $|\overrightarrow{BA}| = 5$.

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c Using the time found in **b**:

i Determine a unit vector $\boldsymbol{c}$ along $\overrightarrow{BA}$.

ii Determine a unit vector $\boldsymbol{d}$ perpendicular to $\overrightarrow{BA}$.

Hint: The vector $y\hat{\boldsymbol{i}} - x\hat{\boldsymbol{j}}$ is perpendicular to $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.

iii Express $6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ in the form $p\boldsymbol{c} + q\boldsymbol{d}$.

30 **a** Sketch the graph of the Cartesian relation corresponding to the vector equation

$$\boldsymbol{r}(\theta) = \cos(\theta)\,\hat{\boldsymbol{i}} - \sin(\theta)\,\hat{\boldsymbol{j}}, \quad 0 < \theta < \frac{\pi}{2}$$

b A particle P describes a circle of radius 16 cm about the origin. It completes the circle every π seconds. At $t = 0$, P is at the point $(16, 0)$ and is moving in a clockwise direction. It can be shown that $\overrightarrow{OP} = a\cos(nt)\,\hat{\boldsymbol{i}} + b\sin(nt)\,\hat{\boldsymbol{j}}\;\; n > 0$. Determine the values of:

i a **ii** b **iii** n **iv** State the velocity and acceleration of P at time t.

c A second particle Q has position vector given by $\overrightarrow{OQ} = 8\sin(t)\,\hat{\boldsymbol{i}} + 8\cos(t)\,\hat{\boldsymbol{j}}$, where measurements are in centimetres. Obtain an expression for:

i $\overrightarrow{PQ}$ **ii** $|\overrightarrow{PQ}|^2$

d Determine the minimum distance between P and Q.

31 At time t, a particle has velocity $\boldsymbol{v} = (2\cos t)\hat{\boldsymbol{i}} - (4\sin t\,\cos t)\hat{\boldsymbol{j}}$, $t \geq 0$. At time $t = 0$, it is at the point with position vector $3\hat{\boldsymbol{j}}$.

a Determine the position of the particle at time t.

b Determine the position of the particle when it first comes to rest.

c **i** Determine the Cartesian equation of the path of the particle.

ii Sketch the path of the particle.

d Express $|\boldsymbol{v}|^2$ in terms of $\cos t$ and, without using calculus, determine the maximum speed of the particle.

e Give the time at which the particle is at rest for the second time.

f **i** Show that the distance, d, of the particle from the origin at time t is given by $d^2 = \cos^2(2t) + 2\cos(2t) + 6$.

ii Determine the time(s) at which the particle is closest to the origin.

32 A golfer hits a ball from a point referred to as the origin with a velocity of $a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + 20\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are unit vectors horizontally forwards, horizontally to the right and vertically upwards respectively. After being hit, the ball is subject to an acceleration $2\hat{\boldsymbol{j}} - 10\hat{\boldsymbol{k}}$. (All distances are measured in metres and all times in seconds.) Determine:

a the velocity of the ball at time t

b the position vector of the ball at time t

c the time of flight of the ball

d the values of a and b if the golfer wishes to hit a *direct* hole-in-one, where the position vector of the hole is $100\hat{\boldsymbol{i}}$

e the angle of projection of the ball relative to the horizontal plane if a hole-in-one is achieved.

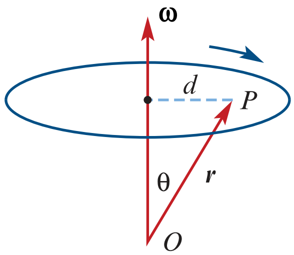
Review

CU

33 The rotation of a wheel can be represented by a vector $\boldsymbol{\omega}$ as shown in the diagram:

- The vector $\boldsymbol{\omega}$ points along the axis of rotation such that the rotation appears anticlockwise if you look at the wheel from O in the direction of $\boldsymbol{\omega}$.
- The magnitude of $\boldsymbol{\omega}$ is the angular velocity of the wheel, $\omega > 0$.

Let $\boldsymbol{r} = \overrightarrow{OP}$ be the position vector of a point on the wheel, let d be the distance of P from the axis of rotation, and let θ be the angle between $\boldsymbol{r}$ and $\boldsymbol{\omega}$. Then the speed of the wheel at P is



$$v = d\omega = |\boldsymbol{r}| \sin\theta \cdot |\boldsymbol{\omega}| = |\boldsymbol{r} \times \boldsymbol{\omega}|$$

The velocity vector at P is given by

$$\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{\omega}$$

Suppose that the wheel has radius 0.35 m, the centre of the wheel is at $2\hat{\boldsymbol{k}}$ and $\boldsymbol{\omega} = 15\hat{\boldsymbol{k}}$. Determine the velocity of the wheel at each of the following points:

a $0.35\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}$ **b** $0.35\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ **c** $-0.35\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}$ **d** $-0.35\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

e $0.2\hat{\boldsymbol{i}} - 0.2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ **f** $-0.2\hat{\boldsymbol{i}} - 0.2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

34 Particles P and Q have variable position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, given by

$$\boldsymbol{p}(t) = \cos(t)\,\hat{\boldsymbol{i}} + \sin(t)\,\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$$
$$\boldsymbol{q}(t) = \cos(2t)\,\hat{\boldsymbol{i}} - \sin(2t)\,\hat{\boldsymbol{j}} + \tfrac{1}{2}\hat{\boldsymbol{k}}$$

where $0 \le t \le 2\pi$.

a **i** For $\boldsymbol{p}(t)$, describe the path.
ii Determine the distance of particle P from the origin at time t.
iii Determine the velocity of particle P at time t.
iv Show that the vector $\cos(t)\,\hat{\boldsymbol{i}} + \sin(t)\,\hat{\boldsymbol{j}}$ is perpendicular to the velocity vector of P for any value of t.
v Determine the acceleration, $\ddot{\boldsymbol{p}}(t)$, at time t.

b **i** Determine the vector $\overrightarrow{PQ}$ at time t.
ii Show that the distance between P and Q at time t is $\sqrt{\frac{17}{4} - 2\cos(3t)}$.
iii Determine the maximum distance between the particles.
iv Determine the times at which this maximum occurs.
v Determine the minimum distance between the particles.
vi Determine the times at which this minimum occurs.

c **i** Show that $\boldsymbol{p}(t) \cdot \boldsymbol{q}(t) = \cos(3t) - \frac{1}{2}$.
ii Determine an expression for $\cos(\angle POQ)$.
iii Determine the greatest magnitude of angle POQ.

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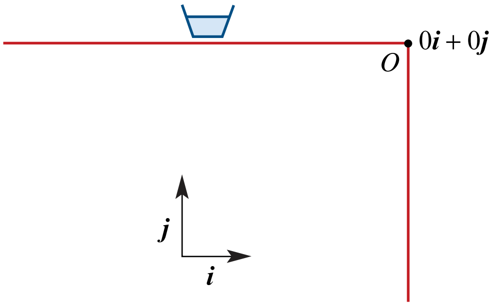
35 A golfer hits a ball from an origin, O, aiming at a hole, H, which is 200 metres away at the end of a horizontal fairway. The initial velocity of the ball is $\boldsymbol{v}(0) = 35\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 24.5\hat{\boldsymbol{k}}$, where the unit vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are chosen such that $\hat{\boldsymbol{i}}$ is in the direction of $\overrightarrow{OH}$ and $\hat{\boldsymbol{k}}$ is in the upwards direction. The ball lands on the fairway at point L. While in the air, the ball is subject only to gravity, so its acceleration is $\boldsymbol{a}(t) = -9.8\hat{\boldsymbol{k}}$ m/s^2.

- **a** Determine the position vector of the ball at time t seconds after being hit.
- **b** Determine the length of time, in seconds, that the ball is in the air.
- **c** Determine the distance, to the nearest metre, along the fairway from L to H.
- **d** Correct to one decimal place, determine the speed of the ball in m/s when it lands.

36 Particles A and B move such that, at any time $t \geq 0$, their position vectors are $\boldsymbol{r}_A = 2t\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_B = (4 - 4\sin(\alpha t))\hat{\boldsymbol{i}} + 4\cos(\alpha t)\hat{\boldsymbol{j}}$, where α is a positive constant.

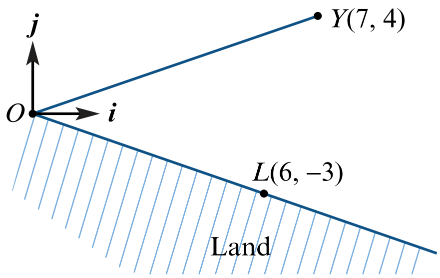
- **a** Determine the speed of B in terms of α.
- **b** Determine the Cartesian equations of the paths of A and B.
- **c** On the same set of axes, sketch the paths of A and B, showing directions of travel.
- **d** Determine the coordinates of the points where the paths of A and B cross.
- **e** Determine the least value of α, correct to two decimal places, for which particles A and B will collide.

37 A bartender slides a glass along a bar for a customer to collect. Unfortunately, the customer has turned to speak to a friend. The glass slides over the edge of the bar with a horizontal velocity of 2 m/s. Assume that air resistance is negligible and that the acceleration due to gravity is 9.8 m/s^2 in a downwards direction.



- **a**
  - **i** Give the acceleration of the glass as a vector expression.
  - **ii** Give the vector expression for the velocity of the glass at time t seconds, where t is measured from when the glass leaves the bar.
  - **iii** Give the position of the glass with respect to the edge of the bar, O, at time t seconds.
- **b** It is 0.8 m from O to the floor directly below. Determine:
  - **i** the time it takes for the glass to hit the floor
  - **ii** the horizontal distance from the bar where the glass hits the floor.

38 A yacht is returning to its marina at O. At noon, the yacht is at Y. The yacht takes a straight-line course to O. Point L is the position of a navigation sign on the shore. Coordinates represent distances east and north of the marina, measured in kilometres.



CU

a **i** Write down the position vector of the navigation sign L.

ii Determine the unit vector in the direction of $\overrightarrow{OL}$.

b Determine the vector resolute of $\overrightarrow{OY}$ in the direction of $\overrightarrow{OL}$ and hence determine the coordinates of the point on shore closest to the yacht at noon.

c The yacht sails towards O. The position vector at time t hours after 12 p.m. is given by $\boldsymbol{r}(t) = (7 - \frac{7}{2}t)\hat{\boldsymbol{i}} + (4 - 2t)\hat{\boldsymbol{j}}$.

i Determine an expression for $\overrightarrow{LP}$, where P is the position of the yacht at time t.

ii Determine the time when the yacht is closest to the navigation sign.

iii Determine the closest distance between the sign and the yacht.

Multiple-choice questions

Technology-free multiple-choice questions

1 A particle moves in a plane such that, at time t, its position is $\boldsymbol{r}(t) = 2t^2\hat{\boldsymbol{i}} + (3t - 1)\hat{\boldsymbol{j}}$. Its acceleration at time t is given by

A $\frac{2}{3}t^3\hat{\boldsymbol{i}} + (\frac{3}{2}t^2 - t)\hat{\boldsymbol{j}}$ **B** $4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **C** $0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$ **D** $4\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$

2 The position vector of a particle at time t, $t \geq 0$, is given by $\boldsymbol{r} = \sin(3t)\hat{\boldsymbol{i}} - 2\cos(t)\hat{\boldsymbol{j}}$. The speed of the particle when $t = \pi$ is

A $2\sqrt{2}$ **B** $\sqrt{5}$ **C** 0 **D** 3

3 A particle moves with constant velocity $5\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Its initial position is $3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{k}}$. Its position vector at time t is given by

A $(3t + 5)\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + (2 - 6t)\hat{\boldsymbol{k}}$ **B** $(5t + 3)\hat{\boldsymbol{i}} - 4t\hat{\boldsymbol{j}} + (2t - 6)\hat{\boldsymbol{k}}$

C $5t\hat{\boldsymbol{i}} - 4t\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$ **D** $-5t\hat{\boldsymbol{i}} - 4t\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$

4 A particle moves with its position vector defined with respect to time t by the vector function $\boldsymbol{r}(t) = (2t^3 - 1)\hat{\boldsymbol{i}} + (2t^2 + 3)j + 6t\hat{\boldsymbol{k}}$. The acceleration when $t = 1$ is given by

A $12\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$ **B** $12\hat{\boldsymbol{i}}$ **C** $2\sqrt{10}$ **D** $12\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

5 The position vector of a particle at time t seconds is $\boldsymbol{r}(t) = (t^2 - 4t)(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$, measured in metres from a fixed point. The distance in metres travelled in the first 4 seconds is

A 0 **B** $4\sqrt{3}$ **C** $8\sqrt{3}$ **D** 4

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6 The initial position, velocity and constant acceleration of a particle are given by $3\hat{\boldsymbol{i}}$, $2\hat{\boldsymbol{j}}$ and $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ respectively. The position vector of the particle at time t is given by

A $(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})t + 3\hat{\boldsymbol{i}}$
B $t^2\hat{\boldsymbol{i}} - \frac{1}{2}t^2\hat{\boldsymbol{j}}$
C $(t^2 + 3)\hat{\boldsymbol{i}} + (2t - \frac{1}{2}t^2)\hat{\boldsymbol{j}}$
D $3\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$

7 The position of a particle at time $t = 0$ is $\boldsymbol{r}(0) = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. The position of the particle at time $t = 3$ is $\boldsymbol{r}(3) = 7\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$. The average velocity for the interval $[0, 3]$ is

A $\frac{1}{3}(8\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$
B $\frac{1}{3}(21\hat{\boldsymbol{i}} + 21\hat{\boldsymbol{j}} - 12\hat{\boldsymbol{k}})$
C $2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$
D $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

8 A particle is moving so its velocity vector at time t is $\dot{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, where $\boldsymbol{r}(t)$ is the position vector at time t. If $\boldsymbol{r}(0) = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, then $\boldsymbol{r}(t)$ is equal to

A $(3t + 1)\hat{\boldsymbol{i}} + (3t^2 + 1)\hat{\boldsymbol{j}}$
B $2t^2\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$
C $5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$
D $(t^2 + 3)\hat{\boldsymbol{i}} + (3t + 1)\hat{\boldsymbol{j}}$

9 The velocity of a particle is given by the vector $\dot{\boldsymbol{r}}(t) = t\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}}$. At time $t = 0$, the position of the particle is given by $\boldsymbol{r}(0) = 3\hat{\boldsymbol{i}}$. The position of the particle at time t is given by

A $\boldsymbol{r}(t) = \frac{1}{2}t^2\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}}$
B $\boldsymbol{r}(t) = \frac{1}{2}(t^2 + 3)\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}}$
C $\boldsymbol{r}(t) = (\frac{1}{2}t^2 + 3)\hat{\boldsymbol{i}} + (e^t - 1)\hat{\boldsymbol{j}}$
D $\boldsymbol{r}(t) = (\frac{1}{2}t^2 + 3)\hat{\boldsymbol{i}} + e^t\hat{\boldsymbol{j}}$

10 A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos(\pi t)\hat{\boldsymbol{i}} + 3\sin(\pi t)\hat{\boldsymbol{j}}$. With respect to a set of Cartesian axes, the gradient of the curve at the point $(\sqrt{3}, 1.5)$ is

A $-(\pi\hat{\boldsymbol{i}} + 3\sqrt{3}\pi\hat{\boldsymbol{j}})$
B $\pi\hat{\boldsymbol{i}} + 3\sqrt{3}\pi\hat{\boldsymbol{j}}$
C $-\dfrac{3\sqrt{3}}{2}\pi$
D $-\dfrac{3\sqrt{3}}{2}$

11 A particle is moving at a constant speed of 6 m/s around a circle centred at the origin. The acceleration is given by $\ddot{\boldsymbol{r}} = -9\boldsymbol{r}$, where $\boldsymbol{r}$ is the position of the particle at time t seconds. The period of the motion, T s, and radius of the circle, r m, are given by

A $T = 3, \quad r = 6$
B $T = \dfrac{2\pi}{3}, \quad r = 4$
C $T = 6\pi, \quad r = 9$
D $T = \dfrac{2\pi}{3}, \quad r = 2$

12 A ball is thrown with speed 40 m/s at an angle of $\alpha°$ to the horizontal, where $\tan(\alpha°) = \dfrac{4}{3}$. The magnitude of the horizontal component of its velocity is

A 8 m/s
B 64 m/s
C 0 m/s
D 24 m/s

13 A projectile is fired from the origin O with an initial velocity of $\boldsymbol{u} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, where $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors in the horizontal (x) and vertical (y) directions respectively. The Cartesian equation of the projectile's path is

A $y = 2x - \dfrac{gx^2}{2}$
B $4y = 2x - \dfrac{gx^2}{2}$
C $4y = 2x + \dfrac{gx^2}{2}$
D $y = 6x - gx^2$

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Review

Technology-active multiple-choice questions

14 An object is projected horizontally from the top of an 80 m high cliff, and strikes the ground 1330 m from the base of the cliff. The object's initial speed is closest to

A 330 m/s **B** 9.8 m/s **C** 82 m/s **D** 170 m/s

15 The position vector of a projectile at time t seconds relative to a point O on the ground is $\boldsymbol{r} = 10t\hat{\boldsymbol{i}} + (19.6t - 4.9t^2)\hat{\boldsymbol{j}}$ where $\hat{\boldsymbol{i}}$ is horizontal and $\hat{\boldsymbol{j}}$ is vertically upwards. The maximum height reached by the projectile in metres is

A 4.9 **B** 10 **C** 19.6 **D** 20

16 The position vector of a particle moving in a circle is given by $\boldsymbol{r} = \cos(3t)\hat{\boldsymbol{i}} + \sin(3t)\hat{\boldsymbol{j}}$. The acceleration of the particle is

A $-\boldsymbol{r}$ **B** 3 **C** $-9\boldsymbol{r}$ **D** $-3\boldsymbol{r}$

17 An object is projected horizontally from a building 123 m high with a speed of 12 m/s. The horizontal distance, correct to the nearest metre, travelled when it hits the ground is

A 5 m **B** 55 m **C** 60 m **D** 70 m

18 The position vector of a particle that is moving along a curve at time t is given by $\boldsymbol{r}(t) = 5\cos(t)\hat{\boldsymbol{i}} + 12\sin(t)\hat{\boldsymbol{j}}$ $t \geq 0$. When the speed of the particle first reaches 10 m/s, the value of t correct to two decimal places is

A 2.22 seconds **B** 1.23 seconds **C** 0.65 seconds **D** 0.91 seconds

19 The position vector $\boldsymbol{r}(t)$ m of a particle moving along a curve at time t seconds is given by $\boldsymbol{r}(t) = 6\cos(2t)\hat{\boldsymbol{i}} + 2\sin(2t)\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$, $t \geq 0$. The minimum speed of this particle is

A 4 ms^{-1} **B** $6\sqrt{2}\text{ ms}^{-1}$ **C** $4\sqrt{6}\text{ ms}^{-1}$ **D** $2\sqrt{5}\text{ ms}^{-1}$

20 A particle is projected from a horizontal plane at an angle of elevation of 30° with a speed of 98 ms^{-1}. The maximum height of the particle is ($g = 9.8\text{ ms}^{-2}$)

A 95.5 m **B** 122.5 m **C** 132.5 m **D** 128.5 m

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8

Matrix algebra and systems of equations

Chapter contents

We start this chapter by revising matrix algebra from Specialist Mathematics Units 1 & 2. From the previous chapter, we know that:

- a line in two-dimensional space has a Cartesian equation of the form $ax + by = c$
- a plane in three-dimensional space has a Cartesian equation of the form $ax + by + cz = d$.

These are both called linear equations, since the power of each variable is 1.

In this chapter, we consider systems of simultaneous linear equations such as

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \qquad\qquad \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

The system on the left represents two lines, and the system on the right represents three planes. We will use elimination methods to solve such systems of equations, and give geometric interpretations of the solutions.

Chapters 8 and 9 cover Unit 3 Topic 5: Further matrices.

8A Revision of matrix algebra

Learning intentions

- To revise matrix arithmetic.

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

The size of a matrix

Matrices vary in size. The **size** of the matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix.

The sizes of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

An $m \times n$ matrix has m rows and n columns.

Storing information in matrices

The use of matrices to store information is demonstrated by the following example.

Four exporters A, B, C and D sell refrigerators (r), dishwashers (d), microwave ovens (m) and televisions (t). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

$$\begin{array}{cc} & \begin{array}{cccc} r & d & m & t \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 120 & 95 & 370 & 250 \\ 430 & 380 & 950 & 900 \\ 60 & 50 & 150 & 100 \\ 200 & 100 & 470 & 50 \end{bmatrix} \begin{array}{l} \text{row 1} \\ \text{row 2} \\ \text{row 3} \\ \text{row 4} \end{array} \\ & \begin{array}{cccc} \text{column 1} & \text{column 2} & \text{column 3} & \text{column 4} \end{array} \end{array}$$

From this matrix it can be seen that:

- Exporter A sold 120 refrigerators, 95 dishwashers, 370 microwave ovens, 250 televisions.
- Exporter B sold 430 refrigerators, 380 dishwashers, 950 microwave ovens, 900 televisions.

The entries for the sales of refrigerators are in column 1.

The entries for the sales of exporter A are in row 1.

Entries and equality

We will use uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ to denote matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 3×4 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal**, and we can write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a 'super number'.

Addition, subtraction and multiplication by a real number

Addition of matrices

If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same size, then the sum $\mathbf{A} + \mathbf{B}$ is the matrix obtained by adding together the corresponding entries of the two matrices.

For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

Multiplication of a matrix by a real number

If $\mathbf{A}$ is any matrix and k is a real number, then the product $k\mathbf{A}$ is the matrix obtained by multiplying each entry of $\mathbf{A}$ by k.

For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Note: If a matrix is added to itself, then the result is twice the matrix, i.e. $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. Similarly, for any natural number n, the sum of n matrices each equal to $\mathbf{A}$ is $n\mathbf{A}$.

If $\mathbf{B}$ is any matrix, then $-\mathbf{B}$ denotes the product $(-1)\mathbf{B}$.

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Subtraction of matrices

If **A** and **B** are matrices of the same size, then **A** − **B** is defined to be the sum

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} + (-1)\mathbf{B}$$

For two matrices **A** and **B** of the same size, the difference **A** − **B** can be found by subtracting corresponding entries.

Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**, and will be denoted by **O**.

For any $m \times n$ matrix **A** and the $m \times n$ zero matrix **O**, we have

$$\mathbf{A} + \mathbf{O} = \mathbf{A} \qquad \text{and} \qquad \mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

TI-84

Using the TI-Nspire CX non-CAS

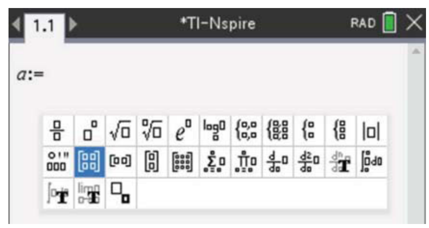
The matrix template

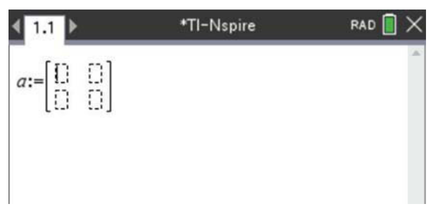
Matrices can be assigned (or stored) as variables for further computations.

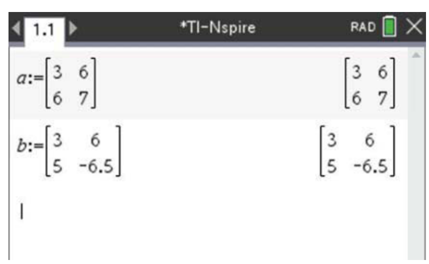
Assign matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as follows:

- In a **Calculator** page, type $a :=$ and then enter the matrix.
- The simplest way to enter a 2×2 matrix is using the 2×2 matrix template as shown. (Access the templates using either or ctrl menu > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.
- Use the touchpad arrows (or tab) to move between the entries of the matrix.

Assign the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ similarly.

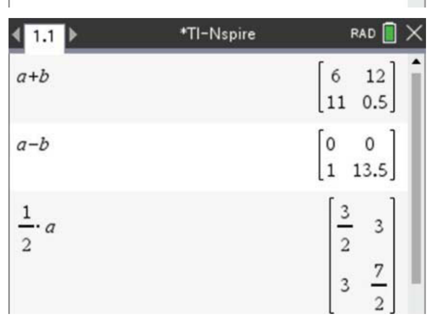






Operations on matrices

Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** − **B** and k**A** can easily be determined.



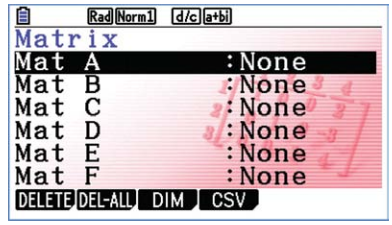
Using the Casio

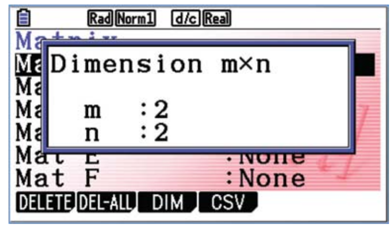
Storing matrices

To store matrices $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$:

- In **Run-Matrix** mode, go to the Matrix Editor screen by selecting ▸ **Mat** F3.
- Press EXE to select Matrix A.
- Specify the size of Matrix A:

  2 EXE 2 EXE EXE

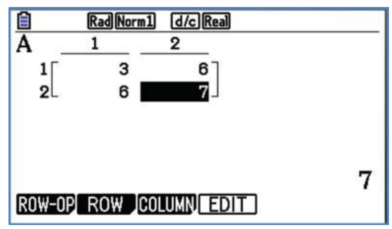




- Input the entries of Matrix A:

  3 EXE 6 EXE 6 EXE 7 EXE

- Press EXIT to return to the Matrix Editor screen.
- Use the cursor key ▼ to move down to Matrix B, and then store the matrix **B** similarly.
- Press EXIT.

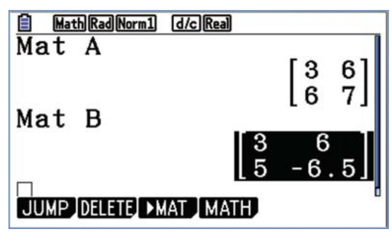


Accessing stored matrices

To access a stored matrix, use **Mat** (SHIFT 2) followed by the matrix name.

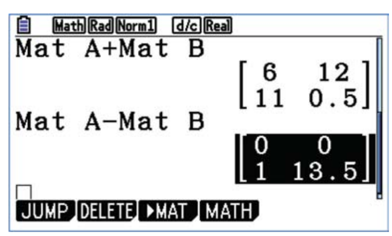
- For example, to view Matrix A, press:

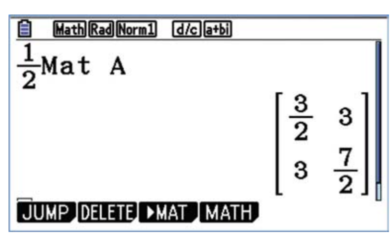
  SHIFT 2 ALPHA X,θ,T EXE



Operations on stored matrices

The matrices $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$ and $\frac{1}{2}\mathbf{A}$ can now be found as shown.





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Multiplication of matrices

We have seen the definition for multiplication of a matrix by a real number. The definition for multiplication of matrices is less straightforward. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

$$\begin{aligned} \text{Then} \quad \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

> To determine the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add the resulting products.

Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Example 1

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, determine $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×2 matrix and $\mathbf{B}$ is a 2×1 matrix. Therefore the product $\mathbf{AB}$ is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

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Example 2

Matrix **X** shows the number of cars of models a and b bought by four dealers A, B, C, D. Matrix **Y** shows the cost in dollars of cars a and b. Determine **XY** and explain what it represents.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

Solution

X is a 4×2 matrix and **Y** is a 2×1 matrix. Therefore **XY** is a 4×1 matrix.

$$\mathbf{XY} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array}\begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

$$= \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

The matrix **XY** shows that dealer A spent \$110 000, dealer B spent \$116 000, dealer C spent \$154 000 and dealer D spent \$58 000.

Skill-sheet

Exercise 8A

SF

1 At a certain school there are 220 girls and 150 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 125 in Year 8, 135 and 102 in Year 9, 112 and 91 in Year 10, 86 and 83 in Year 11, and 48 and 53 in Year 12. Summarise this information in a matrix.

2 The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5
Player B points 8, rebounds 2, assists 3
Player C points 4, rebounds 1, assists 1
Player D points 14, rebounds 8, assists 60
Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

SF

3 Let $\mathbf{A} = \begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$.

Write down the values of x and y if the matrices $\mathbf{A}$ and $\mathbf{B}$ are equal.

4 Let $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$.

Determine $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

5 If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, determine matrices $\mathbf{X}$ and $\mathbf{Y}$ such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

6 Matrices $\mathbf{X}$ and $\mathbf{Y}$ show the production of four models of cars a, b, c, d at two factories P, Q in successive weeks. Determine $\mathbf{X} + \mathbf{Y}$ and describe what this sum represents.

Week 1: $\mathbf{X} = \begin{array}{c} \\ P \\ Q \end{array} \!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 150 & 90 & 100 & 50 \\ 100 & 0 & 75 & 0 \end{bmatrix} \end{array}$ Week 2: $\mathbf{Y} = \begin{array}{c} \\ P \\ Q \end{array} \!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 160 & 90 & 120 & 40 \\ 100 & 0 & 50 & 0 \end{bmatrix} \end{array}$

Example 1 **7** Let $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Determine the products $\mathbf{AX}$, $\mathbf{BX}$, $\mathbf{AY}$, $\mathbf{IX}$, $\mathbf{AC}$, $\mathbf{CA}$, $(\mathbf{AC})\mathbf{X}$, $\mathbf{C}(\mathbf{BX})$, $\mathbf{AI}$, $\mathbf{IB}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{AA} = \mathbf{A}^2$, $\mathbf{BB} = \mathbf{B}^2$, $\mathbf{A}(\mathbf{CA})$ and $\mathbf{A}^2\mathbf{C}$.

CF

8 Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a Show that $ad - bc = 1$.

b What is the product matrix if the order of multiplication on the left-hand side is reversed?

SF

9 Choose any three 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Determine $\mathbf{A}(\mathbf{B} + \mathbf{C})$, $\mathbf{AB} + \mathbf{AC}$ and $(\mathbf{B} + \mathbf{C})\mathbf{A}$.

10 Determine 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Example 2 **11** It takes John 5 minutes to drink a milkshake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.

a Determine the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in fast-food economics.

b Two friends join John. Determine $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

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8B Inverses and determinants for 2×2 matrices

Learning intentions

- To be able to determine the inverse and determinant of a 2×2 matrix.

Identities

A matrix with the same number of rows and columns is called a **square matrix**. For square matrices of a given size (e.g. 2×2), a multiplicative identity $\mathbf{I}$ exists.

For 2×2 matrices, the **identity matrix** is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$. This result holds for any 2×2 matrix $\mathbf{A}$.

Inverses

Given a 2×2 matrix $\mathbf{A}$, is there a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$?

For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2x + 3u & 2y + 3v \\ x + 4u & y + 4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad 2x + 3u = 1 \quad \text{and} \quad 2y + 3v = 0$

$\quad\; x + 4u = 0 \qquad\qquad y + 4v = 1$

These simultaneous equations can be solved to determine x, y, u, v and hence $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

In general:

If $\mathbf{A}$ is a square matrix and if a matrix $\mathbf{B}$ can be found such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$

then $\mathbf{A}$ is said to be **invertible** and $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$.

We leave it as an exercise to show that the inverse of an invertible matrix is unique.

We will denote the inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$.

For an invertible matrix $\mathbf{A}$, we have

$$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

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The inverse of a general 2×2 matrix

Now consider $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $\quad ax + bu = 1$ and $ay + bv = 0$

$\quad cx + du = 0 \qquad cy + dv = 1$

These form two pairs of simultaneous equations, the first for x, u and the second for y, v.

The first pair of equations gives

$(ad - bc)x = d \qquad$ (eliminating u)

$(bc - ad)u = c \qquad$ (eliminating x)

These two equations can be solved for x and u provided $ad - bc \neq 0$:

$$x = \frac{d}{ad - bc} \qquad \text{and} \qquad u = \frac{c}{bc - ad} = \frac{-c}{ad - bc}$$

In a similar way, we obtain

$$y = \frac{-b}{ad - bc} \qquad \text{and} \qquad v = \frac{-a}{bc - ad} = \frac{a}{ad - bc}$$

We have established the following result.

Inverse of a 2×2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{(provided } ad - bc \neq 0\text{)}$$

The determinant

The quantity $ad - bc$ that appears in the formula for $\mathbf{A}^{-1}$ has a name: the **determinant** of $\mathbf{A}$. This is denoted $\det(\mathbf{A})$.

Determinant of a 2×2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

A 2×2 matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

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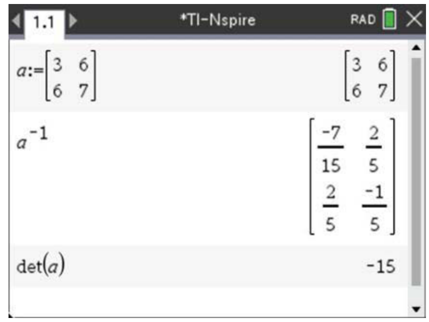
TI-84

Using the TI-Nspire CX non-CAS

- The inverse of a matrix is obtained by raising the matrix to the power of -1.
- The determinant command (menu > **Matrix & Vector** > **Determinant**) is used as shown.

Hint: You can also type in det(a).



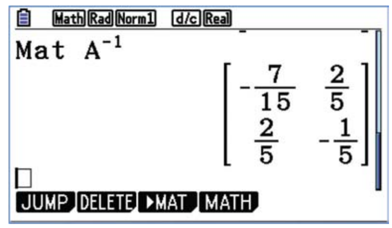
Using the Casio

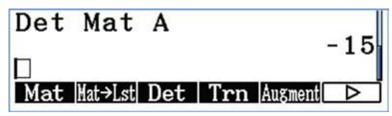
- To determine the inverse of a stored matrix, raise it to the power of -1. For example, to determine $\mathbf{A}^{-1}$:

  SHIFT 2 ALPHA X,θ,T SHIFT) EXE

- To determine the determinant of a stored matrix, use the **Matrix operations** menu OPTN F2. Select **Det** F3; select **Mat** F1; then enter the matrix name.

(Here $\mathbf{A} = \begin{bmatrix}3 & 6\\6 & 7\end{bmatrix}$ has been stored as Matrix A.)





Example 3

For the matrices $\mathbf{A} = \begin{bmatrix}5 & 2\\3 & 1\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}\frac{1}{2} & 0\\0 & 1\end{bmatrix}$, determine:

a det($\mathbf{A}$)

b $\mathbf{A}^{-1}$

c det($\mathbf{B}$)

d $\mathbf{B}^{-1}$

Solution

a $$\det(\mathbf{A}) = 5 \times 1 - 2 \times 3 = -1$$

b $$\mathbf{A}^{-1} = \frac{1}{-1}\begin{bmatrix}1 & -2\\-3 & 5\end{bmatrix} = \begin{bmatrix}-1 & 2\\3 & -5\end{bmatrix}$$

c $$\det(\mathbf{B}) = \tfrac{1}{2} \times 1 - 0 \times 0 = \tfrac{1}{2}$$

d $$\mathbf{B}^{-1} = 2\begin{bmatrix}1 & 0\\0 & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$$

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Example 4

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, determine:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

c $\mathbf{X}$, if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

d $\mathbf{Y}$, if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

b $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

c $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the left) by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

d $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the right) by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$

Exercise 8B

Skill-sheet

Example 3

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, determine:

a $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

SF

2 Determine the inverse of each of the following invertible matrices (where $k \neq 0$):

a $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

CF

3 If the matrix $\mathbf{A}$ is invertible, show that the inverse is unique.

4 Let $\mathbf{A}$ and $\mathbf{B}$ be the invertible matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

a Determine $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

b Determine $\mathbf{AB}$ and hence determine, if possible, $(\mathbf{AB})^{-1}$.

c From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, determine the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

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Example 4

5 Let $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

a Determine $\mathbf{A}^{-1}$.

b If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, determine $\mathbf{X}$.

c If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, determine $\mathbf{Y}$.

SF

6 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

a Determine $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$.

b Determine $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

7 For what values of m does the matrix $\mathbf{B} = \begin{bmatrix} m & 4 \\ 2 & m+2 \end{bmatrix}$ have an inverse?

8 Let $\mathbf{B} = \begin{bmatrix} m & m \\ 4 & 2m \end{bmatrix}\begin{bmatrix} 2 & m \\ 4 & 3 \end{bmatrix}$. Determine the values of m for which $\mathbf{B}^{-1}$ does not exist.

9 Assume that $\mathbf{A}$ is a 2×2 matrix such that $a_{12} = a_{21} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Show that $\mathbf{A}$ is invertible and determine $\mathbf{A}^{-1}$. CF

10 Let $\mathbf{A}$ be an invertible 2×2 matrix, let $\mathbf{B}$ be a 2×2 matrix and assume that $\mathbf{AB} = \mathbf{O}$. Show that $\mathbf{B} = \mathbf{O}$. SF

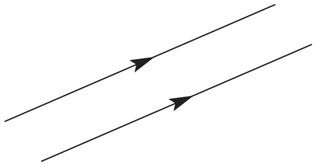
11 Determine all 2×2 matrices such that $\mathbf{A}^{-1} = \mathbf{A}$. CF

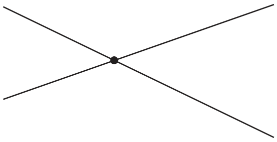
8C Simultaneous linear equations with two variables

Learning intentions

- To consider solutions for simultaneous linear equations with two variables.

In the plane, two distinct straight lines are either parallel or meet at a point.





There are three cases for a system of two linear equations in two variables.

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	Example	Solutions	Geometry
Case 1	$2x + y = 5$ $x - y = 4$	Unique solution: $x = 3,\ y = -1$	Two lines meeting at a point
Case 2	$2x + y = 5$ $2x + y = 7$	No solutions	Distinct parallel lines
Case 3	$2x + y = 5$ $4x + 2y = 10$	Infinitely many solutions	Two copies of the same line

Example 5

Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

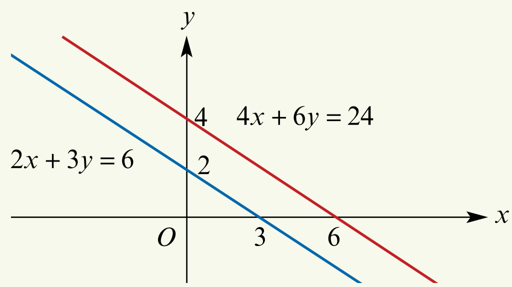
Solution

First write the two equations in the form $y = mx + c$. They become

$y = -\frac{2}{3}x + 2$ and $y = -\frac{2}{3}x + 4$

Both lines have gradient $-\frac{2}{3}$. The y-axis intercepts are 2 and 4 respectively.

The equations have no solution as they correspond to distinct parallel lines.



Example 6

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

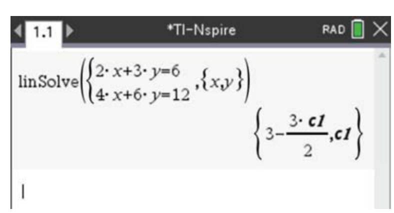
Solution

As the two lines coincide, each of the infinitely many points on the line is a solution to this system of equations. As the two lines coincide, each of the infinitely many points on the line is a solution to this system of equations. To describe these solutions, we can let $y = \lambda$ where λ is *any* real number. Then $x = \dfrac{6 - 3\lambda}{2}$. We can write the solutions as $x = \dfrac{6 - 3\lambda}{2}$ and $y = \lambda$, for $\lambda \in \mathbb{R}$. (This gives a parametric description of the line.)

Using the TI-Nspire CX non-CAS

Simultaneous linear equations can be solved in a **Calculator** application.

- Use menu > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen.



The solution to this system of equations is given by the calculator as shown. The variable $c1$ takes the place of λ.

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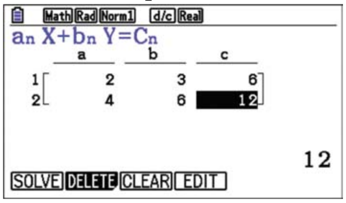
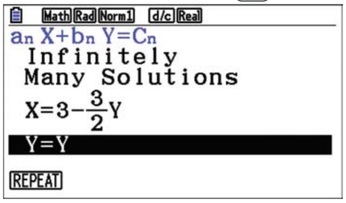
Using the Casio

- Select **Equation** mode (MENU) (ALPHA) (X,θ,T).
- Select **Simultaneous** (F1), then select two unknowns (F1).
- Enter the coefficients of the two equations in the table as shown; select **Solve** (F1).





Example 7

Consider the simultaneous linear equations $(m-2)x + y = 2$ and $mx + 2y = k$.

Determine the values of m and k such that the system of equations has:

a a unique solution **b** no solution **c** infinitely many solutions.

Solution

$$(m-2)x + y = 2 \qquad (1)$$

$$mx + 2y = k \qquad (2)$$

Multiply equation (1) by 2 and subtract from equation (2):

$$(m - 2(m-2))x = k - 4$$

$$\therefore \qquad x = \frac{4-k}{m-4} \qquad (\text{for } m \neq 4)$$

Substitute in (1):

$$\begin{aligned} y &= 2 - (m-2)x \\ &= 2 - (m-2)\frac{4-k}{m-4} \\ &= \frac{k(m-2) - 2m}{m-4} \end{aligned}$$

For $m \neq 4$, we obtain the solution

$$x = \frac{4-k}{m-4} \quad \text{and} \quad y = \frac{k(m-2)-2m}{m-4}$$

a There is a unique solution if $m \neq 4$ and k is any real number.

b If $m = 4$, the equations become

$$2x + y = 2 \quad \text{and} \quad 4x + 2y = k$$

There is no solution if $m = 4$ and $k \neq 4$.

c If $m = 4$ and $k = 4$, there are infinitely many solutions as the equations are the same.

Note: Alternatively, this example can be solved by using the determinant of a 2×2 matrix. This approach will be demonstrated in Section 8D

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Exercise 8C

SF

1 Solve each of the following pairs of simultaneous linear equations:

a $3x + 2y = 6$
$x - y = 7$

b $2x + 6y = 0$
$y - x = 2$

c $4x - 2y = 7$
$5x + 7y = 1$

d $2x - y = 6$
$4x - 7y = 5$

2 For each of the following, state whether the simultaneous equations have no solution, one solution or infinitely many solutions:

a $3x + 2y = 6$
$3x - 2y = 12$

b $x + 2y = 6$
$2x + 4y = 12$

c $x - 2y = 3$
$2x - 4y = 12$

Example 5

SF

3 Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 10$ have no solution.

Example 6

4 The simultaneous equations $x - y = 6$ and $2x - 2y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Example 7

CF

5 Consider the simultaneous equations $3x + my = 5$ and $(m + 2)x + 5y = m$. Determine the value of m for which these equations have:

a infinitely many solutions

b no solutions.

6 Determine the value of m for which the following simultaneous equations have no solution:

$$(m + 3)x + my = 12$$
$$(m - 1)x + (m - 3)y = 7$$

7 Consider the simultaneous equations $mx + 2y = 8$ and $4x - (2 - m)y = 2m$.

a Determine the values of m for which there are:

i no solutions

ii infinitely many solutions.

b Solve the equations in terms of m, for suitable values of m.

8 **a** Solve the simultaneous equations $2x - 3y = 4$ and $x + ky = 2$, where k is a constant.

b Determine the value of k for which there is not a unique solution.

CF

9 Determine the values of b and c for which the equations $x + 5y = 4$ and $2x + by = c$ have:

a a unique solution

b an infinite set of solutions

c no solution.

10 For each of the following systems of equations:

i Determine the values of $b \in \mathbb{R}$ for which there is a unique solution.

ii Determine the values of $b \in \mathbb{R}$ for which there are infinitely many solutions.

iii Determine the values of $b \in \mathbb{R}$ for which there are no solutions.

iv In the cases where solutions exist, express the solutions in terms of b.

a $x + y = 4$
$2x + 2y = b$

b $x + y = 4$
$2x + y = b$

c $x + y = 4$
$bx + y = 8$

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8D Using matrix algebra for systems of linear equations in two variables

Learning intentions

- ▶ To find solutions for simultaneous linear equations with two variables using matrices.

In Section 8A, we looked at methods for solving a system of two linear equations in two variables. In some cases, we can use an inverse matrix to determine the solution. In all cases, we can use a determinant to reveal whether or not there is a unique solution.

Simultaneous equations with a unique solution

For example, consider the pair of simultaneous equations

$$3x - 2y = 5$$
$$5x - 3y = 9$$

This can be written as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$. The determinant of $\mathbf{A}$ is $3(-3) - (-2)5 = 1$.

Since the determinant is non-zero, the inverse matrix exists:

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Now multiply both sides of the original matrix equation on the left by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{I}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix} \qquad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations. We can check this by substituting $x = 3$ and $y = 2$ into the two equations.

Example 8

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

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Solution

If $\mathbf{AX} = \mathbf{K}$, then

$$\begin{aligned}\mathbf{X} &= \mathbf{A}^{-1}\mathbf{K}\\ &= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\end{aligned}$$

Example 9

Solve the following simultaneous equations:

$$\begin{aligned}3x - 2y &= 6\\ 7x + 4y &= 7\end{aligned}$$

Solution

The matrix equation is

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$. Then $\mathbf{A}^{-1} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$.

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 38 \\ -21 \end{bmatrix}$$

Simultaneous equations without a unique solution

If a pair of simultaneous linear equations in two variables corresponds to two parallel lines, then the corresponding matrix is singular or non-invertible.

For example, the following pair of simultaneous equations has no solution:

$$\begin{aligned}x + 2y &= 3\\ -2x - 4y &= 6\end{aligned}$$

The associated matrix equation is

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The determinant of the matrix $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is $1(-4) - 2(-2) = 0$, so the matrix has no inverse.

Example 10

Determine the values of m such that the simultaneous equations

$$\begin{aligned}mx + 2y &= 4\\ x + (m-1)y &= 2\end{aligned}$$

do not have a unique solution.

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Solution

The simultaneous equations can be written as the matrix equation

$$\begin{bmatrix} m & 2 \\ 1 & m-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} m & 2 \\ 1 & m-1 \end{bmatrix}$. Then

$$\begin{aligned} \det(\mathbf{A}) &= m(m-1) - 2 \\ &= m^2 - m - 2 \\ &= (m+1)(m-2) \end{aligned}$$

The equations do not have a unique solution if $\det(\mathbf{A}) = 0$, i.e. if $m = -1$ or $m = 2$.

Notes:

- If $m = -1$, then the equations are $-x + 2y = 4$ and $x - 2y = 2$. These equations represent distinct parallel lines; there are no solutions.
- If $m = 2$, then the equations are $2x + 2y = 4$ and $x + y = 2$. These equations represent the same line; there are infinitely many solutions.

Skill-sheet

Exercise 8D

Example 8

1 Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solve the system $\mathbf{AX} = \mathbf{K}$, where:

a $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **b** $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

SF

Example 9

2 Use matrices to solve each of the following pairs of simultaneous equations:

a $-2x + 4y = 6$
$3x + y = 1$

b $-x + 2y = -1$
$-x + 4y = 2$

c $3x + 2y = 17$
$4x + 5y = 32$

d $2x + 3y = 17$
$4x + 5y = 32$

3 Use matrices to determine the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

CF

4 Consider the following system of linear equations:

$$\sqrt{3}a - b = 8$$
$$a + \sqrt{3}b = 12$$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

b Determine $\mathbf{A}^{-1}$.

c Hence solve the system of equations.

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CF

5 Two children spend their pocket money buying some books and some LED photo clip lights. One child spends \$120 and buys four books and four LED photo clip lights. The other child spends \$114 and buys three LED photo clip lights and five books. Set up a system of simultaneous equations and use matrices to determine the cost of a single book and a single LED photo clip light.

Example 10

6 Determine the values of m such that the simultaneous equations $mx + 4y = 10$ and $2x + (m - 2)y = 4$ do not have a unique solution.

7 Consider the system

$2x - 3y = 3 \qquad 4x - 6y = 6$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

b Is $\mathbf{A}$ an invertible matrix?

c Can any solutions be found for this system of equations?

d How many pairs does the solution set contain?

8E Inverses and determinants for $n \times n$ matrices

Learning intentions

- ▶ To use a calculator to find the determinant and inverse of $n \times n$ matrices.

In the remaining sections of this chapter our attention turns to $n \times n$ matrices and systems of linear equations in n unknowns. Much of the work in these two sections will be completed with the use of technology.

An $n \times n$ matrix $\mathbf{A}$ can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Here a_{ij} is the entry in row i and column j of $\mathbf{A}$.

We will concentrate on 3×3 matrices, but the techniques used for larger square matrices are similar.

Identities

For 3×3 matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For each 3×3 matrix $\mathbf{A}$, we have $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$.

Similarly, for 4×4 matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In general, the $n \times n$ identity matrix has 1s along the main diagonal (top-left to bottom-right) and 0s everywhere else.

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Inverses

Recall that, if **A** is a square matrix and there exists a matrix **B** such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$

then **B** is called the inverse of **A**. When it exists, the inverse of a square matrix **A** is unique and is denoted by $\mathbf{A}^{-1}$.

The following fact is helpful when checking whether two given matrices are inverses of each other.

Let **A** and **B** be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

Example 11

Let $\mathbf{A} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix}$.

Determine the product **AB**, and hence determine $\mathbf{A}^{-1}$.

Solution

$$\mathbf{AB} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix} \begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Hence $\mathbf{A}^{-1} = \mathbf{B}$.

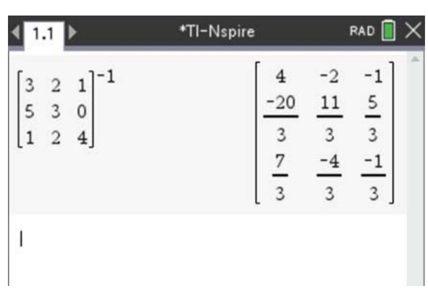
In this course, you are expected to use technology to determine the inverse of a 3×3 matrix.

Example 12

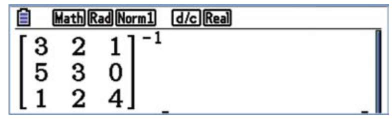
Using your calculator, determine the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$.

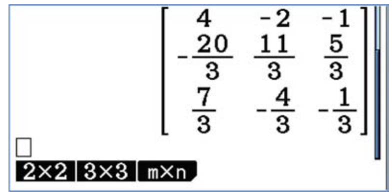
Using the TI-Nspire CX non-CAS

The inverse of a matrix is obtained by raising the matrix to the power of -1.



Using the Casio





- In **Run-Matrix** mode, you can access a template for entering a 3×3 matrix by selecting **Math** (F4), **Matrix** (F1), then **3×3** (F2).
- Input the entries of the matrix:

  (3) ► (2) ► (1) ►
  (5) ► (3) ► (0) ►
  (1) ► (2) ► (4) ►

- Raise the matrix to the power of -1:

  (SHIFT) ( )) (EXE)

Note: Matrices entered in this way are not stored for future calculations.

The determinant

In Section 8B, we defined the determinant of a 2×2 matrix. The definition was motivated by the formula for the inverse of a 2×2 matrix. We saw that a 2×2 matrix has an inverse if and only if its determinant is non-zero.

In fact, the determinant is defined for all square matrices. In this course, you are expected to use technology to determine the determinant of an $n \times n$ matrix when $n \geq 3$.

The determinant has the following important property.

Determinant of an $n \times n$ matrix

An $n \times n$ matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

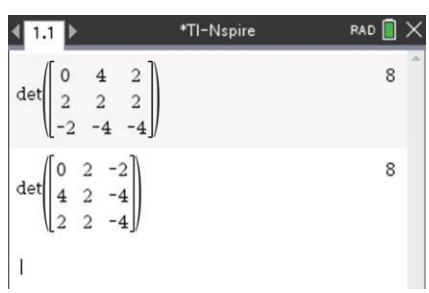
Example 13

Using a calculator, determine the determinant of:

a $\begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix}$

Using the TI-Nspire CX non-CAS

The determinant command ((menu) > **Matrix & Vector** > **Determinant**) is used as shown. Alternatively, type det(.



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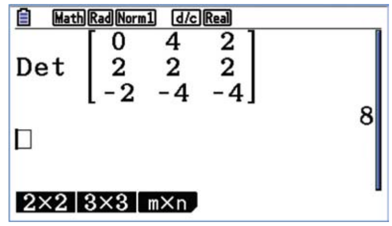
Using the Casio

a
- In **Run-Matrix** mode, go to the **Matrix operations** menu OPTN F2.
- Select **Det** F3. Press EXIT twice.
- To access the 3×3 matrix template, select **Math** F4, **Matrix** F1, then **3×3** F2.
- Input the entries of the matrix as shown.



b Similarly, we can determine $\det \begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix} = 8$.

Exercise 8E

SF

1 Show that the inverse of $\mathbf{B} = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 2 & -1 & 1 \\ -3 & 1 & \frac{1}{2} \end{bmatrix}$ is the matrix $\mathbf{A} = \begin{bmatrix} 3 & -1 & -1 \\ 8 & -3 & -2 \\ 2 & 0 & 0 \end{bmatrix}$.

Example 11

2 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -4 \\ -2 & 5 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 19 & -17 & -11 \\ 6 & -5 & -2 \\ 8 & -9 & -5 \end{bmatrix}$.

Determine the product $\mathbf{AB}$, and hence determine $\mathbf{A}^{-1}$.

3 Let $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$. Determine $\mathbf{A}^2$, and hence determine $\mathbf{A}^{-1}$.

4 Let $\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$. Determine $\mathbf{A}^2$, and hence determine $\mathbf{A}^{-1}$.

Example 12

5 Use your calculator to determine the inverse of each of the following matrices:

a $\begin{bmatrix} 5 & -2 & 3 \\ 1 & 6 & -4 \\ 2 & 2 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} -5 & 8 & -3 \\ 3 & -6 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 12 & 1 & 2 & 6 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 9 & 1 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ 5 & 1 & 0 & 1 \end{bmatrix}$

Example 13

6 Using a calculator, determine the determinant of:

a **i** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

b **i** $\begin{bmatrix} 2 & 4 & 8 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 4 & 4 \\ 6 & 4 & 2 \end{bmatrix}$

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8F Systems of linear equations with more than two variables

Learning intentions

- To use a calculator to determine the solutions for simultaneous linear equations with more than two variables.

Linear equations in three variables

Consider the general system of three linear equations in three variables:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

In this section and the next, we look at how to solve such systems of simultaneous equations. In some cases, this can be done easily by elimination, as shown in Examples 14 and 15. Other cases require a more systematic method, which is introduced in the final section of this chapter.

Example 14

Solve the following system of three equations in three variables:

$$2x + y + z = -1 \quad (1)$$
$$3y + 4z = -7 \quad (2)$$
$$6x + z = 8 \quad (3)$$

Solution

Subtract (1) from (3):

$$4x - y = 9 \quad (4)$$

Subtract (2) from $4 \times (3)$:

$$24x - 3y = 39$$
$$8x - y = 13 \quad (5)$$

Subtract (4) from (5) to obtain $4x = 4$. Hence $x = 1$.

Substitute in (4) to determine $y = -5$, and substitute in (3) to determine $z = 2$.

Explanation

The aim is first to eliminate z and obtain two simultaneous equations in x and y only.

Having obtained equations (4) and (5), we solve for x and y. Then substitute to determine z.

It should be noted that, just as for two linear equations in two variables, there is a geometric interpretation for three linear equations in three variables. There is only a unique solution if the three equations represent three planes intersecting at a point.

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Example 15

Solve the following simultaneous linear equations for x, y and z:

$$x - y + z = 6, \qquad 2x + z = 4, \qquad 3x + 2y - z = 6$$

Solution

$$\begin{aligned} x - y + z &= 6 \qquad (1) \\ 2x + z &= 4 \qquad (2) \\ 3x + 2y - z &= 6 \qquad (3) \end{aligned}$$

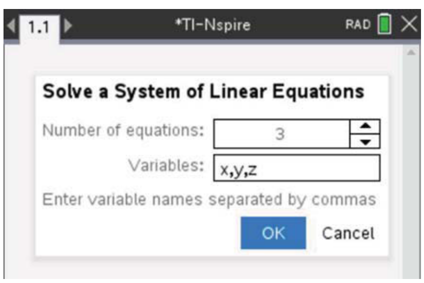
Eliminate z to determine two simultaneous equations in x and y:

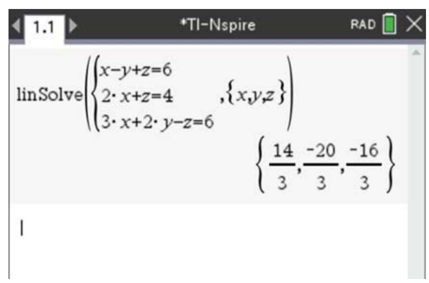
$$\begin{aligned} x + y &= -2 \qquad (4) \qquad \text{subtracted (1) from (2)} \\ 5x + 2y &= 10 \qquad (5) \qquad \text{added (2) to (3)} \end{aligned}$$

Solve to determine $x = \dfrac{14}{3}$, $y = -\dfrac{20}{3}$, $z = -\dfrac{16}{3}$.

Using the TI-Nspire CX non-CAS

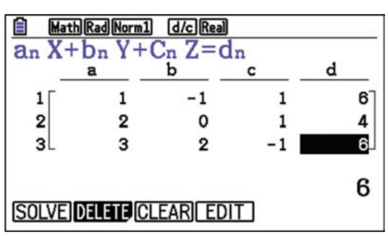
Use the simultaneous equations template (menu > **Algebra** > **Solve System of Linear Equations**) as shown.

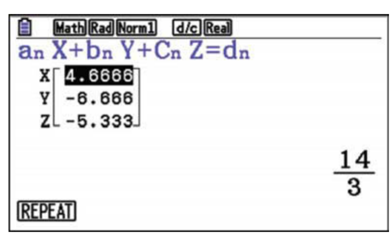




Using the Casio

- Select **Equation** mode MENU ALPHA X,θ,T.
- Select **Simultaneous** F1, then select three unknowns F2.
- Enter the coefficients of the three equations in the table as shown; select **Solve** F1.





Geometric interpretation of linear equations in three variables

We have seen in Chapter 6 that an equation of the form

$$ax + by + cz = d$$

defines a plane in three-dimensional space (provided a, b and c are not all zero).

The solution of a system of three linear equations in three variables can correspond to:

- a point
- a line
- a plane.

There also may be no solution. The situations are as shown in the following diagrams. Examples 14 and 15 provide examples of three planes intersecting at a point (Diagram 1).

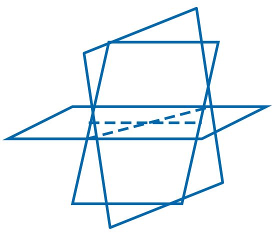


Diagram 1:
Intersection at a point

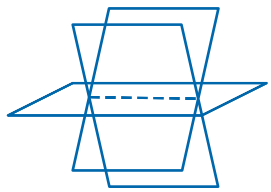


Diagram 2:
Intersection in a line

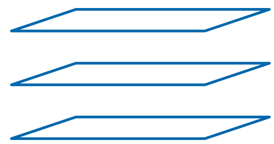


Diagram 3:
No intersection

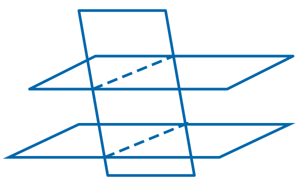


Diagram 4:
No common intersection

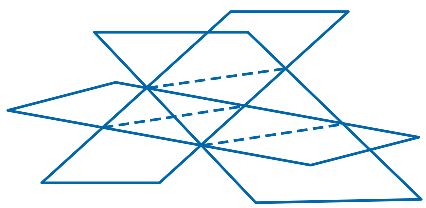


Diagram 5:
No common intersection

Example 16

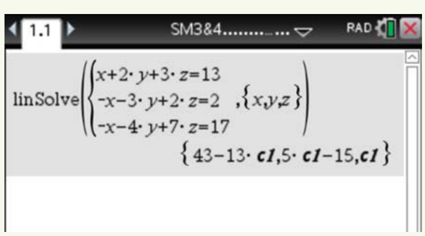
The simultaneous equations $x + 2y + 3z = 13$, $-x - 3y + 2z = 2$ and $-x - 4y + 7z = 17$ have infinitely many solutions.

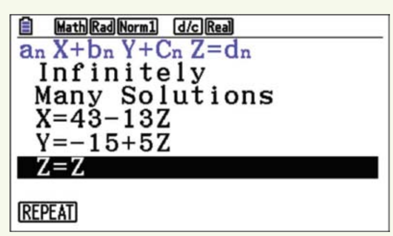
a Describe these solutions through the use of a parameter. (Use a calculator.)

b Give a geometric interpretation of the solution.

Solution

a We can use a calculator to determine all the solutions in terms of a parameter λ.





The solutions are given by $x = 43 - 13\lambda$, $y = 5\lambda - 15$ and $z = \lambda$, for $\lambda \in \mathbb{R}$.

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b The system of equations represents three planes that intersect along a line. (This is the situation shown in Diagram 2.)

The line of intersection has parametric equations $x = 43 - 13\lambda$, $y = -15 + 5\lambda$, $z = \lambda$.

A vector equation of this line is $\boldsymbol{r} = 43\boldsymbol{i} - 15\boldsymbol{j} + \lambda(-13\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k})$, $\lambda \in \mathbb{R}$.

We will investigate the geometric interpretation of systems of linear equations further in Section 8H.

Linear equations in more than three variables

In general, we can consider a system of m linear equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Such a system of equations has a geometric interpretation involving the intersection of 'hyperplanes' in n-dimensional space.

A system of linear equations in several variables can be solved using a calculator or using the method introduced in the next section. Applications of linear equations in economics and biology can involve hundreds or even thousands of variables.

Exercise 8F

Example 14, 15 SF

1 Solve each of the following systems of simultaneous equations:

a $2x + 3y - z = 12$
$2y + z = 7$
$2y - z = 5$

b $x + 2y + 3z = 13$
$-x - y + 2z = 2$
$-x + 3y + 4z = 26$

c $x + y = 5$
$y + z = 7$
$z + x = 12$

d $x - y - z = 0$
$5x + 20z = 50$
$10y - 20z = 30$

Example 16 CF

2 Consider the simultaneous equations $x + 2y - 3z = 4$ and $x + y + z = 6$.

a Subtract the second equation from the first to determine y in terms of z.

b Let $z = \lambda$. Solve the equations to give the solution in terms of λ.

SF

3 Solve each of the following pairs of simultaneous equations, giving your answer in terms of a parameter λ. Use the technique introduced in Question 2.

a $x - y + z = 4$
$-x + y + z = 6$

b $2x - y + z = 6$
$x - z = 3$

c $4x - 2y + z = 6$
$x + y + z = 4$

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CF

4 Consider the simultaneous equations

$$\begin{aligned} x + 2y + 3z &= 13 \qquad (1) \\ -x - 3y + 2z &= 2 \qquad (2) \\ -x - 4y + 7z &= 17 \qquad (3) \end{aligned}$$

a Add equation (2) to equation (1) and subtract equation (2) from equation (3).

b Comment on the equations obtained in part **a**.

c Let $z = \lambda$ and determine y in terms of λ.

d Substitute for z and y in terms of λ in equation (1) to determine x in terms of λ.

5 The following system of equations has infinitely many solutions:

$$\begin{aligned} x + y + z + w &= 4 \\ x + 3y + 3z &= 2 \\ x + y + 2z - w &= 6 \end{aligned}$$

Describe the family of solutions and give the unique solution when $w = 6$.

SF

6 Determine all solutions for each of the following systems of equations:

a $3x - y + z = 4$
$x + 2y - z = 2$
$-x + y - z = -2$

b $x - y - z = 0$
$3y + 3z = -5$

c $2x - y + z = 0$
$y + 2z = 2$

8G Using matrix algebra for systems of linear equations in more than two variables

Learning intentions

- ► To consider solutions for simultaneous linear equations with more than two variables using matrix algebra when the associated inverse exists.

Linear equations in three variables

Consider the general system of three linear equations in three variables:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

This can be written as a matrix equation:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

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Define the matrices

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then the matrix equation becomes

$$\mathbf{AX} = \mathbf{B}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, we can multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{AX}) &= \mathbf{A}^{-1}\mathbf{B} \\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \qquad \text{(where } \mathbf{I} \text{ is the } 3 \times 3 \text{ identity matrix)} \\ \therefore \qquad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

Hence, if the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. You can use your calculator to determine the inverse matrix $\mathbf{A}^{-1}$.

Example 17

Use matrix methods to solve the following system of three equations in three variables:

$$\begin{aligned} 2x + y + z &= -1 \\ 3y + 4z &= -7 \\ 6x + z &= 8 \end{aligned}$$

Solution

Define the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 6 & 0 & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix}$$

Then the system of equations can be written as a matrix equation:

$$\mathbf{AX} = \mathbf{B}$$

Multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore \qquad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

Use your calculator to determine $\mathbf{A}^{-1}\mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

The solution is $x = 1$, $y = -5$ and $z = 2$.

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Example 18

Let $\mathbf{A} = \begin{bmatrix} 5 & -2 & -4 \\ -1 & 6 & 4 \\ 1 & -2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & -4 \\ -1 & 2 & 7 \end{bmatrix}$.

a Evaluate **AB** and hence determine the inverse of **A**.

b Hence solve the following system of equations:

$$\begin{aligned} 5x - 2y - 4z &= 24 \\ -x + 6y + 4z &= 12 \\ x - 2y &= 0 \end{aligned}$$

Solution

a We have

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 5 & -2 & -4 \\ -1 & 6 & 4 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & -4 \\ -1 & 2 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} \\ &= 12\mathbf{I} \end{aligned}$$

Hence $\mathbf{A}^{-1} = \dfrac{1}{12}\mathbf{B}$.

b The system of equations can be written as

$$\mathbf{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 12 \\ 0 \end{bmatrix}$$

Therefore

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \mathbf{A}^{-1}\begin{bmatrix} 24 \\ 12 \\ 0 \end{bmatrix} \\ &= \frac{1}{12}\mathbf{B}\begin{bmatrix} 24 \\ 12 \\ 0 \end{bmatrix} \\ &= \mathbf{B}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 4 \\ 1 & 1 & -4 \\ -1 & 2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \end{aligned}$$

The solution is $x = 6$, $y = 3$ and $z = 0$.

Linear equations in more than three variables

More generally, we can consider a system of n linear equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

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Such a system of equations can be written as a matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If the $n \times n$ matrix has an inverse, then the system of equations has a unique solution. This solution can be found using the inverse matrix in the same way as for a system of three linear equations in three variables.

Exercise 8G

Example 17 SF

1 Solve each of the following systems of linear equations by writing as a matrix equation and using an inverse matrix:

a $\begin{aligned} x - y - z &= 0 \\ 5y + 20z &= 50 \\ 10y - 20z &= 30 \end{aligned}$

b $\begin{aligned} x + 2y + 3z &= 13 \\ -x - y + 2z &= 2 \\ -x + 3y + 4z &= 26 \end{aligned}$

c $\begin{aligned} x + y &= 1 \\ y - z &= 5 \\ x + y + z &= 12 \end{aligned}$

d $\begin{aligned} x - y - z &= 0 \\ 5x + 20z &= 50 \\ 10y - 20z &= 30 \end{aligned}$

e $\begin{aligned} x + y - z &= 3 \\ x - z + w &= 0 \\ 2x - y - z + 3w &= 1 \\ 4x - 2y - 3z + 4w &= 0 \end{aligned}$

f $\begin{aligned} x - y - z - w &= 5 \\ x + 2y + 3z + w &= -2 \\ 2x + 2z + 3w &= 3 \\ 3x + y + 2w &= 1 \end{aligned}$

Example 18 CF

2 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -7 & 5 & 4 \\ 8 & 8 & -8 \\ 5 & -7 & 4 \end{bmatrix}$.

a Evaluate **AB** and hence determine the inverse of **A**.

b Hence solve the following system of equations:

$$\begin{aligned} x + 2y + 3z &= 19 \\ 3x + 2y + z &= -3 \\ 4x + y + 4z &= 0 \end{aligned}$$

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 8 \\ -4 & -4 & -9 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -5 & -2 & -4 \\ -4 & -7 & -8 \\ 4 & 4 & 5 \end{bmatrix}$.

a Evaluate **AB** and hence determine the inverse of **A**.

b Hence solve the following system of equations:

$$\begin{aligned} -5x - 2y - 4z &= 18 \\ -4x - 7y - 8z &= 12 \\ 4x + 4y + 5z &= 3 \end{aligned}$$

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4 Three crop sprays are manufactured by combining chemicals A, B and C as follows:

Spray P One barrel of spray P contains 1 unit of A, 3 units of B and 4 units of C.

Spray Q One barrel of spray Q contains 3 units each of A, B and C.

Spray R One barrel of spray R contains 2 units of A and 5 units of B.

To control a certain crop disease, a farmer requires 6 units of chemical A, 10 units of chemical B and 6 units of chemical C. How much of each type of spray should the farmer use?

CF

8H Using augmented matrices for systems of equations

Learning intentions

- To consider solutions for simultaneous linear equations using augmented matrices..

We now introduce a more systematic method for solving simultaneous linear equations, which is called **Gaussian elimination**. We will focus on the case where there are three variables, but the method applies in general.

We can represent a system of linear equations by an **augmented matrix**. Here is an example of an augmented matrix and the corresponding system of equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \qquad \begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Our aim is to form a new system of equations that is equivalent to this system but has a simpler form, so that we can easily 'read off' the solutions.

We are allowed to use the following operations on the augmented matrix. These operation correspond to rearranging, scaling, or combining the original linear equations, as shown in the following example.

Elementary row operations

- Interchange two rows.
- Multiply a row by a non-zero number.
- Add a multiple of one row to another row.

We apply the elementary row operations to form a new augmented matrix such that all the entries below the main diagonal (top left to bottom right) are zero.

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Step 1 Form the augmented matrix for the original equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \qquad \begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Step 2 Subtract $2 \times$ row 1 from row 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right] \quad R_2' = R_2 - 2 \times R_1 \qquad \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Step 3 Subtract $3 \times$ row 1 from row 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \quad R_3' = R_3 - 3 \times R_1 \qquad \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Step 4 Multiply row 2 by 3 and multiply row 3 by 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 6 & -21 & -51 \\ 0 & 6 & -22 & -54 \end{array}\right] \quad \begin{array}{l} R_2' = 3 \times R_2 \\ R_3' = 2 \times R_3 \end{array} \qquad \begin{aligned} x + y + 2z &= 9 \\ 6y - 21z &= -51 \\ 6y - 22z &= -54 \end{aligned}$$

Step 5 Subtract row 2 from row 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 6 & -21 & -51 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3' = R_3 - R_2 \qquad \begin{aligned} x + y + 2z &= 9 && (1) \\ 6y - 21z &= -51 && (2) \\ -z &= -3 && (3) \end{aligned}$$

We can now determine the solution: $z = 3$ from equation (3), so $y = 2$ from (2) and $x = 1$ from (1).

Row-echelon form

The first non-zero entry of a row is called the **row leader**. In our final augmented matrix, the row leaders are 1, 6 and −1.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 6 & -21 & -51 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

This augmented matrix is said to be in **row-echelon form**: each successive row leader is further to the right, and so each row leader has only 0s below.

If we continue with elementary row operations, we can obtain an even simpler form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

This augmented matrix is in **reduced row-echelon form**: each successive row leader is further to the right, each row leader is 1, and each row leader has only 0s above and below.

Using row operations on augmented matrices provides us with a technique for solving simultaneous linear equations that can be applied in all cases.

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We illustrate the method by investigating some different cases for a system of three equations in three variables.

Three planes intersecting at a point

Example 19

The simultaneous equations $x + 2z = 6$, $-3x + 4y + 6z = 30$ and $-x - 2y + 3z = 8$ have a unique solution.

a Determine this solution.

b Give a geometric interpretation of the solution.

Solution

Step 1 Form the augmented matrix for the original equations:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 6 \\ -3 & 4 & 6 & 30 \\ -1 & -2 & 3 & 8 \end{array}\right] \qquad \begin{aligned} x + 2z &= 6 \\ -3x + 4y + 6z &= 30 \\ -x - 2y + 3z &= 8 \end{aligned}$$

Step 2 Add $3 \times$ row 1 to row 2, and add row 1 to row 3:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 6 \\ 0 & 4 & 12 & 48 \\ 0 & -2 & 5 & 14 \end{array}\right] \begin{array}{l} \\ R_2' = R_2 + 3 \times R_1 \\ R_3' = R_3 + R_1 \end{array} \qquad \begin{aligned} x + 2z &= 6 \\ 4y + 12z &= 48 \\ -2y + 5z &= 14 \end{aligned}$$

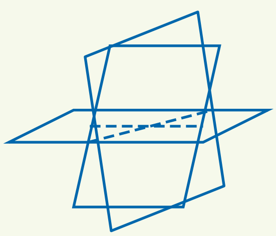
Step 3 Add $\frac{1}{2} \times$ row 2 to row 3:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 6 \\ 0 & 4 & 12 & 48 \\ 0 & 0 & 11 & 38 \end{array}\right] \begin{array}{l} \\ \\ R_3' = R_3 + \frac{1}{2} \times R_2 \end{array} \qquad \begin{aligned} x + 2z &= 6 \\ 4y + 12z &= 48 \\ 11z &= 38 \end{aligned}$$

a From the final system of equations, we obtain the solution

$$z = \frac{38}{11}, \quad y = \frac{18}{11}, \quad x = -\frac{10}{11}$$

b The equations represent three planes that intersect at the point $\left(-\frac{10}{11}, \frac{18}{11}, \frac{38}{11}\right)$.



Note: If only one row operation is applied at each step, then the resulting system will be equivalent to the original system (i.e. the solution set will be the same).
Applying two row operations in one step can sometimes change the system. However, it is safe to use one row to modify two other rows. (In this example, we used row 1 to modify both row 2 and row 3 in Step 2.)

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Three planes intersecting in a line

We now show how this method can be used when there are infinitely many solutions by repeating Example 16.

Example 20

The simultaneous equations $x + 2y + 3z = 13$, $-x - 3y + 2z = 2$ and $-x - 4y + 7z = 17$ have infinitely many solutions.

a Describe these solutions through the use of a parameter.

b Give a geometric interpretation of the solution.

Solution

Step 1 Form the augmented matrix for the original equations:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ -1 & -3 & 2 & 2 \\ -1 & -4 & 7 & 17 \end{array}\right] \qquad \begin{aligned} x + 2y + 3z &= 13 \\ -x - 3y + 2z &= 2 \\ -x - 4y + 7z &= 17 \end{aligned}$$

Step 2 Use row 1 to obtain 0s below the row leader of row 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & -1 & 5 & 15 \\ 0 & -2 & 10 & 30 \end{array}\right] \quad \begin{array}{l} \\ R_2' = R_2 + R_1 \\ R_3' = R_3 + R_1 \end{array}$$

Step 3 Obtain 1 as the row leader of row 2:

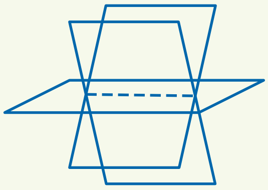
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & -5 & -15 \\ 0 & -2 & 10 & 30 \end{array}\right] \quad R_2' = -R_2$$

Step 4 Use row 2 to obtain 0s above and below the row leader of row 2:

$$\left[\begin{array}{ccc|c} 1 & 0 & 13 & 43 \\ 0 & 1 & -5 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1' = R_1 - 2R_2 \\ \\ R_3' = R_3 + 2R_2 \end{array} \qquad \begin{aligned} x + 13z &= 43 \\ y - 5z &= -15 \\ 0 &= 0 \end{aligned}$$

a We can determine the solutions from the final system of equations. Let $z = \lambda$. Then $y = -15 + 5\lambda$ and $x = 43 - 13\lambda$.

b The equations represent three planes that intersect along the line given by the parametric equations $x = 43 - 13\lambda$, $y = -15 + 5\lambda$, $z = \lambda$.



Note: To obtain the solutions from the final augmented matrix, we use a parameter for each variable without a row leader in its column. We then use the final equations to express the other variables in terms of these parameters.

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Three planes with no common intersection

Example 21

Consider the system of equations $2x + y - z = 1$, $x - y + z = 3$ and $x + 5y - 5z = 2$.

a Show that this system of equations has no solution.

b Give a geometric interpretation of these equations.

Solution

a

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \end{array}\right] \qquad \begin{aligned} 2x + y - z &= 1 \\ x - y + z &= 3 \\ x + 5y - 5z &= 2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \end{array}\right] \quad R_1' = \tfrac{1}{2}R_1$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{5}{2} \\ 0 & \frac{9}{2} & -\frac{9}{2} & \frac{3}{2} \end{array}\right] \quad \begin{aligned} R_2' &= R_2 - R_1 \\ R_3' &= R_3 - R_1 \end{aligned}$$

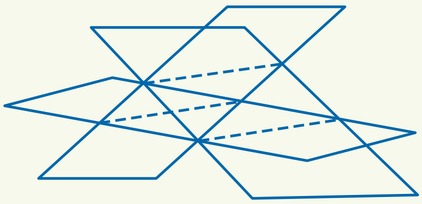
$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{5}{3} \\ 0 & 1 & -1 & \frac{1}{3} \end{array}\right] \quad \begin{aligned} R_2' &= -\tfrac{2}{3}R_2 \\ R_3' &= \tfrac{2}{9}R_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{5}{3} \\ 0 & 0 & 0 & 2 \end{array}\right] \quad R_3' = R_3 - R_2 \qquad \begin{aligned} x + \tfrac{1}{2}y - \tfrac{1}{2}z &= \tfrac{1}{2} \\ y - z &= -\tfrac{5}{3} \\ 0 &= 2 \end{aligned}$$

The final row corresponds to the equation $0x + 0y + 0z = 2$. There are no values of x, y and z that satisfy this equation, as $0 \neq 2$. So the system of equations has no solutions.

b The equations represent three planes that do not have any common intersection.

(Each pair of planes is non-parallel and therefore intersects in a line, but there is no point in common to the three planes.)



Further examples

So far we have seen three cases for a system of linear equations in three variables:

- unique solution (planes intersect at a point)
- infinitely many solutions with one parameter (planes intersect in a line)
- no solutions (planes have no common intersection).

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A system of linear equations in three variables can also represent multiple copies of the same plane. For example:

$$x + 2y + 4z = 1$$
$$2x + 4y + 8z = 2$$

These two planes coincide, and the solution is the set of all points on the plane. In this case we need to use two parameters. A possible form of the solution is

$$x = \lambda, \quad y = \mu, \quad z = \tfrac{1}{4}(1 - \lambda - 2\mu)$$

for $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

Example 22

Consider the system of equations

$$x + 2y - z = 2$$
$$2x + 5y - (a + 2)z = 3$$
$$-x + (a - 5)y + z = 1$$

a Represent the system of equations as an augmented matrix in row-echelon form.

b Determine the values of a for which:

i there is a unique solution

ii there are infinitely many solutions

iii there are no solutions.

c **i** Determine the solution, in terms of a, when a satisfies the conditions of **b i**.

ii Determine the solutions when a satisfies the conditions of **b ii**.

Solution

a
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 5 & -(a+2) & 3 \\ -1 & a-5 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -a & -1 \\ 0 & a-3 & 0 & 3 \end{array}\right] \begin{array}{l} \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -a & -1 \\ 0 & 0 & a(a-3) & a \end{array}\right] \begin{array}{l} \\ \\ R_3' = R_3 - (a-3)R_2 \end{array}$$

b From the last row of the augmented matrix, we have $a(a - 3)z = a$.

i There is a unique solution if $a \neq 0$ and $a \neq 3$.

ii There are infinitely many solutions if $a = 0$.

iii There are no solutions if $a = 3$. In this case, the last row tells us that $0 = 3$.

c **i** If $a \neq 0$ and $a \neq 3$, then the solution is $z = \dfrac{1}{a-3}$, $y = \dfrac{3}{a-3}$, $x = \dfrac{2a-11}{a-3}$.

ii If $a = 0$, then we have $y = -1$. The first equation becomes $x - z = 4$. Let $z = \lambda$. Then $x = \lambda + 4$. The solutions give the line defined by $x = \lambda + 4$, $y = -1$, $z = \lambda$.

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Skill-sheet

Exercise 8H

SF

Example 19

1 Solve each of the following systems of linear equations using augmented matrices:

a $$\begin{aligned} 2x - y + z &= -7 \\ x + 2y - 2z &= 7 \\ -2x + y + 3z &= -1 \end{aligned}$$

b $$\begin{aligned} x + y - 3z &= 6 \\ 2x + 3y + 2z &= 2 \\ x - 2y + z &= -6 \end{aligned}$$

c $$\begin{aligned} x + y + z &= 10 \\ 2x - y - 5z &= 3 \\ x + 2y - z &= 1 \end{aligned}$$

Example 20

2 Solve each of the following systems of linear equations using augmented matrices. Express the solutions using a parameter.

a $$\begin{aligned} x + 2y - 5z &= 7 \\ x + y - 2z &= 5 \\ 2x - 3y + 11z &= 0 \end{aligned}$$

b $$\begin{aligned} x + 2y &= 10 \\ 3x + 2y - 4z &= 18 \\ y + z &= 3 \end{aligned}$$

c $$\begin{aligned} 2x - y + z &= 0 \\ y + 2z &= 2 \end{aligned}$$

Example 21

3 Show that the system of equations

$$\begin{aligned} 3x - y - 2z &= 0 \\ x - y - z &= -1 \\ 2x + 8y + 3z &= 10 \end{aligned}$$

has no solution.

4 For each of the following systems of equations:

i Determine whether the system has a unique solution, infinitely many solutions or no solutions.

ii Determine the solutions if they exist.

a $$\begin{aligned} x + y + 2z &= 11 \\ 2x + 4y - 3z &= -2 \\ 3x + 6y - 5z &= -5 \end{aligned}$$

b $$\begin{aligned} 2x - y + z &= 4 \\ x - 3y &= 3 \\ 3x - 2y + z &= 7 \end{aligned}$$

c $$\begin{aligned} 2x - 3y + 5z &= 7 \\ 9x - y + z &= -1 \\ x + 5y + 4z &= 0 \end{aligned}$$

d $$\begin{aligned} x + y + z &= 7 \\ 2x - y + 3z &= 15 \\ x - 8y + 4z &= 6 \end{aligned}$$

CF

Example 22

5 **a** For which values of a will the following system have a unique solution, infinitely many solutions or no solutions?

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

b Determine the solutions (if any) in terms of a in each case.

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6 **a** For which values of a will the following system have a unique solution, infinitely many solutions or no solutions? CF

$$x + y + z = 4$$
$$2x + 3y + 3z = 10$$
$$x + y + (a^2 - 3)z = a + 2$$

b Determine the solutions (if any) in terms of a in each case.

7 For each of the following pairs of planes, determine a vector equation of the line of intersection: SF

a $2x + 3y + 3z = 10, \quad x + 3y + 2z = 4$

b $2x - 5y - z = -3, \quad x + 3y + z = 7$

8 **a** For which value of a will the following system of equations, corresponding to three planes, have solutions? CF

$$2x + y + 3z = 1$$
$$x - 3y - z = 5$$
$$3x - 2y + 2z = a$$

b For this value of a, explain how these planes intersect with each other.

c For all other values of a, explain geometrically why the third plane does not intersect at the line of intersection of the other two planes.

9 Explain why the following planes do not have any common points of intersection: SF

$$x - 3y + 3z = 6$$
$$2x - 6y + 4z = 10$$
$$x - 3y + 2z = 4$$

10 Solve each of the following systems of linear equations using augmented matrices:

a
$$\begin{aligned} 2x + 2y + 4z &= 0 \\ -y - 3z + w &= 0 \\ 3x + y + z + 2w &= 0 \\ x + 3y - 2z - 2w &= 0 \end{aligned}$$

b
$$\begin{aligned} x + 2y + 4z + w &= 0 \\ 2x + 3y + z + w &= 0 \\ 3x - y + 2z + w &= 0 \end{aligned}$$

c
$$\begin{aligned} 5x + y + 4z + w &= 0 \\ 2z - w &= 0 \\ z + w &= 0 \\ 7w &= 0 \end{aligned}$$

11 Determine an equation of the form $ax + by + cz + d = 0$ that describes the plane containing the three points $(1, 2, -1)$, $(2, 3, 1)$ and $(3, -1, 2)$ by solving a system of three linear equations in a, b, c and d. CF

12 Determine an equation of the form $ax + by + cz + d = 0$ that describes the plane containing the three points $(5, 4, 3)$, $(4, 3, 1)$ and $(1, 5, 4)$ by solving a system of three linear equations in a, b, c and d.

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Review

Chapter summary

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.
- The **size** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.

Basic operations on matrices

- Addition is defined for two matrices only when they have the same size. The sum is found by adding corresponding entries.
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$
  Subtraction is performed in a similar way.
- If **A** is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of **A** by k.
$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$
- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To determine the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  Note that the product **AB** is defined if and only if the number of columns of **A** is the same as the number of rows of **B**.

Identities, inverses and determinants

- For each natural number n, there is an $n \times n$ identity matrix **I**. This matrix satisfies $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for all $n \times n$ matrices **A**.
- The 2×2 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- If **A** is a square matrix and there exists a matrix **B** such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **B** is called the **inverse** of **A** and is denoted by $\mathbf{A}^{-1}$.
- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of **A** is given by
$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad-bc \neq 0)$$
  - the determinant of **A** is given by
$$\det(\mathbf{A}) = ad - bc$$

- Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.
- The **determinant** is defined for all square matrices. A square matrix has an inverse if and only if its determinant is non-zero.

General system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Linear equations in two variables

- A linear equation of the form $ax + by = c$ represents a line in two-dimensional space (provided a and b are not both zero).
- There are three cases for a system of two linear equations in two variables:
  - unique solution (lines intersect at a point)
  - infinitely many solutions (lines coincide)
  - no solutions (lines are parallel).

Linear equations in three variables

- A linear equation of the form $ax + by + cz = d$ represents a plane in three-dimensional space (provided a, b and c are not all zero).
- There are four cases for a system of linear equations in three variables:
  - unique solution (planes intersect at a point)
  - infinitely many solutions with one parameter (planes intersect in a line)
  - infinitely many solutions with two parameters (planes coincide)
  - no solutions (planes have no common intersection).

Systems of linear equations, solved by using the inverse matrix

- A system of three linear equations in three variables has the form:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

- This can be written as a matrix equation $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- If the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

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Review

Augmented matrices

- A system of three linear equations in three variables can be represented by an augmented matrix as shown:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

- Each of the following **row operations** produces an equivalent system of equations (i.e. the solution set is the same):
  - Interchange two rows.
  - Multiply a row by a non-zero number.
  - Add a multiple of one row to another row.
- An augmented matrix is in **row-echelon form** if each successive row leader is further to the right, and so each row leader has only 0s below.
- An augmented matrix is in **reduced row-echelon form** if each successive row leader is further to the right, each row leader is 1, and each row leader has only 0s above and below.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

8A **1** I have reviewed matrix algebra and the use of a calculator for matrices. ☐

See Example 1, Example 2 and Questions 7 and 11

8B **2** I can determine the inverse and determinant of a 2×2 matrix. ☐

See Example 3, Example 4 and Questions 1 and 5

8C **3** I can analyse and solve simultaneous equations in 2 variables. ☐

See Example 5, Example 6, Example 7 and Questions 3, 4 and 5

8D **4** I can use matrix algebra and the inverse to solve simultaneous equations in two variables. ☐

See Example 8, Example 9, Example 10 and Questions 1, 2 and 8

8E **5** I can use a calculator to determine the determinant and inverse of an $n \times n$ matrix. ☐

See Example 11, Example 12, Example 13, and Questions 2, 5 and 6

8F **6** I can use a calculator to solve simultaneous linear equations with n variables. ☐

See Example 14, Example 15, Example 16 and Questions 1 and 2

8G **7** I can use matrix algebra to solve simultaneous linear equations in n variables with the use of a calculator. ☐

See Example 17, Example 18and Questions 1 and 2

8H **8** I can work with augmented matrices to solve simultaneous linear equations. ☐

See Example 19, Example 20, Example 21, Example 22 and Questions 1, 2 and 3

Short-response questions

Technology-free short-response questions

SF

1 If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, determine:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$

b $\mathbf{A}^2 - \mathbf{B}^2$

CF

2 Consider the following system of linear equations:

$$mx + 4y = a$$
$$2x + (m + 2)y = -1$$

a Write this system as a matrix equation of the form $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{X}$ and $\mathbf{B}$ are 2×1 matrices.

b Determine the values of m for which the system has a unique solution.

c Determine the values of m and a for which the system has infinitely many solutions.

3 Consider the following system of linear equations:

$$2mx - 6y = 2m + 6$$
$$4x + (m + 7)y = 1$$

a Determine the value of m for which there are infinitely many solutions.

b Determine the values of m for which there is a unique solution.

SF

4 Determine the 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

5 If $\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$, determine $\mathbf{A}^2$ and hence determine $\mathbf{A}^{-1}$.

6 Let $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$.

a Determine $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.

b Use part **a** to solve the simultaneous equations $4x + 2y = 4$ and $3x + 3y = 12$.

SF

7 Determine the values of k for which the following system of equations has a unique solution:

$$2x - 3y = 4$$
$$x + ky = 2$$

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8 Determine the values of a for which the following system of equations has a unique solution: SF

$$\begin{aligned} ax + 3y + z &= 2 \\ 5x - y - z &= 1 \\ x + 4y + 2z &= 3 \end{aligned}$$

9 The augmented matrix of a system of equations has been converted to row-echelon form as follows. CF

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & a-2 & 0 & a^2-2a \\ 0 & 0 & a+1 & 3 \end{array}\right]$$

Determine the values of a for which:

a there are no solutions

b there are infinitely many solutions

c there is a unique solution.

10 Determine the equation of the parabola $y = ax^2 + bx + c$ that passes through the three points $(-2, 40)$, $(1, 7)$ and $(3, 15)$.

11 Show that the following system of equations has a solution if and only if $a = 6$. SF

$$\begin{aligned} 2x + 3y + 4z &= 3 \\ x + y - 8z &= 1 \\ 5x + 6y - 20z &= a \end{aligned}$$

12 Consider the system of equations CF

$$\begin{aligned} x + y - z &= 1 \\ x + 2y + az &= 3 \\ x + ay + 2z &= 4 \end{aligned}$$

Determine the values of a for which there is:

a a unique solution

b no solution.

13 **a** Determine a vector equation of the line of intersection of the planes with Cartesian equations $x - y + z = 1$ and $x + y - z = 3$. SF

b Determine the points of intersection of this line with the sphere $x^2 + y^2 + z^2 = 9$.

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Review

CF

14 Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 4 & -4 \\ 1 & -6 & 14 \\ 3 & -2 & 2 \end{bmatrix}$.

a Determine $\mathbf{AB}$ and hence determine $\mathbf{A}^{-1}$.

b Use $\mathbf{A}^{-1}$ to solve the system of equations

$$2x + 4z = 8$$
$$5x + y + 3z = 16$$
$$2x + y + z = 32$$

15 **a** Consider the system of equations

$$2x + 5y = 4$$
$$3x + 2y = 8$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Determine $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.

iii Solve the system of equations.

iv Interpret your solution geometrically.

b Consider the system of equations

$$3x + 2y = 3$$
$$9x + 6y = 8$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Determine $\det(\mathbf{A})$ and explain why $\mathbf{A}^{-1}$ does not exist.

c Interpret your findings in part **b** geometrically.

Technology-active short-response questions

CF

16 Given the following linear equations:

$$2x + 3y - z = 1$$
$$x - y - 2z = 1$$
$$\lambda x + y + 4z = 5,$$

a Find the value of λ for which this set of equations do not have a unique solution.

b This set of equations has a unique solution where $z = 3$. Determine the corresponding value for λ.

17 Given the following linear equations:

$$x + y - 2z = 6$$
$$x + 2y + z = 4$$
$$x + 3y + \lambda z = \mu,$$

a Find the value of λ for which the set of equations does not have a unique solution.

b For this value of λ, determine the value of μ for which the set of equations has an infinite number of solutions.

c For these values of λ and μ, determine a vector equation to represent the solutions of this set of equations.

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Review

18 The augmented matrix represents a set of linear equations for the set (x, y, z)

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 5 \\ b & -2 & 2 & 8 \\ 1 & 3 & b & -3 \end{array}\right]$$

a Find the values of b for which this set of equations do not have a unique solution.

b For $b = 0$, find the solution for this set of equations.

CF

19 Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors: Brad, Flynn and Lina.
The matrix shows the number of dresses, pants and shirts that one worker, for each of the contractors, can produce in a week.

$$\begin{array}{r|ccc|} & \text{Brad} & \text{Flynn} & \text{Lina} \\ \text{Dresses} & 5 & 6 & 10 \\ \text{Pants} & 3 & 4 & 5 \\ \text{Shirts} & 2 & 6 & 5 \end{array}$$

The number produced varies because of the different equipment used by the contractors.
The warehouse requires 310 dresses, 175 pants and 175 shirts in a week.

a Write down a system of linear equations in three variables (the numbers x, y and z of workers for the contractors Brad, Flynn and Lina respectively).

b Write this system as an augmented matrix.

c How many workers should each contractor employ to meet the requirement exactly?

CU

20 A quadratic function f has a rule of the form $f(x) = ax^2 + bx + c$. It is known that the points $(2, 0)$ and $(1, 1)$ are on the graph of f.

a Write down two linear equations satisfied by a, b and c.

b Determine a and b in terms of c.

c Determine the values of a, b and c if $f(-1) = 4$.

21 A quartic function f has a rule of the form $f(x) = ax^4 + bx^3 + cx^2 + dx$. The graph has a stationary point at $(1, 1)$ and passes through the point $(-1, 4)$.

a Write down three linear equations satisfied by a, b, c and d.

b Determine a, b and c in terms of d.

c Determine the value of d for which the graph has a stationary point where $x = 4$.

22 **a** Solve the simultaneous linear equations $x + 2y - z = 2$ and $2x - y + 3z = -1$.

b A third equation is $3x + p^2y - z = p + 4$. Determine the values of p for which the system of three equations has:

i no solution

ii a unique solution

iii infinitely many solutions.

23 **a** Determine a vector equation of the line of intersection of the two planes given by the equations $x - y - z = 1$ and $2x + 4y + z = 5$.

b A third plane has equation $3x + p^2y - z = p + 4$.

i Determine the values of p for which the three planes intersect at a point, and give the coordinates of this point in terms of p.

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CU

ii Determine the value of p for which the three planes intersect in a line.

iii Determine the value of p for which the three planes intersect at the point $(1, 1, -1)$.

c Determine the axial intercepts of the sphere $(x-1)^2 + (y-1)^2 + (z+1)^2 = 3$.

d Determine the coordinates of the points of intersection of the line found in part **a** with the sphere.

CF

24 A factory makes and assembles three products, P, Q and R, each requiring different quantities of three components, a, b and c. The following matrix $\mathbf{A}$ represents the required quantities of components for each product, and the matrix $\mathbf{K}$ represents the average daily production of components at the factory.

$$\mathbf{A} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{K} = \begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} 95 \\ 80 \\ 40 \end{bmatrix}$$

a Determine the inverse of $\mathbf{A}$ by evaluating $\mathbf{AB}$, where $\mathbf{B} = \begin{bmatrix} 2 & 5 & -8 \\ 6 & -15 & 16 \\ -4 & 10 & -4 \end{bmatrix}$.

b Assume that the factory uses all components that are produced. Determine the rate of assembly of P, Q and R at the factory, expressed as the average number of products per day.

CU

25 At a particular store at a farmers' market over an hour period Alice sold 2 kilograms of bananas, 3 kilograms of apples and 1 kilogram of peaches for total sales of \$65. Buddy sold 4 kilograms of bananas, 6 kilograms of apples and 3 kilograms of peaches for total sales of \$140. Paul sold 8 kilograms of bananas, 8 kilograms of apples and 5 kilograms of peaches for total sales of \$250. What is the cost per kilogram of each item?

26 A manufacturer assembles three types of solar panel A, B and C requiring various numbers of components x, y and z. The matrix here shows the required number of components for each type of solar panel

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 6 & 10 & 20 \\ 8 & 15 & 5 \\ 11 & 15 & 5 \end{bmatrix} \end{array}$$

If the total components for A cost \$180 , total components for B cost \$225 and total components for C costs \$270, what is the cost of each of the components?

27 Merry inherited \$25,000 and invested part of it in a bank account, part in shares, and part in bonds. After one year the value had risen to \$26,730 from the three investments. The bank account increased by 6% of the original amount invested, the share market increased by 7% of the original amount invested and the bonds increased by 8% of the original amount invested . There was \$2,000 more invested in shares and than in bonds. Determine the amount Merry invested in each type of investment.

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Review

CU

28 A three digit number $\overline{abc}$ can be expressed as $100a + 10b + c$. The number is equal to 17 times the sum of its digits. If 198 is added to the number the digits are reversed. Also, $a + c = b - 1$.

a Determine the digits.

b Are there more solutions if we do not use $a + c = b - 1$? Remember that the a, b and c are digits.

29 Suppose that $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices.

a Prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

b Hence prove that if both $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $\mathbf{AB}$ is invertible.

30 Let $\mathbf{A} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$.

a Show that $\mathbf{A}^2 = \mathbf{A}$.

b Hence, using simple matrix algebra, show that $\mathbf{A}$ cannot have an inverse.

c Let $\mathbf{E} = \mathbf{I} + \lambda\mathbf{A}$, where $\lambda \in \mathbb{R}$. Write $\mathbf{E}$ as a single 3×3 matrix.

d Let $\mathbf{F} = \mathbf{I} + \mu\mathbf{A}$, where $\mu \in \mathbb{R}$. Determine μ in terms of λ if $\mathbf{E}$ and $\mathbf{F}$ are inverse matrices.

e Determine $\mathbf{E}$ and $\mathbf{F}$ if $\lambda = 3$.

f Hence solve the following system of equations:

$$\begin{aligned} 7x - 6y - 12z &= 6 \\ -3x + 10y + 12z &= 8 \\ 3x - 6y - 8z &= 0 \end{aligned}$$

Multiple-choice questions

Technology-free multiple-choice questions

1 Consider the following four matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 3 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 2 & 4 & 5 \\ -1 & 3 & 6 \\ 4 & 0 & 1 \end{bmatrix}$$

Which one of the following matrix products is *not* defined?

A **DB** **B** **BA** **C** **CB** **D** **AC**

2 The system of linear equations

$$4x + (m-1)y = 6$$
$$mx + 3y = 14$$

has a unique solution for

A $m = -3$
B all positive values of m
C all negative values of m
D $m \neq 4$ and $m \neq -3$

3 If both **A** and **B** are $m \times n$ matrices, where $m \neq n$, then $\mathbf{A} + \mathbf{B}$ is

A an $m \times n$ matrix
B an $m \times m$ matrix
C an $n \times n$ matrix
D a $2m \times 2n$ matrix

4 The determinant of the matrix $\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$ is

A 6 **B** 0 **C** −6 **D** 8

5 The inverse of the matrix $\begin{bmatrix} 5 & -7 \\ 2 & -3 \end{bmatrix}$ is

A $\begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} -3 & 7 \\ -2 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} -5 & 7 \\ -2 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & -7 \\ 2 & -5 \end{bmatrix}$

6 Consider the system of simultaneous linear equations

$$bx + 3y = 0$$
$$4x + (b+1)y = 0$$

where b is a real constant. This system has infinitely many solutions for

A $b \in \mathbb{R}$
B $b \in \{-3, 4\}$
C $b \in \mathbb{R} \setminus \{-3, 4\}$
D $b \in \{-4, 3\}$

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Review

7 The simultaneous equations

$$(a-1)x + 5y = 7$$
$$3x + (a-3)y = a$$

have a unique solution for

A $a \in \mathbb{R} \setminus \{-2, 6\}$
B $a \in \mathbb{R} \setminus \{0\}$
C $a \in \mathbb{R} \setminus \{6\}$
D $a = 6$

8 The solution of the two simultaneous equations $ax - 5by = 11$ and $4ax + 10by = 2$ for x and y, in terms of a and b, is

A $x = -\dfrac{10}{a},\ y = -\dfrac{21}{5b}$
B $x = \dfrac{4}{a},\ y = -\dfrac{7}{5b}$
C $x = \dfrac{13}{5a},\ y = -\dfrac{42}{25b}$
D $x = \dfrac{13}{2a},\ y = -\dfrac{9}{10b}$

9 Cara, Mai and Luke have purchased identical pencils and, while playing, got them all mixed up. Now the children need to divide 22 pencils between themselves. Luckily, Cara remembers that she had three more pencils than Mai did. Luke remembers that he had as many pencils as the other two combined. How many pencils did Cara have?

A 2 **B** 5 **C** 7 **D** 3

10 The system of equations

$$x - 2y + z = 3$$
$$2x - 4y + 2z = 7$$
$$x + 3y - z = 4$$

can be interpreted geometrically as

A three planes that intersect at a point
B three planes that intersect in a line
C three parallel planes
D three planes with no common intersection, where two of the planes are parallel

11 A system of linear equations corresponds to the augmented matrix in row-echelon form as shown.

$$x + 2y - 3z = 3$$
$$2x - y - 4z = -2$$
$$-2x + 5y + z = 7$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -5 & 2 & -8 \\ 0 & 0 & -7 & -7 \end{array}\right]$$

A solution of this system of equations is

A $x = 2,\ y = 2,\ z = 1$
B $x = 3,\ y = 1,\ z = 1$
C $x = 3,\ y = 3,\ z = 2$
D $x = 2,\ y = 2,\ z = -1$

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12 A system of linear equations corresponds to the augmented matrix in row-echelon form as shown.

$$\begin{aligned} x + 2y - z &= -3 \\ 3x + 5y + kz &= -4 \\ 9x + (k+13)y + 6z &= 9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & 1 & -k-3 & -5 \\ 0 & 0 & k^2 - 2k & 5k + 11 \end{array}\right]$$

The system has a unique solution for

A $k \in \mathbb{R} \setminus \{-3, 0, 2\}$ **B** $k = -3$

C $k = 0$ or $k = -2$ **D** $k \in \mathbb{R} \setminus \{0, 2\}$

13 A system of linear equations corresponds to the given augmented matrix after some elementary row operations have taken place.

$$\begin{aligned} x - 2y + z &= 11 \\ 3x + y - 2z &= -2 \\ 4x + 3y + 4z &= -11 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 11 \\ 0 & 7 & m & -35 \\ 0 & 11 & n & -55 \end{array}\right]$$

The values of m and n are

A $m = -5,\ n = 0$ **B** $m = 5,\ n = 1$

C $m = 3,\ n = -1$ **D** $m = 0,\ n = 5$

14 The planes with equations $x - y - z = 0$ and $2x + 4y - z = 0$ intersect along a straight line through the origin. A vector equation of this line is

A $t(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}),\quad t \in \mathbb{R}$ **B** $t(\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}),\quad t \in \mathbb{R}$

C $\frac{t}{6}(5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}),\quad t \in \mathbb{R}$ **D** $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}),\quad t \in \mathbb{R}$

15 The planes $x - y - z = 0$ and $2x + 4y + z = 0$ intersect along a straight line through the origin. This line intersects the sphere with equation $x^2 + y^2 + z^2 = 6$ at the points

A $(\sqrt{6}, 0, 0),\ (0, \sqrt{6}, 0)$ **B** $(2, 1, 1),\ (2, 1, -1)$

C $(0, \sqrt{6}, 0),\ (0, -\sqrt{6}, 0)$ **D** $(1, -1, 2),\ (-1, 1, -2)$

16 The system of linear equations

$$\begin{aligned} (m-1)a + 5b &= 7 \\ 3a + (m-3)b &= 0.7m \end{aligned}$$

has infinitely many solutions for

A $m \neq 0$ and $m \neq -2$ **B** $m \neq 0$

C $m \neq 6$ **D** $m = 6$

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Review

17 The augmented matrices given below represent four sets of linear equations. Which one has no solutions?

A $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 1 & 0 \\ 7 & 0 & 1 & 0 \end{array}\right]$

B $\left[\begin{array}{ccc|c} 1 & 5 & 1 & 4 \\ 0 & 2 & -1 & 3 \\ 0 & -4 & 2 & 6 \end{array}\right]$

C $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -1 & -3 & -8 \\ 0 & 0 & 7 & -6 \end{array}\right]$

D $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 2 & 4 & -2 & 10 \\ -1 & -2 & 1 & -5 \end{array}\right]$

18 Matrix **A** is a 5×5 matrix.

- Matrix **B** is a row matrix.
- Matrix **C** is a column matrix.

Which one of the matrix products below could result in a 1×1matrix?

A **ACB** **B** **ABC** **C** **CAB** **D** **BAC**

Technology-active multiple-choice questions

19 The table below shows information about two matrices **A** and **B**.

	Order	Rule
A	3×3	$a_{ij} = ij$
B	3×3	$b_{ij} = i + j$

The sum **A** + **B** is

A $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

B $\begin{bmatrix} 3 & 5 & 7 \\ 3 & 4 & 5 \\ 3 & 6 & 19 \end{bmatrix}$

C $\begin{bmatrix} 3 & 2 & 7 \\ 3 & 4 & 5 \\ 14 & 15 & 16 \end{bmatrix}$

D $\begin{bmatrix} 3 & 5 & 7 \\ 5 & 8 & 11 \\ 7 & 11 & 15 \end{bmatrix}$

20 Consider the matrix equation

$$3 \times \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} + \mathbf{X} = \begin{bmatrix} 14 & 12 \\ 18 & 22 \end{bmatrix}$$

Which one of the following is the matrix **X**?

A $\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$

21 Matrix $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and matrix $\mathbf{Z} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \end{bmatrix}$ with a, b and c all different.

The smallest value of n such that $\mathbf{P}^n\mathbf{Z} = \mathbf{Z}$ is

A 2 **B** 3 **C** 4 **D** 5

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22 Let $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 2 \\ 0 & 6 \end{bmatrix}$

If $\mathbf{X}$ is a 2×2 matrix and $\mathbf{AXB} = \mathbf{C}$ then $\mathbf{X}$ is equal to

A $\frac{1}{28}\begin{bmatrix} 48 & 57 \\ -8 & -20 \end{bmatrix}$

B $\frac{1}{28}\begin{bmatrix} 22 & 23 \\ -24 & -48 \end{bmatrix}$

C $\frac{1}{28}\begin{bmatrix} -24 & 12 \\ 46 & -2 \end{bmatrix}$

D $\frac{1}{28}\begin{bmatrix} 30 & 98 \\ -12 & -56 \end{bmatrix}$

23 Let $\mathbf{A} = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 6 & 0 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

If $\mathbf{XA} = \mathbf{B}$ then $35\mathbf{X}$ is equal to

A $\begin{bmatrix} -22 & 52 & 46 \\ 18 & 2 & -9 \\ 8 & -3 & 31 \end{bmatrix}$

B $\begin{bmatrix} -22 & 33 & -6 \\ 41 & -9 & 8 \\ 49 & -21 & 42 \end{bmatrix}$

C $\begin{bmatrix} 18 & 15 & 6 \\ 27 & 9 & 8 \\ 7 & -3 & 6 \end{bmatrix}$

D $\begin{bmatrix} 7 & 42 & -7 \\ \frac{133}{3} & 21 & \frac{7}{3} \\ 14 & -\frac{77}{3} & \frac{77}{2} \end{bmatrix}$

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9

Applications of matrices

Chapter contents

- **Dominance matrices** can be used to determine rankings. In many groups of individuals or animals, there is a definite "pecking order" or dominance relation between any two members of the group. That is, given any two individuals A and B, either A dominates B or B dominates A. For example, we can determine a ranking of the sports teams competing in a round-robin tournament.
- **Leslie matrices** are used by biologists and ecologists to model changes over time in animal populations and in the distribution of age groups within these populations.They are one of the most common models of population growth used by demographers. This model describes the growth of the female portion of a human or animal population.

Note: The additional online chapter *Modelling with matrices* is available in the Interactive Textbook. This chapter provides extra coverage of dominance and Leslie matrices, and also introduces eigenvalues, Markov chains and Leontief matrices.

Chapters 8 and 9 cover Unit 3 Topic 5: Further matrices

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9A Dominance matrices

Learning intentions

- To be able to use dominance matrices for ranking.

You will find that this topic involves problem-solving and modelling skills. Decisions have to be made about which method to use and how to interpret the results.

A round-robin example

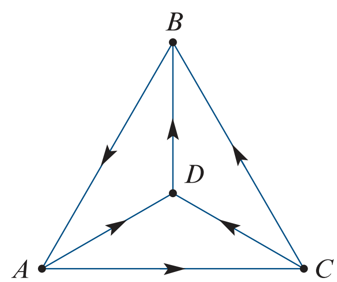
In a round-robin tournament, each team plays every other team exactly once. We will assume that none of the games results in a draw. Our aim is to rank the teams.

For example, if a round-robin rugby competition is held between four schools, A, B, C and D, then there will be six games:

$$A \text{ vs } B, \quad A \text{ vs } C, \quad A \text{ vs } D, \quad B \text{ vs } C, \quad B \text{ vs } D, \quad C \text{ vs } D$$

Suppose that the results of this competition are as follows:

- A defeats C and D
- B defeats A
- C defeats B and D
- D defeats B.



These results are also shown in the diagram. For example, the arrow pointing from B to A indicates that B defeats A.

The one-step dominance matrix

We represent the results of the rugby competition as a 4×4 matrix:

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The entry in row i and column j of $\mathbf{M}$ is 1 if school i defeats school j, and 0 otherwise.

For example, the first row of the matrix shows that school A defeats schools C and D; the second row of the matrix shows that school B defeats school A.

We find the total of each row of the matrix to obtain a dominance score for each school:

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Score} \\ 2 \\ 1 \\ 2 \\ 1 \end{array}$$

These scores indicate that the dominant school is either A or C.

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The two-step dominance matrix

To try to distinguish between schools A and C, we first calculate $\mathbf{M}^2$:

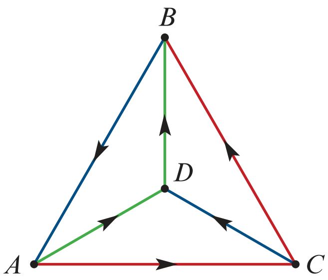
$$\mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Consider the calculation of entry a_{AB} in row A and column B of $\mathbf{M}^2$:

$$a_{AB} = 0 \times 0 + 0 \times 0 + 1 \times 1 + 1 \times 1 = 2$$

This calculation reflects the following two facts:

- A beat C, and C beat B.
  (This is shown in red; we can write $A \to C \to B$.)
- A beat D, and D beat B.
  (This is shown in green; we can write $A \to D \to B$.)



So entry a_{AB} of $\mathbf{M}^2$ is the number of two-step paths from A to B.

The total of the first row of $\mathbf{M}^2$ is the total number of two-step paths starting at A. This is a measure of 'two-step dominance'.

Combining one- and two-step dominance

We can combine the results from the one- and two-step dominance calculations. We do this by adding $\mathbf{M}$ and $\mathbf{M}^2$, and then finding the total of each row of the resulting matrix:

$$\mathbf{M} + \mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Score} \\ 5 \\ 3 \\ 4 \\ 2 \end{array}$$

Hence the dominant school is A.

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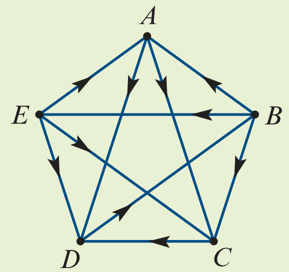
Example 1

Five friends – Ann, Bea, Cat, Deb and Eve – competed in a round-robin tennis tournament. The results were as follows:



- Ann defeated Cat and Deb
- Bea defeated Ann, Cat and Eve
- Cat defeated Deb
- Deb defeated Bea
- Eve defeated Ann, Cat and Deb.

Using this information:

a Construct a one-step dominance matrix, $\mathbf{D}$.

b Construct a two-step dominance matrix, $\mathbf{D}^2$.

c Use the dominance scores from the matrix $\mathbf{D} + \mathbf{D}^2$ to rank the five players.

Solution

a

$$\mathbf{D} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Score} \\ 2 \\ 3 \\ 1 \\ 1 \\ 3 \end{array}$$

Note: Using the one-step dominance matrix, we see that Bea and Eve are equal first.

b

$$\mathbf{D}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix}$$

c

$$\mathbf{D} + \mathbf{D}^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{Score} \\ 4 \\ 9 \\ 2 \\ 4 \\ 7 \end{array}$$

The matrix $\mathbf{D} + \mathbf{D}^2$ gives the following ranking:

Rank	Player	Score
First	Bea	9
Second	Eve	7
Equal third	Ann and Deb	4
Fifth	Cat	2

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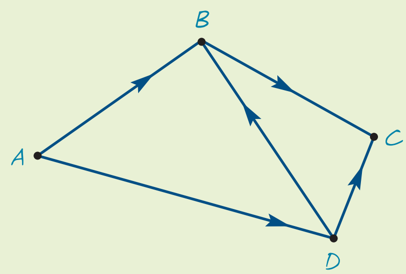
Example 2

Four people, A, B, C and D, have been asked to form a committee to decide on the location of a new toxic waste dump.



From previous experience, it is known that:

- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of C and B.

a Use the graph to construct a dominance matrix that takes into account both one-step and two-step dominances.

b From this matrix, determine who is the most influential person on the committee.

Solution

a Construct the one-step dominance matrix D.

$$D = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{One-step} \\ 2 \\ 1 \\ 0 \\ 2 \end{array}$$

Construct the two-step dominance matrix D^2.

$$D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Two-step} \\ 3 \\ 0 \\ 0 \\ 1 \end{array}$$

Form the sum $T = D + D^2$.

$$T = D + D^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \textit{Total} \\ 5 \\ 1 \\ 0 \\ 3 \end{array}$$

b The person with the highest total dominance score is the most influential.
Person A is the most influential person with a total dominance score of 5.

Further approaches

In general, a **dominance matrix**, $\mathbf{D}$, is an $n \times n$ matrix that represents a competition between the members of a group of size n. Entry d_{ij} in row i and column j of $\mathbf{D}$ is given by

$$d_{ij} = \begin{cases} 1 & \text{if member } i \text{ defeats member } j \\ 0 & \text{otherwise} \end{cases}$$

We have seen that we can rank the members of the group by using the dominance scores from the matrix $\mathbf{D} + \mathbf{D}^2$. However, many other approaches are possible.

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Higher-order powers You do not need to stop at two-step dominance. If the matrix $\mathbf{D} + \mathbf{D}^2$ does not distinguish between all the members of the group, then you can also calculate the three-step dominance matrix $\mathbf{D}^3$ and use the dominance scores from the matrix $\mathbf{D} + \mathbf{D}^2 + \mathbf{D}^3$.

Weighting higher-order powers In working out a ranking, it is often appropriate to give less weighting to the more indirect two-step and three-step contributions. For example, you may decide to base your ranking on the dominance scores from the matrix

$$\mathbf{D} + \frac{1}{2}\mathbf{D}^2 + \frac{1}{3}\mathbf{D}^3$$

Or you may use different weightings if it seems appropriate to the particular context.

Exercise 9A

SF

1 The following dominance matrix, $\mathbf{M}$, gives the results of a series of squash matches between five friends, where $m_{ij} = 1$ if player i beat player j.

$$\mathbf{M} = \begin{array}{r} \\ \text{Ash} \\ \text{Ben} \\ \text{Carl} \\ \text{Dot} \\ \text{Elle} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Ash} & \text{Ben} & \text{Carl} & \text{Dot} & \text{Elle} \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

a How many matches were there?

b Describe the outcomes of the matches.

c Use the dominance scores from the matrix $\mathbf{M}$ to give a ranking of the players.

Example 1, 2

2 Five chess players – A, B, C, D and E – competed in a round-robin chess tournament. The results were as follows:

- A defeated B and D
- B defeated C and E
- C defeated A and D
- D defeated B
- E defeated A, C and D.

Using this information:

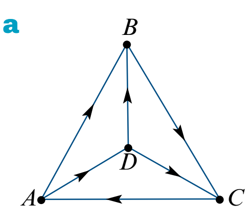
a Construct a one-step dominance matrix, $\mathbf{M}$.

b Use the dominance scores from the matrix $\mathbf{M} + \mathbf{M}^2$ to rank the players.

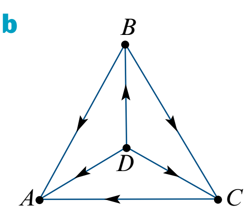
Example 2

3 For each of the graphs below construct the corresponding 4×4 dominance matrix.

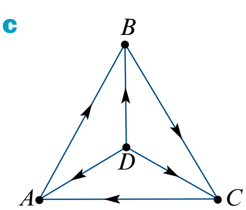
a



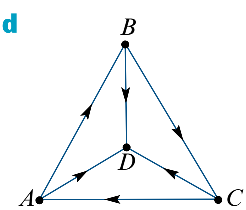
b



c



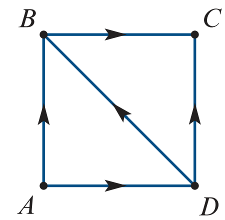
d



SF

4 A committee of four people – A, B, C and D – will decide on the location of a new toxic waste dump. From previous experience, it is known that:

- A influences the decisions of B and D
- B influences the decisions of C
- C influences the decisions of no one
- D influences the decisions of B and C.



Using this information:

a Construct a matrix that takes into account both one-step and two-step dominance.

b From this matrix, determine who is the most influential person on the committee.

CF

5 The following table gives the results of the first round of games at a chess club.

Game	A vs B	C vs D	A vs D	B vs C	B vs D	A vs C
Winner	A	C	D	B	B	A

a Create a one-step dominance matrix, $\mathbf{D}$.

b Create a ranking of the four players using $\mathbf{D}$ and $\mathbf{D}^2$.

SF

6 Four schools – A, B, C and D – compete in a round-robin hockey tournament. The results are summarised by the following dominance matrix:

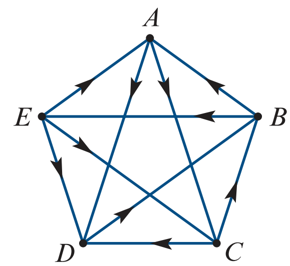
$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Create an ordering of the four schools using $\mathbf{M}$, $\mathbf{M}^2$ and $\mathbf{M}^3$.

CF

7 Five teams competed in a football tournament. Each team played every other team once. The results are shown in the diagram. (For example, the arrow pointing from A to C indicates that A defeated C.)

Create a rank order for the five teams by taking into consideration both one-step and two-step dominance.



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9B Leslie matrices

Learning intentions

- To use and interpret Leslie matrices to analyse population growth.

Leslie matrices are used to construct discrete models of population growth. In particular, they are used to model changes in the sizes of different age groups within a population.

We will consider continuous models of population growth in Chapter 13.

The general setting

Leslie matrices were developed by Patrick Holt Leslie (1900–1972) while he was working in the Bureau of Animal Population at the University of Oxford. They are now used by biologists and ecologists to model changes over time in various animal populations.

Age groups First the population is divided into age groups. Each age range has the same length, and together they cover the life span of the population. For example, in a study of a human population, we could use a time period of 10 years and consider eleven age groups as follows:

Age group (i)	1	2	3	4	$\cdots$	10	11
Age range (years)	0–10	10–20	20–30	30–40	$\cdots$	90–100	100–110

Note: Only the females of the species are counted in the population, as they are the ones who give birth to the new members of the population.

A **Leslie matrix** is a matrix that can be used to describe the way population changes over time. It takes into account two factors for the females in each age group: the *birth rate*, b_i, and *survival rate*, s_i, where i is the number of the age group.

Birth rates We ignore migration, and so the population growth is entirely due to new female births. The birth rate, b_i, for age group m is the average number of female offspring from a mother in age group m during one time period. For example, average birth rate of women in age group 4 (20 – 30 years) might be 1.7 female children for the 10 year period.

Survival rates The survival rate, s_i, for age group i is the proportion of the population in age group i that progress to age group $i + 1$. Note that $0 \le s_i \le 1$. For example, the survival rate for age group 2 might be 0.95, that is 95% of females in this 10 – 20 year age group would survive to progress to age group 3, 20 – 30 years.

Note: The survival rate of the last age group (100 – 110) is taken to be 0.

A simple example

We start with a simple example where the life span of the species is 9 years. We will divide the population into three age groups, and therefore we use a time period of 3 years.

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Age group(i)	1	2	3
Age range (years)	0–3	3–6	6–9

A Leslie matrix for three age groups is a 3×3 matrix of the form

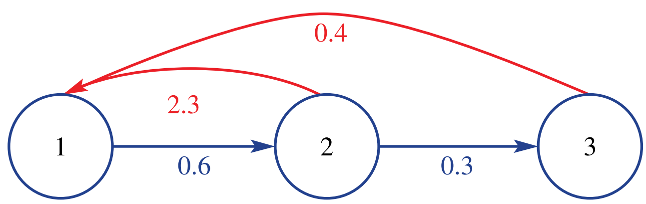
$$\mathbf{L} = \begin{bmatrix} b_1 & b_2 & b_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix}$$

Suppose that the survival rates are $s_1 = 0.6$ and $s_2 = 0.3$, and that the birth rates are $b_1 = 0$, $b_2 = 2.3$ and $b_3 = 0.4$. Then the Leslie matrix is

$$\begin{array}{rcll} & \text{From age group } i & & \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} & & \\ \mathbf{L} = & \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} & \begin{array}{l} 1 \text{ birth rate} \\ 2 \text{ survival rate} \\ 3 \text{ survival rate} \end{array} & \text{To age group } i+1 \end{array}$$

Life cycle transition diagram

The above Leslie matrix can be represented by a diagram which we will refer to as a **Life cycle transition diagram**.



The population state matrix

The **population state matrix** is a column matrix that lists the number in each age group at a given time.

The **initial population state matrix** is denoted by $\mathbf{S}_0$. Suppose that for our example, initially the population has 400 females in each age group. We represent the initial population state matrix, $\mathbf{S}_0$, as a 3×1 column matrix as shown below.

$$\begin{array}{rcc} & & \text{Age group} \\ \mathbf{S}_0 = & \begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

We can now use the Leslie matrix, $\mathbf{L}$, in combination with the initial state matrix $\mathbf{S}_0$ to generate the state matrix after one time period, $\mathbf{S}_1$, to determine the size of each age group after one (3-year time) time period as follows:

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$$\mathbf{S}_1 = \mathbf{L}\mathbf{S}_0 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}\begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix}$$

Thus after one time period, there are 1080 females in age group 1, 240 in age group 2 and 120 in age group 3 and the over-all population size has increased from 400 + 400 + 400 = 1200 to 1080 + 240 + 120 = 1440. Similarly, to determine the number in each age group after two time periods we calculate $\mathbf{N}_2$ from $\mathbf{N}_1$ as follows:

$$\mathbf{S}_2 = \mathbf{L}\mathbf{S}_1 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}\begin{bmatrix} 1080 \\ 240 \\ 120 \end{bmatrix} = \begin{bmatrix} 600 \\ 648 \\ 72 \end{bmatrix}$$

Thus, after two time periods, there are 600 females in age group 1, 648 in age group 2 and 72 in age group 3 and the over-all population size has increased to 1320.

Finding the population matrix S_k after k-time periods.

To speed up the process we can make use of the explicit formula for the state matrix N_k after k-time periods. Notice that there is a pattern when calculating the population state matrices:

$$\begin{aligned} \mathbf{S}_1 &= \mathbf{L}\mathbf{S}_0 \\ \mathbf{S}_2 &= \mathbf{L}\mathbf{S}_1 = \mathbf{L}^2\mathbf{S}_0 \\ \mathbf{S}_3 &= \mathbf{L}\mathbf{S}_2 = \mathbf{L}^3\mathbf{S}_0 \\ &\vdots \end{aligned}$$

In general, we can determine the population matrix $\mathbf{S_n}$ using the rule

$$\mathbf{S_n} = \mathbf{L}^n\mathbf{S}_0$$

Using this rule, to determine $\mathbf{S}_3$, we have

$$\mathbf{S}_3 = \mathbf{L}^3\mathbf{S}_0 = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}^3\begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 1519.2 \\ 360 \\ 194.4 \end{bmatrix}$$

Continuing in this way, we can estimate the change over time in the total population and in the distribution of the age groups.

Change in the population over time

Time period	0	1	2	3	4	5
Age 0–3 years	400	1080	600	1519.2	905.76	2139.70
Age 3–6 years	400	240	648	360.0	911.52	543.46
Age 6–9 years	400	120	72	194.4	108.00	273.46
Total	1200	1440	1320	2073.6	1925.28	2956.61

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Leslie matrices

An $m \times m$ **Leslie matrix** has the form

$$\mathbf{L} = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

where:

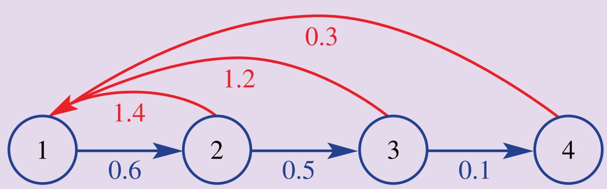
- m is the number of age groups being considered
- s_i, the survival rate, is the proportion of the population in age group i that progress to age group $i + 1$
- b_i, the birth rate, is the average number of female offspring from a mother in age group i during one time period.

Leslie matrix and its interpretation

From age group

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \mathbf{L} = & \begin{bmatrix} 0 & 1.4 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \quad \text{To age group}$$

This is a Leslie matrix with 4 age groups. The corresponding life-cycle transition diagram is shown here.



Recursive rules

The population matrix $\mathbf{S_n}$ is an $m \times 1$ matrix representing the size of each age group after the n time periods. This is calculated using a recursive formula

$\mathbf{S}_0$ is the initial state matrix, $\mathbf{S_{n+1}} = \mathbf{LS_n}$

or the explicit rule

$$\mathbf{S_n} = \mathbf{L^n S}_0$$

Example 3

Use the Leslie matrix and initial state matrix below to answer the following questions.

From age group

$$L = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 0 & 1.8 & 2.6 & 0.1 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \text{To age group} \qquad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a Write down

i The birth rate for age group 2 **ii** The survival rate for age group 3

b Complete life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

$\mathbf{S}_1$, $\mathbf{S}_5$ and $\mathbf{S}_{20}$

d Given that $\mathbf{S_n} = \begin{bmatrix} 9.53 \\ 2.42 \\ 1.22 \\ 0.16 \end{bmatrix}$, determine $\mathbf{S_{n+1}}$

Solution

a **i** The birth rate for age group 2 is given in the matrix position, row 1, column 2.

Birth rate for age group 2 = 1.8

ii The survival rate for age group 3 is given in the matrix position, row 4, column 3.

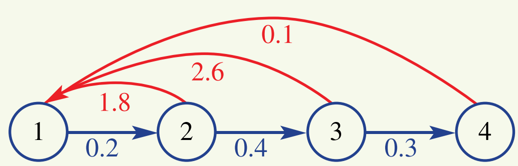
Survival rate for age group 3 = 0.3

b **Survival rates**

$s_1 = 0.2, s_2 = 0.4, s_3 = 0.3$

Birth rates

$b_2 = 1.8, b_3 = 2.6, b_4 = 0.1$



c $\mathbf{S}_1 = \mathbf{L}\mathbf{S}_0$

$\mathbf{S}_5 = \mathbf{L}^5\mathbf{S}_0$

$\mathbf{S}_{20} = \mathbf{L}^{20}\mathbf{S}_0$

Using calculator.

$$\mathbf{S}_1 = \begin{bmatrix} 0 \\ 200 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{S}_5 = \begin{bmatrix} 149.76 \\ 26.4 \\ 16.64 \\ 8.64 \end{bmatrix}, \ \mathbf{S}_{20} = \begin{bmatrix} 3.84 \\ 0.97 \\ 0.49 \\ 0.19 \end{bmatrix}$$

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d $\mathbf{S_{n+1}} = \mathbf{LS_n}$

(Further investigation would reveal that the population continues to decrease over time.)

$$\mathbf{S_{n+1}} = \begin{bmatrix} 7.54 \\ 1.91 \\ 0.97 \\ 0.37 \end{bmatrix}$$

Example 4

Information about a population of female goats is given in the following table.

Age group (years)	0 – 1	1 – 2	2 – 3	3 – 4	4 – 5
Initial population	10	25	40	20	15
Birth rates	0	0.2	0.9	0	0
Survival rates	0.6	0.7	0.5	0.2	0

a Write down the initial population state matrix, $\mathbf{S}_0$.

b Write down the Leslie matrix.

c Construct a time life-cycle transition diagram for this Leslie matrix.

d Determine the number of 3 – 4 year old female goats in the population after 3 years. Round your answer to the nearest whole number.

Solution

a Enter the initial population numbers into a 5×1 matrix

$$\mathbf{S}_0 = \begin{bmatrix} 10 \\ 25 \\ 40 \\ 20 \\ 15 \end{bmatrix}$$

b Enter the birthrates and survival rates into a 5×5 matrix.

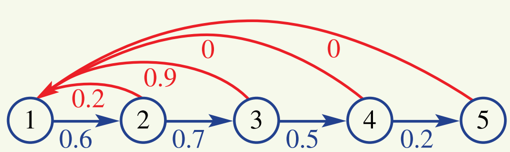
$$\mathbf{L} = \begin{bmatrix} 0 & 0.2 & 0.9 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

c **Survival rates**

$s_1 = 0.6, s_2 = 0.7, s_3 = 0.5,$
$s_4 = 0.2$

Birth rates

$b_2 = 0, 2, b_3 = 0.9, b_4 = 0,$
$b_5 = 0$



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d $\mathbf{S}_3 = \mathbf{L}^3\mathbf{S}_0$

$$\begin{bmatrix} 0 & 0.2 & 0.9 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}^3 \begin{bmatrix} 10 \\ 25 \\ 40 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 8.7 \\ 10.17 \\ 17.22 \\ 2.1 \\ 1.75 \end{bmatrix}$$

There are two goats in the 3-4 year old age group in this population.

Long term (limiting) behaviour of population numbers

The following examples demonstrate numerical techniques for modelling the use of Leslie matrices.

Example 5

Consider the following Leslie matrix L and initial population matrix S_0:

$$\mathbf{L} = \begin{bmatrix} 0 & 4 & 4 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

a Determine

i $\mathbf{S}_5$ **ii** $\mathbf{S}_{10}$ **iii** $\mathbf{S}_{50}$

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine $\mathbf{S}_{25}$ and $\mathbf{S}_{26}$. Divide each age group population for $\mathbf{S}_{26}$ by the corresponding age group population for $\mathbf{S}_{25}$ and show that $\mathbf{S}_{26} \approx 1.1915\mathbf{S}_{25}$ and comment.

Solution

a **i** $\mathbf{S}_5 = \begin{bmatrix} 1000 \\ 250 \\ 62.5 \end{bmatrix}$ **ii** $\mathbf{S}_{10} = \begin{bmatrix} 2500 \\ 531.25 \\ 218.75 \end{bmatrix}$ **iii** $\mathbf{S}_{50} = \begin{bmatrix} 2\,777\,063 \\ 582\,688.05 \\ 244521.18 \end{bmatrix}$

Finding the total populations according to this model

i $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{S}_5 = [1312.5]$. The population is approximately 1312 after 5 years.

ii $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{S}_{10} = [3250]$. The population is approximately 3250 after 10 years.

iii $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{S}_{50} = [3\,604\,272.2]$. The population is approximately 3 604 272 after 50 years.

A numerical investigation reveals that the population increases without bound.

b We calculate $\mathbf{S}_{25}$ and $\mathbf{S}_{26}$:

$$\mathbf{S}_{25} = \mathbf{L}^{25}\mathbf{S}_0 = \begin{bmatrix} 34\,781.25 \\ 7297.85 \\ 3062.50 \end{bmatrix}, \quad \mathbf{S}_{26} = \mathbf{L}^{26}\mathbf{S}_0 = \begin{bmatrix} 41\,441.41 \\ 8695.31 \\ 3648.93 \end{bmatrix}$$

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Then we can find the rate of increase in each age group during the 26th time period:

$$\frac{41\,441.41}{34\,781.25} \approx \frac{8695.31}{7297.85} \approx \frac{3648.93}{3062.50} \approx 1.1915$$

This suggests that the age-group proportions have stabilised after the 25 time periods.

The long-term growth rate is approximately 1.19. (That is, after a certain stage, the population is increasing by 19% each time period.)

Note: Try different entries in S_0 to see if you get the same behaviour. The long-term growth rate is largely dependent on the Leslie matrix L.

Limiting behaviour of Leslie matrices

Often we will find that, after a long enough time, the proportion of the population in each age group does not change from one time period to the next. This happens if we can find a real number k such that $\mathbf{LS_n} = k\mathbf{S_n}$ for some sufficiently large n. This does not happen with every Leslie matrix as we see in Example 7.

Example 6

Consider the following Leslie matrix L and initial population matrix S_0:

$$\mathbf{L} = \begin{bmatrix} 0 & 2.3 & 0.4 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_0 = \begin{bmatrix} 1600 \\ 800 \\ 200 \end{bmatrix}.$$

a Determine

i $\mathbf{S}_{10}$ **ii** $\mathbf{S}_{12}$ **iii** $\mathbf{S}_{15}$

b The rate of increase of the population is a constant and each of the age group populations increase in the same way. Find this rate by comparing $\mathbf{S}_{14}$ and $\mathbf{S}_{15}$.

c Confirm the ratio of the age group populations stays constant for $\mathbf{S}_0, \mathbf{S}_1$ and $\mathbf{S}_{10}$ at 8 : 4 : 1.

Solution

a **i** $\mathbf{S}_{10} = \begin{bmatrix} 9906.78 \\ 4953.39 \\ 1238.35 \end{bmatrix}$ **ii** $\mathbf{S}_{14} = \begin{bmatrix} 20542.70 \\ 10271.35 \\ 2567.84 \end{bmatrix}$ **iii** $\mathbf{S}_{15} = \begin{bmatrix} 24651.24 \\ 12325.62 \\ 3081.4043 \end{bmatrix}$

b By comparing $\mathbf{S}_{14}$ and $\mathbf{S}_{15}$,

$$\frac{24651.24}{20542.70} = \frac{12325.62}{10271.35} = \frac{3081.4043}{2567.84} \approx 1.2$$

we find that the growth rate is 1.2.

c $8 : 4 : 1 = 1600 : 800 : 200 \approx 9906.78 : 4953.39 : 1238.35$

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Example 7

Consider the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{S}_0$:

$$L = \begin{bmatrix} 0 & 0 & b_3 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

Investigate the long-term behaviour of the population if:

a $b_3 = 8$ **b** $b_3 = 4$ **c** $b_3 = 10$

Solution

a Let $b_3 = 8$. Use your calculator to store the matrices $\mathbf{L}$ and $\mathbf{S}_0$. Then compute:

$$\mathbf{S}_1 = \mathbf{L}\mathbf{S}_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad \mathbf{S}_2 = \mathbf{L}^2\mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \\ 125 \end{bmatrix}, \quad \mathbf{S}_3 = \mathbf{L}^3\mathbf{S}_0 = \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

The population will continue to cycle through these three states; this is because $\mathbf{L}^3 = \mathbf{I}$.

b Let $b_3 = 4$. Then a numerical investigation suggests that the population decreases over the long term:

$$\mathbf{S}_1 = \mathbf{L}\mathbf{S}_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad S_5 = \mathbf{L}^5\mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \\ 62.5 \end{bmatrix}, \quad \mathbf{S}_{50} = \mathbf{L}^{50}\mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \\ 0.0019 \end{bmatrix}$$

c Let $b_3 = 10$. Then a numerical investigation suggests that the population increases over the long term:

$$\mathbf{S}_1 = \mathbf{L}\mathbf{S}_0 = \begin{bmatrix} 0 \\ 250 \\ 0 \end{bmatrix}, \quad \mathbf{S}_5 = \mathbf{L}^5\mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \\ 156.25 \end{bmatrix}, \quad \mathbf{S}_{50} = \mathbf{L}^{50}\mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \\ 4440.89 \end{bmatrix}$$

Note: A population can increase, decrease, become constant or oscillate.

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Exercise 9B

Example 3

SF

1 Use the Leslie matrix and initial state matrix below to answer the following questions.

From age group

$$\mathbf{L} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1.9 & 2.1 & 1.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \text{ To age group} \qquad \mathbf{S}_0 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$$

a Write down

i The birth rate for age group 2 **ii** The survival rate for age group 3

b Complete the life cycle diagram for this Leslie matrix.

c Evaluate the following population state matrices.

i $\mathbf{S}_1$ **ii** $\mathbf{S}_3$ **iii** $\mathbf{S}_{20}$

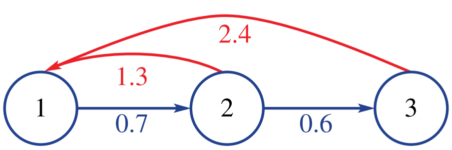
d Given that $\mathbf{S}_7 = \begin{bmatrix} 2613 \\ 1200 \\ 485 \\ 168 \end{bmatrix}$ determine $\mathbf{S}_8$. Give your values correct to the nearest whole number.

2 Complete the life cycle diagram corresponding to each of the following Leslie matrices:

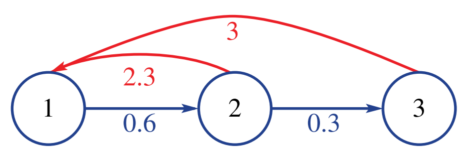
a $\begin{bmatrix} 0 & 2.9 & 3.1 & 2.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0.42 \\ 0.6 & 0 & 0 \\ 0 & 0.75 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 0 & 3 & 8 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \end{bmatrix}$

3 Construct the Leslie matrix corresponding to each life cycle diagram.

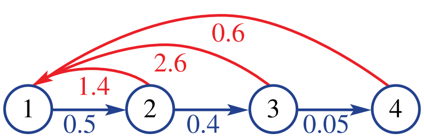
a



b



c



Example 4

4 Information about a population of female kangaroos in a particular area is given in the following table.

Age group (years)	0 – 4	4 – 8	8 – 12	12 – 16	16 – 20
Initial population	15	20	30	15	10
Birth rates	0	0.2	0.9	1.1	0
Survival rates	0.8	0.9	0.7	0.8	0

a Write down the initial population state matrix, $\mathbf{S}_0$.

b Write down the Leslie matrix.

c Complete the life cycle diagram for this Leslie matrix.

d Determine the population state matrix after

i one year, $(\mathbf{S}_1)$ **ii** after 5 years, $(\mathbf{S}_5)$.

e Determine the number of 4 – 8 year old female kangaroos in the population after 5 years.

f State the initial total population.

g Use multiplication of state matrices by the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ to determine the total population after

i one year **ii** 5 years **iii** 10 years.

h It is suggested that the population is increasing by about 10% per annum. Calculate each of the following and comment

i 1.1×90 **ii** $1.1^5 \times 90$ **iii** $1.1^{10} \times 90$.

5 Information about a population of female locusts is given in the following table.

Stage	Eggs	Nymphs	Adults
Initial population	0	0	50
Birth rates	0	0	1000
Survival rates	0.02	0.05	0

a Write down the initial population state matrix, $\mathbf{S}_0$.

b Write down the Leslie matrix.

c Construct a time life-cycle transition diagram for this Leslie matrix.

d Determine the population state matrix after

i one year $(\mathbf{S}_1)$ **ii** 3 years $(\mathbf{S}_3)$ **iii** 4 years $(\mathbf{S}_4)$.

e If the initial population is now:

Stage	Eggs	Nymphs	Adults
Initial population	50	100	50

determine the populations of each after

i one year $(\mathbf{S}_1)$ **ii** 3 years $(\mathbf{S}_3)$ **iii** 4 years $(\mathbf{S}_4)$.

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Example 5

6 Consider the following Leslie matrix $\mathbf{L}$ and initial population state matrix $\mathbf{S}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & 2 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_0 = \begin{bmatrix} 204 \\ 96 \\ 23 \end{bmatrix}$$

a Determine

i $\mathbf{S}_5$ **ii** $\mathbf{S}_{10}$ **iii** $\mathbf{S}_{20}$

Premultiply each of these state matrices by $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ to calculate the total populations at each of these stages and comment.

b Determine $\mathbf{S}_{20}$ and $\mathbf{S}_{21}$. Divide each age group population for $\mathbf{S}_{21}$ by the corresponding age group population for $\mathbf{S}_{20}$ and show that $\mathbf{S}_{21} \approx 1.057\mathbf{S}_{20}$ and comment.

Example 6

7 A Leslie matrix that models a certain population of female animals is

$$\mathbf{L} = \begin{bmatrix} 0 & 2.5 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

where the animals have a maximum life span of 9 years, and the population has been divided into three age groups of 3 years each.

a Assume that each age group initially consists of 400 females. What is the number of females in each age group after:

i 3 years **ii** 6 years **iii** 9 years

b Now assume that the initial population is 1200 and $\mathbf{S}_0 = \begin{bmatrix} 767 \\ 362 \\ 71 \end{bmatrix}$. Determine $\mathbf{S_n}$ after

i 3 years **ii** 6 years **iii** 9 years

c Calculate

i $1.27\mathbf{S}_0$ **ii** $1.27^2\mathbf{S}_0$ **iii** $1.27^3\mathbf{S}_0$

Compare these answers to the answers of part **b**.

Example 7

8 Consider the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{S}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_0 = \begin{bmatrix} 1200 \\ 0 \\ 0 \end{bmatrix}$$

a Determine:

i $\mathbf{LS}_0$ **ii** $\mathbf{L}^2\mathbf{S}_0$ **iii** $\mathbf{L}^3\mathbf{S}_0$

b Comment on these results in terms of the population behaviour. Try using a different initial population matrix S_0.

c Now investigate for each of the following Leslie matrices. Comment on population increase or decrease.

i $\mathbf{L} = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$ **ii** $\mathbf{L} = \begin{bmatrix} 0 & 0 & 15 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$

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CU

9 A Leslie matrix that models a certain population of female insects is

$$\mathbf{L} = \begin{bmatrix} 0 & 3 & 2 & 2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

where the insects have a maximum life span of 4 months, and the population has been divided into four age groups of 1 month each.
Assume that each age group initially consists of 400 females. What is the number of females in each age group after:

a 1 month **b** 2 months **c** 3 months?

10 For a certain species of fish, we consider three age groups each of one year in length. These fish reproduce only during their third year and then die. Assume that 20% of fish survive their first year and that 50% of these survivors make it to reproduction age. The initial population consists of 1000 newborns.

a Investigate what happens for each of the following values of b_3:

i $b_3 = 10$ **ii** $b_3 = 15$ **iii** $b_3 = 6$

b For $b_3 = 20$, determine the long-term growth rate.

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Review

Chapter summary

Dominance matrices

- A **dominance matrix**, $\mathbf{D}$, is an $n \times n$ matrix that represents a competition between the members of a group of size n. Entry d_{ij} in row i and column j of $\mathbf{D}$ is given by

$$d_{ij} = \begin{cases} 1 & \text{if member } i \text{ defeats member } j \\ 0 & \text{otherwise} \end{cases}$$

- We find the total of each row of $\mathbf{D}$ to obtain a dominance score for each group member. We can also find dominance scores using matrices such as $\mathbf{D} + \mathbf{D}^2$ or $\mathbf{D} + \frac{1}{2}\mathbf{D}^2 + \frac{1}{3}\mathbf{D}^3$.

Leslie matrices

- An $m \times m$ **Leslie matrix** has the form

$$\mathbf{L} = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{m-1} & 0 \end{bmatrix}$$

where:

- m is the number of age groups being considered
- s_i is the proportion of the population in age group i that progress to age group $i + 1$
- b_i is the average number of female offspring from a mother in age group i during one time period.

- The population matrix $\mathbf{S_n}$ is an $m \times 1$ matrix representing the size of each age group after the n time periods. This is calculated using a recursive formula

$\mathbf{S}_0$ is the initial state matrix, $\mathbf{S_{n+1}} = \mathbf{LS_n}$

or the explicit rule

$\mathbf{S_n} = \mathbf{L^n}\mathbf{S}_0$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

9A **1** I can work with dominance matrices to achieve rankings. ☐

See Example 1, Example 2 and Questions 1 and 3

9B **2** I can use a Leslie matrix to investigate the growth of population . ☐

See Example 3, Example 4 and Questions 1 and 4

9B **3** I can use a Leslie matrix to investigate long-term growth in a population. ☐

See Example 5, Example 6, Example 7 and Questions 6, 7 and 8

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Short-response questions (Technology-active)

1 A certain population can be modelled by the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{N}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & 0.1 & 0.2 & 0.2 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_0 = \begin{bmatrix} 200 \\ 200 \\ 200 \\ 200 \end{bmatrix}$$

a Calculate $\mathbf{S}_1$ and $\mathbf{S}_2$.

b Determine the rate of decrease of the population.

c How many time periods does it take for the total population to drop below 100?

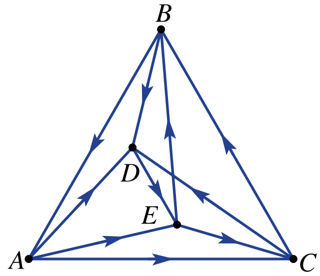
2 Players A, B, C, D and E competed in a tennis tournament. The one-step dominance matrix $\mathbf{M}$ and two-step dominance matrix $\mathbf{M}^2$ are given below:

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{M}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 2 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

a Describe the outcomes of the 10 matches.

b Determine $\mathbf{M} + \mathbf{M}^2$ and hence rank the five players.

3 The diagram on the right shows the outcomes of a round-robin chess tournament between five players, A, B, C, D and E. For example, the arrow pointing from B to A indicates that B defeated A.



a Determine the one-step dominance matrix, $\mathbf{D}$.

b Rank the players based on the matrix $\mathbf{D} + \frac{1}{2}\mathbf{D}^2$.

4 A particular species of insect has a maximum life span of 3 months. The female insects reproduce only in the third month of life. When we consider three age groups, each of length one month, the Leslie matrix takes the following form:

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & m^3 \\ a^3 & 0 & 0 \\ 0 & b^3 & 0 \end{bmatrix}$$

It can be seen that the long-term growth rate of the population is mab.

Review

CU

a Express c and d in terms of m, a and b if

$$\mathbf{L}\begin{bmatrix}1\\c\\d\end{bmatrix} = mab\begin{bmatrix}1\\c\\d\end{bmatrix}$$

b Consider the situation when $a = b = \frac{1}{2}$ and $m = 1$.

i Write down the long-term growth rate mab.

ii Assume that the initial female population is 7000. Determine $\mathbf{S}_0$ given that the sizes of the three age groups are in the ratio $1 : c : d$. (Use the expressions for c and d from part **a**.)

iii Determine $\mathbf{S}_1$ and $\mathbf{S}_2$.

iv Describe the situation. After how long will the total female population drop below 10?

c Consider the situation when $a = b = \frac{1}{2}$ and $m = 4$.

i Write down the long-term growth rate mab.

ii Assume that the initial female population is 7300. Determine $\mathbf{S}_0$ given that the sizes of the three age groups are in the ratio $1 : c : d$.

iii Determine $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ and $\mathbf{S}_4$. Comment.

d Consider the situation when $a = b = \frac{1}{2}$ and $m = 8$. Describe the fluctuation in the population over time.

5 Five types of dog food are compared by a group of veterinarians. The comparisons are done in pairs. The results can be seen in the following dominance matrix:

CF

$$\begin{array}{r c} & \begin{array}{ccccc}\text{DogDelite} & \text{CanineCandy} & \text{Doggy+} & \text{SuperDog} & \text{EasyFeed}\end{array} \\ \begin{array}{r}\text{DogDelite}\\ \text{CanineCandy}\\ \text{Doggy+}\\ \text{SuperDog}\\ \text{EasyFeed}\end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

a Write down all the preferences from the matrix. For example: DogDelite is preferred to Doggy+.

b Determine a ranking of the five dog foods. (Do not consider powers of the dominance matrix.)

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6 The following table represents a study of a particular population of marsupials, which has been divided into eight age groups. The table gives the initial population, birth rate and survival rate for each age group.

Age group	1	2	3	4	5	6	7	8
Initial population	0	100	100	50	0	0	0	0
Birth rate	0	0.1	0.9	0.2	0	0	0	0
Survival rate	0.98	0.95	0.95	0.9	0.7	0.5	0.1	0

a Write down the Leslie matrix for this population.

b Calculate $\mathbf{S}_2$ and $\mathbf{S}_3$.

c Estimate the long-term growth rate of the population.

7 The life cycle of a type of insect can be descibed by a Leslie matrix. The stages of life which are used are

- Egg E
- Juvenile J
- Young adult Y
- adult A

From stage of life

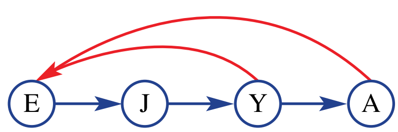
$$\mathbf{L} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} E & J & Y & A \end{array}$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 20 & 30 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}\begin{array}{c} E \\ J \\ Y \\ A \end{array} \quad \text{To next stage of life}$$

The initial population is described by the 4×1 state matrix $\mathbf{S}_0 = \begin{bmatrix} 897 \\ 438 \\ 43 \\ 2 \end{bmatrix}\begin{array}{c} E \\ J \\ Y \\ A \end{array}$

The time period in this model is one week and the state matrix after n weeks can be determined by $\mathbf{S_{n+1}} = \mathbf{LS_n}$.

a From the Leslie matrix complete the life-cycle diagram.



b How many insect eggs are there after one week?

c How many weeks pass before there are more than 1000 eggs?

d What percentage of young adult insects become adult insects each week?

e How many insects of every type (including eggs) are there after

i 8 weeks **ii** 9 weeks

Give answers correct to the nearest whole number.

f It is known that after some weeks the rate of increase per week of the entire population (including eggs) is very close to being a constant. Use the results of **f** to give an estimate of this rate per week as a percentage correct to the nearest percent.(That is, in the form $a\%$, where a is a whole number)

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Review

Multiple-choice questions (technology-active)

1 Four teams, A, B, C and D, competed in a round-robin competition where each team played each of the other teams once. There were no draws. The results are shown in the matrix below.

$$\text{winner}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \textit{loser} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & x & y & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & z & 1 & 0 \end{bmatrix} \end{array}$$

The values of x, y and z are

A $x = 0, y = 0, z = 0$

B $x = 0, y = 1, z = 0$

C $x = 1, y = 0, z = 0$

D $x = 1, y = 0, z = 1$

2 Four teams, X, Y, Z and W, competed in a round-robin competition where each team played each of the other teams once. There were no draws. The results are shown in the matrix below.

$$\text{winner}\quad \begin{array}{c} \\ X \\ Y \\ Z \\ W \end{array}\begin{array}{c} \text{loser} \\ \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ a & 0 & 0 & 1 \\ 0 & 1 & 0 & c \\ b & 0 & 1 & 0 \end{bmatrix} \end{array}$$

A '1' in the matrix shows that the team named in that row defeated the team named in that column. In this matrix, the values of a, b and c are

A $a = 1, b = 0, c = 0$

B $a = 0, b = 1, c = 0$

C $a = 1, b = 0, c = 1$

D $a = 0, b = 1, c = 1$

3 Four soccer teams, X, Y, Z and W, compete in a round-robin competition. In each game, there is a winner and a loser. To decide the winner of the tournament, the sum of the one-step dominance matrix, $\mathbf{D}$, and the two-step dominance matrix, $\mathbf{D}^2$, is found. This sum is,

$$\mathbf{D} + \mathbf{D}^2 = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Which one of the following is the correct one-step dominance for this tournament?

A $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

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4 The matrix gives the results of a table tennis round robin competition between five players: A, B, C, D and E. A '1' indicates a win of 'row' over 'column'.

$$\text{Winner}\quad \begin{array}{c} \text{Loser} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \end{array}$$

When the sum of the one-step and two-step dominances is used to rank the players in this competition, the ranking is:

A B, E, D, A, C **B** B, E, C, D, A
C B, E, D, C, A **D** E, B, D, A, C

5 The Leslie matrix for a certain endangered species is:

$$\mathbf{L} = \begin{bmatrix} 0.9 & 2.5 & 0.4 \\ 0.3 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$$

Some of the species were moved into a sanctuary. The initial female population in the sanctuary is given by

$$\mathbf{S}_0 = \begin{bmatrix} 130 \\ 40 \\ 20 \end{bmatrix}$$

The expected size of the total female population after 7 years is closest to

A 1000 **B** 1500 **C** 2000 **D** 2500

6 The Leslie matrix $\mathbf{L} = \begin{bmatrix} 0 & 2 & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$ satisfies the matrix equation

$$\mathbf{L}\begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 2 \end{bmatrix}$$

The values of b, c and d are

A $b = 2, d = \frac{1}{4}, c = \frac{1}{2}$ **B** $b = 2, d = \frac{1}{2}, c = \frac{1}{4}$
C $b = 4, d = \frac{1}{4}, c = \frac{1}{2}$ **D** $b = 4, d = \frac{1}{2}, c = \frac{1}{4}$

7 A population of birds is modelled by using the Leslie matrix

$$\mathbf{L} = \begin{bmatrix} 0 & 2 & 1.5 \\ 0.44 & 0 & 0 \\ 0 & 0.55 & 0 \end{bmatrix}$$

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Review

The growth has reached the point where the rates of growth of the different age groups of the population are constant and the state matrix at this point is $\mathbf{S}_\ell = \begin{bmatrix} 1000 \\ 400 \\ 200 \end{bmatrix}$. The rate of growth per time period is

A 10% **B** 11%

C 12% **D** 13%

8 There are five hens in a coop. Their owner calls them Alpha, Beta, Gamma, Delta and Epsilon. There is a pecking order in the coop, and the following dominance matrix, $\mathbf{M}$, was formed by the owner:

$$\mathbf{M} = \begin{array}{r} \\ \text{Alpha} \\ \text{Beta} \\ \text{Gamma} \\ \text{Delta} \\ \text{Epsilon} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Alpha} & \text{Beta} & \text{Gamma} & \text{Delta} & \text{Epsilon} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Based on the matrix $\mathbf{M} + \mathbf{M}^2$, which of the following best describes the pecking order in the coop?

A Beta, Epsilon, Delta, Alpha, Gamma **B** Beta, Epsilon, Gamma, Delta, Alpha

C Beta, Epsilon, Delta, Gamma, Alpha **D** Epsilon, Beta, Delta, Alpha, Gamma

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10 Revision of Unit 3

Revision

10A Short-response questions

Technology-free short-response questions

Unit 3 Topic 1: Further complex numbers

SF

1 Determine all solutions of $z^4 - z^2 - 12 = 0$ for $z \in \mathbb{C}$.

2 Consider $z = \dfrac{\sqrt{3} - i}{1 - i}$. Determine $\operatorname{Arg} z$.

3 Let $P(z) = z^5 - 6z^3 - 2z^2 + 17z - 10$. Given that $P(1) = P(2) = 0$, solve the equation $P(z) = 0$ for $z \in \mathbb{C}$.

4 **a** Solve the equation $z^3 - 2z^2 + 2z - 1 = 0$ for $z \in \mathbb{C}$.

b Write the solutions in polar form.

c Show the solutions on an Argand diagram.

5 The point $(-1, 3)$ is rotated about the origin by angle $\dfrac{\pi}{4}$ anticlockwise. By multiplying two complex numbers, determine the image of the point.

6 Let $z = \sqrt{3} + i$. Plot z, z^2 and z^3 on an Argand diagram.

CF

7 Simplify $\dfrac{\cos(2\theta) + i\sin(2\theta)}{\cos(3\theta) + i\sin(3\theta)}$, writing your answer in Cartesian form.

8 **a** Show that $z - 1 - i$ is a factor of $f(z) = z^3 - (5 + i)z^2 + (17 + 4i)z - 13 - 13i$.

b Hence factorise $f(z)$.

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Revision

9 Let $f(z) = z^2 + aiz + b$, where a and b are real numbers.

a Use the quadratic formula to show that the equation $f(z) = 0$ has imaginary solutions if and only if $b \geq -\dfrac{a^2}{4}$. (Imaginary solutions have no real part.)

b Hence solve each of the following:

i $z^2 + 2iz + 1 = 0$ **ii** $z^2 - 2iz - 1 = 0$ **iii** $z^2 + 2iz - 2 = 0$

CF

10 **a** If the equation $z^3 + az^2 + bz + c = 0$ has solutions $-1 + i$, -1 and $-1 - i$, determine the values of a, b and c.

b If $\sqrt{3} + i$ and $-2i$ are two of the solutions to the equation $z^3 = w$, where w is a complex number, determine the third solution.

11 Solve the equation $z^5 = 1 + i$ for z, giving your solutions in polar form. Illustrate the solutions on an Argand diagram.

12 A circle on an Argand diagram has equation $|z - c| = r$. The circle passes through the points $2 + i$, $2 - i$ and i. Determine the values of c and r.

13 The equation $z^3 - 5z^2 + 16z + k = 0$ has a solution $z = 1 + ai$, where $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$. Determine the values of a and k.

14 Consider the complex numbers $z_1 = \sqrt{3} - i$ and $z_2 = -1 - i$.

a Express z_1 and z_2 in polar form.

b Determine $\dfrac{z_1}{z_2}$ in polar form.

c Determine the complex conjugate of $\dfrac{z_1}{z_2}$ in polar form.

d On a single Argand diagram, sketch the graphs of:

i $|z - z_1| = 2$ **ii** $\operatorname{Arg}(z - z_2) = \dfrac{\pi}{4}$

15 Let $z_1 = 1 + \sqrt{3}i$.

CU

a **i** Write z_1 in polar form.

ii Write z_1^n in polar form, where n is an integer.

iii Determine the integer values of n for which z_1^n is real.

iv Determine the integer values of n for which z_1^n is imaginary.

b Express z_1^2 and z_1^3 in Cartesian form.

c Given that $z_1 = 1 + \sqrt{3}i$ is a solution of the equation $2z^3 + az^2 + bz + 20 = 0$, determine the values of the real numbers a and b.

d For these values of a and b, solve the equation $2z^3 + az^2 + bz + 20 = 0$.

16 The vertices A, B, C and D of a square, taken anticlockwise, are drawn on an Argand diagram. The points A and B are $-1 + 4i$ and -3 respectively.

a Determine the complex numbers corresponding to C and D.

b Determine the complex number corresponding to the centre of the square.

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CU

17 Let $z, w \in \mathbb{C}$. Prove that $|z+w|^2 - |z-\overline{w}|^2 = 4\,\mathrm{Re}(z)\,\mathrm{Re}(w)$.

18 Let ω be a cube root of unity with $\omega \neq 1$, and let n be a natural number.

a Prove that if n is a multiple of 3, then $1 + \omega^n + \omega^{2n} = 3$.

b Prove that if n is not a multiple of 3, then $1 + \omega^n + \omega^{2n} = 0$.

Unit 3 Topic 2: Mathematical induction and trigonometric proofs

SF

19 Prove by mathematical induction that, for every positive integer n:

a $7^n + 2$ is divisible by 3 **b** $5^{2n} + 3n - 1$ is divisible by 9.

20 Use induction to prove that

$$1 \times 4 + 2 \times 5 + 3 \times 6 + \cdots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$$

for each natural number n.

CF

21 Prove by mathematical induction that, for every positive integer n:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

22 Prove by mathematical induction that

$$\sum_{r=1}^{n} r(2r+1) = \frac{1}{6}n(n+1)(4n+5)$$

for all positive integers n.

23 **a** Prove by mathematical induction that $(1+i)^{4n} = (-4)^n$, where n is a natural number.

b Now give a proof of the same result by using De Moivre's theorem.

24 Use induction to prove that $3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by 17, for all $n \in \mathbb{N}$.

CU

25 Use mathematical induction to prove that, if $\mathrm{cis}(\theta) \neq 1$, then

$$1 + \mathrm{cis}(\theta) + \mathrm{cis}(2\theta) + \mathrm{cis}(3\theta) + \cdots + \mathrm{cis}(n\theta) = \frac{1 - \mathrm{cis}((n+1)\theta)}{1 - \mathrm{cis}(\theta)}$$

for each natural number n.

26 Prove by induction that $m^4 - 1$ is divisible by 16 for all odd integers, $m \geq 3$.

27 Consider the sequence of numbers $0, 3, 15, 42, \ldots$. It is generated by the recursive formula $t_n = t_{n-1} + 3n^2$ in the following way:

$t_0 = 0$ $\quad t_1 = t_0 + 3 \times 1^2 = 3$ $\quad t_2 = t_1 + 3 \times 2^2 = 15$ $\quad$ etc.

Prove that $t_n = \frac{1}{2} \times n(2n+1)(n+1)$

28 Prove that $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ for all $n \in \mathbb{N}$.

29 Prove by induction that $\displaystyle\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2 \sin x}$

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Revision

30 Prove by induction that $\cos 2x \times \cos 2^2x \times \cdots \times \cos 2^n x = \dfrac{\sin 2^{n+1}x}{2^n \sin 2x}$, where $\sin 2x \neq 0$ for all positive integer n. CU

Unit 3 Topic 3: Vectors in two and three dimensions

31 Points $A(2, 1, 2)$, $B(-3, 2, 5)$ and $C(4, 5, -2)$ are three vertices of a parallelogram. The fourth vertex of the parallelogram is at point D. Show that there are three possible locations for the point D, and determine their coordinates. SF

32 Resolve the vector $3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ into two components: one parallel to the vector $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and the other perpendicular to it.

33 Let O be the origin and consider points $A(2, 2, 1)$ and $B(1, 2, 1)$.

a Determine $\overrightarrow{AB}$.

b Determine $\cos(\angle AOB)$.

c Determine the area of triangle AOB.

34 Consider the vectors $\boldsymbol{a} = -2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + m\hat{\boldsymbol{k}}$, $\boldsymbol{b} = \hat{\boldsymbol{i}} - \dfrac{3}{2}\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

a Determine the values of m for which $|\boldsymbol{a}| = \sqrt{38}$.

b Determine the value of m such that $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$.

c Determine $-2\boldsymbol{b} + 3\boldsymbol{c}$.

35 Points A and B have position vectors $\overrightarrow{OA} = \hat{\boldsymbol{i}} + \sqrt{3}\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{k}}$. Point P lies on AB with $\overrightarrow{AP} = \lambda\overrightarrow{AB}$.

a Show that $\overrightarrow{OP} = (1 + 2\lambda)\hat{\boldsymbol{i}} + \sqrt{3}(1 - \lambda)\hat{\boldsymbol{j}} - 4\lambda\hat{\boldsymbol{k}}$.

b Hence determine λ if OP is the bisector of $\angle AOB$.

36 **a** Determine a unit vector perpendicular to the line $2y + 3x = 6$.

b Let A be the point $(2, -5)$ and let P be the point on the line $2y + 3x = 6$ such that AP is perpendicular to the line. Determine:

i $\overrightarrow{AP}$ **ii** $|\overrightarrow{AP}|$

37 Let $z = 1 + \sqrt{2}i$ and define the two complex numbers CF

$$p = z + \frac{1}{z} \quad \text{and} \quad q = z - \frac{1}{z}$$

a Express p and q in Cartesian form.

On an Argand diagram, let P and Q be the points representing p and q respectively. Let O be the origin, let M be the midpoint of PQ and let G be the point on the line segment OM with $OG = \frac{2}{3}OM$.

b Determine each of the following vectors in terms of $\boldsymbol{a} = \overrightarrow{OP}$ and $\boldsymbol{b} = \overrightarrow{OQ}$:

i $\overrightarrow{PQ}$ **ii** $\overrightarrow{OM}$ **iii** $\overrightarrow{OG}$ **iv** $\overrightarrow{GP}$ **v** $\overrightarrow{GQ}$

c Prove that $\angle PGQ$ is a right angle.

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Revision

Vector equations of lines and planes

SF

38 For each of the following, determine a vector equation of the line through the two points:

a $(0,0,0),\ (3,0,4)$ **b** $(0,2,1),\ (-1,3,4)$ **c** $(3,2,4),\ (0,4,-2)$

39 For each of the following, determine a vector equation of the plane that contains the three points:

a $(0,0,0),\ (1,2,3),\ (1,3,5)$

b $(2,-3,5),\ (3,-2,6),\ (1,-2,4)$

c $(3,2,4),\ (0,4,-2),\ (3,6,0)$

CF

40 **a** Determine the perpendicular distance between the parallel planes with equations $2x+2y+z=6$ and $2x+2y+z=10$.

b Determine the perpendicular distance from the point $P(1,0,1)$ to the plane with equation $x+2y+3z=6$.

c Determine the distance from the point $P(1,2,3)$ to the line given by $\dfrac{x+1}{2}=y=z-1$.

41 Determine the point of intersection of the lines ℓ_1 and ℓ_2 given by

$$\ell_1:\quad \boldsymbol{r}=2t\hat{\boldsymbol{i}}+(2-2t)\hat{\boldsymbol{j}}+(3-4t)\hat{\boldsymbol{k}},\qquad t\in\mathbb{R}$$
$$\ell_2:\quad \boldsymbol{r}=(3+s)\hat{\boldsymbol{i}}+(s-1)\hat{\boldsymbol{j}}+(4s-3)\hat{\boldsymbol{k}},\qquad s\in\mathbb{R}$$

42 Determine the coordinates of the point where the line through $(0,1,0)$ and $(1,0,1)$ meets the plane with equation:

a $x+y+z=1$ **b** $x+y+z=3$ **c** $x-y+z=1$

43 Consider the vectors $\boldsymbol{a}=2m\hat{\boldsymbol{i}}+3m\hat{\boldsymbol{j}}+6m\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$, where $m\in\mathbb{R}^+$.

a Given that $\boldsymbol{a}$ is a unit vector, determine the exact value of m.

b Using the value of m from part **a**, determine:

i $\boldsymbol{a}\cdot\boldsymbol{b}$ **ii** $\boldsymbol{a}\times\boldsymbol{b}$

44 Determine the length of the perpendicular from the point with coordinates $(4,0,1)$ to the plane with equation $3x+6y+2z=-7$.

45 **a** Determine the value of a for which the three planes $2x-y+5z=7$, $5x+3y-z=4$ and $3x+4y-6z=a$ intersect in a line.

b Determine a vector equation of this line.

46 Plane Π_1 has equation $\boldsymbol{r}\cdot(2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}})=1$. Plane Π_2 has equation $\boldsymbol{r}\cdot(-\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}})=2$. These two planes intersect in a line, ℓ.

a Determine the cosine of the acute angle between the planes Π_1 and Π_2.

b Determine a vector equation of the line ℓ.

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47 Consider the points $A(4, 1, -1)$, $B(0, 3, 3)$, $C(-4, -1, 1)$ and $D(0, -3, -3)$.

a Show that these points all lie on the plane with equation $x - 2y + 2z = 0$.

b Show that $ABCD$ is a square.

c Write a vector equation of the line through the point $P(0, 8, 5)$ that is perpendicular to the plane.

48 Determine the cosine of the acute angle between each of the following pairs of planes:

a $2x + 3y - 2z = 0, \quad x - y - z = 4$ **b** $4x + 3y + 2z = 5, \quad 2x - 4y + 3z = 6$

49 Describe the line that passes through the points $A(3, 5, 9)$ and $B(1, 9, 10)$ using:

a a vector equation **b** Cartesian equations **c** parametric equations.

50 Point A has coordinates $(2, 2, 1)$ and point B has coordinates $(1, 2, 1)$, relative to an origin O.

a Determine $\overrightarrow{AB}$.

b Determine $\cos(\angle AOB)$.

c Determine the area of triangle AOB.

CU

51 Points A, B, C and D have position vectors $\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $-\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $4\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ respectively.

a Prove that the triangle ABC is right-angled.

b Prove that the triangle ABD is isosceles.

c Show that BD passes through the midpoint, E, of AC and determine the ratio $BE : ED$.

52 The coordinates of the vertices of a parallelogram are $A(2, 1, 2)$, $B(-3, 2, 5)$, $C(4, 5, -2)$ and $D(9, 4, -5)$.

a Find the coordinates of the point where the diagonals of parallelogram $ABCD$ meet.

b Find $\cos(\angle BAC)$.

c Find the area of parallelogram $ABCD$.

53 $OABC$ is a parallelogram shown in the diagram and A and B are located at $(1, 6, 7)$ and $(10, 0, 10)$ respectively.

a Determine the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

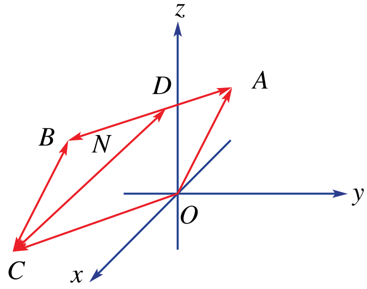
b D is a point on AB such that $\overrightarrow{AD} = \dfrac{1}{3}\overrightarrow{AB}$.

i Determine the vector $\overrightarrow{OD}$.

N is a point on CD such that $\overrightarrow{DN} = \dfrac{2}{5}\overrightarrow{DC}$.

ii Determine the vector $\overrightarrow{ON}$.

iii Show that N lies on the line OB.



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54 A parallelogram $OABC$ has one vertex at the origin O and two other vertices at the points $A(0, 2, 7)$ and $B(0, 3, 9)$. Determine the area of $OABC$.

55 Determine the area of the triangle PQR with vertices:

a $P(2, 2, 0)$, $Q(0, 0, 0)$ and $R(4, 3, 1)$ **b** $P(3, 2, 0)$, $Q(1, 2, 0)$ and $R(2, 3, 1)$

56 Determine a if the triangle with vertices $A(a, 1, 2)$, $B(1, 0, 1)$ and $C(0, 1, 1)$ has area 1.

Unit 3 Topic 4: Vector calculus

CF

57 The position of a particle at time t seconds, relative to an origin O, is given by

$$\boldsymbol{r}(t) = \sin(t)\,\hat{\boldsymbol{i}} + \frac{1}{2}\sin(2t)\,\hat{\boldsymbol{j}}, \quad t \geq 0$$

a Determine the velocity of the particle at time t.

b Determine the acceleration at time t.

c Determine an expression for the distance of the particle from the origin at time t in terms of $\sin(t)$.

d Determine an expression for the speed of the particle at time t in terms of $\sin(t)$.

e Determine the Cartesian equation of the path of the particle.

58 A particle moves according to the vector function $\boldsymbol{r}(t) = 3\sin(2t)\,\hat{\boldsymbol{i}} - 3\cos(2t)\,\hat{\boldsymbol{j}}$, where $\boldsymbol{r}(t)$ m is the position of the particle at time t seconds $(t \geq 0)$.

a Determine the Cartesian equation of the path of the particle. Describe the path, the direction of motion and the initial position of the particle.

b Determine the velocity and acceleration of the particle at time t seconds.

c Determine the speed of the particle.

59 The position vector of a particle moving relative to the origin at time t seconds is given by $\boldsymbol{r}(t) = 2\sec(t)\,\hat{\boldsymbol{i}} + \frac{1}{2}\tan(t)\,\hat{\boldsymbol{j}}$, for $t \in \left[0, \frac{\pi}{2}\right)$.

a Determine the Cartesian equation of the path.

b Determine the velocity of the particle at time t.

c Determine the speed of the particle when $t = \frac{\pi}{3}$.

60 A particle moves such that, at time t seconds, the velocity, $\boldsymbol{v}$ m/s, is given by $\boldsymbol{v} = e^{2t}\hat{\boldsymbol{i}} - e^{-2t}\hat{\boldsymbol{k}}$. Given that, at $t = 0$, the position of the particle is $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, determine the position at $t = \log_e 2$.

61 A particle has acceleration, $\boldsymbol{a}$ m/s^2, given by $\boldsymbol{a} = -g\hat{\boldsymbol{j}}$, where $\hat{\boldsymbol{j}}$ is a unit vector vertically upwards. Let $\hat{\boldsymbol{i}}$ be a horizontal unit vector in the plane of the particle's motion. The particle is projected from the origin with an initial speed of 20 m/s at an angle of 60° to the horizontal.

a Prove that the velocity, in m/s, at t seconds is given by $\boldsymbol{v} = 10\hat{\boldsymbol{i}} + (10\sqrt{3} - gt)\hat{\boldsymbol{j}}$.

b Hence determine the Cartesian equation of the path of the particle.

Revision

CF

62 The velocity, $\boldsymbol{v}$, of a particle at time t seconds is given by

$$\boldsymbol{v}(t) = -2\sin(2t)\,\hat{\boldsymbol{i}} + 2\cos(2t)\,\hat{\boldsymbol{j}}, \quad 0 \le t \le 2\pi$$

The particle moves in the horizontal plane. Let $\hat{\boldsymbol{i}}$ be the unit vector in the easterly direction and $\hat{\boldsymbol{j}}$ be the unit vector in the northerly direction. Determine:

a the position vector, $\boldsymbol{r}(t)$, given that $\boldsymbol{r}(0) = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

b the Cartesian equation of the path of the particle

c the time(s) when the particle is moving in the westerly direction.

63 A particle is projected from the origin such that its position vector, $\boldsymbol{r}(t)$ metres, after t seconds is given by

$$\boldsymbol{r}(t) = 14\sqrt{3}t\,\hat{\boldsymbol{i}} + \left(14t - \frac{g}{2}t^2\right)\hat{\boldsymbol{j}}$$

where $\hat{\boldsymbol{i}}$ is the unit vector in the direction of the x-axis, horizontally, and $\hat{\boldsymbol{j}}$ is the unit vector in the direction of the y-axis, vertically. The x-axis represents ground level. Determine:

a the time (in seconds) taken for the particle to reach the ground, in terms of g

b the Cartesian equation of the parabolic path

c the maximum height reached by the particle (in metres), in terms of g.

64 A particle travels on a path given by the Cartesian equation $y = x^2 + 2x$.

a Show that one possible vector representing the position of the particle is

$$\boldsymbol{r}(t) = (t - 1)\hat{\boldsymbol{i}} + (t^2 - 1)\hat{\boldsymbol{j}}$$

b Show that another possible vector representing the position of the particle is

$$\boldsymbol{r}(t) = (e^{-t} - 1)\hat{\boldsymbol{i}} + (e^{-2t} - 1)\hat{\boldsymbol{j}}$$

c Two particles travel simultaneously. At time $t \ge 0$, the positions of the two particles are given by

$$\boldsymbol{r}_1(t) = (t - 1)\hat{\boldsymbol{i}} + (t^2 - 1)\hat{\boldsymbol{j}}$$
$$\boldsymbol{r}_2(t) = (e^{-t} - 1)\hat{\boldsymbol{i}} + (e^{-2t} - 1)\hat{\boldsymbol{j}}$$

i Determine the initial positions of the two particles.

ii Show that the particles travel in opposite directions along the path $y = x^2 + 2x$.

iii Determine, correct to two decimal places, the point at which the two particles collide.

CU

65 Two particles A and B are projected simultaneously from a point O at ground level. The particles travel in the same plane. They are both projected with the same speed, but particle A is projected at an angle of $\alpha°$ to the horizontal and particle B is projected at an angle of $(90 - \alpha)°$ to the horizontal.

a Prove that the two particles return to ground level at the same point.

b Prove that, at any time during the flight, the line joining the two particles is inclined at 45° to the horizontal.

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CU

66 A hiker starts from a point defined by the position vector $-7\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and travels at the rate of 6 km/h along a line parallel to the vector $4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$. The units in the frame of reference are in kilometres.

a Determine the vector which represents the displacement of the hiker in 1 hour.

b Determine, in terms of position vectors, the position of the hiker after:

i 1 hour **ii** 2 hours **iii** t hours.

c The path of a cyclist along a straight road is defined simultaneously by the vector equation $\boldsymbol{b}(t) = (7t - 4)\hat{\boldsymbol{i}} + (9t - 1)\hat{\boldsymbol{j}}$.

i Determine the position of the hiker when she reaches the road.

ii Determine the time taken by the hiker to reach the road.

iii Determine, in terms of t, the distance between the hiker and the cyclist t seconds after the start.

iv Determine the shortest distance between the hiker and the cyclist, correct to two decimal places.

Unit 3 Topic 5: Further matrices

SF

67 Write the simultaneous equations $3x - 2y = 6$ and $2x + 5y = 7$ as a matrix equation. Hence solve for x and y.

68 Solve the following system of linear equations:

$$\begin{aligned} 2x - y + z &= 0 \\ y + 2z &= 1 \\ 2x + 5z &= 2 \end{aligned}$$

69 **a** Determine the determinant of the matrix $\mathbf{A} = \begin{bmatrix} k-2 & 6 \\ 1 & 2k \end{bmatrix}$ in terms of k.

b Hence determine the values of k for which the system of equations

$$\begin{aligned} (k-2)x + 6y &= 6 \\ x + 2ky &= 8 \end{aligned}$$

has a unique solution.

CF

70 For which value(s) of a will the following system of equations have no solutions?

$$\begin{aligned} ax + y + 2z &= 4 \\ x + 2y + z &= -3 \\ 2x - y - 2z &= 1 \end{aligned}$$

71 Consider the following system of equations:

$$\begin{aligned} ax + y + 2z &= 4 \\ 2x + 2y + 3z &= 1 \\ 2x - y - 4z &= b \end{aligned}$$

Determine the values of a and b such that this system has infinitely many solutions. Give the solutions in this case.

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72 Consider the simultaneous equations $ax + by = 3$ and $bx + ay = 4$, where a and b are constants. If this pair of equations has no solutions, then how must a and b be related? CF

Technology-active short-response questions

Unit 3 Topic 1: Further complex numbers

73 **a** Let $S_1 = \{z : |z| \le 2\}$ and $T_1 = \{z : \text{Im}(z) + \text{Re}(z) \ge 4\}$. CF

i On the same diagram, sketch S_1 and T_1, clearly indicating which boundary points are included.

ii Let $d = |z_1 - z_2|$, where $z_1 \in S_1$ and $z_2 \in T_1$. Determine the minimum value of d.

b Let $S_2 = \{z : |z - 1 - i| \le 1\}$ and $T_2 = \{z : |z - 2 - i| \le |z - i|\}$.

i On the same diagram, sketch S_2 and T_2, clearly indicating which boundaries are included.

ii If z belongs to $S_2 \cap T_2$, determine the maximum and minimum values of $|z|$.

74 **a** For $\alpha = 1 - \sqrt{3}i$, write the product of $z - \alpha$ and $z - \overline{\alpha}$ as a quadratic expression in z with real coefficients, where $\overline{\alpha}$ denotes the complex conjugate of α.

b **i** Express α in polar form.

ii Determine α^2 and α^3.

iii Show that α is a solution of the equation $z^3 - z^2 + 2z + 4 = 0$, and determine all three solutions of this equation.

c On an Argand diagram, plot the three points corresponding to the three solutions. Let A be the point in the first quadrant, let B be the point on the real axis and let C be the third point.

i Determine the lengths AB and CB.

ii Describe the triangle ABC.

75 **a** Determine the linear factors of $z^2 + 4$.

b Express $z^4 + 4$ as the product of two quadratic factors in $\mathbb{C}$.

c Show that:

i $(1 + i)^2 = 2i$ **ii** $(1 - i)^2 = -2i$

d Use the results of **c** to factorise $z^4 + 4$ into linear factors.

e Hence factorise $z^4 + 4$ into two quadratic factors with real coefficients.

76 Let $z = 2 + i$.

a Express z^3 in the form $x + yi$, where x and y are integers.

b Let the polar form of $z = 2 + i$ be $r(\cos\alpha + i\sin\alpha)$. Using the polar form of z^3, but without evaluating α, determine the value of:

i $\cos(3\alpha)$ **ii** $\sin(3\alpha)$

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77 The cube roots of unity are often denoted by 1, w and w^2, where $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

a **i** Illustrate these three numbers on an Argand diagram.

ii Show that $(w^2)^2 = w$.

b By factorising $z^3 - 1$, show that $w^2 + w + 1 = 0$.

c Evaluate:

i $(1 + w)(1 + w^2)$ **ii** $(1 + w^2)^3$

d Form the quadratic equation whose solutions are:

i $2 + w$ and $2 + w^2$ **ii** $3w - w^2$ and $3w^2 - w$

e Determine the possible values of the expression $1 + w^n + w^{2n}$ for $n \in \mathbb{N}$.

78 **a** Let $z^5 - 1 = (z - 1)P(z)$, where $P(z)$ is a polynomial. Determine $P(z)$ by division.

b Show that $z = \operatorname{cis}\left(\frac{2\pi}{5}\right)$ is a solution of the equation $z^5 - 1 = 0$.

c Hence determine another complex solution of the equation $z^5 - 1 = 0$.

d Determine all the complex solutions of $z^5 - 1 = 0$.

e Hence factorise $P(z)$ as a product of two quadratic polynomials with real coefficients.

79 **a** Two complex variables w and z are related by $w = \frac{az + b}{z + c}$, where $a, b, c \in \mathbb{R}$. Given that $w = 3i$ when $z = -3i$ and that $w = 1 - 4i$ when $z = 1 + 4i$, determine the values of a, b and c.

b Let $z = x + yi$. Show that there is a unique circle of centre $(4, 0)$ such that whenever $w = \bar{z}$, z lies on this circle, and state the radius of this circle.

80 **a** Use De Moivre's theorem to show that $(1 + i\tan\theta)^5 = \frac{\operatorname{cis}(5\theta)}{\cos^5(\theta)}$.

b Hence determine expressions for $\cos(5\theta)$ and $\sin(5\theta)$ in terms of $\tan\theta$ and $\cos\theta$.

c Show that $\tan(5\theta) = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.

d Use the result of **c** and an appropriate substitution to show that $\tan\left(\frac{\pi}{5}\right) = (5 - 2\sqrt{5})^{\frac{1}{2}}$.

81 **a** Express, in terms of θ, the solutions α and β of the equation $z + z^{-1} = 2\cos\theta$.

b If P and Q are points on the Argand diagram representing $\alpha^n + \beta^n$ and $\alpha^n - \beta^n$ respectively, show that PQ is of constant length for $n \in \mathbb{N}$.

82 Let S and T be the subsets of the complex plane given by

$$S = \left\{ z : \sqrt{2} \le |z| \le 3 \text{ and } \frac{\pi}{2} < \operatorname{Arg} z \le \frac{3\pi}{4} \right\}$$

$$T = \{ z : z\bar{z} + 2\operatorname{Re}(iz) \le 0 \}$$

a Sketch S on an Argand diagram.

b Determine all the elements of S of the form $z = x + yi$, where x and y are integers.

c On a separate diagram, sketch $S \cap T$.

Unit 3 Topic 3: Vectors in two and three dimensions

83 Points O, A B and P are such that $\overrightarrow{OA} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$, $\overrightarrow{OB} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\overrightarrow{OP} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$. CF

- **a** Determine the magnitude of angle AOB. (Give your answer in degrees correct to two decimal places.)
- **b** Determine the exact area of the parallelogram $OAPB$.

84 **a** Points A, B and P are collinear with B between A and P. The points A, B and P have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{r}$ respectively, relative to an origin O. If $\overrightarrow{AP} = \frac{3}{2}\overrightarrow{AB}$:

- **i** express $\overrightarrow{AP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
- **ii** express $\boldsymbol{r}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b The points A, B and C have position vectors $\hat{\boldsymbol{i}}$, $2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ respectively.

- **i** Determine $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
- **ii** Show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have equal magnitudes.
- **iii** Show that AB and BC are perpendicular.
- **iv** Determine the position vector of D such that $ABCD$ is a square.

c The triangle OAB is such that O is the origin, $\overrightarrow{OA} = 8\hat{\boldsymbol{i}}$ and $\overrightarrow{OB} = 10\hat{\boldsymbol{j}}$. The point P with position vector $\overrightarrow{OP} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ is equidistant from O, A and B and is at a distance of 2 above the triangle. Determine x, y and z.

85 $OACB$ is a trapezium with OB parallel to AC and $AC = 2OB$. Point D is the point of trisection of OC nearer to O.

a If $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$, determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{BD}$ **iii** $\overrightarrow{DA}$

b Hence prove that A, D and B are collinear.

86 **a** If $\boldsymbol{a} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 12\hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$, determine: CU

- **i** the magnitude of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ to the nearest degree
- **ii** the vector resolute of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$
- **iii** real numbers x, y and z such that $x\boldsymbol{a} + y\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 30\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$.

b In triangle OAB, $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. Points P and Q are such that P is the point of trisection of AB nearer to B and $\overrightarrow{OQ} = 1.5\overrightarrow{OP}$.

- **i** Determine an expression for $\overrightarrow{AQ}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- **ii** Show that $\overrightarrow{OA}$ is parallel to $\overrightarrow{BQ}$.

87 **a** Show that if $2a + b - c = 0$ and $a - 4b - 2c = 0$, then $a : b : c = 2 : -1 : 3$.

b Assume that the vector $x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ is perpendicular to both $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$. Establish two equations in x, y and z, and determine the ratio $x : y : z$.

c Hence, or otherwise, determine any vector $\boldsymbol{v}$ which is perpendicular to both $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$.

d Show that the vector $4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$ is also perpendicular to vector $\boldsymbol{v}$.

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e Determine the values of s and t such that $4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$ can be expressed in the form $s(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$.

f Show that any vector $\boldsymbol{r} = s(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ is perpendicular to vector $\boldsymbol{v}$ (where $s \in \mathbb{R}$ and $t \in \mathbb{R}$).

88 In the quadrilateral $ABCD$, the points X and Y are the midpoints of the diagonals AC and BD respectively.

a Show that $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BX}$.

b Show that $\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{DC} = 4\overrightarrow{YX}$.

89 The position vectors of the vertices of a triangle ABC, relative to a given origin O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Let P and Q be points on the line segments AB and AC respectively such that $AP : PB = 1 : 2$ and $AQ : QC = 2 : 1$. Let R be the point on the line segment PQ such that $PR : RQ = 2 : 1$.

a Prove that $\overrightarrow{OR} = \frac{4}{9}\boldsymbol{a} + \frac{1}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$.

b Let M be the midpoint of AC. Prove that R lies on the median BM.

c Determine $BR : RM$.

90 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O. The point C lies on AB between A and B, and is such that $AC : CB = 2 : 1$, and D is the midpoint of OC. The line AD meets OB at E.

a Determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OC}$ **ii** $\overrightarrow{AD}$

b Determine the ratios:

i $OE : EB$ **ii** $AE : ED$

91 The position vectors of the vertices A, B and C of a triangle, relative to an origin O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. The side BC is extended to D so that $BC = CD$. The point X divides side AB in the ratio $2 : 1$, and the point Y divides side AC in the ratio $4 : 1$. That is, $AX : XB = 2 : 1$ and $AY : YC = 4 : 1$.

a Express in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

i $\overrightarrow{OD}$ **ii** $\overrightarrow{OX}$ **iii** $\overrightarrow{OY}$

b Show that D, X and Y are collinear.

92 Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be non-zero vectors in three dimensions such that

$$\boldsymbol{a} \times \boldsymbol{b} = 3\boldsymbol{a} \times \boldsymbol{c}$$

a Show that there exists $k \in \mathbb{R}$ such that $\boldsymbol{b} - 3\boldsymbol{c} = k\boldsymbol{a}$.

b Given that $|\boldsymbol{a}| = |\boldsymbol{c}| = 1$, $|\boldsymbol{b}| = 3$ and the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ is $\arccos\left(\frac{1}{3}\right)$, determine:

i $\boldsymbol{b} \cdot \boldsymbol{c}$ **ii** $|\boldsymbol{b} - 3\boldsymbol{c}|$ **iii** the possible values of k.

c Hence determine the cosine of the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{c}$.

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CU

93 **a** Let A, B and C be points in three-dimensional space with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. Given that A, B and C are not collinear, prove that the plane ABC can be represented by the vector equation

$$\boldsymbol{r} = \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \nu\boldsymbol{c}, \quad \text{where } \lambda, \mu, \nu \in \mathbb{R} \text{ with } \lambda + \mu + \nu = 1$$

b For each of the following, write down a vector equation of the plane ABC in the form established in part **a**:

i $A(1, 1, 1)$, $B(1, -1, 1)$, $C(1, 1, -1)$ **ii** $A(1, 1, 1)$, $B(-1, -2, 3)$, $C(2, 1, -2)$

c Determine a vector equation (using just one parameter $t \in \mathbb{R}$) for the line of intersection of the two planes given by

$$\boldsymbol{r}_1 = \lambda_1\hat{\boldsymbol{i}} + 2\mu_1\hat{\boldsymbol{j}} + 3\nu_1\hat{\boldsymbol{k}}, \quad \text{where } \lambda_1, \mu_1, \nu_1 \in \mathbb{R} \text{ with } \lambda_1 + \mu_1 + \nu_1 = 1$$

$$\boldsymbol{r}_2 = 2\lambda_2\hat{\boldsymbol{i}} + \mu_2\hat{\boldsymbol{j}} + 2\nu_2\hat{\boldsymbol{k}}, \quad \text{where } \lambda_2, \mu_2, \nu_2 \in \mathbb{R} \text{ with } \lambda_2 + \mu_2 + \nu_2 = 1$$

94 A vector equation of a plane Π is $\boldsymbol{r} \cdot \boldsymbol{n} = k$.

a Let ℓ be a line with vector equation $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$, $t \in \mathbb{R}$. Given that $\boldsymbol{b} \cdot \boldsymbol{n} \neq 0$, show that the plane Π meets the line ℓ at the point with position vector

$$\frac{(\boldsymbol{b} \cdot \boldsymbol{n})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{b} + k\boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{n}}$$

b Let P be a point, with position vector $\boldsymbol{p}$, such that P does not lie on the plane Π.

i Using part **a**, express the position vector of the point where the plane Π meets the line through P perpendicular to Π in terms of $\boldsymbol{p}$, $\boldsymbol{n}$ and k.

ii Express the distance from the point P to the plane Π in terms of $\boldsymbol{p}$, $\boldsymbol{n}$ and k.

95 Points A, B and C have position vectors $\overrightarrow{OA} = -\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\overrightarrow{OB} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}$ and $\overrightarrow{OC} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

a Determine the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Determine $\overrightarrow{AB} \times \overrightarrow{AC}$.

c Using part **b**, determine a Cartesian equation of the plane Π through points A, B and C.

Let D be the point with position vector $\overrightarrow{OD} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

d Determine a vector equation of the line through D perpendicular to the plane Π.

e Determine the position vector of the point of intersection of this line with the plane Π.

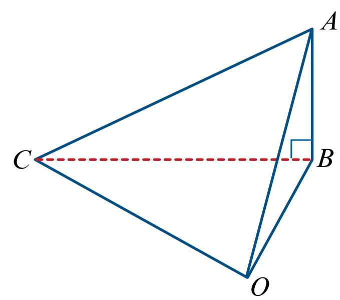
f Determine the shortest distance from the point D to the plane Π

96 In the tetrahedron shown, $\overrightarrow{OB} = \hat{\boldsymbol{i}}$, $\overrightarrow{OC} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\overrightarrow{BA} = \sqrt{\lambda}\hat{\boldsymbol{k}}$.

a Express $\overrightarrow{OA}$ and $\overrightarrow{CA}$ in terms of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, $\hat{\boldsymbol{k}}$ and $\sqrt{\lambda}$.

b Determine the magnitude of $\angle CBO$ to the nearest degree.

c Determine the value of λ, if the magnitude of $\angle OAC$ is 30°.



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CU

97 Let $ABCD$ be a regular tetrahedron. The intersection point of the perpendicular bisectors of the edges of a triangle is called the circumcentre of the triangle. Let X, Y, Z and W be the circumcentres of faces ABC, ACD, ABD and BCD respectively. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are the position vectors of the four vertices.

- **a** Determine the position vectors of X, Y, Z and W.
- **b** Determine the vectors $\overrightarrow{DX}$, $\overrightarrow{BY}$, $\overrightarrow{CZ}$ and $\overrightarrow{AW}$.
- **c** Let P be a point on DX such that $\overrightarrow{DP} = \frac{3}{4}\overrightarrow{DX}$. Determine the position vector of P.
- **d** Hence determine the position vectors of the points Q, R and S on BY, CZ and AW respectively such that $\overrightarrow{BQ} = \frac{3}{4}\overrightarrow{BY}$, $\overrightarrow{CR} = \frac{3}{4}\overrightarrow{CZ}$ and $\overrightarrow{AS} = \frac{3}{4}\overrightarrow{AW}$.
- **e** Explain the geometric significance of results **c** and **d**.

Unit 3 Topic 4: Vector calculus

SF

98 A particle is projected at an angle of $\alpha°$ to the horizontal with a speed of 15 m/s. It reaches its greatest height after $\frac{1}{3}$ second. Determine $\alpha°$, correct to two decimal places.

99 A particle is projected with a speed of 30 m/s from a point P, which is 2 m above the ground. The angle of projection is 45° upwards from the horizontal. Determine the horizontal distance from P travelled by the particle when it hits the ground (to two decimal places).

CF

100 **a** The velocity vector of a particle P at time t is $\dot{\boldsymbol{r}}_1(t) = 3\cos(2t)\,\hat{\boldsymbol{i}} + 4\sin(2t)\,\hat{\boldsymbol{j}}$, where $\boldsymbol{r}_1(t)$ is the position relative to O at time t. Determine:

- **i** $\boldsymbol{r}_1(t)$, given that $\boldsymbol{r}_1(0) = -2\hat{\boldsymbol{j}}$
- **ii** the acceleration vector at time t
- **iii** the times when the position and velocity vectors are perpendicular
- **iv** the Cartesian equation of the path.

b At time t, a second particle Q has a position vector (relative to O) given by $\boldsymbol{r}_2(t) = \frac{3}{2}\sin(2t)\,\hat{\boldsymbol{i}} + 2\cos(2t)\,\hat{\boldsymbol{j}} + (a - t)\hat{\boldsymbol{k}}$. Determine the possible values of a in order for the particles to collide.

101 Determine the Cartesian equation of the curve with the following vector equation:

$$\boldsymbol{r}(\theta) = (1 - \cos(2\theta))\,\hat{\boldsymbol{i}} + (2 + \sin(2\theta))\,\hat{\boldsymbol{j}} \quad \text{for } \theta \geq 0$$

102 A particle is moving in an elliptical path such that its position, $\boldsymbol{r}(t)$, at time t is given by

$$\boldsymbol{r}(t) = 3\cos t\,\hat{\boldsymbol{i}} + 4\sin t\,\hat{\boldsymbol{j}}$$

- **a** Calculate the position of the particle at times $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
- **b** Is the particle moving clockwise or anticlockwise?
- **c** Determine the Cartesian equation of the path of the particle.

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103 The acceleration vector, $\ddot{\boldsymbol{r}}(t)$ m/s^2, of a particle at time t seconds is given by CF

$$\ddot{\boldsymbol{r}}(t) = -16(\cos(4t)\,\hat{\boldsymbol{i}} + \sin(4t)\,\hat{\boldsymbol{j}})$$

- **a** Determine the position vector, $\boldsymbol{r}(t)$ m, given that $\dot{\boldsymbol{r}}(0) = 4\hat{\boldsymbol{j}}$ and $\boldsymbol{r}(0) = \hat{\boldsymbol{j}}$.
- **b** Show that the path of the particle is a circle and state the position vector of its centre.
- **c** Show that the acceleration is always perpendicular to the velocity.

104 Two particles A and B are projected simultaneously from a point O at ground level. The particles travel in the x–y plane, where the unit of distance is metres and the positive y-direction is vertically up. Particles A and B are projected at angles of 30° and 60° to the positive x-direction respectively. They both have the same initial speed of 20 m/s. CU

- **a** Give the position vector of each of the particles at time t seconds (in terms of g).
- **b** For each particle, determine the position vector of the point of maximum height.
- **c** Determine the distance between the two particles after 1 second, correct to two decimal places.

105 Two particles A and B are moving with constant velocities on a horizontal plane:

$$\dot{\boldsymbol{r}}_A(t) = 9\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} \quad \text{and} \quad \dot{\boldsymbol{r}}_B(t) = 5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$$

where time t is measured in seconds and distance is measured in metres.

- **a** Determine the speeds of the two particles.
- **b** Given that $\boldsymbol{r}_A(3) = 29\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_B(3) = 18\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}$, determine $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$.
- **c** Determine $\overrightarrow{AB}$ in terms of t.
- **d** Determine $|\overrightarrow{AB}|$ in terms of t.
- **e** Determine the time when the two particles are closest to each other.

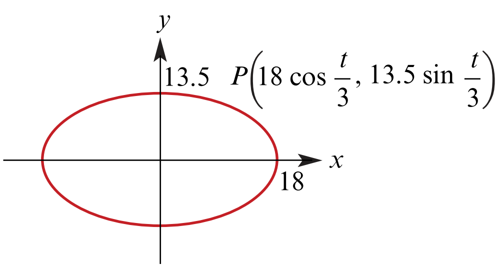
106 The position vector of a particle at time t seconds is given by $\boldsymbol{r}_1(t) = 2t\,\hat{\boldsymbol{i}} - (t^2 + 2)\hat{\boldsymbol{j}}$, where distances are measured in metres.

- **a** What is the average velocity of the particle for the interval [0, 10]?
- **b** By differentiation, determine the velocity at time t.
- **c** In what direction is the particle moving when $t = 3$?
- **d** When is the particle moving with minimum speed?
- **e** At what time is the particle moving at the average velocity for the first 10 seconds?
- **f** A second particle has its position at time t given by $\boldsymbol{r}_2(t) = (t^3 - 4)\hat{\boldsymbol{i}} - 3t\hat{\boldsymbol{j}}$. Are the two particles coincident at any time t?

107 An ice-skater describes an elliptic path. His position at time t seconds is given by CU

$$\boldsymbol{r} = 18\cos\left(\frac{t}{3}\right)\hat{\boldsymbol{i}} + 13.5\sin\left(\frac{t}{3}\right)\hat{\boldsymbol{j}}$$

When $t = 0$, $\boldsymbol{r} = 18\hat{\boldsymbol{i}}$.



- **a** How long does the skater take to go around the path once?

b **i** Determine the velocity of the ice-skater at $t = 2\pi$.

ii Determine the acceleration of the ice-skater at $t = 2\pi$.

c **i** Determine an expression for the speed of the ice-skater at time t.

ii At what time is his speed greatest?

d Prove that the acceleration satisfies $\ddot{\boldsymbol{r}} = k\boldsymbol{r}$, and hence determine when the acceleration has a maximum magnitude.

108 Two trains, T_1 and T_2, are moving on perpendicular tracks that cross at the point O. Relative to O, the position vectors of T_1 and T_2 at time t are given by $\boldsymbol{r}_1 = Vt\,\hat{\boldsymbol{i}}$ and $\boldsymbol{r}_2 = 2V(t - t_0)\hat{\boldsymbol{j}}$ respectively, where V and t_0 are positive constants.

a **i** Which train goes through O first?

ii How much later does the other train go through O?

b **i** Show that the trains are closest together when $t = \dfrac{4t_0}{5}$.

ii Calculate their distance apart at this time.

iii Draw a diagram to show the positions of the trains at this time. Also show the directions in which they are moving.

109 A stone is to be projected from ground level over two walls that are a distance of d m apart. Both walls have a height of h m. Let g m/s^2 be the acceleration due to gravity.

a Determine the minimum initial speed of the stone, in terms of d, g and h, such that the stone will go over the two walls.

Hints: First consider the motion from the top of the first wall.
Remember that the horizontal component of the velocity is constant.

b When the stone is projected over the two walls with the minimum possible initial speed, determine the cosine of the angle of projection in terms of d and h.

110 A particle is fired from the top of a cliff h m above sea level with an initial velocity of V m/s inclined at an angle α above the horizontal. Let $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ define the horizontal and vertically upwards vectors in the plane of the particle's path.

a Define:

i the initial position vector of the particle

ii the particle's initial velocity.

b The acceleration vector of the particle under gravity is given by $\boldsymbol{a} = -g\hat{\boldsymbol{j}}$. Determine:

i the velocity vector of the particle t seconds after it is projected

ii the corresponding position vector.

c Use the velocity vector to determine the time at which the particle reaches its highest point.

d Show that the time at which the particle hits the sea is given by

$$t = \frac{V\sin\alpha + \sqrt{(V\sin\alpha)^2 + 2gh}}{g}$$

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111 An aircraft takes off from the end of a runway in a southerly direction and climbs at an angle of $\tan^{-1}(\frac{1}{2})$ to the horizontal at a speed of $225\sqrt{5}$ km/h. CU

a Show that, t seconds after take-off, the position vector $\boldsymbol{r}_1$ of the aircraft with respect to the end of the runway is given by $\boldsymbol{r}_1 = \dfrac{t}{16}(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}})$, where $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ are vectors of magnitude 1 km in the directions south, east and vertically upwards respectively.

b At time $t = 0$, a second aircraft, flying horizontally south-east at $720\sqrt{2}$ km/h, has position vector $-1.2\hat{\boldsymbol{i}} + 3.2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

i Determine its position vector $\boldsymbol{r}_2$ at time t in terms of $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$.

ii Show that there will be a collision and state the time at which it will occur.

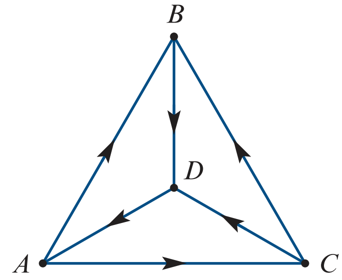
Unit 3 Topic 5: Further matrices

112 Suppose that a person weighing 85 kg burns 360 calories per hour when walking (W), 520 calories per hour when cycling (C), and 1000 calories per hour when running (R). A fitness fanatic weighing 85 kg plans an exercise program according to the following matrix (with exercise times given in hours): SF

$$\begin{array}{c c} & \begin{array}{ccc} W & C & R \end{array} \\ \begin{array}{c} \text{Sat} \\ \text{Sun} \\ \text{Wed} \\ \text{Thu} \end{array} & \begin{bmatrix} 1 & 1 & 0.5 \\ 0.5 & 1 & 1.5 \\ 1.5 & 0 & 1 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

Using matrix multiplication, determine the number of calories burned each day and the total number of calories burned during the program.

113 This diagram gives the results of a round-robin tennis competition between four friends – Amina, Bessie, Carl and Dylan. For example, the arrow from A to B indicates that Amina won against Bessie.



a Write down the one-step dominance matrix $\mathbf{M}$ that represents the results of this competition.

b Rank the four players by using $\mathbf{M} + \mathbf{M}^2$.

114 Let $\mathbf{A} = \begin{bmatrix} 2 & 6 & 3 \\ 1 & 3 & 0 \\ 2 & 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -6 & 9 \\ 0 & 6 & -3 \\ 4 & -8 & 0 \end{bmatrix}$. CF

a Determine the product $\mathbf{AB}$, and hence determine $\mathbf{A}^{-1}$.

b Use $\mathbf{A}^{-1}$ to solve the system of equations.

$$\begin{aligned} 2x + 6y + 3z &= 12 \\ x + 3y &= 24 \\ x + y &= 18 \end{aligned}$$

115 Consider the following system of linear equations:

$$\begin{aligned} x + ay - z &= 0 \\ 2x + y + z &= k \\ x - y + z &= 2 \end{aligned}$$

a Show that, if $a \neq 5$, then there is a unique solution. Determine this solution in terms of a and k.

b Given that $a = 5$, determine the values of k for which there are no solutions.

c Given that $a = 5$, determine the values of k for which there are infinitely many solutions.

116 Consider the system of equations

$$\begin{aligned} x + y + 2z &= a \\ x + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

a Determine the relationship between a, b and c if the system has at least one solution.

b Under the conditions found in part **a**, determine a vector equation of the line of intersection of the three planes defined by these three equations.

117 The general equation of a circle in the Cartesian plane is $x^2 + y^2 + ax + by + c = 0$.

a The three points $(3, -1)$, $(-1, -2)$ and $(4, -5)$ lie on a circle.

i Write down three linear equations in a, b and c.

ii Represent these three equations as an augmented matrix.

iii Determine the centre and radius of the circle.

b Consider the three points $(3, -1)$, $(-1, -2)$ and $(0, k)$.

i By substituting these values into the equation $x^2 + y^2 + ax + by + c = 0$, write down three linear equations in a, b and c.

ii Represent these three equations as an augmented matrix.

iii Determine the values of k for which these three points lie on a circle.

c Determine the values of k for which the points $(3, -1)$, $(-1, -2)$ and $(1, k)$ lie on a circle.

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118 When a new infectious disease was first noticed in a particular country, there were 100 people already infected. As the disease spread, the country's health authority collected the following information: CU

- The duration of the disease is at most 3 weeks. People who contract the disease either die during these 3 weeks or else recover during the third week.
- The survival rate of people who have the disease is 90% in the first week and 80% in the second week.
- People who contract the disease are not infectious during the first week. Then the rate of infection is 90% in the second week(meaning that an infected person passes on the infection to an average of 0.9 other people) and 70% in the third week.

Using this information:

a Construct a Leslie matrix **L** for the disease based on three one-week stages.

b Assume that, when the disease is first noticed, the 100 infected people are all in the second week of the disease. Write down the initial population matrix $\mathbf{P}_0$.

For each of the following, give answers to the nearest whole number.

c Determine how the disease is spreading by using the Leslie matrix **L** to determine $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. Comment.

d Determine $\mathbf{P}_{40}$, and then use $\mathbf{P}_{40}$ to determine $\mathbf{P}_{41}$. Verify that, in these two population matrices, the sizes of the three groups are in nearly the same ratio. Hence estimate the growth rate of the disease at this stage.

e Suppose that the health authority had taken immediate action to reduce the rate of infection in the third week to 35%. Repeat parts **a**–**d**. Would this action have been sufficient to eradicate the disease? Give evidence for your answer.

f Now suppose that the health authority had taken more drastic action and reduced the rate of infection in the third week to 10%. Repeat parts **a**–**d**. Comment.

10B Multiple-choice questions

1 For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.
The statement $P(1)$ is

A $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

B $0^2 = \dfrac{0(0+1)(2 \times 0 + 1)}{6}$

C $1^2 + 2^2 + \cdots + n^2 = \dfrac{1}{6}$

D $1^2 = \dfrac{1(1+1)(2 \times 1 + 1)}{6}$

2 For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $n^2 - n + 41$ is a prime number.
Which one of the following is correct?

A $P(2)$ is not true

B $P(3)$ is not true

C $P(5)$ is not true

D $P(41)$ is not true

3 The hyperbola $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$ has asymptotes with the equations

A $y = \frac{3}{4}x + \frac{8}{3}$ and $y = \frac{3}{4}x + \frac{2}{3}$

B $y = \frac{3}{4}x + \frac{10}{3}$ and $y = \frac{3}{4}x + \frac{2}{3}$

C $y = \frac{4}{3}x + \frac{10}{3}$ and $y = -\frac{4}{3}x + \frac{2}{3}$

D $y = \frac{4}{3}x + \frac{10}{3}$ and $y = -\frac{4}{3}x + \frac{10}{3}$

4 A circle has a diameter with endpoints at $(4, -2)$ and $(-2, -2)$. The equation of the circle is

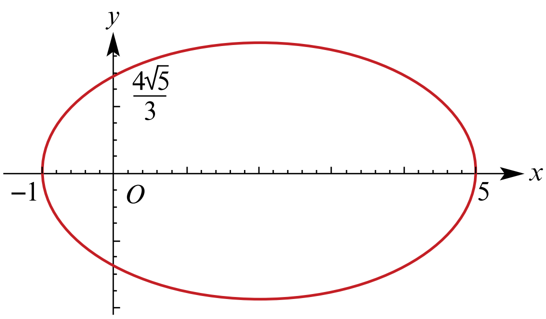
A $(x-1)^2 + (y-2)^2 = 3$

B $(x-1)^2 + (y+2)^2 = 3$

C $(x+1)^2 + (y-2)^2 = 6$

D $(x-1)^2 + (y+2)^2 = 9$

5 The ellipse shown has its centre on the x-axis. Its equation is



A $\frac{(x+2)^2}{9} + \frac{y^2}{16} = 1$

B $\frac{(x-2)^2}{9} + \frac{y^2}{16} = 1$

C $\frac{(x+2)^2}{3} + \frac{y^2}{4} = 1$

D $\frac{(x-2)^2}{3} + \frac{y^2}{4} = 1$

6 The ellipse with equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$ has x-axis intercepts with coordinates

A $(-3, -5)$ and $(3, 5)$

B $(-5, -3)$ and $(5, 3)$

C $(0, -3)$ and $(0, 3)$

D $(-3, 0)$ and $(3, 0)$

7 The circle defined by the equation $x^2 + y^2 - 6x + 8y = 0$ has centre

A $(2, 4)$ **B** $(-5, 9)$ **C** $(4, -3)$ **D** $(3, -4)$

8 If the line $x = k$ is a tangent to the circle with equation $(x-1)^2 + (y+2)^2 = 1$, then k is equal to

A 1 or 3 **B** −1 or −3 **C** 0 or −2 **D** 0 or 2

9 The curve with equation $x^2 - 2x = y^2$ is

A an ellipse with centre $(1, 0)$

B a hyperbola with centre $(1, 0)$

C a circle with centre $(1, 0)$

D an ellipse with centre $(-1, 0)$

10 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$, $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = -3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$, then $\boldsymbol{a} - 2\boldsymbol{b} - \boldsymbol{c}$ equals

A $3\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} - 12\hat{\boldsymbol{k}}$

B $-3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 12\hat{\boldsymbol{k}}$

C $4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$

D $-4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

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11 A vector of magnitude 6 and with direction opposite to $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ is

A $6\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}} + 12\hat{\boldsymbol{k}}$ **B** $-6\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$

C $-3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$ **D** $-2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$

12 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$, then the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

A $7(-2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}})$ **B** $\frac{1}{7}(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}})$

C $-\frac{1}{7}(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ **D** $-\frac{7}{11}(2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$

13 If $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, then a vector which is not perpendicular to $\boldsymbol{a}$ is

A $\frac{1}{35}(3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ **B** $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$

C $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}}$ **D** $-3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 34\hat{\boldsymbol{k}}$

14 The magnitude of vector $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ is

A $\sqrt{17}$ **B** 35 **C** 17 **D** $\sqrt{35}$

15 Let $\boldsymbol{u} = 2\hat{\boldsymbol{i}} - a\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{v} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - b\hat{\boldsymbol{k}}$. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular to each other when

A $a = 2$ and $b = -1$ **B** $a = -2$ and $b = 10$

C $a = \frac{1}{2}$ and $b = -5$ **D** $a = 0$ and $b = 0$

16 Let $\boldsymbol{u} = \hat{\boldsymbol{i}} + a\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ and $\boldsymbol{v} = b\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel to each other when

A $a = -\frac{8}{3}$ and $b = -\frac{3}{4}$ **B** $a = -\frac{3}{2}$ and $b = -\frac{3}{4}$

C $a = -\frac{8}{3}$ and $b = -\frac{4}{3}$ **D** none of these

17 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. Then the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is

A $-\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ **B** $-5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 5\hat{\boldsymbol{k}}$ **C** $5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ **D** $\frac{5}{3}\hat{\boldsymbol{i}} + \frac{2}{3}\hat{\boldsymbol{j}} + \frac{5}{3}\hat{\boldsymbol{k}}$

18 If points A, B and C are such that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, which one of the following statements must be true?

A Either $\overrightarrow{AB}$ or $\overrightarrow{BC}$ is a zero vector.

B Vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have the same magnitude.

C The vector resolute of $\overrightarrow{AC}$ in the direction of $\overrightarrow{AB}$ is $\overrightarrow{AB}$.

D The vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$ is $\overrightarrow{AC}$.

19 If $\boldsymbol{u} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{v} = 4\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$, then $\boldsymbol{u} \cdot \boldsymbol{v}$ equals

A $4\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ **B** $5\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$ **C** -5 **D** 19

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20 If $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, then the scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

A $\dfrac{10}{49}(6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$ **B** $\dfrac{10}{7}$

C $\dfrac{10}{49}$ **D** $\dfrac{\sqrt{10}}{7}$

21 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$. The unit vector in the direction of $\boldsymbol{a} - \boldsymbol{b}$ is

A $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ **B** $\dfrac{1}{\sqrt{65}}(5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}})$

C $\dfrac{1}{3}(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$ **D** $\dfrac{1}{9}(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

22 If the points P, Q and R are collinear with $\overrightarrow{OP} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $\overrightarrow{OQ} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\overrightarrow{OR} = 2\hat{\boldsymbol{i}} + p\hat{\boldsymbol{j}} + q\hat{\boldsymbol{k}}$, then

A $p = -3$ and $q = 2$ **B** $p = -\dfrac{7}{2}$ and $q = 2$

C $p = -\dfrac{1}{2}$ and $q = 0$ **D** $p = 3$ and $q = -2$

23 If $\tan\alpha = \dfrac{3}{4}$ and $\tan\beta = \dfrac{4}{3}$, where both α and β are acute, then $\sin(\alpha + \beta)$ equals

A $\dfrac{24}{25}$ **B** $\dfrac{7}{25}$ **C** 0 **D** 1

24 If $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\boldsymbol{x} = \hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ and $\boldsymbol{x} = s\boldsymbol{a} + t\boldsymbol{b}$, then the scalars s and t are given by

A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$

C $s = 1$ and $t = -1$ **D** $s = 1$ and $t = 1$

25 Given that $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and the points O, P and Q are not collinear, which one of the following points, whose position vectors are given, is not collinear with P and Q?

A $\dfrac{1}{2}\boldsymbol{p} + \dfrac{1}{2}\boldsymbol{q}$ **B** $3\boldsymbol{p} - 2\boldsymbol{q}$ **C** $\boldsymbol{p} - \boldsymbol{q}$ **D** $\dfrac{1}{3}\boldsymbol{p} + \dfrac{2}{3}\boldsymbol{q}$

26 Assume that $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$, $t \in \mathbb{R}$, is a vector equation of a line that does not pass through the origin. Which one of the following is *not* the position vector of a point on the line?

A $\boldsymbol{a}$ **B** $\boldsymbol{b}$ **C** $\boldsymbol{a} + \boldsymbol{b}$ **D** $\boldsymbol{a} - \boldsymbol{b}$

27 The two lines given by the vector equations $\boldsymbol{r} = 9\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \lambda(3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$, for $\lambda \in \mathbb{R}$, and $\boldsymbol{s} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \mu(3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$, for $\mu \in \mathbb{R}$, intersect at the point with coordinates

A $(12, -3)$ **B** $(6, -1)$ **C** $(0, -3)$ **D** $(3, 0)$

28 The plane with vector equation $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 2$ contains the point

A $(1, -1, 1)$ **B** $(-1, 1, 0)$ **C** $(0, 1, 1)$ **D** $(2, 0, 0)$

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29 For the straight line ℓ given by the vector equation

$$\boldsymbol{r} = -\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}), \quad t \in \mathbb{R}$$

which one of the following is true?

A The line ℓ is perpendicular to the vector $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$.

B The line ℓ passes through the point $(-2, -3, 6)$.

C The line ℓ passes through the origin.

D The line ℓ lies in the plane with equation $x + y - z = -1$.

30 A system of linear equations has an augmented matrix in row-echelon form as shown.

$$\begin{aligned} 2x + y + z &= 3 \\ x + 2y - 2z &= 4 \\ x - 4y + 8z &= -6 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{5}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The set of all solutions of this system of equations can be described as

A $x = 0,\ y = 0,\ z = 0$

B $x = \frac{2}{3},\ y = \frac{5}{3},\ z = 0$

C $x = \frac{4}{3},\ y = -\frac{5}{3},\ z = 0$

D $x = \frac{2 - 4\lambda}{3},\ y = \frac{5 + 5\lambda}{3},\ z = \lambda,\ \lambda \in \mathbb{R}$

31 For the system of linear equations shown

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + kz &= 4 \\ 2x + 3ky + 2z &= 6 \end{aligned}$$

which one of the following is true?

A For all values of k, this system of equations has no solutions.

B There is only one value of k for which this system has a unique solution.

C There are exactly two values of k for which this system has a unique solution.

D There are exactly two values of k for which this system does not have a unique solution.

32 Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be non-zero vectors in three dimensions. If $\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u} \times \boldsymbol{v}|$, then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is

A 0 **B** $\frac{\pi}{6}$ **C** $\frac{\pi}{4}$ **D** $\frac{\pi}{3}$

33 $\cos^2\theta + 3\sin^2\theta$ equals

A $3 - 2\cos(2\theta)$

B $2 - \cos\theta$

C $2\cos(2\theta) - 1$

D none of these

34 Assume that the two vector equations $\boldsymbol{r}_1 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$, $t \in \mathbb{R}$, and $\boldsymbol{r}_2 = \boldsymbol{a}_2 + s\boldsymbol{d}_2$, $s \in \mathbb{R}$, represent the same line ℓ, where ℓ does not pass through the origin. Which one of the following is *not* true?

A $\boldsymbol{d}_1 = k\boldsymbol{d}_2$ for some $k \in \mathbb{R}$

B $\boldsymbol{a}_2 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$ for some $t \in \mathbb{R}$

C $\boldsymbol{d}_2 = \boldsymbol{a}_1 + t\boldsymbol{d}_1$ for some $t \in \mathbb{R}$

D $\boldsymbol{a}_2 - \boldsymbol{a}_1 = k\boldsymbol{d}_2$ for some $k \in \mathbb{R}$

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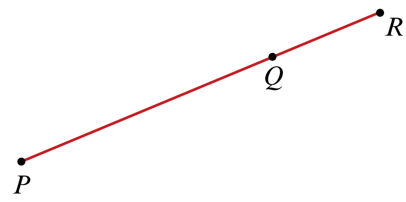
35 PQR is a straight line and $PQ = 2QR$.
If $\overrightarrow{OQ} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\overrightarrow{OR} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, then $\overrightarrow{OP}$ is equal to



A $-\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ **B** $7\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$
C $4\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}}$ **D** $-4\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}$

36 If $\overrightarrow{OP} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\overrightarrow{PQ} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, then $|\overrightarrow{OQ}|$ equals

A $3\sqrt{2}$ **B** 6 **C** 9 **D** 4

37 If $z_1 = 2 - i$ and $z_2 = 3 + 4i$, then $\left|\dfrac{z_2}{z_1}\right|^2$ equals

A $\sqrt{5}$ **B** 5 **C** $\dfrac{125}{9}$ **D** $\left(\dfrac{2+11i}{5}\right)^2$

38 If $z = -1 - \sqrt{3}i$, then $\operatorname{Arg} z$ equals

A $-\dfrac{2\pi}{3}$ **B** $-\dfrac{5\pi}{6}$ **C** $\dfrac{2\pi}{3}$ **D** $\dfrac{5\pi}{6}$

39 The vectors $p\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3p\hat{\boldsymbol{k}}$ and $p\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ are perpendicular when p is equal to

A 0 only **B** 3 only **C** 0 or 3 **D** 1 or 2

40 One solution of the equation $z^3 - 5z^2 + 17z - 13 = 0$ is $2 + 3i$. The other solutions are

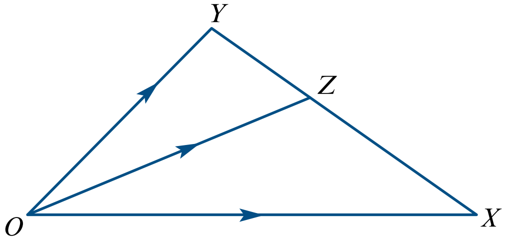
A $-2 - 3i$ and 1 **B** $2 - 3i$ and 1
C $-2 + 3i$ and -1 **D** $2 - 3i$ and -1

41 The value of $\dfrac{(\cos 60^\circ + i\sin 60^\circ)^4}{(\cos 30^\circ + i\sin 30^\circ)^2}$ is

A -1 **B** i **C** $-i$ **D** $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

42 If $3\overrightarrow{OX} + 4\overrightarrow{OY} = 7\overrightarrow{OZ}$, then $\dfrac{XZ}{ZY}$ equals

A $\dfrac{3}{5}$ **B** $\dfrac{3}{4}$
C 1 **D** $\dfrac{4}{3}$



43 If $x + yi = \dfrac{1}{3+4i}$, where x and y are real, then

A $x = \dfrac{3}{25}$ and $y = -\dfrac{4}{25}$ **B** $x = \dfrac{3}{25}$ and $y = \dfrac{4}{25}$
C $x = -\dfrac{3}{7}$ and $y = \dfrac{4}{7}$ **D** $x = \dfrac{1}{3}$ and $y = \dfrac{1}{4}$

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44 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + p\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then p equals

A $-\frac{7}{3}$ **B** -2 **C** $-\frac{5}{3}$ **D** 2

45 Let $z = \frac{1}{1-i}$. If $r = |z|$ and $\theta = \operatorname{Arg} z$, then

A $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **B** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$

C $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **D** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

46 If $u = 3\operatorname{cis}\left(\frac{\pi}{4}\right)$ and $v = 2\operatorname{cis}\left(\frac{\pi}{2}\right)$, then uv is equal to

A $6\operatorname{cis}\left(\frac{\pi^2}{8}\right)$ **B** $6\operatorname{cis}^2\left(\frac{\pi^2}{8}\right)$ **C** $5\operatorname{cis}\left(\frac{3\pi}{4}\right)$ **D** $6\operatorname{cis}\left(\frac{3\pi}{4}\right)$

47 The modulus of $12 - 5i$ is

A 119 **B** 7 **C** 13 **D** $\sqrt{119}$

48 When $\sqrt{3} - i$ is divided by $-1 - i$, the modulus and the principal argument of the quotient are

A $2\sqrt{2}$ and $\frac{7\pi}{12}$ **B** $\sqrt{2}$ and $-\frac{11\pi}{12}$

C $\sqrt{2}$ and $\frac{7\pi}{12}$ **D** $2\sqrt{2}$ and $-\frac{11\pi}{12}$

49 Let z be a complex number such that $|z + 4i| = 3$. Then the smallest and largest possible values of $|z + 3|$ are

A 2 and 8 **B** 3 and 4 **C** 4 and 8 **D** 5 and 8

50 Let $P(z)$ be a quadratic polynomial with real coefficients. Which one of the following is *not* possible?

A $P(z)$ has two real roots **B** $P(z)$ has two imaginary roots

C $P(z)$ has one real and one non-real root **D** $P(z)$ has two non-real roots

51 The product of the complex numbers $\frac{1-i}{\sqrt{2}}$ and $\frac{\sqrt{3}+i}{2}$ has argument

A $-\frac{5\pi}{12}$ **B** $-\frac{\pi}{12}$ **C** $\frac{\pi}{12}$ **D** $\frac{5\pi}{12}$

52 If $\tan\theta = \frac{1}{3}$, then $\tan(2\theta)$ equals

A $\frac{3}{5}$ **B** $\frac{2}{3}$ **C** $\frac{3}{4}$ **D** $\frac{4}{5}$

53 The modulus of $1 + \cos(2\theta) + i\sin(2\theta)$, where $0 < \theta < \frac{\pi}{2}$, is

A $4\cos^2\theta$ **B** $4\sin^2\theta$ **C** $2\cos\theta$ **D** $2\sin\theta$

54 An expression for an argument of $1 + \cos\theta + i\sin\theta$ is

A $2\cos\left(\frac{\theta}{2}\right)$ **B** $2\sin\left(\frac{\theta}{2}\right)$ **C** θ **D** $\frac{\theta}{2}$

55 A quadratic equation with solutions $2 + 3i$ and $2 - 3i$ is

A $x^2 + 4x + 13 = 0$ **B** $x^2 - 4x + 13 = 0$

C $x^2 + 4x - 13 = 0$ **D** $x^2 + 4x - 5 = 0$

56 The subset of the complex plane defined by the equation $|z - 2| - |z + 2| = 0$ is

A a circle **B** an ellipse

C a straight line **D** the empty set

57 The subset of the complex plane defined by the equation $|z - (2 - i)| = 6$ is

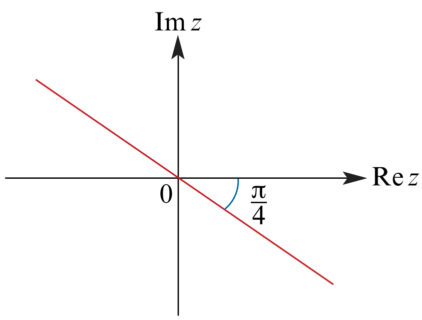
A a circle with centre at $-2 + i$ and radius 6

B a circle with centre at $2 - i$ and radius 6

C a circle with centre at $2 - i$ and radius 36

D a circle with centre at $-2 + i$ and radius 36

58 The line shown can be represented by the set



A $\left\{z : \operatorname{Arg} z = \frac{\pi}{4}\right\}$

B $\left\{z : \operatorname{Arg} z = -\frac{\pi}{4}\right\}$

C $\left\{z : \operatorname{Arg} z = \frac{7\pi}{4}\right\}$

D $\{z : \operatorname{Im} z + \operatorname{Re} z = 0\}$

59 The subset of the complex plane defined by the equation $|z - 2| - |z - 2i| = 0$ is

A a circle **B** an ellipse

C a straight line **D** the empty set

60 Which one of the following subsets of the complex plane is *not* a circle?

A $\{z : |z - i| = 2\}$ **B** $\{z : z\bar{z} + 2\operatorname{Re}(iz) = 0\}$

C $\{z : |z - 1| = 2\}$ **D** $\{z : |z| = 2i\}$

61 Which one of the following subsets of the complex plane is *not* a line?

A $\{z : \operatorname{Im}(z) = 0\}$ **B** $\{z : \operatorname{Im}(z) + \operatorname{Re}(z) = 1\}$

C $\{z : z + \bar{z} = 4\}$ **D** $\left\{z : \operatorname{Arg}(z) = \frac{\pi}{4}\right\}$

62 Points P, Q, R and M are such that $\overrightarrow{PQ} = 5\hat{\boldsymbol{i}}$, $\overrightarrow{PR} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\overrightarrow{RM}$ is parallel to $\overrightarrow{PQ}$ so that $\overrightarrow{RM} = \lambda\hat{\boldsymbol{i}}$, where λ is a constant. The value of λ for which angle RQM is a right angle is

A 0 **B** $\frac{19}{4}$ **C** $\frac{21}{4}$ **D** 10

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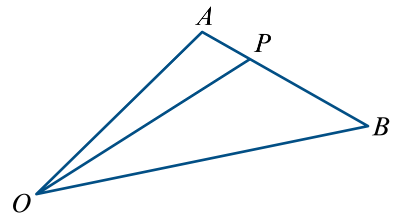
63 In this diagram, $\overrightarrow{OA} = 6\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 8\hat{\boldsymbol{k}}$, $\overrightarrow{OB} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ and $AP : PB = 1 : 2$.



The vector $\overrightarrow{OP}$ is equal to

A $\frac{7}{3}\hat{\boldsymbol{j}} + \frac{4}{3}\hat{\boldsymbol{k}}$ **B** $3\hat{\boldsymbol{i}} + \frac{7}{3}\hat{\boldsymbol{j}} + \frac{4}{3}\hat{\boldsymbol{k}}$

C $3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$ **D** $3\hat{\boldsymbol{i}} + \frac{2}{3}\hat{\boldsymbol{j}} + \frac{14}{3}\hat{\boldsymbol{k}}$

64 In an Argand diagram, O is the origin, P is the point $(2, 1)$ and Q is the point $(1, 2)$. If P represents the complex number z and Q the complex number α, then α equals

A $i\bar{z}$ **B** $-\bar{z}$ **C** $-i\bar{z}$ **D** $z\bar{z}$

65 In an Argand diagram, the points that represent the complex numbers z, $-\bar{z}$, z^{-1} and $-\overline{(z^{-1})}$ necessarily lie at the vertices of a

A trapezium **B** rectangle

C parallelogram **D** rhombus

66 A projectile is launched at an angle of 60° to the horizontal with an initial speed of 30 m/s. What is the magnitude of the horizontal component of the projectile's displacement at the end of 2 seconds?

A 30 m **B** 40 m **C** 10 m **D** 20 m

67 A particle is moving so its velocity vector at time t is $\dot{\boldsymbol{r}}(t) = 2t\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, where $\boldsymbol{r}(t)$ is the position vector of the particle at time t. If $\boldsymbol{r}(0) = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, then $\boldsymbol{r}(t)$ is equal to

A $2\hat{\boldsymbol{i}}$ **B** $5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

C $(3t + 1)\hat{\boldsymbol{i}} + (3t^2 + 1)\hat{\boldsymbol{j}}$ **D** $(t^2 + 3)\hat{\boldsymbol{i}} + (3t + 1)\hat{\boldsymbol{j}}$

68 The simultaneous equations $x - 3y = 5$ and $-2x + y = 7$ can be written in the form of a matrix equation as

A $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

C $\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

69 The matrix that corresponds to a dilation of factor 3 from the y-axis followed by a reflection in the x-axis is

A $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

70 The position of a particle at time $t = 0$ is $\boldsymbol{r}(0) = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, and its position at time $t = 2$ is $\boldsymbol{r}(2) = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$. The average velocity for the interval $[0, 2]$ is

A $\frac{1}{2}(6\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}})$ **B** $\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

C $24\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ **D** $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

71 A particle moves in the x–y plane such that its position vector $\boldsymbol{r}$ at time t seconds is given by $\boldsymbol{r} = 2t^2\hat{\boldsymbol{i}} + t^3\hat{\boldsymbol{j}}$ metres. When $t = 1$, the speed of the particle (in m/s) is

A $\dfrac{3}{4}$ **B** $\sqrt{5}$ **C** 5 **D** 7

72 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{x}}(t) = 2\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$. If the velocity of the particle at time $t = 0$ is described by the vector $2\hat{\boldsymbol{i}}$, then the velocity at time t is

A $\dot{\boldsymbol{x}}(t) = 2t\,\hat{\boldsymbol{i}} + \frac{1}{2}t^2\hat{\boldsymbol{j}}$ **B** $\dot{\boldsymbol{x}}(t) = (2t+2)\hat{\boldsymbol{i}} + \frac{1}{2}t^2\hat{\boldsymbol{j}}$

C $\dot{\boldsymbol{x}}(t) = 2\hat{\boldsymbol{i}} + (2\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}})t$ **D** $\dot{\boldsymbol{x}}(t) = 2(2\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}})$

73 A particle has its position in metres from a given point at time t seconds defined by the vector $\boldsymbol{r}(t) = 4t\,\hat{\boldsymbol{i}} - \frac{1}{3}t^2\hat{\boldsymbol{j}}$. The magnitude of the displacement in the third second is

A 4 m **B** $3\frac{2}{3}$ m **C** $4\frac{1}{3}$ m **D** $6\frac{2}{3}$ m

74 The position of a particle at time t seconds is given by $\boldsymbol{r}(t) = (t^2 - 2t)(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, measured in metres from a fixed point. The distance travelled by the particle in the first 2 seconds is

A 0 m **B** 2 m **C** −2 m **D** 6 m

75 The position of a particle at time t seconds is given by the vector

$$\boldsymbol{r}(t) = \left(\frac{1}{3}t^3 - 4t^2 + 15t\right)\hat{\boldsymbol{i}} + \left(t^3 - \frac{15}{2}t^2\right)\hat{\boldsymbol{j}}$$

When the particle is instantaneously at rest, its acceleration vector is given by

A $15\hat{\boldsymbol{i}}$ **B** $-18\hat{\boldsymbol{j}}$ **C** $2\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}}$ **D** $-8\hat{\boldsymbol{i}} - 15\hat{\boldsymbol{j}}$

76 A particle moves with its position defined with respect to time t by the vector function $\boldsymbol{r}(t) = (3t^3 - t)\hat{\boldsymbol{i}} + (2t^2 + 1)\hat{\boldsymbol{j}} + 5t\hat{\boldsymbol{k}}$. When $t = \frac{1}{2}$, the magnitude of the acceleration is

A 17 **B** $4\sqrt{3}$ **C** $4\sqrt{5}$ **D** none of these

77 The velocity of a particle is given by the vector $\dot{\boldsymbol{r}}(t) = \sin(t)\,\hat{\boldsymbol{i}} + \cos(2t)\,\hat{\boldsymbol{j}}$. At time $t = 0$, the position of the particle is given by the vector $6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. The position of the particle at time t is given by

A $(7 - \cos t)\hat{\boldsymbol{i}} + (\frac{1}{2}\sin(2t) - 4)\hat{\boldsymbol{j}}$ **B** $(5 - \cos t)\hat{\boldsymbol{i}} + (\frac{1}{2}\sin(2t) - 3)\hat{\boldsymbol{j}}$

C $(5 + \cos t)\hat{\boldsymbol{i}} + (2\sin(2t) - 4)\hat{\boldsymbol{j}}$ **D** $(6 + \cos t)\hat{\boldsymbol{i}} + (2\sin(2t) - 4)\hat{\boldsymbol{j}}$

78 The initial position, velocity and constant acceleration of a particle are given by $2\hat{\boldsymbol{i}}$, $3\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ respectively. The position of the particle at time t is given by

A $(4 + t)\hat{\boldsymbol{i}} + (3 - \frac{1}{2}t^2)\hat{\boldsymbol{j}}$ **B** $2\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}}$

C $2t\,\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}}$ **D** $(2 + \frac{1}{2}t^2)\hat{\boldsymbol{i}} + (3t - \frac{1}{2}t^2)\hat{\boldsymbol{j}}$

79 Consider a square matrix $\mathbf{P}$. If $\mathbf{P}^2 = 9\mathbf{I}$, then $\mathbf{P}^{-1}$ equals

A $\dfrac{1}{9}\mathbf{P}$ **B** $\dfrac{1}{3}\mathbf{P}$ **C** $\dfrac{1}{3}\mathbf{I}$ **D** $3\mathbf{P}$

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80 Which of the following could be the dominance matrix representing the outcomes from a round-robin tournament between three teams? (Assume that there are no draws.)

A $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

81 An object slides across a smooth horizontal tabletop of height h m at a constant speed of u m/s. It slides off the edge of the tabletop and hits the floor a distance of x m away. What is the relationship between x and h?

A $h = \dfrac{ux}{g}$ **B** $x = \dfrac{u^2}{2gh}$ **C** $x = \dfrac{u^2}{gh}$ **D** $x = u\sqrt{\dfrac{2h}{g}}$

82 The Cartesian equation of a sphere with centre $(1, -2, -3)$ and radius 7 is

A $(x-1)^2 + (y+2)^2 + (z+3)^2 = 7$ **B** $(x+1)^2 + (y-2)^2 + (z-3)^2 = 49$

C $(x-1)^2 + (y+2)^2 + (z+3)^2 = 49$ **D** $(x+1)^2 + (y-2)^2 + z-3)^2 = 7$

83 A plane is represented by the equation $3y + 4z = 6$. A vector normal to this plane is

A $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$ **B** $\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$ **D** $\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$

84 The augmented matrix shown is produced when a Gaussian elimination technique is used to solve a certain system of equations with three variables x, y and z. The columns of the matrix correspond to these variables in the usual way.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 4 & -3 & 16 \\ 0 & 0 & -5 & 20 \end{array}\right]$$

The solution of the equations will be

A $x = 1, y = 1, z = 4$ **B** $x = 2, y = 1, z = 3$

C $x = 16, y = 1, z = -4$ **D** $x = 9, y = 0, z = -4$

85 The points $A(2, 4, 7)$ and $B(-3, 6, 1)$ are points in three-dimensional space. The position vector of the midpoint M of AB is

A $\overrightarrow{OM} = -\dfrac{1}{2}\hat{i} + 5\hat{j} + 4\hat{k}$ **B** $\overrightarrow{OM} = -\dfrac{5}{2}\hat{i} + \hat{j} - 3\hat{k}$

C $\overrightarrow{OM} = -\hat{i} + 10\hat{j} + 2\hat{k}$ **D** $\overrightarrow{OM} = -5\hat{i} + 2\hat{j} - 6\hat{k}$

86 An equation of a line passing through the points $A(2, 6, 8)$ and $B(4, -6, 10)$ is

A $4\hat{i} - 6\hat{j} + 10\hat{k} + t(2\hat{i} + 12\hat{j} + 2\hat{k}) \quad t \in \mathbb{R}$

B $x = 2 + 2t, y = 6 + 12t, z = 8 + 2t \quad t \in \mathbb{R}$

C $4\hat{i} - 6\hat{j} + 10\hat{k} + t(2\hat{i} - 12\hat{j} + 2\hat{k}) \quad t \in \mathbb{R}$

D $\dfrac{x-2}{2} = \dfrac{y-6}{12} = \dfrac{z-8}{2}$

Technology-active multiple-choice questions

87 If $\boldsymbol{u} = 2\hat{\boldsymbol{i}} - \sqrt{2}\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{v} = \hat{\boldsymbol{i}} + \sqrt{2}\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, then the angle between the direction of $\boldsymbol{u}$ and $\boldsymbol{v}$, correct to two decimal places, is

A 92.05° **B** 87.95° **C** 79.11° **D** 100.89°

88 Let $\boldsymbol{a} = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}$. Then $\boldsymbol{a} \times \boldsymbol{b}$ is equal to

A $\begin{bmatrix} 2 \\ 80 \\ 29 \end{bmatrix}$ **B** -26 **C** $\begin{bmatrix} 28 \\ -6 \\ -48 \end{bmatrix}$ **D** $\begin{bmatrix} -2 \\ -80 \\ -29 \end{bmatrix}$

89 The angle correct to the nearest degree between the vector $\begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix}$ and the z-axis is

A 56° **B** 127° **C** 34° **D** 110°

90 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$. Then $|\boldsymbol{a} \times \boldsymbol{b}|$ is equal to

A $3\sqrt{6}$ **B** -34 **C** $2\sqrt{385}$ **D** $8\sqrt{6}$

91 The area of triangle ABC with vertices $A(1, -2, 3)$, $B(6, 4, -2)$ and $C(-2, 4, 1)$ correct to two decimal places is

A 62.31 **B** 28.52 **C** 25.04 **D** 31.52

92 The Leslie matrix for a certain species is:

$$\mathbf{L} = \begin{bmatrix} 0.8 & 2.4 & 0.3 \\ 0.25 & 0 & 0 \\ 0 & 0.50 & 0 \end{bmatrix}$$

Some of the species were moved to another location. The initial female population in this locationis given by

$$\mathbf{S}_0 = \begin{bmatrix} 140 \\ 50 \\ 30 \end{bmatrix}$$

The expected size of the total female population after 5 years is closest to

A 700 **B** 790 **C** 810 **D** 920

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93 The Leslie matrix $\mathbf{L} = \begin{bmatrix} 0 & 2.5 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}$ satisfies the matrix equation $\mathbf{L}\begin{bmatrix} 64 \\ 16 \\ 8 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 8 \end{bmatrix}$.

The values of b, c and d are

A $a = 2, b = \dfrac{1}{2}, c = \dfrac{1}{4}$
B $a = 8, b = \dfrac{1}{16}, c = \dfrac{1}{64}$
C $a = 2, b = \dfrac{1}{2}, c = \dfrac{1}{4}$
D $a = 3, b = \dfrac{1}{4}, c = \dfrac{1}{2}$

94 A population of insects is modelled by using the Leslie matrix

$$\mathbf{L} = \begin{bmatrix} 0 & 1.26 & 2 \\ 0.614 & 0 & 0 \\ 0 & 0.55 & 0 \end{bmatrix}$$

The growth has reached the point where the rates of growth of the different age groups of the population are nearly constant and the state matrix at this point is $\mathbf{S}_\ell = \begin{bmatrix} 864 \\ 456 \\ 215 \end{bmatrix}$.

The approximate rate of growth per time period is

A 16% **B** 12.5% **C** 12% **D** 18%

95 A quadrilateral $OABC$ has vertices $O(0, 0, 0)$, $A(2, 5, -6)$, $B(3, -3, -4)$ and $C(2, -16, 4)$. The size of $\angle ABC$ correct to the nearest degree is

A 61° **B** 159° **C** 123° **D** 67°

96 Consider the matrix equation $\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}\mathbf{X} = \begin{bmatrix} 3 & -3 & 2 \\ 7 & -4 & 7 \\ 7 & -1 & 2 \end{bmatrix}$. Then $\mathbf{X}$ is equal to

A $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$
B $\begin{bmatrix} -1 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

C $\dfrac{1}{11}\begin{bmatrix} -1 & 4 & -2 \\ -3 & 1 & 5 \\ 6 & -2 & 1 \end{bmatrix}$
D $\dfrac{1}{11}\begin{bmatrix} 18 & 5 & -19 \\ 47 & 10 & -27 \\ 8 & 23 & -17 \end{bmatrix}$

97 A vector normal to the plane which contains the vectors $\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ is

A $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \\ 6 \\ 12 \end{bmatrix}$ **C** $\begin{bmatrix} 0 \\ -12 \\ 6 \end{bmatrix}$ **D** $\begin{bmatrix} 12 \\ 0 \\ 6 \end{bmatrix}$

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98 The coordinates of a point A in 3-dimensional space are $(3, 2, 6)$. The altitude angle of $\overrightarrow{OA}$ in radians correct to two decimal places is

A 0.44 **B** 0.71 **C** 1.03 **D** 0.59

99 The imaginary part of $\left(\text{cis}\left(\frac{\pi}{5}\right)\right)^{-3}$ correct to two decimal places is

A -0.31 **B** 1.61 **C** 0.03 **D** -0.95

100 Two objects, A and B, move in three-dimensional space such that their positions,over time, t, are described by the following vectors until they collide.

$$\boldsymbol{r}_1 = (2 + 4t^2)\hat{\boldsymbol{i}} + (3 + 2t)\hat{\boldsymbol{j}} + (t - 2)\hat{\boldsymbol{k}} \qquad \boldsymbol{r}_2 = 6t\hat{\boldsymbol{i}} + (4 + t)\hat{\boldsymbol{j}} + (t^3 - 2t^2)\hat{\boldsymbol{k}}$$

The objects will collide when

A $t = 1$ **B** $t = 2$ **C** $t = 3$ **D** $t = 4$

101 Let $u = 1 - 2i$ and $v = 3 - 4i$.
$\text{Im}(u^3 - |v|)$ is equal to

A 2 **B** 16 **C** 37 **D** 38

102 Let $z = 1 - 3i$. Then, correct to two decimal places, $\text{Arg}(z^3)$ is equal to

A 3.74 **B** -1.95 **C** 2.54 **D** 0.97

103 One solution of the equation $z^2 - (2 + i)z = 1 - 7i$ is $z = 3 - i$. The other solution is

A $1 + 2i$ **B** $1 - 2i$ **C** $-1 + 2i$ **D** $-(1 + 2i)$

104 Let $z^5 + 1 + i = 0$. Then one possible solution to this equation is

A $2^{\frac{1}{5}}\,\text{cis}\left(\frac{3\pi}{20}\right)$ **B** $2^{\frac{1}{5}}\,\text{cis}\left(\frac{7\pi}{20}\right)$ **C** $2^{\frac{1}{10}}\,\text{cis}\left(\frac{7\pi}{20}\right)$ **D** $2^{\frac{1}{10}}\,\text{cis}\left(\frac{\pi}{4}\right)$

105 Let $z = -1 + 2i$ and $\omega = 2 + 3i$
Then, correct to two decimal places, $|z\bar{\omega} + iz|$ is equal to

A 9.49 **B** 8.25 **C** 6.32 **D** 7.44

106 Mathematical induction can be used to prove $3^{3n+1} + 9 \times 2^{n+3}$ is divisible by 25 for all $n \in \mathbb{N}$. To prove the inductive step we would have to show that

A $3^{3k+1} + 11 \times 2^{k+3} = 25m$ for some $m \in \mathbb{N}$

B $3^{3k+1} + 9 \times 2^{k+3} = 25m$ for some $m \in \mathbb{N}$

C $3^{3(k+1)+1} + 72 \times 2^{(k+1)} = 25m$ for some $m \in \mathbb{N}$

D $3^{3k+4} + 9 \times 2^{k+4} = 25m$ for some $m \in \mathbb{N}$

107 Mathematical induction can be used to prove $(1 + i\tan\theta)^n = \dfrac{\text{cis}\, n\theta}{\cos^n\theta}$ for all $n \in \mathbb{N}$. To prove the inductive step we would have to show that

A $(1 + i\tan\theta)^{k+1} = \dfrac{\text{cis}\,(k+1)\theta}{\cos^{(k+1)}\theta}$ **B** $(1 + i\tan\theta)^{k} = \dfrac{\text{cis}\, k\theta}{\cos^{(k+1)}\theta}$

C $(1 + i\tan\theta)^{k+1} = \dfrac{i\,\text{cis}\,(k+1)\theta}{\cos^{(k+1)}\theta}$ **D** $(i\tan\theta)^{k+1} = -\dfrac{\text{cis}\, k\theta}{i\cos^{k}\theta}$

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108 Mathematical induction can be used to prove to prove

$$\sum_{r=1}^{n} \sin((2r-1)A) = \sin^2(nA)\operatorname{cosec}(A).$$

To prove the inductive step we would have to show that

A $\displaystyle\sum_{r=1}^{k} \sin((2r+1)A) = \sin^2(kA)\operatorname{cosec}(A)$

B $\displaystyle\sum_{r=1}^{k+1} \sin((2r-1)A) = \sin^2((k+1)A)\operatorname{cosec}(A)$

C $\displaystyle\sum_{r=1}^{k+1} \sin((2(r+1)-1)A) = \sin^2((k+1)A)\operatorname{cosec}(A)$

D $\displaystyle\sum_{r=1}^{k-1} \sin((2r-1)A) = \sin^2((2k-1)A)\operatorname{cosec}(A)$

109 Matrix $\mathbf{M}$ represents the results of a competition involving 4 teams.

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Key: A lost to B and D but won against C

Using the ranking model $\mathbf{M} + \mathbf{M}^2$, the teams that placed first, second and third respectively are

A B, D and A

B D, B and A

C B, D and C

D D, B and C

110 The Leslie matrix $\mathbf{L}$ below represents the changes in a population of rodents in a particular environment. $\mathbf{P}_0$ represents the initial female population of rodents in their different age groups.

$$\mathbf{L} = \begin{bmatrix} 0 & 10 & 12 & 8 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0.12 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix} \qquad \mathbf{P}_0 = \begin{bmatrix} 1000 \\ 50 \\ 30 \\ 10 \end{bmatrix}$$

After 5 years, the total female population would be closest to

A 1390 **B** 1290 **C** 1320 **D** 1550

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11

Integration techniques

Chapter contents

Integration is used in many areas of this course. In the next chapter, integration is used to find areas and volumes. In Chapter 13, it is used to help solve differential equations, which are of great importance in mathematical modelling.

In the first section of this chapter, we briefly revise integration from Mathematical Methods Units 3 & 4. We also use the modulus function to give an anti-derivative of $f(x) = x^{-1}$ that applies for negative values of x as well as for positive values.

In the remainder of the chapter, we introduce techniques for integrating many more functions. We will use the inverse trigonometric functions, trigonometric identities, partial fractions and two techniques which can be described as 'reversing' the chain rule and the product rule.

This chapter covers Unit 4 Topic 1: Integration techniques.

11A Determining definite integrals and using the modulus function

Learning intentions

- To be able to evaluate definite integrals.
- To be able to differentiate $\ln|x|$ and use the modulus function with integration of functions f with rule $f(x) = \dfrac{1}{ax+b}$.

In this chapter, our focus is on developing techniques for calculating definite integrals using anti-differentiation.

Definite integrals

For a continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ denotes the signed area enclosed by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

By the fundamental theorem of calculus, we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any anti-derivative of f.

Note: The symbol $\int$ is called the **integral sign**, the numbers a and b are called the **limits** or **endpoints** of the integral, and the function f is called the **integrand**.

Example 1

Evaluate each of the following integrals:

a $\int_0^{\frac{\pi}{2}} \cos(3x)\,dx$ **b** $\int_0^1 e^{2x} - e^x\,dx$ **c** $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$ **d** $\int_0^1 \sqrt{2x+1}\,dx$

Solution

a
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(3x)\,dx &= \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{2}}\\ &= \frac{1}{3}\left(\sin\left(\frac{3\pi}{2}\right) - \sin 0\right)\\ &= \frac{1}{3}(-1-0)\\ &= -\frac{1}{3}\end{aligned}$$

b
$$\begin{aligned}\int_0^1 e^{2x} - e^x\,dx &= \left[\frac{1}{2}e^{2x} - e^x\right]_0^1\\ &= \frac{1}{2}e^2 - e^1 - \left(\frac{1}{2}e^0 - e^0\right)\\ &= \frac{e^2}{2} - e - \left(\frac{1}{2} - 1\right)\\ &= \frac{e^2}{2} - e + \frac{1}{2}\end{aligned}$$

c
$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$
$$\begin{aligned}\therefore\ \int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx &= \left[\frac{1}{2}\tan(2x)\right]_0^{\frac{\pi}{8}}\\ &= \frac{1}{2}\left(\tan\left(\frac{\pi}{4}\right) - \tan 0\right)\\ &= \frac{1}{2}\end{aligned}$$

d
$$\begin{aligned}\int_0^1 \sqrt{2x+1}\,dx &= \int_0^1 (2x+1)^{\frac{1}{2}}\,dx\\ &= \left[\frac{1}{2\times\frac{3}{2}}(2x+1)^{\frac{3}{2}}\right]_0^1\\ &= \frac{1}{3}\left((2+1)^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)\\ &= \frac{1}{3}\left(3^{\frac{3}{2}} - 1\right)\\ &= \frac{1}{3}(3\sqrt{3}-1)\end{aligned}$$

The modulus function

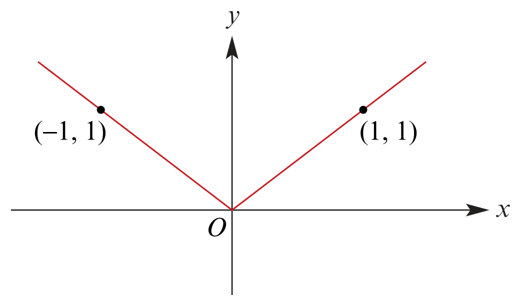
The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

The graph of the function $y = |x|$ is shown on the right.

This graph is symmetric about the y-axis, since $|x| = |-x|$.



Example 2

Evaluate each of the following:

a **i** $|-3 \times 2|$ **ii** $|-3| \times |2|$

b **i** $\left|\dfrac{-4}{2}\right|$ **ii** $\dfrac{|-4|}{|2|}$

c **i** $|-6 + 2|$ **ii** $|-6| + |2|$

Solution

a **i** $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|ab| = |a|\,|b|$

b **i** $\left|\dfrac{-4}{2}\right| = |-2| = 2$ **ii** $\dfrac{|-4|}{|2|} = \dfrac{4}{2} = 2$ Note: $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

c **i** $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|a + b| \leq |a| + |b|$

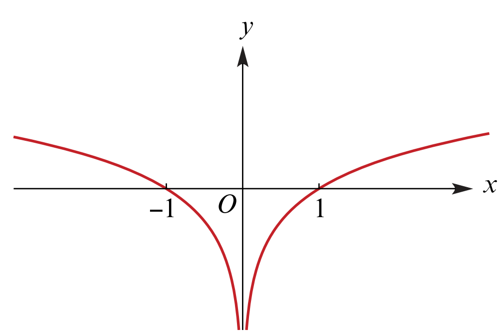
The function $y = \ln|x|$

The graph of the function

$$y = \ln|x|, \quad x \neq 0$$

is shown on the right.

This function is very important in this course. We find its derivative in the next example.



Example 3

a Determine $\frac{d}{dx}(\ln|x|)$ for $x \neq 0$.

b Determine $\frac{d}{dx}(\ln|\sec x|)$ for $x \notin \left\{ \frac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}$.

Solution

a Let $y = \ln|x|$.

If $x > 0$, then $y = \ln x$, so

$$\frac{dy}{dx} = \frac{1}{x}$$

If $x < 0$, then $y = \ln(-x)$, so the chain rule gives

$$\frac{dy}{dx} = \frac{1}{-x} \times (-1) = \frac{1}{x}$$

Hence

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \text{for } x \neq 0$$

b Let $y = \ln|\sec x|$

$$= \ln\left|\frac{1}{\cos x}\right|$$

$$= \ln\left(\frac{1}{|\cos x|}\right)$$

$$= -\ln|\cos x|$$

Let $u = \cos x$. Then $y = -\ln|u|$.

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= -\frac{1}{u} \times (-\sin x)$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

From Example 3a, we have:

If $f(x) = \ln|x|$, then $f'(x) = \frac{1}{x}$ for $x \neq 0$.

The general anti-derivative of x^{-1}

Using the derivative of $\ln|x|$, we now obtain the anti-derivative

$$\int \frac{1}{x}\, dx = \ln|x| + c \qquad \text{for } x \neq 0$$

More generally, we can use the chain rule to show that

$$\frac{d}{dx}(\ln|ax+b|) = \frac{a}{ax+b}$$

This gives the following anti-derivative:

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\ln|ax+b| + c \qquad \text{for } ax+b \neq 0$$

Example 4

a Determine an anti-derivative of $\dfrac{1}{4x+2}$ for $x \neq -\dfrac{1}{2}$.

b Evaluate $\displaystyle\int_0^1 \frac{1}{4x+2}\,dx$.

c Evaluate $\displaystyle\int_{-2}^{-1} \frac{1}{4x+2}\,dx$.

Solution

a $\dfrac{1}{4x+2}$ is of the form $\dfrac{1}{ax+b}$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\therefore \quad \int \frac{1}{4x+2}\,dx = \frac{1}{4}\ln|4x+2| + c$$

b
$$\begin{aligned}\int_0^1 \frac{1}{4x+2}\,dx &= \left[\frac{1}{4}\ln|4x+2|\right]_0^1 \\ &= \frac{1}{4}(\ln 6 - \ln 2) \\ &= \frac{1}{4}\ln 3\end{aligned}$$

c
$$\begin{aligned}\int_{-2}^{-1} \frac{1}{4x+2}\,dx &= \left[\frac{1}{4}\ln|4x+2|\right]_{-2}^{-1} \\ &= \frac{1}{4}(\ln|-2| - \ln|-6|) \\ &= \frac{1}{4}\ln\left(\frac{1}{3}\right) \\ &= -\frac{1}{4}\ln 3\end{aligned}$$

Exercise 11A

SF

Example 1

1 Evaluate each of the following integrals:

a $\displaystyle\int_{-1}^{1} e^x - e^{-x}\,dx$ **b** $\displaystyle\int_0^2 3x^2 + 2x + 4\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{2}} \sin(2x)\,dx$

d $\displaystyle\int_2^3 \frac{3}{x^3}\,dx$ **e** $\displaystyle\int_0^{\frac{\pi}{4}} \cos(x) + 2x\,dx$ **f** $\displaystyle\int_0^1 e^{3x} + x\,dx$

g $\displaystyle\int_0^{\frac{\pi}{2}} \cos(4x)\,dx$ **h** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left(\frac{x}{2}\right)dx$ **i** $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$

Example 2

2 Evaluate each of the following:

a $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$

d $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

Example 3b

3 For each of the following, find the derivative with respect to x:

a $\ln|2x+1|$ **b** $\ln|-2x+1|$ **c** $\ln|\sin x|$

d $\ln|\sec x + \tan x|$ **e** $\ln|\operatorname{cosec} x + \tan x|$ **f** $\ln|\tan(\frac{1}{2}x)|$

g $\ln|\operatorname{cosec} x - \cot x|$ **h** $\ln|x + \sqrt{x^2-4}|$ **i** $\ln|x + \sqrt{x^2+4}|$

Example 4

4 **a** Determine an anti-derivative of $\dfrac{1}{2x-5}$ for $x \neq \dfrac{5}{2}$.

b Evaluate $\displaystyle\int_0^1 \frac{1}{2x-5}\,dx$. **c** Evaluate $\displaystyle\int_{-2}^{-1} \frac{1}{2x-5}\,dx$.

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5 Evaluate each of the following integrals:

a $\int_0^1 \frac{1}{3x+2}\,dx$ **b** $\int_{-3}^{-1} \frac{1}{3x-2}\,dx$ **c** $\int_{-1}^{0} \frac{1}{4-3x}\,dx$

d $\int_{-3}^{-2} \frac{1}{x-2}\,dx$ **e** $\int_{1}^{3} \frac{1}{2x+4}\,dx$ **f** $\int_{-1}^{1} \frac{1}{6-3x}\,dx$

g $\int_{-2}^{-1} \frac{3x+1}{x}\,dx$ **h** $\int_{-3}^{-2} \frac{3x+1}{x+1}\,dx$ **i** $\int_{-1}^{0} \frac{2x+1}{x-3}\,dx$

11B Derivatives of inverse trigonometric functions

Learning intentions

- To be able to differentiate the inverse trigonometric functions

The inverse trigonometric functions – arcsine, arccosine and arctangent – were introduced in Section 1D

To find the derivatives of these functions, we will use the result

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

This was used in Mathematical Methods Units 3 & 4 to find the derivative of the natural logarithm function $y = \ln x$ (which is the inverse of the natural exponential function $y = e^x$).

The derivative of $\sin^{-1}(x)$

If $f(x) = \sin^{-1}(x)$, then $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

Proof Let $y = \sin^{-1}(x)$, where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The equivalent form is $x = \sin y$ and so $\dfrac{dx}{dy} = \cos y$.

Thus $\dfrac{dy}{dx} = \dfrac{1}{\cos y}$ and $\cos y \neq 0$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The Pythagorean identity is used to express $\dfrac{dy}{dx}$ in terms of x:

$$\sin^2 y + \cos^2 y = 1$$
$$\cos^2 y = 1 - \sin^2 y$$
$$\cos y = \pm\sqrt{1 - \sin^2 y}$$

Therefore $\cos y = \sqrt{1-\sin^2 y}$ since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so $\cos y > 0$

$= \sqrt{1-x^2}$ since $x = \sin y$

Hence $\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$

The derivative of $\cos^{-1}(x)$

If $f(x) = \cos^{-1}(x)$, then $f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

Proof Let $y = \cos^{-1}(x)$, where $x \in [-1, 1]$ and $y \in [0, \pi]$.

The equivalent form is $x = \cos y$ and so $\dfrac{dx}{dy} = -\sin y$.

Thus $\dfrac{dy}{dx} = \dfrac{-1}{\sin y}$ and $\sin y \neq 0$ for $y \in (0, \pi)$.

Using the Pythagorean identity yields

$$\sin y = \pm\sqrt{1-\cos^2 y}$$

Therefore

$$\begin{aligned} \sin y &= \sqrt{1-\cos^2 y} && \text{since } y \in (0,\pi) \text{ and so } \sin y > 0 \\ &= \sqrt{1-x^2} && \text{since } x = \cos y \end{aligned}$$

Hence $\dfrac{dy}{dx} = \dfrac{-1}{\sin y} = \dfrac{-1}{\sqrt{1-x^2}}$

The derivative of $\tan^{-1}(x)$

If $f(x) = \tan^{-1}(x)$, then $f'(x) = \dfrac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Proof Let $y = \tan^{-1}(x)$, where $x \in \mathbb{R}$ and $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Then $x = \tan y$. Therefore $\dfrac{dx}{dy} = \sec^2 y$, giving $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$.

Using the Pythagorean identity $\tan^2 y + 1 = \sec^2 y$, we have

$$\begin{aligned} \frac{dy}{dx} = \frac{1}{\sec^2 y} &= \frac{1}{1+\tan^2 y} \\ &= \frac{1}{1+x^2} && \text{since } x = \tan y \end{aligned}$$

For $a > 0$, the following results can be obtained using the chain rule.

Inverse trigonometric functions

- If $f(x) = \sin^{-1}\left(\dfrac{x}{a}\right)$, then $f'(x) = \dfrac{1}{\sqrt{a^2-x^2}}$ for $x \in (-a, a)$.
- If $f(x) = \cos^{-1}\left(\dfrac{x}{a}\right)$, then $f'(x) = \dfrac{-1}{\sqrt{a^2-x^2}}$ for $x \in (-a, a)$.
- If $f(x) = \tan^{-1}\left(\dfrac{x}{a}\right)$, then $f'(x) = \dfrac{a}{a^2+x^2}$ for $x \in \mathbb{R}$.

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Proof We show how to obtain the first result; the remaining two are left as an exercise.

Let $y = \sin^{-1}\left(\frac{x}{a}\right)$. Then by the chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \times \frac{1}{a} = \frac{1}{\sqrt{a^2\left(1 - \frac{x^2}{a^2}\right)}} = \frac{1}{\sqrt{a^2 - x^2}}$$

Example 5

Differentiate each of the following with respect to x:

a $\sin^{-1}\left(\frac{x}{3}\right)$ **b** $\cos^{-1}(4x)$ **c** $\tan^{-1}\left(\frac{2x}{3}\right)$ **d** $\sin^{-1}(x^2 - 1)$

Solution

a Let $y = \sin^{-1}\left(\frac{x}{3}\right)$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$$

b Let $y = \cos^{-1}(4x)$ and $u = 4x$.

By the chain rule:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - u^2}} \times 4$$

$$= \frac{-4}{\sqrt{1 - 16x^2}}$$

c Let $y = \tan^{-1}\left(\frac{2x}{3}\right)$ and $u = \frac{2x}{3}$.

By the chain rule:

$$\frac{dy}{dx} = \frac{1}{1 + u^2} \times \frac{2}{3}$$

$$= \frac{1}{1 + \left(\frac{2x}{3}\right)^2} \times \frac{2}{3}$$

$$= \frac{9}{4x^2 + 9} \times \frac{2}{3}$$

$$= \frac{6}{4x^2 + 9}$$

d Let $y = \sin^{-1}(x^2 - 1)$ and $u = x^2 - 1$.

By the chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - u^2}} \times 2x$$

$$= \frac{2x}{\sqrt{1 - (x^2 - 1)^2}}$$

$$= \frac{2x}{\sqrt{1 - (x^4 - 2x^2 + 1)}}$$

$$= \frac{2x}{\sqrt{2x^2 - x^4}}$$

$$= \frac{2x}{\sqrt{x^2}\sqrt{2 - x^2}}$$

$$= \frac{2x}{|x|\sqrt{2 - x^2}}$$

Hence $\frac{dy}{dx} = \frac{2}{\sqrt{2 - x^2}}$ for $0 < x < \sqrt{2}$

and $\frac{dy}{dx} = \frac{-2}{\sqrt{2 - x^2}}$ for $-\sqrt{2} < x < 0$

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Skill-sheet

Exercise 11B

Example 5

SF

1 Determine the derivative of each of the following with respect to x:

a $\sin^{-1}\left(\frac{x}{2}\right)$ **b** $\cos^{-1}\left(\frac{x}{4}\right)$ **c** $\tan^{-1}\left(\frac{x}{3}\right)$ **d** $\sin^{-1}(3x)$ **e** $\cos^{-1}(2x)$

f $\tan^{-1}(5x)$ **g** $\sin^{-1}\left(\frac{3x}{4}\right)$ **h** $\cos^{-1}\left(\frac{3x}{2}\right)$ **i** $\tan^{-1}\left(\frac{2x}{5}\right)$ **j** $\sin^{-1}(0.2x)$

2 Determine the derivative of each of the following with respect to x:

a $\sin^{-1}(x+1)$ **b** $\cos^{-1}(2x+1)$ **c** $\tan^{-1}(x+2)$ **d** $\cos^{-1}(1-3x)$

e $2\sin^{-1}\left(\frac{3x+1}{2}\right)$ **f** $-4\cos^{-1}\left(\frac{5x-3}{2}\right)$ **g** $5\tan^{-1}\left(\frac{1-x}{2}\right)$ **h** $-\sin^{-1}(x^2)$

3 Determine the derivative of each of the following with respect to x:

a $y=\cos^{-1}\left(\frac{3}{x}\right)$ for $x>3$ **b** $y=\sin^{-1}\left(\frac{5}{x}\right)$ for $x>5$ **c** $y=\cos^{-1}\left(\frac{3}{2x}\right)$ for $x>\frac{3}{2}$

4 For a positive constant a, find the derivative of each of the following:

a $\sin^{-1}(ax)$ **b** $\cos^{-1}(ax)$ **c** $\tan^{-1}(ax)$

5 Determine the second derivative of each of the following:

a $4\sin^{-1}(x)$ **b** $\tan^{-1}(x)$ **c** $3\sin^{-1}\left(\frac{x}{4}\right)$ **d** $\cos^{-1}(3x)$ **e** $2\tan^{-1}\left(\frac{2x}{3}\right)$

CF

6 Let $f(x)=3\sin^{-1}\left(\frac{x}{2}\right)$.

a **i** Determine the maximal domain of f. **ii** Determine the range of f.

b Determine the derivative of $f(x)$, and state the domain for which the derivative exists.

c Sketch the graph of $y=f'(x)$, labelling the turning points and the asymptotes.

7 Let $f(x)=4\cos^{-1}(3x)$.

a **i** Determine the maximal domain of f. **ii** Determine the range of f.

b Determine the derivative of $f(x)$, and state the domain for which the derivative exists.

c Sketch the graph of $y=f'(x)$, labelling the turning points and the asymptotes.

8 Let $f(x)=2\tan^{-1}\left(\frac{x+1}{2}\right)$.

a **i** Determine the maximal domain of f. **ii** Determine the range of f.

b Find the derivative of $f(x)$.

c Sketch the graph of $y=f'(x)$, labelling the turning points and the asymptotes.

SF

9 Differentiate each of the following with respect to x:

a $(\sin^{-1}x)^2$ **b** $\sin^{-1}x+\cos^{-1}x$ **c** $\sin(\cos^{-1}x)$

d $\cos(\sin^{-1}x)$ **e** $e^{\sin^{-1}x}$ **f** $\tan^{-1}(e^x)$

10 Determine, correct to two decimal places where necessary, the gradient of the graph of each of the following functions at the value of x indicated:

a $y=\sin^{-1}\left(\frac{x}{3}\right)$, $x=1$ **b** $y=2\cos^{-1}(3x)$, $x=0.1$ **c** $y=3\tan^{-1}(2x+1)$, $x=1$

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11 For each of the following, determine the value(s) of a from the given information:

a $f(x) = 2\sin^{-1} x, \quad f'(a) = 4$

b $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right), \quad f'(a) = -10$

c $f(x) = \tan^{-1}(3x), \quad f'(a) = 0.5$

d $f(x) = \sin^{-1}\left(\frac{x+1}{2}\right), \quad f'(a) = 20$

e $f(x) = 2\cos^{-1}\left(\frac{2x}{3}\right), \quad f'(a) = -8$

f $f(x) = 4\tan^{-1}(2x-1), \quad f'(a) = 1$

12 Determine, in the form $y = mx + c$, the equation of the tangent to the graph of:

a $y = \sin^{-1}(2x)$ at $x = \frac{1}{4}$

b $y = \tan^{-1}(2x)$ at $x = \frac{1}{2}$

c $y = \cos^{-1}(3x)$ at $x = \frac{1}{6}$

d $y = \cos^{-1}(3x)$ at $x = \frac{1}{2\sqrt{3}}$

13 Let $f(x) = \cos^{-1}\left(\frac{6}{x}\right)$.

a Determine the maximal domain of f.

b Determine $f'(x)$ and show that $f'(x) > 0$ for $x > 6$.

c Sketch the graph of $y = f(x)$ and label endpoints and asymptotes.

11C Anti-derivatives involving inverse trigonometric functions

Learning intentions

- To be able to use the inverse trigonometric functions to integrate suitable expressions.

In the previous section, we established the following rules for differentiation of inverse trigonometric functions:

- If $f(x) = \sin^{-1}\left(\frac{x}{a}\right)$, then $f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$ for $x \in (-a, a)$.
- If $f(x) = \cos^{-1}\left(\frac{x}{a}\right)$, then $f'(x) = \frac{-1}{\sqrt{a^2 - x^2}}$ for $x \in (-a, a)$.
- If $f(x) = \tan^{-1}\left(\frac{x}{a}\right)$, then $f'(x) = \frac{a}{a^2 + x^2}$ for $x \in \mathbb{R}$.

From these results, we obtain the following anti-derivatives:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\, dx = \cos^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in (-a, a)$$

$$\int \frac{a}{a^2 + x^2}\, dx = \tan^{-1}\left(\frac{x}{a}\right) + c \qquad \text{for } x \in \mathbb{R}$$

Note: It follows that $\sin^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{x}{a}\right)$ must be constant for $x \in (-a, a)$.

By substituting $x = 0$, we can see that $\sin^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{x}{a}\right) = \frac{\pi}{2}$ for all $x \in (-a, a)$.

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Example 6

Determine an anti-derivative of each of the following:

a $\dfrac{1}{\sqrt{9-x^2}}$ **b** $\dfrac{1}{\sqrt{9-4x^2}}$ **c** $\dfrac{1}{9+4x^2}$

Solution

a $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + c$

b
$$\begin{aligned}\int \frac{1}{\sqrt{9-4x^2}}\,dx &= \int \frac{1}{2\sqrt{\frac{9}{4}-x^2}}\,dx\\ &= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx\\ &= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c\end{aligned}$$

c
$$\begin{aligned}\int \frac{1}{9+4x^2}\,dx &= \int \frac{1}{4(\frac{9}{4}+x^2)}\,dx\\ &= \frac{2}{3}\int \frac{\frac{3}{2}}{4(\frac{9}{4}+x^2)}\,dx\\ &= \frac{1}{6}\int \frac{\frac{3}{2}}{\frac{9}{4}+x^2}\,dx\\ &= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c\end{aligned}$$

Example 7

Evaluate each of the following definite integrals:

a $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ **b** $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$ **c** $\displaystyle\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx$

Solution

a
$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx &= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1\\ &= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0\\ &= \frac{\pi}{6}\end{aligned}$$

b
$$\begin{aligned}\int_0^2 \frac{1}{4+x^2}\,dx &= \frac{1}{2}\int_0^2 \frac{2}{4+x^2}\,dx\\ &= \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2\\ &= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0)\\ &= \frac{\pi}{8}\end{aligned}$$

c
$$\begin{aligned}\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx &= \int_0^1 \frac{3}{2\sqrt{\frac{9}{4}-x^2}}\,dx\\ &= \frac{3}{2}\int_0^1 \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx\\ &= \frac{3}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^1\\ &= \frac{3}{2}\sin^{-1}\left(\frac{2}{3}\right)\\ &\approx 1.095\end{aligned}$$

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Exercise 11C

Example 6

1 Determine each of the following integrals:

a $\int \frac{1}{\sqrt{9-x^2}}\,dx$ **b** $\int \frac{1}{5+x^2}\,dx$ **c** $\int \frac{1}{1+t^2}\,dt$ **d** $\int \frac{5}{\sqrt{5-x^2}}\,dx$

e $\int \frac{3}{16+x^2}\,dx$ **f** $\int \frac{1}{\sqrt{16-4x^2}}\,dx$ **g** $\int \frac{10}{\sqrt{10-t^2}}\,dt$ **h** $\int \frac{1}{9+16t^2}\,dt$

i $\int \frac{1}{\sqrt{5-2x^2}}\,dx$ **j** $\int \frac{7}{3+y^2}\,dy$

Example 7

2 Evaluate each of the following:

a $\int_0^1 \frac{2}{1+x^2}\,dx$ **b** $\int_0^{\frac{1}{2}} \frac{3}{\sqrt{1-x^2}}\,dx$ **c** $\int_0^1 \frac{5}{\sqrt{4-x^2}}\,dx$ **d** $\int_0^5 \frac{6}{25+x^2}\,dx$

e $\int_0^{\frac{3}{2}} \frac{3}{9+4x^2}\,dx$ **f** $\int_0^2 \frac{1}{8+2x^2}\,dx$ **g** $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}\,dx$ **h** $\int_0^{\frac{3\sqrt{2}}{4}} \frac{1}{\sqrt{9-4x^2}}\,dx$

i $\int_0^{\frac{1}{3}} \frac{3}{\sqrt{1-9y^2}}\,dy$ **j** $\int_0^2 \frac{1}{1+3x^2}\,dx$

11D Integration by substitution

Learning intentions

- To be able to use substitution for integration.

In this section, we introduce the technique of substitution. The substitution will result in one of the forms for integrands covered in Sections 11A and 11C.

First consider the following example.

Example 8

Differentiate each of the following with respect to x:

a $(2x^2+1)^5$ **b** $\cos^3 x$ **c** e^{3x^2}

Solution

a Let $y = (2x^2+1)^5$ and $u = 2x^2+1$.

Then $y = u^5$, $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 4x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= 5u^4\cdot 4x\\ &= 20u^4x\\ &= 20x(2x^2+1)^4\end{aligned}$$

b Let $y = \cos^3 x$ and $u = \cos x$.

Then $y = u^3$, $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = -\sin x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx}\\ &= 3u^2\cdot(-\sin x)\\ &= 3\cos^2 x\cdot(-\sin x)\\ &= -3\cos^2 x\,\sin x\end{aligned}$$

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c Let $y = e^{3x^2}$ and $u = 3x^2$.

Then $y = e^u$, $\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = 6x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx} = e^u \cdot 6x \\ &= 6xe^{3x^2}\end{aligned}$$

This example suggests that a 'converse' of the chain rule can be used to obtain a method for anti-differentiating functions of a particular form.

- From Example 8a: $\int 20x(2x^2+1)^4\,dx = (2x^2+1)^5 + c$

  This is of the form: $\int 5h'(x)\,(h(x))^4\,dx = (h(x))^5 + c$ where $h(x) = 2x^2 + 1$

- From Example 8b: $\int -3\cos^2 x \sin x\,dx = \cos^3 x + c$

  This is of the form: $\int 3h'(x)\,(h(x))^2\,dx = (h(x))^3 + c$ where $h(x) = \cos x$

- From Example 8c: $\int 6xe^{3x^2}\,dx = e^{3x^2} + c$

  This is of the form: $\int h'(x)\,e^{h(x)}\,dx = e^{h(x)} + c$ where $h(x) = 3x^2$

This suggests a method that can be used for integration.

e.g. $\displaystyle\int 2x(x^2+1)^5\,dx = \frac{(x^2+1)^6}{6} + c$ $\quad[h(x) = x^2 + 1]$

$\displaystyle\int \cos x \sin x\,dx = \frac{\sin^2 x}{2} + c$ $\quad[h(x) = \sin x]$

A formalisation of this idea provides a method for integrating functions of this form.

Let $y = \int f(u)\,du$, where $u = g(x)$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\,\frac{du}{dx} \\ &= f(u)\cdot\frac{du}{dx}\end{aligned}$$

$$\therefore \quad y = \int f(u)\,\frac{du}{dx}\,dx$$

This gives the following technique for integration.

Integration by substitution

$$\int f(u)\,\frac{du}{dx}\,dx = \int f(u)\,du$$

This is also called the **change of variable rule**.

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Example 9

Determine an anti-derivative of each of the following:

a $\sin x \cos^2 x$ **b** $5x^2(x^3-1)^{\frac{1}{2}}$ **c** $3xe^{x^2}$

Solution

a $\int \sin x \cos^2 x \, dx$

Let $u = \cos x$. Then $f(u) = u^2$ and $\dfrac{du}{dx} = -\sin x$.

$$\begin{aligned}
\therefore \quad \int \sin x \cos^2 x \, dx &= -\int \cos^2 x \cdot (-\sin x) \, dx \\
&= -\int f(u) \frac{du}{dx} \, dx \\
&= -\int f(u) \, du \\
&= -\int u^2 \, du \\
&= -\frac{u^3}{3} + c \\
&= -\frac{\cos^3 x}{3} + c
\end{aligned}$$

b $\int 5x^2(x^3-1)^{\frac{1}{2}} \, dx$

Let $u = x^3 - 1$.

Then $f(u) = u^{\frac{1}{2}}$ and $\dfrac{du}{dx} = 3x^2$.

$$\begin{aligned}
\therefore \quad &\int 5x^2(x^3-1)^{\frac{1}{2}} \, dx \\
&= \frac{5}{3}\int (x^3-1)^{\frac{1}{2}} \cdot 3x^2 \, dx \\
&= \frac{5}{3}\int u^{\frac{1}{2}} \frac{du}{dx} \, dx \\
&= \frac{5}{3}\int u^{\frac{1}{2}} \, du \\
&= \frac{5}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c \\
&= \frac{10}{9}u^{\frac{3}{2}} + c \\
&= \frac{10}{9}(x^3-1)^{\frac{3}{2}} + c
\end{aligned}$$

c $\int 3xe^{x^2} \, dx$

Let $u = x^2$.

Then $f(u) = e^u$ and $\dfrac{du}{dx} = 2x$.

$$\begin{aligned}
\therefore \quad &\int 3xe^{x^2} \, dx \\
&= \frac{3}{2}\int e^u \cdot 2x \, dx \\
&= \frac{3}{2}\int e^u \frac{du}{dx} \, dx \\
&= \frac{3}{2}\int e^u \, du \\
&= \frac{3}{2}e^u + c \\
&= \frac{3}{2}e^{x^2} + c
\end{aligned}$$

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Example 10

Determine an anti-derivative of each of the following:

a $\dfrac{2}{x^2 + 2x + 6}$

b $\dfrac{3}{\sqrt{9 - 4x - x^2}}$

Solution

a Completing the square gives

$$\begin{aligned} x^2 + 2x + 6 &= x^2 + 2x + 1 + 5 \\ &= (x+1)^2 + 5 \end{aligned}$$

Therefore

$$\int \frac{2}{x^2 + 2x + 6}\,dx = \int \frac{2}{(x+1)^2 + 5}\,dx$$

Let $u = x + 1$. Then $\dfrac{du}{dx} = 1$ and hence

$$\begin{aligned} \int \frac{2}{(x+1)^2 + 5}\,dx &= \int \frac{2}{u^2 + 5}\,du \\ &= \frac{2}{\sqrt{5}} \int \frac{\sqrt{5}}{u^2 + 5}\,du \\ &= \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c \end{aligned}$$

b Completing the square gives

$$\begin{aligned} 9 - 4x - x^2 &= -(x^2 + 4x - 9) \\ &= -((x+2)^2 - 13) \\ &= 13 - (x+2)^2 \end{aligned}$$

Therefore

$$\int \frac{3}{\sqrt{9 - 4x - x^2}}\,dx = \int \frac{3}{\sqrt{13 - (x+2)^2}}\,dx$$

Let $u = x + 2$. Then $\dfrac{du}{dx} = 1$ and hence

$$\begin{aligned} \int \frac{3}{\sqrt{13 - (x+2)^2}}\,dx &= \int \frac{3}{\sqrt{13 - u^2}}\,du \\ &= 3\sin^{-1}\left(\frac{u}{\sqrt{13}}\right) + c \\ &= 3\sin^{-1}\left(\frac{x+2}{\sqrt{13}}\right) + c \end{aligned}$$

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Linear substitutions

Anti-derivatives of expressions such as

$$(2x+3)\sqrt{3x-4}, \quad \frac{2x+5}{\sqrt{3x-4}}, \quad \frac{2x+5}{(x+2)^2}, \quad (2x+4)(x+3)^{20}, \quad x^2\sqrt{3x-1}$$

can be found using a linear substitution.

Example 11

Determine an anti-derivative of each of the following:

a $(2x+1)\sqrt{x+4}$ **b** $\dfrac{2x+1}{(1-2x)^2}$ **c** $x^2\sqrt{3x-1}$

Solution

a $\displaystyle\int (2x+1)\sqrt{x+4}\,dx$

Let $u = x+4$. Then $\dfrac{du}{dx} = 1$ and $x = u-4$.

$$\begin{aligned}\therefore \quad \int (2x+1)\sqrt{x+4}\,dx &= \int (2(u-4)+1)\,u^{\frac{1}{2}}\,du \\ &= \int (2u-7)u^{\frac{1}{2}}\,du \\ &= \int 2u^{\frac{3}{2}} - 7u^{\frac{1}{2}}\,du \\ &= 2\left(\frac{2}{5}u^{\frac{5}{2}}\right) - 7\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= \frac{4}{5}(x+4)^{\frac{5}{2}} - \frac{14}{3}(x+4)^{\frac{3}{2}} + c\end{aligned}$$

b $\displaystyle\int \frac{2x+1}{(1-2x)^2}\,dx$

Let $u = 1-2x$. Then $\dfrac{du}{dx} = -2$ and $2x = 1-u$.

Therefore

$$\begin{aligned}\int \frac{2x+1}{(1-2x)^2}\,dx &= -\frac{1}{2}\int \frac{2-u}{u^2}(-2)\,dx \\ &= -\frac{1}{2}\int \frac{2-u}{u^2}\frac{du}{dx}\,dx \\ &= -\frac{1}{2}\int 2u^{-2} - u^{-1}\,du \\ &= -\frac{1}{2}(-2u^{-1} - \ln|u|) + c \\ &= u^{-1} + \frac{1}{2}\ln|u| + c \\ &= \frac{1}{1-2x} + \frac{1}{2}\ln|1-2x| + c\end{aligned}$$

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c $\int x^2\sqrt{3x-1}\,dx$

Let $u = 3x - 1$. Then $\dfrac{du}{dx} = 3$.

We have $x = \dfrac{u+1}{3}$ and so $x^2 = \dfrac{(u+1)^2}{9}$.

Therefore

$$\begin{aligned}
\int x^2\sqrt{3x-1}\,dx &= \int \frac{(u+1)^2}{9}\sqrt{u}\,dx \\
&= \frac{1}{27}\int (u+1)^2\,u^{\frac{1}{2}}\,(3)\,dx \\
&= \frac{1}{27}\int (u^2+2u+1)\,u^{\frac{1}{2}}\,\frac{du}{dx}\,dx \\
&= \frac{1}{27}\int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du \\
&= \frac{1}{27}\left(\frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + c \\
&= \frac{2}{27}u^{\frac{3}{2}}\left(\frac{1}{7}u^2 + \frac{2}{5}u + \frac{1}{3}\right) + c \\
&= \frac{2}{2835}(3x-1)^{\frac{3}{2}}\left(15(3x-1)^2 + 42(3x-1) + 35\right) + c \\
&= \frac{2}{2835}(3x-1)^{\frac{3}{2}}(135x^2 + 36x + 8) + c
\end{aligned}$$

Exercise 11D

Example 9

SF

1 Find each of the following:

a $\int 2x(x^2+1)^3\,dx$

b $\int \dfrac{x}{(x^2+1)^2}\,dx$

c $\int \cos x \sin^3 x\,dx$

d $\int \dfrac{\cos x}{\sin^2 x}\,dx$

e $\int (2x+1)^5\,dx$

f $\int 5x\sqrt{9+x^2}\,dx$

g $\int x(x^2-3)^5\,dx$

h $\int \dfrac{x+1}{(x^2+2x)^3}\,dx$

i $\int \dfrac{2}{(3x+1)^3}\,dx$

j $\int \dfrac{1}{\sqrt{1+x}}\,dx$

k $\int (x^2-2x)(x^3-3x^2+1)^4\,dx$

l $\int \dfrac{3x}{x^2+1}\,dx$

m $\int \dfrac{3x}{2-x^2}\,dx$

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Example 10

2 Determine an anti-derivative of each of the following:

a $\dfrac{1}{x^2 + 2x + 2}$ **b** $\dfrac{1}{x^2 - x + 1}$ **c** $\dfrac{1}{\sqrt{21 - 4x - x^2}}$

d $\dfrac{1}{\sqrt{10x - x^2 - 24}}$ **e** $\dfrac{1}{\sqrt{40 - x^2 - 6x}}$ **f** $\dfrac{1}{3x^2 + 6x + 7}$

Example 11

3 Determine an anti-derivative of each of the following:

a $x\sqrt{2x + 3}$ **b** $x\sqrt{1 - x}$ **c** $6x(3x - 7)^{-\frac{1}{2}}$

d $(2x + 1)\sqrt{3x - 1}$ **e** $\dfrac{2x - 1}{(x - 1)^2}$ **f** $(x + 3)\sqrt{3x + 1}$

g $(x + 2)(x + 3)^{\frac{1}{3}}$ **h** $\dfrac{5x - 1}{(2x + 1)^2}$ **i** $x^2\sqrt{x - 1}$

j $\dfrac{x^2}{\sqrt{x - 1}}$

11E Definite integrals by substitution

Learning intentions

► To be able to evaluate definite integrals using substitution.

Example 12

Evaluate $\int_0^4 3x\sqrt{x^2 + 9}\, dx$.

Solution

Let $u = x^2 + 9$. Then $\dfrac{du}{dx} = 2x$ and so

$$\begin{aligned}
\int 3x\sqrt{x^2 + 9}\, dx &= \frac{3}{2}\int \sqrt{x^2 + 9} \cdot 2x\, dx \\
&= \frac{3}{2}\int u^{\frac{1}{2}} \frac{du}{dx}\, dx \\
&= \frac{3}{2}\int u^{\frac{1}{2}}\, du \\
&= \frac{3}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c \\
&= u^{\frac{3}{2}} + c \\
&= (x^2 + 9)^{\frac{3}{2}} + c
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \int_0^4 3x\sqrt{x^2 + 9}\, dx &= \left[(x^2 + 9)^{\frac{3}{2}}\right]_0^4 \\
&= 25^{\frac{3}{2}} - 9^{\frac{3}{2}} \\
&= 125 - 27 = 98
\end{aligned}$$

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In a definite integral which involves the change of variable rule, it is not necessary to return to an expression in x if the values of u corresponding to each of the limits of x are found.

For the previous example:

- $x = 0$ implies $u = 9$
- $x = 4$ implies $u = 25$

Therefore the integral can be evaluated as

$$\frac{3}{2}\int_9^{25} u^{\frac{1}{2}}\,du = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^{25} = 125 - 27 = 98$$

Example 13

Evaluate the following:

a $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$

b $\int_0^1 2x^2 e^{x^3}\,dx$

Solution

a $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos x\,(\cos^2 x)\,dx$

$$= \int_0^{\frac{\pi}{2}} \cos x\,(1 - \sin^2 x)\,dx$$

Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

When $x = \frac{\pi}{2}$, $u = 1$ and when $x = 0$, $u = 0$.

Therefore the integral becomes

$$\int_0^1 (1 - u^2)\,du = \left[u - \frac{u^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

b $\int_0^1 2x^2 e^{x^3}\,dx$

Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$.

When $x = 1$, $u = 1$ and when $x = 0$, $u = 0$.

We have

$$\frac{2}{3}\int_0^1 e^{x^3}\cdot(3x^2)\,dx = \frac{2}{3}\int_0^1 e^u\,du$$

$$= \frac{2}{3}[e^u]_0^1$$

$$= \frac{2}{3}(e^1 - e^0)$$

$$= \frac{2}{3}(e - 1)$$

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Skill-sheet

Exercise 11E

SF

Example 12, 13

1 Evaluate each of the following definite integrals:

a $\int_0^3 x\sqrt{x^2+16}\,dx$ **b** $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\,dx$ **c** $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$

d $\int_3^4 x(x-3)^{17}\,dx$ **e** $\int_0^1 x\sqrt{1-x}\,dx$ **f** $\int_e^{e^2} \dfrac{1}{x\ln x}\,dx$

g $\int_0^4 \dfrac{1}{\sqrt{3x+4}}\,dx$ **h** $\int_{-1}^1 \dfrac{e^x}{e^x+1}\,dx$ **i** $\int_0^{\frac{\pi}{4}} \dfrac{\sin x}{\cos^3 x}\,dx$

j $\int_0^1 \dfrac{2x+3}{x^2+3x+4}\,dx$ **k** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{\cos x}{\sin x}\,dx$ **l** $\int_{-4}^{-3} \dfrac{2x}{1-x^2}\,dx$

m $\int_{-2}^{-1} \dfrac{e^x}{1-e^x}\,dx$

11F Using trigonometric identities for integration

Learning intentions

▶ To be able to use trigonometric identities for integration.

Products of sines and cosines

Integrals of the form $\int \sin^m x \cos^n x\,dx$, where m and n are non-negative integers, can be considered in the following three cases.

Case A: the power of sine is odd

If m is odd, write $m = 2k+1$. Then

$$\begin{aligned}\sin^{2k+1} x &= (\sin^2 x)^k \sin x\\ &= (1-\cos^2 x)^k \sin x\end{aligned}$$

and the substitution $u = \cos x$ can now be made.

Case B: the power of cosine is odd

If m is even and n is odd, write $n = 2k+1$. Then

$$\begin{aligned}\cos^{2k+1} x &= (\cos^2 x)^k \cos x\\ &= (1-\sin^2 x)^k \cos x\end{aligned}$$

and the substitution $u = \sin x$ can now be made.

Case C: both powers are even

If both m and n are even, then the identity $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, $\cos^2 x = \frac{1}{2}(1+\cos(2x))$ or $\sin(2x) = 2\sin x \cos x$ can be used.

Also note that $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + c$. The identity $\tan^2 x + 1 = \sec^2 x$ is used in the following example.

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Example 14

Find:

a $\int \cos^2 x \, dx$ **b** $\int \tan^2 x \, dx$ **c** $\int \sin(2x)\cos(2x) \, dx$

d $\int \cos^4 x \, dx$ **e** $\int \sin^3 x \cos^2 x \, dx$

Solution

a Use the identity $\cos(2x) = 2\cos^2 x - 1$. Rearranging gives

$$\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$$

$$\begin{aligned} \therefore \quad \int \cos^2 x \, dx &= \frac{1}{2}\int \cos(2x) + 1 \, dx \\ &= \frac{1}{2}\left(\frac{1}{2}\sin(2x) + x\right) + c \\ &= \frac{1}{4}\sin(2x) + \frac{x}{2} + c \end{aligned}$$

b Use the identity $\tan^2 x + 1 = \sec^2 x$. This gives $\tan^2 x = \sec^2 x - 1$ and so

$$\begin{aligned} \int \tan^2 x \, dx &= \int \sec^2 x - 1 \, dx \\ &= \tan x - x + c \end{aligned}$$

c Use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$.

Let $\theta = 2x$. Then $\sin(4x) = 2\sin(2x)\cos(2x)$ and so $\sin(2x)\cos(2x) = \frac{1}{2}\sin(4x)$.

$$\begin{aligned} \therefore \quad \int \sin(2x)\cos(2x) \, dx &= \frac{1}{2}\int \sin(4x) \, dx \\ &= \frac{1}{2}\left(-\frac{1}{4}\cos(4x)\right) + c \\ &= -\frac{1}{8}\cos(4x) + c \end{aligned}$$

d $\cos^4 x = (\cos^2 x)^2 = \left(\dfrac{\cos(2x) + 1}{2}\right)^2 = \dfrac{1}{4}(\cos^2(2x) + 2\cos(2x) + 1)$

As $\cos(4x) = 2\cos^2(2x) - 1$, this gives

$$\begin{aligned} \cos^4 x &= \frac{1}{4}\left(\frac{\cos(4x) + 1}{2} + 2\cos(2x) + 1\right) \\ &= \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8} \end{aligned}$$

$$\begin{aligned} \therefore \quad \int \cos^4 x \, dx &= \int \frac{1}{8}\cos(4x) + \frac{1}{2}\cos(2x) + \frac{3}{8} \, dx \\ &= \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c \end{aligned}$$

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e $\int \sin^3 x \cos^2 x \, dx = \int \sin x (\sin^2 x) \cos^2 x \, dx$

$= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$

Now let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$. We obtain

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= -\int (-\sin x)(1-u^2)(u^2) \, dx \\ &= -\int (1-u^2) u^2 \frac{du}{dx} \, dx \\ &= -\int u^2 - u^4 \, du \\ &= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c \\ &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c\end{aligned}$$

Skill-sheet

Exercise 11F

SF

1 Determine an anti-derivative of each of the following:

a $\sin^2 x$ **b** $\sin^4 x$ **c** $2\tan^2 x$
d $2\sin(3x)\cos(3x)$ **e** $\sin^2(2x)$ **f** $\tan^2(2x)$
g $\sin^2 x \cos^2 x$ **h** $\cos^2 x - \sin^2 x$ **i** $\cot^2 x$
j $\cos^3(2x)$

2 Determine an anti-derivative of each of the following:

a $\sec^2 x$ **b** $\sec^2(2x)$ **c** $\sec^2(\frac{1}{2}x)$
d $\sec^2(kx)$ **e** $\tan^2(3x)$ **f** $1 - \tan^2 x$
g $\tan^2 x - \sec^2 x$ **h** $\operatorname{cosec}^2\left(x - \frac{\pi}{2}\right)$

3 Evaluate each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ **b** $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx$ **c** $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx$
d $\int_0^{\frac{\pi}{4}} \cos^4 x \, dx$ **e** $\int_0^{\pi} \sin^3 x \, dx$ **f** $\int_0^{\frac{\pi}{2}} \sin^2(2x) \, dx$
g $\int_0^{\frac{\pi}{3}} \sin^2 x \cos^2 x \, dx$ **h** $\int_0^{1} \sin^2 x + \cos^2 x \, dx$

4 Determine an anti-derivative of each of the following:

a $\cos^3 x$ **b** $\sin^3\left(\frac{x}{4}\right)$ **c** $\cos^2(4\pi x)$
d $7\cos^7 t$ **e** $\cos^3(5x)$ **f** $8\sin^4 x$
g $\sin^2 x \cos^4 x$ **h** $\cos^5 x$

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11G Partial fractions

Learning intentions

- To be able to use partial fractions for integration.

A **rational function** has a rule of the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials. For example:

$$f(x) = \frac{4x + 2}{x^2 - 1}$$

- If the degree of $g(x)$ is less than the degree of $h(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then $f(x)$ is an **improper fraction**.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x + 2}{x^2 - 1} = \frac{3}{x - 1} + \frac{1}{x + 1}$$

We will see that this is a useful technique for integration.

Proper fractions

For proper fractions, the method used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have two distinct linear factors.

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax + b}$.

To resolve an algebraic fraction into its partial fractions:

Step 1 Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.

Step 2 Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.

Step 3 Find the values of the introduced constants A and B by substituting appropriate values for x or by equating coefficients.

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Example 15

Resolve $\dfrac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \qquad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$. Then

$$3x+5 = A(x+3) + B(x-1) \qquad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$

$$\therefore \quad A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$

$$\therefore \quad B = 1$$

Hence $\dfrac{3x+5}{(x-1)(x+3)} = \dfrac{2}{x-1} + \dfrac{1}{x+3}$.

Explanation

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Notes:

- You could substitute any values of x to find A and B in this way, but these values simplify the calculations.
- The method of equating coefficients could also be used here.

Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 16

Express $\dfrac{x^5+2}{x^2-1}$ as partial fractions.

Solution

Step 1: Divide through

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2 - 1 & x^5 + 2 \\ & x^5 - x^3 \\ \hline & \quad x^3 + 2 \\ & \quad x^3 - x \\ \hline & \qquad x + 2 \end{array}$$

Therefore

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}$$

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Step 2: Resolve the proper fraction

Now consider the resulting proper fraction:

$$\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

This gives the equation

$$x + 2 = A(x+1) + B(x-1)$$

Let $x = -1$: $\quad 1 = -2B$

$$\therefore \quad B = -\frac{1}{2}$$

Let $x = 1$: $\quad 3 = 2A$

$$\therefore \quad A = \frac{3}{2}$$

So we have

$$\frac{x+2}{x^2-1} = \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$$

Hence the original fraction can be expressed as

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$$

Summary of partial fractions

- Examples of resolving a proper fraction into partial fractions:
  - **Distinct linear factors**

$$\frac{3x-4}{(2x-3)(x+5)} = \frac{A}{2x-3} + \frac{B}{x+5}$$

Using partial fractions for integration

We now use partial fractions to help perform integration.

Example 17

Determine $\int \frac{3x+5}{(x-1)(x+3)}\,dx$.

Solution

In Example 15, we found that

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

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Therefore

$$\int \frac{3x+5}{(x-1)(x+3)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{1}{x+3}\,dx$$
$$= 2\ln|x-1| + \ln|x+3| + c$$

Using the logarithm rules:

$$\int \frac{3x+5}{(x-1)(x+3)}\,dx = \ln\big((x-1)^2\,|x+3|\big) + c$$

Improper fractions

If the degree of the numerator is greater than or equal to the degree of the denominator, then division must take place first.

Example 18

Determine $\int \frac{x^5+2}{x^2-1}\,dx$.

Solution

In Example 18, we divided through to find that

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}$$

Expressing as partial fractions:

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$$

Hence

$$\int \frac{x^5+2}{x^2-1}\,dx = \int x^3 + x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\,dx$$
$$= \frac{x^4}{4} + \frac{x^2}{2} + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$
$$= \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2}\ln\left(\frac{|x-1|^3}{|x+1|}\right) + c$$

Exercise 11G

Example 15 **1** Resolve the following rational expressions into partial fractions:

a $\dfrac{5x+1}{(x-1)(x+2)}$ **b** $\dfrac{-1}{(x+1)(2x+1)}$ **c** $\dfrac{3x-2}{x^2-4}$

d $\dfrac{4x+7}{x^2+x-6}$ **e** $\dfrac{7-x}{(x-4)(x+1)}$

SF

Example 16 **2** Resolve $\dfrac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

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Example 17

3 Decompose $\dfrac{9}{(x-10)(x-1)}$ into partial fractions and find its anti-derivatives.

4 Decompose each of the following into partial fractions and find their anti-derivatives:

a $\dfrac{7}{(x-2)(x+5)}$ b $\dfrac{x+3}{x^2-3x+2}$ c $\dfrac{2x+1}{(x+1)(x-1)}$

Example 18

d $\dfrac{2x^2}{x^2-1}$ e $\dfrac{2x+1}{x^2+4x-12}$ f $\dfrac{4x-2}{(x-2)(x+4)}$

5 Determine an anti-derivative of each of the following:

a $\dfrac{2x-3}{x^2-5x+6}$ b $\dfrac{5x+1}{(x-1)(x+2)}$ c $\dfrac{x^3-2x^2-3x+9}{x^2-4}$

d $\dfrac{4x+10}{x^2+5x+4}$ e $\dfrac{x^3+x^2-3x+3}{x+2}$ f $\dfrac{x^3+3}{x^2-x}$

6 Evaluate the following:

a $\displaystyle\int_1^2 \frac{1}{x(x+1)}\,dx$ b $\displaystyle\int_0^1 \frac{1}{(x+1)(x+2)}\,dx$ c $\displaystyle\int_2^3 \frac{x-2}{(x-1)(x+2)}\,dx$

d $\displaystyle\int_0^1 \frac{x^2}{x^2+3x+2}\,dx$ e $\displaystyle\int_2^3 \frac{x+7}{(x+3)(x-1)}\,dx$ f $\displaystyle\int_2^3 \frac{x+2}{x(x+4)}\,dx$

g $\displaystyle\int_0^1 \frac{1-4x}{3+x-2x^2}\,dx$ h $\displaystyle\int_1^2 \frac{1}{x(x-4)}\,dx$ i $\displaystyle\int_{-3}^{-2} \frac{1-4x}{(x+6)(x+1)}\,dx$

11H Integration by parts

Learning intentions

- To be able to complete integrations using integration by parts.

The product rule is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate both sides with respect to x:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

By rearranging this equation, we obtain the following technique for integration.

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Note: We can use integration by parts to find an integral $\displaystyle\int u\frac{dv}{dx}\,dx$ if the integral $\displaystyle\int v\frac{du}{dx}\,dx$ is easier to find.

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Example 19

Determine an anti-derivative of each of the following:

a $x\cos x$ **b** xe^x **c** $\arcsin x$

Solution

a Let $u = x$ and $\dfrac{dv}{dx} = \cos x$.

Then $\dfrac{du}{dx} = 1$ and $v = \sin x$. (Choose v to be the simplest anti-derivative of $\dfrac{dv}{dx}$.)

So
$$\begin{aligned}\int x\cos x\,dx &= \int u\frac{dv}{dx}\,dx\\ &= uv - \int v\frac{du}{dx}\,dx\\ &= x\sin x - \int \sin x\,dx\\ &= x\sin x + \cos x + c\end{aligned}$$

b $\int xe^x\,dx$

Let $u = x$ and $\dfrac{dv}{dx} = e^x$. Then $\dfrac{du}{dx} = 1$ and $v = e^x$.

So
$$\begin{aligned}\int xe^x\,dx &= \int u\frac{dv}{dx}\,dx\\ &= uv - \int v\frac{du}{dx}\,dx\\ &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + c\end{aligned}$$

c $\int \arcsin x\,dx$

Let $u = \arcsin x$ and $\dfrac{dv}{dx} = 1$. Then $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ and $v = x$.

So
$$\begin{aligned}\int \arcsin x\,dx &= \int u\frac{dv}{dx}\,dx\\ &= uv - \int v\frac{du}{dx}\,dx\\ &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx\\ &= x\arcsin x + \sqrt{1-x^2} + c\end{aligned}$$

Note: We can find $\int \dfrac{x}{\sqrt{1-x^2}}\,dx$ by using the substitution $w = 1 - x^2$.

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Using integration by parts more than once

In some cases, we need to use integration by parts more than once.

Example 20

Determine $\int x^2 e^x \, dx$.

Solution

Let $u = x^2$ and $\dfrac{dv}{dx} = e^x$. Then $\dfrac{du}{dx} = 2x$ and $v = e^x$.

So
$$\begin{aligned}\int x^2 e^x \, dx &= \int u \frac{dv}{dx} \, dx \\ &= uv - \int v \frac{du}{dx} \, dx \\ &= x^2 e^x - \int 2xe^x \, dx \\ &= x^2 e^x - 2(xe^x - e^x) + c \qquad \text{(using Example 19b)} \\ &= (x^2 - 2x + 2)e^x + c\end{aligned}$$

Using integration by parts by solving for the unknown integral

Integration by parts can be applied to expressions of the form $e^{ax} \sin(bx)$ and $e^{ax} \cos(bx)$ in a different way. Again, we use integration by parts twice. We form an equation which we can solve for the unknown integral.

Example 21

Determine $\int e^x \cos x \, dx$.

Solution

Let $u = e^x$ and $\dfrac{dv}{dx} = \cos x$. Then $\dfrac{du}{dx} = e^x$ and $v = \sin x$.

So, using integration by parts, we obtain

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx \qquad (1)$$

Similarly, we can use integration by parts to obtain

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx \qquad (2)$$

Substitute (2) in (1) and then rearrange:

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx\right)$$

$$\therefore \quad 2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + c$$

Now dividing by 2 and renaming the constant gives

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c$$

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Using integration by parts for definite integrals

We can also use integration by parts to evaluate definite integrals.

$$\int_a^b u\,\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\,\frac{du}{dx}\,dx$$

Example 22

Evaluate $\int_1^2 \ln x\,dx$.

Solution

Let $u = \ln x$ and $\frac{dv}{dx} = 1$. Then $\frac{du}{dx} = \frac{1}{x}$ and $v = x$.

We have

$$\begin{aligned}\int_1^2 \ln x\,dx &= [x\ln x]_1^2 - \int_1^2 1\,dx\\ &= [x\ln x]_1^2 - [x]_1^2\\ &= 2\ln 2 - (2-1)\\ &= 2\ln 2 - 1\end{aligned}$$

Exercise 11H

Example 19

1 Determine an anti-derivative of each of the following:

a xe^{-x} **b** $\ln x$ **c** $x\sin x$ **d** $\arccos x$
e $x\cos(3x)$ **f** $x\sec^2 x$ **g** $x\tan^2 x$ **h** $\arcsin(2x)$
i $\arctan x$ **j** $(x+1)e^{-x}$ **k** $x\arctan x$ **l** $x\ln x$
m $x^2\ln x$ **n** $x^{-\frac{1}{2}}\ln x$ **o** $(x+3)e^x$ **p** $x^5\ln x$
q xe^{2x+1} **r** $x\ln(2x)$

Example 20

2 Determine an anti-derivative of each of the following:

a x^2e^{-x} **b** $x^2\sin x$

Example 21

3 Determine an anti-derivative of each of the following:

a $e^x\sin x$ **b** $e^{2x}\cos(3x)$ **c** $e^{3x}\sin x$ **d** $e^x\sin\left(\frac{x}{2}\right)$

Example 22

4 Evaluate each of the following:

a $\int_0^2 xe^{2x}\,dx$ **b** $\int_0^{2\pi} x\sin(4x)\,dx$ **c** $\int_0^{\frac{\pi}{4}} x\cos(4x)\,dx$
d $\int_0^1 2xe^{3x}\,dx$ **e** $\int_0^{\pi} (4x-3)\sin\left(\frac{x}{4}\right)dx$ **f** $\int_0^1 x^2e^{3x-1}\,dx$
g $\int_1^2 \ln(3x)\,dx$ **h** $\int_0^2 x^2e^{2x}\,dx$ **i** $\int_1^3 x^2\ln x\,dx$

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11I Further techniques and miscellaneous exercises

Learning intentions

- To further explore techniques for integration.

In this section, the different techniques are arranged so that a choice must be made of the most suitable one for a particular problem. Often there is more than one appropriate choice.

The relationship between a function and its derivative is also exploited. This is illustrated in the following example.

Example 23

a Find the derivative of $\sin^{-1}(x) + x\sqrt{1-x^2}$.

b Hence evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.

Solution

a Let $y = \sin^{-1}(x) + x\sqrt{1-x^2}$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \left(\sqrt{1-x^2} + \frac{(-x)x}{\sqrt{1-x^2}}\right) \qquad \text{(using the product rule for } x\sqrt{1-x^2}\text{)}$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1-x^2-x^2}{\sqrt{1-x^2}}$$

$$= \frac{2(1-x^2)}{\sqrt{1-x^2}}$$

$$= 2\sqrt{1-x^2}$$

b From part **a**, we have

$$\int 2\sqrt{1-x^2}\,dx = \sin^{-1}(x) + x\sqrt{1-x^2} + c$$

$$\therefore \quad \int_0^{\frac{1}{2}} 2\sqrt{1-x^2}\,dx = \Big[\sin^{-1}(x) + x\sqrt{1-x^2}\Big]_0^{\frac{1}{2}}$$

$$\therefore \quad \int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\sin^{-1}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\sqrt{1-\left(\tfrac{1}{2}\right)^2} - \left(\sin^{-1}(0) + 0\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Skill-sheet

Exercise 11I

SF

1 If $\int_0^1 \frac{1}{(x+1)(x+2)}\,dx = \ln p$, determine p.

2 Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 x \cos x\,dx$.

3 Evaluate $\int_0^1 \frac{e^{2x}}{1+e^x}\,dx$.

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4 Evaluate $\int_0^{\frac{\pi}{3}} \sin^3 x \cos x \, dx$.

5 Evaluate $\int_3^4 \frac{x}{(x-2)(x+1)} \, dx$.

6 Determine c if $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x} \, dx = \ln c$.

7 Determine an anti-derivative of $\sin(3x)\cos^5(3x)$.

8 If $\int_4^6 \frac{2}{x^2-4} \, dx = \ln p$, find p.

9 If $\int_5^6 \frac{3}{x^2-5x+4} \, dx = \ln p$, find p.

10 Determine an anti-derivative of each of the following:

a $\frac{\cos x}{\sin^3 x}$ **b** $x(4x^2+1)^{\frac{3}{2}}$ **c** $\sin^2 x \cos^3 x$ **d** $\frac{e^x}{e^{2x}-2e^x+1}$

11 Evaluate $\int_0^3 \frac{x}{\sqrt{25-x^2}} \, dx$.

12 Determine an anti-derivative of each of the following:

a $\frac{1}{(x+1)^2+4}$ **b** $\frac{1}{\sqrt{1-9x^2}}$ **c** $\frac{1}{\sqrt{1-4x^2}}$ **d** $\frac{1}{(2x+1)^2+9}$

CF

Example 23

13 Let $f(x) = \sin^{-1}\left(\frac{1}{\sqrt{x}}\right)$ for $x > 1$.

a Determine $f'(x)$.

b Using the result of part **a**, find $\int_2^4 \frac{1}{x\sqrt{x-1}} \, dx$.

14 For each of the following, use an appropriate substitution to find an expression for the anti-derivative in terms of $f(x)$:

a $\int f'(x)(f(x))^2 \, dx$ **b** $\int \frac{f'(x)}{(f(x))^2} \, dx$

c $\int \frac{f'(x)}{f(x)} \, dx$, where $f(x) > 0$ **d** $\int f'(x)\sin(f(x)) \, dx$

15 If $y = x\sqrt{4-x}$, find $\frac{dy}{dx}$ and simplify. Hence evaluate $\int_0^2 \frac{8-3x}{\sqrt{4-x}} \, dx$.

16 Determine a, b and c such that $\frac{2x^3-11x^2+20x-13}{(x-2)^2} = ax+b+\frac{c}{(x-2)^2}$ for all $x \neq 2$.

Hence find $\int \frac{2x^3-11x^2+20x-13}{(x-2)^2} \, dx$.

SF

17 Evaluate each of the following:

a $\int_0^{\frac{\pi}{4}} \sin^2(2x)\,dx$ **b** $\int_{-1}^{0}(14-2x)\sqrt{x^2-14x+1}\,dx$

c $9\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx$ **d** $\int_e^{e^2} \frac{1}{x\ln x}\,dx$

e $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ **f** $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}\,dx$

18 Determine $\int \sin x\cos x\,dx$ using:

a the substitution $u = \sin x$

b the identity $\sin(2x) = 2\sin x\cos x$.

CF

19 **a** If $y = \ln(x+\sqrt{x^2+1})$, find $\frac{dy}{dx}$. Hence find $\int \frac{1}{\sqrt{x^2+1}}\,dx$.

b If $y = \ln(x+\sqrt{x^2-1})$, find $\frac{dy}{dx}$. Hence show that $\int_2^7 \frac{1}{\sqrt{x^2-1}}\,dx = \ln(2+\sqrt{3})$.

SF

20 Determine an anti-derivative of each of the following:

a $\frac{1}{4+x^2}$ **b** $\frac{1}{4-x^2}$ **c** $\frac{4+x^2}{x}$ **d** $\frac{x}{4+x^2}$

e $\frac{x^2}{4+x^2}$ **f** $\frac{1}{1+4x^2}$ **g** $x\sqrt{4+x^2}$ **h** $x\sqrt{4+x}$

i $\frac{1}{\sqrt{4-x}}$ **j** $\frac{1}{\sqrt{4-x^2}}$ **k** $\frac{x}{\sqrt{4-x}}$ **l** $\frac{x}{\sqrt{4-x^2}}$

21 Evaluate each of the following definite integrals:

a $\int_1^2 xe^{2x}\,dx$ **b** $\int_1^e x\ln x\,dx$ **c** $\int_0^{\pi}(x-\pi)\sin x\,dx$

22 Determine constants c and d such that $\int_2^3 \frac{x^3-x+2}{x^2-1}\,dx = c + \ln d$.

CF

23 **a** Differentiate $f(x) = \sin(x)\cos^{n-1}(x)$.

b Hence verify that $n\int \cos^n x\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx$.

c Hence evaluate:

i $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$ **ii** $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$

iii $\int_0^{\frac{\pi}{2}} \cos^4 x\sin^2 x\,dx$ **iv** $\int_0^{\frac{\pi}{4}} \sec^4(x)\,dx$

SF

24 Determine:

a $\int \frac{x}{(x+1)^n}\,dx,\ n \neq 1$ and $n \neq 2$ **b** $\int_1^2 x(x-1)^n\,dx,\ n \neq -1$ and $n \neq -2$

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CF

25 **a** Evaluate $\int_0^1 (1 + ax)^2 \, dx$.

b For what value of a is the value of this integral a minimum?

26 **a** Differentiate $\dfrac{a \sin x - b \cos x}{a \cos x + b \sin x}$ with respect to x.

b Hence evaluate $\int_0^{\frac{\pi}{2}} \dfrac{1}{(a \cos x + b \sin x)^2} \, dx$.

27 Let $U_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, where $n \in \mathbb{Z}$ with $n > 1$.

a Express $U_n + U_{n-2}$ in terms of n.

b Hence show that $U_6 = \dfrac{13}{15} - \dfrac{\pi}{4}$.

28 **a** Simplify $\dfrac{1}{1 + \tan x} + \dfrac{1}{1 + \cot x}$.

b Let $\varphi = \dfrac{\pi}{2} - \theta$. Show that $\int_0^{\frac{\pi}{2}} \dfrac{1}{1 + \tan \theta} \, d\theta = \int_0^{\frac{\pi}{2}} \dfrac{1}{1 + \cot \varphi} \, d\varphi$.

c Use these results to evaluate $\int_0^{\frac{\pi}{2}} \dfrac{1}{1 + \tan \theta} \, d\theta$.

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Chapter summary

Using the modulus function

$$\int \frac{1}{x}\,dx = \ln|x| + c \qquad \text{for } x \neq 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c \qquad \text{for } ax+b \neq 0$$

Inverse trigonometric functions

$$f(x) = \sin^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{1}{\sqrt{a^2 - x^2}} \qquad \text{for } x \in (-a, a)$$

$$f(x) = \cos^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{-1}{\sqrt{a^2 - x^2}} \qquad \text{for } x \in (-a, a)$$

$$f(x) = \tan^{-1}\left(\frac{x}{a}\right), \qquad f'(x) = \frac{a}{a^2 + x^2} \qquad \text{for } x \in \mathbb{R}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

Integration by substitution

- The change of variable rule is

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du \qquad \text{where } u \text{ is a function of } x$$

- **Linear substitution**
  A linear substitution can be used to find anti-derivatives of expressions such as

$$(2x+3)\sqrt{3x-4}, \qquad \frac{2x+5}{\sqrt{3x-4}} \qquad \text{and} \qquad \frac{2x+5}{(x+2)^2}$$

  Consider $\int f(x)\,g(ax+b)\,dx$.
  Let $u = ax + b$. Then $x = \dfrac{u-b}{a}$ and so

$$\begin{aligned}\int f(x)\,g(ax+b)\,dx &= \int f\left(\frac{u-b}{a}\right)g(u)\,dx \\ &= \frac{1}{a}\int f\left(\frac{u-b}{a}\right)g(u)\,du\end{aligned}$$

- Definite integration involving the change of variable rule:
  Let $u = g(x)$. Then

$$\int_a^b f(u)\frac{du}{dx}\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Trigonometric identities

$$\sin(2x) = 2\sin x \cos x$$

$$\begin{aligned}\cos(2x) &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

$$\tan^2 x + 1 = \sec^2 x$$

Partial fractions

- A **rational function** has a rule of the form

  $$f(x) = \frac{g(x)}{h(x)}$$

  where $g(x)$ and $h(x)$ are polynomials.
  - If the degree of $g(x)$ is less than the degree of $h(x)$, then $f(x)$ is a **proper fraction**.
  - If the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then $f(x)$ is an **improper fraction**.
- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**.
- Examples of resolving a proper fraction into partial fractions:
  - **Distinct linear factors**

    $$\frac{3x-4}{(2x-3)(x+5)} = \frac{A}{2x-3} + \frac{B}{x+5}$$
- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$.
- If $f(x)$ is an improper fraction, then the division must be performed first. Write $f(x)$ in the form

  $$\frac{g(x)}{h(x)} = q(x) + \frac{r(x)}{h(x)}$$

  where the degree of $r(x)$ is less than the degree of $h(x)$.

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

11A **1** I can evaluate definite integrals of functions covered in Mathematical Methods Units 3&4. ☐

See Example 1 and Question 1

11A **2** I can carry out calculations involving the modulus function. ☐

See Example 2 and Question 2

11A **3** I can differentiate expressions involving the modulus function ☐

See Example 3 and Question 3

11A **4** I can calculate definite integrals involving functions f with rule of the form $f(x) = \dfrac{1}{ax+b}$. ☐

See Example 4 and Question 4

11B **5** I can determine the derivative of an inverse trigonometric function. ☐

See Example 5 and Question 1

11C **6** I can use the inverse trigonometric functions to determine the antiderivative of expressions of the form $\dfrac{1}{\sqrt{a^2-x^2}}$ and $\dfrac{a}{a^2+x^2}$. ☐

See Example 6, Example 7 and Questions 1 and 2.

11D **7** I can use substitution to determine antiderivatives.. ☐

See Example 8, Example 9, Example 10, Example 11 and Questions 1, 2 and 3

11E **8** I can use the use substitution to evaluate definite integrals. ☐

See Example 13 and Question 1

11F **9** I can use trigonometric identities with integration. ☐

See Example 14 and Questions 1, 2, 3 and 4

11G **10** I can use partial fractions with integration. ☐

See Example 15, Example 16, Example 17, Example 18 and Questions 1, 2, 3 and 4

11H **11** I can determine integrals using integration by parts. ☐

See Example 19 and Questions 1, 2, 3 and 4

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Review

Short-response questions

Technology-free short-response questions

SF

1 Evaluate each of the following integrals:

a $\int_{-2}^{-1} \frac{1}{2x-3}\,dx$ **b** $\int_{-1}^{0} \frac{1}{2-3x}\,dx$ **c** $\int_{-2}^{0} \frac{2x+5}{x-1}\,dx$

2 Determine $\frac{dy}{dx}$ if:

a $y = \sin^{-1}(2x-1)$ **b** $y = \cos^{-1}(2x+1)$ **c** $y = 1 - \tan^{-1}(1-x)$

d $y = \cos(\sin^{-1} x)$ **e** $y = \tan(\tan^{-1} x)$ **f** $y = \sin^{-1}(e^x)$

3 Given that $f(x) = \tan^{-1}\left(\frac{1}{x-1}\right)$, find $f''(0)$.

4 Find an anti-derivative of each of the following:

a $\cos^3(2x)$ **b** $\frac{2x+3}{4x^2+1}$ **c** $\frac{1}{1-4x^2}$ **d** $\frac{x}{\sqrt{1-4x^2}}$

e $\frac{x^2}{1-4x^2}$ **f** $x\sqrt{1-2x^2}$ **g** $\sin^2\left(x-\frac{\pi}{3}\right)$ **h** $\frac{x}{\sqrt{x^2-2}}$

i $\sin^2(3x)$ **j** $\sin^3(2x)$ **k** $x\sqrt{x+1}$ **l** $\frac{1}{1+\cos(2x)}$

m $\frac{e^{3x}+1}{e^{3x+1}}$ **n** $\frac{x}{x^2-1}$ **o** $\sin^2 x \cos^2 x$ **p** $\frac{x^2}{1+x}$

5 Evaluate each of the following integrals:

a $\int_0^{\frac{1}{2}} x(1-x^2)^{\frac{1}{2}}\,dx$ **b** $\int_0^{\frac{1}{2}} (1-x^2)^{-1}\,dx$ **c** $\int_0^{\frac{1}{2}} x(1+x^2)^{\frac{1}{2}}\,dx$

d $\int_1^2 \frac{1}{6x+x^2}\,dx$ **e** $\int_0^1 \frac{2x^2+3x+2}{x^2+3x+2}\,dx$ **f** $\int_0^1 \frac{1}{\sqrt{4-3x}}\,dx$

g $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ **h** $\int_0^{\frac{\pi}{2}} \sin^2(2x)\,dx$ **i** $\int_{-\pi}^{\pi} \sin^2 x \cos^2 x\,dx$

j $\int_0^{\frac{\pi}{2}} \sin^2(2x)\cos^2(2x)\,dx$ **k** $\int_0^{\frac{\pi}{4}} \frac{2\cos x - \sin x}{2\sin x + \cos x}\,dx$ **l** $\int_{-1}^{2} x^2\sqrt{x^3+1}\,dx$

6 **a** Show that $\frac{x}{x^2+2x+3} = \frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right) - \frac{1}{x^2+2x+3}$.

b Hence find $\int \frac{x}{x^2+2x+3}\,dx$.

CF

7 **a** Differentiate $\sin^{-1}(\sqrt{x})$ and hence find $\int \frac{1}{\sqrt{x(1-x)}}\,dx$.

b Differentiate $\sin^{-1}(x^2)$ and hence find $\int \frac{2x}{\sqrt{1-x^4}}\,dx$.

8 CF **a** Determine $\frac{d}{dx}(x\sin^{-1}x)$ and hence find $\int \sin^{-1}x\,dx$.

b Determine $\frac{d}{dx}(x\ln x)$ and hence find $\int \ln x\,dx$.

c Determine $\frac{d}{dx}(x\tan^{-1}x)$ and hence find $\int \tan^{-1}x\,dx$.

9 SF Determine an anti-derivative of each of the following:

a $\sin(2x)\cos(2x)$ **b** $x^2(x^3+1)^2$ **c** $\dfrac{\cos\theta}{(3+2\sin\theta)^2}$

d xe^{1-x^2} **e** $\tan^2(x+3)$ **f** $\dfrac{2x}{\sqrt{6+2x^2}}$

g $\tan^2 x\sec^2 x$ **h** $\sec^3 x\tan x$ **i** $\tan^2(3x)$

10 Evaluate the following:

a $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$ **b** $\int_1^8 (13-5x)^{\frac{1}{3}}\,dx$ **c** $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$

d $\int_1^2 (3-y)^{\frac{1}{2}}\,dy$ **e** $\int_0^{\pi} \sin^2 x\,dx$ **f** $\int_{-2}^{-1} \dfrac{x^2+1}{x^2+3x}\,dx$

11 CF Find the derivative of $\left(x^2+\dfrac{1}{x}\right)^{\frac{1}{2}}$ and hence evaluate $\int_1^2 (2x-x^{-2})\left(x^2+\dfrac{1}{x}\right)^{-\frac{1}{2}}\,dx$.

12 Let $f(x)=\dfrac{4x^2+16x}{(x-2)^2(x^2+4)}$.

a Given that $f(x)=\dfrac{a}{x-2}+\dfrac{6}{(x-2)^2}-\dfrac{bx+4}{x^2+4}$, find a and b.

b Given that $\int_{-2}^{0} f(x)\,dx=\dfrac{c-\pi-\ln d}{2}$, find c and d.

13 SF Determine an anti-derivative of each of the following:

a $e^{-2x}\cos(2x+3)$ **b** $x\sec^2 x$ **c** $e^{3x}\cos\left(\dfrac{x}{2}\right)$

14 Evaluate each of the following definite integrals:

a $\int_1^2 x^2\ln x\,dx$ **b** $\int_1^2 \dfrac{\ln x}{x}\,dx$ **c** $\int_0^1 xe^{-2x}\,dx$

Technology-active short-response questions

15 SF Evaluate each of the following approximately using your calculator. Give your answer correct to four decimal places.

a $\int_0^{\frac{\pi}{2}} \sin x^2\,dx$ **b** $\int_0^1 e^{x^2}\,dx$ **c** $\dfrac{1}{\sqrt{2\pi}}\int_{-100}^{100} e^{-\frac{1}{2}x^2}\,dx$

d $\int_2^3 \dfrac{1}{\ln(x)}\,dx$ **e** $\int_0^{\frac{1}{2}} \dfrac{1}{\sqrt{1-x^4}}\,dx$ **f** $\int_0^{\frac{1}{2}} \dfrac{2x}{\sqrt{1-x^4}}\,dx$

g $\int_0^{\frac{\pi}{2}} \dfrac{1}{\sqrt{1-\frac{1}{4}\sin^2\theta}}\,d\theta$ **h** $\int_0^{\frac{\pi}{2}} \cos x^2\,dx$

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Review

16 If $F(x)$ is an antiderivative of $2x^2 \ln(x^2)$ and $F(1) = -0.4444$, correct to four decimal places, then calculate $F(4)$ correct to two decimal places.

17 If $\int_0^a \cos^2\theta \, d\theta = 0.6$ where $0 < a < \frac{\pi}{2}$ then determine the value of a correct to two decimal places.

SF

Multiple-choice questions

Technology-free multiple-choice questions

1 If $y = \cos^{-1}\left(\frac{4}{x}\right)$ and $x > 4$, then $\frac{dy}{dx}$ is equal to

A $\frac{-1}{\sqrt{16 - x^2}}$ **B** $\frac{-4}{\sqrt{1 - 16x^2}}$ **C** $\frac{-4x}{\sqrt{x^2 - 16}}$ **D** $\frac{4}{x\sqrt{x^2 - 16}}$

2 $\int \frac{\sin x}{\cos^2 x} \, dx$ is equal to

A $\frac{1}{\cos x} + c$ **B** $\frac{-1}{\cos x} + c$ **C** $\frac{3}{\cos^3 x} + c$ **D** $\frac{-3}{\cos^3 x} + c$

3 $\int_3^5 \frac{1}{(x-2)(x-1)} \, dx$ is equal to

A $\ln\left(\frac{3}{2}\right)$ **B** $\ln\left(\frac{3}{8}\right)$ **C** $\frac{101}{144}$ **D** $\ln(6)$

4 An anti-derivative of $x\sqrt{4 - x}$ is

A $\frac{2x}{3}(4 - x)^{\frac{3}{2}}$ **B** $\frac{x^2}{3}(4 - x)^{\frac{3}{2}}$

C $\frac{8}{3}(4 - x)^{\frac{3}{2}} - \frac{2}{5}(4 - x)^{\frac{5}{2}}$ **D** $\frac{2}{5}(4 - x)^{\frac{5}{2}} - \frac{8}{3}(4 - x)^{\frac{3}{2}}$

5 The definite integral $\int_0^{\frac{\pi}{2}} (x + 1) \sin x \, dx$ is equal to

A 2 **B** $(\frac{\pi}{2} + 2)$ **C** $\frac{\pi}{2}$ **D** 1

6 If $\int_0^m \tan x \sec^2 x \, dx = \frac{3}{2}$, where $m \in \left(0, \frac{\pi}{2}\right)$, then the value of m is

A 0.5 **B** 1 **C** $\frac{\pi}{3}$ **D** $\frac{\pi}{6}$

7 An anti-derivative of $\tan(2x)$ is

A $\frac{1}{2}\sec^2(2x)$ **B** $\frac{1}{2}\ln(\cos(2x))$

C $\frac{1}{2}\ln(\sec(2x))$ **D** $\frac{1}{2}\ln(\sin(2x))$

8 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2 + \cos(2x)} \, dx$ is equal to

A $\frac{1}{\sqrt{2}}$ **B** $\ln\left(\frac{1}{\sqrt{2}}\right)$ **C** $\ln 2$ **D** $\frac{1}{2}\ln 2$

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9 $\int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$ written as an integral with respect to u, where $u = \cos x$, is

A $\int_{\frac{1}{2}}^{1} u^3 \, du$ **B** $\int_0^{\frac{\pi}{3}} u^3 \, du$

C $\int_1^{\frac{1}{2}} u^3\sqrt{1-u^2} \, du$ **D** $\int_{\frac{1}{2}}^{0} u^3\sqrt{1-u^2} \, du$

10 An anti-derivative of $\dfrac{2}{\sqrt{1-16x^2}}$ is

A $\sin^{-1}\left(\dfrac{x}{4}\right)$ **B** $\dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{4}\right)$ **C** $\sin^{-1}(4x)$ **D** $\dfrac{1}{2}\sin^{-1}(4x)$

11 An anti-derivative of $\dfrac{1}{9+4x^2}$ is

A $\dfrac{1}{9}\tan^{-1}\left(\dfrac{2x}{9}\right)$ **B** $\dfrac{1}{3}\tan^{-1}\left(\dfrac{2x}{3}\right)$ **C** $\dfrac{1}{6}\tan^{-1}\left(\dfrac{2x}{3}\right)$ **D** $9\tan^{-1}\left(\dfrac{2x}{9}\right)$

12 If $\dfrac{d}{dx}(xf(x)) = xf'(x) + f(x)$ and $xf'(x) = \dfrac{1}{1+x^2}$, then an anti-derivative of $f(x)$ is

A $xf(x) - \tan^{-1}(x)$ **B** $\ln(x^2+1)$

C $\dfrac{1}{2x}\ln(x^2+1)$ **D** $f(x) - \tan^{-1}(x)$

13 If $F'(x) = f(x)$, then an anti-derivative of $3f(3-2x)$ is

A $\dfrac{3}{2}F(3-2x)$ **B** $-\dfrac{3}{4}(3-2x)^2$

C $\dfrac{3}{4}(3-2x)^2$ **D** $-\dfrac{3}{2}F(3-2x)$

14 Let $y = \sin^{-1}\left(\dfrac{x}{2}\right)$ for $x \in [0, 1]$. Then $\dfrac{d^2y}{dx^2}$ is equal to

A $\cos^{-1}\left(\dfrac{x}{2}\right)$ **B** $x(4-x^2)^{-\frac{3}{2}}$

C $\dfrac{-x}{\sqrt{4-x^2}}$ **D** $\dfrac{-x}{\sqrt{4-x^2}\,(4-x^2)}$

15 If $y = \tan^{-1}\left(\dfrac{1}{3x}\right)$, then $\dfrac{dy}{dx}$ is equal to

A $\dfrac{1}{3(1+x^2)}$ **B** $\dfrac{-1}{3(1+x^2)}$ **C** $\dfrac{1}{3(1+9x^2)}$ **D** $\dfrac{-3}{9x^2+1}$

16 The substitution $u = \sin x$ is made to the integral $\int \cos^3 x \sin^n x \, dx$. If the resulting integral is $\int u^9 - u^{11} \, du$, then the value of n is

A 10 **B** 2 **C** 9 **D** 11

17 Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f(0) = 2$, $f(1) = 3$, $f'(0) = 6$ and $f'(1) = 10$. The value of $\int_0^1 5xf''(x) \, dx$ is

A -16 **B** 45 **C** 10 **D** 36

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Technology-active multiple-choice questions

18 $\int_0^a \frac{2}{\sqrt{4 - x^2}}\, dx = 0.3,\ 0 < a < 2$. The value of a, correct to three decimal places, for this to be true is

A 0.300 **B** 0.302 **C** 0.298 **D** 0.299

19 $\int_{-a}^a \frac{2}{\sqrt{4 - x^2}}\, dx = 0.3,\ 0 < a < 2$. The value of a, correct to three decimal places, for this to be true is

A 0.150 **B** 0.598 **C** 0.154 **D** 0.600

20 The value of $\int_0^2 \cos^2 x - \sin^2 x\, dx$, correct to four decimal places, is

A -0.0348 **B** 0.0349 **C** -0.3784 **D** 2.0000

21 If $F(x)$ is an antiderivative of $x^2(\ln(x))^3$ and $F(1) = 1.9259$, correct to four decimal places, then $F(3)$ correct to two decimal places is equal to

A 2.03 **B** 7.66 **C** 8.12 **D** 17.85

22 Let f and g be functions such that $g(x) = f(x) + 7x$ for all real numbers x. If $\int_5^{11} f(x)\, dx = 6.4$ then the value of $\int_5^{11} f(x) + g(x)\, dx$ is

A 355.2 **B** 336.8 **C** 348.8 **D** 356.4

12

Applications of integral calculus

Chapter contents

In this chapter we revisit the **fundamental theorem of calculus**. We will apply this theorem to the new functions introduced in this course, and use the integration techniques developed in the previous chapter.

We then study two further applications of integration:

- **Volume of a solid of revolution**
  We will see how to find the volume of a solid formed by revolving a bounded region defined by a curve around both the x- and y-axes. You will be able to apply this technique to derive the formula for the volume of a sphere, which you have used for several years.
- **Exponential probability distribution**
  We will use integration to investigate the exponential probability distribution, which is often used to model the time between the occurrence of random events.

We finish this chapter by introducing **Simpson's rule**, which is an efficient numerical technique for approximating definite integrals.

This chapter covers Unit 4 Topic 2: Applications of integral calculus.

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12A The fundamental theorem of calculus

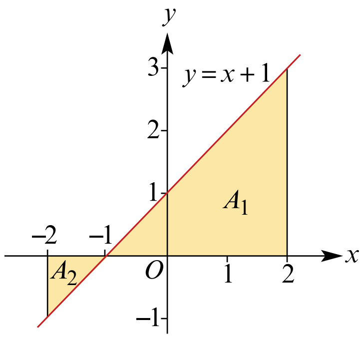
Learning intentions

- To be able to determine areas defined by some of the functions introduced earlier in this course.

In this section we review integration from Mathematical Methods Units 3 & 4. We consider the graphs of some of the functions introduced in earlier chapters, and the areas of regions defined through these functions. It may be desirable to use a graphing package or a calculator to help with the graphing in this section.

Signed area

Consider the graph of $y = x + 1$ shown to the right.



$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

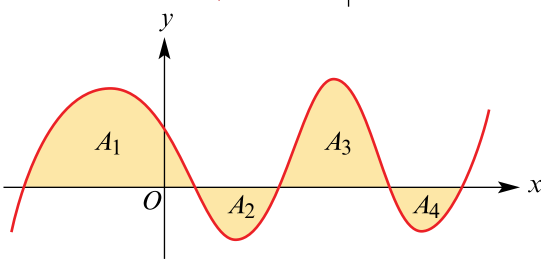
The total area is $A_1 + A_2 = 5$.

The **signed area** is $A_1 - A_2 = 4$.

Regions above the x-axis have **positive signed area**.

Regions below the x-axis have **negative signed area**.

The total area of the shaded region shown adjacent is $A_1 + A_2 + A_3 + A_4$.



The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.

The definite integral

Let f be a continuous function on a closed interval $[a, b]$. The signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by

$$\int_a^b f(x)\,dx$$

and is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$.

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Notes:

- If $f(x) \geq 0$ for all $x \in [a, b]$, the area between $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- If $f(x) \leq 0$ for all $x \in [a, b]$, the area between $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.

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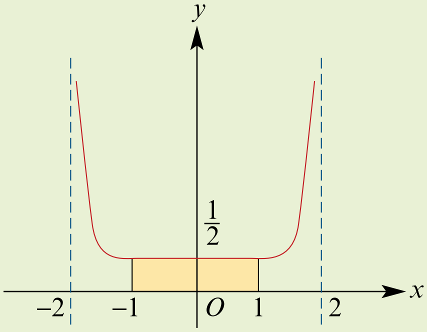
Example 1

The graph of $y = \dfrac{1}{\sqrt{4 - x^2}}$ is shown.

Determine the area of the shaded region.



Solution

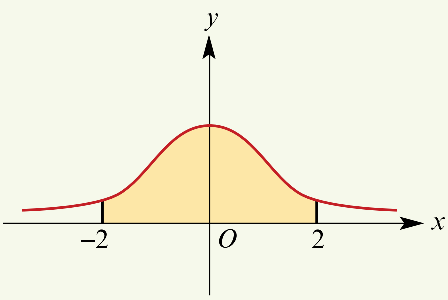
$$\begin{aligned}
\text{Area} &= \int_{-1}^{1} \frac{1}{\sqrt{4 - x^2}}\, dx \\
&= 2\int_{0}^{1} \frac{1}{\sqrt{4 - x^2}}\, dx \qquad \text{(by symmetry)} \\
&= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 \\
&= 2\sin^{-1}\left(\frac{1}{2}\right) \\
&= 2 \times \frac{\pi}{6} \\
&= \frac{\pi}{3}
\end{aligned}$$

Example 2

Determine the area under the graph of $y = \dfrac{6}{4 + x^2}$ between $x = -2$ and $x = 2$.

Solution

$$\begin{aligned}
\text{Area} &= 6\int_{-2}^{2} \frac{1}{4 + x^2}\, dx \\
&= \frac{6}{2}\int_{-2}^{2} \frac{2}{4 + x^2}\, dx \\
&= 6\int_{0}^{2} \frac{2}{4 + x^2}\, dx \qquad \text{(by symmetry)} \\
&= 6\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 \\
&= 6\tan^{-1}(1) \\
&= 6 \times \frac{\pi}{4} \\
&= \frac{3\pi}{2}
\end{aligned}$$



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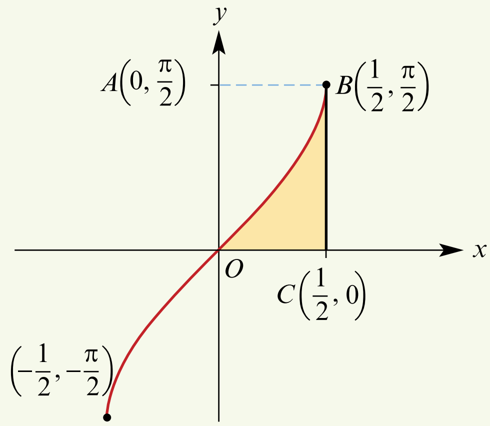
Example 3

Sketch the graph of $f(x) = \sin^{-1}(2x)$, $x \in [-\frac{1}{2}, \frac{1}{2}]$. Shade the region defined by the inequalities $0 \le x \le \frac{1}{2}$ and $0 \le y \le f(x)$. Determine the area of this region.

Solution

$$\text{Area} = \int_0^{\frac{1}{2}} \sin^{-1}(2x)\, dx$$

Note: This definite integral can be evaluated using integration by parts. Here we use a simpler method to find the area.



$$\begin{aligned}\text{Area} &= \text{area rectangle } OABC - \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin y\, dy \\ &= \frac{\pi}{4} - \frac{1}{2}\Big[-\cos y\Big]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

Example 4

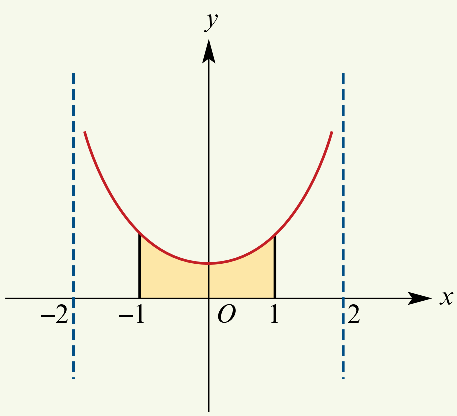
Sketch the graph of $y = \dfrac{1}{4 - x^2}$. Shade the region for the area determined by $\displaystyle\int_{-1}^{1} \frac{1}{4 - x^2}\, dx$ and find this area.

Solution

$$\begin{aligned}\text{Area} &= \int_{-1}^{1} \frac{1}{4 - x^2}\, dx \\ &= \frac{1}{4}\int_{-1}^{1} \frac{1}{2 - x} + \frac{1}{2 + x}\, dx\end{aligned}$$

By symmetry:

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{1} \frac{1}{2 - x} + \frac{1}{2 + x}\, dx \\ &= \frac{1}{2}\left[\ln\left(\frac{2 + x}{2 - x}\right)\right]_0^1 \\ &= \frac{1}{2}(\ln 3 - \ln 1) \\ &= \frac{1}{2}\ln 3\end{aligned}$$



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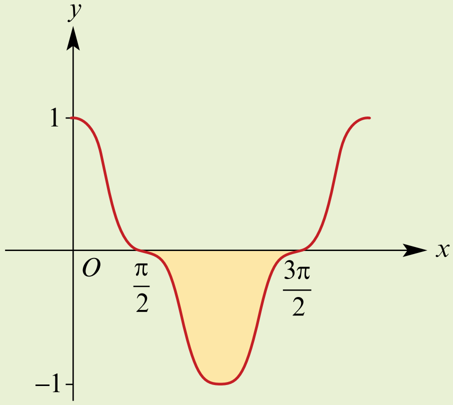
Example 5

The graph of $y = \cos^3 x$ is shown.

Determine the area of the shaded region.



Solution

$$\text{Area} = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 x\, dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \cos^2 x\, dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,(1 - \sin^2 x)\, dx$$

Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$.

When $x = \dfrac{\pi}{2}$, $u = 1$. When $x = \dfrac{3\pi}{2}$, $u = -1$.

$$\therefore \quad \text{Area} = -\int_{1}^{-1} (1 - u^2)\, du$$

$$= -\left[u - \frac{u^3}{3}\right]_1^{-1}$$

$$= -\left(-1 + \frac{1}{3} - \left(1 - \frac{1}{3}\right)\right)$$

$$= \frac{4}{3}$$

Properties of the definite integral

- $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
- $\int_a^a f(x)\, dx = 0$
- $\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx$
- $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
- $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

Exercise 12A

SF

Example 1

1 Sketch the graph of $f(x) = \dfrac{1}{\sqrt{9-4x^2}}$, $x \in \left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$.

Find the area of the region defined by the inequalities $0 \le y \le f(x)$ and $-1 \le x \le 1$.

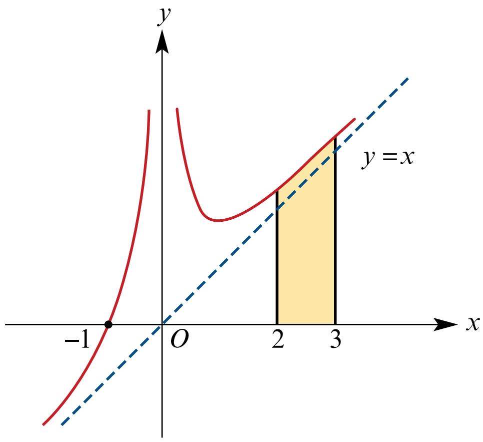
Example 2

2 Sketch the graph of $f(x) = \dfrac{9}{4+x^2}$, $x \in \mathbb{R}$.

Find the area of the region defined by the inequalities $0 \le y \le f(x)$ and $-2 \le x \le 2$.

3 The graph of $f(x) = x + \dfrac{1}{x^2}$ is as shown.

Find the area of the shaded region.



4 Sketch the graph of $f(x) = x + \dfrac{2}{x}$. Shade the region for which the area is determined by the integral $\int_1^2 f(x)\,dx$ and evaluate this integral.

Example 3

5 For each of the following:

i sketch the appropriate graph and shade the required region

ii evaluate the integral.

a $\int_0^1 \tan^{-1} x\,dx$ **b** $\int_0^{\frac{1}{2}} \cos^{-1}(2x)\,dx$ **c** $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{-1}(2x)\,dx$

d $\int_0^1 2\sin^{-1} x\,dx$ **e** $\int_0^2 \sin^{-1}\left(\dfrac{x}{2}\right)dx$ **f** $\int_{-1}^2 \sin^{-1}\left(\dfrac{x}{2}\right)dx$

Example 4

6 Sketch the graph of $g(x) = \dfrac{4}{9-x^2}$. Determine the area of the region with $-2 \le x \le 2$ and $0 \le y \le g(x)$.

7 For the curve with equation $y = -1 + \dfrac{2}{x^2+1}$, find:

a the coordinates of its turning point **b** the equation of its asymptote

c the area enclosed by the curve and the x-axis.

8 Consider the graph of $y = x - \dfrac{4}{x+3}$.

a Determine the coordinates of the intercepts with the axes.

b Determine the equations of all asymptotes.

c Sketch the graph.

d Determine the area bounded by the curve, the x-axis and the line $x = 8$.

SF

9 **a** State the implied domain of the function g with rule $g(x) = \dfrac{1}{(1 - x)(x - 2)}$.

b Sketch the graph of $y = g(x)$, indicating the equation of any asymptotes and the coordinates of the turning points.

c State the range of g.

d Determine the area of the region bounded by the graph of $y = g(x)$, the x-axis and the lines $x = 4$ and $x = 3$.

10 Sketch the graph of $f(x) = \dfrac{-3}{\sqrt{1 - x^2}}$, $x \in (-1, 1)$. Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{-3}{\sqrt{1 - x^2}}\, dx$.

11 Determine the area of the region enclosed by the curve $y = \dfrac{1}{\sqrt{4 - x^2}}$, the x-axis and the lines $x = 1$ and $x = \sqrt{2}$.

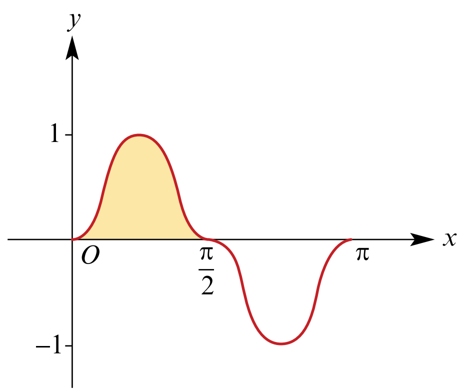
12 Sketch the curve with equation $y = \tan^{-1} x$. Determine the area enclosed between this curve, the line $x = \sqrt{3}$ and the x-axis.

CF

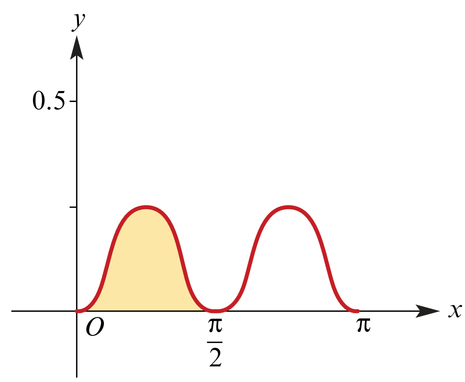
13 Determine the area between the curve $y = \dfrac{2 \ln x}{x}$ and the x-axis from $x = 1$ to $x = e$.

Example 5

14 The graph of $y = \sin^3(2x)$ for $x \in [0, \pi]$ is as shown. Determine the area of the shaded region.



15 The graph of $y = \sin x \cos^2 x$ for $x \in [0, \pi]$ is as shown. Determine the area of the shaded region.



16 Sketch the curve with equation $y = \dfrac{2x}{x + 3}$, showing clearly how the curve approaches its asymptotes. On your diagram, shade the finite region bounded by the curve and the lines $x = 0$, $x = 3$ and $y = 2$. Find the area of this region.

17 **a** Show that the curve $y = \dfrac{3}{(2x + 1)(1 - x)}$ has only one turning point.

b Determine the coordinates of this point and determine its nature.

c Sketch the curve.

d Determine the area of the region enclosed by the curve and the line $y = 3$.

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12B Area of a region between two curves

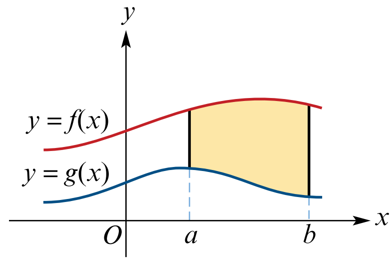
Learning intentions

- To be able to determine the area of the region bounded by two curves.

Let f and g be continuous functions on the interval $[a, b]$ such that

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b]$$

Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ can be found by evaluating



$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

Example 6

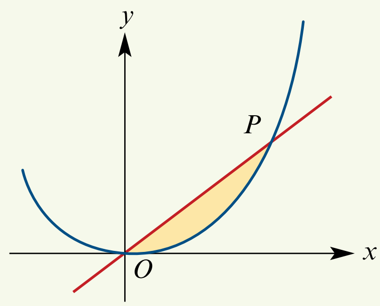
Determine the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

We first find the coordinates of the point P:

$$x^2 = 2x$$
$$x(x-2) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 2$$

Therefore the coordinates of P are $(2, 4)$.



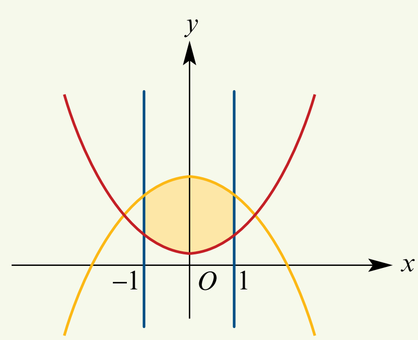
$$\begin{aligned}\text{Required area} &= \int_0^2 2x - x^2\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= 4 - \frac{8}{3} = \frac{4}{3}\end{aligned}$$

Example 7

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution

$$\begin{aligned}\text{Required area} &= \int_{-1}^1 4 - x^2 - (x^2 + 1)\,dx \\ &= \int_{-1}^1 3 - 2x^2\,dx \\ &= \left[3x - \frac{2x^3}{3}\right]_{-1}^1 \\ &= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right) \\ &= \frac{14}{3}\end{aligned}$$



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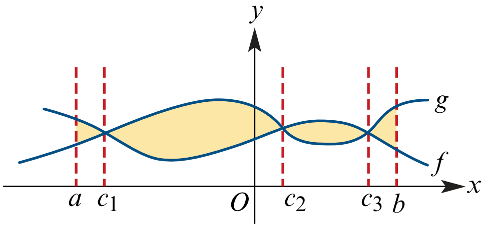
In the two examples considered so far in this section, the graph of one function is 'above' the graph of the other for all of the interval considered.

What happens when the graphs cross?



To find the area of the shaded region, we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$

The absolute value function could also be used here:

$$\left|\int_a^{c_1} f(x) - g(x)\,dx\right| + \left|\int_{c_1}^{c_2} f(x) - g(x)\,dx\right| + \left|\int_{c_2}^{c_3} f(x) - g(x)\,dx\right| + \left|\int_{c_3}^{b} f(x) - g(x)\,dx\right|$$

With a graphics calculator the approximate area can be found by evaluating $\int_a^b |f(x) - g(x)|\,dx$. That is, use the graph of the absolute value function applied to $f(x) - g(x)$.

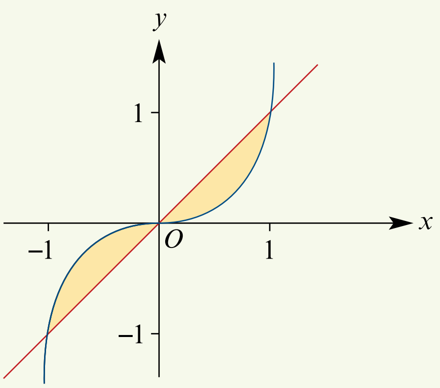
Example 8

Determine the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution

The graphs intersect where $f(x) = g(x)$:

$$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ \therefore \quad x &= 0 \text{ or } x = \pm 1 \end{aligned}$$



We see that:

- $f(x) \geq g(x)$ for $-1 \leq x \leq 0$
- $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by

$$\begin{aligned} \int_{-1}^{0} f(x) - g(x)\,dx + \int_0^1 g(x) - f(x)\,dx &= \int_{-1}^{0} x^3 - x\,dx + \int_0^1 x - x^3\,dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 \\ &= -\left(-\frac{1}{4}\right) + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

Note: The result could also be obtained by observing the symmetry of the graphs, finding the area of the region where both x and y are non-negative, and then multiplying by 2.

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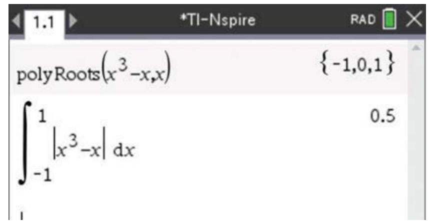
TI-84

Using the TI-Nspire CX non-CAS

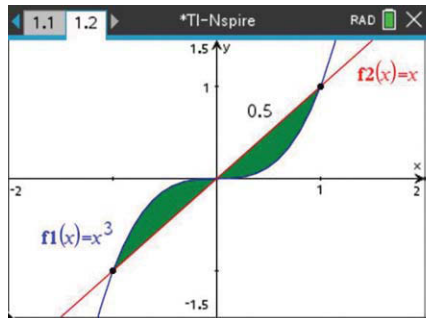
Method 1: Using a Calculator application

- Enter the integral as shown.
  (Use the 2D-template palette (template key) for the definite integral and the absolute value.)



Method 2: Using a Graphs application

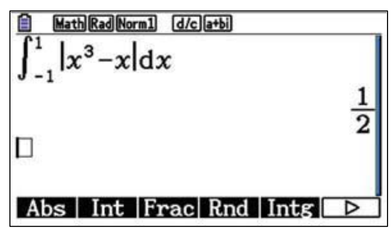
- Enter the functions $f1(x) = x^3$ and $f2(x) = x$ as shown.
- To find the area of the bounded region, use (menu) > **Analyze Graph** > **Bounded Area** and click on the lower and upper intersections of the graphs.



Using the Casio

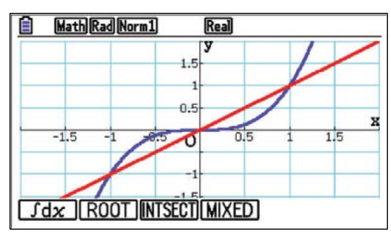
Method 1: Using Run-Matrix mode

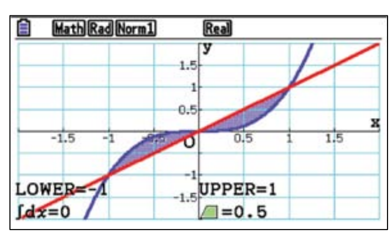
- Go to the **Calculation** menu (OPTN) (F4), and select the integral template **∫dx** (F4).
- To obtain the absolute value, go to the **Numeric** menu (OPTN) (F6) (F4), and select **Abs** (F1).
- Enter the integrand and endpoints as shown.
- Hence the enclosed area is $\frac{1}{2}$ square unit.



Method 2: Using Graph mode

- Plot the graphs of $y = x^3$ and $y = x$.
- Adjust the View Window (SHIFT) (F3) as shown.
- Go to **G-Solve** (SHIFT) (F5) and select **∫dx** (F6) (F3), then **Intersection** (F3).
- Select the first and third intersection points by pressing (EXE) ► ► (EXE).
- Hence the enclosed area is 0.5 square units.





Note: Here the absolute value function is used to simplify the process of finding areas with a graphics calculator. This technique is not helpful when doing these problems by hand.

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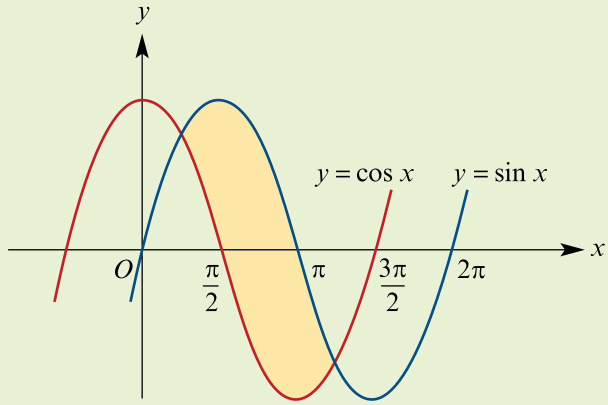
Example 9

Determine the area of the shaded region.



Solution

First find the x-coordinates of the two points of intersection.

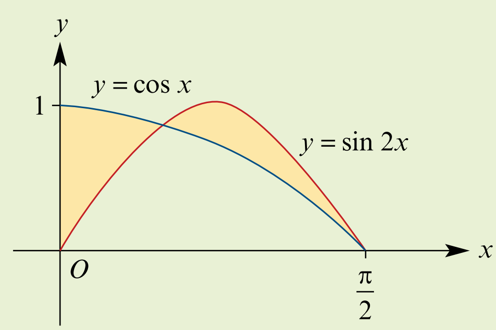
If $\sin x = \cos x$, then $\tan x = 1$ and so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

$$\begin{aligned}
\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx \\
&= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{4}{\sqrt{2}} = 2\sqrt{2}
\end{aligned}$$

The area is $2\sqrt{2}$ square units.

Example 10

Determine the area of the shaded region.



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Solution

First determine the points of intersection:

$$\begin{aligned}\cos x &= \sin(2x)\\ \cos x &= 2\sin x\cos x\\ 0 &= \cos x\,(2\sin x - 1)\end{aligned}$$

$$\therefore \quad \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

Therefore $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$ for $x \in \left[0, \frac{\pi}{2}\right]$.

$$\begin{aligned}\text{Area} &= \int_0^{\frac{\pi}{6}} \cos x - \sin(2x)\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(2x) - \cos x\,dx\\ &= \left[\sin x + \frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{6}} + \left[-\frac{1}{2}\cos(2x) - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\\ &= \left(\frac{1}{2} + \frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - 1 - \left(-\frac{1}{4} - \frac{1}{2}\right)\right)\\ &= \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

Skill-sheet

Exercise 12B

Example 6

1 Determine the coordinates of the points of intersection of the two curves with equations $y = x^2 - 2x$ and $y = -x^2 + 8x - 12$. Find the area of the region enclosed between the two curves. CF

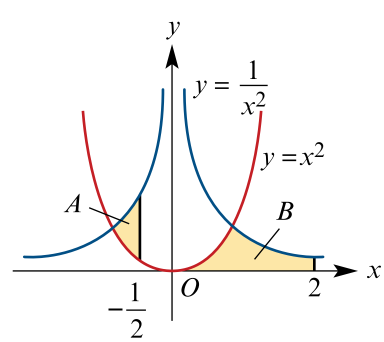
Example 7

2 Determine the area of the region enclosed by the graphs of $y = -x^2$ and $y = x^2 - 2x$.

3 Determine the area of:

a region A

b region B



4 Let $f(x) = x^2 - 4$. Sketch the graphs of $y = f(x)$ and $y = \frac{16}{f(x)}$ on the same set of axes. Find the area of the region bounded by the two graphs and the lines $x = 1$ and $x = -1$.

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CF

5 The area of the region bounded by $y = \frac{12}{x}$, the x-axis, $x = 1$ and $x = a$ is 24. Find the value of a.

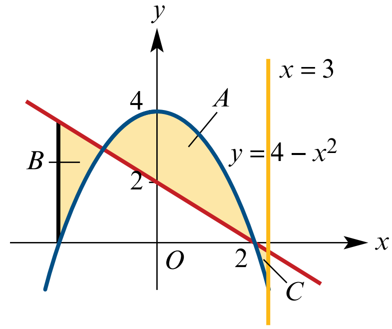
Example 8

6 Determine the area of:

a region A

b region B

c region C



Example 9, 10

7 For each of the following, find the area of the region enclosed by the lines and curves. Draw a sketch graph and shade the appropriate region for each example.

a $y = 2\sin x$ and $y = \sin(2x)$, for $0 \le x \le \pi$

b $y = \sin(2x)$ and $y = \cos x$, for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$

c $y = \sqrt{x}$, $y = 6 - x$ and $y = 1$

d $y = \frac{2}{1 + x^2}$ and $y = 1$

e $y = \sin^{-1} x$, $x = \frac{1}{2}$ and $y = 0$

f $y = \cos(2x)$ and $y = 1 - \sin x$, for $0 \le x \le \pi$

g $y = \frac{1}{3}(x^2 + 1)$ and $y = \frac{3}{x^2 + 1}$

8 Evaluate each of the following. (Draw the appropriate graph first.)

a $\int_1^e \ln x \, dx$

b $\int_{\frac{1}{2}}^1 \ln(2x) \, dx$

Hint: You can use the inverse relationship $y = \ln x \Leftrightarrow x = e^y$. First find the area between the curve and the y-axis.

9 Let $f(x) = xe^x$.

a Determine the derivative of f.

b Determine the values of x such that $f'(x) = 0$.

c Sketch the curve $y = f(x)$.

d Determine the equation of the tangent to this curve at $x = -1$.

e Determine the area of the region bounded by this tangent, the curve and the y-axis.

10 Let P be the point with coordinates $(1, 1)$ on the curve with equation $y = 1 + \ln x$.

a Determine the equation of the normal to the curve at P.

b Determine the area of the region enclosed by the normal, the curve and the x-axis.

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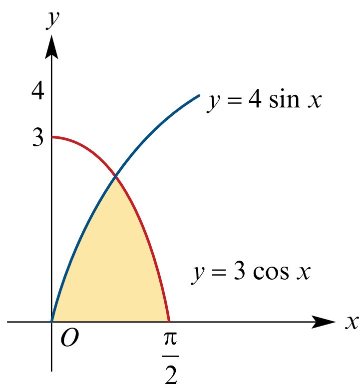
CF

11 **a** Find the coordinates of the points of intersection of the curves with equations $y = (x - 1)(x - 2)$ and $y = \dfrac{3(x - 1)}{x}$.

b Sketch the two curves on the one set of axes.

c Find the area of the region bounded by the two curves for $1 \le x \le 3$.

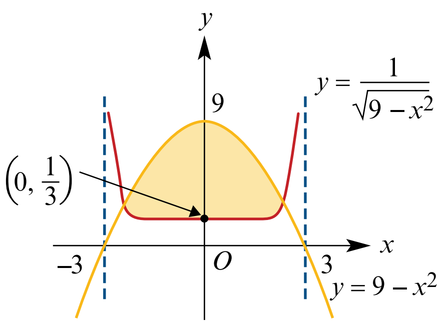
12 Show that the area of the shaded region is 2.



13 The graphs of $y = 9 - x^2$ and $y = \dfrac{1}{\sqrt{9 - x^2}}$ are as shown.

a Determine the coordinates of the points of intersection of the two graphs.

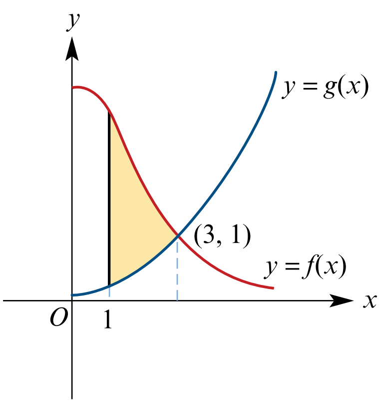
b Determine the area of the shaded region.



14 Determine the area enclosed by the graphs of $y = x^2$ and $y = x + 2$.

15 Consider the functions $f(x) = \dfrac{10}{1 + x^2}$ for $x \ge 0$ and $g(x) = e^{x-3}$ for $x \ge 0$.

The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point $(3, 1)$. Find, correct to three decimal places, the area of the region enclosed by the two graphs and the line with equation $x = 1$.



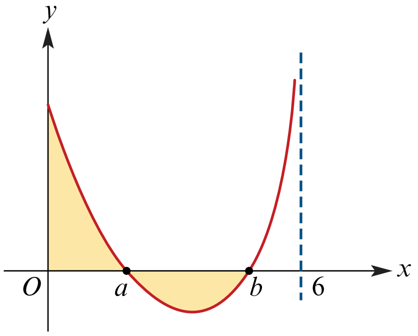
16 The graph of the function

$$f(x) = \frac{8\sqrt{5}}{\sqrt{36 - x^2}} - x, \quad x \in [0, 6)$$

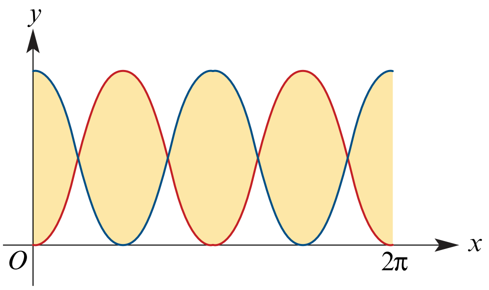
is shown.

a Determine the values of a and b.

b Determine the total area of the shaded regions.



17 The graphs of $y = \cos^2 x$ and $y = \sin^2 x$ are shown for $0 \le x \le 2\pi$. Determine the total area of the shaded regions.



12C Integration using a graphics calculator

Learning intentions

- To be able to use a graphics calculator for integration.

In Chapter 15, we discussed methods of integration by rule. However, for many functions it is not possible to determine an antiderivative by rule. In this section, we use a graphics calculator to evaluate definite integrals numerically.

Using a graphics calculator to find approximations of definite integrals

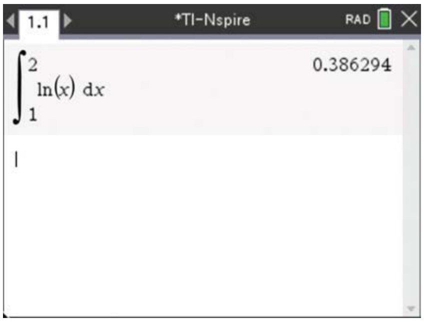
Example 11

Use a graphics calculator to evaluate $\int_1^2 \ln x \, dx$ correct to two decimal places.

Using the TI-Nspire CX non-CAS

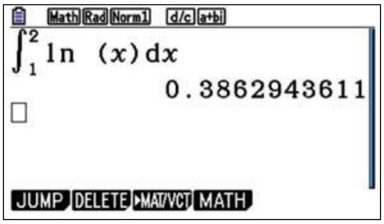
- Use menu > **Calculus** > **Numerical Integral** and complete as shown.
- Hence the integral is 0.39, correct to two decimal places.

Note: The definite-integral template can also be accessed from the 2D-template palette or by using shift +.



Using the Casio

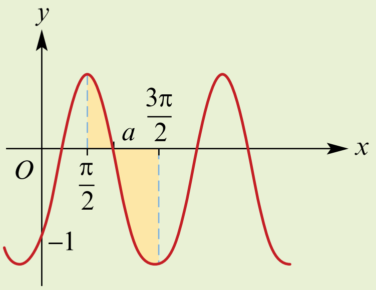
- In **Run-Matrix** mode, go to the **Calculation** menu OPTN F4 and select $\int$**dx** F4.
- Enter the integrand and endpoints as shown.
- Hence the integral is 0.39, correct to two decimal places.



Example 12

The graph of $y = e^{\sin x} - 2$ is as shown.

Using a graphics calculator, find the area of the shaded regions.



Solution

Using a graphics calculator, first find the value of a, which is approximately 2.37575.

$$\begin{aligned}\text{Required area} &= \int_{\frac{\pi}{2}}^{a} (e^{\sin x} - 2)\, dx - \int_{a}^{\frac{3\pi}{2}} (e^{\sin x} - 2)\, dx \\ &= 0.369\,213\ldots + 2.674\,936\ldots \\ &= 3.044\,149\ldots\end{aligned}$$

The area is approximately 3.044 square units. You can also do this by using your calculator with $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |e^{\sin x} - 2|\, dx$ to evaluate this integral.

Using the fundamental theorem of calculus

We have used the fundamental theorem of calculus to find areas using antiderivatives. We can also use the theorem to define antiderivatives using area functions.

If F is an anti-derivative of a continuous function f, then

$$F(b) - F(a) = \int_a^b f(x)\, dx$$

Using a dummy variable t, we can write

$$F(x) - F(a) = \int_a^x f(t)\, dt$$

$$\therefore \qquad F(x) = F(a) + \int_a^x f(t)\, dt$$

So if we define a function G by

$$G(x) = \int_a^x f(t)\, dt$$

then F and G differ by a constant. Hence G is also an antiderivative of f.

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Example 13

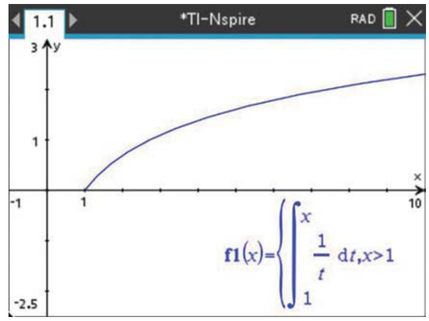
Plot the graph of $F(x) = \int_1^x \frac{1}{t}\, dt$ for $x > 1$.

Using the TI-Nspire CX non-CAS

In a **Graphs** page, enter the function

$$f1(x) = \int_1^x \frac{1}{t}\, dt \mid x > 1$$

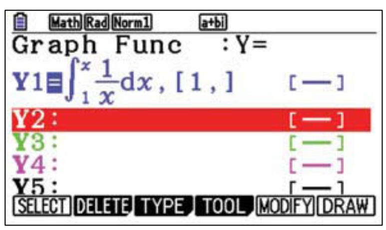
Note: The integral template can be obtained from the 2D-template palette ⊞.

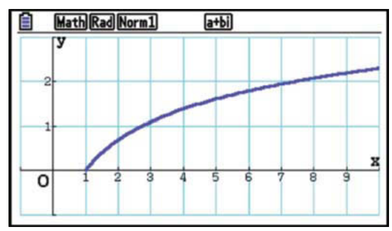


Using the Casio

In **Graph** mode, enter the function $y = \int_1^x \frac{1}{x}\, dx$ with domain $[1, \infty)$ as shown.

Hint: To obtain the integral template in **Graph** mode, use OPTN F2 F3.





Note: The natural logarithm function can be defined by $\ln(x) = \int_1^x \frac{1}{t}\, dt$.

The number e can then be defined to be the unique real number a such that $\ln(a) = 1$.

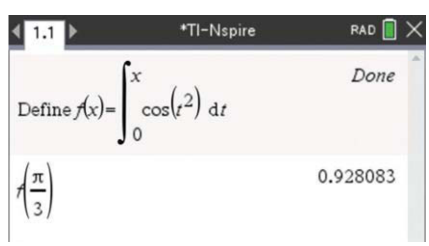
Example 14

Use a graphics calculator to find an approximate value of $\int_0^{\frac{\pi}{3}} \cos(x^2)\, dx$ and to plot the graph of $f(x) = \int_0^x \cos(t^2)\, dt$ for $-\frac{\pi}{4} \le x \le \pi$.

Using the TI-Nspire CX non-CAS

Method 1: Using a Calculator application

- Use menu > **Actions** > **Define** to define the function as shown and evaluate for $x = \frac{\pi}{3}$.



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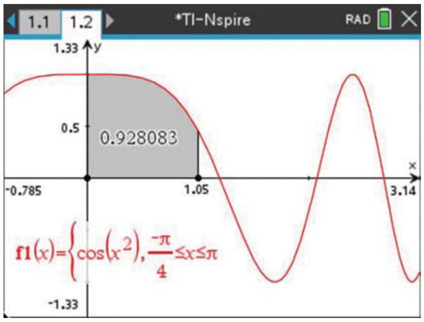
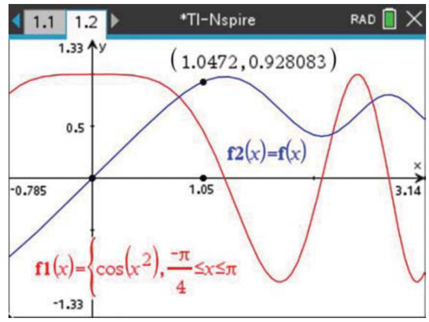
Method 2: Using a Graphs application

- Plot the graph of $f1(x) = \cos(x^2)$ for $-\frac{\pi}{4} \le x \le \pi$.
- To find the required area, use the integral measurement tool from (menu) > **Analyze Graph** > **Integral**. Type in the lower limit 0 and press (enter). Move to the right, type in the upper limit $\pi/3$ and press (enter).

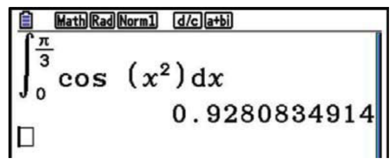




Using the Casio

Method 1: Using Run-Matrix mode

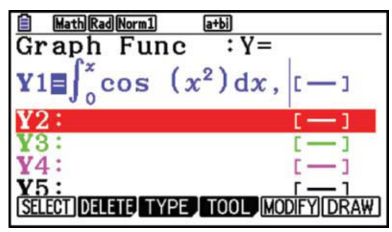
- Go to the **Calculation** menu (OPTN) (F4), and select the integral template **∫dx** (F4).
- Enter the integrand and endpoints as shown.



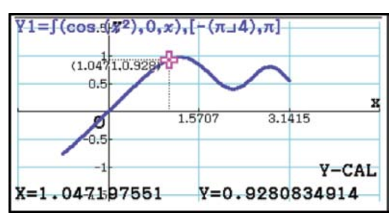
Method 2: Using Graph mode

- Enter the function $y = \int_0^x \cos(x^2)\, dx$ with domain $\left[-\frac{\pi}{4}, \pi\right]$.

Hint: To obtain the integral template in **Graph** mode, use (OPTN) (F2) (F3).



- Select **Draw** (F6).
- Adjust the View Window (SHIFT) (F3) if required.
- To find the value at $x = \frac{\pi}{3}$, go to the **G-Solve** menu (SHIFT) (F5) and select **y-Cal** (F6) (F1).



Exercise 12C

Example 11

1 Using a calculator, evaluate each of the following correct to two decimal places:

a $\int_0^2 e^{\sin x}\, dx$ **b** $\int_0^\pi x \sin x\, dx$ **c** $\int_1^3 (\ln x)^2\, dx$ **d** $\int_{-1}^1 \cos(e^x)\, dx$

e $\int_{-1}^2 \frac{e^x}{e^x + e^{-x}}\, dx$ **f** $\int_0^2 \frac{x}{x^4+1}\, dx$ **g** $\int_1^2 x \ln x\, dx$ **h** $\int_{-1}^1 x^2 e^x\, dx$

i $\int_0^1 \sqrt{1+x^4}\, dx$ **j** $\int_0^{\frac{\pi}{2}} \sin(x^2)\, dx$

SF

Example 12

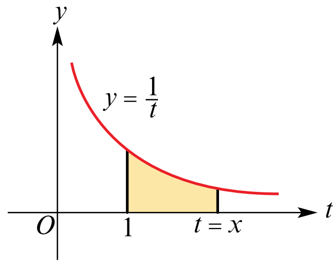
2 Give the approximate total area of the regions contained between the graph and the x-axis. Give your answer correct to three decimal places:

a $f(x) = \sin(x^2),\ 0 \le x \le 2.5$

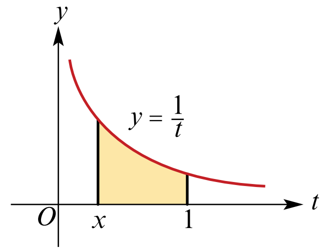
b $f(x) = \sin(e^x),\ 0 \le x \le 2$

3 In each of the following, the rule of the function is defined as an area function. Find $f(x)$ in each case.

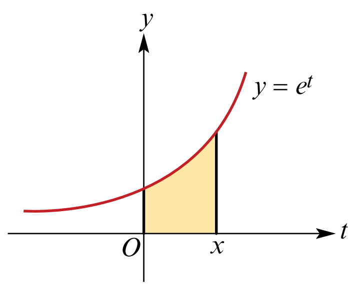
a $f(x) = \int_1^x \frac{1}{t}\,dt$, for $x > 1$



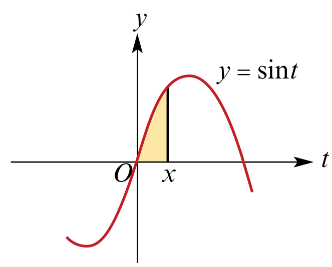
b $f(x) = \int_x^1 \frac{1}{t}\,dt$, for $0 < x < 1$



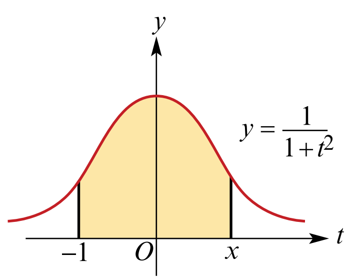
c $f(x) = \int_0^x e^t\,dt$, for $x \in \mathbb{R}$



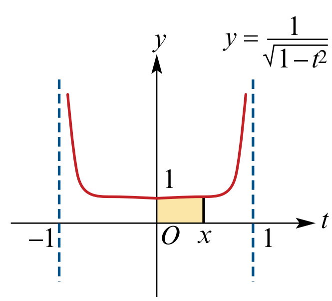
d $f(x) = \int_0^x \sin t\,dt$, for $x \in \mathbb{R}$



e $f(x) = \int_{-1}^x \frac{1}{1+t^2}\,dt$, for $x \in \mathbb{R}$



f $f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$, for $-1 < x < 1$



Example 13

4 Use a calculator to plot the graph of each of the following:

a $f(x) = \int_0^x \tan^{-1} t\,dt$

b $f(x) = \int_0^x e^{t^2}\,dt$

c $f(x) = \int_0^x \sin^{-1} t\,dt$

d $f(x) = \int_0^x \sin(t^2)\,dt$

e $f(x) = \int_1^x \frac{\sin t}{t}\,dt,\ x > 1$

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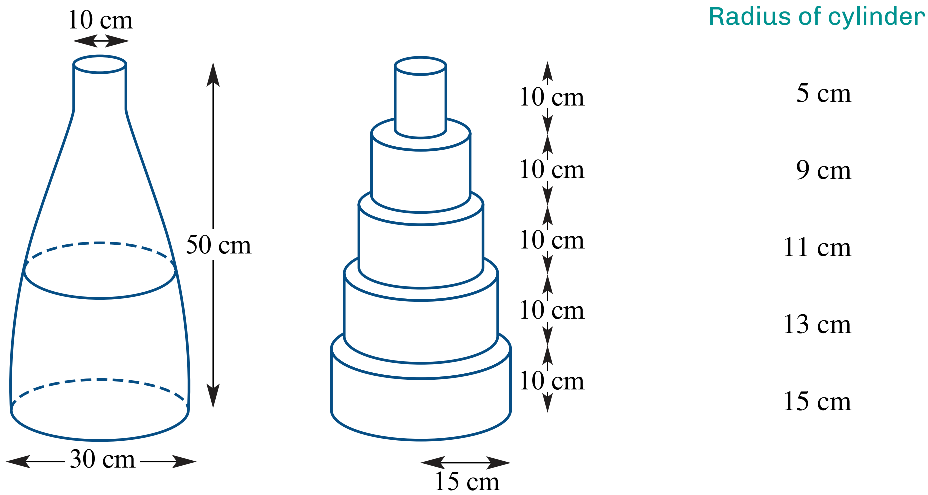
12D Volumes of solids of revolution

Learning intentions

- ▶ To be able to determine the volume of a solid of revolution.

A large glass flask has a shape as illustrated in the figure below. The volume of the flask can be approximated by finding the volume of series of cylinders.



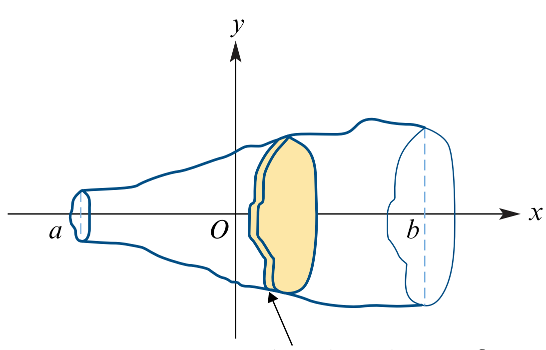
$$\therefore \quad \text{Volume of flask} \approx \pi(15^2 + 13^2 + 11^2 + 9^2 + 5^2) \times 10$$
$$\approx 19\,509.29 \text{ cm}^3$$
$$\approx 19 \text{ litres}$$

This estimate can be improved by taking increasingly thin cylinders.

In Mathematical Methods Units 3 & 4, it was shown that areas defined by well-behaved functions can be determined as the limit of a sum.

This can also be done for volumes. The volume of a typical thin slice is $A\delta x$, and the approximate total volume is

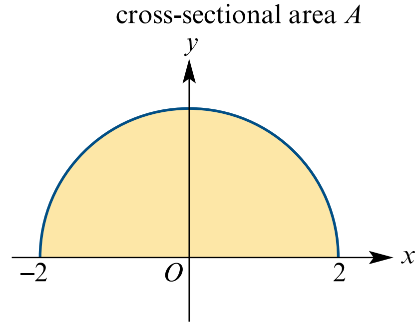
$$\sum_{x=a}^{x=b} A\delta x$$



Volume of a sphere

Consider the graph of $f(x) = \sqrt{4 - x^2}$.

If the shaded region is rotated around the x-axis, it will form a sphere of radius 2.



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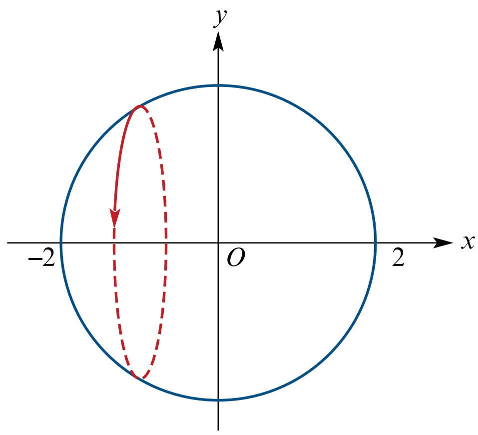
Divide the interval $[-2, 2]$ into n subintervals $[x_{i-1}, x_i]$ with $x_0 = -2$ and $x_n = 2$.



The volume of a typical slice (a cylinder) is approximately $\pi\big(f(c_i)\big)^2(x_i - x_{i-1})$, where $c_i \in [x_{i-1}, x_i]$.

The total volume will be approximated by the sum of the volumes of these slices. As the number of slices n gets larger and larger:

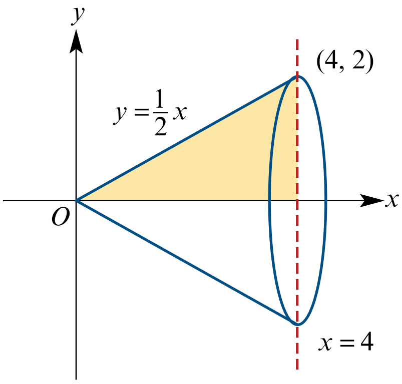
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\big(f(c_i)\big)^2(x_i - x_{i-1})$$

It has been seen that the limit of such a sum is an integral and therefore:

$$\begin{aligned}
V &= \int_{-2}^{2} \pi\big(f(x)\big)^2\, dx \\
&= \int_{-2}^{2} \pi(4 - x^2)\, dx \\
&= \pi\left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\
&= \pi\left(8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right) \\
&= \pi\left(16 - \frac{16}{3}\right) \\
&= \frac{32\pi}{3}
\end{aligned}$$

Volume of a cone

If the region between the line $y = \frac{1}{2}x$, the line $x = 4$ and the x-axis is rotated around the x-axis, then a solid in the shape of a cone is produced.



The volume of the cone is given by:

$$\begin{aligned}
V &= \int_{0}^{4} \pi y^2\, dx \\
&= \int_{0}^{4} \pi\left(\frac{1}{2}x\right)^2\, dx \\
&= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_{0}^{4} \\
&= \frac{\pi}{4} \times \frac{64}{3} \\
&= \frac{16\pi}{3}
\end{aligned}$$

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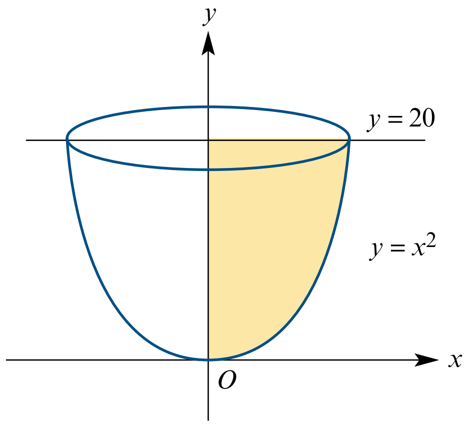
Solids of revolution

In general, the solid formed by rotating a region about a line is called a **solid of revolution**.

For example, if the region between the graph of $y = x^2$, the line $y = 20$ and the y-axis is rotated about the y-axis, then a solid in the shape of the top of a wine glass is produced.



Volume of a solid of revolution

- **Rotation about the x-axis**
  If the region to be rotated is bounded by the curve with equation $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis, then

$$V = \int_{x=a}^{x=b} \pi y^2 \, dx$$
$$= \pi \int_a^b (f(x))^2 \, dx$$

- **Rotation about the y-axis**
  If the region to be rotated is bounded by the curve with equation $x = f(y)$, the lines $y = a$ and $y = b$ and the y-axis, then

$$V = \int_{y=a}^{y=b} \pi x^2 \, dy$$
$$= \pi \int_a^b (f(y))^2 \, dy$$

Example 15

Find the volume of the solid of revolution formed by rotating the curve $y = x^3$ about:

a the x-axis for $0 \le x \le 1$

b the y-axis for $0 \le y \le 1$

Solution

a
$$V = \pi \int_0^1 y^2 \, dx$$
$$= \pi \int_0^1 x^6 \, dx$$
$$= \pi \left[\frac{x^7}{7} \right]_0^1$$
$$= \frac{\pi}{7}$$

b
$$V = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 y^{\frac{2}{3}} \, dy$$
$$= \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^1$$
$$= \frac{3\pi}{5}$$

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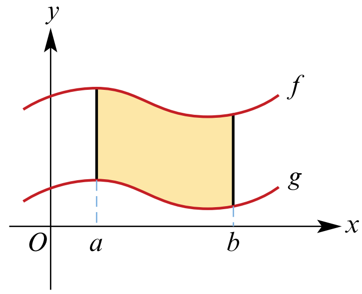
Regions bounded by the graphs of two functions

If the shaded region is rotated about the x-axis, then the volume V is given by

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx$$



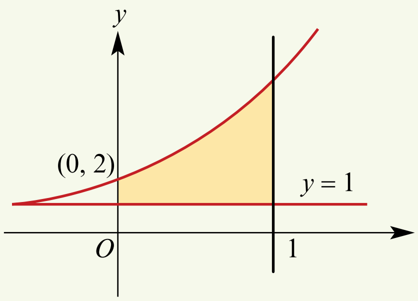
Example 16

Determine the volume of the solid of revolution when the region bounded by the graphs of $y = 2e^{2x}$, $y = 1$, $x = 0$ and $x = 1$ is rotated around the x-axis.

Solution

The volume is given by

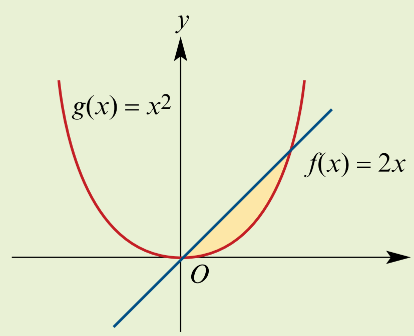
$$\begin{aligned} V &= \pi \int_0^1 4e^{4x} - 1 \, dx \\ &= \pi \left[e^{4x} - x \right]_0^1 \\ &= \pi(e^4 - 1 - (1)) \\ &= \pi(e^4 - 2) \end{aligned}$$



Note: Here $f(x) = 2e^{2x}$ and $g(x) = 1$.

Example 17

The shaded region is rotated around the x-axis.
Determine the volume of the resulting solid.



Solution

The graphs meet where $2x = x^2$, i.e. at the points with coordinates $(0, 0)$ and $(2, 4)$.

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 (f(x))^2 - (g(x))^2 \, dx \\ &= \pi \int_0^2 4x^2 - x^4 \, dx \\ &= \pi \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 \\ &= \pi \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{64\pi}{15} \end{aligned}$$

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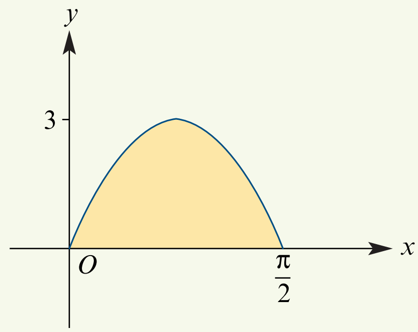
Example 18

A solid is formed when the region bounded by the x-axis and the graph of $y = 3\sin(2x)$, $0 \le x \le \frac{\pi}{2}$, is rotated around the x-axis. Determine the volume of this solid.

Solution

$$\begin{aligned}
V &= \pi \int_0^{\frac{\pi}{2}} (3\sin(2x))^2 \, dx \\
&= \pi \int_0^{\frac{\pi}{2}} 9\sin^2(2x) \, dx \\
&= 9\pi \int_0^{\frac{\pi}{2}} \sin^2(2x) \, dx \\
&= 9\pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(4x)) \, dx \\
&= \frac{9\pi}{2} \int_0^{\frac{\pi}{2}} 1 - \cos(4x) \, dx \\
&= \frac{9\pi}{2} \left[x - \frac{1}{4}\sin(4x) \right]_0^{\frac{\pi}{2}} \\
&= \frac{9\pi}{2} \left(\frac{\pi}{2} \right) \\
&= \frac{9\pi^2}{4}
\end{aligned}$$

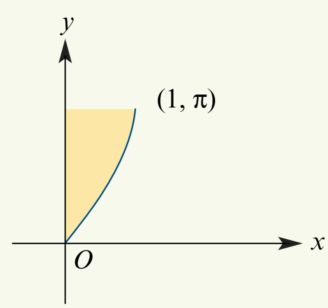


Example 19

The curve $y = 2\sin^{-1} x$, $0 \le x \le 1$, is rotated around the y-axis to form a solid of revolution. Determine the volume of this solid.

Solution

$$\begin{aligned}
V &= \pi \int_0^{\pi} \sin^2\left(\frac{y}{2}\right) dy \\
&= \frac{\pi}{2} \int_0^{\pi} 1 - \cos y \, dy \\
&= \frac{\pi}{2} \big[y - \sin y \big]_0^{\pi} \\
&= \frac{\pi^2}{2}
\end{aligned}$$



Skill-sheet

Exercise 12D

Example 15

1 Determine the area of the region bounded by the x-axis and the curve whose equation is $y = 4 - x^2$. Also find the volume of the solid formed when this region is rotated about the y-axis. CF

CF

2 Determine the volume of the solid of revolution when the region bounded by the given curve, the x-axis and the given lines is rotated about the x-axis:

a $f(x) = \sqrt{x},\ x = 4$

b $f(x) = 2x + 1,\ x = 0,\ x = 4$

c $f(x) = 2x - 1,\ x = 4$

d $f(x) = \sin x,\ 0 \le x \le \dfrac{\pi}{2}$

e $f(x) = e^x,\ x = 0,\ x = 2$

f $f(x) = \sqrt{9 - x^2},\ -3 \le x \le 3$

3 The hyperbola $x^2 - y^2 = 1$ is rotated around the x-axis to form a surface of revolution. Find the volume of the solid enclosed by this surface between $x = 1$ and $x = \sqrt{3}$.

4 Determine the volumes of the solids generated by rotating about the x-axis each of the regions bounded by the following curves and lines:

a $y = \dfrac{1}{x},\ y = 0,\ x = 1,\ x = 4$

b $y = x^2 + 1,\ y = 0,\ x = 0,\ x = 1$

c $y = \sqrt{x},\ y = 0,\ x = 2$

d $y = \sqrt{a^2 - x^2},\ y = 0$

e $y = \sqrt{9 - x^2},\ y = 0$

f $y = \sqrt{9 - x^2},\ y = 0,\ x = 0,$ given $x \ge 0$

Example 16, 17

5 The region bounded by the line $y = 5$ and the curve $y = x^2 + 1$ is rotated about the x-axis. Determine the volume generated.

Example 18

6 The region, for which $x \ge 0$, bounded by the curves $y = \cos x$ and $y = \sin x$ and the y-axis is rotated around the x-axis, forming a solid of revolution. By using the identity $\cos(2x) = \cos^2 x - \sin^2 x$, obtain a volume for this solid.

7 The region enclosed by $y = \dfrac{4}{x^2}$, $x = 4$, $x = 1$ and the x-axis is rotated about the x-axis. Determine the volume generated.

8 The region enclosed by $y = x^2$ and $y^2 = x$ is rotated about the x-axis. Determine the volume generated.

9 A region is bounded by the curve $y = \sqrt{6 - x}$, the straight line $y = x$ and the positive x-axis. Determine the volume of the solid of revolution formed by rotating this figure about the x-axis.

10 The region bounded by the x-axis, the line $x = \dfrac{\pi}{2}$ and the curve $y = \tan\left(\dfrac{x}{2}\right)$ is rotated about the x-axis. Prove that the volume of the solid of revolution is $\dfrac{\pi}{2}(4 - \pi)$.

Hint: Use the result that $\tan^2\left(\dfrac{x}{2}\right) = \sec^2\left(\dfrac{x}{2}\right) - 1$.

11 Sketch the graphs of $y = \sin x$ and $y = \sin(2x)$ for $0 \le x \le \dfrac{\pi}{2}$. Show that the area of the region bounded by these graphs is $\dfrac{1}{4}$ square unit, and the volume formed by rotating this region about the x-axis is $\dfrac{3}{16}\pi\sqrt{3}$ cubic units.

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CF

12 Let V be the volume of the solid formed when the region enclosed by $y = \frac{1}{x}$, $y = 0$, $x = 4$ and $x = b$, where $0 < b < 4$, is rotated about the x-axis. Determine the value of b for which $V = 3\pi$.

13 Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x+1}$, $y = \sqrt{3x}$, $y = 0$ and $x = 1$ is rotated about the x-axis.

Example 19 **14** Find the volumes of the solids formed when the following regions are rotated around the y-axis:

a $x^2 = 4y^2 + 4$ for $0 \le y \le 1$ **b** $y = \ln(2 - x)$ for $0 \le y \le 2$

15 a Determine the area of the region bounded by the curve $y = e^x$, the tangent at the point $(1, e)$ and the y-axis.

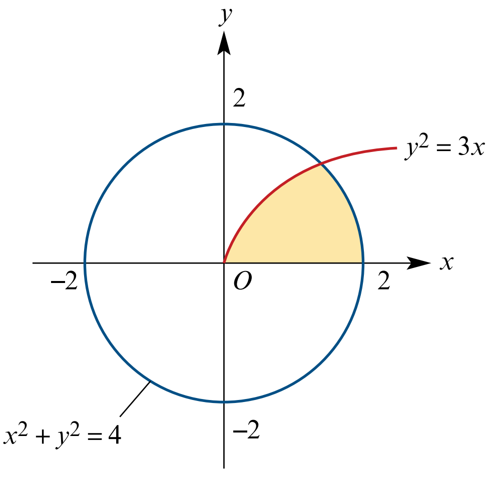
b Determine the volume of the solid formed by rotating this region through a complete revolution about the x-axis.

16 The region defined by the inequalities $y \ge x^2 - 2x + 4$ and $y \le 4$ is rotated about the line $y = 4$. Determine the volume generated.

17 The region enclosed by $y = \cos\left(\frac{x}{2}\right)$ and the x-axis, for $0 \le x \le \pi$, is rotated about the x-axis. Determine the volume generated.

18 Determine the volume generated by revolving the region enclosed between the parabola $y = 3x - x^2$ and the line $y = 2$ about the x-axis.

19 The shaded region is rotated around the x-axis to form a solid of revolution. Determine the volume of this solid.



20 The region enclosed between the curve $y = e^x - 1$, the x-axis and the line $x = \ln 2$ is rotated around the x-axis to form a solid of revolution. Find the volume of this solid.

21 Show that the volume of the solid of revolution formed by rotating about the x-axis the region bounded by the curve $y = e^{-2x}$ and the lines $x = 0$, $y = 0$ and $x = \ln 2$ is $\frac{15\pi}{64}$.

22 Determine the volume of the solid generated by revolving about the x-axis the region bounded by the graph of $y = 2\tan x$ and the lines $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ and $y = 0$.

23 The region bounded by the parabola $y^2 = 4(1 - x)$ and the y-axis is rotated about:

a the x-axis **b** the y-axis.

Prove that the volumes of the solids formed are in the ratio 15 : 16.

24 The region bounded by the graph of $y = \dfrac{1}{\sqrt{x^2 + 9}}$, the x-axis, the y-axis and the line $x = 4$ is rotated about:

a the x-axis **b** the y-axis.

Determine the volume of the solid formed in each case.

25 A bucket is defined by rotating the curve with equation

$$y = 40\ln\left(\frac{x - 20}{10}\right), \quad 0 \le y \le 40$$

about the y-axis (Assume a base has been added). If x and y are measured in centimetres, determine the maximum volume of liquid that the bucket could hold. Give the answer to the nearest cm^3.

26 An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Determine the volume of the solid generated when the region bounded by the ellipse is rotated about:

a the x-axis **b** the y-axis.

27 The diagram shows part of the curve $y = \dfrac{12}{x}$.

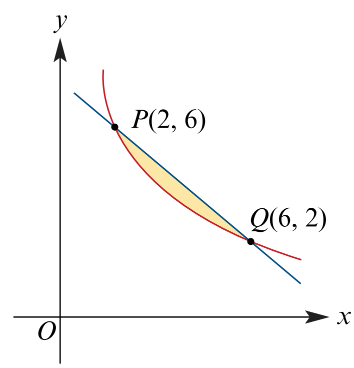
Points $P(2, 6)$ and $Q(6, 2)$ lie on the curve.

Determine:

a the equation of the line PQ

b the volume obtained when the shaded region is rotated about:

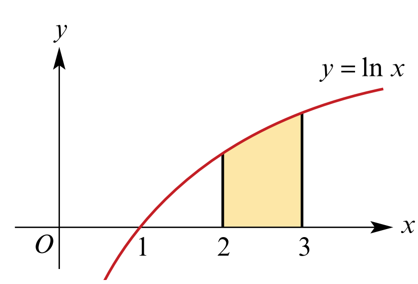
i the x-axis **ii** the y-axis.



28 **a** Sketch the graph of $y = 2x + \dfrac{9}{x}$.

b Find the volume generated when the region bounded by the curve $y = 2x + \dfrac{9}{x}$ and the lines $y = 0$, $x = 1$ and $x = 3$ is rotated about the x-axis.

29 The region shown is rotated about the x-axis to form a solid of revolution. Determine the volume of the solid, correct to three decimal places.



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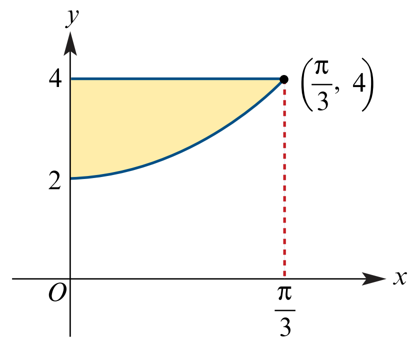
30 The graphs of $y = 2\sec x$ and $y = 4$ are shown for $0 \le x \le \frac{\pi}{3}$.
The shaded region is rotated about the x-axis to form a solid of revolution. Calculate the exact volume of this solid.



CF

12E The exponential probability distribution

Learning intentions

- To be able to determine probabilities using the exponential distribution.

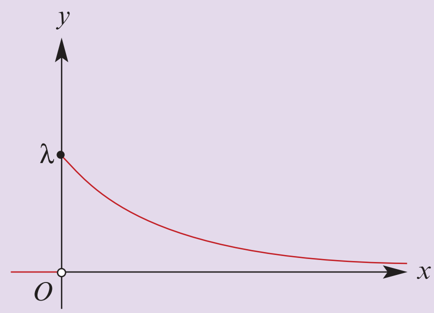
Continuous probability distributions are introduced in Mathematical Methods Units 3 & 4. In this section, we investigate the exponential probability distribution, which is often used to model the time between the occurrence of random events.

Exponential distribution

For $\lambda > 0$, an exponential random variable X with parameter λ has a probability density function given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The graph of $y = f(x)$ is shown on the right.



To verify that f is a probability density function, we need to show that:

1 $f(x) \ge 0$ for all x

2 the area under the graph of f is equal to 1.

The first condition is clearly satisfied. To check the second condition, we evaluate

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_0^{\infty} \lambda e^{-\lambda x}\,dx \\
&= \lim_{k\to\infty} \int_0^k \lambda e^{-\lambda x}\,dx \\
&= \lim_{k\to\infty} \left[-e^{-\lambda x}\right]_0^k \\
&= \lim_{k\to\infty} \left((-e^{-\lambda k}) - (-e^0)\right) \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

Thus f satisfies the two conditions for a probability density function.

We can use the probability density function f to determine probabilities associated with an exponential random variable, as shown in the following example.

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Example 20

The time, X minutes, that a shop assistant waits before the next customer arrives is known to be exponentially distributed, with probability density function given by

$$f(x) = \begin{cases} 0.2e^{-0.2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the probability that he will wait more than 8 minutes for the next customer to arrive.

Solution

$$\begin{aligned} P(X > 8) &= \int_8^\infty 0.2e^{-0.2x}\,dx \\ &= \lim_{k\to\infty} \left[-e^{-0.2x}\right]_8^k \\ &= e^{-1.6} \approx 0.2019 \end{aligned}$$

The mean and standard deviation of an exponential random variable

Let X be an exponentially distributed random variable with parameter λ. We can find the expected value of X using integration by parts:

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^\infty x \cdot f(x)\,dx \\ &= \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx \\ &= \lim_{k\to\infty} \left(\left[-xe^{-\lambda x}\right]_0^k - \int_0^k -e^{-\lambda x}\,dx \right) && \text{use } u = x \text{ and } v = -e^{-\lambda x} \\ &= \lim_{k\to\infty} \left(-ke^{-\lambda k} - \left[\frac{1}{\lambda}e^{-\lambda x}\right]_0^k \right) \\ &= \frac{1}{\lambda} && \text{since } \lim_{k\to\infty} ke^{-\lambda k} = 0 \end{aligned}$$

To find the standard deviation of X, we first need to find $\mathrm{E}(X^2)$. This time we use integration by parts twice:

$$\begin{aligned} \mathrm{E}(X^2) &= \int_{-\infty}^\infty x^2 \cdot f(x)\,dx \\ &= \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\,dx \\ &= \lim_{k\to\infty} \left[-x^2 e^{-\lambda x} - \frac{2}{\lambda}xe^{-\lambda x} - \frac{2}{\lambda^2}e^{-\lambda x} \right]_0^k \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Therefore $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \dfrac{1}{\lambda^2}$ and so $\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = \dfrac{1}{\lambda}$.

We have used unbounded integrals in the above discussion. In general unbounded integrals are not studied in this course and their use is therefore restricted.

Mean and standard deviation of an exponential random variable

For an exponentially distributed random variable X with parameter λ:

$$\mathrm{E}(X) = \frac{1}{\lambda} \quad \text{and} \quad \mathrm{sd}(X) = \frac{1}{\lambda}$$

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Example 21

For the situation in Example 20, determine the mean and standard deviation of the time that the shop assistant waits for a customer.

Solution

$$\mathrm{E}(X) = \frac{1}{\lambda} = \frac{1}{0.2} = 5 \text{ minutes}$$

$$\mathrm{sd}(X) = \frac{1}{\lambda} = \frac{1}{0.2} = 5 \text{ minutes}$$

Example 22

The time, T minutes, that it takes a librarian to locate a book is exponentially distributed with a mean of 3 minutes. Determine:

a the probability density function of T

b the probability that it takes her less than 2 minutes to find a book.

Solution

a Since T is exponentially distributed, we have $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

We are given that $\mathrm{E}(T) = \dfrac{1}{\lambda} = 3$ and therefore $\lambda = \dfrac{1}{3}$.

Hence $f(t) = \dfrac{1}{3}e^{-\frac{t}{3}}$ for $t \geq 0$.

b
$$\begin{aligned} P(T < 2) &= \int_0^2 \frac{1}{3}e^{-\frac{t}{3}}\,dt \\ &= \Big[-e^{-\frac{t}{3}}\Big]_0^2 \\ &= 1 - e^{-\frac{2}{3}} \\ &\approx 0.4866 \end{aligned}$$

The cumulative distribution function of an exponential random variable

Again, let X be an exponentially distributed random variable with parameter λ. For $x \geq 0$, the cumulative distribution function of X is given by

$$\begin{aligned} F(x) = P(X \leq x) &= \int_0^x \lambda e^{-\lambda t}\,dt \\ &= \Big[-e^{-\lambda t}\Big]_0^x \\ &= 1 - e^{-\lambda x} \end{aligned}$$

Cumulative distribution for an exponential random variable

For an exponentially distributed random variable X with parameter λ:

$$F(x) = P(X \leq x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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Example 23

For the situation in Example 20, find the cumulative distribution function of X, and hence find the probability that the shop assistant waits between 5 and 10 minutes for a customer.

Solution

Since $\lambda = 0.2$, we have $F(x) = 1 - e^{-0.2x}$ for $x \geq 0$. Therefore

$$\begin{aligned} P(5 \leq X \leq 10) &= P(X \leq 10) - P(X < 5) \\ &= F(10) - F(5) \\ &= (1 - e^{-2}) - (1 - e^{-1}) \\ &\approx 0.2325 \end{aligned}$$

Exercise 12E

SF

Example 20

1 Let X be an exponential random variable with probability density function

$$f(x) = \begin{cases} 0.5e^{-0.5x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Determine $P(X > 1)$.

b Determine $P(X < 2)$.

2 Suppose that X is an exponential random variable with probability density function

$$f(x) = \begin{cases} \dfrac{1}{7}e^{-\frac{x}{7}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a Determine $P(X > 3)$.

b Determine $P(7 < X < 14)$.

Example 21

3 Suppose that X is an exponentially distributed random variable with probability density function given by

$$f(x) = \begin{cases} 0.1e^{-0.1x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Determine $\mathrm{E}(X)$ and $\mathrm{sd}(X)$.

b Determine $P(X > \mathrm{E}(X))$.

CF

Example 22

4 Customers at a checkout wait on average 6 minutes to be served. If the time, T minutes, that a customer waits to be served is exponentially distributed, find:

a the probability density function of T

b the probability that a customer waits less than 3 minutes to be served

c the probability that a customer waits more than 10 minutes to be served, given that they have already waited 5 minutes.

5 The time, X seconds, that it takes to locate a card in a file of records has an exponential distribution with a mean of 20 seconds. Determine:

a $P(X \leq 30)$

b $P(X \geq 20)$

c $P(20 \leq X \leq 30)$

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CF

6 The lifetime of a certain kind of battery is an exponential random variable with a mean of 250 hours. What is the probability that such a battery will last at most 200 hours?

7 The time (in hours) required to repair a machine is an exponentially distributed random variable with mean 1.5 hours.

a What is the probability that a repair takes more than 2 hours?

b What is the probability that a repair takes at least 10 hours, given that it takes at least 9 hours?

8 The random variable X represents the time (in minutes) between the arrival of customers at an ATM. If X has an exponential distribution with parameter $\lambda = 0.2$, determine:

a the expected time between two successive arrivals

b the standard deviation of the time between successive arrivals

c the probability $P(X \leq 2.5)$.

9 Suppose that X is an exponentially distributed random variable with mean 0.5.

a Determine $P(X > 2)$.

b Determine $\text{Var}(X)$.

c Determine $P(X > 2 \mid X < 3)$.

Example 23

10 Let X be an exponentially distributed random variable with parameter $\lambda = \frac{1}{4}$.

a Find the cumulative distribution function $F(x) = P(X \leq x)$.

b Hence find the median of X. That is, find m such that $P(X \leq m) = 0.5$.

11 Lacey receives four phone calls per hour on average. If the time between phone calls is exponentially distributed, find the probability that she will wait no longer than 10 minutes for her next phone call.

12 Suppose that the length of time that an electric light bulb lasts, X hours, is an exponential random variable with cumulative distribution function given by

$$F(x) = 1 - e^{-\frac{x}{500}} \quad \text{for } x \geq 0$$

Determine:

a the probability that a bulb lasts more than 300 hours

b the mean lifetime of a bulb

c the median lifetime of a bulb.

13 The time (in minutes) between telephone calls received at a pizza restaurant is exponentially distributed with a mean of 4 minutes. Find the median time between calls.

14 Suppose that the amount of time that customers spend in a bank is exponentially distributed with a mean of 10 minutes.

a What is the probability that a customer will spend more than 15 minutes in the bank?

b What is the probability that a customer will spend more than 15 minutes in the bank, given that she is still in the bank after 10 minutes?

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12F Simpson's rule

Learning intentions

- To be able to use Simpson's rule to approximate areas under a curve.

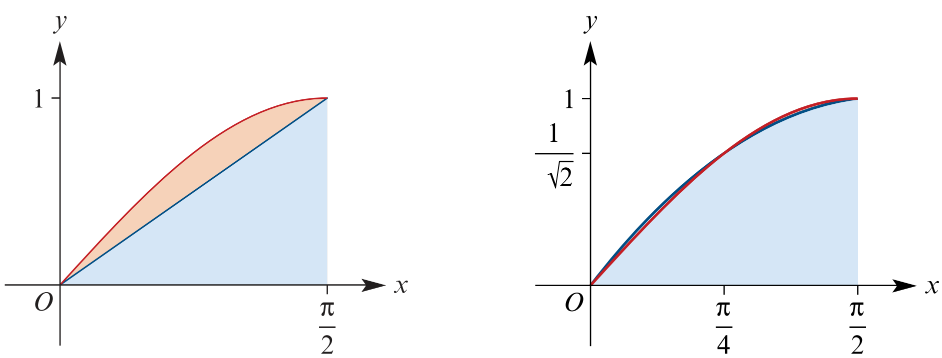
You have met the trapezoidal rule in Mathematical Methods Units 3 & 4. This rule is used to estimate the area under a curve by taking many trapezoidal strips. However, we can also think of the trapezoidal rule as approximating the curve by using many straight-line segments.

Similarly, we can think of Simpson's rule as a method for estimating a definite integral by approximating the curve using parabolas.

For example, consider the area under the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$.

Approximation by line segment

Approximation by parabola



The exact area under the sine curve is $\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$.

- In the graph on the left, the sine curve is approximated by the line segment joining the points $(0, 0)$ and $\left(\frac{\pi}{2}, 1\right)$. The area under the line segment is 0.7854.
- In the graph on the right, the sine curve is approximated by the parabola through the points $(0, 0)$, $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{\pi}{2}, 1\right)$. The area under the parabola is 1.0023.

The basic form of Simpson's rule

For a continuous function f on an interval $[a, b]$, we consider three points on the graph of f, where the three x-values are

$$x_0 = a, \quad x_1 = \frac{a+b}{2} \quad \text{and} \quad x_2 = b$$

Simpson's rule is

$$\int_a^b f(x)\, dx \approx \frac{b-a}{6}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big)$$

Note: This estimate comes from the unique parabola (or straight line) through the three points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$; see Question 14 in Exercise 12F.

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For example, Simpson's rule gives

$$\int_a^b 1\,dx \approx \frac{b-a}{6}(1+4+1) = b-a$$

$$\int_a^b x\,dx \approx \frac{b-a}{6}(a+2(a+b)+b) = \frac{b-a}{6}(3a+3b) = \frac{1}{2}(b^2-a^2)$$

$$\int_a^b x^2\,dx \approx \frac{b-a}{6}(a^2+(a+b)^2+b^2) = \frac{b-a}{6}(2a^2+2ab+2b^2) = \frac{1}{3}(b^3-a^3)$$

Note that, in each of these three cases, the approximation given by Simpson's rule is in fact the exact value of the definite integral. More generally:

Simpson's rule gives the exact value of the definite integral $\int_a^b f(x)\,dx$ whenever f is a polynomial function of degree at most 3.

Note: You will prove this fact for polynomial functions of degree at most 2 in Question 14.

Example 24

Use Simpson's rule to estimate the integral $\int_2^5 x^3\,dx$.

Solution

Simpson's rule gives

$$\int_2^5 x^3\,dx \approx \frac{5-2}{6}\left(2^3 + 4\left(\frac{7}{2}\right)^3 + 5^3\right)$$
$$= \frac{609}{4}$$

Note: You can easily check that here Simpson's rule gives the exact value of the integral.

In the next example, we use Simpson's rule to estimate an integral that we can calculate exactly, so that we can check its accuracy.

Example 25

Use Simpson's rule to estimate the integral $\int_0^1 e^x\,dx$.

Solution

Simpson's rule gives

$$\int_0^1 e^x\,dx \approx \frac{1}{6}(e^0 + 4e^{0.5} + e^1)$$
$$\approx 1.718861$$

Note: The exact value of the integral is $e-1 \approx 1.718282$, correct to six decimal places.

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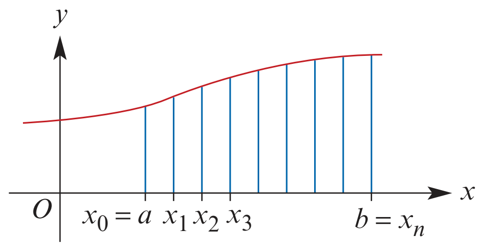
The general form of Simpson's rule

In the basic form of Simpson's rule, the interval $[a, b]$ is divided into two equal subintervals $[x_0, x_1]$ and $[x_1, x_2]$. We can obtain a more accurate estimate of the definite integral by using more subintervals.

Let n be a positive even number. Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ as shown.



The width of each subinterval is

$$w = \frac{b-a}{n}$$

We can now state a more general form of Simpson's rule as follows.

Simpson's rule for n subintervals (where n is even)

$$\int_a^b f(x)\,dx$$

$$\approx \frac{w}{3}\Big(f(x_0) + 4[f(x_1) + f(x_3) + \cdots + f(x_{n-1})] + 2[f(x_2) + f(x_4) + \cdots + f(x_{n-2})] + f(x_n)\Big)$$

Note: To derive this rule, use the basic form of Simpson's rule to estimate each term in the sum $\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$.

Example 26

Use Simpson's rule with 10 subintervals to estimate the integral $\int_0^1 e^x\,dx$.

Solution

Using Simpson's rule with $n = 10$ and $w = \dfrac{1-0}{10} = 0.1$, we obtain

$$\int_0^1 e^x\,dx \approx \frac{0.1}{3}\Big(e^0 + 4[e^{0.1} + e^{0.3} + e^{0.5} + e^{0.7} + e^{0.9}] + 2[e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8}] + e^1\Big)$$

$$\approx 1.718283$$

Note: The exact value is $e - 1 \approx 1.718282$, correct to six decimal places. This estimate is much closer than the estimate obtained in Example 25 using two subintervals.

Exercise 12F

Example 24

1 Use Simpson's rule to estimate the integral $\int_2^4 f(x)\,dx$ for each table of values:

a

x	2	3	4
$f(x)$	8	6	4

b

x	2	3	4
$f(x)$	2	5	8

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Example 25

2 Use Simpson's rule to estimate the integral $\int_0^1 \cos x \, dx$.

SF

3 Five values of a function f are known, as shown in the table.

x	3	4	5	6	7
$f(x)$	2	5	8	10	11

Use these values to estimate:

a $\int_3^5 f(x)\,dx$ **b** $\int_5^7 f(x)\,dx$ **c** $\int_3^7 f(x)\,dx = \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

Example 26

4 Give an approximate value for the integral $\int_0^1 x^4\,dx$ by using:

a Simpson's rule with two subintervals (answer to four decimal places)

b Simpson's rule with four subintervals (answer to four decimal places).

5 Determine an approximation to each of the following integrals by using the basic form of Simpson's rule ($n = 2$). Give answers to five decimal places.

a $\int_1^2 \frac{1}{x}\,dx$ **b** $\int_0^1 \ln(x+4)\,dx$ **c** $\int_0^1 \cos(2x)\,dx$

6 The following integrals cannot be evaluated exactly using elementary functions. Find estimates of these integrals by using Simpson's rule with strips of width $w = 0.25$.

a $\int_1^2 x^x\,dx$ **b** $\int_0^1 \frac{\sin x}{x-2}\,dx$ **c** $\int_0^1 \cos(\sqrt{x})\,dx$

7 In this question, give each answer to four decimal places.

CF

a Use Simpson's rule with $n = 8$ to find an approximation of $\int_0^1 e^{-\frac{1}{2}x^2}\,dx$.

b Hence find an approximation of $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx$.

c The probability density function f of the standard normal random variable X has the rule $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Use part **b** to find an approximation of $P(-1 < X < 1)$.

8 Prove that Simpson's rule gives the exact value of the integral $\int_a^b x^3\,dx$ by showing that

CU

$$\frac{b-a}{6}\left(a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3\right) = \frac{1}{4}(b^4 - a^4)$$

9 Consider the cubic function $f(x) = x^3 + 7$. The three points $(-2, -1)$, $(1, 8)$ and $(4, 71)$ lie on the graph of f.

a Let $g(x) = ax^2 + bx + c$. Determine the values of a, b and c such that the points $(-2, -1)$, $(1, 8)$ and $(4, 71)$ lie on the graph of g.

b Determine the exact values of $\int_{-2}^4 f(x)\,dx$ and $\int_{-2}^4 g(x)\,dx$.

c Sketch the graphs of f and g on the one set of axes for $x \in [-2, 4]$.

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CF

10 **a** Show that $\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$.

b Use Simpson's rule with four subintervals to approximate $\int_0^1 \frac{1}{1+x^2}\,dx$.

c Hence find an approximation of π.

11 An object is moving in a straight line. Its velocity is recorded each second, as shown in the following table.

Time (s)	0	1	2	3	4
Velocity (m/s)	2.5	1.5	2	2.5	3

Determine an estimate of the distance travelled by the object during the first 4 seconds.

12 The graph of $y = e^{\frac{x}{12}} + 4$, $x \in [0, 6]$, is rotated about the x-axis to produce the shape of a bowl.

a Write down the definite integral that gives the volume of this bowl.

b Use integration to find this volume. (Answer correct to three decimal places.)

c Use Simpson's rule (with $n = 2$) to approximate the volume. (Answer to three decimal places.)

d Find the percentage error in this approximation. (Answer to one significant figure.)

13 The region enclosed by the graphs of $y = \sin\left(\frac{\pi x}{2}\right)$ and $y = x$ is rotated about the x-axis to form a solid of revolution. In this question, give your answers to four decimal places.

a Use integration to find the volume of the solid.

b Use Simpson's rule with $n = 2$ to find an approximation for this volume. Comment.

CU

14 Consider a quadratic function f with rule $f(x) = ax^2 + bx + c$. Assume that the three points $(-w, y_0)$, $(0, y_1)$ and (w, y_2) lie on the graph of f, where $w > 0$.

a Show that $y_0 + y_2 = 2aw^2 + 2c$ and $y_1 = c$.

b Show that $\int_{-w}^{w} f(x)\,dx = \frac{2}{3}aw^3 + 2cw$.

c Use the results of parts **a** and **b** to show that

$$\int_{-w}^{w} f(x)\,dx = \frac{w}{3}(y_0 + 4y_1 + y_2)$$

Note: This shows that Simpson's rule is exact when f is a quadratic function. (It is not a restriction to consider an interval $[-w, w]$ centred at the origin, as we can first apply a translation parallel to the x-axis.)

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Review

Chapter summary

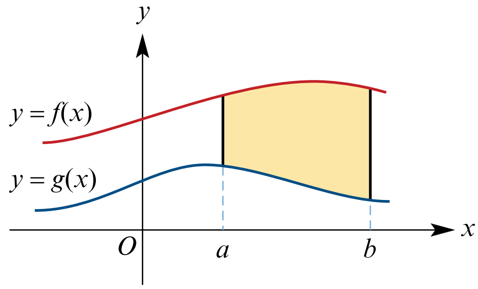
Fundamental theorem of calculus

- If f is a continuous function on an interval $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any anti-derivative of f.
- If f is a continuous function and the function G is defined by $G(x) = \int_a^x f(t)\,dt$, then G is an anti-derivative of f.

Areas of regions between curves

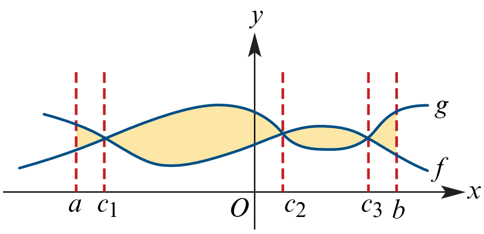
- If f and g are continuous functions such that $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region bounded by the curves and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x) - g(x)\,dx$$



- For graphs that cross, consider intervals. For example, the area of the shaded region is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx$$
$$+ \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$



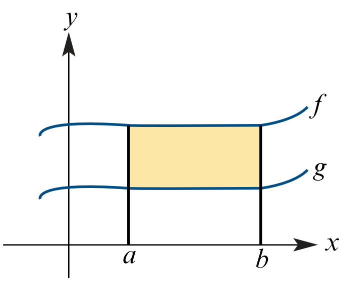
Volumes of solids of revolution

- If the region to be rotated about the x-axis is bounded by the curve with equation $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis, then the volume V is given by

$$V = \int_a^b \pi y^2\,dx = \pi \int_a^b (f(x))^2\,dx$$

- If the shaded region is rotated about the x-axis, then the volume V is given by

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2\,dx$$



Exponential distribution

- For $\lambda > 0$, an exponential random variable X with parameter λ has a probability density function given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The mean and standard deviation of X are given by

$$\mathrm{E}(X) = \frac{1}{\lambda} \quad \text{and} \quad \mathrm{sd}(X) = \frac{1}{\lambda}$$

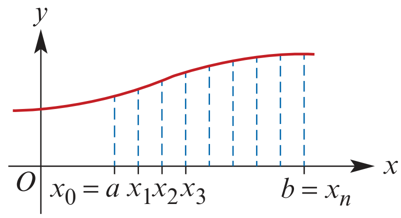
Simpson's rule

- Let n be a positive even number. Divide the interval $[a, b]$ on the x-axis into n equal subintervals

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \dots,\ [x_{n-1}, x_n]$$

each of width

$$w = \frac{b-a}{n}$$



- The definite integral $\int_a^b f(x)\,dx$ can be approximated by

$$\frac{w}{3}\Big(f(x_0) + 4[f(x_1) + f(x_3) + \dots + f(x_{n-1})] + 2[f(x_2) + f(x_4) + \dots + f(x_{n-2})] + f(x_n)\Big)$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

12A **1** I can evaluate definite integrals of functions using techniques covered in the previous chapters. ☐

See Example 1, Example 2, Example 3, Example 4, Example 5 and Questions 1, 2, 5, 6 and 14

12B **2** I can determine the area of a region between two curves. ☐

See Example 6, Example 7, Example 8, Example 9 Example 10 and Questions 1, 2, 6 and 7

12C **3** I can use a graphics calculator to evaluate definite integrals. ☐

See Example 11, Example 12, Example 13, Example 14 and Questions 1 and 3

12D **4** I can calculate the volume of solids of revolution. ☐

See Example 15, Example 16, Example 17, Example 18, Example 19 and Questions 1, 5, 6 and 14

12E **5** I can calculate probabilities using the exponential probability distribution. ☐

See Example 20, Example 21, Example 22, Example 23 and Questions 1, 3, 4 and 10

12F **6** I can approximate definite integrals using Simpson's rule. ☐

See Example 24, Example 25, Example 26 and Questions 1, 2 and 4

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Review

Short-response questions

Technology-free short-response questions

SF

1 Calculate the area of the region enclosed by the graph of $y = \dfrac{x}{\sqrt{x-2}}$ and the line $y = 3$.

2 **a** If $y = 1 - \cos x$, determine the value of $\int_0^{\frac{\pi}{2}} y\,dx$. By sketching a graph, show the regions whose area this integral calculates.

b Hence find $\int_0^1 x\,dy$.

CF

3 Find the volume of revolution of each of the following. (Rotation is about the x-axis.)

a $y = \sec x$ between $x = 0$ and $x = \dfrac{\pi}{4}$

b $y = \sin x$ between $x = 0$ and $x = \dfrac{\pi}{4}$

c $y = \cos x$ between $x = 0$ and $x = \dfrac{\pi}{4}$

d the region between $y = x^2$ and $y = 4x$

e $y = \sqrt{1+x}$ between $x = 0$ and $x = 8$

4 Determine the volume generated when the region bounded by the curve $y = 1 + \sqrt{x}$, the x-axis and the lines $x = 1$ and $x = 4$ is rotated about the x-axis.

5 The region S in the first quadrant of the Cartesian plane is bounded by the axes, the line $x = 3$ and the curve $y = \sqrt{1 + x^2}$. Find the volume of the solid formed when S is rotated:

a about the x-axis

b about the y-axis.

6 Sketch the graph of $y = \sec x$ for $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$. Find the volume of the solid of revolution obtained by rotating this curve about the x-axis for $x \in \left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$.

7 **a** Find the coordinates of the points of intersection of the graphs of $y^2 = 8x$ and $y = 2x$.

b Find the volume of the solid formed when the area enclosed by these graphs is rotated about the x-axis.

8 **a** On the one set of axes, sketch the graphs of $y = 1 - x^2$ and $y = x - x^3 = x(1 - x^2)$. (Turning points of the second graph do not have to be determined.)

b Determine the area of the region enclosed between the two graphs.

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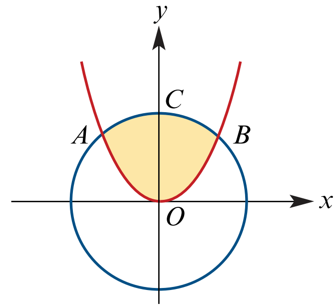
Review

CF

9 The curves $y = x^2$ and $x^2 + y^2 = 2$ meet at the points A and B.



a Determine the coordinates of A, B and C.

b Determine the volume of the solid of revolution formed by rotating the shaded region about the x-axis.

10 **a** Sketch the graph of $y = 2x - x^2$ for $y \geq 0$.

b Determine the area of the region enclosed between this curve and the x-axis.

c Determine the volume of the solid of revolution formed by rotating this region about the x-axis.

11 **a** Let the graph of $f(x) = x^2$, for $x \in [0, b]$, be rotated:

i around the x-axis to define a solid of revolution, and find the volume of this solid in terms of b (where the region rotated is between the curve and the x-axis)

ii around the y-axis to define a solid of revolution, and find the volume of this solid in terms of b (where the region rotated is between the curve and the y-axis).

b For what value of b are the two volumes equal?

12 **a** Sketch the graph of $y = \dfrac{1}{4x^2 + 1}$.

b Determine $\dfrac{dy}{dx}$ and hence find the equation of the tangent to this curve at $x = \frac{1}{2}$.

c Determine the area of the region bounded by the curve and the tangent to the curve at $x = \frac{1}{2}$.

13 Let $f(x) = x$ and $g(x) = \dfrac{9}{x}$.

a On the same set of axes, sketch the graphs of $f + g$ and $f - g$.

b Determine the area of the region bounded by the two graphs sketched in part **a** and the lines $x = 1$ and $x = 3$.

14 Sketch the graph of $y = x - 5 + \dfrac{4}{x}$. Determine the area of the region bounded by this graph and the x-axis.

15 Sketch the graph of $y = \dfrac{1}{2 + x - x^2}$. Determine the area of the region bounded by this graph and the line $y = \frac{1}{2}$.

Review

16 The probability density function of an exponentially distributed random variable X is given by SF

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Determine the exact value of $P(X > 1)$.

17 Suppose that X is an exponential random variable with probability density function CF

$$f(x) = \begin{cases} \frac{1}{8}e^{-\frac{x}{8}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a Determine the cumulative distribution function of X.

b Hence determine the exact value of the median of X.

18 Use Simpson's rule to estimate the integral $\int_1^3 f(x)\,dx$ for each table of values: SF

a

x	1	2	3
$f(x)$	6	7	5

b

x	1	2	3
$f(x)$	4	5	7

19 Five values of a function f are known, as shown in the table.

x	5	6	7	8	9
$f(x)$	4	7	10	12	13

Apply the general form of Simpson's rule to estimate $\int_5^9 f(x)\,dx$.

Technology-active short-response questions

20 **a** Sketch the curve with equation CF

$$y = 1 - \frac{1}{x+2}$$

b Determine the area of the region bounded by the x-axis, the curve and the lines $x = 0$ and $x = 2$.

c Determine the volume of the solid of revolution formed when this region is rotated around the x-axis.

21 Let $f(x) = x\tan^{-1} x$. CU

a Determine $f'(x)$.

b Hence find $\int_0^1 \tan^{-1} x\,dx$.

c Use the result of part **b** to find the area of the region bounded by $y = \tan^{-1} x$, $y = \frac{\pi}{4}$ and the y-axis.

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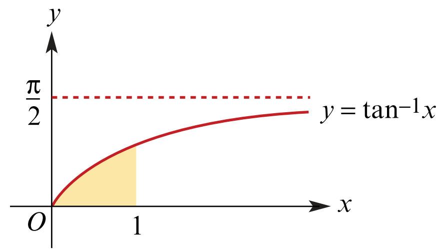
d Let $g(x) = (\tan^{-1} x)^2$.

i Determine $g'(x)$.

ii Show that $g'(x) > 0$ for $x > 0$.

iii Sketch the graph of g.

e Find the volume of the solid of revolution formed when the shaded region shown is rotated around the y-axis.



22 **a** **i** Differentiate $x \ln x$ and hence find $\int \ln x \, dx$.

ii Differentiate $x(\ln x)^2$ and hence find $\int (\ln x)^2 \, dx$.

b Sketch the graph of the function f with domain $[-2, 2]$ and rule

$$f(x) = \begin{cases} e^x & x \in [0, 2] \\ e^{-x} & x \in [-2, 0) \end{cases}$$

c The interior of a wine glass is formed by rotating the curve $y = e^x$ from $x = 0$ to $x = 2$ about the y-axis. If the units are in centimetres find, correct to two significant figures, the volume of liquid that the glass contains when full.

23 The lifetime of an electronic component, T years, is an exponentially distributed random variable with probability density function given by

$$f(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

a Determine $P(T > 2)$.

b Determine $P(T > 2 \mid T < 3)$.

c Determine the mean, μ, and the standard deviation, σ, of T. Hence evaluate

$$P(\mu - 2\sigma < T < \mu + 2\sigma)$$

d Determine the lifetime, L years, which a typical component is 80% certain to exceed.

e If five components are sold, what is the probability that at least one of them will have a lifetime of less than L years?

Review

CF

CU

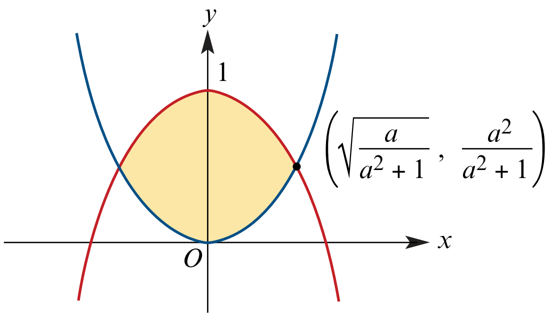
24 A bowl is modelled by rotating the curve $y = x^2$ for $0 \le x \le 1$ around the y-axis.

a Determine the volume of the bowl.

b **i** Determine the volume of liquid in the bowl when the depth of liquid is $\frac{1}{2}$.

ii Determine the depth of liquid in the bowl when it is half full.

25 The curves $y = ax^2$ and $y = 1 - \frac{x^2}{a}$ are shown, where $a > 0$.



a Show that the area enclosed by the two curves is $\frac{4}{3}\sqrt{\frac{a}{a^2+1}}$.

b **i** Find the value of a which gives the maximum area.

ii Find the maximum area.

c Determine the volume of the solid formed when the region bounded by these curves is rotated about the y-axis.

26 **a** On the same set of axes, sketch the graphs of $y = 3\sec^2 x$ and $y = 16\sin^2 x$ for $0 \le x \le \frac{\pi}{4}$.

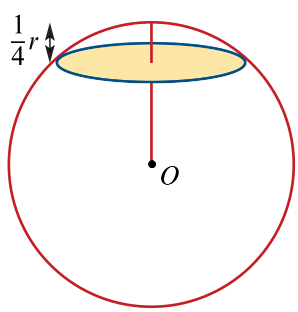
b Determine the coordinates of the point of intersection of these two curves.

c Determine the area of the region bounded by the two curves and the y-axis.

27 The curves $cy^2 = x^3$ and $y^2 = ax$ (where $a > 0$ and $c > 0$) intersect at the origin, O, and at a point P in the first quadrant. The areas of the regions enclosed by the curves OP, the x-axis and the vertical line through P are A_1 and A_2 respectively for the two curves. The volumes of the two solids formed by rotating these regions about the x-axis are V_1 and V_2 respectively. Show that $A_1 : A_2 = 3 : 5$ and $V_1 : V_2 = 1 : 2$.

28 **a** Find the area of the circle formed when a sphere is cut by a plane at a distance y from the centre, where $y < r$.

b By integration, prove that the volume of a 'cap' of height $\frac{1}{4}r$ cut from the top of the sphere, as shown in the diagram, is $\frac{11\pi r^3}{192}$.



29 Consider the section of a hyperbola with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $a \le x \le 2a$ (where $a > 0$). Determine the volume of the solid formed when the region bounded by the hyperbola and the line with equation $x = 2a$ is rotated about:

a the x-axis

b the y-axis.

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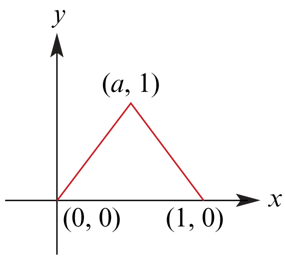
30 **a** Show that the line $y = \frac{3x}{2}$ does not meet the curve $y = \frac{1}{\sqrt{1 - x^2}}$.

b Determine the area of the region bounded by the curve with equation $y = \frac{1}{\sqrt{1 - x^2}}$ and the lines $y = \frac{3x}{2}$, $x = 0$ and $x = \frac{1}{2}$.

c Determine the volume of the solid of revolution formed by rotating the region defined in part **b** about the x-axis. Express your answer in the form $\pi(a + \ln b)$.

31 **a** For $0 \le a \le 1$, let T_a be the triangle whose vertices are $(0, 0)$, $(1, 0)$ and $(a, 1)$. Find the volume of the solid of revolution when T_a is rotated about the x-axis.



b For $0 \le k \le 1$, let T_k be the triangle whose vertices are $(0, 0)$, $(k, 0)$ and $(0, \sqrt{1 - k^2})$. The triangle T_k is rotated about the x-axis. What value of k gives the maximum volume? What is the maximum volume?

32 A model for a bowl is formed by rotating a section of the graph of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ around the x-axis to form a solid of revolution. The cubic is chosen to pass through the points with coordinates $(0, 0)$, $(5, 1)$, $(10, 2.5)$ and $(30, 10)$.

a **i** Write down the four simultaneous equations that can be used to determine the coefficients a, b, c and d.

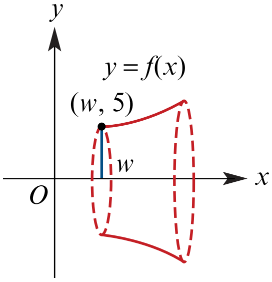
ii Using a calculator, or otherwise, find the values of a, b, c and d. (Exact values should be stated.)

b Find the area of the region enclosed by the curve, the x-axis and the line $x = 30$.

c **i** Write the expression that can be used to determine the volume of the solid of revolution when the section of the curve $0 \le x \le 30$ is rotated around the x-axis.

ii Use a calculator to determine this volume.

d Using the initial design, the bowl is unstable.
The designer is very fond of the cubic $y = f(x)$, and modifies the design so that the base of the bowl has radius 5 units. Using a calculator:



i find the value of w such that $f(w) = 5$, $0 < w < 30$.

ii find the new volume, correct to four significant figures.

e A mathematician looks at the design and suggests that it may be more pleasing to the eye if the base is chosen to occur at a point where $x = p$ and $f''(p) = 0$. Determine the values of coordinates of the point $(p, f(p))$.

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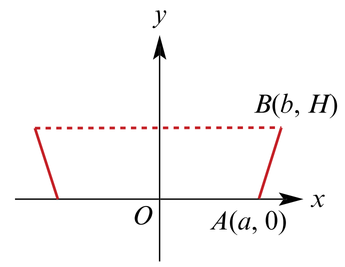
Review

CU

33 A model of a bowl is formed by rotating the line segment AB about the y-axis to form a solid of revolution.



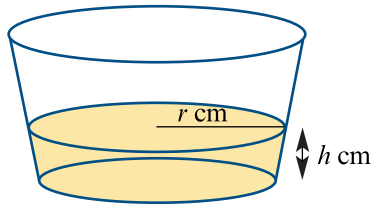
a Determine the volume, V cm^3, of the bowl in terms of a, b and H. (Units are centimetres.)

b If the bowl is filled with water to a height $\dfrac{H}{2}$, find the volume of water.

c Find an expression for the volume of water in the bowl when the radius of the water surface is r cm. (The constants a, b and H are to be used.)

d Determine $\dfrac{dV}{dr}$.

e Determine an expression for the depth of the water, h cm, in terms of r.



34 Simpson's rule

In this question, we consider intervals centred at the origin. Fix a positive number w. The basic form of Simpson's rule gives an estimate of $\int_{-w}^{w} f(x)\,dx$ by using only three function values:

$$f(-w), \quad f(0) \quad \text{and} \quad f(w)$$

Now suppose that k, ℓ and m are real constants such that

$$\int_{-w}^{w} f(x)\,dx = k\,f(-w) + \ell\,f(0) + m\,f(w) \qquad (\star)$$

for every polynomial function f of degree at most 2.

a By using $f(x) = 1$ in $(\star)$, find an equation relating w, k, ℓ and m.

b By using $f(x) = x$ in $(\star)$, find another equation relating w, k, ℓ and m.

c By using $f(x) = x^2$ in $(\star)$, find another equation relating w, k, ℓ and m.

d Solve the system of three equations from parts **a**, **b** and **c** to show that

$$k = m = \frac{w}{3} \quad \text{and} \quad \ell = \frac{4w}{3}$$

Note: These values of k, ℓ and m give the basic form of Simpson's rule. For the proof that these values make equation $(\star)$ hold for every polynomial function f of degree 2, see Question 14 in Exercise 12F.

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Review

Multiple-choice questions

Technology-free multiple-choice questions

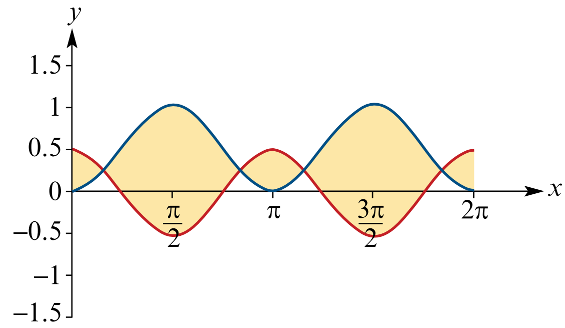
1 The graphs of $y = \sin^2 x$ and $y = \frac{1}{2}\cos(2x)$ are shown in the diagram. The total area of the shaded regions is equal to

A $\int_0^{2\pi} \sin^2 x - \frac{1}{2}\cos(2x)\, dx$

B $4\int_0^{\frac{\pi}{6}} \frac{1}{2}\cos(2x) - \sin^2 x\, dx$
$+ 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x - \frac{1}{2}\cos(2x)\, dx$

C 3.14

D π



2 The volume of the solid of revolution formed when the region bounded by the axes, the line $x = 1$ and the curve with equation $y = \dfrac{1}{\sqrt{4 - x^2}}$ is rotated about the x-axis is

A $\dfrac{\pi^2}{6}$ **B** $\dfrac{\pi^2}{3}$ **C** $\dfrac{\pi}{4}\ln(3)$ **D** $\pi\sqrt{3}\ln(3)$

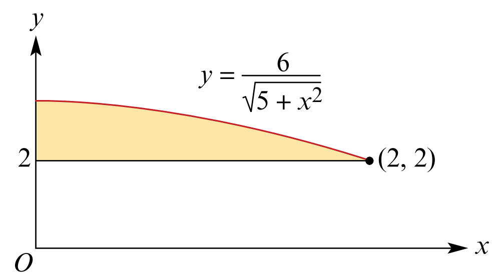
3 The shaded region shown below is enclosed by the curve $y = \dfrac{6}{\sqrt{5 + x^2}}$, the straight line $y = 2$ and the y-axis. The region is rotated about the x-axis to form a solid of revolution. The volume of this solid, in cubic units, is given by

A $\pi\int_0^2 \left(\dfrac{6}{\sqrt{5 + x^2}} - 2\right)^2 dx$

B $6\pi\tan^{-1}\left(\dfrac{2}{5}\right)$

C $\dfrac{36\pi}{\sqrt{5}}\tan^{-1}\left(\dfrac{2}{\sqrt{5}}\right)$

D $\pi\int_0^2 \left(\dfrac{6}{\sqrt{5 + x^2}}\right)^2 - 4\, dx$



4 A help desk receives an average of 10 queries per hour. If the time between receiving queries is exponentially distributed, then the probability that it will be more than 10 minutes until the next query is equal to

A e^{-1} **B** $e^{-\frac{5}{3}}$ **C** e^{-60} **D** $1 - e^{-1}$

Review

5 Let X be an exponentially distributed random variable with a mean of 0.5. For $x \geq 0$, the probability density function of X is given by

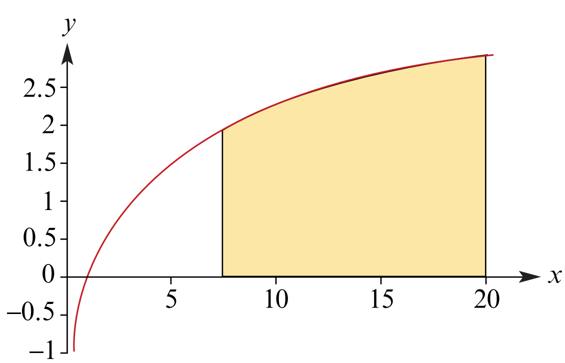
A $f(x) = 2e^{-2x}$
B $f(x) = 0.5e^{-2x}$
C $f(x) = 1 - 0.5e^{-\frac{x}{2}}$
D $f(x) = 0.5e^{-\frac{x}{2}}$

6 Let X be an exponentially distributed random variable with a mean of 4. For $x \geq 0$, the cumulative distribution function of X is given by

A $F(x) = 1 - 4e^{-4x}$
B $F(x) = 1 - e^{-4x}$
C $F(x) = 0.25e^{-0.25x}$
D $F(x) = 1 - e^{-0.25x}$

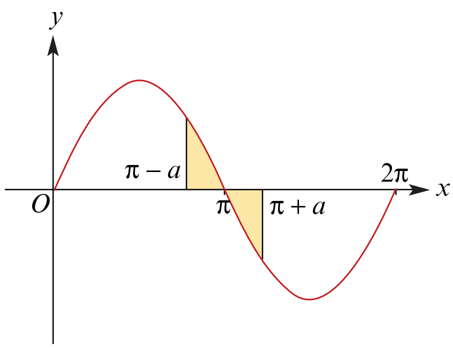
7 The shaded region in the diagram is bounded by the lines $x = e^2$ and $x = e^3$, the x-axis and the graph of $y = \ln x$. The volume of the solid of revolution formed by rotating this region about the x-axis is equal to

A $\pi \int_2^3 e^{2x}\, dx$
B $\pi \int_7^{20} (\ln x)^2\, dx$
C $\pi \int_{e^2}^{e^3} (\ln x)^2\, dx$
D $\pi(e^3 - e^2)$



8 The graph represents the function $y = \sin x$ where $0 \leq x \leq 2\pi$. The total area of the shaded regions is

A $1 - \cos a$
B $-2\sin a$
C $2(1 - \cos a)$
D 0



9 The area of the region enclosed between the curve with equation $y = \sin^3 x$, $x \in [0, a]$, the x-axis and the line with equation $x = a$, where $0 < a < \dfrac{\pi}{2}$, is

A $\dfrac{2}{3} - \dfrac{1}{3}\sin^3 a$
B $\left(\dfrac{2}{3} - \dfrac{1}{3}\sin^2 a\right)\cos a + \dfrac{2}{3}$
C $\dfrac{1}{3}\cos^3 a \, \sin a$
D $\dfrac{2}{3} - \cos a + \dfrac{1}{3}\cos^3 a$

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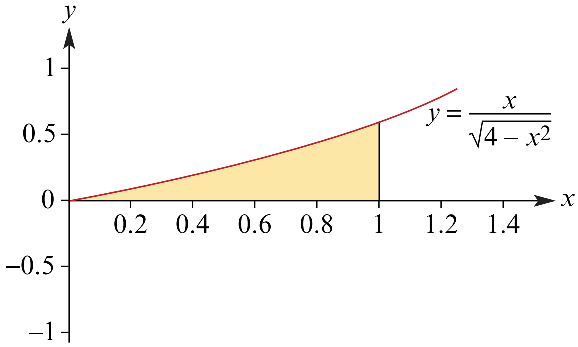
10 The shaded region shown is rotated around the x-axis to form a solid of revolution. The volume of the solid of revolution is



A $1 - \ln(\frac{1}{3})$

B $\pi(\ln 3 - 1)$

C 0.099

D $\pi(-1 + \ln(\frac{1}{3}))$

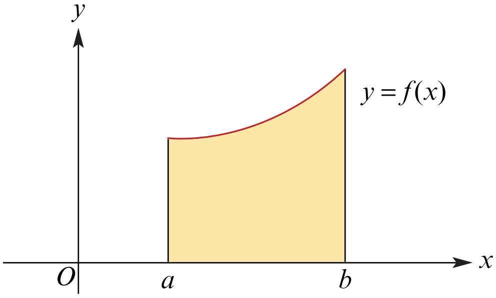
11 The shaded region shown in the diagram is rotated around the x-axis to form a solid of revolution, where $f'(x) > 0$ and $f''(x) > 0$ for all $x \in [a, b]$ and the volume of the solid of revolution is V cubic units. Which of the following statements is false?

A $V < \pi(f(b))^2(b - a)$

B $V > \pi(f(a))^2(b - a)$

C $V = \pi \int_a^b (f(x))^2 \, dx$

D $V = \pi\left((F(b))^2 - (F(a))^2\right)$, where $F'(x) = f(x)$



12 The area of the region bounded by the curve $y = \cos\left(\frac{x}{2}\right)$, the x-axis and the lines $x = 0$ and $x = \pi$ is

A 0 **B** 1 **C** 2 **D** π

13 The region bounded by the coordinate axes and the graph of $y = \cos x$, for $0 \le x \le \frac{\pi}{2}$, is rotated about the y-axis to form a solid of revolution. The volume of the solid is given by

A $\pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$ **B** $\pi \int_0^1 \cos^{-1} y \, dx$

C $\pi \int_0^{\frac{\pi}{2}} (\cos^{-1} y)^2 \, dy$ **D** $\pi \int_0^1 (\cos^{-1} y)^2 \, dy$

Technology-active multiple-choice questions

14 Using Simpson's rule with strips of width $w = 1$, the area under the curve $y = 2^x$ between $x = 0$ and $x = 2$ is estimated as

A $\frac{5}{3}$ **B** 4 **C** $\frac{13}{3}$ **D** 4.328

15 If X is an exponentially distributed random variable with a mean of 4, then $P(X < 2)$ is closest to

A 0.6065 **B** 0.3935 **C** 0.1516 **D** 0.3023

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Review

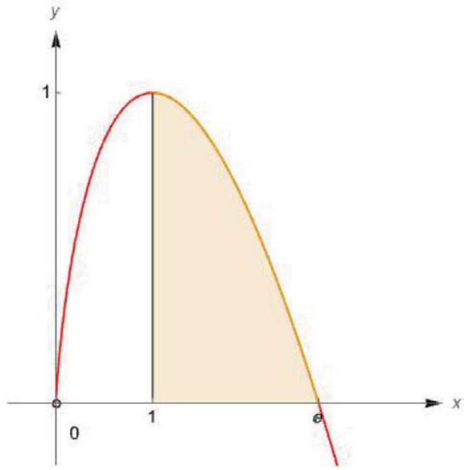
16 The area of the region enclosed between the curve with equation $y = x - x\ln x$, the x-axis and the line with equation $x = 1$ is

A $\dfrac{e^2 - 3}{4}$

B $2e$

C $\dfrac{e^2 - 5}{6}$

D $\dfrac{e^3 - 7}{8}$



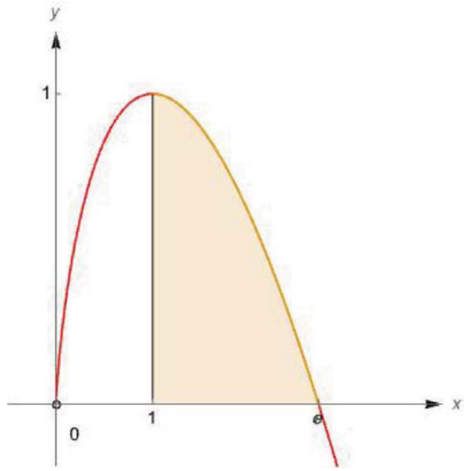
17 The region enclosed between the curve with equation $y = x - x\ln x$, the x-axis and the line with equation $x = 1$ is rotated around the x-axis to form a solid of revolution. The volume correct to three decimal places is

A 0.858

B 2.628

C 0.876

D 2.696



18 Using Simpson's rule with 4 equal intervals gives an approximation of $\int_2^4 e^x \, dx$ correct to three decimal places as

A 47.209 **B** 47.225 **C** 47.210 **D** 47.208

19 The number of days ahead travellers purchase their airline tickets can be modelled by an exponential distribution with the mean amount of time equal to 10 days. Find the probability correct to three decimal places that a traveller will purchase a ticket fewer than 5 days in advance.

A 0.607 **B** 0.421 **C** 0.561 **D** 0.393

20 The time between emergency calls to a small suburban fire station follows an exponential distribution with an average rate of a calls per day. The probability of a call in the next 15 minutes is 0.0186. The value of a is closest to.

A 1.2 **B** 1.3 **C** 1.4 **D** 1.8

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13

Rates of change and differential equations

Chapter contents

A differential equation arises when there is a relationship between an unknown function and its derivatives. These equations describe how the rate of change of a quantity is related to the quantity itself. They commonly appear in various fields of science and engineering to model physical phenomena, such as motion, growth, decay, and many other dynamical systems.

For example, we know that the rate of decay of a radioactive substance is proportional to the mass m of substance remaining at time t. We can write this as a differential equation:

$$\frac{dm}{dt} = -km$$

where k is a constant. What we would really like is an expression for the mass m at time t. In this chapter we will determine that the general solution to this differential equation is $m = Ae^{-kt}$.

Chapters 13 covers Unit 4 Topic 3: Rates of change and differential equations.

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13A Implicit differentiation

Learning intentions

- To use implicit differentiation to determine the derivative of an implicitly defined curve.

It is not always possible to determine a single equation that expresses a dependent variable in terms of the independent variable. The equations of a unit circle $x^2 + y^2 = 1$ is one simple example. **Implicit differentiation** is a technique used to differentiate equations where the dependent variable is not explicitly expressed in terms of the independent variable.

Outlining the approach

To perform implicit differentiation, we differentiate both sides of the equation with respect to the independent variable, treating the dependent variable as a function of the independent variable. This involves applying the chain rule whenever the dependent variable appears.

For example, consider the curve with equation $x = y^2$. If we could solve this for y (which we won't always be able to do) we would determine y as some function of x. That is: $y = y(x)$. Therefore

$$x = (y(x))^2$$

The next step is to differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}x = \frac{d}{dx}(y(x))^2$$

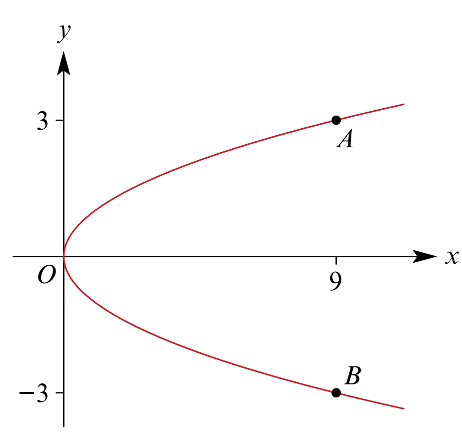
The derivative of the left-hand side is simply equal to 1. We differentiate the right-hand side using the chain rule:

$$\begin{aligned} \frac{d}{dx}x &= \frac{d}{dx}(y(x))^2 \\ 1 &= 2y(x)\frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2y} \end{aligned}$$

To illustrate this result, the curve $x = y^2$ is shown on the right. There are two points on the curve, A and B, for which $x = 9$.

At A, we have $y = 3$ and so $\dfrac{dy}{dx} = \dfrac{1}{6}$.

At B, we have $y = -3$ and so $\dfrac{dy}{dx} = -\dfrac{1}{6}$.



We will usually just write y instead of $y(x)$ however we must remember to regard y as a function of x when differentiating.

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Example 1

For each of the following, determine $\frac{dy}{dx}$ by implicit differentiation:

a $x^3 = y^2$

b $xy = 2x + 1$

Solution

a We note that $y = y(x)$ and differentiate both sides with respect to x:

$$\frac{d}{dx}(x^3) = \frac{d}{dx}(y^2)$$

$$3x^2 = 2y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

b We note that $y = y(x)$ and differentiate both sides with respect to x:

$$\therefore \quad \frac{d}{dx}(xy) = \frac{d}{dx}(2x + 1)$$

$$\frac{d}{dx}(xy) = 2$$

Use the product rule on the left-hand side:

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{2 - y}{x}$$

Example 2

Determine $\frac{dy}{dx}$ if $x^2 + y^2 = 1$.

Solution

Note that $x^2 + y^2 = 1$ leads to

$$y = \pm\sqrt{1 - x^2} \quad \text{or} \quad x = \pm\sqrt{1 - y^2}$$

So y is not a function of x, and x is not a function of y. Implicit differentiation should be used. Since $x^2 + y^2 = 1$ is the unit circle, we can also determine the derivative geometrically.

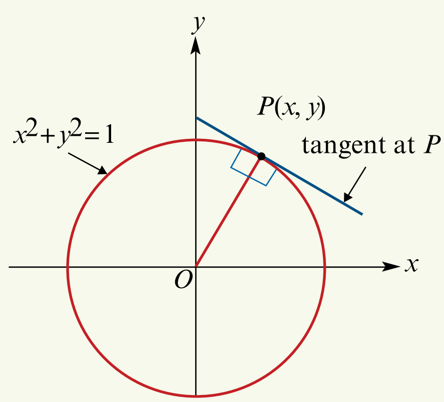
Method 1: Using geometry

Let $P(x, y)$ be a point on the unit circle with $x \neq 0$.

The gradient of OP is $\frac{\text{rise}}{\text{run}} = \frac{y}{x}$.

Since the radius is perpendicular to the tangent for a circle, the gradient of the tangent is $-\frac{x}{y}$, provided $y \neq 0$.

That is, $\frac{dy}{dx} = -\frac{x}{y}$.



From the graph, when $y = 0$ the tangents are parallel to the y-axis, hence $\frac{dy}{dx}$ is not defined.

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Method 2: Using implicit differentiation

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{(differentiate both sides with respect to } x\text{)}$$

$$\therefore \quad 2y\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y} \qquad \text{for } y \neq 0$$

Example 3

Given $xy - y - x^2 = 0$, determine $\dfrac{dy}{dx}$.

Solution

Method 1: Expressing y as a function of x

$$xy - y - x^2 = 0$$

$$y(x-1) = x^2$$

$$y = \frac{x^2}{x-1}$$

Therefore $\quad y = x + 1 + \dfrac{1}{x-1} \qquad (\text{for } x \neq 1)$

Hence $\quad \dfrac{dy}{dx} = 1 - \dfrac{1}{(x-1)^2}$

$$= \frac{(x-1)^2 - 1}{(x-1)^2}$$

$$= \frac{x^2 - 2x}{(x-1)^2} \qquad (\text{for } x \neq 1)$$

Method 2: Using implicit differentiation

$$xy - y - x^2 = 0$$

$$\therefore \quad \frac{d}{dx}(xy) - \frac{dy}{dx} - \frac{d}{dx}(x^2) = \frac{d}{dx}(0) \qquad \text{(differentiate both sides with respect to } x\text{)}$$

$$\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) - \frac{dy}{dx} - 2x = 0 \qquad \text{(product rule)}$$

$$x\frac{dy}{dx} - \frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx}(x-1) = 2x - y$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x-y}{x-1} \qquad (\text{for } x \neq 1)$$

By substituting $y = \dfrac{x^2}{x-1}$ we can confirm that the two results are identical.

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Example 4

Consider the curve with equation $2x^2 - 2xy + y^2 = 5$.

a Determine $\dfrac{dy}{dx}$.

b Determine the gradient of the tangent to the curve at the point $(1, 3)$.

Solution

a Neither x nor y can be expressed as a function, so implicit differentiation must be used.

$$2x^2 - 2xy + y^2 = 5$$

$$\frac{d}{dx}(2x^2) - \frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$4x - \left(2x \cdot \frac{dy}{dx} + y \cdot 2\right) + 2y\frac{dy}{dx} = 0 \qquad \text{(by the product and chain rules)}$$

$$4x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 4x$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 4x$$

$$\therefore \quad \frac{dy}{dx} = \frac{2y - 4x}{2y - 2x}$$

$$= \frac{y - 2x}{y - x} \qquad \text{(for } x \neq y\text{)}$$

b When $x = 1$ and $y = 3$, the gradient is $\dfrac{3 - 2}{3 - 1} = \dfrac{1}{2}$.

Skill-sheet

Exercise 13A

Example 1, 2

1 For each of the following, determine $\dfrac{dy}{dx}$ using implicit differentiation:

a $x^2 - 2y = 3$ **b** $x^2y = 1$ **c** $x^3 + y^3 = 1$

d $y^3 = x^2$ **e** $x - \sqrt{y} = 2$ **f** $xy - 2x + 3y = 0$

g $y^2 = 4ax$ **h** $4x + y^2 - 2y - 2 = 0$

SF

Example 3

2 Determine $\dfrac{dy}{dx}$ for each of the following:

a $(x + 2)^2 - y^2 = 4$ **b** $\dfrac{1}{x} + \dfrac{1}{y} = 1$ **c** $y = (x + y)^2$

d $x^2 - xy + y^2 = 1$ **e** $y = x^2e^y$ **f** $\sin y = \cos^2 x$

g $\sin(x - y) = \sin x - \sin y$ **h** $y^5 - x\sin y + 3y^2 = 1$

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Example 4

3 For each of the following, determine the equation of the tangent at the indicated point:

a $y^2 = 8x$ at $(2, -4)$

b $x^2 - 9y^2 = 9$ at $\left(5, \frac{4}{3}\right)$

c $xy - y^2 = 1$ at $\left(\frac{17}{4}, 4\right)$

d $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at $(0, -3)$

CF

4 Determine $\frac{dy}{dx}$ in terms of x and y, given that $\ln(y) = \ln(x) + 1$.

SF

5 Determine the gradient of the curve $x^3 + y^3 = 9$ at the point $(1, 2)$.

6 A curve is defined by the equation $x^3 + y^3 + 3xy - 1 = 0$. Determine the gradient of the curve at the point $(2, -1)$.

7 Given that $\tan x + \tan y = 3$, determine the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{4}$.

CF

8 Determine the gradient at the point $(1, -3)$ on the curve with equation $y^2 + xy - 2x^2 = 4$.

9 Consider the curve with equation $x^3 + y^3 = 28$.

a Obtain an expression for $\frac{dy}{dx}$.

b Show that $\frac{dy}{dx}$ cannot be positive.

c Calculate the value of $\frac{dy}{dx}$ when $x = 1$.

10 The equation of a curve is $2x^2 + 8xy + 5y^2 = -3$. Determine the equations of the two tangents that are parallel to the x-axis.

11 The equation of a curve C is $x^3 + xy + 2y^3 = k$, where k is a constant.

a Determine $\frac{dy}{dx}$ in terms of x and y.

b The curve C has a tangent parallel to the y-axis. Show that the y-coordinate at the point of contact satisfies $216y^6 + 4y^3 + k = 0$.

c Hence show that $k \le \frac{1}{54}$.

d Determine the possible value(s) of k in the case where $x = -6$ is a tangent to C.

12 The equation of a curve is $x^2 - 2xy + 2y^2 = 4$.

a Determine an expression for $\frac{dy}{dx}$ in terms of x and y.

b Determine the coordinates of each point on the curve at which the tangent is parallel to the x-axis.

13 Consider the curve with equation $y^2 + x^3 = 1$.

a Determine $\frac{dy}{dx}$ in terms of x and y.

b Determine the points where $\frac{dy}{dx} = 0$.

c Describe the behaviour as $x \to -\infty$.

d Express y in terms of x.

e Determine the coordinates of the points of inflection of the curve.

f Use a calculator to help you sketch the graph of $y^2 + x^3 = 1$.

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13B An introduction to differential equations

Learning intentions

- To understand the concept of differential equations and verify solutions by substitution.

A differential equation contains derivatives of a particular function. For example,

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = \frac{y}{y+1}$$

A solution to a differential equation is a clearly defined function that, when differentiated according to the rules of the equation, produces a result that satisfies the equation.

For example, if $\frac{dy}{dx} = \cos x$, then $y = \int \cos x \, dx$ and so $y = \sin x + c$.

Here $y = \sin x + c$ is the **general solution** of the differential equation $\frac{dy}{dx} = \cos x$.

This example displays features typical of such solutions. Solutions of differential equations are the result of an integral, and therefore produce a family of functions.

To obtain a **particular solution**, we require further information, which is usually given as an ordered pair belonging to the function or relation. For equations with second derivatives, we need two items of information.

Verifying a solution of a differential equation

We can verify that a particular expression is a solution of a differential equation by substitution. This is demonstrated in the following examples.

We will use the following notation to denote the y-value for a given x-value:

$y(0) = 3$ will mean that when $x = 0$, $y = 3$.

Example 5

a Verify that $y = Ae^x - x - 1$ is a solution of the differential equation $\frac{dy}{dx} = x + y$.

b Hence determine the particular solution of the differential equation given that $y(0) = 3$.

Solution

a Let $y = Ae^x - x - 1$. We need to check that $\frac{dy}{dx} = x + y$.

$$\begin{aligned}\text{LHS} &= \frac{dy}{dx} \\ &= Ae^x - 1\end{aligned}$$

$$\begin{aligned}\text{RHS} &= x + y \\ &= x + Ae^x - x - 1 \\ &= Ae^x - 1\end{aligned}$$

Hence LHS = RHS and so $y = Ae^x - x - 1$ is a solution of $\frac{dy}{dx} = x + y$.

b $y(0) = 3$ means that when $x = 0$, $y = 3$.

Substituting in the solution $y = Ae^x - x - 1$ verified in **a**:

$$3 = Ae^0 - 0 - 1$$
$$3 = A - 1$$
$$\therefore \quad A = 4$$

The particular solution is $y = 4e^x - x - 1$.

Example 6

Verify that $y = e^{2x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$.

Solution

Let $\quad y = e^{2x}$

Then $\quad \dfrac{dy}{dx} = 2e^{2x}$

and $\quad \dfrac{d^2y}{dx^2} = 4e^{2x}$

Now consider the differential equation:

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y \\ &= 4e^{2x} + 2e^{2x} - 6e^{2x} \qquad \text{(from above)} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Example 7

Verify that $y = ae^{2x} + be^{-3x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$.

Solution

Let $\quad y = ae^{2x} + be^{-3x}$

Then $\quad \dfrac{dy}{dx} = 2ae^{2x} - 3be^{-3x}$

and $\quad \dfrac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x}$

So

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y \\ &= (4ae^{2x} + 9be^{-3x}) + (2ae^{2x} - 3be^{-3x}) - 6(ae^{2x} + be^{-3x}) \\ &= 4ae^{2x} + 9be^{-3x} + 2ae^{2x} - 3be^{-3x} - 6ae^{2x} - 6be^{-3x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

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Example 8

Determine the constants a and b if $y = e^{4x}(2x + 1)$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$$

Solution

Let $\quad y = e^{4x}(2x + 1)$

Then
$$\begin{aligned}\frac{dy}{dx} &= 4e^{4x}(2x + 1) + 2e^{4x}\\ &= 2e^{4x}(4x + 2 + 1)\\ &= 2e^{4x}(4x + 3)\end{aligned}$$

and
$$\begin{aligned}\frac{d^2y}{dx^2} &= 8e^{4x}(4x + 3) + 4 \times 2e^{4x}\\ &= 8e^{4x}(4x + 3 + 1)\\ &= 8e^{4x}(4x + 4)\\ &= 32e^{4x}(x + 1)\end{aligned}$$

If $y = e^{4x}(2x + 1)$ is a solution of the differential equation, then

$$\frac{d^2y}{dx^2} - a\frac{dy}{dx} + by = 0$$

i.e. $\quad 32e^{4x}(x + 1) - 2ae^{4x}(4x + 3) + be^{4x}(2x + 1) = 0$

We can divide through by e^{4x} (since $e^{4x} > 0$):

$$32x + 32 - 8ax - 6a + 2bx + b = 0$$

i.e. $\quad (32 - 8a + 2b)x + (32 - 6a + b) = 0$

Thus

$$32 - 8a + 2b = 0 \qquad (1)$$
$$32 - 6a + b = 0 \qquad (2)$$

Multiply (2) by 2 and subtract from (1):

$$-32 + 4a = 0$$

Hence $a = 8$ and $b = 16$.

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Exercise 13B

Example 5

1 For each of the following, verify that the given function or relation is a solution of the differential equation. Hence determine the particular solution from the given information.

	Differential equation	Function or relation	Added information
a	$\dfrac{dy}{dt} = 2y + 4$	$y = Ae^{2t} - 2$	$y(0) = 2$
b	$\dfrac{dy}{dx} = \ln\lvert x\rvert$	$y = x\ln\lvert x\rvert - x + c$	$y(1) = 3$
c	$\dfrac{dy}{dx} = \dfrac{1}{y}$	$y = \sqrt{2x + c}$	$y(1) = 9$
d	$\dfrac{dy}{dx} = \dfrac{y+1}{y}$	$y - \ln\lvert y + 1\rvert = x + c$	$y(3) = 0$
e	$\dfrac{d^2y}{dx^2} = 6x^2$	$y = \dfrac{x^4}{2} + Ax + B$	$y(0) = 2,\ y(1) = 2$
f	$\dfrac{d^2y}{dx^2} = 4y$	$y = Ae^{2x} + Be^{-2x}$	$y(0) = 3,\ y(\ln 2) = 9$
g	$\dfrac{d^2x}{dt^2} + 9x = 18$	$x = A\sin(3t) + B\cos(3t) + 2$	$x(0) = 4,\ x\left(\dfrac{\pi}{2}\right) = -1$

Example 6, 7

2 For each of the following, verify that the given function is a solution of the differential equation:

a $\dfrac{dy}{dx} = 2y, \quad y = 4e^{2x}$

b $\dfrac{dy}{dx} = -4xy^2, \quad y = \dfrac{1}{2x^2}$

c $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}, \quad y = x\ln\lvert x\rvert + x$

d $\dfrac{dy}{dx} = \dfrac{2x}{y^2}, \quad y = \sqrt[3]{3x^2 + 27}$

e $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0, \quad y = e^{-2x} + e^{3x}$

f $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 16y = 0, \quad y = e^{4x}(x + 1)$

g $\dfrac{d^2y}{dx^2} = -n^2y, \quad y = a\sin(nx)$

h $\dfrac{d^2y}{dx^2} = n^2y, \quad y = e^{nx} + e^{-nx}$

i $\dfrac{dy}{dx} = \dfrac{1 + y^2}{1 + x^2}, \quad y = \dfrac{x + 1}{1 - x}$

j $y\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2, \quad y = \dfrac{4}{x + 1}$

Example 8

3 If the differential equation $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 10y = 0$ has a solution $y = ax^n$, determine the possible values of n.

4 Determine the constants a, b and c if $y = a + bx + cx^2$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 4x^2$.

5 Determine the constants a and b if $x = t(a\cos(2t) + b\sin(2t))$ is a solution of the differential equation $\dfrac{d^2x}{dt^2} + 4x = 2\cos(2t)$.

6 Determine the constants a, b, c and d if $y = ax^3 + bx^2 + cx + d$ is a solution to the differential equation $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^3$.

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13C Differential equations involving a function of the independent variable

Learning intentions

- To determine general solutions of differential equations through anti-differentiation.
- To solve differential equations with initial conditions to determine particular solutions.

In this section we solve differential equations of the following two forms:

$$\frac{dy}{dx} = f(x) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x)$$

Solving differential equations of the form $\frac{dy}{dx} = f(x)$

The simplest differential equations are those of the form

$$\frac{dy}{dx} = f(x)$$

Such a differential equation can be solved provided an anti-derivative of $f(x)$ can be found.

If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\,dx$.

Example 9

Determine the general solution of each of the following:

a $\frac{dy}{dx} = x^4 - 3x^2 + 2$

b $\frac{dy}{dt} = \sin(2t)$

c $\frac{dx}{dt} = e^{-3t} + \frac{1}{t}$

d $\frac{dx}{dy} = \frac{1}{1+y^2}$

Solution

a $\frac{dy}{dx} = x^4 - 3x^2 + 2$

$\therefore\ y = \int x^4 - 3x^2 + 2\,dx$

$\therefore\ y = \frac{x^5}{5} - x^3 + 2x + c$

b $\frac{dy}{dt} = \sin(2t)$

$\therefore\ y = \int \sin(2t)\,dt$

$\therefore\ y = -\frac{1}{2}\cos(2t) + c$

c $\frac{dx}{dt} = e^{-3t} + \frac{1}{t}$

$\therefore\ x = \int e^{-3t} + \frac{1}{t}\,dt$

$\therefore\ x = -\frac{1}{3}e^{-3t} + \ln|t| + c$

d $\frac{dx}{dy} = \frac{1}{1+y^2}$

$\therefore\ x = \int \frac{1}{1+y^2}\,dy$

$\therefore\ x = \tan^{-1}(y) + c$

This can also be written as $y = \tan(x - c)$.

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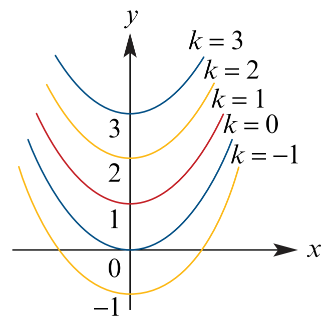
Families of solution curves

Solving a differential equation requires determining an equation that connects the variables, but does not contain a derivative. By solving differential equations, it is possible to determine what function or functions might model a particular situation or physical law.

If $\frac{dy}{dx} = x$, then it follows that $y = \frac{1}{2}x^2 + k$, where k is a constant.

The **general solution** of the differential equation $\frac{dy}{dx} = x$ can be given as $y = \frac{1}{2}x^2 + k$.

If different values of the constant k are taken, then a family of curves is obtained. This differential equation represents the family of curves $y = \frac{1}{2}x^2 + k$, where $k \in \mathbb{R}$.



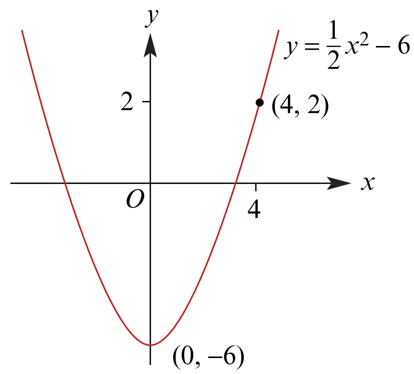
For **particular solutions** of a differential equation, a particular curve from the family can be distinguished by selecting a specific point of the plane through which the curve passes.

For instance, the particular solution of $\frac{dy}{dx} = x$ for which $y = 2$ when $x = 4$ can be thought of as the solution curve of the differential equation that passes through the point $(4, 2)$.

From above:

$$\begin{aligned} y &= \tfrac{1}{2}x^2 + k \\ \therefore \quad 2 &= \tfrac{1}{2} \times 16 + k \\ 2 &= 8 + k \\ \therefore \quad k &= -6 \end{aligned}$$

Thus the solution is $y = \frac{1}{2}x^2 - 6$.



Example 10

a Determine the family of curves with gradient given by e^{2x}. That is, determine the general solution of the differential equation $\frac{dy}{dx} = e^{2x}$.

b Determine the equation of the curve that has gradient e^{2x} and passes through $(0, 3)$.

Solution

a $\frac{dy}{dx} = e^{2x}$

$$\begin{aligned} \therefore \quad y &= \int e^{2x}\,dx \\ &= \tfrac{1}{2}e^{2x} + c \end{aligned}$$

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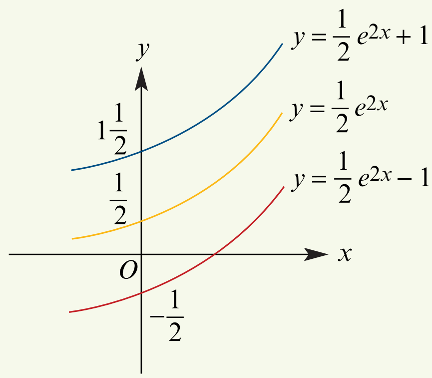
The general solution $y = \frac{1}{2}e^{2x} + c$ represents a family of curves, since c can take any real number value. The diagram shows some of these curves.



b Substituting $x = 0$ and $y = 3$ in the general equation $y = \frac{1}{2}e^{2x} + c$, we have

$$3 = \frac{1}{2}e^{0} + c$$

$$\therefore \quad c = \frac{5}{2}$$

The equation is $y = \frac{1}{2}e^{2x} + \frac{5}{2}$.

Solving differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

These differential equations are similar to those discussed above, with anti-differentiation being applied twice.

Let $p = \frac{dy}{dx}$. Then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = f(x)$.

The technique involves first determining p as the solution of the differential equation $\frac{dp}{dx} = f(x)$, and then substituting p into $\frac{dy}{dx} = p$ and solving this differential equation.

Example 11

Determine the general solution of each of the following:

a $\frac{d^2y}{dx^2} = 10x^3 - 3x + 4$

b $\frac{d^2y}{dx^2} = \cos(3x)$

c $\frac{d^2y}{dx^2} = e^{-x}$

d $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}}$

Solution

a Let $p = \frac{dy}{dx}$.

Then $\frac{dp}{dx} = 10x^3 - 3x + 4$

$$\therefore \quad p = \frac{5x^4}{2} - \frac{3x^2}{2} + 4x + c$$

$$\therefore \quad \frac{dy}{dx} = \frac{5x^4}{2} - \frac{3x^2}{2} + 4x + c$$

$$\therefore \quad y = \frac{x^5}{2} - \frac{x^3}{2} + 2x^2 + cx + d, \quad \text{where } c, d \in \mathbb{R}$$

b $\dfrac{d^2y}{dx^2} = \cos(3x)$

Let $p = \dfrac{dy}{dx}$. Then $\dfrac{dp}{dx} = \cos(3x)$.

Thus
$$\begin{aligned} p &= \int \cos(3x)\, dx \\ &= \frac{1}{3}\sin(3x) + c \end{aligned}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3}\sin(3x) + c$$

$$\begin{aligned} \therefore \quad y &= \int \frac{1}{3}\sin(3x) + c\, dx \\ &= -\frac{1}{9}\cos(3x) + cx + d, \quad \text{where } c, d \in \mathbb{R} \end{aligned}$$

The p substitution can be omitted:

c
$$\frac{d^2y}{dx^2} = e^{-x}$$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \int e^{-x}\, dx \\ &= -e^{-x} + c \end{aligned}$$

$$\begin{aligned} \therefore \quad y &= \int -e^{-x} + c\, dx \\ &= e^{-x} + cx + d \qquad (c, d \in \mathbb{R}) \end{aligned}$$

d
$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{x+1}}$$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \int (x+1)^{-\frac{1}{2}}\, dx \\ &= 2(x+1)^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \therefore \quad y &= \int 2(x+1)^{\frac{1}{2}} + c\, dx \\ &= \frac{4}{3}(x+1)^{\frac{3}{2}} + cx + d \qquad (c, d \in \mathbb{R}) \end{aligned}$$

Example 12

Consider the differential equation $\dfrac{d^2y}{dx^2} = \cos^2 x$.

a Determine the general solution.

b Determine the solution given that $\dfrac{dy}{dx} = 0$ when $x = 0$ and that $y(0) = -\dfrac{1}{8}$.

Solution

a Now $\dfrac{d^2y}{dx^2} = \cos^2 x$

$$\therefore \quad \frac{dy}{dx} = \int \cos^2 x\, dx$$

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Use the trigonometric identity $\cos(2x) = 2\cos^2 x - 1$:

$$\frac{dy}{dx} = \int \cos^2 x \, dx$$

$$= \int \frac{1}{2}(\cos(2x) + 1)\, dx$$

$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$$

$$\therefore \quad y = \int \frac{1}{4}\sin(2x) + \frac{1}{2}x + c \, dx$$

Hence $y = -\frac{1}{8}\cos(2x) + \frac{1}{4}x^2 + cx + d$ is the general solution.

b First use $\frac{dy}{dx} = 0$ when $x = 0$. We have

$$\frac{dy}{dx} = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c \qquad \text{(from } \mathbf{a}\text{)}$$

$$0 = \frac{1}{4}\sin 0 + 0 + c \qquad \text{(substituting given condition)}$$

$$\therefore \quad c = 0$$

$$\therefore \quad y = -\frac{1}{8}\cos(2x) + \frac{1}{4}x^2 + d$$

Now using $y(0) = -\frac{1}{8}$, substitute and determine:

$$-\frac{1}{8} = -\frac{1}{8}\cos 0 + 0 + d$$

$$\therefore \quad d = 0$$

Hence $y = -\frac{1}{8}\cos(2x) + \frac{1}{4}x^2$ is the solution.

Exercise 13C

Example 9

SF

1 Determine the general solution of each of the following differential equations:

a $\frac{dy}{dx} = x^2 - 3x + 2$

b $\frac{dy}{dx} = \frac{x^2 + 3x - 1}{x}$

c $\frac{dy}{dx} = (2x + 1)^3$

d $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

e $\frac{dy}{dt} = \frac{1}{2t - 1}$

f $\frac{dy}{dt} = \sin(3t - 2)$

g $\frac{dy}{dt} = \tan(2t)$

h $\frac{dx}{dy} = e^{-3y}$

i $\frac{dx}{dy} = \frac{1}{\sqrt{4 - y^2}}$

j $\frac{dx}{dy} = -\frac{1}{(1 - y)^2}$

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Example 11

2 Determine the general solution of each of the following differential equations:

a $\dfrac{d^2y}{dx^2} = 5x^3$ **b** $\dfrac{d^2y}{dx^2} = \sqrt{1-x}$ **c** $\dfrac{d^2y}{dx^2} = \sin\left(2x + \dfrac{\pi}{4}\right)$

d $\dfrac{d^2y}{dx^2} = e^{\frac{x}{2}}$ **e** $\dfrac{d^2y}{dx^2} = \dfrac{1}{\cos^2 x}$ **f** $\dfrac{d^2y}{dx^2} = \dfrac{1}{(x+1)^2}$

Example 10

3 Determine the solution for each of the following differential equations:

a $\dfrac{dy}{dx} = \dfrac{1}{x^2}$, given that $y = \dfrac{3}{4}$ when $x = 4$

b $\dfrac{dy}{dx} = e^{-x}$, given that $y(0) = 0$

c $\dfrac{dy}{dx} = \dfrac{x^2-4}{x}$, given that $y = \dfrac{3}{2}$ when $x = 1$

d $\dfrac{dy}{dx} = \dfrac{x}{x^2-4}$, given that $y(2\sqrt{2}) = \ln 2$

e $\dfrac{dy}{dx} = x\sqrt{x^2-4}$, given that $y = \dfrac{1}{4\sqrt{3}}$ when $x = 4$

f $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{4-x^2}}$, given that $y(1) = \dfrac{\pi}{3}$

g $\dfrac{dy}{dx} = \dfrac{1}{4-x^2}$, given that $y = 2$ when $x = 0$

h $\dfrac{dy}{dx} = \dfrac{1}{4+x^2}$, given that $y(2) = \dfrac{3\pi}{8}$

i $\dfrac{dy}{dx} = x\sqrt{4-x}$, given that $y = -\dfrac{8}{15}$ when $x = 0$

j $\dfrac{dy}{dx} = \dfrac{e^x}{e^x+1}$, given that $y(0) = 0$

SF

Example 12

4 Determine the solution for each of the following differential equations:

a $\dfrac{d^2y}{dx^2} = e^{-x} - e^x$, given that $y(0) = 0$ and that $\dfrac{dy}{dx} = 0$ when $x = 0$

b $\dfrac{d^2y}{dx^2} = 2 - 12x$, given that when $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 0$

c $\dfrac{d^2y}{dx^2} = 2 - \sin(2x)$, given that when $x = 0$, $y = -1$ and $\dfrac{dy}{dx} = \dfrac{1}{2}$

d $\dfrac{d^2y}{dx^2} = 1 - \dfrac{1}{x^2}$, given that $y(1) = \dfrac{3}{2}$ and that $\dfrac{dy}{dx} = 0$ when $x = 1$

e $\dfrac{d^2y}{dx^2} = \dfrac{2x}{(1+x^2)^2}$, given that when $x = 0$, $\dfrac{dy}{dx} = 0$ and that when $x = 1$, $y = 1$

f $\dfrac{d^2y}{dx^2} = 24(2x+1)$, given that $y(-1) = -2$ and that $\dfrac{dy}{dx} = 6$ when $x = -1$

g $\dfrac{d^2y}{dx^2} = \dfrac{x}{(4-x^2)^{\frac{3}{2}}}$, given that when $x = 0$, $\dfrac{dy}{dx} = \dfrac{1}{2}$ and when $x = -2$, $y = -\dfrac{\pi}{2}$

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5 Determine the family of curves defined by each of the following differential equations:

a $\dfrac{dy}{dx} = 3x + 4$ **b** $\dfrac{d^2y}{dx^2} = -2x$ **c** $\dfrac{dy}{dx} = \dfrac{1}{x-3}$

6 Determine the equation of the curve defined by each of the following:

a $\dfrac{dy}{dx} = 2 - e^{-x}, \quad y(0) = 1$ **b** $\dfrac{dy}{dx} = x + \sin(2x), \quad y(0) = 4$

c $\dfrac{dy}{dx} = \dfrac{1}{2-x}, \quad y(3) = 2$

13D Separation of variables

Learning intentions

- To solve first-order differential equations using separation of variables.

Separation of variables is a method that allows us to treat the derivative $\dfrac{dy}{dx}$ as though it were a fraction that can be separated. For example, if we are to solve $\dfrac{dy}{dx} = \dfrac{x^2}{2y}$, then the first step is to separate the fraction

$$\frac{dy}{dx} = \frac{x^2}{2y}$$

$$\therefore 2y\,dy = x^2\,dx.$$

This is formally meaningless, but we integrate both sides of this to obtain our solution

$$\int 2y\,dy = \int x^2\,dx$$

$$\therefore y^2 = \frac{x^3}{3} + c.$$

It is worth emphasising that the initial separation of the variables, as shown in the example, is not mathematically rigorous, but it serves as a convenient way to visualise the separation process.

More generally, a first-order differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx} = f(x)\,g(y)$$

To see why this method works, if $g(y) \neq 0$ we can divide both sides by $g(y)$ to give

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrating both sides with respect to x:

$$\int f(x)\,dx = \int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int \frac{1}{g(y)}\,dy$$

If $\dfrac{dy}{dx} = f(x)\,g(y)$, then $\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$.

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Example 13

Solve the differential equation $\dfrac{dy}{dx} = e^{2x}(1 + y^2)$.

Solution

We first separate the variables and then integrate

$$\frac{dy}{dx} = e^{2x}(1 + y^2)$$

$$\frac{1}{y^2 + 1}\,dy = e^{2x}\,dx$$

$$\int \frac{1}{y^2 + 1}\,dy = \int e^{2x}\,dx$$

$$\tan^{-1} y = \frac{1}{2}e^{2x} + c$$

$$\Rightarrow \qquad y = \tan\left(\frac{1}{2}e^{2x} + c\right)$$

Example 14

Given that $y(0) = 1$, determine the solution of the differential equation

$$\frac{dy}{dx} = \frac{\sin^2 x}{y^2}$$

Solution

We again separate the variables and then integrate, giving

$$\frac{dy}{dx} = \frac{\sin^2 x}{y^2}$$

$$\int y^2\,dy = \int \sin^2 x\,dx$$

$$\frac{y^3}{3} = \int \sin^2 x\,dx$$

$$= \int \frac{1}{2}(1 - \cos(2x))\,dx$$

$$= \frac{1}{2}\int (1 - \cos(2x))\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + c$$

This gives the general solution. To determine the particular solution, we note that $y(0) = 1$ from which we determine that $c = \dfrac{1}{3}$. Therefore

$$\frac{y^3}{3} = \frac{x}{2} - \frac{1}{4}\sin(2x) + \frac{1}{3}$$

Making y the subject gives

$$y = \sqrt[3]{\frac{3x}{2} - \frac{3}{4}\sin(2x) + 1}$$

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Example 15

A tank contains 30 litres of a solution of a chemical in water. The concentration of the chemical is reduced by running pure water into the tank at a rate of 1 litre per minute and allowing the solution to run out of the tank at a rate of 2 litres per minute. The tank contains x litres of the chemical at time t minutes after the dilution starts.

a Show that $\dfrac{dx}{dt} = \dfrac{-2x}{30 - t}$.

b Determine the general solution of this differential equation.

c Determine the fraction of the original chemical still in the tank after 20 minutes.

Solution

a The solution is flowing out at 2 litres per minute and water is flowing in at 1 litre per minute. So every minute the volume decreases by 1 litre. Therefore at time t minutes, the volume of solution in the tank is $30 - t$ litres.

At time t minutes, the fraction of the solution which is the chemical is $\dfrac{x}{30 - t}$.

Hence the rate of flow of the chemical out of the tank is $2 \cdot \dfrac{x}{30 - t}$.

Therefore $\dfrac{dx}{dt} = \dfrac{-2x}{30 - t}$.

b Using separation of variables, we have

$$\frac{dx}{dt} = \frac{-2x}{30 - t}$$

$$\frac{1}{x}\,dx = \frac{-2}{30 - t}\,dt$$

$$\int \frac{1}{x}\,dx = \int \frac{-2}{30 - t}\,dt$$

$$\Rightarrow \qquad \ln x = 2\ln(30 - t) + c$$

Let A_0 be the initial amount of chemical in the solution. So $x = A_0$ when $t = 0$, and therefore

$$c = \ln(A_0) - 2\ln(30) = \ln\left(\frac{A_0}{900}\right)$$

Hence

$$\ln x = 2\ln(30 - t) + \ln\left(\frac{A_0}{900}\right)$$

$$\ln x = \ln\left(\frac{A_0}{900}(30 - t)^2\right)$$

$$\Rightarrow \qquad x = \frac{A_0}{900}(30 - t)^2$$

c When $t = 20$, $x = \dfrac{1}{9}A_0$.

The amount of chemical is one-ninth of the original amount.

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Constant solutions

When undertaking separation of variables, be careful that you do not lose solutions when dividing. For example, the differential equation $\frac{dy}{dx} = y - 2$ has a constant solution $y = 2$. To determine all solutions we follow this strategy:

1 **Determine constant solutions** Check for any values of y that make $g(y) = 0$. These correspond to constant solutions.

2 **Separate variables** Rewrite the differential equation in the form $\frac{1}{g(y)}\,dy = f(x)\,dx$.

3 **Integrate** Integrate both sides of the equation to give $\int \frac{1}{g(y)}\,dy = f(x)\,dx$

4 **Solve for y** Solve the resulting equation for y if possible.

5 **Use intitial conditions** If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

We will follow this five step method in the example below.

Example 16

Determine the general solution of $\frac{dy}{dx} = -\frac{y^2}{\sqrt{1-x^2}}$.

Solution

1 If $y = 0$, then we obtain a constant solution of this differential equation.

2 Now suppose $y \neq 0$. Then we separate the variables giving

$$\frac{dy}{dx} = -\frac{y^2}{\sqrt{1-x^2}}$$

$$-\frac{1}{y^2}\,dy = \frac{1}{\sqrt{1-x^2}}\,dx$$

3 Now integrate both sides to give

$$\int -y^{-2}\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$y^{-1} = \sin^{-1} x + c$$

4 Solving for y gives

$$y = \frac{1}{\sin^{-1} x + c}$$

5 There are no initial conditions.

Differential equations involving a function of the dependent variable

Many important differentiable equations have the form

$$\frac{dy}{dx} = g(y).$$

These can also be solved using the separation of variables as shown in the next example.

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Example 17

Determine the general solution of each of the following differential equations:

a $\dfrac{dy}{dx} = 2y + 1$, for $y > -\frac{1}{2}$

b $\dfrac{dy}{dx} = 1 - y^2$, for $-1 < y < 1$

Solution

a We separate variables and then integrate:

$$\begin{aligned}
\frac{dy}{dx} &= 2y + 1 \\
\frac{1}{2y+1}\,dy &= dx \\
\int \frac{1}{2y+1}\,dy &= \int 1\,dx \\
\tfrac{1}{2}\ln(2y+1) &= x + c && \text{as } y > -\tfrac{1}{2} \\
\ln(2y+1) &= 2x + k && \text{where } k = 2c \\
2y + 1 &= e^{2x+k} \\
y &= \tfrac{1}{2}e^{2x+k} - \tfrac{1}{2} \\
\Rightarrow \qquad y &= Ae^{2x} - \tfrac{1}{2} && \text{where } A = \tfrac{1}{2}e^{k}
\end{aligned}$$

b We separate variables and then integrate using partial fractions:

$$\begin{aligned}
\frac{dy}{dx} &= 1 - y^2 \\
dx &= \frac{1}{1-y^2}\,dy \\
\int 1\,dx &= \int \frac{1}{1-y^2}\,dy \\
x &= \int \frac{1}{2(1-y)} + \frac{1}{2(1+y)}\,dy \\
&= -\tfrac{1}{2}\ln(1-y) + \tfrac{1}{2}\ln(1+y) + c \qquad (\text{since } -1 < y < 1).
\end{aligned}$$

Therefore

$$\begin{aligned}
x - c &= \tfrac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \\
e^{2(x-c)} &= \frac{1+y}{1-y}
\end{aligned}$$

Let $A = e^{-2c}$. Then

$$\begin{aligned}
Ae^{2x} &= \frac{1+y}{1-y} \\
Ae^{2x}(1-y) &= 1 + y \\
Ae^{2x} - 1 &= y(1 + Ae^{2x}) \\
\therefore \qquad y &= \frac{Ae^{2x} - 1}{Ae^{2x} + 1}
\end{aligned}$$

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Skill-sheet

Exercise 13D

Example 13

1 Using separation of variables, determine the general solution of each of the following. Remember to determine constant solutions.

a $\frac{dy}{dx} = yx$ **b** $\frac{dy}{dx} = \frac{x}{y}$ **c** $\frac{4}{x^2}\frac{dy}{dx} = y$ **d** $\frac{dy}{dx} = \frac{1}{xy}$

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2 Using separation of variables, determine the general solution of each of the following differential equations:

a $\frac{dy}{dx} = 3y - 5, \quad y > \frac{5}{3}$ **b** $\frac{dy}{dx} = 1 - 2y, \quad y > \frac{1}{2}$ **c** $\frac{dy}{dx} = e^{2y-1}$

d $\frac{dy}{dx} = \cos^2 y, \quad |y| < \frac{\pi}{2}$ **e** $\frac{dy}{dx} = \cot y, \quad y \in \left(0, \frac{\pi}{2}\right)$ **f** $\frac{dy}{dx} = y^2 - 1, \quad |y| < 1$

g $\frac{dy}{dx} = 1 + y^2$ **h** $\frac{dy}{dx} = \frac{1}{5y^2 + 2y}$ **i** $\frac{dy}{dx} = \sqrt{y}, \quad y > 0$

3 Determine the solution for each of the following differential equations:

a $\frac{dy}{dx} = y$, given that $y = e$ when $x = 0$ **b** $\frac{dy}{dx} = y + 1$, given that $y(4) = 0$

c $\frac{dy}{dx} = 2y$, given that $y = 1$ when $x = 1$ **d** $\frac{dy}{dx} = 2y + 1$, given that $y(0) = -1$

e $\frac{dy}{dx} = \frac{e^y}{e^y + 1}$, if $y = 0$ when $x = 0$ **f** $\frac{dy}{dx} = \sqrt{9 - y^2}$, given that $y(0) = 3$

g $\frac{dy}{dx} = 9 - y^2$, if $y = 0$ when $x = \frac{7}{6}$ **h** $\frac{dy}{dx} = 1 + 9y^2$, given that $y\left(\frac{-\pi}{12}\right) = -\frac{1}{3}$

i $\frac{dy}{dx} = \frac{y^2 + 2y}{2}$, given that $y = -4$ when $x = 0$

4 For each of the following, determine the equation for the family of curves:

a $\frac{dy}{dx} = \frac{1}{y^2}$ **b** $\frac{dy}{dx} = 2y - 1, \quad y > \frac{1}{2}$

Example 14

5 **a** Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, given that $y(1) = 1$.

b Solve the differential equation $\frac{dy}{dx} = \frac{y}{x}$, given that $y(1) = 1$.

c Sketch the graphs of both solutions on the one set of axes.

6 Solve $(1 + x^2)\frac{dy}{dx} = 4xy$ if $y = 2$ when $x = 1$.

7 Determine the equation of the curve which satisfies the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and passes through the point $(2, 3)$.

8 Solve the differential equation $\frac{dy}{dx} = \frac{x + 1}{3 - y}$ and describe the solution curves.

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9 Determine the general solution of the differential equation $y^2 \dfrac{dy}{dx} = \dfrac{1}{x^3}$.

10 Determine the general solution of the differential equation $x^3 \dfrac{dy}{dx} = y^2(x-3),\ y \neq 0$.

11 Determine the general solution of each of the following:

a $\dfrac{dy}{dx} = y(1+e^x)$ **b** $\dfrac{dy}{dx} = 9x^2y$ **c** $\dfrac{4}{y^3}\dfrac{dy}{dx} = \dfrac{1}{x}$

d $\dfrac{dy}{dx} = \dfrac{\log_e x}{yx}$ **e** $\dfrac{dy}{dx} = yxe^{x^2}$ **f** $\dfrac{dy}{dx} = 2y^2x\sqrt{1-x^2}$

12 Solve each of the following differential equations:

a $y\dfrac{dy}{dx} = 1 + x^2, \quad y(0) = 1$ **b** $x^2\dfrac{dy}{dx} = \cos^2 y, \quad y(1) = \dfrac{\pi}{4}$

13 Determine the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2 - x}{y^2 - y}$.

CF

Example 15 **14** A tank contains 50 litres of a solution of a chemical in water. The concentration of the chemical is reduced by running pure water into the tank at a rate of 2 litres per minute and allowing the solution to run out of the tank at a rate of 4 litres per minute. The tank contains x litres of the chemical at time t minutes after the dilution starts.

a Show that $\dfrac{dx}{dt} = \dfrac{-4x}{50-2t}$.

b Determine the general solution of this differential equation.

c Determine the fraction of the original chemical still in the tank after 10 minutes.

15 Bacteria in a tank of water increase at a rate proportional to the number present. Water is drained out of the tank, initially containing 100 litres, at a steady rate of 2 litres per hour. Let N be the number of bacteria present at time t hours after the draining starts.

a Show that $\dfrac{dN}{dt} = kN - \dfrac{2N}{100-2t}$.

b If $k = 0.6$ and at $t = 0$, $N = N_0$, determine in terms of N_0 the number of bacteria after 24 hours.

16 Determine the general solution of these differential equations. Remember to determine constant solutions.

a $\dfrac{dy}{dx} = xy^2$ **b** $\dfrac{dy}{dx} = y\sin x - \sin x$ **c** $\dfrac{dy}{dx} = 2x(1-y)^2$

17 Solve the differential equation $x\dfrac{dy}{dx} = y + x^2y$, given that $y = 2\sqrt{e}$ when $x = 1$.

18 Determine y in terms of x if $\dfrac{dy}{dx} = (1+y)^2\sin^2(x)\cos(x)$ given that $y = 2$ when $x = 0$.

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13E Applications of differential equations

Learning intentions

- To model and solve differential equations that describe real-world phenomena.

Many differential equations arise from scientific or economic situations and are constructed from observations and data obtained from experiment. For example, the following two results from science are described by differential equations:

- **Newton's law of cooling** The rate at which a body cools is proportional to the difference between its temperature and the temperature of its immediate surroundings.
- **Radioactive decay** The rate at which a radioactive substance decays is proportional to the mass of the substance remaining.

These two results will be investigated further in worked examples in this section.

Example 18

The table gives the observed rate of change of a variable x with respect to time t.

t	0	1	2	3	4
$\dfrac{dx}{dt}$	0	2	8	18	32

a Construct the differential equation which applies to this situation.

b Solve the differential equation to determine x in terms of t, given that $x = 2$ when $t = 0$.

Solution

a From the table, it can be established that $\dfrac{dx}{dt} = 2t^2$.

b Therefore $x = \int 2t^2 \, dt = \dfrac{2t^3}{3} + c$.

When $t = 0$, $x = 2$. This gives $2 = 0 + c$ and so $c = 2$. Hence $x = \dfrac{2t^3}{3} + 2$.

Differential equations can also be constructed from statements, as shown in the following.

Example 19

The population of a city is P at time t years from a certain date. The population increases at a rate that is proportional to the square root of the population at that time.

a Construct the differential equation that applies to this situation.

b Solve the appropriate differential equation and sketch the population-time graph.

Solution

a As $\dfrac{dP}{dt} \propto \sqrt{P}$, the differential equation is $\dfrac{dP}{dt} = k\sqrt{P}$. Since the population is increasing, we have $k > 0$.

b No initial conditions are given so we determine a general solution by separating

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variables:

$$\frac{dP}{dt} = k\sqrt{P}$$

$$dt = \frac{1}{k}P^{-\frac{1}{2}}\,dP$$

$$\int 1\,dt = \int \frac{1}{k}P^{-\frac{1}{2}}\,dP$$

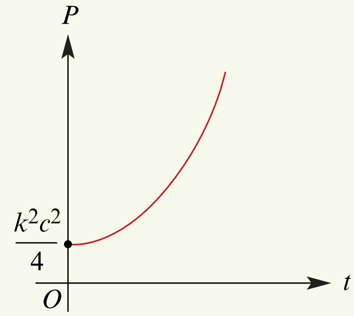
$$t = \frac{2}{k}P^{\frac{1}{2}} + c$$

Rearranging to make P the subject:

$$t = \frac{2}{k}\sqrt{P} + c$$

$$\therefore \quad P = \frac{k^2}{4}(t-c)^2$$

The graph is a section of a parabola with vertex at $(c, 0)$.



Example 20

In another city, with population P at time t years after a certain date, the population increases at a rate proportional to the population at that time.

a Construct the differential equation that applies to this situation.

b Solve the appropriate differential equation and sketch the population-time graph.

Solution

a As $\frac{dP}{dt} \propto P$ the differential equation is $\frac{dP}{dt} = kP$ where $k > 0$.

b We solve by separating variables:

$$\frac{dP}{dt} = kP$$

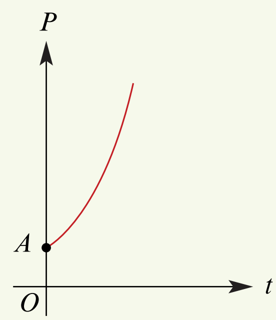
$$\frac{1}{P}\,dP = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln P = kt + c$$

$$P = e^{kt+c}$$

$$= Ae^{kt} \qquad (\text{where } A = e^c)$$



This is the general solution. The graph is a section of the exponential curve $P = Ae^{kt}$.

The following example uses Newton's law of cooling.

Example 21

An iron bar is placed in a room which has a temperature of 20°C. The iron bar initially has a temperature of 80°C. It cools to 70°C in 5 minutes. Let T be the temperature of the bar at time t minutes.

a Construct a differential equation.

b Solve this differential equation.

c Sketch the graph of T against t.

d How long does it take the bar to cool to 40°C?

Solution

a Newton's law of cooling yields

$$\frac{dT}{dt} = -k(T - 20) \quad \text{where } k \in \mathbb{R}^+$$

Note the use of the negative sign as the temperature is decreasing.

b We solve by separating variables:

$$\begin{aligned} \frac{dT}{dt} &= -k(T - 20) \\ \frac{1}{T - 20}\,dT &= -k\,dt \\ \int \frac{1}{T - 20}\,dT &= -\int k\,dt \\ \ln(T - 20) &= -kt + c && (\text{as } T > 20) \\ T - 20 &= e^{-kt+c} \\ T &= 20 + Ae^{-kt} && (\text{where } A = e^c) \end{aligned}$$

When $t = 0, T = 80$. This gives $A = 60$. Therefore $T = 20 + 60e^{-kt}$. Finally, when $t = 5, T = 70$. Therefore

$$\begin{aligned} 70 &= 20 + 60e^{-5k} \\ e^{-5k} &= \frac{5}{6} \\ k &= -\frac{1}{5}\ln\left(\frac{5}{6}\right) \end{aligned}$$

Substituting this value of k into the equation gives

$$\begin{aligned} T &= 20 + 60e^{\frac{1}{5}\ln\left(\frac{5}{6}\right)t} \\ &= 20 + 60\left(e^{\ln\left(\frac{5}{6}\right)^{\frac{1}{5}}}\right)^t \\ &= 20 + 60\left(\tfrac{5}{6}\right)^{\frac{t}{5}} \end{aligned}$$

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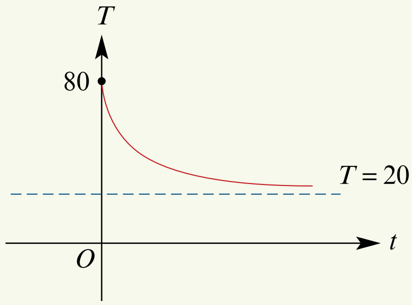
c



d When $T = 40$, we have

$$20 + 60\left(\tfrac{5}{6}\right)^{\frac{t}{5}} = 40$$
$$\left(\tfrac{5}{6}\right)^{\frac{t}{5}} = \tfrac{1}{3}$$
$$\tfrac{t}{5}\ln\left(\tfrac{5}{6}\right) = \ln\left(\tfrac{1}{3}\right)$$
$$t = \frac{5\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{5}{6}\right)} \approx 30.1$$

The bar reaches a temperature of 40°C after 30.1 minutes.

Example 22

Suppose that a tank containing liquid has a vent at the top and an outlet at the bottom through which the liquid drains.

Torricelli's law states that if, at time t seconds after opening the outlet, the depth of the liquid is h m and the surface area of the liquid is A m^2, then

$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{A} \quad \text{where } k > 0$$

(The constant k depends on factors such as the viscosity of the liquid and the cross-sectional area of the outlet.)

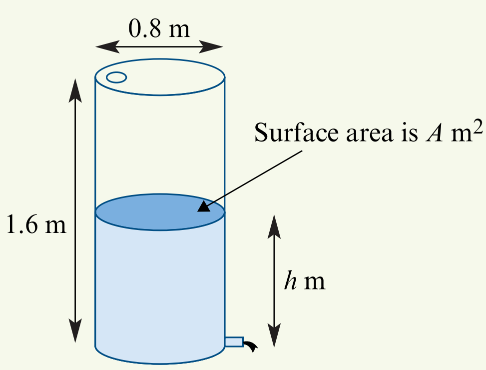
a A cylindrical tank is initially full, with a height of 1.6 m and a radius length of 0.4 m. Apply Torricelli's law to construct the appropriate differential equation. Use $k = 0.025$.

b Solve this differential equation to determine how many seconds it will take for the tank to empty to the nearest second.

Solution

a We start by drawing a diagram. Since the surface area is a circle with constant area $A = \pi \times 0.4^2$, we have

$$\frac{dh}{dt} = \frac{-0.025\sqrt{h}}{\pi \times 0.4^2}$$
$$= \frac{-0.025\sqrt{h}}{0.16\pi}$$
$$= \frac{-5\sqrt{h}}{32\pi}$$



b We solving the differential equation by separating variables:

$$\begin{aligned}
\frac{dh}{dt} &= \frac{-5\sqrt{h}}{32\pi} \\
\frac{1}{\sqrt{h}}\,dh &= \frac{-5}{32\pi}\,dt \\
\int h^{-\frac{1}{2}}\,dh &= \int \frac{-5}{32\pi}\,dt \\
2h^{\frac{1}{2}} &= \frac{-5}{32\pi}t + c \\
2\sqrt{h} &= \frac{-5}{32\pi}t + c
\end{aligned}$$

The tank is initially full, so when $t = 0$, $h = 1.6$. Therefore

$$c = 2\sqrt{1.6}$$

So the particular solution for this differential equation can be found by solving for h. This gives

$$\begin{aligned}
2\sqrt{h} &= -\frac{5}{32\pi}t + 2\sqrt{1.6} \\
h &= \left(-\frac{5}{64\pi}t + \sqrt{1.6}\right)^2
\end{aligned}$$

Now we determine the time when the tank is empty. That is, we determine t when $h = 0$. By substitution:

$$\begin{aligned}
\left(-\frac{5}{64\pi}t + \sqrt{1.6}\right)^2 &= 0 \\
\frac{5}{64\pi}t &= \sqrt{1.6} \\
t &= \frac{64\pi}{5}\sqrt{1.6} \\
&\approx 50.9.
\end{aligned}$$

It will take approximately 51 seconds to empty this tank.

Difference of rates

Consider the following situations:

- An object is being heated, but at the same time is subject to cooling.
- A population is increasing due to births, but at the same time is diminishing due to deaths.
- A liquid is being poured into a container, while at the same time the liquid is flowing out.

In each of these situations:

rate of change = (rate of increase) − (rate of decrease)

For example, if water is flowing into a container at 8 litres per minute and at the same time water is flowing out of the container at 6 litres per minute, then the overall rate of change is $\frac{dV}{dt} = 8 - 6 = 2$, where the volume of water in the container is V litres at time t minutes.

Example 23

A certain radioactive isotope decays at a rate that is proportional to the mass, m kg, present at any time t years. The rate of decay is $2m$ kg per year. The isotope is formed as a byproduct from a nuclear reactor at a constant rate of 0.5 kg per year. None of the isotope was present initially.

a Construct a differential equation.
b Solve the differential equation.
c Sketch the graph of m against t.
d How much isotope is there after two years?

Solution

a The differential equation is

$$\begin{aligned}\frac{dm}{dt} &= \text{(rate of increase)} - \text{(rate of decrease)}\\ &= 0.5 - 2m\\ &= \frac{1-4m}{2}\end{aligned}$$

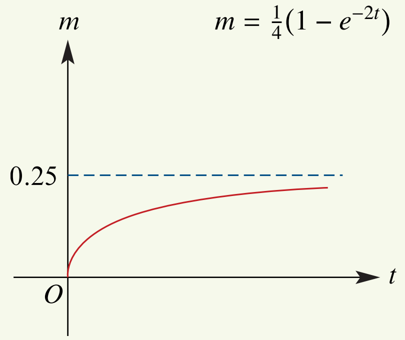
b By separating variables, we determine that

$$\begin{aligned}\frac{dm}{dt} &= \frac{1-4m}{2}\\ dt &= \frac{2}{1-4m}\,dm\\ \int 1\,dt &= \int \frac{2}{1-4m}\,dm\\ t &= -\frac{1}{2}\ln|1-4m| + c\\ -2t-2c &= \ln|1-4m|\\ 1-4m &= Ae^{-2t} \quad (\text{where } A = \pm e^{-2c})\end{aligned}$$

When $t = 0$, $m = 0$ and therefore $A = 1$. So

$$\begin{aligned}1-4m &= e^{-2t}\\ e^{-2t} &= 1-4m\\ \therefore \quad m &= \tfrac{1}{4}(1-e^{-2t})\end{aligned}$$

c



d When $t = 2$,

$$\begin{aligned}m &= \tfrac{1}{4}(1-e^{-4})\\ &= 0.245\ldots\end{aligned}$$

After two years, the mass of the isotope is 0.245 kg.

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Example 24

Pure oxygen is pumped into a 50-litre tank of air at 5 litres per minute. The oxygen is well mixed with the air in the tank. The mixture is removed at the same rate.

a Construct a differential equation, given that plain air contains 23% oxygen.

b To the closest second, how long is required for the mixture to contain 50% oxygen?

Solution

a Let Q litres be the volume of oxygen in the tank at time t minutes.
When $t = 0$, $Q = 50 \times 0.23 = \frac{23}{2}$.

$$\begin{aligned}\frac{dQ}{dt} &= \text{(rate of inflow)} - \text{(rate of outflow)}\\ &= 5 - \left(\frac{Q}{50} \times 5\right).\end{aligned}$$

By determineing a common denominator, we conclude that

$$\frac{dQ}{dt} = \frac{50 - Q}{10}$$

b Separating variables, we determine that

$$\begin{aligned}dt &= \frac{10}{50 - Q}\,dQ\\ \int 1\,dt &= \int \frac{10}{50 - Q}\,dQ\\ \therefore \quad t &= -10\ln|50 - Q| + c\\ &= -10\ln(50 - Q) + c \qquad (\text{as } Q < 50)\end{aligned}$$

When $t = 0$, $Q = \frac{23}{2}$. Therefore

$$c = 10\ln\left(\frac{77}{2}\right)$$

$$\therefore \quad t = 10\ln\left(\frac{77}{2(50 - Q)}\right)$$

When the mixture is 50% oxygen, we have $Q = 25$ and so

$$\begin{aligned}t &= 10\ln\left(\frac{77}{2 \cdot 25}\right)\\ &= 10\ln\left(\frac{77}{50}\right)\\ &= 4.317\ldots\end{aligned}$$

The tank contains 50% oxygen after 4 minutes and 19 seconds.

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Exercise 13E

Example 18

SF

1 Each of the following tables gives the results of an experiment where a rate of change was found to be a linear function of time, i.e. $\frac{dx}{dt} = at + b$. For each table, set up a differential equation and solve it using the additional information.

a

t	0	1	2	3
$\frac{dx}{dt}$	1	3	5	7

and $x(0) = 3$

b

t	0	1	2	3
$\frac{dx}{dt}$	-1	2	5	8

and $x(1) = 1$

c

t	0	1	2	3
$\frac{dx}{dt}$	8	6	4	2

and $x(2) = -3$

CF

2 For each of the following, construct (but do not attempt to solve) a differential equation:

a A family of curves is such that the gradient at any point (x, y) is the reciprocal of the y-coordinate (for $y \neq 0$).

b A family of curves is such that the gradient at any point (x, y) is the square of the reciprocal of the y-coordinate (for $y \neq 0$).

c The rate of increase of a population of size N at time t years is inversely proportional to the square of the population.

d A particle moving in a straight line is x m from a fixed point O after t seconds. The rate at which the particle is moving is inversely proportional to the distance from O.

e The rate of decay of a radioactive substance is proportional to the mass of substance remaining. Let m kg be the mass of the substance at time t minutes.

f The gradient of the normal to a curve at any point (x, y) is three times the gradient of the line joining the same point to the origin.

Example 19, 20

3 A city, with population P at time t years after a certain date, has a population that increases at a rate proportional to the population at that time.

a **i** Set up a differential equation to describe this situation.

ii Solve to obtain a general solution.

b If the initial population was 1000 and after two years the population had risen to 1100:

i determine the population after five years

ii sketch a graph of P against t.

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CF

4 An island has a population of rabbits of size P at time t years after 1 January 2010. Due to a virus, the population is decreasing at a rate proportional to the square root of the population at that time.

a **i** Set up a differential equation to describe this situation.

ii Solve to obtain a general solution.

b If the population was initially 15 000 and decreased to 13 500 after five years:

i determine the population after 10 years

ii sketch a graph of P against t.

5 A city has population P at time t years from a certain date. The population increases at a rate inversely proportional to the population at that time.

a **i** Set up a differential equation to describe this situation.

ii Solve to obtain a general solution.

b Initially the population was 1 000 000, but after four years it had risen to 1 100 000.

i Determine an expression for the population in terms of t.

ii Sketch the graph of P against t.

6 A curve has the property that its gradient at any point is one-tenth of the y-coordinate at that point. It passes through the point $(0, 10)$. Determine the equation of the curve.

Example 21

7 A body at a temperature of 80°C is placed in a room which is kept at a constant temperature of 20°C. After 20 minutes, the temperature of the body is 60°C. Assuming Newton's law of cooling, determine the temperature after a further 20 minutes.

8 If the thermostat in an electric heater fails, the rate of increase in its temperature, $\frac{d\theta}{dt}$, is 0.01θ K per minute, where the temperature θ is measured in kelvins (K) and the time t in minutes. If the heater is switched on at a room temperature of 300 K and the thermostat does not function, what is the temperature of the heater after 10 minutes?

9 The rate of decay of a radioactive substance is proportional to the amount Q of matter present at any time t. The differential equation for this situation is $\frac{dQ}{dt} = -kQ$, where k is a constant. Given that $Q = 50$ when $t = 0$ and that $Q = 25$ when $t = 10$, determine the time t at which $Q = 10$.

10 The rate of decay of a substance is km, where k is a positive constant and m is the mass of the substance remaining. Show that the half-life (i.e. the time in which the amount of the original substance remaining is halved) is given by $\frac{1}{k}\ln 2$.

11 The concentration, x grams per litre, of salt in a solution at time t minutes is given by $\frac{dx}{dt} = \frac{20 - 3x}{30}$.

a If the initial concentration was 2 grams per litre, solve the differential equation, giving x in terms of t.

b Determine the time taken, to the nearest minute, for the salt concentration to rise to 6 grams per litre.

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SF

12 If $\frac{dy}{dx} = 10 - \frac{y}{10}$ and $y = 10$ when $x = 0$, determine y in terms of x. Sketch the graph of the equation for $x \geq 0$.

CF

13 The number n of bacteria in a colony grows according to the law $\frac{dn}{dt} = kn$, where k is a positive constant. If the number increases from 4000 to 8000 in four days, determine, to the nearest hundred, the number of bacteria after three days more.

14 A town had a population of 10 000 in 2000 and 12 000 in 2010. If the population is N at a time t years after 2000, determine the predicted population in the year 2020 assuming:

a $\frac{dN}{dt} \propto N$ **b** $\frac{dN}{dt} \propto \frac{1}{N}$ **c** $\frac{dN}{dt} \propto \sqrt{N}$

15 For each of the following, construct a differential equation, but do not solve it:

a Water is flowing into a tank at a rate of 0.3 m^3 per hour. At the same time, water is flowing out through a hole in the bottom of the tank at a rate of $0.2\sqrt{V}$ m^3 per hour, where V m^3 is the volume of the water in the tank at time t hours. (Determine an expression for $\frac{dV}{dt}$.)

b A tank initially contains 200 litres of pure water. A salt solution containing 5 kg of salt per litre is added at the rate of 10 litres per minute, and the mixed solution is drained simultaneously at the rate of 12 litres per minute. There is m kg of salt in the tank after t minutes. (Determine an expression for $\frac{dm}{dt}$.)

c A partly filled tank contains 200 litres of water in which 1500 grams of salt have been dissolved. Water is poured into the tank at a rate of 6 L/min. The mixture, which is kept uniform by stirring, leaves the tank through a hole at a rate of 5 L/min. There is x grams of salt in the tank after t minutes. (Determine an expression for $\frac{dx}{dt}$.)

Example 23

16 A certain radioactive isotope decays at a rate that is proportional to the mass, m kg, present at any time t years. The rate of decay is m kg per year. The isotope is formed as a byproduct from a nuclear reactor at a constant rate of 0.25 kg per year. None of the isotope was present initially.

a Construct a differential equation.

b Solve the differential equation.

c Sketch the graph of m against t.

d How much isotope is there after two years?

Example 24

17 A tank holds 100 litres of water in which 20 kg of sugar was dissolved. Water runs into the tank at the rate of 1 litre per minute. The solution is continually stirred and, at the same time, the solution is being pumped out at 1 litre per minute. At time t minutes, there is m kg of sugar in the solution.

a At what rate is the sugar being removed at time t minutes?

b Set up a differential equation to represent this situation.

c Solve the differential equation.

d Sketch the graph of m against t.

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CF

18 A tank holds 100 litres of pure water. A sugar solution containing 0.25 kg per litre is being run into the tank at the rate of 1 litre per minute. The liquid in the tank is continuously stirred and, at the same time, liquid from the tank is being pumped out at the rate of 1 litre per minute. After t minutes, there is m kg of sugar dissolved in the solution.

- **a** At what rate is the sugar being added to the solution at time t?
- **b** At what rate is the sugar being removed from the tank at time t?
- **c** Construct a differential equation to represent this situation.
- **d** Solve this differential equation.
- **e** Determine the time taken for the concentration in the tank to reach 0.1 kg per litre.
- **f** Sketch the graph of m against t.

19 A laboratory tank contains 100 litres of a 20% serum solution (i.e. 20% of the contents is pure serum and 80% is distilled water). A 10% serum solution is then pumped in at the rate of 2 litres per minute, and an amount of the solution currently in the tank is drawn off at the same rate.

- **a** Set up a differential equation to show the relation between x and t, where x litres is the amount of pure serum in the tank at time t minutes.(Assume that at all times the contents of the tank form a uniform solution.)
- **b** How long will it take for there to be an 18% solution in the tank?

20 A tank initially contains 400 litres of water in which is dissolved 10 kg of salt. A salt solution of concentration 0.2 kg/L is poured into the tank at the rate of 2 L/min. The mixture, which is kept uniform by stirring, flows out at the rate of 2 L/min.

- **a** If the mass of salt in the tank is x kg after t minutes, set up and solve the differential equation for x in terms of t.
- **b** If instead the mixture flows out at 1 L/min, set up (but do not solve) the differential equation for the mass of salt in the tank.

21 A tank contains 20 litres of water with 10 kg of dissolved salt. Pure water is poured in and released at 2 litres per minute, with uniform mixing. Let x kg be the mass of salt in the tank at time t minutes.

- **a** Construct a differential equation representing this information, expressing $\dfrac{dx}{dt}$ as a function of x.
- **b** Solve the differential equation.
- **c** Sketch the mass–time graph.
- **d** How long will it take the original mass of salt to be halved?

22 A country's population N at time t years after 1 January 2010 changes according to the differential equation $\dfrac{dN}{dt} = 0.1N - 5000$. (There is a 10% growth rate and 5000 people leave the country every year.)

- **a** Given that the population was 5 000 000 at the start of 2010, determine N in terms of t.
- **b** In which year will the country have a population of 10 million?

CF

23 An initial amount of \$10 000 is invested at an interest rate of 3% p.a.

a Determine the value of the investment after two years (to the nearest cent) if the interest is compounded:

i yearly **ii** half yearly **iii** quarterly **iv** monthly **v** daily.

In financial mathematics, we often assume that interest is compounded continuously. Then the interest rate becomes a rate of growth, and the investment is modelled by the differential equation

$$\frac{dA}{dt} = \frac{rA}{100}, \quad t > 0$$

where \$$A$ is the value of the investment after t years and r% p.a. is the rate of growth.

b Suppose that \$10 000 is invested at a rate of growth of 3% p.a. Solve the differential equation to determine the value of the investment after two years.

c Comment on your answers to parts **a** and **b**.

24 Treasury bonds provide a way for the government to borrow money. The government sells bonds to investors, who are promised a fixed amount of money after a fixed period of time (e.g. \$100 000 after 10 years). The bonds are sold by tender or on the financial market, and so the initial value of the bonds is determined by investors.

a State and solve a differential equation that models this situation, where \$$A_0$ is the initial value of the bond, \$$A$ is the value of the bond after t years and r% p.a. is the rate of growth in the value of the bond.

b An investor pays \$80 000 for a bond of \$100 000 after 10 years. Determine the rate of growth for this investment.

c Another investor wants to achieve a rate of growth of 3.5% p.a. on her investment. How much is she willing to pay for this bond?

25 An investor pays \$85 000 for a treasury bond of \$100 000 after five years.

a Determine the rate of growth expected on this investment.

b Determine the value of the investment after two years according to this investor's model.

c After two years, the investor sells the bond for \$91 000. Determine the actual rate of growth achieved on the investment.

26 An initial deposit of \$10 000 is made into an account that earns interest of 8% p.a., compounded continuously. Money is withdrawn continuously from the account at a rate of \$1000 per year, until there is no money left in the account. Let \$$A$ be the amount of money in the account after t years.

a Explain why $\dfrac{dA}{dt} = 0.08A - 1000$.

b Determine an expression for A in terms of t.

c When will there be no money left in the account?

d What is the total amount of money withdrawn from the account?

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13F The logistic differential equation

Learning intentions

- To apply and solve the logistic differential equation to model population growth.

In the previous section, we modelled the growth of a population, P, over time, t, using a differential equation of the form

$$\frac{dP}{dt} = kP$$

The solution is $P = P_0 e^{kt}$, where P_0 is the initial population.

This exponential growth model can be appropriate for a short time. However, it is not realistic over a long period of time since this model implies that the population will grow without limit. But a population will usually be limited by the available resources, such as food and space.

We need a model which acknowledges that there is an upper limit to growth.

Example 25

A population grows according to the differential equation

$$\frac{dP}{dt} = 0.025P\left(1 - \frac{P}{1000}\right), \quad 0 < P < 1000$$

where P is the population at time t. When $t = 0$, $P = 20$.

a Determine the population P at time t.

b Sketch the graph of P against t.

c Determine the population P when the rate of growth is at a maximum.

Solution

a We first write this differential equation as

$$\frac{dP}{dt} = \frac{P(1000 - P)}{40\,000}.$$

We solve by separating variables and then using partial fractions:

$$dt = \frac{40\,000}{P(1000 - P)}\,dP$$

$$\int 1\,dt = \int \frac{40\,000}{P(1000 - P)}\,dP$$

$$t = 40\int \frac{1}{P} + \frac{1}{1000 - P}\,dP$$

$$= 40\ln|P| - \ln|1000 - P| + c$$

$$= 40\ln\left(\frac{P}{1000 - P}\right) + c \qquad (\text{as } 0 < P < 1000)$$

$$\Rightarrow e^{\frac{t-c}{40}} = \frac{P}{1000 - P}$$

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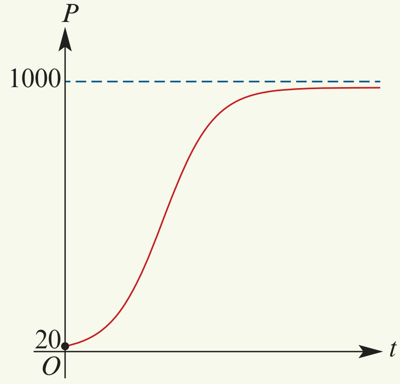
By letting $A = e^{\frac{-c}{40}}$ we determine that

$$Ae^{\frac{t}{40}} = \frac{P}{1000 - P}$$

When $t = 0$, $P = 20$. This implies that $A = \frac{1}{49}$, and so

$$\begin{aligned}(1000 - P)e^{\frac{t}{40}} &= 49P \\ 1000e^{\frac{t}{40}} &= 49P + Pe^{\frac{t}{40}} \\ 1000e^{\frac{t}{40}} &= P\left(49 + e^{\frac{t}{40}}\right) \\ \implies P &= \frac{1000e^{\frac{t}{40}}}{49 + e^{\frac{t}{40}}}\end{aligned}$$

b



c The maximum rate of increase occurs at the point of inflection on the graph. We have

$$\frac{dP}{dt} = \frac{1000P - P^2}{40\,000}$$

The chain rule gives

$$\frac{d^2P}{dt^2} = \frac{1000 - 2P}{40\,000} \cdot \frac{dP}{dt}$$

Since $0 < P < 1000$, we have $\frac{dP}{dt} \neq 0$.

Therefore $\frac{d^2P}{dt^2} = 0$ implies $P = 500$.

Note: Since $\frac{dP}{dt}$ is a quadratic in P, the maximum rate of increase occurs at the vertex of the parabola, which is midway between its intercepts at $P = 0$ and $P = 1000$.

Logistic differential equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad 0 < P < K$$

This differential equation can be used to model a population P at time t, where:

- the constant r is called the **growth parameter**
- the constant K is called the **carrying capacity**.

Notes:

- As in the example, we can show that the solution of this differential equation is

$$P(t) = \frac{P_0K}{P_0 + (K - P_0)e^{-rt}} = \frac{P_0Ke^{rt}}{P_0e^{rt} + (K - P_0)} \quad \text{where } P_0 = P(0)$$

- The carrying capacity K is the upper limit on the population: the rate of increase approaches 0 as P approaches K; the population P approaches K as $t \to \infty$.
- The maximum rate of increase occurs when $P = \frac{K}{2}$.

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Exercise 13F

SF CF

1 Solve the differential equation $\frac{dP}{dt} = P(1 - P)$, where $P(0) = 2$.

Example 25

2 A population grows according to the differential equation

$$\frac{dP}{dt} = 0.02P\left(1 - \frac{P}{500}\right), \quad 0 < P < 500$$

where P is the population at time t. When $t = 0$, $P = 100$.

a Determine the population P at time t.
b Sketch the graph of P against t.
c Determine the population P when the rate of growth is at a maximum.

3 Let $P(t)$ be the population of a species of fish in a lake after t years. Suppose that $P(t)$ is modelled by a logistic differential equation with a growth parameter of $r = 0.3$ and a carrying capacity of $K = 10\,000$.

a Write down the logistic differential equation for this situation.
b If $P(0) = 2500$, solve the differential equation for $P(t)$.
c Sketch the graph of $P(t)$ against t.
d Determine the number of fish in the lake after 5 years.
e Determine the time that it will take for there to be 5000 fish in the lake.

4 A population of wasps is growing according to the logistic differential equation, where P is the number of wasps after t months. If the carrying capacity is 500 and the growth parameter is 0.1, what is the maximum possible growth rate for the population?

5 A population of bacteria grows according to the differential equation

$$\frac{dP}{dt} = 0.05P(1 - 0.001P), \quad P_0 = 300, \quad 0 < P < 1000$$

Determine the population P at time t.

6 Suppose that t weeks after the start of an epidemic in a certain community, the number of people who have caught the disease, $P(t)$, is given by the logistic function

$$P(t) = \frac{2000}{5 + 395e^{-\frac{4t}{5}}}$$

a How many people had the disease when the epidemic began?
b Approximately how many people in total will get the disease?
c When was the disease spreading most rapidly?
d How fast was the disease spreading at the peak of the epidemic?
e At what rate was the disease spreading when 300 people had caught the disease?

7 Consider the differential equation $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{1000}\right)$. For each of the following cases, solve the differential equation and sketch the graph of P against t:

a $P_0 = 1500$ and $P > 1000$
b $P_0 = 200$ and $0 < P < 1000$
c $P_0 = 1000$

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CF

8 A population of rabbits grows in a way described by the logistic differential equation

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{25\,000}\right)$$

where P is the number of rabbits after t months, and the initial population is $P_0 = 2000$.

a Solve the differential equation for P.

b How many rabbits are there after:

i 6 months **ii** 5 years?

c After how many months is the population increasing most rapidly?

d How long does it take for the population to reach 20 000?

e Sketch the graph of P against t.

CU

9 Consider the differential equation

$$\frac{dy}{dx} = -\left(1 - \frac{y}{K_1}\right)\left(1 - \frac{y}{K_2}\right)$$

where K_1 and K_2 are positive constants. Taking $K_1 = 5$ and $K_2 = 10$, solve the differential equation for each of the following cases:

a $y(0) = 20,\ \ y > 10$

b $y(0) = 8,\ \ 5 < y < 10$

c $y(0) = 3,\ \ 0 < y < 5$

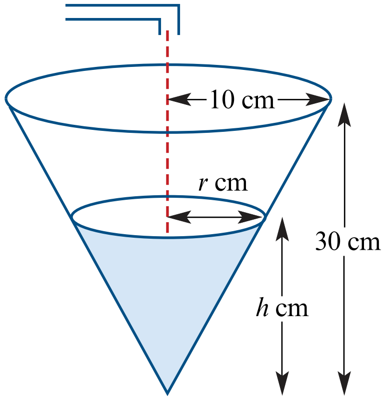
13G Related rates

Learning intentions

- To use the chain rule to solve real-world problems involving related rates.
- To determine the gradient at a point on a parametric curve.

Consider the situation of a right circular cone being filled from a tap. At time t seconds:

- the volume of water in the cone is V cm^3
- the height of the water in the cone is h cm
- the radius of the circular water surface is r cm.

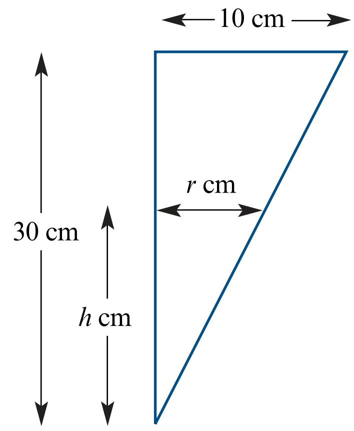


As the water flows in, the values of V, h and r change:

- $\dfrac{dV}{dt}$ is the rate of change of volume with respect to time
- $\dfrac{dh}{dt}$ is the rate of change of height with respect to time
- $\dfrac{dr}{dt}$ is the rate of change of radius with respect to time.

It is clear that these rates are related to each other. The chain rule is used to establish these relationships.

For example, if the height of the cone is 30 cm and the radius of the cone is 10 cm, then similar triangles yield



$$\frac{r}{h} = \frac{10}{30}$$

$$\therefore \quad h = 3r$$

Then the chain rule is used:

$$\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$$

$$= 3 \cdot \frac{dr}{dt}$$

The volume of a cone is given in general by $V = \frac{1}{3}\pi r^2 h$. Since $h = 3r$, we have

$$V = \pi r^3$$

Therefore by using the chain rule again:

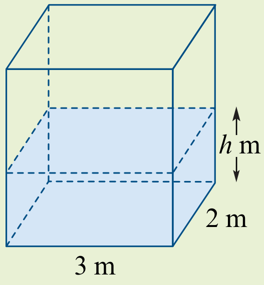
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= 3\pi r^2 \cdot \frac{dr}{dt}$$

The relationships between the rates have been established.

Example 26

A rectangular prism is being filled with water at a rate of 0.00042 m^3/s. Determine the rate at which the height of the water is increasing.



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Solution

Let t be the time in seconds after the prism begins to fill. Let V m^3 be the volume of water at time t, and let h m be the height of the water at time t.

We are given that $\dfrac{dV}{dt} = 0.00042$ and $V = 6h$.

Using the chain rule, the rate at which the height is increasing is

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

Since $V = 6h$, we have $\dfrac{dV}{dh} = 6$ and so $\dfrac{dh}{dV} = \dfrac{1}{6}$.

$$\begin{aligned}\text{Thus } \frac{dh}{dt} &= \frac{dh}{dV}\frac{dV}{dt}\\ &= \frac{1}{6} \times 0.00042\\ &= 0.00007 \text{ m/s}\end{aligned}$$

i.e. the height is increasing at a rate of 0.00007 m/s.

Example 27

As Steven's ice block melts, it forms a circular puddle on the floor. The radius of the puddle increases at a rate of 3 cm/min. When its radius is 2 cm, determine the rate at which the area of the puddle is increasing.

Solution

The area, A, of a circle is given by $A = \pi r^2$, where r is the radius of the circle.

The rate of increase of the radius is $\dfrac{dr}{dt} = 3$ cm/min.

Using the chain rule, the rate of increase of the area is

$$\begin{aligned}\frac{dA}{dt} &= \frac{dA}{dr}\frac{dr}{dt}\\ &= 2\pi r \times 3\\ &= 6\pi r\end{aligned}$$

When $r = 2$, $\dfrac{dA}{dt} = 12\pi$.

Hence the area of the puddle is increasing at 12π cm^2/min.

Example 28

A metal cube is being heated so that the side length is increasing at the rate of 0.02 cm per hour. Calculate the rate at which the volume is increasing when the side length is 5 cm.

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Solution

Let x be the length of a side of the cube. Then the volume is $V = x^3$.

We are given that $\dfrac{dx}{dt} = 0.02$ cm/h.

The rate of increase of volume is found using the chain rule:

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dx}\frac{dx}{dt}\\ &= 3x^2 \times 0.02\\ &= 0.06x^2\end{aligned}$$

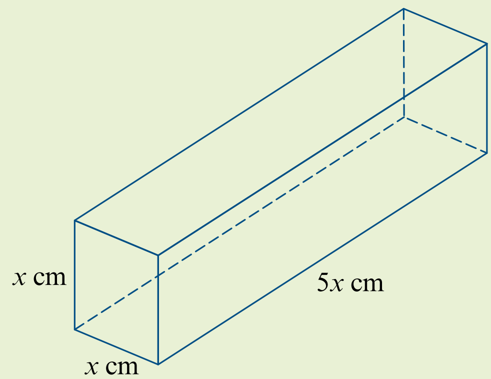
When $x = 5$, the volume of the cube is increasing at a rate of 1.5 cm^3/h.

Example 29

The diagram shows a rectangular block of ice that is x cm by x cm by $5x$ cm.

a Express the total surface area, A cm^2, in terms of x and then determine $\dfrac{dA}{dx}$.

b If the ice is melting such that the total surface area is decreasing at a constant rate of 4 cm^2/s, calculate the rate of decrease of x when $x = 2$.



Solution

a
$$\begin{aligned}A &= 4 \times 5x^2 + 2 \times x^2\\ &= 22x^2\end{aligned}$$

$$\frac{dA}{dx} = 44x$$

b The surface area is decreasing, so $\dfrac{dA}{dt} = -4$.

By the chain rule:

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dA}\frac{dA}{dt}\\ &= \frac{1}{44x} \times (-4)\\ &= -\frac{1}{11x}\end{aligned}$$

When $x = 2$, $\dfrac{dx}{dt} = -\dfrac{1}{22}$ cm/s.

Note: The rates of change of the lengths of the edges are $-\frac{1}{22}$ cm/s, $-\frac{1}{22}$ cm/s and $-\frac{5}{22}$ cm/s. The negative signs indicate that the lengths are decreasing.

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Parametric equations

Parametric equations were introduced in Chapter 2. For example:

- The unit circle can be described by the parametric equations $x = \cos t$ and $y = \sin t$.
- The parabola $y^2 = 4ax$ can be described by the parametric equations $x = at^2$ and $y = 2at$.

In general, a parametric curve is specified by a pair of equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

For a point $(f(t), g(t))$ on the curve, we can consider the gradient of the tangent to the curve at this point. By the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

This gives the following result.

Gradient at a point on a parametric curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{provided } \frac{dx}{dt} \neq 0$$

Note: A curve defined by parametric equations is not necessarily the graph of a function. However, each value of t determines a point on the curve, and we can use this technique to determine the gradient of the curve at this point (given the tangent exists).

Example 30

A curve has parametric equations

$$x = 2t - \ln(2t) \quad \text{and} \quad y = t^2 - \ln(t^2)$$

Determine:

a $\dfrac{dx}{dt}$ **b** $\dfrac{dy}{dt}$ **c** $\dfrac{dy}{dx}$

Solution

a $x = 2t - \ln(2t)$

$$\frac{dx}{dt} = 2 - \frac{1}{t} = \frac{2t-1}{t}$$

b $y = t^2 - \ln(t^2)$

$$\frac{dy}{dt} = 2t - \frac{2}{t} = \frac{2t^2-2}{t}$$

c
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t^2-2}{t} \times \frac{t}{2t-1} = \frac{2t^2-2}{2t-1}$$

Example 31

For the curve defined by the given parametric equations, determine the gradient of the tangent at a point $P(x, y)$ on the curve, in terms of the parameter t:

a $x = 16t^2$ and $y = 32t$ **b** $x = 2\sin(3t)$ and $y = -2\cos(3t)$

Solution

a $x = 16t^2$ and so $\dfrac{dx}{dt} = 32t$

$y = 32t$ and so $\dfrac{dy}{dt} = 32$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{32}{32t} = \frac{1}{t}$$

The gradient of the tangent at the point $P(16t^2, 32t)$ is $\dfrac{1}{t}$, for $t \neq 0$.

b $x = 2\sin(3t)$ and so $\dfrac{dx}{dt} = 6\cos(3t)$

$y = -2\cos(3t)$ and so $\dfrac{dy}{dt} = 6\sin(3t)$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6\sin(3t)}{6\cos(3t)} = \tan(3t)$$

The gradient of the tangent at the point $P(2\sin(3t), -2\cos(3t))$ is $\tan(3t)$.

Exercise 13G

Example 26, 27

1 The radius of a spherical balloon is 2.5 m and its volume is increasing at a rate of 0.1 m^3/min.

- **a** At what rate is the radius increasing?
- **b** At what rate is the surface area increasing?

CF

Example 28

2 When a wine glass is filled to a depth of x cm, it contains V cm^3 of wine, where $V = 4x^{\frac{3}{2}}$. If the depth is 9 cm and wine is being poured into the glass at 10 cm^3/s, at what rate is the depth changing?

SF

3 Variables x and y are connected by the equation $y = 2x^2 + 5x + 2$. Given that x is increasing at the rate of 3 units per second, determine the rate of increase of y with respect to time when $x = 2$.

Example 29

CF

4 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume, V cm^3, of the water is given by

$$V = \frac{1}{3}\pi x^2(18 - x)$$

Water is poured into the bowl at a rate of 3 cm^3/s. Determine the rate at which the water level is rising when the depth is 2 cm.

SF

5 Variables p and v are linked by the equation $pv = 1500$. Given that p is increasing at the rate of 2 units per minute, determine the rate of decrease of v at the instant when $p = 60$.

6 A circular metal disc is being heated so that the radius is increasing at the rate of 0.01 cm per hour. Determine the rate at which the area is increasing when the radius is 4 cm.

7 The area of a circle is increasing at the rate of 4 cm^2 per second. At what rate is the circumference increasing at the instant when the radius is 8 cm?

Example 30

8 A curve has parametric equations $x = \frac{1}{1+t^2}$ and $y = \frac{t}{1+t^2}$.

a Determine $\frac{dy}{dt}$ and $\frac{dx}{dt}$.

b Determine $\frac{dy}{dx}$.

9 A curve has parametric equations $x = 2t + \sin(2t)$ and $y = \cos(2t)$. Determine $\frac{dy}{dx}$.

Example 31

10 A curve has parametric equations $x = t - \cos t$ and $y = \sin t$. Determine the equation of the tangent to the curve when $t = \frac{\pi}{6}$.

11 A point moves along the curve $y = x^2$ such that its velocity parallel to the x-axis is a constant 2 cm/s (i.e. $\frac{dx}{dt} = 2$). Determine its velocity parallel to the y-axis (i.e. $\frac{dy}{dt}$) when:

a $x = 3$

b $y = 16$

SF

12 Variables x and y are related by $y = \frac{2x-6}{x}$. They are given by $x = f(t)$ and $y = g(t)$, where f and g are functions of time. Determine $f'(t)$ when $y = 1$, given that $g'(t) = 0.4$.

13 A particle moves along the curve

$$y = 10\cos^{-1}\left(\frac{x-5}{5}\right)$$

in such a way that its velocity parallel to the x-axis is a constant 3 cm/s. Determine its velocity parallel to the y-axis when:

a $x = 6$

b $y = \frac{10\pi}{3}$

14 The radius, r cm, of a sphere is increasing at a constant rate of 2 cm/s. Determine, in terms of π, the rate at which the volume is increasing at the instant when the volume is 36π cm^3.

15 Liquid is poured into a container at a rate of 12 cm^3/s. The volume of liquid in the container is V cm^3, where $V = \frac{1}{2}(h^2 + 4h)$ and h is the height of the liquid in the container. Determine, when $V = 16$:

a the value of h

b the rate at which h is increasing

16 The area of an ink blot, which is always circular in shape, is increasing at a rate of 3.5 cm^2/s. Determine the rate of increase of the radius when the radius is 3 cm.

17 A tank in the shape of a prism has constant cross-sectional area A cm^2. The amount of water in the tank at time t seconds is V cm^3 and the height of the water is h cm. Determine the relationship between $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

CF

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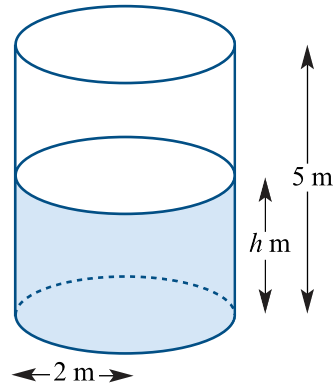
18 A cylindrical tank 5 m high with base radius 2 m is initially full of water. Water flows out through a hole at the bottom of the tank at the rate of $\sqrt{h}$ m^3/h, where h metres is the depth of the water remaining in the tank after t hours. Determine:



a $\dfrac{dh}{dt}$

b **i** $\dfrac{dV}{dt}$ when $V = 10\pi$ m^3

ii $\dfrac{dh}{dt}$ when $V = 10\pi$ m^3

CF

19 For the curve defined by the parametric equations $x = 2\cos t$ and $y = \sin t$, determine the equation of the tangent to the curve at the point:

a $\left(\sqrt{2}, \dfrac{\sqrt{2}}{2}\right)$

b $(2\cos t, \sin t)$, where t is any real number.

20 For the curve defined by the parametric equations $x = 2\sec\theta$ and $y = \tan\theta$, determine the equation of:

a the tangent at the point where $\theta = \dfrac{\pi}{4}$

b the normal at the point where $\theta = \dfrac{\pi}{4}$

c the tangent at the point $(2\sec\theta, \tan\theta)$.

SF

21 For the curve with parametric equations $x = 2\sec t - 3$ and $y = 4\tan t + 2$, determine:

a $\dfrac{dy}{dx}$

b the equation of the tangent to the curve when $t = \dfrac{\pi}{4}$.

CF

22 A curve is defined by the parametric equations $x = \sec t$ and $y = \tan t$.

a Determine the equation of the normal to the curve at the point $(\sec t, \tan t)$.

b Let A and B be the points of intersection of the normal to the curve with the x-axis and y-axis respectively, and let O be the origin. Determine the area of $\triangle OAB$.

c Determine the value of t for which the area of $\triangle OAB$ is $4\sqrt{3}$.

23 A curve is specified by the parametric equations $x = e^{2t} + 1$ and $y = 2e^t + 1$ for $t \in \mathbb{R}$.

a Determine the gradient of the curve at the point $(e^{2t} + 1, 2e^t + 1)$.

b State the domain of the relation.

c Sketch the graph of the relation.

d Determine the equation of the tangent at the point where $t = \ln\left(\dfrac{1}{2}\right)$.

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13H Differential equations with related rates

Learning intentions

- To construct and solve differential equations using related rates in various contexts.

We now use related rates for constructing and solving differential equations in a variety of different situations.

Example 32

For the variables x, y and t, it is known that $\frac{dx}{dt} = \tan t$ and $y = 3x$.

a Determine $\frac{dy}{dt}$ as a function of t.

b Determine the solution of the resulting differential equation.

Solution

a We are given that $y = 3x$ and $\frac{dx}{dt} = \tan t$.

Using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = 3\tan t$$

b $\frac{dy}{dt} = \frac{3\sin t}{\cos t}$

Let $u = \cos t$. Then $\frac{du}{dt} = -\sin t$.

$$\therefore \quad y = -3\int \frac{1}{u}\,du$$

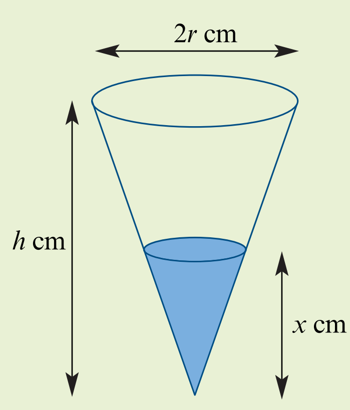
$$= -3\ln|u| + c$$

$$\therefore \quad y = -3\ln|\cos t| + c$$

Example 33

An inverted cone has height h cm and radius length r cm. It is being filled with water, which is flowing from a tap at k litres per minute. The depth of water in the cone is x cm at time t minutes.

Construct an appropriate differential equation for $\frac{dx}{dt}$ and solve it, given that initially the cone was empty.



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Solution

Let V cm^3 be the volume of water at time t minutes.

Since k litres is equal to $1000k$ cm^3, the given rate of change is $\frac{dV}{dt} = 1000k$, where $k > 0$.

To determine an expression for $\frac{dx}{dt}$, we can use the chain rule:

$$\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt} \qquad (1)$$

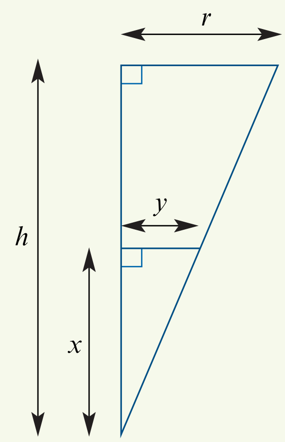
To determine $\frac{dx}{dV}$, we first need to establish the relationship between x and V.

The formula for the volume of a cone gives

$$V = \frac{1}{3}\pi y^2 x \qquad (2)$$

where y cm is the radius length of the surface when the depth is x cm.

By similar triangles:



$$\frac{y}{r} = \frac{x}{h}$$

$$\therefore \quad y = \frac{rx}{h}$$

So $\quad V = \frac{1}{3}\pi \cdot \frac{r^2x^2}{h^2} \cdot x \qquad$ (substitution into (2))

$$\therefore \quad V = \frac{\pi r^2}{3h^2} \cdot x^3$$

$\therefore \quad \frac{dV}{dx} = \frac{\pi r^2}{h^2} \cdot x^2 \qquad$ (by differentiation)

$$\therefore \quad \frac{dx}{dV} = \frac{h^2}{\pi r^2} \cdot \frac{1}{x^2}$$

So $\quad \frac{dx}{dt} = \frac{h^2}{\pi r^2} \cdot \frac{1}{x^2} \cdot 1000k \qquad$ (substitution into (1))

$\therefore \quad \frac{dx}{dt} = \frac{1000kh^2}{\pi r^2} \cdot \frac{1}{x^2} \qquad$ where $k > 0$

To solve this differential equation:

$$\frac{dt}{dx} = \frac{\pi r^2}{1000kh^2} \cdot x^2$$

$$\therefore \quad t = \frac{\pi r^2}{1000kh^2} \int x^2\, dx$$

$$= \frac{\pi r^2}{1000kh^2} \cdot \frac{x^3}{3} + c$$

$$\therefore \quad t = \frac{\pi r^2 x^3}{3000kh^2} + c$$

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The cone was initially empty, so $x = 0$ when $t = 0$, and therefore $c = 0$.

$$\therefore \quad t = \frac{\pi r^2 x^3}{3000kh^2}$$

$$\therefore \quad x^3 = \frac{3000kh^2 t}{\pi r^2}$$

Hence $x = \sqrt[3]{\dfrac{3000kh^2 t}{\pi r^2}}$ is the solution of the differential equation.

Skill-sheet

Exercise 13H

CF

1 Construct, but do not solve, a differential equation for each of the following:

a An inverted cone with depth 50 cm and radius 25 cm is initially full of water, which drains out at 0.5 litres per minute. The depth of water in the cone is h cm at time t minutes. (Determine an expression for $\dfrac{dh}{dt}$.)

b A tank with a flat bottom and vertical sides has a constant horizontal cross-section of A m^2. The tank has a tap in the bottom through which water is leaving at a rate of $c\sqrt{h}$ m^3 per minute, where h m is the height of the water in the tank and c is a constant. Water is being poured into the tank at a rate of Q m^3 per minute. (Determine an expression for $\dfrac{dh}{dt}$.)

c Water is flowing at a constant rate of 0.3 m^3 per hour into a tank. At the same time, water is flowing out through a hole in the bottom of the tank at the rate of $0.2\sqrt{V}$ m^3 per hour, where V m^3 is the volume of the water in the tank at time t hours. It is known that $V = 6\pi h$, where h m is the height of the water at time t. (Determine an expression for $\dfrac{dh}{dt}$.)

d A cylindrical tank 4 m high with base radius 1.5 m is initially full of water. The water starts flowing out through a hole at the bottom of the tank at the rate of $\sqrt{h}$ m^3 per hour, where h m is the depth of water remaining in the tank after t hours. (Determine an expression for $\dfrac{dh}{dt}$.)

Example 32

SF

2 For the variables x, y and t, it is known that $\dfrac{dx}{dt} = \sin t$ and $y = 5x$.

a Determine $\dfrac{dy}{dt}$ as a function of t.

b Determine the solution of the resulting differential equation.

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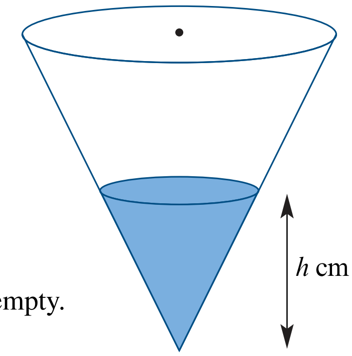
Example 33

3 A conical tank has a radius length at the top equal to its height. Water, initially with a depth of 25 cm, leaks out through a hole in the bottom of the tank at the rate of $5\sqrt{h}$ cm^3 per minute, where the depth is h cm at time t minutes.

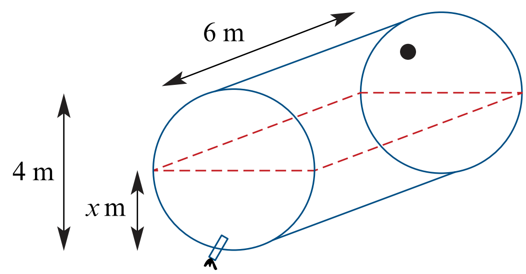


a Construct a differential equation expressing $\frac{dh}{dt}$ as a function of h, and solve it.

b Hence determine how long it will take for the tank to empty.

4 A cylindrical tank is lying on its side. The tank has a hole in the top, and another in the bottom so that the water in the tank leaks out. The depth of water is x m at time t minutes and

$$\frac{dx}{dt} = \frac{-0.025\sqrt{x}}{A}$$



where A m^2 is the surface area of the water at time t minutes.

a Construct the differential equation expressing $\frac{dx}{dt}$ as a function of x only.

b Solve the differential equation given that initially the tank was full.

c Determine how long it will take to empty the tank.

5 A spherical drop of water evaporates so that the volume remaining is V mm^3 and the surface area is A mm^2 when the radius is r mm at time t seconds.
Given that $\frac{dV}{dt} = -2A^2$:

a Construct the differential equation expressing $\frac{dr}{dt}$ as a function of r.

b Solve the differential equation given that the initial radius was 2 mm.

c Sketch the graph of A against t and the graph of r against t.

6 A water tank of uniform cross-sectional area A cm^2 is being filled by a pipe which supplies Q litres of water every minute. The tank has a small hole in its base through which water leaks at a rate of kh litres every minute, where h cm is the depth of water in the tank at time t minutes. Initially the depth of the water is h_0 cm.

a Construct the differential equation expressing $\frac{dh}{dt}$ as a function of h.

b Solve the differential equation if $Q > kh_0$.

c Determine the time taken for the depth to reach $\frac{Q + kh_0}{2k}$.

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13I Using a definite integral to solve a differential equation

Learning intentions

- To understand the use of definite integrals to determine solutions to differential equations.

There are many situations in which an exact solution to a differential equation $\frac{dy}{dx} = f(x)$ is not required. Indeed, in some cases it may not even be possible to obtain an exact solution. For such differential equations, an approximate solution can be found by numerically evaluating a definite integral.

To solve the differential equation $\frac{dy}{dx} = f(x)$, we start with the fundamental theorem of calculus, which we recast in an equivalent form:

$$y(x) - y(a) = \int_a^x \frac{dy}{dt}\, dt$$

Adding $y(a)$ to both sides gives the useful expression,

$$\begin{aligned} y(x) &= \int_a^x \frac{dy}{dt}\, dt + y(a) \\ &= \int_a^x f(t)\, dt + y(a). \end{aligned}$$

This final formula serves two purposes. First, it often allows us to solve a differential equation efficiently. Second, it allows us to solve a differential equation when we cannot determine a formula for the anti-derivative.

Example 34

Solve the differential equation $\frac{dy}{dx} = x^2 + 2$, given that $y = 7$ when $x = 1$.

Solution

Algebraic method

$$\frac{dy}{dx} = x^2 + 2$$

$$\therefore \quad y = \frac{x^3}{3} + 2x + c$$

Since $y = 7$ when $x = 1$, we have

$$7 = \frac{1}{3} + 2 + c$$

$$\therefore \quad c = \frac{14}{3}$$

$$\therefore \quad y = \frac{x^3}{3} + 2x + \frac{14}{3}$$

Using a definite integral

$$\begin{aligned} y &= \int_1^x (t^2 + 2)\, dt + y(1) \\ &= \left[\frac{t^3}{3} + 2t\right]_1^x + 7 \\ &= \left(\frac{x^3}{3} + 2x\right) - \left(\frac{1}{3} + 2\right) + 7 \\ &= \frac{x^3}{3} + 2x + \frac{14}{3} \end{aligned}$$

Example 35

Using a definite integral, solve the differential equation $\frac{dy}{dx} = \cos x$ at $x = \frac{\pi}{6}$, given that $y = \frac{1}{2}$ at $x = 0$.

Solution

$$\frac{dy}{dx} = \cos x$$

$$\begin{aligned}
\therefore \quad y &= \int_0^{\frac{\pi}{6}} \cos t \, dt + y(0) \\
&= [\sin t]_0^{\frac{\pi}{6}} + \frac{1}{2} \\
&= \sin\left(\frac{\pi}{6}\right) + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

Example 36

Solve the differential equation $f'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ at $x = 1$, given that $f(0) = 0.5$.

Give your answer correct to four decimal places.

Solution

Calculus methods are not available for this differential equation and, since an approximate answer is acceptable, the use of a graphics calculator is appropriate. We determine that

$$\begin{aligned}
f(1) &= \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt + f(0) \\
&= \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt + 0.5 \\
&\approx 0.8413,
\end{aligned}$$

which is correct to four decimal places.

Exercise 13I

Example 35, 36

SF

1 For each of the following, use a calculator to determine values correct to four decimal places:

a $\frac{dy}{dx} = \sqrt{\cos x}$ and $y = 1$ when $x = 0$. Determine y when $x = \frac{\pi}{4}$.

b $\frac{dy}{dx} = \frac{1}{\sqrt{\cos x}}$ and $y = 1$ when $x = 0$. Determine y when $x = \frac{\pi}{4}$.

c $\frac{dy}{dx} = \ln(x^2)$ and $y = 2$ when $x = 1$. Determine y when $x = e$.

d $\frac{dy}{dx} = \sqrt{\ln x}$ and $y = 2$ when $x = 1$. Determine y when $x = e$.

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13J Slope field for a differential equation

Learning intentions

- To be able to use slope fields to visualise differential equations.
- To be able to plot slope fields for differential equations using a graphing calculator.
- To use slope fields to visualise particular solution curves given initial conditions.

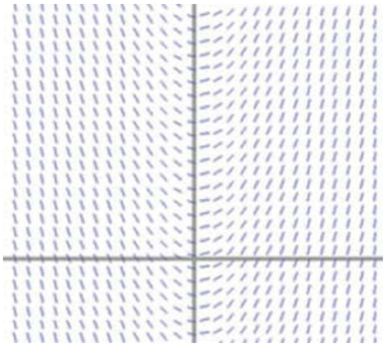
Consider a differential equation of the form $\frac{dy}{dx} = f(x)$.

The **slope field** of this differential equation assigns to each point $P(x, y)$ in the plane (for which x is in the domain of f) the number $f(x)$, which is the gradient of the solution curve through P.

For the differential equation $\frac{dy}{dx} = 2x$, a gradient value is assigned for each point $P(x, y)$.

- For $(1, 3)$ and $(1, 5)$, the gradient value is 2.
- For $(-2, 5)$ and $(-2, -2)$, the gradient value is -4.

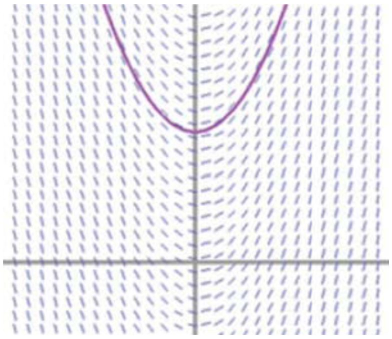
A slope field can, of course, be represented in a graph. The slope field for $\frac{dy}{dx} = 2x$ is shown opposite.



When initial conditions are given, a particular solution curve can be drawn.

Here the solution curve with $y = 2$ when $x = 0$ has been superimposed on the slope field for $\frac{dy}{dx} = 2x$.

Changing the initial conditions changes the particular solution.



A slope field is defined similarly for any differential equation of the form $\frac{dy}{dx} = f(x, y)$.

Example 37

a Use a calculator to plot the slope field for the differential equation $\frac{dy}{dt} = y$.

b On the plot of the slope field, plot the graphs of the particular solutions for:

i $y = 2$ when $t = 0$ **ii** $y = -3$ when $t = 1$.

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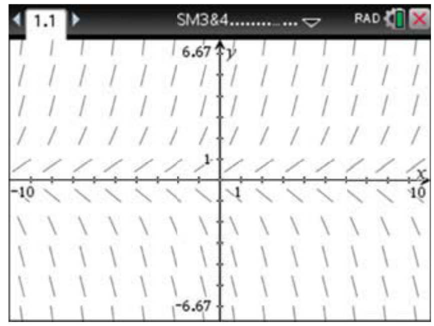
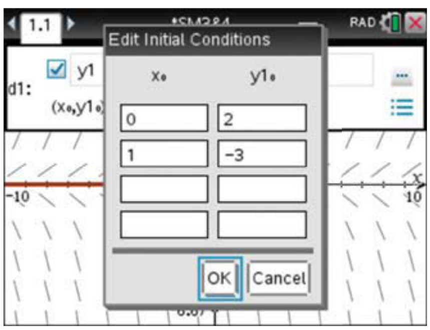
Using the TI-Nspire CX non-CAS

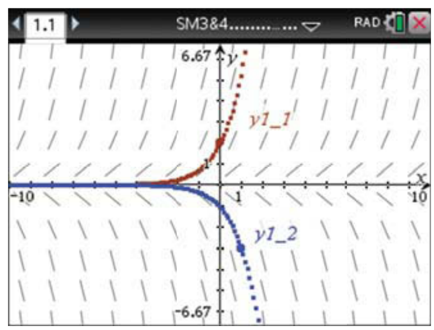
a
- In a **Graphs** application, select (menu) > **Graph Entry/Edit** > **Diff Eq**.
- Enter the differential equation as $y1' = y1$.
- Press (enter) to plot the slope field.

Note: The notation must match when entering the differential equation. (Here $y1$ is used for y.)



b In the graph entry line, you have the option of adding several initial conditions.
- To show the graph entry line, press (tab) or double click in an open area.
- Arrow up to $y1'$ and add the first set of initial conditions: $x = 0$ and $y1 = 2$.
- Click on the green 'plus' icon to add more initial conditions: $x = 1$ and $y1 = -3$.
- Select OK to plot the solution curves for the given initial conditions.





Note: You can grab the initial point and drag to show differing initial conditions.

The differential equation $\dfrac{dy}{dt} = y$ can be solved analytically in the usual manner.

Write $\dfrac{dt}{dy} = \dfrac{1}{y}$. Then $t = \ln|y| + c$, which implies $|y| = e^{t-c} = Ae^t$.

- If $y = 2$ when $t = 0$, then $A = 2$ and therefore $y = 2e^t$, as $y > 0$.
- If $y = -3$ when $t = 1$, then $A = 3e^{-1}$ and therefore $y = -3e^{t-1}$, as $y < 0$.

Example 38

Use a calculator to plot the slope field for the differential equation

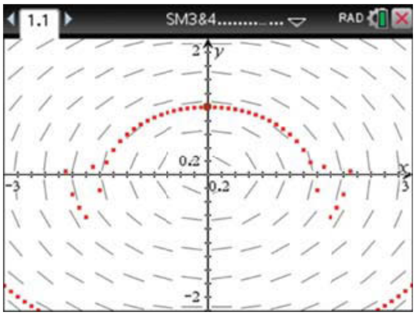
$$\frac{dy}{dx} = -\frac{x}{2y}$$

and show the solution for the initial condition $x = 0$, $y = 1$.

Using the TI-Nspire CX non-CAS

- In a **Graphs** application, select (menu) > **Graph Entry/Edit** > **Diff Eq**.
- Enter the differential equation as $y1' = -\dfrac{x}{2y1}$.
- Enter the initial conditions $x = 0$ and $y1 = 1$.
- Press (enter).

Note: Set the window to $-3 \le x \le 3$ and $-2 \le y \le 2$.



Exercise 13J

Example 37

1 For each of the following differential equations, sketch a slope field graph and the solution curve for the given initial conditions, using $-3 \le x \le 3$ and $-3 \le y \le 3$. Use calculus to solve the differential equation in each case.

a $\dfrac{dy}{dx} = 3x^2$, given $y = 0$ when $x = 1$

b $\dfrac{dy}{dx} = \sin x$, given $y = 0$ when $x = 0$ (use radian mode)

c $\dfrac{dy}{dx} = e^{-2x}$, given $y = 1$ when $x = 0$

d $\dfrac{dy}{dx} = y^2$, given $y = 1$ when $x = 1$

e $\dfrac{dy}{dx} = y^2$, given $y = -1$ when $x = 1$

f $\dfrac{dy}{dx} = y(y-1)$, given $y = -1$ when $x = 0$

g $\dfrac{dy}{dx} = y(y-1)$, given $y = 2$ when $x = 0$

h $\dfrac{dy}{dx} = \tan x$, given $y = 0$ when $x = 0$

Example 38

2 For each of the following differential equations, sketch a slope field graph and the solution curve for the given initial conditions, using $-3 \le x \le 3$ and $-3 \le y \le 3$. Do not attempt to solve the differential equations by calculus methods.

a $\dfrac{dy}{dx} = -\dfrac{x}{y}$, where at $x = 0$, $y = \pm 1$

b $\dfrac{dy}{dx} = -\dfrac{x}{y}$ where at $x = \dfrac{1}{2}$, $y = \dfrac{\sqrt{3}}{2}$

Review

Chapter summary

Differential equations

- A differential equation is an equation that contains at least one derivative.
- A solution of a differential equation is a function that satisfies the differential equation when it and its derivatives are substituted. The general solution is the family of functions that satisfy the differential equation.

Differential equation	Method of solution
$\frac{dy}{dx} = f(x)$	$\frac{dy}{dx} = f(x)$ $\therefore\ y = \int f(x)\,dx$ $= F(x) + c, \quad \text{where } F'(x) = f(x)$
$\frac{d^2y}{dx^2} = f(x)$	$\frac{d^2y}{dx^2} = f(x)$ $\frac{dy}{dx} = \int f(x)\,dx$ $= F(x) + c, \quad \text{where } F'(x) = f(x)$ $\therefore\ y = \int F(x) + c\,dx$ $= G(x) + cx + d, \quad \text{where } G'(x) = F(x)$
$\frac{dy}{dx} = f(x)\,g(y)$	$\frac{dy}{dx} = f(x)\,g(y)$ $\frac{1}{g(y)}\,dy = f(x)\,dx$ $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

- Slope field

  The slope field of a differential equation

  $$\frac{dy}{dx} = f(x, y)$$

  assigns to each point $P(x, y)$ in the plane (for which $f(x, y)$ is defined) the number $f(x, y)$, which is the gradient of the solution curve through P.

Implicit differentiation

- Many curves are not defined by a rule of the form $y = f(x)$ or $x = f(y)$; for example, the unit circle $x^2 + y^2 = 1$. Implicit differentiation is used to determine the gradient at a point on such a curve. To do this, we differentiate both sides of the equation with respect to x.

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- Using operator notation:

$$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\,\frac{dy}{dx} \qquad \text{(use of chain rule)}$$

$$\frac{d}{dx}(x^2 y) = x^2\,\frac{dy}{dx} + 2xy \qquad \text{(use of product rule)}$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

13A **1** I can use implicit differentiation to determine the derivatives of an implicit function. ☐

See Example 1, Example 2, Example 3, Question 1 and Question 2

13B **2** I can verify solutions of a differential equation. ☐

See Example 5, Example 6, Example 7, Question 1 and Question 2

13C **3** I can determine general solutions of simple differential equations. ☐

See Example 9, Question 1 and Question 2

13C **4** I can determine particular solutions of simple differential equations. ☐

See Example 10, Example 12, Question 3 and Question 4

13D **5** I can solve first-order differential equations using separation of variables. ☐

See Example 13, Example 14, Question 1 and Question 2

13E **6** I can model and solve differential equations relating to real-world phenomena. ☐

See Example 19, Example 20, Example 21, Question 3, Question 7 and Question 9

13F **7** I can solve the logistic differential equation to model population growth. ☐

See Example 25, Question 2 and Question 3

13G **8** I can use the chain rule to solve real-world problems involving related rates. ☐

See Example 26, Example 27, Example 28, Question 1 and Question 2

13H **9** I can construct and solve differential equations using related rates. ☐

See Example 32, Example 33, Question 1, Question 2 and Question 3

13I **10** I can use the definite integral to solve differential equations. ☐

See Example 34, Example 35, Example 36 and Question 1

13J **11** I can plot a slope field and draw particular solutions of a differential equation. ☐

See Example 37, Example 38, Questions 1 and Question 2

Review

Short-response questions

Technology-free short-response questions

SF

1 Determine the general solution of each of the following differential equations:

a $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2}, \quad x > 0$

b $\dfrac{1}{y} \cdot \dfrac{dy}{dx} = 10, \quad y > 0$

c $\dfrac{d^2y}{dt^2} = \dfrac{1}{2}(\sin(3t) + \cos(2t)), \quad t \ge 0$

d $\dfrac{d^2y}{dx^2} = \dfrac{e^{-x} + e^{x}}{e^{2x}}$

e $\dfrac{dy}{dx} = \dfrac{3-y}{2}, \quad y < 3$

f $\dfrac{dy}{dx} = \dfrac{3-x}{2}$

2 Determine the solution of the following differential equations under the stated conditions:

a $\dfrac{dy}{dx} = \pi\cos(2\pi x)$, if $y = -1$ when $x = \dfrac{5}{2}$

b $\dfrac{dy}{dx} = \cot(2x)$, if $y = 0$ when $x = \dfrac{\pi}{4}$

c $\dfrac{dy}{dx} = \dfrac{1+x^2}{x}$, if $y = 0$ when $x = 1$

d $\dfrac{dy}{dx} = \dfrac{x}{1+x^2}$, if $y(0) = 1$

e $6\dfrac{dy}{dx} = -3y$, if $y = e^{-1}$ when $x = 2$

f $\dfrac{d^2x}{dt^2} = -10$, given that $\dfrac{dx}{dt} = 4$ when $t = 0$ and that $x = 0$ when $t = 4$

CF

3 **a** If $y = x\sin x$ is a solution of the differential equation $x^2\dfrac{d^2y}{dx^2} - kx\dfrac{dy}{dx} + (x^2 - m)y = 0$, determine k and m.

b Show that $y = xe^{2x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 3e^{2x} = 2xe^{2x}$.

4 The curve with equation $y = f(x)$ passes through the point $P\left(\dfrac{\pi}{4}, 3\right)$, with a gradient of 1 at this point, and $f''(x) = 2\sec^2(x)$.

a Determine the gradient of the curve at $x = \dfrac{\pi}{6}$.

b Determine $f''\left(\dfrac{\pi}{6}\right)$.

5 Determine all real values of n such that $y = e^{nx}$ is a solution of $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 15y = 0$.

SF

6 Determine $\dfrac{dy}{dx}$ by implicit differentiation:

a $x^2 + 2xy + y^2 = 1$

b $x^2 + 2x + y^2 + 6y = 10$

c $\dfrac{2}{x} + \dfrac{1}{y} = 4$

d $(x+1)^2 + (y-3)^2 = 1$

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Review

CF

7 A point moves along the curve $y = x^3$ in such a way that its velocity parallel to the x-axis is a constant 3 cm/s. Determine its velocity parallel to the y-axis when:

a $x = 6$

b $y = 8$

8 Consider the differential equation $\frac{dy}{dx} = 4 + y^2$.

a Sketch the slope field of the differential equation for $y = -2, -1, 0, 1, 2$ at $x = -2, -1, 0, 1, 2$.

b If $y = -1$ when $x = 2$, solve the differential equation, giving your answer with y in terms of x.

9 A container of water is heated to boiling point (100°C) and then placed in a room with a constant temperature of 25°C. After 10 minutes, the temperature of the water is 85°C. Newton's law of cooling gives $\frac{dT}{dt} = -k(T - 25)$, where T°C is the temperature of the water at time t minutes after being placed in the room.

a Determine the value of k.

b Determine the temperature of the water 15 minutes after it was placed in the room.

SF

10 Solve the differential equation $\frac{dy}{dx} = 2x\sqrt{25 - x^2}$, for $-5 \leq x \leq 5$, given that $y = 25$ when $x = 4$.

CF

11 If $y = e^x \sin(x)$ is a solution to the differential equation $\frac{d^2y}{dx^2} + k\frac{dy}{dx} + y = e^x \cos(x)$, determine the value of k.

12 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume, V cm^3, is given by

$$V = \frac{\pi}{3}x^2(18 - x)$$

If water is poured into the bowl at the rate of 3 cm^3/s, construct the differential equation expressing $\frac{dx}{dt}$ as a function of x.

13 A circle has area A cm^2 and circumference C cm at time t seconds. If the area is increasing at a rate of 4 cm^2/s, construct the differential equation expressing $\frac{dC}{dt}$ as a function of C.

14 A population of size x is decreasing according to the law $\frac{dx}{dt} = -\frac{x}{100}$, where t denotes the time in days. If initially the population is of size x_0, determine to the nearest day how long it takes for the size of the population to be halved.

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Review

CF

15 Some students put 3 kilograms of soap powder into a water fountain. The soap powder totally dissolved in the 1000 litres of water, thus forming a solution in the fountain. When the soap solution was discovered, clean water was run into the fountain at the rate of 40 litres per minute. The clean water and the solution in the fountain mixed instantaneously and the excess mixture was removed immediately at a rate of 40 litres per minute. If S kilograms was the amount of soap powder in the fountain t minutes after the soap solution was discovered, construct and solve the differential equation to fit this situation.

16 A metal rod that is initially at a temperature of 10°C is placed in a warm room. After t minutes, the temperature, θ°C, of the rod is such that $\dfrac{d\theta}{dt} = \dfrac{30 - \theta}{20}$.

a Solve this differential equation, expressing θ in terms of t.

b Calculate the temperature of the rod after one hour has elapsed, giving the answer correct to the nearest degree.

c Determine the time taken for the temperature of the rod to rise to 20°C, giving the answer correct to the nearest minute.

17 A fire broke out in a forest and, at the moment of detection, covered an area of 0.5 hectares. From an aerial surveillance, it was estimated that the fire was spreading at a rate of increase in area of 2% per hour. If the area of the fire at time t hours is denoted by A hectares:

a Write down the differential equation that relates $\dfrac{dA}{dt}$ and A.

b What would be the area of the fire 10 hours after it is first detected?

c When would the fire cover an area of 3 hectares (to the nearest quarter-hour)?

18 A flexible beam is supported at its ends, which are at the same horizontal level and at a distance L apart. The deflection, y, of the beam, measured downwards from the horizontal through the supports, satisfies the differential equation

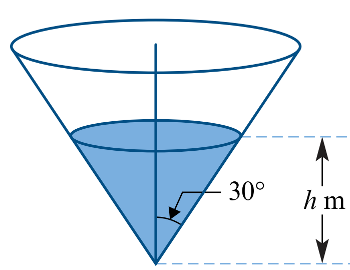
$$16\frac{d^2y}{dx^2} = L - 3x, \quad 0 \le x \le L$$

where x is the horizontal distance from one end. Determine where the deflection has its greatest magnitude, and also the value of this magnitude.

19 A vessel in the shape of a right circular cone has a vertical axis and a semi-vertex angle of 30°.

There is a small hole at the vertex so that liquid leaks out at the rate of $0.05\sqrt{h}$ m^3 per hour, where h m is the depth of liquid in the vessel at time t hours.

Given that the liquid is poured into this vessel at a constant rate of 2 m^3 per hour, set up (but do not attempt to solve) a differential equation for h.



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Technology-active short-response questions

20 A rocket, R, is launched from a point A at ground level and moves vertically upwards. The rocket is observed from a point B at ground level, where B is 10 km from A. When the angle of elevation $\angle ABR$ is $\frac{\pi}{4}$ radians, this angle is increasing at the rate of 0.005 radians per second. Determine the speed of the rocket (in km/s) at this instant.

CF

21 The percentage of radioactive carbon-14 in living matter decays, from the time of death, at a rate proportional to the percentage present.

- **a** If $x\%$ is present t years after death:
  - **i** Construct an appropriate differential equation.
  - **ii** Solve the differential equation, given that carbon-14 has a half-life of 5760 years, i.e. 50% of the original amount will remain after 5760 years.
- **b** A sample was taken from a tree buried by volcanic ash and was found to contain 45.1% of the amount of carbon-14 present in living timber. How long ago did the eruption occur?
- **c** Sketch the graph of x against t.

22 Two chemicals, A and B, are put together in a solution, where they react to form a compound, X. The rate of increase of the mass, x kg, of X is proportional to the product of the masses of *unreacted* A and B present at time t minutes. It takes 1 kg of A and 3 kg of B to form 4 kg of X. Initially, 2 kg of A and 3 kg of B are put together in solution, and 1 kg of X forms in 1 minute.

- **a** Set up the appropriate differential equation expressing $\frac{dx}{dt}$ as a function of x.
- **b** Solve the differential equation.
- **c** Determine the time taken to form 2 kg of X.
- **d** Determine the mass of X formed after 2 minutes.

23 Newton's law of cooling states that the rate of cooling of a body is proportional to the excess of its temperature above that of its surroundings. The body has a temperature of T°C at time t minutes, while the temperature of the surroundings is a constant T_S°C.

- **a** Construct a differential equation expressing $\frac{dT}{dt}$ as a function of T.
- **b** A teacher pours a cup of coffee at lunchtime. The lunchroom is at a constant temperature of 22°C, while the coffee is initially 72°C. The coffee becomes undrinkable (too cold) when its temperature drops below 50°C. After 5 minutes, the temperature of the coffee has fallen to 65°C. Determine correct to one decimal place:
  - **i** the length of time, after it was poured, that the coffee remains drinkable
  - **ii** the temperature of the coffee at the end of 30 minutes.

Review

CF

24 On a cattle station there were p head of cattle at time t years after 1 January 2005. The population naturally increases at a rate proportional to p. Every year 1000 head of cattle are withdrawn from the herd.

a Show that $\dfrac{dp}{dt} = kp - 1000$, where k is a constant.

b If the herd initially had 5000 head of cattle, determine an expression for t in terms of k and p.

c The population increased to 6000 head of cattle after 5 years.

i Show that $5k = \ln\left(\dfrac{6k-1}{5k-1}\right)$.

ii Use a calculator to determine an approximation for the value of k.

d Sketch a graph of p against t.

25 In the main lake of a trout farm, the trout population is N at time t days after 1 January 2015. The number of trout harvested on a particular day is proportional to the number of trout in the lake at that time. Every day 100 trout are added to the lake.

a Construct a differential equation with $\dfrac{dN}{dt}$ in terms of N and k, where k is a constant.

b Initially the trout population was 1000. Determine an expression for t in terms of k and N.

c The trout population decreases to 700 after 10 days. Use a calculator to determine an approximation for the value of k.

d Sketch a graph of N against t.

e If the procedure at the farm remains unchanged, determine the eventual trout population in the lake.

26 A thin horizontal beam, AB, of length L cm, is bent under a load so that the deflection, y cm at a point x cm from the end A, satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{9}{40L^2}(3x - L), \qquad 0 \le x \le L$$

Given that the deflection of the beam and its inclination to the horizontal are both zero at A, determine:

a where the maximum magnitude of deflection occurs

b the magnitude of the maximum deflection.

27 The water in a hot-water tank cools at a rate which is proportional to $T - T_0$, where T°C is the temperature of the water at time t minutes and T_0°C is the temperature of the surrounding air. When $T = 60$, the water is cooling at 1°C per minute. When switched on, the heater supplies sufficient heat to raise the water temperature by 2°C each minute (neglecting heat loss by cooling). If $T = 20$ when the heater is switched on and $T_0 = 20$:

a Construct a differential equation for $\dfrac{dT}{dt}$ as a function of T (where both heating and cooling are taking place).

b Solve the differential equation.

c Determine the temperature of the water 30 minutes after turning on the heater.

d Sketch the graph of T against t.

Review

CF

28 **a** The rate of growth of a population of iguanas on an island is $\frac{dW}{dt} = 0.04W$, where W is the number of iguanas alive after t years. Initially there were 350 iguanas.

i Solve the differential equation.

ii Sketch the graph of W against t.

iii Give the value of W to the nearest integer when $t = 50$.

b If $\frac{dW}{dt} = kW$ and there are initially 350 iguanas, determine the value of k for which the population remains constant.

c A more realistic population model for the iguanas is determined by the logistic differential equation $\frac{dW}{dt} = (0.04 - 0.00005W)W$. Initially there were 350 iguanas.

i Solve the differential equation.

ii Sketch the graph of W against t.

iii Determine the population after 50 years.

29 A hospital patient is receiving a drug at a constant rate of R mg per hour through a drip. At time t hours, the amount of the drug in the patient is x mg. The rate of loss of the drug from the patient is proportional to x.

a When $t = 0$, $x = 0$:

i Show that $\frac{dx}{dt} = R - kx$, where k is a positive constant.

ii Determine an expression for x in terms of t, k and R.

b If $R = 50$ and $k = 0.05$:

i Sketch the graph of x against t.

ii Determine the time taken for there to be 200 mg in the patient, correct to two decimal places.

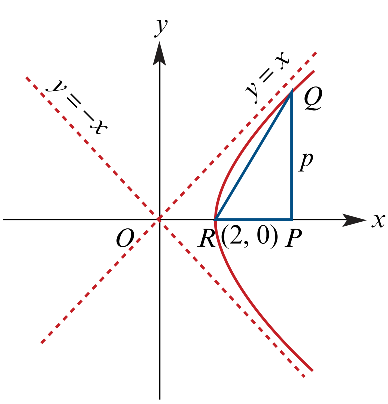
c When the patient contains 200 mg of the drug, the drip is disconnected.

i Assuming that the rate of loss remains the same, determine the time taken for the amount of the drug in the patient to fall to 100 mg, correct to two decimal places.

ii Sketch the graph of x against t, showing the rise to 200 mg and fall to 100 mg.

CU

30 The diagram shows part of the curve $x^2 - y^2 = 4$. The line segment PQ is parallel to the y-axis, and R is the point $(2, 0)$. The length of PQ is p.



a Determine the area, A, of triangle PQR in terms of p.

b **i** Determine $\frac{dA}{dp}$.

ii Use your calculator to help sketch the graph of A against p.

iii Determine the value of p for which $A = 50$ (correct to two decimal places).

iv Prove that $\frac{dA}{dp} \geq 0$ for all p.

Review

CU

c Point Q moves along the curve and point P along the x-axis so that PQ is always parallel to the y-axis and p is increasing at a rate of 0.2 units per second. Determine the rate at which A is increasing, correct to three decimal places, when:

i $p = 2.5$ **ii** $p = 4$ **iii** $p = 50$ **iv** $p = 80$

(Use calculus to obtain the rate.)

31 An ellipse is described by the parametric equations

$$x = 3\cos\theta \quad \text{and} \quad y = 2\sin\theta$$

a Show that the tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ has equation $2x\cos\theta + 3y\sin\theta = 6$.

b The tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ meets the line with equation $x = 3$ at a point T.

i Determine the coordinates of the point T.

ii Let A be the point with coordinates $(-3, 0)$ and let O be the origin. Prove that OT is parallel to AP.

c The tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ meets the x-axis at Q and the y-axis at R.

i Determine the midpoint M of the line segment QR in terms of θ.

ii Determine the locus of M as θ varies.

d $W(-3\sin\theta, 2\cos\theta)$ and $P(3\cos\theta, 2\sin\theta)$ are points on the ellipse.

i Determine the equation of the tangent to the ellipse at W.

ii Determine the coordinates of Z, the point of intersection of the tangents at P and W, in terms of θ.

iii Determine the locus of Z as θ varies.

32 The volume, V litres, of water in a pool at time t minutes is given by the rule

$$V = -3000\pi\,(\ln(1 - h) + h)$$

where h metres is the depth of water in the pool at time t minutes.

a **i** Determine $\dfrac{dV}{dh}$ in terms of h.

ii Sketch the graph of $\dfrac{dV}{dh}$ against h for $0 \le h \le 0.9$.

b The maximum depth of the pool is 90 cm.

i Determine the maximum volume of the pool to the nearest litre.

ii Sketch the graphs of $y = -3000\pi\ln(1 - x)$ and $y = -3000\pi x$. Use addition of ordinates to sketch the graph of V against h for $0 \le h \le 0.9$.

c If water is being poured into the pool at 15 litres/min, determine the rate at which the depth of the water is increasing when $h = 0.2$, correct to two significant figures.

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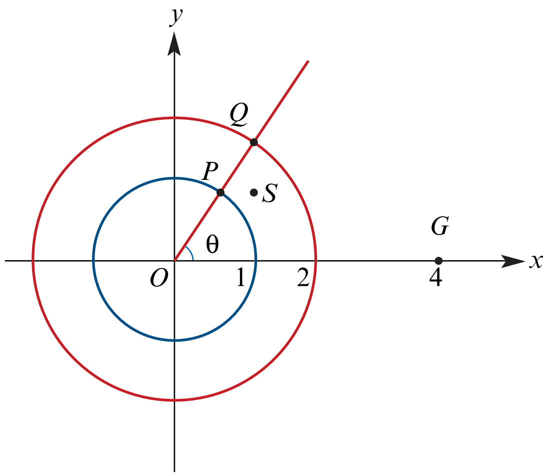
33 An electronic game appears on a flat screen, part of which is shown in the diagram. Concentric circles of radii one unit and two units appear on the screen.

Points P and Q move around the circles so that O, P and Q are collinear and OP makes an angle of θ with the x-axis.

A spaceship S moves around between the two circles and a gun is on the x-axis at G, which is 4 units from O.



The spaceship moves so that at any time it is at a point (x, y), where x is equal to the x-coordinate of Q and y is equal to the y-coordinate of P. The player turns the gun and tries to hit the spaceship.

a Determine the Cartesian equation of the path C of S.

b Show that the equation of the tangent to C at the point (u, v) on C is $y = \dfrac{-u}{4v}x + \dfrac{1}{v}$.

c Show that in order to aim at the spaceship at any point on its path, the player needs to turn the gun through an angle of at most 2α, where $\tan\alpha = \dfrac{1}{6}\sqrt{3}$.

Multiple-choice questions

Technology-free multiple-choice questions

1 The equation of the tangent to $x^2 + y^2 = 1$ at the point with coordinates $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ is

A $y = -x$ **B** $y = -x + 2\sqrt{2}$ **C** $y = -x + 1$ **D** $y = -x + \sqrt{2}$

2 If $x = t^2$ and $y = t^3$, then $\dfrac{dx}{dy}$ is equal to

A $\dfrac{3}{2t}$ **B** $\dfrac{2}{3t}$ **C** $\dfrac{3t}{2}$ **D** $\dfrac{2t}{3}$

3 If $\dfrac{dy}{dx} = \dfrac{2x+1}{4}$ and $y = 0$ when $x = 2$, then y is equal to

A $\dfrac{1}{4}(x^2 + x) + \dfrac{1}{2}$ **B** $\dfrac{x(x+1)}{4}$ **C** $\dfrac{1}{4}(x^2 + x) + 2$ **D** $\dfrac{1}{4}(x^2 + x - 6)$

4 For which one of the following differential equations is $y = 2xe^{2x}$ a solution?

A $\dfrac{dy}{dx} - 2y = 0$ **B** $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$

C $\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$ **D** $\dfrac{d^2y}{dx^2} - 4y = 8e^{2x}$

5 The solution of the differential equation $\dfrac{dy}{dx} = y$, where $y = 2$ when $x = 0$, is

A $y = e^{2x}$ **B** $y = e^{\frac{x}{2}}$ **C** $y = 2e^x$ **D** $y = \dfrac{1}{2}e^x$

6 The acceleration, a m/s^2, of an object moving in a straight line at time t seconds is given by $a = \sin(2t)$. If the object has an initial velocity of 4 m/s, then v is equal to

A $2\cos(2t) + 4$ **B** $2\cos(2t) + 2$

C $\int_0^t \sin(2x)\,dx + 4$ **D** $-\dfrac{1}{2}\cos(2t) + 4$

7 Assume that $\dfrac{dy}{dx} = \dfrac{2-y}{4}$ and that $x = 3$ when $y = 1$. The value of x when $y = \dfrac{1}{2}$ is given by

A $x = \displaystyle\int_1^{\frac{1}{2}} \frac{4}{2-t}\,dt + 3$ **B** $x = \displaystyle\int_3^{\frac{1}{2}} \frac{4}{2-t}\,dt + 1$

C $x = \displaystyle\int_1^{\frac{1}{2}} \frac{2-t}{4}\,dt + 3$ **D** $x = \displaystyle\int_3^{\frac{1}{2}} \frac{2-t}{4}\,dt + 1$

8 If $\dfrac{dy}{dx} = \dfrac{1}{5}(y-1)^2$ and $y = 0$ when $x = 0$, then y is equal to

A $\dfrac{5}{1-x} - 5$ **B** $1 + \dfrac{5}{x+5}$

C $\dfrac{x}{x+5}$ **D** $\dfrac{5}{x+5} - 1$

9 The solution of the differential equation $\dfrac{dy}{dx} = e^{-x^2}$, where $y = 4$ when $x = 1$, is

A $y = \displaystyle\int_1^4 e^{-x^2}\,dx$ **B** $y = \displaystyle\int_1^4 e^{-x^2}\,dx + 4$

C $y = \displaystyle\int_1^x e^{-u^2}\,du - 4$ **D** $y = \displaystyle\int_1^x e^{-u^2}\,du + 4$

10 Water is leaking from an initially full container with a depth of 40 cm. The volume, V cm^3, of water in the container is given by $V = \pi(5h^2 + 225h)$, where h cm is the depth of the water at time t minutes.

If water leaks out at the rate of $\dfrac{5\sqrt{h}}{2h+45}$ cm^3/min, then the rate of change of the depth is

A $\dfrac{-\sqrt{h}}{\pi(2h+45)^2}$ cm/min **B** $5\pi(2h+45)$ cm/min

C $\dfrac{\sqrt{h}}{\pi(2h+45)^2}$ cm/min **D** $\dfrac{1}{5\pi(2h+45)}$ cm/min

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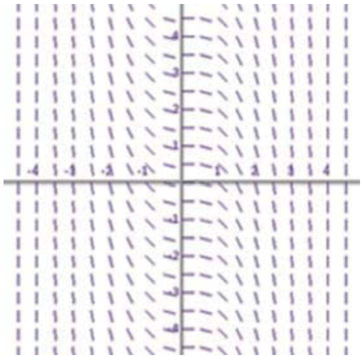
Technology-active multiple-choice questions

11 The rate at which a particular disease spreads through a population of 2000 cattle is proportional to the product of the number of infected cows and the number of non-infected cows. Initially four cows are infected. If N denotes the number of infected cows at time t days, then a differential equation to describe this is

A $\frac{dN}{dt} = kN(2000 - N)$ **B** $\frac{dN}{dt} = k(4 - N)(200 - N)$

C $\frac{dN}{dt} = kN(200 - N)$ **D** $\frac{dN}{dt} = kN^2(2000 - N^2)$

12 The differential equation that best matches the slope field shown is



A $\frac{dy}{dx} = x$ **B** $\frac{dy}{dx} = -x$

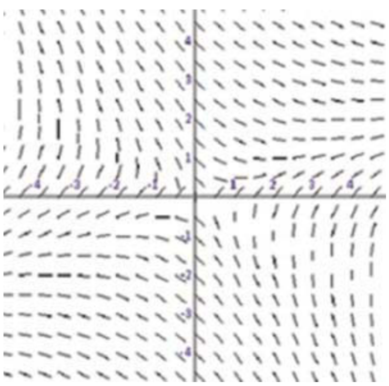
C $\frac{dy}{dx} = x^2$ **D** $\frac{dy}{dx} = -x^2$

13 A tank initially contains 100 litres of pure water. A salt solution containing 5 grams of salt per litre is added at the rate of 5 litres per minute, and the mixed solution is drained simultaneously at the rate of 3 litres per minute. There is m grams of salt in the tank after t minutes. The differential equation which applies to this situation is

A $\frac{dm}{dt} = 25 - \frac{3m}{100 + 2t}$ **B** $\frac{dm}{dt} = 5 - \frac{3m}{100 + 5t}$

C $\frac{dm}{dt} = 25 - \frac{m}{100 + 5t}$ **D** $\frac{dm}{dt} = 5 - \frac{m}{100 - 2t}$

14 The differential equation that best matches the slope field shown is



A $\frac{dy}{dx} = \frac{y}{x}$ **B** $\frac{dy}{dx} = \frac{x}{y}$

C $\frac{dy}{dx} = \frac{x - 2y}{2y + x}$ **D** $\frac{dy}{dx} = -\frac{y}{x}$

15 Suppose that $f'(x) = \sin(x^2)$ and $f(0) = 1$. Correct to two decimal places, $f(1)$ is equal to

A 0.31 **B** 1.31 **C** 2.31 **D** 3.31

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Review

16 Air is pumped into a spherical balloon at a rate of 3 cm^3/sec. Correct to two decimal places, when $r = 13$, the rate at which the surface area is increasing is

A 0.46 cm^2/sec **B** 0.57 cm^2/sec **C** 0.79 cm^2/sec **D** 0.92 cm^2/sec

17 The equation of the line that is tangent to the graph of $x^2 + 2xy + y^2 = 1$ at $(0, 1)$ is

A $y = x - 1$ **B** $y = x + 1$ **C** $y = -x - 1$ **D** $y = -x + 1$

18 If $f(x) = ax + b$ is a particular solution of $2f'(x) + f(x) = 3x + 4$, then

A $a = 3, b = 2$ **B** $a = 3, b = -2$ **C** $a = -3, b = 2$ **D** $a = -3, b = -2$

19 For the variables x, y and t, it is known that $y = \sqrt{x}$ and $\dfrac{dx}{dt} = 3t$. For some $c \in \mathbb{R}$, the relationship between y and t can be expressed in the form

A $2y^2 = 3t^2 + c$ **B** $3y^2 = 2t^2 + c$ **C** $y^2 = 3t^2 + c$ **D** $y^2 = 2t^3 + c$

20 The rate of growth of a population of rabbits on an island is $\dfrac{dP}{dt} = 0.001P(800 - P)$, where P is the population of rabbits alive after t years. If initially there are 100 rabbits, after 5 years, the number of rabbits is approximately

A 509 **B** 609 **C** 709 **D** 809

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14

Modelling motion

Chapter contents

In this chapter, we apply two different approaches to modelling motion in a straight line:

- **Kinematics** This is the study of motion without reference to the cause of motion. In the first four sections of this chapter, we will analyse the motion of a particle if we are given a rule that describes the motion. This builds on your study of motion in a straight line from Mathematical Methods Units 3 & 4.
- **Dynamics** This is the study of forces and their effect on motion. We will learn how to to describe the motion of a particle if we are given the forces that act on the particle. This builds on your study of forces and equilibrium from Specialist Mathematics Units 1 & 2.

This chapter covers Unit 4 Topic 4: Modelling motion.

14A Motion in a straight line

Learning intentions

- ▶ To be able to solve problems involving motion in a straight line with constant acceleration or variable acceleration.

In this chapter, we consider the motion of a particle in a straight line. When referring to the motion of a particle, we may in fact be referring to an object of any size. However, for the purposes of studying its motion, we can assume that all forces acting on the object, causing it to move, are acting through a single point. Hence we can consider the motion of a car or a train in the same way as we would consider the motion of a dimensionless particle.

When studying motion, it is important to make a distinction between vector quantities and scalar quantities:

Vector quantities Position, displacement, velocity and acceleration must be specified by both magnitude and direction.

Scalar quantities Distance, speed and time are specified by their magnitude only.

Since we are considering movement in a straight line, the *direction* of each vector quantity is simply specified by the *sign* of the numerical value.

Position, velocity and acceleration

We begin this section by revising basic concepts from Mathematical Methods Units 3 & 4.

Position

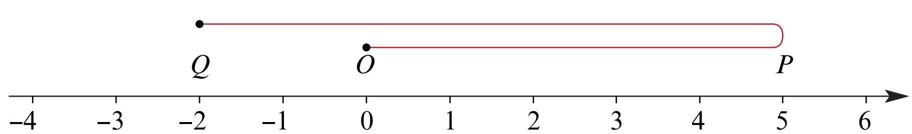
The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.

Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O; while if $x = 3$, the position is 3 m to the right of O.

Displacement and distance

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction). For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$

where $x(t)$ is the position of the particle at time t.

Velocity

The **velocity** of a particle is the instantaneous rate of change of its position with respect to time:

$$\text{velocity } v = \frac{dx}{dt}$$

where x is the position at time t.

Note: Velocity is also denoted by $\dot{x}$ or $\dot{x}(t)$.

Velocity is a vector quantity. For motion in a straight line, the direction is specified by the sign of the numerical value.

If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

Speed and average speed

Speed is a scalar quantity; its value is always non-negative.

- **Speed** is the magnitude of the velocity.
- **Average speed** over a time interval is given by $\dfrac{\text{distance travelled}}{\text{change in time}}$

Acceleration

The **acceleration** of a particle is the instantaneous rate of change of its velocity with respect to time:

$$\text{acceleration } a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

where v and x are the velocity and position at time t respectively.

Note: Acceleration is also denoted by $\ddot{x}$ or $\ddot{x}(t)$.

Acceleration may be positive, negative or zero. Constant zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

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Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

Units of measurement

Common units for velocity (and speed) are:

$$\begin{aligned} &1 \text{ metre per second} &&= 1 \text{ m/s} &&= 1 \text{ m s}^{-1} \\ &1 \text{ centimetre per second} &&= 1 \text{ cm/s} &&= 1 \text{ cm s}^{-1} \\ &1 \text{ kilometre per hour} &&= 1 \text{ km/h} &&= 1 \text{ km h}^{-1} \end{aligned}$$

The first and third units are connected in the following way:

$$1 \text{ km/h} = 1000 \text{ m/h} = \frac{1000}{60 \times 60} \text{ m/s} = \frac{5}{18} \text{ m/s}$$

$$\therefore \quad 1 \text{ m/s} = \frac{18}{5} \text{ km/h}$$

Common units for acceleration are cm/s^2 and m/s^2.

Using integration for motion in a straight line

From Mathematical Methods Units 3 & 4, you are familiar with using differentiation to go from position to velocity to acceleration. You are also familiar with using anti-differentiation and additional information to go from acceleration to velocity to position.

Here we will revisit the use of velocity–time graphs and integration for studying motion in a straight line.

Example 1

A particle starts moving along a straight line. Its velocity, $v(t)$ m/s, after t seconds is given by $v(t) = 2t - 6$.

a Determine the particle's displacement after 4 seconds.

b Determine the distance travelled by the particle in the first 4 seconds.

c Sketch the velocity–time graph and determine:

  i the signed area enclosed by the graph and the t-axis between $t = 0$ and $t = 4$

  ii the area enclosed by the graph and the t-axis between $t = 0$ and $t = 4$.

Solution

a Anti-differentiate $v(t)$ to determine the position, $x(t)$ m, after t seconds:

$$x(t) = t^2 - 6t + c$$

The particle's displacement is its change in position:

$$x(4) - x(0) = (-8 + c) - c = -8$$

The displacement after 4 seconds is -8 m.

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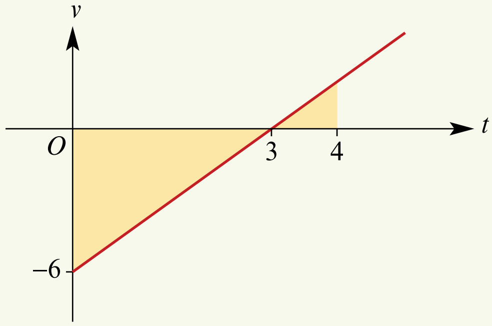
b First determine when the particle is at rest: $v(t) = 0$ implies $2t - 6 = 0$, i.e. $t = 3$.

Now consider the displacement over the two time intervals $[0, 3]$ and $[3, 4]$:

- Displacement over $[0, 3]$: $x(3) - x(0) = (-9 + c) - c = -9$ m
- Displacement over $[3, 4]$: $x(4) - x(3) = (-8 + c) - (-9 + c) = 1$ m

The particle travelled 10 m in the first 4 seconds.

c



i Signed area $= \int_0^4 v(t)\,dt$

$$= \int_0^4 2t - 6\,dt$$
$$= \left[t^2 - 6t\right]_0^4$$
$$= -8 - 0$$
$$= -8$$

ii Area $= -\int_0^3 v(t)\,dt + \int_3^4 v(t)\,dt$

$$= -\int_0^3 2t - 6\,dt + \int_3^4 2t - 6\,dt$$
$$= -\left[t^2 - 6t\right]_0^3 + \left[t^2 - 6t\right]_3^4$$
$$= -(-9 - 0) + (-8 - (-9))$$
$$= 9 + 1$$
$$= 10$$

Note: The displacement is equal to the signed area.
The distance travelled is equal to the area.

Since a particle's position, $x(t)$, is an anti-derivative of its velocity, $v(t)$, the results below follow directly from the fundamental theorem of calculus.

Using definite integrals for motion in a straight line

- The displacement of a particle over a time interval $[t_1, t_2]$ is given by

$$\text{displacement} = x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\,dt$$

- The average velocity of a particle over a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} v(t)\,dt$$

Hence we see that, for motion in a straight line, the following information can be obtained from the velocity–time graph:

- **Acceleration** is given by the gradient.
- **Displacement** is given by the signed area bounded by the graph and the t-axis.
- **Distance travelled** is given by the total area bounded by the graph and the t-axis.

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Example 2

A particle starts from rest and moves in a straight line with acceleration $a = 6t + 8$ m/s^2, where t is the time in seconds ($t \geq 0$).

a Determine the displacement (change in position) over the first 4 seconds of motion.

b Determine the average velocity over the first 4 seconds of motion.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Determine the velocity by anti-differentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

a
$$\begin{aligned} \text{Displacement} &= \int_0^4 v(t)\,dt \\ &= \int_0^4 3t^2 + 8t\,dt \\ &= \Big[t^3 + 4t^2\Big]_0^4 \\ &= 128 \text{ m} \end{aligned}$$

b
$$\begin{aligned} \text{Average velocity} &= \frac{\text{change in position}}{\text{change in time}} \\ &= \frac{128}{4} \\ &= 32 \text{ m/s} \end{aligned}$$

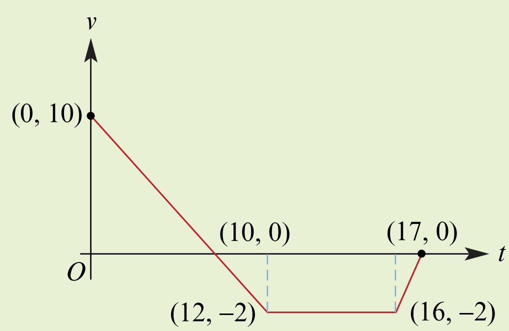
Example 3

The graph shows the motion of a particle.

a Describe the motion.

b Determine the distance travelled.

Velocity is measured in m/s and time in seconds.



Solution

a The particle decelerates uniformly from an initial velocity of 10 m/s. After 10 seconds, it is instantaneously at rest before it accelerates uniformly in the opposite direction for 2 seconds, until its velocity reaches −2 m/s. It continues to travel in this direction with a constant velocity of −2 m/s for a further 4 seconds. Finally, it decelerates uniformly until it comes to rest after 17 seconds.

b Distance travelled = total area

$$= \left(\tfrac{1}{2} \times 10 \times 10\right) + \left(\tfrac{1}{2} \times 2 \times 2\right) + (4 \times 2) + \left(\tfrac{1}{2} \times 1 \times 2\right) = 61 \text{ m}$$

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Constant acceleration

If an object is moving due to a constant force (for example, gravity), then its acceleration is constant. There are several useful formulas that apply in this situation.

Formulas for constant acceleration

When analysing the motion of a particle moving in a straight line with constant acceleration a, the following formulas can be utilised. Here, u represents the initial velocity, v denotes the final velocity, s represents the displacement, and t is the time taken:

1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$

Proof 1 We can write

$$\frac{dv}{dt} = a$$

where a is a constant and v is the velocity at time t. By anti-differentiating with respect to t, we obtain

$$v = at + c$$

where the constant c is the initial velocity. We denote the initial velocity by u, and therefore $v = u + at$.

2 We now write

$$\frac{dx}{dt} = v = u + at$$

where x is the position at time t. By anti-differentiating again, we have

$$x = ut + \frac{1}{2}at^2 + d$$

where the constant d is the initial position. The particle's displacement (change in position) is given by $s = x - d$, and so we obtain the second equation.

3 Transform the first equation $v = u + at$ to make t the subject:

$$t = \frac{v - u}{a}$$

Now substitute this into the second equation:

$$s = ut + \frac{1}{2}at^2$$

$$s = \frac{u(v - u)}{a} + \frac{a(v - u)^2}{2a^2}$$

$$\begin{aligned} 2as &= 2u(v - u) + (v - u)^2 \\ &= 2uv - 2u^2 + v^2 - 2uv + u^2 \\ &= v^2 - u^2 \end{aligned}$$

4 Similarly, the fourth equation can be derived from the first and second equations.

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These four formulas are very useful, but it must be remembered that they only apply when the acceleration is constant.

When approaching problems involving constant acceleration, it is a good idea to list the quantities you are given, establish which quantity or quantities you require, and then use the appropriate formula. Ensure that all quantities are converted to compatible units.

Example 4

An object is moving in a straight line with uniform acceleration. Its initial velocity is 12 m/s and after 5 seconds its velocity is 20 m/s. Determine:

a the acceleration

b the distance travelled during the first 5 seconds

c the time taken to travel a distance of 200 m.

Solution

We are given $u = 12$, $v = 20$ and $t = 5$.

a Determine a using

$$\begin{aligned} v &= u + at \\ 20 &= 12 + 5a \\ a &= 1.6 \end{aligned}$$

The acceleration is 1.6 m/s^2.

b Determine s using

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 12(5) + \frac{1}{2}(1.6)5^2 = 80 \end{aligned}$$

The distance travelled is 80 m.

Note: Since the object is moving in one direction, the distance travelled is equal to the displacement.

c We are now given $a = 1.6$, $u = 12$ and $s = 200$.

Determine t using $s = ut + \frac{1}{2}at^2$

$$\begin{aligned} 200 &= 12t + \frac{1}{2} \times 1.6 \times t^2 \\ 200 &= 12t + \frac{4}{5}t^2 \\ 1000 &= 60t + 4t^2 \\ 250 &= 15t + t^2 \end{aligned}$$

$$t^2 + 15t - 250 = 0$$

$$(t - 10)(t + 25) = 0$$

$$\therefore \quad t = 10 \text{ or } t = -25$$

As $t \geq 0$, the only allowable solution is $t = 10$.

The object takes 10 s to travel a distance of 200 m.

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Example 5

A body is moving in a straight line with uniform acceleration and an initial velocity of 12 m/s. If the body stops after 20 metres, determine the acceleration of the body.

Solution

We are given $u = 12$, $v = 0$ and $s = 20$.

Determine a using

$$\begin{aligned} v^2 &= u^2 + 2as \\ 0 &= 144 + 2 \times a \times 20 \\ 0 &= 144 + 40a \\ \therefore \quad a &= -\frac{144}{40} \end{aligned}$$

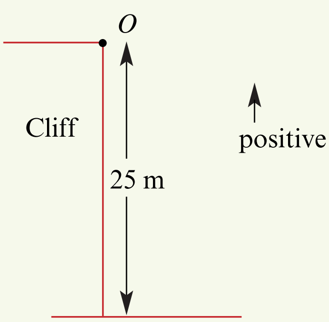
The acceleration is $-\frac{18}{5}$ m/s^2.

Example 6

A stone is thrown vertically upwards from the top of a cliff which is 25 m high. The velocity of projection of the stone is 22 m/s. Determine the time it takes to reach the base of the cliff. (Give the answer correct to two decimal places.)

Solution

Take the origin at the top of the cliff and vertically upwards as the positive direction.



We are given $s = -25$, $u = 22$ and $a = -9.8$.

Determine t using

$$\begin{aligned} s &= ut + \tfrac{1}{2}at^2 \\ -25 &= 22t + \tfrac{1}{2} \times (-9.8) \times t^2 \\ -25 &= 22t - 4.9t^2 \end{aligned}$$

Therefore

$$4.9t^2 - 22t - 25 = 0$$

By the quadratic formula:

$$t = \frac{22 \pm \sqrt{22^2 - 4 \times 4.9 \times (-25)}}{2 \times 4.9}$$

$$\therefore \quad t = 5.429\ldots \text{ or } t = -0.9396\ldots$$

But $t \geq 0$, so the only allowable solution is $t = 5.429\ldots$.

It takes 5.43 seconds for the stone to reach the base of the cliff.

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Skill-sheet

Exercise 14A

CF

1 The position of a particle travelling in a horizontal line, relative to a point O on the line, is x metres at time t seconds. The position is described by $x = 1 + 2t - t^2$, $t \geq 0$.

a Determine the displacement of the particle in the third second.

b Determine the average velocity in the first 4 seconds.

c Determine the relation between velocity, v m/s, and time, t s.

d Determine the velocity of the particle when $t = 3$.

e Determine when and where the particle changes direction.

f Determine the distance travelled in the first 4 seconds.

g Determine the particle's average speed for the first 4 seconds.

2 A particle moves in a straight line. Relative to a fixed point O on the line, the particle's position, x m, at time t seconds is given by $x = t(t - 3)^2$. Determine:

a the velocity of the particle after 2 seconds

b the values of t for which the particle is instantaneously at rest

c the acceleration of the particle after 4 seconds.

3 A particle moving in a straight line has position given by $x = 2t^3 - 4t^2 - 100$. Determine the time(s) when the particle has zero velocity.

4 A particle moving in a straight line passes through a fixed point O on the line with a velocity of 30 m/s. The acceleration, a m/s^2, of the particle at time t seconds after passing O is given by $a = 13 - 6t$. Determine:

a the velocity of the particle 3 seconds after passing O

b the time taken to reach the maximum distance from O in the initial direction of motion

c the value of this maximum distance.

Example 1

5 A particle moves in a straight line. The velocity of the particle, v m/s, at time t seconds is given by $v = -t^2 + 6t - 5$ for $t \geq 0$.

a Sketch the velocity–time graph for the motion of the particle from $t = 0$ to $t = 5$.

b By calculating a total signed area, determine the particle's displacement after 5 seconds.

c By calculating a total area, determine the distance travelled by the particle in the first 5 seconds.

Example 2

6 A particle moving in a straight line passes through a fixed point O with velocity 8 m/s. Its acceleration, a m/s^2, at time t seconds after passing O is given by $a = 12 - 6t$.

a Determine the displacement of the particle over the first 2 seconds of motion.

b Determine the average velocity over the first 2 seconds of motion.

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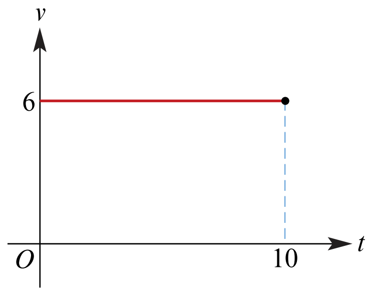
Example 3

7 Each of the following graphs shows the motion of a particle. For each graph:

i describe the motion **ii** determine the distance travelled.

Velocity is measured in m/s and time in seconds.

a



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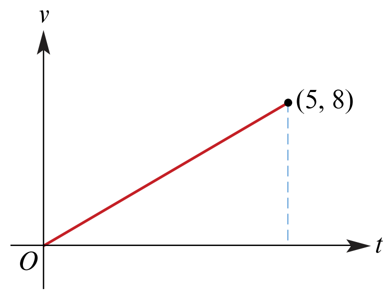

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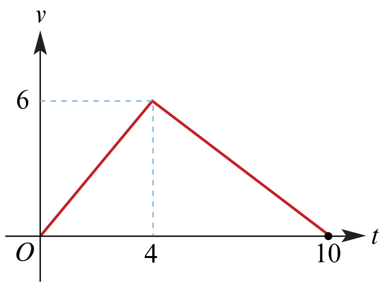

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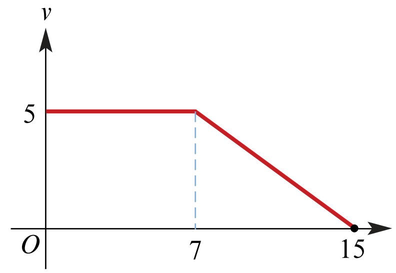

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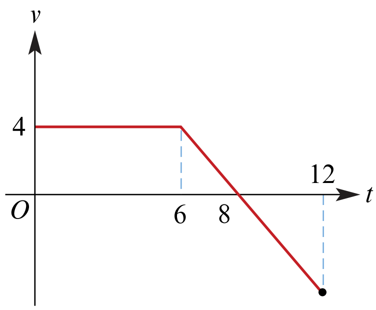

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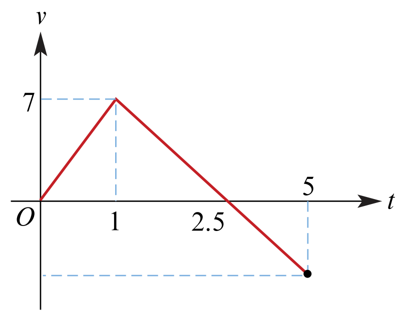

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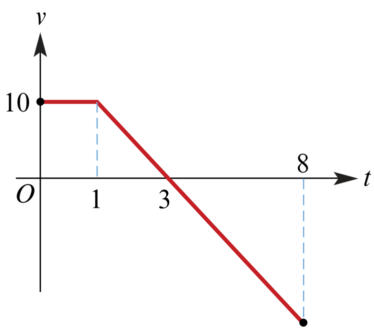

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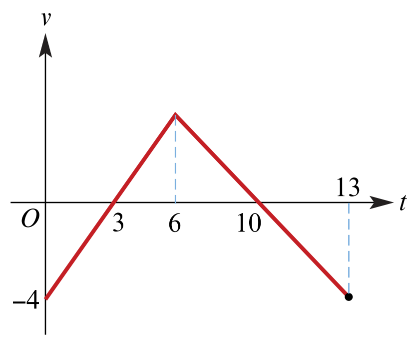

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8 A particle moves in a straight line such that, at time t seconds after passing through a fixed point O, its velocity, v m/s, is given by $v = 3t^2 - 30t + 72$. Determine:

- **a** the initial acceleration of the particle
- **b** the two values of t for which the particle is instantaneously at rest
- **c** the distance travelled by the particle between these two times.
- **d** the total distance moved by the particle between $t = 0$ and $t = 7$.

9 An object moves in a line with acceleration, $\ddot{x}$ m/s^2, given by $\ddot{x} = \dfrac{1}{(2t+3)^2}$. If the object starts from rest at the origin, determine the position–time relationship.

10 A particle moves in a line with acceleration, $\ddot{x}$ m/s^2, given by $\ddot{x} = \dfrac{2t}{(1+t^2)^2}$. If the initial velocity is 0.5 m/s, determine the distance travelled in the first $\sqrt{3}$ seconds.

11 An object moves in a line with velocity, $\dot{x}$ m/s, given by $\dot{x} = \dfrac{t}{1+t^2}$. The object starts from the origin. Determine:

- **a** the initial velocity
- **b** the maximum velocity
- **c** the distance travelled in the third second
- **d** the position–time relationship
- **e** the acceleration–time relationship
- **f** the average acceleration over the third second
- **g** the minimum acceleration.

Constant acceleration

12 An object with constant acceleration starts with a velocity of 15 m/s. At the end of the eleventh second, its velocity is 48 m/s. What is its acceleration?

13 A car accelerates uniformly from 5 km/h to 41 km/h in 10 seconds. Express this acceleration in:

- **a** km/h^2
- **b** m/s^2

Example 4

14 An object is moving in a straight line with uniform acceleration. Its initial velocity is 10 m/s and after 5 seconds its velocity is 25 m/s. Determine:

- **a** the acceleration
- **b** the distance travelled during the first 5 seconds
- **c** the time taken to travel a distance of 100 m.

Example 5

15 A body moving in a straight line has uniform acceleration and an initial velocity of 20 m/s. If the body stops after 40 metres, determine the acceleration of the body.

16 A particle starts from a fixed point O with an initial velocity of -10 m/s and a uniform acceleration of 4 m/s^2. Determine:

- **a** the displacement of the particle from O after 6 seconds
- **b** the velocity of the particle after 6 seconds
- **c** the time when the velocity is zero
- **d** the distance travelled in the first 6 seconds.

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Example 6 **17** **a** A stone is thrown vertically upwards from ground level at 21 m/s. The acceleration due to gravity is 9.8 m/s^2.

i What is its height above the ground after 2 seconds?

ii What is the maximum height reached by the stone?

b If the stone is thrown vertically upwards from a cliff 17.5 m high at 21 m/s:

i How long will it take to reach the ground at the base of the cliff?

ii What is the velocity of the stone when it hits the ground?

18 A basketball is thrown vertically upwards with a velocity of 14 m/s. The acceleration due to gravity is 9.8 m/s^2. Determine:

a the time taken by the ball to reach its maximum height

b the greatest height reached by the ball

c the time taken for the ball to return to the point from which it is thrown.

19 A car sliding on ice is decelerating at the rate of 0.1 m/s^2. Initially the car is travelling at 20 m/s. Determine:

a the time taken before it comes to rest

b the distance travelled before it comes to rest.

20 An object is dropped from a point 100 m above the ground. The acceleration due to gravity is 9.8 m/s^2. Determine:

a the time taken by the object to reach the ground

b the velocity at which the object hits the ground.

21 An object is projected vertically upwards from a point 50 m above ground level. (Acceleration due to gravity is 9.8 m/s^2.) If the initial velocity is 10 m/s, determine:

a the time the object takes to reach the ground (correct to two decimal places)

b the object's velocity when it reaches the ground.

22 A particle travels in a straight line with a constant velocity of 4 m/s for 12 seconds. It is then subjected to a constant acceleration in the opposite direction for 20 seconds, which returns the particle to its original position. Determine the acceleration of the particle.

23 A child slides from rest down a slide 4 m long. The child undergoes constant acceleration and reaches the end of the slide travelling at 2 m/s. Determine:

a the time taken to go down the slide

b the child's acceleration.

CF

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14B Differential equations of the form $v = f(x)$ and $a = f(v)$

Learning intentions

- To be able to solve problems involving **motion in a straight line** with information given in one of the forms $v = f(x)$ or $a = f(v)$.

Example 7

The velocity of a particle moving along a straight line is inversely proportional to its position relative to point O. The particle is initially 1 m from point O and is 2 m from point O after 1 second.

a Determine an expression for the particle's position, x m, at time t seconds.

b Determine an expression for the particle's velocity, v m/s, at time t seconds.

Solution

a The information can be written as

$$v = \frac{k}{x} \quad \text{for } k \in \mathbb{R}^+, \quad x(0) = 1 \quad \text{and} \quad x(1) = 2$$

This gives

$$\frac{dx}{dt} = \frac{k}{x}$$

$$\therefore \quad \int \frac{x}{k}\,dx = \int 1\,dt$$

$$\therefore \quad t = \int \frac{x}{k}\,dx$$

$$= \frac{x^2}{2k} + c$$

Since $x(0) = 1$: $\quad 0 = \dfrac{1}{2k} + c \quad (1)$

Since $x(1) = 2$: $\quad 1 = \dfrac{4}{2k} + c \quad (2)$

Subtracting (1) from (2) yields $1 = \dfrac{3}{2k}$ and therefore $k = \dfrac{3}{2}$.

Substituting in (1) yields $c = -\dfrac{1}{2k} = -\dfrac{1}{3}$.

Now $\quad t = \dfrac{x^2}{3} - \dfrac{1}{3}$

$$x^2 = 3t + 1$$

$$\therefore \quad x = \pm\sqrt{3t+1}$$

But when $t = 0$, $x = 1$ and therefore

$$x = \sqrt{3t+1}$$

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b $x = \sqrt{3t+1}$ implies

$$v = \frac{dx}{dt} = 3 \times \frac{1}{2} \times \frac{1}{\sqrt{3t+1}}$$

$$= \frac{3}{2\sqrt{3t+1}}$$

Example 8

A body moving in a straight line has an initial velocity of 25 m/s and its acceleration, a m/s^2, is given by $a = -k(50 - v)$, where k is a positive constant and v m/s is its velocity. Determine v in terms of t and sketch the velocity–time graph for the motion.

(The motion stops when the body is instantaneously at rest for the first time.)

Solution

$$a = -k(50 - v)$$

$$\frac{dv}{dt} = -k(50 - v)$$

$$\int 1 \, dt = \int \frac{1}{-k(50 - v)} \, dv$$

$$t = -\frac{1}{k} \int \frac{1}{50 - v} \, dv$$

$$= -\frac{1}{k}\big(-\ln|50 - v|\big) + c$$

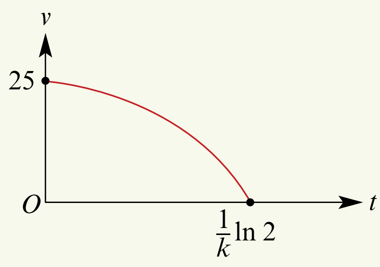
$$\therefore \quad t = \frac{1}{k} \ln(50 - v) + c \qquad \text{(Note that } v \le 25 \text{ since } a < 0.\text{)}$$

When $t = 0$, $v = 25$, and so $c = -\dfrac{1}{k} \ln 25$.

Thus $\quad t = \dfrac{1}{k} \ln\left(\dfrac{50 - v}{25}\right)$

$$e^{kt} = \frac{50 - v}{25}$$

$$\therefore \quad v = 50 - 25e^{kt}$$



Example 9

The acceleration, a, of an object moving along a line is given by $a = -(v + 1)^2$, where v is the velocity of the object at time t. Also $v(0) = 10$ and $x(0) = 0$, where x is the position of the object at time t. Determine:

a an expression for the velocity of the object in terms of t

b an expression for the position of the object in terms of t.

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Solution

a $a = -(v+1)^2$ gives

$$\frac{dv}{dt} = -(v+1)^2$$

$$\int 1\,dt = \int \frac{-1}{(v+1)^2}\,dv$$

$$t = -\int \frac{1}{(v+1)^2}\,dv$$

$$\therefore \quad t = \frac{1}{v+1} + c$$

Since $v(0) = 10$, we obtain $c = -\frac{1}{11}$ and so

$$t = \frac{1}{v+1} - \frac{1}{11}$$

This can be rearranged as

$$v = \frac{11}{11t+1} - 1$$

b $\frac{dx}{dt} = v = \frac{11}{11t+1} - 1$

$$\therefore \quad x = \int \frac{11}{11t+1} - 1\,dt$$

$$= \ln|11t+1| - t + c$$

Since $x(0) = 0$, $c = 0$ and therefore $x = \ln|11t+1| - t$.

Exercise 14B

Example 7

1 A particle moves in a line such that the velocity, $\dot{x}$ m/s, is given by $\dot{x} = \frac{1}{2x-4}$, $x > 2$. If $x = 3$ when $t = 0$, determine:

a the position at 24 seconds

b the distance travelled in the first 24 seconds.

CF

2 A particle moves in a straight line such that its velocity, v m/s, and position, x m, are related by $v = 1 + e^{-2x}$.

a Determine x in terms of time t seconds ($t \geq 0$), given that $x = 0$ when $t = 0$.

b Hence determine the acceleration when $t = \ln 5$.

Example 8

3 An object moves in a straight line such that its acceleration, a m/s^2, and velocity, v m/s, are related by $a = 3 + v$. If the object is initially at rest at the origin, determine:

a v in term of t

b a in terms of t

c x in terms of t

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CF

4 An object falls from rest with acceleration, a m/s^2, given by $a = g - kv$, $k > 0$. Determine:

a an expression for the velocity, v m/s, at time t seconds

b the terminal velocity, i.e. the limiting velocity as $t \to \infty$.

Example 9

5 A body is projected along a horizontal surface. Its deceleration is $0.3(v^2 + 1)$, where v m/s is the velocity of the body at time t seconds. If the initial velocity is $\sqrt{3}$ m/s, determine:

a an expression for v in terms of t

b an expression for x m, the displacement of the body from its original position, in terms of t.

6 The velocity, v m/s, and acceleration, a m/s^2, of an object t seconds after it is dropped from rest are related by $a = \dfrac{450 - v}{50}$ for $v < 450$. Express v in terms of t.

7 The brakes are applied in a car travelling in a straight line. The acceleration, a m/s^2, of the car is given by $a = -0.4\sqrt{225 - v^2}$. If the initial velocity of the car was 12 m/s, determine an expression for v, the velocity of the car, in terms of t, the time after the brakes were first applied.

8 An object moves in a straight line such that its velocity is directly proportional to x m, its position relative to a fixed point O on the line. The object starts 5 m to the right of O with a velocity of 2 m/s.

a Express x in terms of t, where t is the time after the motion starts.

b Determine the position of the object after 10 seconds.

9 The velocity, v m/s, and the acceleration, a m/s^2, of an object t seconds after it is dropped from rest are related by the equation $a = \dfrac{1}{50}(500 - v)$, $0 \le v < 500$.

a Express t in terms of v.

b Express v in terms of t.

10 A particle is travelling in a horizontal straight line. The initial velocity of the particle is u and the acceleration is given by $-k(2u - v)$, where v is the velocity of the particle at any instant and k is a positive constant. Determine the time taken for the particle to come to rest.

11 A boat is moving at 8 m/s. When the boat's engine stops, its acceleration is given by $\dfrac{dv}{dt} = -\dfrac{1}{5}v$. Express v in terms of t and determine the velocity when $t = 4$.

12 A particle, initially at a point O, slows down under the influence of an acceleration, a m/s^2, such that $a = -kv^2$, where v m/s is the velocity of the particle at any instant. Its initial velocity is 30 m/s and its initial acceleration is -20 m/s^2. Determine:

a its velocity at time t seconds

b its position relative to the point O when $t = 10$.

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14C Other expressions for acceleration

Learning intentions

- To be able to solve problems involving two new expressions for acceleration.

In the earlier sections of this chapter, we have written acceleration as $\dfrac{dv}{dt}$ and $\dfrac{d^2x}{dt^2}$. In this section, we use two further expressions for acceleration.

Expressions for acceleration

$$a = v\frac{dv}{dx} \quad \text{and} \quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Proof Using the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\,\frac{dx}{dt} = \frac{dv}{dx}\,v$$

Using the chain rule again:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} = v\,\frac{dv}{dx} = a$$

The different expressions for acceleration are useful in different situations:

Given	Initial conditions	Useful form
$a = f(t)$	in terms of t and v	$a = \dfrac{dv}{dt}$
$a = f(v)$	in terms of t and v	$a = \dfrac{dv}{dt}$
$a = f(v)$	in terms of x and v	$a = v\dfrac{dv}{dx}$
$a = f(x)$	in terms of x and v	$a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

Note: In the last case, it is also possible to use $a = v\dfrac{dv}{dx}$ and separation of variables.

Example 10

An object travels in a line such that the velocity, v m/s, is given by $v^2 = 4 - x^2$. Determine the acceleration at $x = 1$.

Solution

Given $v^2 = 4 - x^2$, we can use implicit differentiation to obtain:

$$\frac{d}{dx}(v^2) = \frac{d}{dx}(4 - x^2)$$

$$2v\frac{dv}{dx} = -2x$$

$$a = -x$$

So, at $x = 1$, $a = -1$. The acceleration at $x = 1$ is -1 m/s^2.

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Example 11

An object moves in a line so that the acceleration, $\ddot{x}$ m/s^2, is given by $\ddot{x} = 1 + v$. Its velocity at the origin is 1 m/s. Determine the position of the object when its velocity is 2 m/s.

Solution

Since we are given a as a function of v and initial conditions involving x and v, it is appropriate to use the form $a = v\dfrac{dv}{dx}$.

Now
$$\ddot{x} = 1 + v$$
$$v\frac{dv}{dx} = 1 + v$$
$$\int 1\,dx = \int \frac{1+v}{v}\,dv$$
$$\therefore \quad x = \int \frac{v}{1+v}\,dv$$
$$= \int 1 - \frac{1}{1+v}\,dv$$
$$\therefore \quad x = v - \ln|1+v| + c$$

Since $v = 1$ when $x = 0$, we have
$$0 = 1 - \ln 2 + c$$
$$\therefore \quad c = \ln 2 - 1$$

Hence
$$x = v - \ln|1+v| + \ln 2 - 1$$
$$= v + \ln\left(\frac{2}{1+v}\right) - 1 \qquad (\text{as } v > 0)$$

Now, when $v = 2$,
$$x = 2 + \ln\left(\tfrac{2}{3}\right) - 1$$
$$= 1 + \ln\left(\tfrac{2}{3}\right)$$
$$\approx 0.59$$

So, when the velocity is 2 m/s, the position is 0.59 m.

Example 12

A particle is moving in a straight line. Its acceleration, a m/s^2, is described by $a = -\sqrt{x}$, where x m is its position with respect to an origin O. Determine a relation between v and x which describes the motion, given that $v = 2$ m/s when the particle is at the origin.

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Solution

Given $\qquad a = -\sqrt{x}$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -x^{\frac{1}{2}}$$

$$\frac{1}{2}v^2 = -\frac{2}{3}x^{\frac{3}{2}} + c$$

When $x = 0$, $v = 2$, and therefore $c = 2$.

Thus $\qquad \frac{1}{2}v^2 = 2 - \frac{2}{3}x^{\frac{3}{2}}$

$\therefore \qquad v^2 = \frac{4}{3}\left(3 - x^{\frac{3}{2}}\right)$

Note: This problem can also be solved using $a = v\,\dfrac{dv}{dx}$ and separation of variables.

Example 13

An object falls from a hovering helicopter over the ocean 1000 m above sea level. Determine the velocity of the object when it hits the water:

a neglecting air resistance

b assuming due to air resistance the acceleration is $-g + 0.2v^2$.

Solution

a An appropriate starting point is $\ddot{y} = -9.8$.

Since the initial conditions involve y and v, use $\ddot{y} = \dfrac{d}{dy}\left(\dfrac{1}{2}v^2\right)$.

Now $\qquad \dfrac{d}{dy}\left(\dfrac{1}{2}v^2\right) = -9.8$

$$\frac{1}{2}v^2 = -9.8y + c$$

Using $v = 0$ at $y = 1000$ gives

$$0 = -9.8 \times 1000 + c$$

$\therefore \qquad c = 9800$

Hence $\qquad \frac{1}{2}v^2 = -9.8y + 9800$

$\therefore \qquad v^2 = -19.6y + 19\,600$

The object is falling, so $v < 0$.

$$v = -\sqrt{19\,600 - 19.6y}$$

At sea level, $y = 0$ and therefore

$$v = -\sqrt{19\,600} = -140$$

The object has a velocity of -140 m/s at sea level (504 km/h).

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b In this case, we have

$$\begin{aligned}\ddot{y} &= -9.8 + 0.2v^2\\ &= \frac{v^2 - 49}{5}\end{aligned}$$

Because of the initial conditions given, use $\ddot{y} = v\dfrac{dv}{dy}$:

$$\begin{aligned} v\frac{dv}{dy} &= \frac{v^2 - 49}{5}\\ \frac{dv}{dy} &= \frac{v^2 - 49}{5v}\\ y &= \int \frac{5v}{v^2 - 49}\, dv\\ &= \frac{5}{2}\int \frac{2v}{v^2 - 49}\, dv\\ \therefore \quad y &= \frac{5}{2}\ln|v^2 - 49| + c\end{aligned}$$

Now, when $v = 0$, $y = 1000$, and so $c = 1000 - \dfrac{5}{2}\ln 49$.

$$\begin{aligned}\therefore \quad y &= \frac{5}{2}\ln|49 - v^2| + 1000 - \frac{5}{2}\ln 49\\ &= \frac{5}{2}\left(\ln|49 - v^2| - \ln 49\right) + 1000\\ &= \frac{5}{2}\ln\left|\frac{49 - v^2}{49}\right| + 1000\end{aligned}$$

Assume that $-7 < v < 7$. Then

$$\begin{aligned} y - 1000 &= \frac{5}{2}\ln\left(1 - \frac{v^2}{49}\right)\\ \frac{2}{5}(y - 1000) &= \ln\left(1 - \frac{v^2}{49}\right)\\ e^{\frac{2}{5}(y-1000)} &= 1 - \frac{v^2}{49}\\ \therefore \quad v^2 &= 49\left(1 - e^{\frac{2}{5}(y-1000)}\right)\end{aligned}$$

But the object is falling and thus $v < 0$. Therefore

$$v = -7\sqrt{1 - e^{\frac{2}{5}(y-1000)}}$$

At sea level, $y = 0$ and therefore

$$v = -7\sqrt{1 - e^{-400}}$$

The object has a velocity of approximately -7 m/s at sea level (25.2 km/h).

Note: If $v < -7$, then $v^2 = 49\left(1 + e^{\frac{2}{5}(y-1000)}\right)$ and the initial conditions are not satisfied.

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Skill-sheet

Exercise 14C

SF

Example 10

1 An object travels in a line such that the velocity, v m/s, is given by $v^2 = 9 - x^2$. Determine the acceleration at $x = 2$.

Example 11, 12

2 For each of the following, a particle moves in a horizontal line such that, at time t seconds, the position is x m, the velocity is v m/s and the acceleration is a m/s^2.

   a If $a = -x$ and $v = 0$ at $x = 4$, determine v at $x = 0$.
   b If $a = 2 - v$ and $v = 0$ when $t = 0$, determine t when $v = -2$.
   c If $a = 2 - v$ and $v = 0$ when $x = 0$, determine x when $v = -2$.

3 The motion of a particle is in a horizontal line such that, at time t seconds, the position is x m, the velocity is v m/s and the acceleration is a m/s^2.

   a If $a = -v^3$ and $v = 1$ when $x = 0$, determine v in terms of x.
   b If $v = x + 1$ and $x = 0$ when $t = 0$, determine:
      i x in terms of t ii a in terms of t iii a in terms of v.

CF

4 An object is projected vertically upwards from the ground with an initial velocity of 100 m/s. Assuming that the acceleration, a m/s^2, is given by $a = -g - 0.2v^2$, determine x in terms of v. Hence determine the maximum height reached.

5 The velocity, v m/s, of a particle moving along a line is given by $v = 2\sqrt{1 - x^2}$. Determine:

   a the position, x m, in terms of time t seconds, given that when $t = 0$, $x = 1$
   b the acceleration, a m/s^2, in terms of x.

6 Each of the following gives the acceleration, a m/s^2, of an object travelling in a line. Given that $v = 0$ and $x = 0$ when $t = 0$, solve for v in each case.

   a $a = \dfrac{1}{1 + t}$ b $a = \dfrac{1}{1 + x}$, $x > -1$ c $a = \dfrac{1}{1 + v}$

7 A particle moves in a straight line from a position of rest at a fixed origin O. Its velocity is v when its displacement from O is x. If its acceleration is $(2 + x)^{-2}$, determine v in terms of x.

8 A particle moves in a straight line and, at time t, its position relative to a fixed origin is x and its velocity is v.

   a If its acceleration is $1 + 2x$ and $v = 2$ when $x = 0$, determine v when $x = 2$.
   b If its acceleration is $2 - v$ and $v = 0$ when $x = 0$, determine the position at which $v = 1$.

Example 13

9 A particle is projected vertically upwards. The speed of projection is 50 m/s. The acceleration of the particle, a m/s^2, is given by $a = -\frac{1}{5}(v^2 + 50)$, where v m/s is the velocity of the particle when it is x m above the point of projection. Determine:

   a the height reached by the particle b the time taken to reach this highest point.

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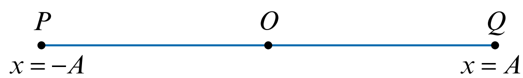
14D Simple harmonic motion

Learning intentions

- To be able to solve problems involving simple harmonic motion.

Simple harmonic motion occurs when a particle is moving along a straight line such that its acceleration is always directed towards a fixed point O on the line and directly proportional to its distance from O. The particle will oscillate about O between two points P and Q.



Situations that can be modelled using simple harmonic motion include the oscillation of a spring, the motion of a simple pendulum and molecular vibration.

Equations of simple harmonic motion

For a particle in simple harmonic motion about the origin O, let x be its position with respect to O at time t. The particle's motion is described by the following equations, where A, ω and α are constants with $A > 0$ and $\omega > 0$:

Acceleration $\ddot{x} = -\omega^2 x$

Velocity $v^2 = \omega^2(A^2 - x^2)$

Position $x = A\sin(\omega t + \alpha)$

We can easily verify that the position function $x = A\sin(\omega t + \alpha)$ is a solution of the differential equation $\ddot{x} = -\omega^2 x$, since we obtain

$$\dot{x} = \omega A\cos(\omega t + \alpha)$$

$$\ddot{x} = -\omega^2 A\sin(\omega t + \alpha) = -\omega^2 x$$

It is more difficult to show that it is the general solution, but we can give an indication as to why this is the case. Starting from $\ddot{x} = -\omega^2 x$, we can write

$$\therefore \quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\omega^2 x$$

$$\therefore \quad \frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$$

The particle will be at rest when $x = A$ or $x = -A$. So $c = \frac{1}{2}\omega^2 A^2$ and we have

$$v^2 = \omega^2(A^2 - x^2)$$

$$\therefore \quad v = \pm\omega\sqrt{A^2 - x^2}$$

For simplicity, we only consider the case where $v > 0$. Then we have

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

$$\frac{dt}{dx} = \frac{1}{\omega\sqrt{A^2 - x^2}}$$

$$\therefore \quad t = \frac{1}{\omega}\arcsin\left(\frac{x}{A}\right) + d$$

This gives $x = A\sin(\omega t + \alpha)$, where $\alpha = -\omega d$.

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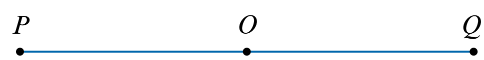
Properties of simple harmonic motion

Consider a particle in simple harmonic motion, with its position x at time t given by $x = A\sin(\omega t + \alpha)$, where $A > 0$ and $\omega > 0$.

Amplitude	A	the particle's maximum distance from the centre of motion
Period	$T = \dfrac{2\pi}{\omega}$	the time taken for one complete cycle
Frequency	$f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$	the number of cycles per unit of time

- The constant ω is the **angular frequency**.
- The constant α is the **phase shift**, and depends on the initial position of the particle. If $x = 0$ when $t = 0$, then we can take $\alpha = 0$.
- Since $\dot{x} = \omega A\cos(\omega t + \alpha)$, the maximum speed of the particle is ωA.
- Since $\ddot{x} = -\omega^2 A\sin(\omega t + \alpha)$, the maximum magnitude of the acceleration is $\omega^2 A$.



	At point P	At point O	At point Q
Position	$x = -A$	$x = 0$	$x = A$
Velocity	$\dot{x} = 0$	$\dot{x} = \pm\omega A$	$\dot{x} = 0$
Acceleration	$\ddot{x} = \omega^2 A$	$\ddot{x} = 0$	$\ddot{x} = -\omega^2 A$

Note: From the velocity equation $v^2 = \omega^2(A^2 - x^2)$, we can see that the speed of the particle is determined by its position. But its velocity is not determined by its position (except at the endpoints), as the particle may be travelling in either direction.

Example 14

A particle is moving in a straight line with simple harmonic motion. Relative to an origin O, its position, x cm, at time t seconds is given by

$$x = 4\sin\left(\frac{2\pi t}{3}\right)$$

Determine:

a the particle's velocity at time t

b the particle's acceleration at time t

c the period of the motion

d the particle's maximum speed.

Solution

a $v = \dfrac{dx}{dt} = \dfrac{8\pi}{3}\cos\left(\dfrac{2\pi t}{3}\right)$

b $a = \dfrac{dv}{dt} = -\dfrac{16\pi^2}{9}\sin\left(\dfrac{2\pi t}{3}\right)$

c $T = \dfrac{2\pi}{\omega} = 2\pi \times \dfrac{3}{2\pi} = 3$

The period is 3 seconds.

d The maximum value of $|v|$ is $\dfrac{8\pi}{3}$.

So the maximum speed is $\dfrac{8\pi}{3}$ cm/s.

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Example 15

A particle moves in a straight line with acceleration given by $\ddot{x} = -9x$, where x is the position of the particle at time t. The particle's initial position and velocity are $x(0) = 0$ and $\dot{x}(0) = 4$. Determine:

a the period and amplitude of the motion

b the particle's position, velocity and acceleration at time t.

Solution

a In this case, we have $\omega^2 = 9$.

So $\omega = 3$ and the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3}$$

To determine the amplitude, we use

$$v^2 = \omega^2(A^2 - x^2)$$

Since $v(0) = 4$ and $x(0) = 0$:

$$16 = 9(A^2 - 0)$$

$$\frac{16}{9} = A^2$$

The amplitude is $A = \frac{4}{3}$.

b We know that

$$\begin{aligned} x &= A\sin(\omega t + \alpha) \\ &= \frac{4}{3}\sin(3t + \alpha) \end{aligned}$$

Since $x(0) = 0$, we can take $\alpha = 0$. So

$$x = \frac{4}{3}\sin(3t)$$

$$\therefore \quad \dot{x} = 4\cos(3t)$$

$$\therefore \quad \ddot{x} = -12\sin(3t)$$

Simple harmonic motion about a point other than the origin

If the centre of motion is at $x = c$, then the equations of simple harmonic motion are:

$$\begin{aligned} \ddot{x} &= -\omega^2(x - c) \\ v^2 &= \omega^2(A^2 - (x - c)^2) \\ x &= c + A\sin(\omega t + \alpha) \end{aligned}$$

Example 16

A particle moves in a straight line such that $v^2 = -(x^2 - 6bx + 5b^2)$, where $b > 0$. Prove that the motion is simple harmonic and determine the period, amplitude and maximum speed.

Solution

$$\begin{aligned} \frac{1}{2}v^2 &= -\frac{1}{2}(x^2 - 6bx + 5b^2) \\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dx}\left(-\frac{1}{2}(x^2 - 6bx + 5b^2)\right) \\ \ddot{x} &= -\frac{1}{2}(2x - 6b) \\ \therefore \qquad \ddot{x} &= -(x - 3b) \end{aligned}$$

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The motion is simple harmonic with $\omega^2 = 1$ and $c = 3b$. The period is 2π.

When $v = 0$:

$$x^2 - 6bx + 5b^2 = 0$$

$$(x - 5b)(x - b) = 0$$

$$\therefore \quad x = 5b \text{ or } x = b$$

The motion takes place between $x = b$ and $x = 5b$. The centre of motion is at $x = 3b$. Therefore the amplitude is $A = 2b$ and the maximum speed is $\omega A = 2b$.

Exercise 14D

CF

Example 14

1 A particle is moving in a straight line such that its position, x metres, at time t seconds is given by $x = 0.5\sin(2\pi t)$. Determine:

a the amplitude of the motion
b the period of oscillation
c the maximum speed
d the maximum acceleration.

Example 15

2 A particle moves in a straight line with acceleration given by $\ddot{x} = -25x$, where x is the position of the particle at time t. The particle's initial position and velocity are $x(0) = 4$ and $\dot{x}(0) = 0$. Determine:

a the period and amplitude of the motion
b the particle's position, velocity and acceleration at time t.

3 A body is oscillating in simple harmonic motion with a frequency of 15 cycles per second. The maximum acceleration of the body is 20 m/s^2. Determine:

a the maximum speed
b the amplitude of the motion
c the average speed for one oscillation.

4 A body is oscillating in simple harmonic motion with an amplitude of 1.7 m. After travelling 0.1 m from a position of rest, the body has a speed of 2 m/s. How much further does the body travel before its speed first reaches 4 m/s? What is the greatest speed that it achieves?

5 A particle moves in a straight line with acceleration given by $\ddot{x} = -\omega^2 x$, where x is its position relative to the origin O. The particle is initially moving towards O. When it is 4 m from O, its speed is 20 m/s and the magnitude of its acceleration is $\frac{20}{3}$ m/s^2. At what distance from O did it start from rest?

6 An object moves in simple harmonic motion with amplitude 2 m and period π seconds. Give an expression to describe the position of the object, x m, relative to the origin at time t seconds from the beginning of motion.

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CF

7 A particle is moving with simple harmonic motion such that its position, x metres, at time t seconds is given by $x = 0.2\cos(4\pi t)$. Calculate:

a the amplitude of the motion
b the period of the motion.

8 A particle moving in simple harmonic motion completes 120 oscillations per minute. Its greatest speed is 5 cm/s. Determine the maximum magnitude of the acceleration.

9 A particle is moving in simple harmonic motion with a period of $\frac{\pi}{12}$ s and an amplitude of 5 cm. Determine the speed of the particle when it is 3 cm from the centre of its motion.

10 A particle is moving along a straight line. Relative to an origin O, its position, x cm, at time t seconds is given by $x = 2\sin\left(\frac{\pi t}{6}\right)$. Determine:

a the amplitude of the motion
b the period of the motion
c the maximum speed of the particle
d the maximum magnitude of the acceleration.

11 A ride on a ferris wheel lasts for 5 minutes. The height, h m, of a particular passenger's foot above the centre is given by $h = 15\sin(10t)^\circ$ at time t seconds from the beginning of the ride. On how many occasions is the passenger's foot 10 metres above the centre during the first minute of the ride? Determine the corresponding values of t.

12 A particle moving in a straight line has acceleration given by $\ddot{x} = -9x$, where x m is the position of the particle at time t seconds. If $x = 5$ and $v = 0$ when $t = 0$, determine:

a the period of the motion
b the maximum speed
c the velocity at time t
d the position at time t.

13 A body moving in simple harmonic motion has position, x cm, at time t seconds given by $x = 10\cos\left(\frac{\pi t}{12}\right)$. Determine:

a the velocity at time t
b the acceleration at time t
c the period of the motion
d the maximum speed
e the maximum magnitude of the acceleration.

14 A particle in simple harmonic motion has position, x cm, at time t seconds given by

$$x = 8\sin\left(\frac{\pi t}{5}\right) \quad \text{for } 0 \le t \le 7$$

Determine the values of t when:

a i $x = 8$ ii $x = 0$ iii $x = -8$ iv $x = 4$
b $|\dot{x}|$ is a maximum
c i $\dot{x} = \frac{4\pi}{5}$ ii $\dot{x} = -\frac{4\pi}{5}$

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SF

15 Given that $\ddot{x} = -9x$, determine x as a function of t for each of the following cases:

a $x = 5$ and $\dot{x} = 0$ when $t = 0$

b $x = 0$ and $\dot{x} = 10$ when $t = 0$

c $x = 5$ and $\dot{x} = 5$ when $t = 0$.

Example 16

CF

16 For a particle moving in a straight line, the velocity v satisfies $v^2 = 36 - 6x - 2x^2$, where x is the position of the particle relative to a fixed point O.

a Show that the motion is simple harmonic motion.

b **i** Determine the period.

ii Determine the amplitude.

iii Determine the maximum speed.

17 A particle is moving in a straight line. At time t seconds, its position, x cm, relative to a fixed point O is given by

$$x = 5\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$$

a Determine the velocity and acceleration in terms of t.

b Determine the velocity and acceleration in terms of x.

c Determine:

i the period of the motion

ii the amplitude of the motion

iii the speed when $x = -\frac{5}{2}$

iv the acceleration when $t = 0$.

18 The position at time t of a particle moving in a straight line is given by $x = 5 - 4\cos^2 t$.

a Show that the motion is simple harmonic motion.

b **i** Determine the centre of motion.

ii Determine the amplitude.

iii Determine the period.

19 A particle is moving in simple harmonic motion with $\ddot{x} = -9x + 9$.

a Determine x at time t, given that $x = 0$ and $\dot{x} = 3$ when $t = 0$.

b Determine the period and amplitude of the motion.

20 For a particle moving in a straight line, the velocity v satisfies $v^2 = 96 + 64x - 32x^2$, where x is the position of the particle relative to a fixed point O.

a Show that the motion is simple harmonic motion.

b **i** Determine the centre of motion.

ii Determine the amplitude.

iii Determine the period.

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14E Force

Learning intentions

- To understand and use the definitions of:
  - mass
  - weight
  - force
  - resultant force

We now begin our study of dynamics, which involves modelling motion caused by forces. **Force** is a word in common usage, and most people have an intuitive idea of its meaning. When a piano or some other object is pushed across the floor, this is done by exerting some force on the piano. A body falls because of the gravitational force exerted on it by the Earth. We can say that force is a vector quantity which is the influence that causes an object to accelerate

We consider different types of forces in the next section. We start with a discussion of some key concepts for the study of dynamics.

Particle model

Throughout this chapter, we use a **particle model**. This means that an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

Measurements

The mks system

The description of motion is dependent on the measurement of length, mass and time. For the rest of this chapter, the principal unit of:

- length will be the metre
- mass will be the kilogram
- time will be the second.

Other units will occur, but it is often advisable to convert these to metres, kilograms and seconds. This system of units is called the mks system.

Note: The **mass** of an object is the amount of matter that it contains. The measurement of the mass of an object does not depend on its position. In mathematics, the terms *mass* and *weight* do not have the same meaning.

Vector and scalar quantities

Throughout this chapter, we have been using vector and scalar quantities:

Vector quantities Position, displacement, velocity and acceleration must be specified by both magnitude and direction.

Scalar quantities Length, mass and time are specified by their magnitude only.

Units of force

One unit of force is the **kilogram weight** (kg wt). If an object on the surface of the Earth has a mass of 1 kg, then the gravitational force acting on the object is 1 kg wt.

The standard unit of force is the **newton** (N). The conversion is 1 kg wt = g N, where g m/s^2 is the acceleration due to gravity. The significance of this unit will be discussed in the next section.

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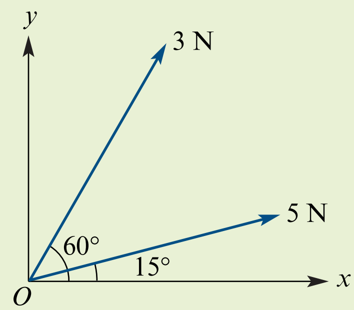
Resultant force

Force is a vector quantity. In this chapter, we represent forces as vectors in two dimensions.

The vector sum of the forces acting at a point is called the **resultant force**.

Example 17

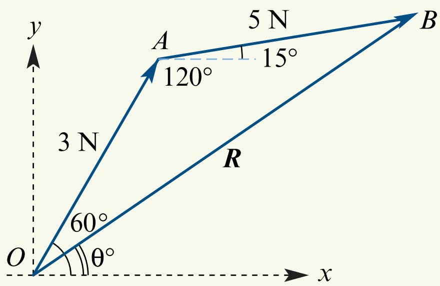
Determine the magnitude and direction of the resultant force of the forces 3 N and 5 N acting on a particle at O as shown in this diagram.



Solution

Method 1: Using trigonometry

The resultant force, $\boldsymbol{R}$, is given by the vector sum. The angle OAB has magnitude 135°.



Using the cosine rule:

$$\begin{aligned}|\boldsymbol{R}|^2 &= 3^2 + 5^2 - 2 \times 3 \times 5 \cos 135^\circ \\ &= 9 + 25 - 30 \cos 135^\circ \\ &= 34 + 30 \times \frac{1}{\sqrt{2}} \\ &= 34 + 15\sqrt{2}\end{aligned}$$

$$\therefore \quad |\boldsymbol{R}| \approx 7.43 \text{ N}$$

The magnitude of the resultant force is 7.43 N, correct to two decimal places.

To describe the direction of the vector, we will determine the angle θ° between the vector and the positive direction of the x-axis.

Let $\angle AOB = (60 - \theta)^\circ$.

Then

$$\begin{aligned}\frac{|\boldsymbol{R}|}{\sin 135^\circ} &= \frac{5}{\sin(60 - \theta)^\circ} \\ \sin(60 - \theta)^\circ &= \frac{5 \sin 135^\circ}{|\boldsymbol{R}|} \\ &= 0.4758\ldots \\ \therefore \quad \theta^\circ &= 31.59^\circ \qquad \text{correct to two decimal places}\end{aligned}$$

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Method 2: Converting to component form

The problem can also be completed by expressing each of the vectors in $\boldsymbol{i}$–$\boldsymbol{j}$ notation.

The vector of magnitude 3 N in component form is

$$3\cos 60^\circ\, \boldsymbol{i} + 3\sin 60^\circ\, \boldsymbol{j}$$

The vector of magnitude 5 N in component form is

$$5\cos 15^\circ\, \boldsymbol{i} + 5\sin 15^\circ\, \boldsymbol{j}$$

The sum is $(6.3296\ldots)\boldsymbol{i} + (3.8921\ldots)\boldsymbol{j}$.

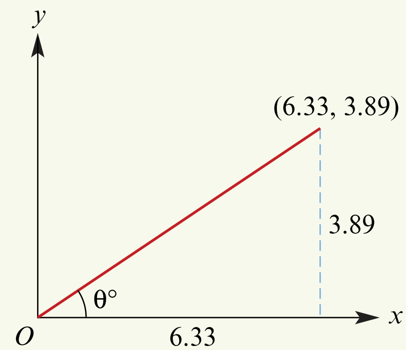
The magnitude of the resultant is 7.43 N, correct to two decimal places.

Determine the direction:

$$\tan\theta^\circ = \frac{3.8921\ldots}{6.3296\ldots}$$

$$\theta = 31.5879\ldots$$

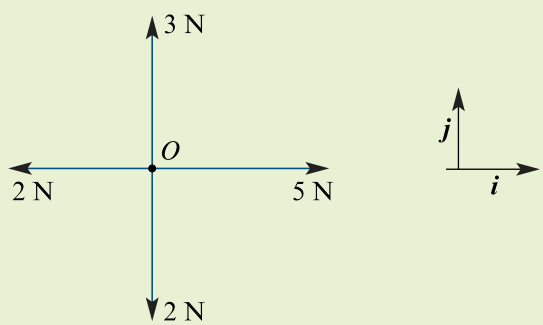
The resultant force is 7.43 N acting in the direction 31.59° anticlockwise from the x-axis.



Example 18

a Four forces are acting on a particle as shown. Express the resultant force in $\boldsymbol{i}$–$\boldsymbol{j}$ form.

b Give the magnitude of the resultant force and the angle that it makes with the $\boldsymbol{i}$-direction.



Solution

a Resultant force $= (5-2)\boldsymbol{i} + (3-2)\boldsymbol{j}$

$= (3\boldsymbol{i} + \boldsymbol{j})$ N

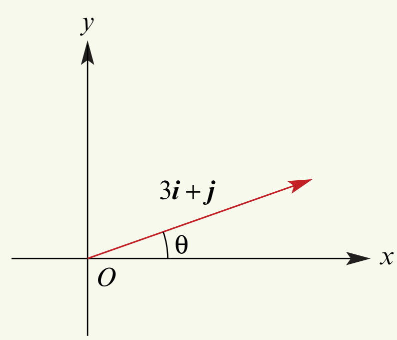
b Magnitude of the force $= \sqrt{3^2 + 1^2}$

$= \sqrt{10}$

The angle with the $\boldsymbol{i}$-direction is given by

$$\tan\theta = \frac{1}{3}$$

$$\therefore \quad \theta = 18.43^\circ$$



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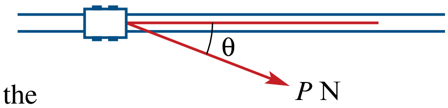
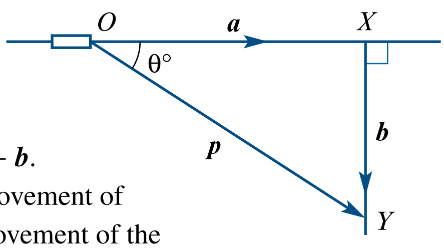
Resolution of a force in a given direction



Consider a model railway trolley, set on smooth straight tracks, pulled by a force of magnitude P N along a horizontal string which makes an angle θ with the direction of the track. (The plan view is shown in the diagram.)

- When $\theta = 0°$, the trolley moves along the track.
- As θ increases, the trolley still moves along the track, but the same force will have a decreasing effect on its motion, i.e. the acceleration of the trolley will be less.
- When $\theta = 90°$, the trolley stays in equilibrium, i.e. if at rest it will not move, unless the force is strong enough to cause it to topple sideways.

A force acting on a body has an influence in all directions except the direction perpendicular to its line of action.



Let the force of P N be represented by the vector $\boldsymbol{p}$.

Let $\boldsymbol{a}$ be the resolute of $\boldsymbol{p}$ in the $\overrightarrow{OX}$ direction and let $\boldsymbol{b}$ be the perpendicular resolute.

From the triangle of vectors, it can be seen that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$.
As the force represented by $\boldsymbol{b}$ does not influence the movement of the trolley along the track, the net effect of $\boldsymbol{P}$ on the movement of the trolley in the direction of the track is $\boldsymbol{a}$. The force represented by $\boldsymbol{a}$ is the resolved part of the force P in the direction of $\overrightarrow{OX}$.

Resolution of a force

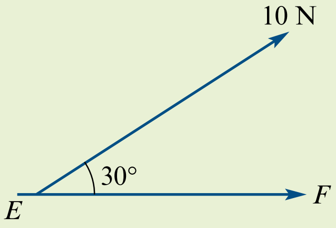
The resolved part of a force of P N in a direction that makes an angle θ with its own line of action is a force of magnitude $P\cos\theta$.

Note: The resolved part is also called the **component** of the force in the given direction.

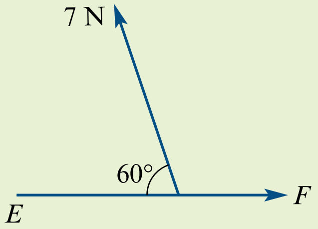
Example 19

Determine the resolved part of each of the following forces in the direction of $\overrightarrow{EF}$:

a



b



Solution

a Resolved part is $10\cos 30° = 5\sqrt{3}$ N.

b Resolved part is $7\cos 120° = -3.5$ N.

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Example 20

Determine the component of the force $\boldsymbol{F} = (3\boldsymbol{i} + 2\boldsymbol{j})$ N in the direction of the vector $2\boldsymbol{i} - \boldsymbol{j}$.

Solution

Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$. The unit vector in the direction of $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{5}}(2\boldsymbol{i} - \boldsymbol{j})$.

Thus $\quad \boldsymbol{F} \cdot \hat{\boldsymbol{a}} = (3\boldsymbol{i} + 2\boldsymbol{j}) \cdot \frac{1}{\sqrt{5}}(2\boldsymbol{i} - \boldsymbol{j}) = \frac{1}{\sqrt{5}}(6 - 2) = \frac{4}{\sqrt{5}}$

and $\quad (\boldsymbol{F} \cdot \hat{\boldsymbol{a}})\,\hat{\boldsymbol{a}} = \frac{4}{\sqrt{5}} \times \frac{1}{\sqrt{5}}(2\boldsymbol{i} - \boldsymbol{j}) = \frac{4}{5}(2\boldsymbol{i} - \boldsymbol{j})$

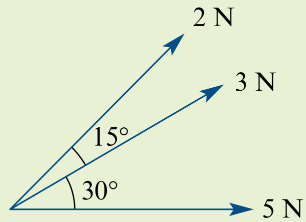
The component of $\boldsymbol{F}$ in the direction of $2\boldsymbol{i} - \boldsymbol{j}$ is $\frac{4}{5}(2\boldsymbol{i} - \boldsymbol{j})$ N.

Example 21

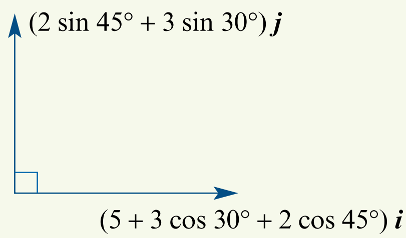
Forces of 5 N, 3 N and 2 N act at a point as shown in the diagram. Determine:

a the magnitude of the resultant of these forces

b the direction of the resultant force with respect to the 5 N force.



Solution



a Let $\boldsymbol{i}$ be in the direction of the 5 N force.

The sum of the resolved parts in the direction of the 5 N force is

$$(5 + 3\cos 30^\circ + 2\cos 45^\circ)\boldsymbol{i}$$
$$= 9.01\boldsymbol{i} \text{ N correct to two decimal places}$$

The sum of the resolved parts in the direction perpendicular to the 5 N force is

$$(2\sin 45^\circ + 3\sin 30^\circ)\boldsymbol{j}$$
$$= 2.91\boldsymbol{j} \text{ N correct to two decimal places}$$

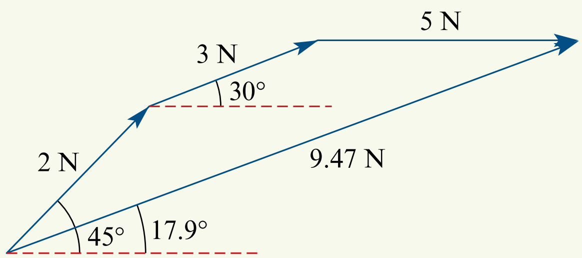
Therefore the magnitude of the resultant force is $\sqrt{9.01^2 + 2.91^2} = 9.47$ N.

b Let θ be the angle that the resultant force makes with the 5 N force. Then

$$\tan\theta = \frac{2.91}{9.01}$$
$$\therefore \quad \theta = 17.9^\circ$$

The resultant force of 9.47 N is inclined at an angle of 17.9° to the 5 N force.

The vector diagram for the resultant is shown here.



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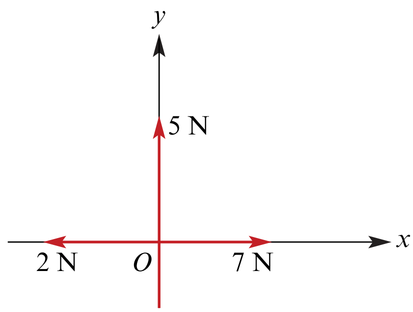
Exercise 14E

Example 18

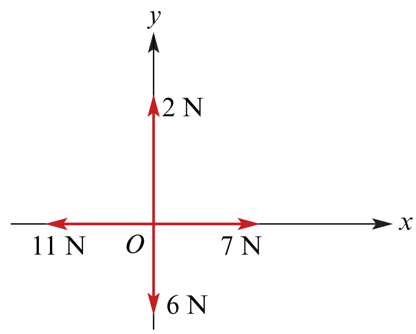
1 Let $\boldsymbol{i}$ be the unit vector in the positive direction of the x-axis and $\boldsymbol{j}$ be the unit vector in the positive direction of the y-axis. For each of the following, determine:

- **i** the resultant force using $\boldsymbol{i}$–$\boldsymbol{j}$ notation
- **ii** the magnitude and direction of the resultant force. (The angle is measured anticlockwise from the $\boldsymbol{i}$-direction.)

a



b



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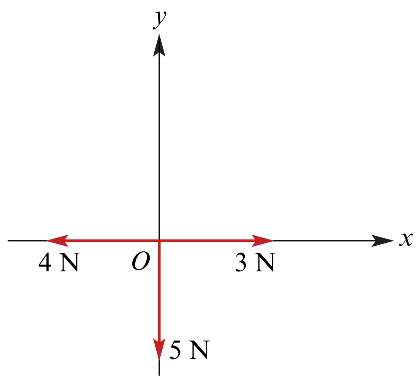

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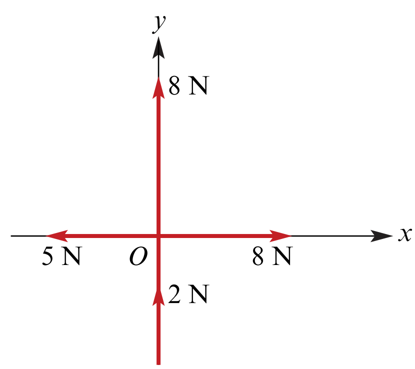
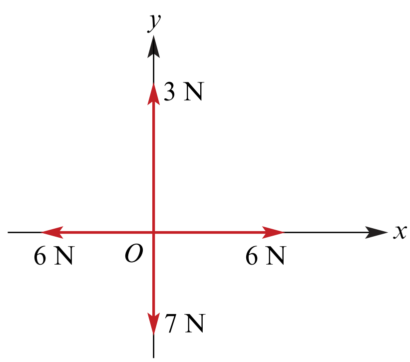
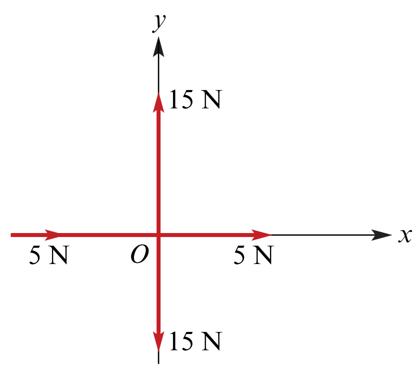


e



f



2 The forces $\boldsymbol{F}_1 = (3\boldsymbol{i} + 2\boldsymbol{j})$ N, $\boldsymbol{F}_2 = (6\boldsymbol{i} - 4\boldsymbol{j})$ N and $\boldsymbol{F}_3 = (2\boldsymbol{i} - \boldsymbol{j})$ N act on a particle. Determine the resultant force acting on the particle.

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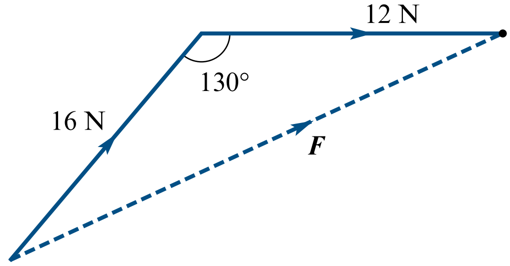
Example 17

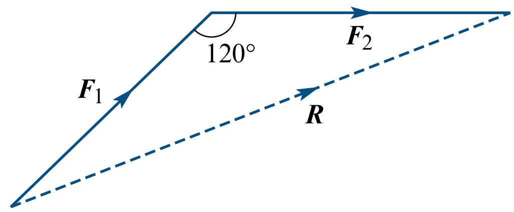
3 Determine the magnitude of $\boldsymbol{F}$, the resultant force of the 16 N and 12 N forces.



4 $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$

$|\boldsymbol{R}| = 16$ N and $|\boldsymbol{F}_1| = 9$ N

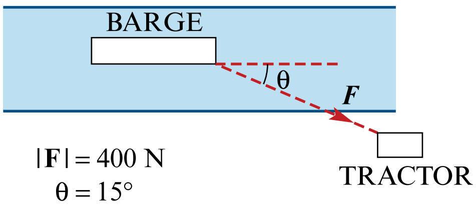
Determine $|\boldsymbol{F}_2|$.



5 $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{F}$ and $\boldsymbol{F} = (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$ N, $\boldsymbol{F}_1 = (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ N and $\boldsymbol{F}_2 = (3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ N. Determine $\boldsymbol{F}_3$.

Example 19

6 A tractor is pulling a barge along a canal with a force of 400 N. The barge is moving parallel to the bank. Determine the component of $\boldsymbol{F}$ in the direction of motion.

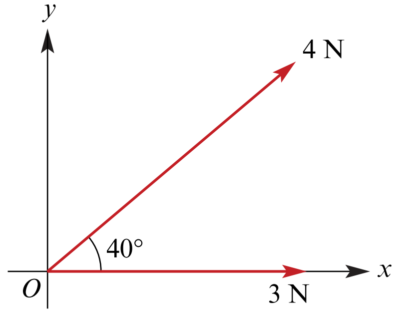


7 Let $\boldsymbol{i}$ be the unit vector in the positive direction of the x-axis and $\boldsymbol{j}$ be the unit vector in the positive direction of the y-axis. For each of the following, determine:

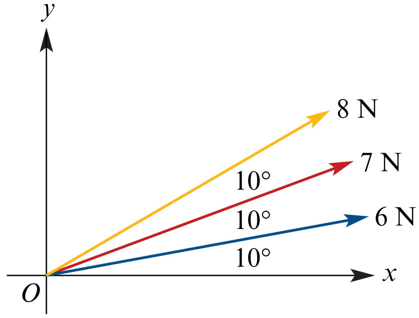
i the resultant force using $\boldsymbol{i}$–$\boldsymbol{j}$ notation

ii the magnitude and direction of the resultant force. (The angle is measured anticlockwise from the $\boldsymbol{i}$-direction.)

a



b



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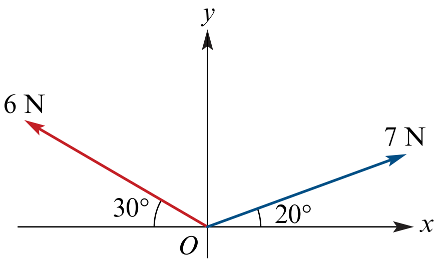
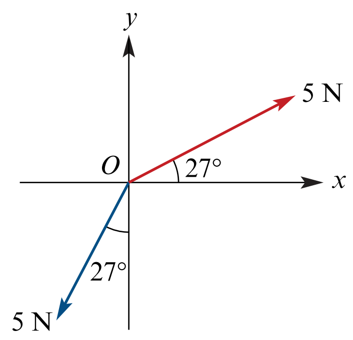
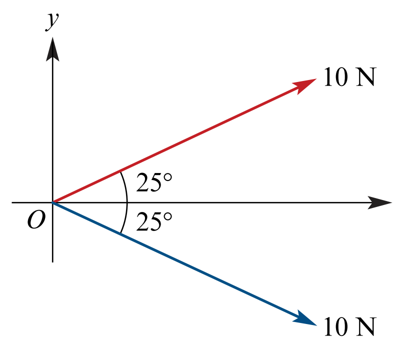
c



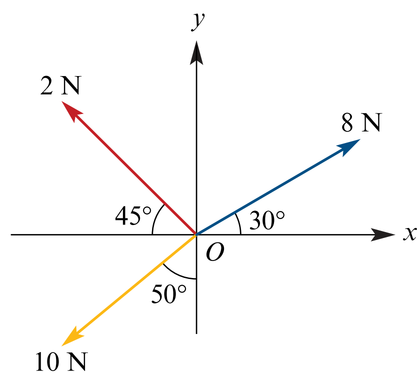
d



e



f

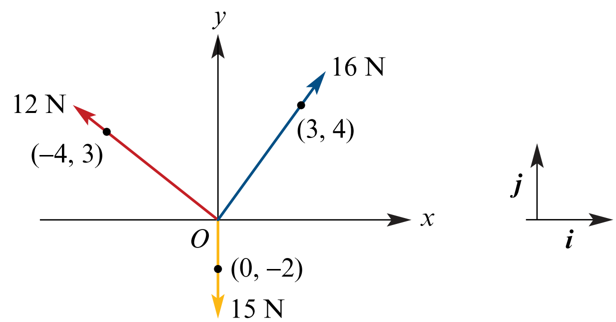


8 For each of the diagrams **a**, **c** and **e** of Question 7, determine the resultant force using a triangle of forces.

9 Three forces are acting at the origin in the directions of the coordinate points as shown in the diagram.

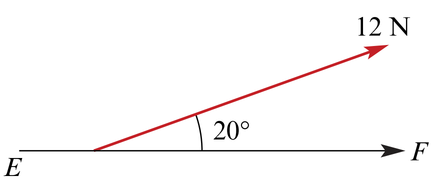
a Determine the resultant force.

b Determine the magnitude and direction of the resultant force.

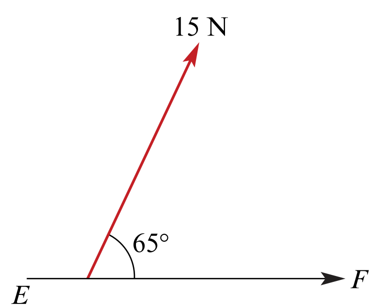


10 Determine the resolved part of each of the following forces in the direction of $\overrightarrow{EF}$:

a



b



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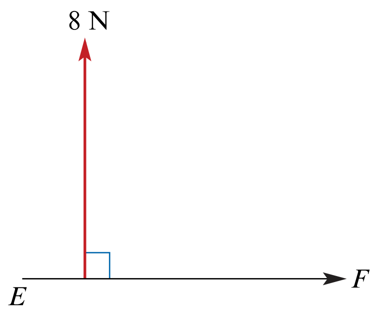
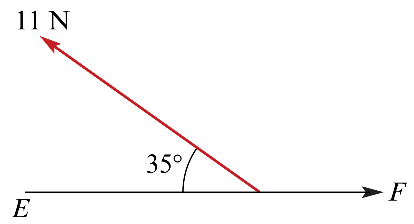
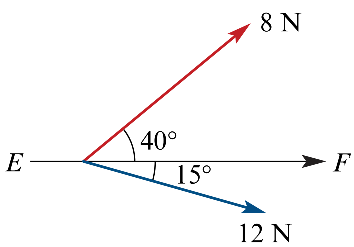
c



d



11 Two forces act on a particle as shown in the diagram.



a Determine the sum of the resolved parts of the forces in the direction of $\overrightarrow{EF}$.

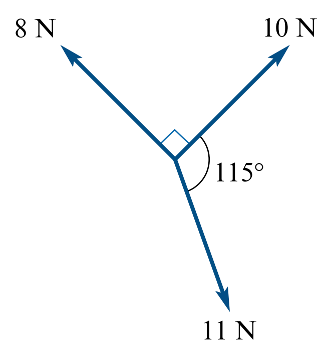
b Determine the sum of the resolved parts of the forces in the direction of the 8 N force.

Example 20

12 **a** Determine the component of the force $(7\boldsymbol{i} + 3\boldsymbol{j})$ N in the direction of the vector $2\boldsymbol{i} - \boldsymbol{j}$.

b Determine the component of the force $(2\boldsymbol{i} - 3\boldsymbol{j})$ N in the direction of the vector $3\boldsymbol{i} + 4\boldsymbol{j}$.

13 Three forces act on a particle as shown in the diagram. Determine the sum of the resolved parts of the forces in the direction of:

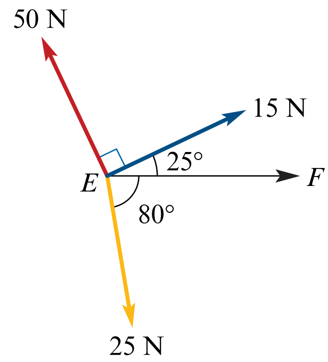


a the 8 N force

b the 10 N force

c the 11 N force.

14 Determine the sum of the resolved parts of the forces in the direction of $\overrightarrow{EF}$ in this diagram.



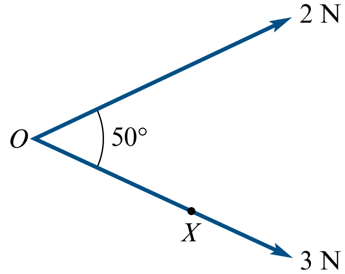
15 A frame is in the shape of a right-angled triangle ABC, where $AB = 6.5$ m, $BC = 6$ m and $AC = 2.5$ m. A force of 10 N acts along $\overrightarrow{BC}$ and a force of 24 N acts along $\overrightarrow{BA}$. Determine the sum of the resolved parts of the two forces in the direction of:

a $\overrightarrow{BC}$ **b** $\overrightarrow{BA}$

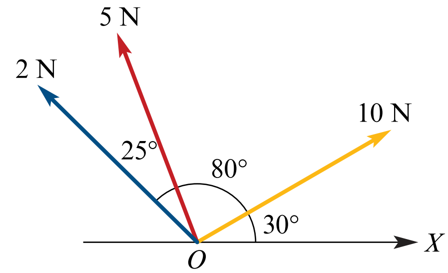
Example 21

16 Determine the magnitude and direction with respect to $\overrightarrow{OX}$ of the resultant of the following forces:

a



b

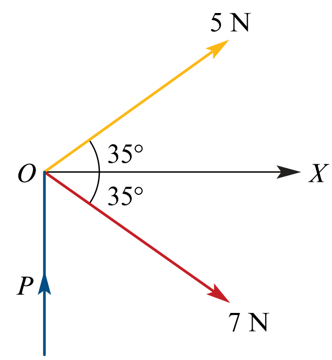


17 Determine the magnitude of the resultant of two forces of 7 N and 10 N acting at an angle of 50° to each other.

18 The angles between the forces of magnitude 8 N, 10 N and P N are 60° and 90° respectively. The resultant acts along the 10 N force. Determine:

a P **b** the magnitude of the resultant.

19 Three forces 5 N, 7 N and P N act on a particle at O. Determine the value of P that will produce a resultant force along $\overrightarrow{OX}$ if the line of action of the P N force is perpendicular to OX.



14F Newton's laws of motion

Learning intentions

To be able to apply Newton's three laws of motion.

Weight

The gravitational force per unit mass due to the Earth is g newtons per kilogram. It varies from place to place on the Earth's surface, having a value of 9.8321 at the poles and 9.7799 at the equator.

In this book, the value 9.8 will be assumed for g, unless otherwise stated.

A mass of m kg on the Earth's surface has a force of m kg wt = mg N acting on it. This force is known as the **weight**.

Momentum

The **momentum** of a particle is defined as the product of its mass and velocity:

momentum = mass × velocity

Let $\boldsymbol{v}$ be the velocity of the particle and m the mass. The momentum, $\boldsymbol{P}$, is a vector quantity. It has the same direction as the velocity:

$$\boldsymbol{P} = m\boldsymbol{v}$$

The units of momentum are kg m/s or kg ms^{-1}.

For example, the momentum of an object of mass 3 kg moving at 2 m/s is 6 kg m/s.

Momentum is the fundamental quantity of motion.

Example 22

a Determine the momentum of a particle of mass 6 kg moving with velocity $(3\boldsymbol{i} + 4\boldsymbol{j})$ m/s.

b Determine the momentum of a 12 kg particle moving with a velocity of 8 m/s in an easterly direction.

Solution

a Momentum = $6(3\boldsymbol{i} + 4\boldsymbol{j})$ kg m/s

b Momentum = 96 kg m/s in an easterly direction

The **change of momentum** is central to Newton's second law of motion. Its importance is introduced through the following example.

Example 23

Determine the change in momentum of a ball of mass 0.5 kg if the velocity changes from 5 m/s to 2 m/s. The ball is moving in the one direction in a straight line.

Solution

Initial momentum = $0.5 \times 5 = 2.5$ kg m/s

Final momentum = $0.5 \times 2 = 1$ kg m/s

Change in momentum = $1 - 2.5 = -1.5$ kg m/s

Newton used this idea of change of momentum to give a formal definition of force. In the example, the resistance force has changed the velocity from 5 m/s to 2 m/s.

We shall see that, in Newton's second law of motion, the rate of change of momentum with respect to time is used to define force.

Newton's three laws of motion

Dynamics is based on Newton's laws of motion, which can be stated as follows.

1 Newton's first law of motion

A particle remains stationary, or in uniform straight-line motion (i.e. in a straight line with constant velocity), unless acted on by some overall external force, i.e. if the resultant force is zero.

2 Newton's second law of motion

A particle acted on by forces whose resultant is not zero will move in such a way that the rate of change of its momentum with respect to time will at any instant be proportional to the resultant force.

3 Newton's third law of motion

If a particle A exerts a force on a second particle B, then B exerts a collinear force of equal magnitude and opposite direction on A.

Implications of Newton's first law of motion

- A force is needed to start an object moving (or to stop it), but once moving the object will continue at a constant velocity without any force being needed.
- If an object is at rest or in uniform straight-line motion, then any forces acting on the object must balance – that is, the resultant force is zero.
- If motion is changing (in speed or direction), then the forces cannot balance – that is, the resultant force is non-zero.

Implications of Newton's second law of motion

Let $\boldsymbol{F}$ represent the resultant force exerted on an object of mass m kg moving at a velocity $\boldsymbol{v}$ m/s in a straight line. Then

$$\boldsymbol{F} = k\frac{d}{dt}(m\boldsymbol{v})$$

Assuming that the mass is a constant:

$$\boldsymbol{F} = km\frac{d}{dt}(\boldsymbol{v}) = km\boldsymbol{a}$$

The newton is the unit of force chosen so that the constant k is equal to 1 when the mass is measured in kilograms and the acceleration in m/s^2. That is, one **newton** is the force which causes a change of momentum of 1 kg m/s per second.

Newton's second law of motion

When measuring force in newtons, mass in kilograms and acceleration in m/s^2, the formula can be written as

$$\boldsymbol{F} = m\boldsymbol{a}$$

Note: The directions of the acceleration and the resultant force are the same.

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Example 24

A stone of mass 16 grams is acted on by a force of 0.6 N. Determine its acceleration?

Solution

First convert to standard units: 16 g = 0.016 kg.

Use the formula $\boldsymbol{F} = m\boldsymbol{a}$:

$$0.6 = 0.016a$$

$$\therefore \quad a = 37.5$$

The acceleration is 37.5 m/s^2.

Note: Force is a vector quantity, but it is often useful to employ only the magnitude of a force in calculations, and the direction is evident from the context.
In the remainder of this chapter, and in particular in diagrams, we often denote the magnitude of a force (for example, $\boldsymbol{F}$) by the same unbolded letter (in this case, F).

Example 25

An ice-hockey puck of mass 150 grams loses speed from 26 m/s to 24 m/s over a distance of 35 m. Determine the uniform force which causes this change in velocity. How much further could the puck travel?

Solution

The retarding force is uniform. Therefore $a = k$, where k is a constant.

Using $a = v\dfrac{dv}{dx}$ and separation of variables, we obtain $\dfrac{1}{2}v^2 = kx + c$.

When $t = 0$, $v = 26$ and $x = 0$. Therefore $c = \dfrac{26^2}{2}$.

When $x = 35$, $v = 24$:

$$\frac{24^2}{2} = 35k + \frac{26^2}{2}$$

$$\therefore \quad k = -\frac{10}{7}$$

Thus the uniform force that is acting is

$$F = ma = 0.15 \times \left(-\frac{10}{7}\right) = -\frac{3}{14}\text{ N}$$

When $v = 0$:

$$kx + c = 0$$

$$\therefore \quad x = \frac{-26^2}{2} \times \left(-\frac{7}{10}\right) = 236.6\text{ m}$$

The puck would travel a further 201.6 m before coming to rest.

Note: Alternatively, we could have used the formula for constant acceleration $v^2 = u^2 + 2as$.

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Example 26

Three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ act on a particle of mass 2 kg, where $\boldsymbol{F}_1 = (2\boldsymbol{i} - 3\boldsymbol{j})$ N and $\boldsymbol{F}_2 = (4\boldsymbol{i} + 2\boldsymbol{j})$ N. The acceleration of the particle is $4\boldsymbol{i}$ m/s^2. Determine $\boldsymbol{F}_3$.

Solution

Newton's second law of motion gives

$$\begin{aligned} \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 &= 2 \times 4\boldsymbol{i} \\ 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{F}_3 &= 8\boldsymbol{i} \\ 6\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{F}_3 &= 8\boldsymbol{i} \\ \therefore \qquad \boldsymbol{F}_3 &= (2\boldsymbol{i} + \boldsymbol{j}) \text{ N} \end{aligned}$$

Implications of Newton's third law of motion

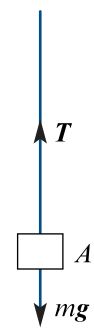
An alternative wording of Newton's third law is:

> If one object exerts a force on another (action force), then the second object exerts a force (reaction force) equal in magnitude but opposite in direction to the first.

It is important to note that the action and reaction forces, which always occur in pairs, act on different objects. If they were to act on the same object, then there would never be accelerated motion, because the resultant force on every object would be zero.

For example:

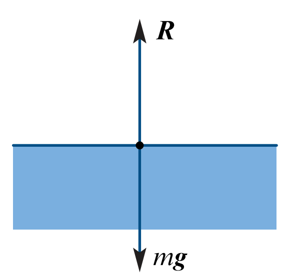
- If a person kicks a door, then the door 'accelerates' open because of the force exerted by the person. At the same time, the door exerts a force on the foot of the person which 'decelerates' the foot.
- For a particle A hanging from a string, the forces $\boldsymbol{T}$ and $m\boldsymbol{g}$ both act on the particle. They are not necessarily equal and opposite forces. In fact, they are equal only if the acceleration of the particle is zero (by Newton's second law). The forces $\boldsymbol{T}$ and $m\boldsymbol{g}$ are not an action–reaction pair of Newton's third law, as they both act on the one particle.
- If a person is pulling horizontally on a rope with a force $\boldsymbol{F}$, then the rope exerts a force of $-\boldsymbol{F}$ on the person.



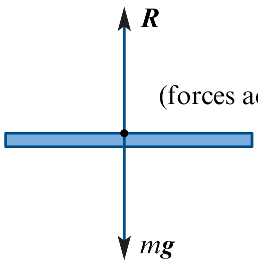
Normal reaction force

If a particle lies on a surface and exerts a force on the surface, then the surface exerts a force, R N, on the particle. If the surface is smooth, this force is taken to act at right angles to the surface and is called the normal reaction force.

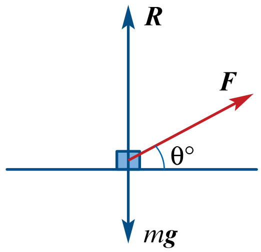
In such a situation, we have $R = mg$.



If the particle is on a platform which is being accelerated upwards at a m/s^2, then $R - mg = ma$.



If a particle of mass m kg lies on a smooth horizontal surface and a force of F N acts at an angle of $\theta°$ to the horizontal, then $R = mg - F\sin\theta°$.

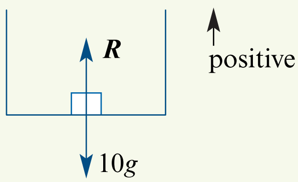


Example 27

A box is on the floor of a lift that is accelerating upwards at 2.5 m/s^2. The mass of the box is 10 kg. Determine the reaction of the floor of the lift on the box.

Solution

Let R be the reaction of the floor on the box.



Newton's second law of motion gives

$$R - 10g = 10 \times 2.5$$

$$\therefore \quad R = 10g + 25$$

$$= 98 + 25$$

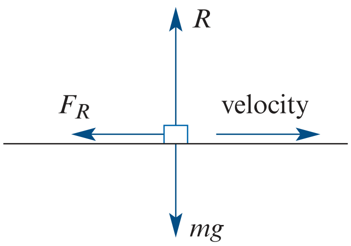
$$= 123 \text{ N}$$

The reaction of the floor of the lift on the box is 123 N.

Friction

For an object moving across a rough surface, there is a resistance force, F_R, due to friction:

- The frictional force acts in the opposite direction to the velocity of the object.
- The magnitude of the frictional force depends on the roughness of the surfaces in contact and on the normal reaction force.



For a smooth surface, we take $F_R = 0$.

Example 28

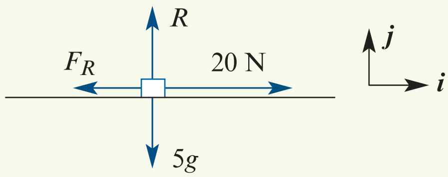
An object of mass 5 kg at rest on a rough horizontal plane is pushed by a horizontal force of 20 N for 5 seconds. The frictional force on the object when it is moving is 16 N.

a How far does the object travel during the first 5 seconds?

b How much further will it move after the pushing force is removed?

Solution

a Use Newton's second law of motion in the $\boldsymbol{i}$-direction (horizontally):



$$(20 - 16)\boldsymbol{i} = 5\boldsymbol{a}$$

$$\boldsymbol{a} = \frac{4}{5}\boldsymbol{i}$$

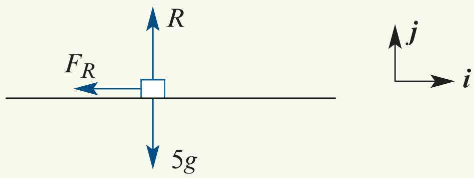
$$\therefore \quad a = 0.8 \text{ m/s}^2$$

After 5 seconds, the velocity is $0.8 \times 5 = 4$ m/s.

The distance travelled is given by

$$s = ut + \tfrac{1}{2}at^2 = \tfrac{1}{2} \times 0.8 \times 5^2$$
$$= 10 \text{ m}$$

b Again use Newton's second law of motion in the $\boldsymbol{i}$-direction:



$$-16\boldsymbol{i} = 5\boldsymbol{a}$$

$$-3.2 = a$$

$$\therefore \quad a = -3.2 \text{ m/s}^2$$

Now use $v^2 = u^2 + 2as$ with $v = 0$:

$$0 = 4^2 - 2 \times 3.2 \times s$$

$$s = \frac{4^2}{2 \times 3.2} = 2.5 \text{ m}$$

The object will come to rest after 2.5 metres.

Exercise 14F

Skill-sheet

Example 22

1 Determine the momentum of each of the following:

a a mass of 2 kg moving with a velocity of 5 m/s

b a mass of 300 g moving with a velocity of 3 cm/s

c a mass of 1 tonne moving with a velocity of 30 km/h

d a mass of 6 kg moving with a velocity of 10 m/s

e a mass of 3 tonnes moving with a velocity of 50 km/h

SF

Example 22 **2** **a** Determine the momentum of a particle of mass 10 kg moving with a velocity of $(\boldsymbol{i} + \boldsymbol{j})$ m/s.

b **i** Determine the momentum of a particle of mass 10 kg moving with a velocity of $(5\boldsymbol{i} + 12\boldsymbol{j})$ m/s.

ii Determine the magnitude of this momentum.

SF

Example 23 **3** Determine the change in momentum when a body of mass 10 kg moving in a straight line changes its velocity from:

a 6 m/s to 3 m/s **b** 6 m/s to 10 m/s **c** −6 m/s to 3 m/s

4 Determine the weight, in newtons, of each of the following:

a a 5 kg bag of potatoes **b** a tractor of mass 3 tonnes

c a tennis ball of mass 60 g

Example 24 **5** **a** A body of mass 8 kg is moving with an acceleration of 4 m/s^2 in a straight line. Determine the resultant force acting on the body.

b A body of mass 10 kg is moving in a straight line. The resultant force acting on the body is 5 N. Determine the magnitude of the acceleration of the body.

6 **a** A force of 10 N acts on a particle of mass m kg and produces an acceleration of 2.5 m/s^2. Determine the value of m.

b A force of F N acts on a particle of 2 kg and produces an acceleration of 3.5 m/s^2. Determine the value of F.

7 What size mass would be accelerated upwards, against gravity, at 1.2 m/s^2 by a vertical force of 96 N?

CF

8 A parachutist of mass 75 kg, whose parachute only partly opens, accelerates downwards at 1 m/s^2. What upwards force must her parachute be providing?

9 In a lift that is accelerating upwards at 2 m/s^2, a spring balance shows the apparent weight of an object to be 2.5 kg wt. What would be the reading if the lift were at rest?

10 An electron of mass 9×10^{-31} kg in a magnetic field has, at a given instant, an acceleration of 6×10^{16} m/s^2. Determine the resultant force on the electron at that instant.

SF

11 A force of $(2\boldsymbol{i} + 10\boldsymbol{j})$ N acts on a body of mass 2 kg. Determine the acceleration of the body.

12 A particle of mass 10 kg is acted on by two forces $(8\boldsymbol{i} + 2\boldsymbol{j})$ N and $(2\boldsymbol{i} - 6\boldsymbol{j})$ N. Determine the acceleration of the particle.

CF

13 In a lift that is accelerating downwards at 1 m/s^2, a spring balance shows the apparent weight of an object to be 2.5 kg wt. What would be the reading if the lift were:

a at rest **b** accelerating upwards at 2 m/s^2?

Example 25 **14** A truck of mass 25 tonnes is travelling at 50 km/h when its brakes are applied. What constant force is required to bring it to rest in 10 seconds?

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Example 26 **15** A particle of mass 16 kg is acted on by three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ in newtons, where $\boldsymbol{F}_1 = -10\boldsymbol{i} - 15\boldsymbol{j}$ and $\boldsymbol{F}_2 = 16\boldsymbol{j}$. If the acceleration of the particle is $0.6\boldsymbol{i}$ m/s^2, determine $\boldsymbol{F}_3$. SF

Example 27 **16** A box of mass 10 kg lies on the horizontal floor of a lift which is accelerating upwards at 1.5 m/s^2. Determine the reaction, in newtons, of the lift floor on the box. CF

17 A particle of mass 5 kg is observed to be travelling in a straight line at a speed of 5 m/s. Three seconds later the particle's speed is 8 m/s in the same direction. Determine the magnitude of the constant force which could produce this change in speed.

18 A particle of mass 4 kg is subjected to forces of $8\boldsymbol{i} + 12\boldsymbol{j}$ newtons and $6\boldsymbol{i} - 4\boldsymbol{j}$ newtons. Determine the acceleration of the particle. SF

19 A lift operator of mass 85 kg stands in a lift which is accelerating downwards at 2 m/s^2. Determine the reaction force of the lift floor on the operator. CF

Example 28 **20** An object of mass 10 kg is being pushed across a rough horizontal table by a horizontal force of magnitude 40 N. The frictional force is 20 N. Determine:

a the acceleration of the object

b the velocity of the object after 10 seconds, if it starts from rest.

21 A reindeer is hauling a heavy sled of mass 300 kg across a rough surface. The reindeer exerts a horizontal force of 600 N on the sled, while the resistance to the sled's motion is 550 N. If the sled is initially at rest, determine the velocity of the sled after 3 seconds.

22 The engine of a train of mass 200 tonnes exerts a force of 8000 kg wt, and the total air and rail resistance is 20 kg wt per tonne. How long will it take the train on level ground to acquire a speed of 30 km/h from rest?

23 One man can push a wardrobe of mass 250 kg with an acceleration of magnitude 0.15 m/s^2. With help from another man pushing just as hard (i.e. with the same force), the wardrobe accelerates at 0.4 m/s^2. How hard is each man pushing and what is the resistance to sliding?

24 What force is necessary to accelerate a train of mass 200 tonnes at 0.2 m/s^2 against a resistance of 20 000 N? What will be the acceleration if the train free-wheels against the same resistance?

25 A puck of mass 0.1 kg is sliding in straight line on an ice-rink. There is a frictional force of 0.025 N. Determine the speed of the puck after 20 seconds if its initial speed is 10 m/s.

26 A load of 200 kg is being raised by a cable. Determine the tension in the cable when:

a the load is lifted at a steady speed of 2 m/s

b the load is lifted with an upwards acceleration of 0.5 m/s^2.

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14G Resolution of forces and inclined planes

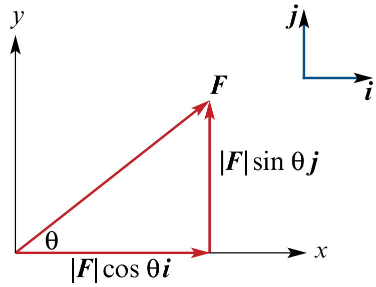
Learning intentions

- To be able to solve problems involving resolution of forces.

If all forces under consideration are acting in the same plane, then these forces and the resultant force can each be expressed as a sum of its $\boldsymbol{i}$- and $\boldsymbol{j}$-components.

If a force $\boldsymbol{F}$ acts at an angle of θ to the x-axis, then $\boldsymbol{F}$ can be written as the sum of two forces, one 'horizontal' and the other 'vertical':

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\boldsymbol{i} + |\boldsymbol{F}|\sin\theta\,\boldsymbol{j}$$



The force $\boldsymbol{F}$ is **resolved** into two components:

- the $\boldsymbol{i}$-component is parallel to the x-axis
- the $\boldsymbol{j}$-component is parallel to the y-axis.

Example 29

A particle at O is acted on by forces of magnitude 3 N and 5 N as in Example 17. If the particle has mass 1 kg, determine the acceleration and state the direction of the acceleration.

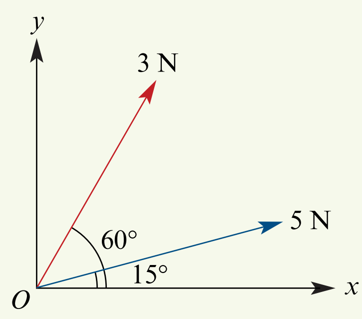
Solution

The resultant force $\boldsymbol{F} = 6.33\boldsymbol{i} + 3.89\boldsymbol{j}$ was found in Example 17.

Using the equation $\boldsymbol{F} = m\boldsymbol{a}$ gives

$$\boldsymbol{a} = 6.33\boldsymbol{i} + 3.89\boldsymbol{j}$$

The direction of the acceleration is the same as the direction of the force, i.e. at 31.59° anticlockwise from the x-axis.



Example 30

A block of mass 10 kg is pulled along a rough horizontal plane by a force of 10 N inclined at 30° to the plane. The frictional force is $0.05R$ N, where R N is the normal reaction force.

a Determine the normal reaction force. **b** Determine the acceleration of the block.

Solution

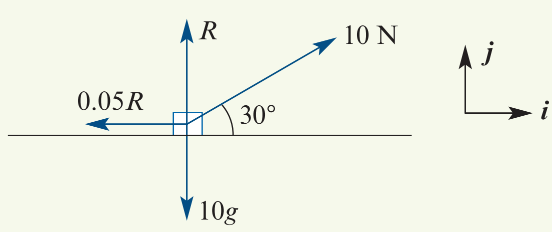
a Resolving in the $\boldsymbol{j}$-direction:

$$(R + 10\cos 60^\circ - 10g)\boldsymbol{j} = \boldsymbol{0}$$

$$\therefore\quad R = 10(g - \cos 60^\circ)$$

$$= 10\left(g - \tfrac{1}{2}\right)$$

The normal reaction force is $10\left(g - \frac{1}{2}\right)$ N.



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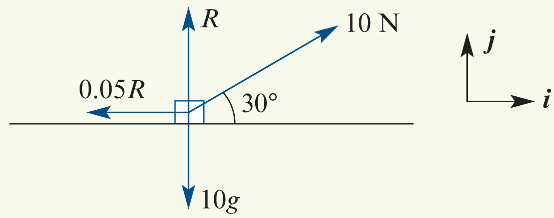
b Resolving in the $\boldsymbol{i}$-direction:

$$(10\cos 30^\circ - 0.05R)\boldsymbol{i} = 10\boldsymbol{a}$$

$$\cos 30^\circ - 0.05(g - \tfrac{1}{2}) = a$$

$$\therefore \quad a \approx 0.4 \text{ m/s}^2$$

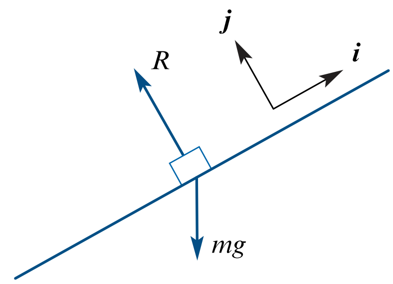
The acceleration of the block is approximately 0.4 m/s^2.



Normal reaction forces for inclined planes

For a mass on a plane that is inclined to the horizontal, the normal reaction force is at right angles to the plane.

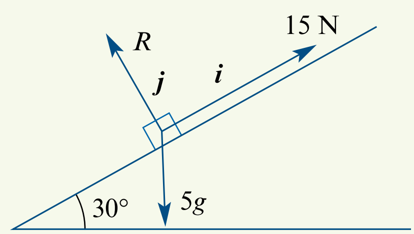
In such a situation, it is often advantageous to choose the direction up the plane to be $\boldsymbol{i}$ and the direction perpendicular from the plane to be $\boldsymbol{j}$.



Example 31

A particle of mass 5 kg lies on a smooth plane inclined at 30° to the horizontal. There is a force of 15 N acting up the plane. Determine the acceleration of the particle down the incline and the reaction force R.

Solution



Resolving in the $\boldsymbol{i}$-direction:

$$15 + 5g\cos 120^\circ = 15 - \frac{49}{2} = -\frac{19}{2}$$

For the $\boldsymbol{i}$-direction:

$$-\frac{19}{2}\boldsymbol{i} = 5\boldsymbol{a}$$

$$\therefore \quad \boldsymbol{a} = -1.9\boldsymbol{i}$$

The acceleration is 1.9 m/s^2 down the plane.

Resolving in the $\boldsymbol{j}$-direction:

$$R + 5g\cos 150^\circ = R - \frac{5\sqrt{3}}{2}g = 0$$

$$\therefore \quad R = \frac{5\sqrt{3}}{2}g$$

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Skill-sheet

Exercise 14G

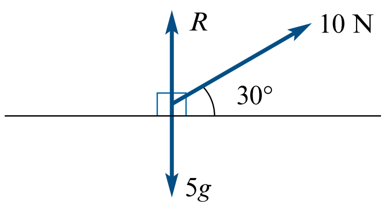
Example 29

1 A particle has mass 1 kg. It is acted on by two forces of magnitudes 3 N and 5 N, which act on the particle at an angle of 50° to each other. Determine the magnitude of the resulting acceleration and state its direction relative to the 5 N force.

Example 30

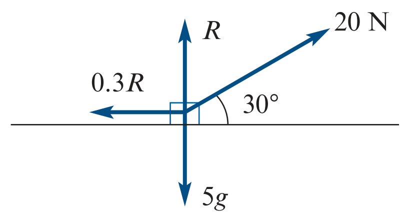
2 Determine the acceleration of a 5 kg mass for each of the following situations:

a



smooth surface

b



rough surface

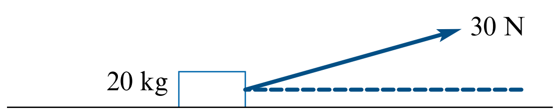
3 A particle slides down a smooth slope of 45°. What is its acceleration?

Example 31

4 A particle of mass 10 kg lies on a plane inclined at 30° to the horizontal. There is a force of 10 N, acting up the plane, that resists motion. Determine the acceleration of the particle down the incline and the reaction force R.

5 A 60 kg woman skis down a slope that makes an angle of 60° with the horizontal. The woman has an acceleration of 8 m/s^2. What is the magnitude of the resistive force?

6 A box of mass 20 kg is pulled along a smooth horizontal table by a force of 30 N acting at an angle of 30° to the horizontal.



a Determine the acceleration of the box.

b Determine the magnitude of the normal reaction of the table on the box.

7 A particle is projected up a smooth plane inclined at 30° to the horizontal. Let $\boldsymbol{i}$ be the unit vector up the plane. Determine the acceleration of the particle.

8 An object is projected up a smooth incline of 20° with an initial velocity of 10 m/s. Determine the distance that it goes up the plane and the velocity with which it returns to its starting point.

9 A particle of mass m kg slides from rest down a rough plane inclined at 60° to the horizontal. The frictional force has a magnitude of $0.4mg$ N. Determine the speed of the particle after it has travelled 5 m.

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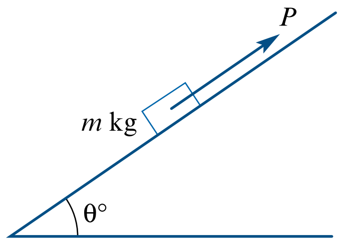
CF

10 A particle of mass m kg is being accelerated up a rough inclined plane at a m/s^2 by a force of P N acting parallel to the plane. The plane is inclined at an angle of $\theta°$ to the horizontal. The frictional force is kR N, where k is a positive constant and R N is the normal reaction force. Determine a in terms of P, θ, m, k and g.



11 A body of mass of M kg is pulled along a rough horizontal plane by a constant force of F newtons, at an inclination of θ. The frictional force is kR N, where k is a positive constant and R N is the normal reaction force. Determine the acceleration of the body if:

a θ is upwards from the horizontal

b θ is downwards from the horizontal.

12 A car of mass 1 tonne coasts at a constant speed down a slope inclined at $\theta°$ to the horizontal, where $\sin\theta° = \frac{1}{20}$. The car can ascend the same slope with a maximum acceleration of 1 m/s^2. Determine:

a the total resistance to the motion (assumed constant)

b the driving force exerted by the engine when the maximum acceleration is reached.

13 A body of mass 5 kg is placed on a smooth horizontal plane and is acted upon by the following horizontal forces:

- a force of 8 N in a direction of 330°
- a force of 10 N in a direction of 090°
- a force of P N in a direction of 180°

Given that the magnitude of the acceleration of the body is 2 m/s^2, calculate the value of P correct to two decimal places.

14 A block of mass 2 kg lies on a rough horizontal table. The frictional force that acts on the block when it moves is $0.5R$ N, where R N is the normal reaction force. Determine the magnitude of the force on the block which, when acting at 45° upwards from the horizontal, produces in the block a horizontal acceleration of $0.25g$ m/s^2.

15 A particle of mass 5 kg is being pulled up a slope inclined at 30° to the horizontal. The pulling force, F newtons, acts parallel to the slope, as does the resistance with a magnitude one-fifth of the magnitude of the normal reaction.

a Determine the value of F such that the acceleration is 1.5 m/s^2 up the slope.

b Also determine the magnitude of the acceleration if this pulling force now acts at an angle of 30° to the slope (i.e. at 60° to the horizontal).

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14H Variable forces

Learning intentions

- To be able to solve problems involving variable forces.

In the previous four sections of this chapter, we have considered constant forces. We now consider variable forces. We will use the expressions for acceleration from Section 14C:

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

where x, v and a are the position, velocity and acceleration at time t respectively.

Note: It was observed in Section 14C that the last form is not really necessary, as the form $a = v\dfrac{dv}{dx}$ can be used instead, together with separation of variables.

Example 32

A body of mass 5 kg, initially at rest, is acted on by a force of $F = (6 - t)^2$ newtons, where $0 \le t \le 6$ (seconds). Determine the speed of the body after 6 seconds and the distance travelled.

Solution

Newton's second law of motion gives

$$F = ma$$

$$(6 - t)^2 = 5a$$

$$a = \frac{1}{5}(6 - t)^2$$

Hence $\quad \dfrac{dv}{dt} = \dfrac{1}{5}(6 - t)^2$

giving $\quad v = \dfrac{1}{5}\displaystyle\int (6 - t)^2\,dt$

$$= \frac{1}{5} \times (-1) \times \frac{(6 - t)^3}{3} + c$$

$$= -\frac{1}{15}(6 - t)^3 + c$$

When $t = 0$, $v = 0$, so $c = \dfrac{72}{5}$.

$\therefore \quad v = -\dfrac{1}{15}(6 - t)^3 + \dfrac{72}{5}$

When $t = 6$, $v = \dfrac{72}{5}$, i.e. the velocity after 6 seconds is $\dfrac{72}{5}$ m/s.

Integrating again with respect to t gives $x = \dfrac{1}{60}(6 - t)^4 + \dfrac{72}{5}t + d$.

When $t = 0$, $x = 0$ and therefore $d = -21.6$.

Hence when $t = 6$, $x = \dfrac{72}{5} \times 6 - 21.6 = 64.8$.

The distance travelled is 64.8 m.

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Example 33

A particle of mass 3 units moves in a straight line and, at time t, its position relative to a fixed origin is x and its velocity is v.

a If the resultant force is $9\cos t$, and $v = 2$ and $x = 0$ when $t = 0$, determine x in terms of t.

b If the resultant force is $3 + 6x$, and $v = 2$ when $x = 0$, determine v when $x = 2$.

Solution

a Using Newton's second law of motion:

$$F = ma$$
$$9\cos t = 3a$$
$$a = 3\cos t$$
$$\frac{dv}{dt} = 3\cos t$$
$$\therefore \quad v = 3\sin t + c$$

When $t = 0$, $v = 2$, so $c = 2$.

Hence $v = 3\sin t + 2$

i.e. $\dfrac{dx}{dt} = 3\sin t + 2$

$$\therefore \quad x = -3\cos t + 2t + d$$

When $t = 0$, $x = 0$ and therefore $d = 3$.

Hence $x = 3 - 3\cos t + 2t$.

b Using Newton's second law of motion:

$$F = ma$$
$$3 + 6x = 3a$$
$$1 + 2x = a$$
$$v\frac{dv}{dx} = 1 + 2x$$
$$\therefore \quad \tfrac{1}{2}v^2 = x + x^2 + c$$

When $x = 0$, $v = 2$. Therefore $c = 2$.

Thus $\tfrac{1}{2}v^2 = x + x^2 + 2$

When $x = 2$:

$$\tfrac{1}{2}v^2 = 2 + 4 + 2$$
$$\therefore \quad v = \pm 4$$

Skill-sheet

Exercise 14H

Example 32

1 A body of mass 10 kg, initially at rest, is acted on by a force of $F = (10 - t)^2$ newtons at time t seconds, where $0 \le t \le 10$. Determine the speed of the body after 10 seconds and the distance travelled.

CF

Example 33

2 A particle of mass 5 kg moves in a straight line and, at time t seconds, its position relative to a fixed origin is x m and its velocity is v m/s.

a If the resultant force acting is $10\sin t$, and $v = 4$ and $x = 0$ when $t = 0$, determine x in terms of t.

b If the resultant force acting is $10 + 5x$, and $v = 4$ when $x = 0$, determine v when $x = 4$.

c If the resultant force acting is $10\cos^2 t$, and $v = 0$ and $x = 0$ when $t = 0$, determine x in terms of t.

3 A body of mass 6 kg, moving initially with a speed of 10 m/s, is acted on by a force $F = \dfrac{100}{(t+5)^2}$ N. Determine the speed reached after 10 seconds and the distance travelled in this time.

4 A particle of unit mass is acted on by a force of magnitude $1 - \sin\left(\frac{t}{4}\right)$, for $0 \le t \le 2\pi$. If the particle is initially at rest, determine an expression for the distance covered at time t.

5 A particle of unit mass is acted on by a force of magnitude $1 - \cos\left(\frac{1}{2}t\right)$, for $0 \le t \le \frac{\pi}{2}$. If the particle is initially at rest, determine an expression for:

a the velocity at time t **b** the displacement at time t.

6 A particle of mass 4 kg is acted on by a resultant force whose direction is constant and whose magnitude at time t seconds is $(12t - 3t^2)$ N. If the particle has an initial velocity of 2 m/s in the direction of the force, determine the velocity at the end of 4 seconds.

7 A particle of mass 1 kg on a smooth horizontal plane is acted on by a horizontal force $\frac{t}{t+1}$ N at time t seconds after it starts from rest. Determine its velocity after 10 seconds.

8 A body of mass 0.5 kg is acted on by a resultant force $e^{-\frac{t}{2}}$ N at time t seconds after the body is at rest.

a If the body starts from rest, determine the velocity, v m/s, at time t seconds.

b Sketch the velocity–time graph.

c If the body moves under the given force for 30 seconds, determine the distance travelled.

9 A body of mass 10 units is accelerated from rest by a force F whose magnitude at time t is given by

$$F(t) = \begin{cases} 14 - 2t & \text{for } 0 \le t \le 5 \\ 100t^{-2} & \text{for } t > 5 \end{cases}$$

Determine:

a the speed of the body when $t = 10$

b the distance travelled by this time.

10 A body of mass m kg moving with a velocity of u m/s ($u > 0$) is acted on by a resultant force kv N (in its initial direction), where v m/s is its velocity at time t seconds and k is a positive constant. Determine the distance travelled after t seconds.

11 A particle of mass m is projected along a horizontal line from O with speed V. It is acted on by a resistance kv when the speed is v. Determine the velocity after the particle has travelled a distance x.

12 A particle of mass m kg at rest on a horizontal plane is acted on by a constant horizontal force b N. The total resistance to motion is cv N, where v m/s is the velocity and c is a constant value. Determine the velocity at time t seconds and the terminal velocity.

CF

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CF

13 A body of mass m is projected vertically upwards with speed u. The force due to air resistance is equal to k times the square of the speed, where k is a constant. Determine the maximum height reached by the particle and the speed when it is returns to the point of projection.

14 A particle of mass 0.2 kg moving on the positive x-axis has position x metres and velocity v m/s at time t seconds. At time $t = 0$, $v = 0$ and $x = 1$. The particle moves under the action of a force of magnitude $\frac{4}{x}$ N in the positive direction of the x-axis. Show that $v = \sqrt{40 \ln x}$.

15 A particle P of unit mass moves on the positive x-axis. At time t, the velocity of the particle is v and the force F acting on the particle is given by

$$F = \begin{cases} \dfrac{50}{25 + v} & \text{for } 0 \le t \le 50 \\[2ex] \dfrac{-v^2}{1000} & \text{for } t > 50 \end{cases}$$

Initially the particle is at rest at the origin O.

a Show that $v = 50$ when $t = 50$.

b Determine the distance of P from O when $v = 50$.

c Determine the distance of P from O when $v = 25$ and $t > 50$.

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Chapter summary

Motion in a straight line

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the origin, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.
- **Displacement** is the change in position (i.e. final position minus initial position).
- **Average velocity** $= \dfrac{\text{change in position}}{\text{change in time}}$
- For a particle moving in a straight line with position x at time t:
  - **velocity** (v) is the rate of change of position with respect to time
  - **acceleration** (a) is the rate of change of velocity with respect to time

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

  - velocity at time t is also denoted by $\dot{x}(t)$
  - acceleration at time t is also denoted by $\ddot{x}(t)$
- Scalar quantities
  - **Distance travelled** means the total distance travelled.
  - **Speed** is the magnitude of the velocity.
  - **Average speed** $= \dfrac{\text{distance travelled}}{\text{change in time}}$
- Velocity–time graphs
  - Acceleration is given by the gradient.
  - Displacement is given by the signed area bounded by the graph and the t-axis.
  - Distance travelled is given by the total area bounded by the graph and the t-axis.
- Constant acceleration

  If acceleration is constant, then the following formulas can be used (for acceleration a, initial velocity u, final velocity v, displacement s and time taken t):

  1 $v = u + at$ **2** $s = ut + \frac{1}{2}at^2$ **3** $v^2 = u^2 + 2as$ **4** $s = \frac{1}{2}(u + v)t$
- **Acceleration** $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$
- Simple harmonic motion

  Simple harmonic motion is a special type of motion in a straight line where the particle is oscillating about a centre point.
  - Centred at the origin:

$$\ddot{x} = -\omega^2 x$$
$$v^2 = \omega^2(A^2 - x^2)$$
$$x = A\sin(\omega t + \alpha)$$

  - Centred at $x = c$:

$$\ddot{x} = -\omega^2(x - c)$$
$$v^2 = \omega^2(A^2 - (x - c)^2)$$
$$x = c + A\sin(\omega t + \alpha)$$

  - Amplitude A
  - Period $T = \dfrac{2\pi}{\omega}$
  - Frequency $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$

Review

Dynamics

- The units of force used are the kilogram weight and the newton.
  1 kg wt = g N, where g m/s^2 is the magnitude of the acceleration due to gravity.
- The vector sum of the forces acting at a point is called the **resultant force**.
- A force acting on a body has an influence in directions other than its line of action, except in the direction perpendicular to its line of action.
- The **resolved part** of a force P N in a direction that makes an angle θ with its own line of action is a force of magnitude $P\cos\theta$.
- If a force is resolved in two perpendicular directions, then the vector sum of the resolved parts is equal to the force itself.
- The **momentum** of a particle is the product of its mass and velocity:

  $$\text{momentum} = \text{mass} \times \text{velocity}$$

  The units of momentum are kg m/s or kg ms^{-1}.
- Newton's second law of motion (assuming constant mass)
  $F = ma$, where force is measured in newtons, mass in kilograms and acceleration in m/s^2.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

14A **1** I can use integration to find the displacement of a particle moving in a straight line given an expression for its velocity as a function of t. ☐

See Example 1 and Question 5

14A **2** I can use integration to find the displacement and velocity of a particle moving in a straight line given an expression for its acceleration as a function of t. ☐

See Example 2 and Question 6

14A **3** I can describe the motion of a particle moving in a straight line from a velocity time graph. ☐

See Example 3 and Question 7

14A **4** I can solve problems involving the motion of a particle moving in a straight line with constant acceleration. ☐

See Example 4, Example 5, Example 6 and Questions 14, 15 and 17

14B **5** I can solve problems involving motion in a straight line with information given in one of the forms $v = f(x)$ or $a = f(v)$. ☐

See Example 7, Example 8, Example 9 and Questions 1, 3 and 5

14C **6** I can solve problems using $a = v\dfrac{dv}{dx}$ and $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$. ☐

See Example 10, Example 11, Example 12, Example 13 and Questions 1, 2 and 9

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14D **7** I can solve problems concerning simple harmonic motion. ☐

See Example 14, Example 15, Example 16 and Questions 1, 2 and 16

14E **8** I can solve problems involving forces including resultant force and resolution of forces. ☐

See Example 17, Example 18, Example 19, Example 20, Example 21 and Questions 1, 3, 6, 12 and 16

14F **9** I can solve problems involving Newton's laws of motion. ☐

See Example 22, Example 23, Example 24, Example 25, Example 26, Example 27, Example 28 and Questions 1, 2, 3, 5, 14, 15, 16 and 20

14G **10** I can solve problems involving Newton's laws of motion, resolution of forces and inclined planes. ☐

See Example 29, Example 30, Example 31 and Questions 1, 2 and 4

14H **11** I can solve problems involving variable forces. ☐

See Example 32, Example 33 and Questions 1 and 2

Short-response questions

Technology-free short-response questions

CF

1 A particle is moving in a straight line with position, x metres, at time t seconds ($t \geq 0$) given by $x = t^2 - 7t + 10$. Determine:

a when its velocity equals zero

b its acceleration at this time

c the distance travelled in the first 5 seconds

d when and where its velocity is -2 m/s.

2 An object moves in a straight line so that its acceleration, a m/s^2, at time t seconds ($t \geq 0$) is given by $a = 2t - 3$. Initially, the position of the object is 2 m to the right of a point O and its velocity is 3 m/s. Determine the position and velocity after 10 seconds.

3 A golf ball is putted across a level putting green with an initial velocity of 8 m/s. Owing to friction, the velocity decreases at the rate of 2 m/s^2. How far will the golf ball roll?

4 A particle moves in a straight line such that after t seconds its position, x metres, relative to a point O on the line is given by $x = \sqrt{9 - t^2}$, $0 \leq t < 3$.

a When is the position $\sqrt{5}$?

b Determine expressions for the velocity and acceleration of the particle at time t.

c Determine the particle's maximum distance from O.

d When is the velocity zero?

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Review

CF

5 A ball is thrown vertically upwards from ground level with an initial velocity of 35 m/s. Let g m/s^2 be the acceleration due to gravity. Determine:

a the velocity, in terms of g, and the direction of motion of the ball after:

i 3 seconds **ii** 5 seconds

b the total distance travelled by the ball, in terms of g, when it reaches the ground again

c the velocity with which the ball strikes the ground.

6 A projectile is fired vertically upwards from a point on the ground, level with the base of a tower 64 m high. The projectile is level with the top of the tower 0.8 seconds after being fired. Let g m/s^2 be the acceleration due to gravity. Determine in terms of g:

a the initial velocity of the projectile

b the time taken to reach its greatest height

c the greatest height

d the length of time for which the projectile is higher than the top of the tower.

7 For a particle moving in a straight line, the velocity v satisfies $v^2 = 128 - 32x - 16x^2$, where x is the position of the particle relative to a fixed point O.

a Show that the motion is simple harmonic motion.

b **i** Find the period. **ii** Find the amplitude. **iii** Find the maximum speed.

8 A man of mass 75 kg is in a lift of mass 500 kg that is accelerating upwards at 2 m/s^2.

a Determine the force exerted by the floor on the man.

b Determine the total tension in the cables raising the lift.

9 A particle of mass 5 kg, starting from rest, moves in a straight line under the action of a force which after t seconds is $\dfrac{20}{(t+1)^2}$ newtons. Determine:

a the acceleration at time t **b** the velocity at time t

c the displacement from its starting point at time t.

10 A car of mass 1 tonne, travelling at 60 km/h on a level road, has its speed reduced to 24 km/h in 5 seconds when the brakes are applied. Determine the total retarding force (assumed constant).

11 A rope will break when its tension exceeds 400 kg wt.

a Calculate the greatest acceleration with which a particle of mass 320 kg can be hauled upwards with this rope.

b Show how the rope might be used to lower a particle of mass 480 kg without breaking.

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Review

CF

12 A particle of mass 3 kg moves in a straight line and, at time t, its position relative to a fixed origin is x and its velocity is v. If the resultant force is $3 + 6x$, and $v = 2$ when $x = 0$, determine v when $x = 2$.

13 A particle of mass 3 kg, moving in a straight line, has initial velocity $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j}$ m/s. It is acted on by a force $\boldsymbol{F} = 3\boldsymbol{i} + 6\boldsymbol{j}$ newtons.

a Determine the acceleration at time t.

b **i** Determine the velocity at time t.

ii Determine the speed at time t.

c Determine the position of the particle at time t if initially the particle is at the origin.

d Determine the equation of the straight line in which the particle is moving.

14 A train that is moving with uniform acceleration is observed to take 20 s and 30 s to travel successive half kilometres. How much farther will it travel before coming to rest if the acceleration remains constant?

15 What force, in newtons, will give a stationary mass of 9000 kg a horizontal velocity of 15 m/s in 1 minute?

16 A train travelling uniformly on the level at the rate of 20 m/s begins an ascent with an angle of elevation of $\theta°$ such that $\sin\theta° = \frac{3}{50}$. The force exerted by the engine is constant throughout, and the resistant force due to friction is also constant. How far up the incline will the train travel before coming to rest?

17 A body of mass m kg is placed in a lift that is moving with an upwards acceleration of f m/s^2. Determine the reaction force of the lift on the body.

18 A 0.05 kg bullet travelling at 200 m/s will penetrate 10 cm into a fixed block of wood. Determine the velocity with which it would emerge if fired through a fixed board 5 cm thick. (Assume that the resistance is uniform and has the same value in both cases.)

19 In a lift accelerated upwards at a m/s^2, a spring balance indicates that an object has a weight of 10 kg wt. When the lift is accelerated downwards at $2a$ m/s^2, the weight of the object appears to be 7 kg wt. Determine:

a the weight of the object

b the upwards acceleration.

Technology-active short-response questions

CU

20 The velocity, v km/h, of a train which moves along a straight track from station A, where it starts at rest, to station B, where it next stops, is given by

$$v = kt(1 - \sin(\pi t))$$

where t hours is the time measured from when the train left station A and k is a positive constant.

a Determine the time that the train takes to travel from A to B.

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CU

b **i** Determine an expression for the acceleration at time t.

ii Determine the interval of time for which the velocity is increasing. (Give your answer correct to two decimal places.)

c Given that the distance from A to B is 20 km, determine the value of k. (Give your answer correct to three significant figures.)

21 A particle A moves along a horizontal line so that its position, x m, relative to a point O is given by $x = 28 + 4t - 5t^2 - t^3$, where t is the time in seconds after the motion starts.

a Determine:

i the velocity of A in terms of t

ii the acceleration of A in terms of t

iii the value of t for which the velocity is zero (to two decimal places)

iv the times when the particle is 28 m to the right of O (to two decimal places)

v the time when the particle is 28 m to the left of O (to two decimal places).

b A second particle B moves along the same line as A. It starts from O at the same time that A begins to move. The initial velocity of B is 2 m/s and its acceleration at time t is $(2 - 6t)$ m/s^2.

i Determine the position of B at time t.

ii Determine the time at which A and B collide.

iii At the time of collision are they going in the same direction?

22 The motion of a bullet through a special shield is modelled by the equation $a = -30(v + 110)^2$, $v \geq 0$, where a m/s^2 is its acceleration and v m/s its velocity t seconds after impact. When $t = 0$, $v = 300$.

a Determine v in terms of t.

b Sketch the graph of v against t.

c Let x m be the penetration into the shield at time t seconds.

i Determine x in terms of t

ii Determine x in terms of v.

iii Determine how far the bullet penetrates the shield before coming to rest.

d Another model for the bullet's motion is $a = -30(v^2 + 11\,000)$, $v \geq 0$. Given that when $t = 0$, $v = 300$:

i Determine t in terms of v.

ii Determine v in terms of t.

iii Sketch the graph of v against t.

iv Determine the distance travelled by the bullet in the first 0.0001 seconds after impact.

23 A buoy of mass 4 kg is held 5 metres below the surface of the water by a vertical cable. There is an upwards buoyancy force of 42 N acting on the buoy.

a Determine the tension in the cable.

b Suddenly the cable breaks. Determine the acceleration of the buoy while it is still in the water.

c The buoy maintains this constant acceleration while it is still in the water. Determine:

i the time taken for it to reach the surface

ii the velocity of the buoy at this time.

d Ignoring air resistance, determine the height above water level that the buoy will reach.

24 Annabelle and Cuthbert are ants on a picnic table. Annabelle falls off the edge of the table at point X. She falls 1.2 m to the ground. (Assume $g = 9.8$ for this question.)

a Assuming that Annabelle's acceleration down is g m/s^2, determine:

i Annabelle's velocity when she hits the ground, correct to two decimal places

ii the time it takes for Annabelle to hit the ground, correct to two decimal places.

b Assume now that Annabelle's acceleration is slowed by air resistance and is given by $(g - t)$ m/s^2, where t is the time in seconds after leaving the table.

i Determine Annabelle's velocity, v m/s, at time t.

ii Determine Annabelle's position, x m, relative to X at time t.

iii Determine the time in seconds, correct to two decimal places, when Annabelle hits the ground.

c When Cuthbert reaches the edge of the table, he observes Annabelle groaning on the ground below. He decides that action must be taken and fashions a parachute from a small piece of potato chip. He jumps from the table and his acceleration is $\frac{g}{2}$ m/s^2 down.

i Determine an expression for x, the distance in metres that Cuthbert is from the ground at time t seconds.

ii Unfortunately, Annabelle is very dizzy and on seeing Cuthbert coming down jumps vertically with joy. Her initial velocity is 1.4 m/s up and her acceleration is g m/s^2 down. She jumps 0.45 seconds after Cuthbert leaves the top of the table. How far above the ground (to the nearest cm) do the two ants collide?

25 The total resistance on a train with the brakes applied is $(a + bv^2)$ per unit mass, where v is its velocity.

a **i** Show that $\dfrac{dv}{dx} = -\dfrac{(a + bv^2)}{v}$, where x is the distance travelled from when the brakes were first applied.

ii If u is the velocity of the train when the brakes are first applied, show that the train comes to rest when $x = \dfrac{1}{2b}\ln\left(1 + \dfrac{bu^2}{a}\right)$.

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Review

CU

b **i** Show that the train stops when $t = \dfrac{1}{\sqrt{ab}} \tan^{-1}\left(\dfrac{\sqrt{b}\,u}{\sqrt{a}}\right)$.

ii Determine the time it takes for the train to stop if $b = 0.005$, $a = 2$ and $u = 25$.

26 A particle of mass m kg falls vertically from rest in a medium in which the resistance is $0.02\ mv^2$ N when the velocity is v m/s.

a Determine the distance, x m, which the particle has fallen in terms of v.

b Determine v in terms of x.

c Sketch the graph of v against x.

27 The velocity, v m/s, of a vehicle moving in a straight line is $v = 125(1 - e^{-0.1t})$ m/s at time t seconds. The mass of the vehicle is 250 kg.

a Determine the acceleration of the vehicle at time t.

b The resultant force acting on the vehicle is $(P - 20v)$ N, where P N is the driving force and $20v$ N is the resistance force.

i Determine P in terms of t.

ii Determine P in terms of v.

iii Determine P when $v = 20$.

iv Determine P when $t = 30$.

c Sketch the graph of P against t.

Multiple-choice questions

Technology-free multiple-choice questions

1 A particle moves in a straight line so that its position, x cm, relative to a point O at time t seconds ($t \geq 0$) is given by $x = t^3 - 9t^2 + 24t - 1$. The position (in cm) of the particle at $t = 3$ is

A 17 **B** 16
C 24 **D** -17

2 A body is projected up from the ground with a velocity of 30 m/s. Its acceleration due to gravity is -10 m/s^2. The body's velocity (in m/s) at time $t = 2$ seconds is

A 10 **B** -10
C 0 **D** 20

3 An object is oscillating along a straight line with acceleration given by $\ddot{x} = -9x$, where x cm is the position of the particle relative to the centre of motion. The maximum speed is 6 cm/s. The period, T s, and amplitude, A cm, of the motion are given by

A $T = 3, \quad A = 6$ **B** $T = \dfrac{2\pi}{3}, \quad A = 4$
C $T = 6\pi, \quad A = 9$ **D** $T = \dfrac{2\pi}{3}, \quad A = 2$

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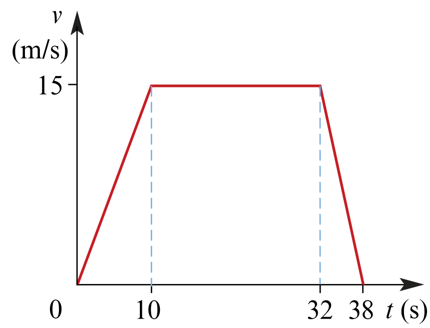
4 The acceleration of a particle $a(t)$ moving in a straight line is not constant. Which function could represent the displacement, $x(t)$ of the particle ?

A e **B** $t - 4$ **C** $t^2 + 3t + 1$ **D** t^3

5 The velocity–time graph shows the motion of a tram between two stops. The distance between the stops, in metres, is



A 300
B 360
C 405
D 450

6 An object is moving in a straight line. Its acceleration, a m/s^2, and its position relative to the origin, x m, are related by $a = -x$, where $-\sqrt{3} \leq x \leq \sqrt{3}$. If the object starts from the origin with a velocity of $\sqrt{3}$ m/s, then its velocity, v m/s, is given by

A $-\sqrt{3 - x^2}$ **B** $\sqrt{3 - x^2}$ **C** $\pm\sqrt{3 - x^2}$ **D** $-\sqrt{x^2 - 3}$

7 The position, x metres, with respect to an origin of a particle travelling in a straight line is given by $x = 2 - 2\cos\left(\frac{3\pi}{2}t - \frac{\pi}{2}\right)$. The velocity (in m/s) at time $t = \frac{8}{3}$ seconds is

A -3π **B** 3π **C** 0 **D** $-\frac{3\pi}{2}$

8 The velocity of a body of mass 3 kg has a horizontal component of magnitude 6 m/s and a vertical component of magnitude 8 m/s. The momentum of the body has a magnitude (in kg m/s) of

A 6 **B** 18 **C** 24 **D** 30

9 A block of mass 10 kg rests on the floor of a lift which is accelerating upwards at 4 m/s^2. Taking the acceleration due to gravity to be $g = 9.8$ m/s^2, the magnitude of the reaction force of the floor of the lift on the block is

A 96 N **B** 60 N **C** 30 N **D** 138 N

10 Two perpendicular forces have magnitudes 8 N and 6 N. The magnitude of the resultant force is

A 14 N **B** 10 N **C** $2\sqrt{7}$ N **D** 2 N

11 A particle moves in a straight line where $x(t)$ and $a(t)$ define respectively, the position and acceleration of the particle at time t, $t > 0$.
If $a(t) = x(t)$, then $x(t)$ could be

A $\sin\left(t + \frac{\pi}{2}\right)$ **B** $\cos t$ **C** $e^t + e^{-t}$ **D** $(t + 2)^2$

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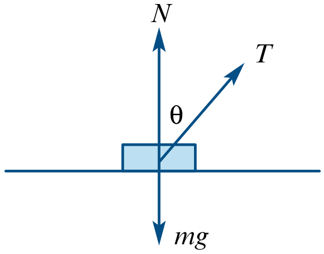
Review

12 A body of mass m kg is being pulled along a smooth horizontal table by a string inclined at $\theta°$ to the vertical. The diagram shows the forces acting on the body. Which one of the following statements is true?

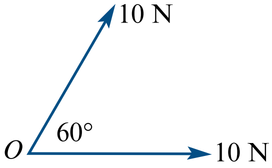


A $N - mg = 0$
B $N + T\sin\theta - mg = 0$
C $N - T\sin\theta - mg = 0$
D $N + T\cos\theta - mg = 0$

13 A boy slides down a smooth slide with an inclination to the horizontal of $\theta°$, where $\sin\theta° = \frac{4}{5}$. Let g m/s^2 be the acceleration due to gravity. Then the boy's acceleration down the slide (in m/s^2) is given by

A $\dfrac{3g}{5}$ **B** $\dfrac{4g}{5}$ **C** $40g$ **D** $\dfrac{200g}{3}$

14 Two forces of magnitude 10 N act on a particle at O as shown. The magnitude of the resultant force in newtons is



A 20 **B** $10\sqrt{3}$
C 0 **D** 10

15 The external resultant force on a body is zero. Which one of the following statements cannot be true?

A The body has constant momentum. **B** The body is moving in a circle.
C The body is moving in a straight line. **D** The body has constant velocity.

Technology-active multiple-choice questions

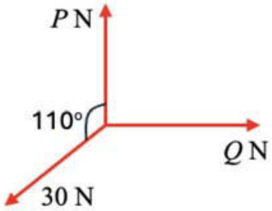
16 At a particular instant a barge, which is driven by a force of magnitude 5400 N, is accelerating at 1.6 m/s^2. The resistance to motion at this instant is 1200 N. The mass of the barge in kilograms is

A 2625 **B** 4125 **C** 3375 **D** 750

17 A solid ball of mass 1.8 kg is dropped vertically through water with an acceleration of 5.6 m/s^2. The magnitude of the force resisting motion of the ball is ($g = 9.8$ m/s^2).

A 7.92 N **B** 8 N **C** 7.56 N **D** 7 N

18 A particle is in equilibrium under the action of three coplanar forces shown in the diagram. The magnitudes of the perpendicular forces P and Q correct to one decimal place are



A $P = 10.3$ N and $Q = 28.2$ N
B $P = 12.3$ N and $Q = 24.5$ N
C $P = 9.3$ N and $Q = 21.2$ N
D $P = 18.2$ N and $Q = 11.8$ N

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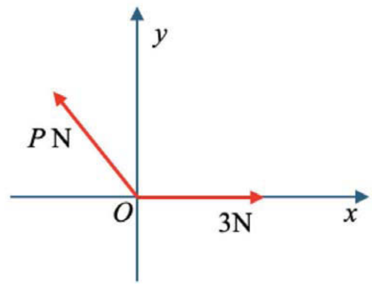
19 A particle of mass 2 kg is at rest at the origin when acted upon by two forces 3 N and P N as shown in the diagram. The particle then moves with an acceleration of 2 ms^{-2} in the direction of the y-axis. The angle between the two vectors is $\theta°$. The value of P and the value of θ, to the nearest degree are



A $P = 5$ and $\theta = 53°$ **B** $P = 5$ and $\theta = 127°$
C $P = \sqrt{13}$ and $\theta = 134°$ **D** $P = 4$ and $\theta = 126°$

20 An object starting at the origin has a velocity given by $v = 10\sin(\pi t)$. The distance that the object travels from $t = 0$ to $t = 1.6$, correct to two decimal places, is

A 1.60 **B** 2.20 **C** 10.53 **D** 6.37

21 A particle moves with simple harmonic motion along the x-axis with centre $x = 0$. It starts from $x = 5$ with zero velocity. When $t = 1$, $x = 2$. The position of the particle, correct to two decimal places, when $t = 2$ is

A $x = 1.00$ **B** $x = -3.40$ **C** $x = -2.01$ **D** $x = 1.02$

15

Statistical inference

Chapter contents

In practice, the reason we analyse samples is to further our understanding of the population from which they are drawn. That is, we know what is in the sample, and from that knowledge we would like to infer something about the population. In this chapter we investigate the distribution of the sample mean, calculated from data drawn from a from a variety of different distributions. We can then apply that knowledge to estimate the value of a population mean when it is unknown, based on the value of a sample mean.

Note: The statistics material in Specialist Mathematics Units 3 & 4 requires a knowledge of probability and statistics from Mathematical Methods Units 3 & 4.

This chapter covers Unit 4 Topic 5: Statistical inference.

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15A Linear combinations of random variables

Learning intentions

- To understand and use the distribution of a linear function of a random variable.
- To determine the mean and variance of a linear function of a random variable.
- To understand and use the distribution of the sum of independent identical random variables.
- To determine the mean and variance of the sum of independent identical random variables.

A linear function of a random variable

Consider a random variable Y which is a linear function of another random variable X. That is, $Y = aX + b$, where a and b are constants. We can consider b as a location parameter and a as a scale parameter.

Discrete random variables

If X is a discrete random variable, then $Y = aX + b$ is also a discrete random variable. We can determine probabilities associated with Y by using the original probability distribution of X, as illustrated in the following example.

Example 1

The probability distribution of X, the number of cars that Matt sells in a week, is given in the following table.

Number of cars sold, x	0	1	2	3	4
$P(X = x)$	0.45	0.25	0.20	0.08	0.02

Suppose that Matt is paid \$750 each week, plus \$1000 commission on each car sold.

a Express S, Matt's weekly salary, as a linear function of X.

b What is the probability distribution of S?

c What is the probability that Matt earns more than \$2000 in any given week?

Solution

a $S = 1000X + 750$

b We can use the rule from part **a** to determine the possible values of S.

Weekly salary, s	750	1750	2750	3750	4750
$P(S = s)$	0.45	0.25	0.20	0.08	0.02

c From the table, we have $P(S > 2000) = 0.20 + 0.08 + 0.02 = 0.30$.

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Continuous random variables

A continuous random variable X has a probability density function f such that:

1 $f(x) \geq 0$ for all x

2 $\int_{-\infty}^{\infty} f(x)\, dx = 1$

And, we have

$$P(X \leq c) = \int_{-\infty}^{c} f(x)\, dx$$

If X is a continuous random variable and $a \neq 0$, then $Y = aX + b$ is also a continuous random variable. If $a > 0$, then

$$P(Y \leq y) = P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right)$$

giving

$$P(Y \leq y) = \int_{-\infty}^{\frac{y-b}{a}} f(x)\, dx$$

Example 2

Assume that the random variable X has density function f given by

$$f(x) = \begin{cases} 1.5(1 - x^2) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

a Determine $P(X \leq 0.5)$.

b Let $Y = 2X + 3$. Determine $P(Y \leq 3.5)$.

Solution

a
$$\begin{aligned} P(X \leq 0.5) &= \int_{0}^{0.5} f(x)\, dx \\ &= \int_{0}^{0.5} 1.5(1 - x^2)\, dx \\ &= 1.5\left[x - \frac{x^3}{3}\right]_0^{0.5} \\ &= 1.5\left(0.5 - \frac{0.5^3}{3}\right) \\ &= 0.6875 \end{aligned}$$

b
$$\begin{aligned} P(Y \leq 3.5) &= \int_{0}^{\frac{3.5-3}{2}} f(x)\, dx \\ &= \int_{0}^{0.25} 1.5(1 - x^2)\, dx \\ &= 1.5\left[x - \frac{x^3}{3}\right]_0^{0.25} \\ &= 1.5\left(0.25 - \frac{0.25^3}{3}\right) \\ &\approx 0.3672 \end{aligned}$$

The mean of a linear function of a random variable

Now we consider the mean of Y, where $Y = aX + b$.

Discrete random variables

For a discrete random variable X, by definition we have

$$\mathrm{E}(X) = \sum_{x} x \cdot P(X = x)$$

Thus $\quad \mathrm{E}(Y) = \mathrm{E}(aX + b)$

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$$= \sum_x (ax+b) \cdot P(X=x)$$
$$= \sum_x ax \cdot P(X=x) + \sum_x b \cdot P(X=x)$$
$$= a\sum_x x \cdot P(X=x) + b\sum_x P(X=x)$$
$$= a\mathrm{E}(X) + b \qquad \text{since } \sum_x P(X=x) = 1$$

Continuous random variables

Similarly, for a continuous random variable X, we have

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Thus $\quad \mathrm{E}(Y) = \mathrm{E}(aX+b)$
$$= \int_{-\infty}^{\infty} (ax+b) \cdot f(x)\, dx$$
$$= \int_{-\infty}^{\infty} ax \cdot f(x)\, dx + \int_{-\infty}^{\infty} b \cdot f(x)\, dx$$
$$= a\int_{-\infty}^{\infty} x \cdot f(x)\, dx + b\int_{-\infty}^{\infty} f(x)\, dx$$
$$= a\mathrm{E}(X) + b \qquad \text{since } \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Mean of a linear function of a random variable

If X is a random variable and $Y = aX + b$, where a and b are constants, then

$$\mathrm{E}(Y) = \mathrm{E}(aX+b) = a\mathrm{E}(X) + b$$

The variance of a linear function of a random variable

What can we say about the variance of Y, where $Y = aX + b$? Whether the random variable X is discrete or continuous, we have

$$\mathrm{Var}(aX+b) = \mathrm{E}[(aX+b)^2] - [\mathrm{E}(aX+b)]^2$$

Now $\quad [\mathrm{E}(aX+b)]^2 = [a\mathrm{E}(X)+b]^2$
$$= (a\mu + b)^2$$
$$= a^2\mu^2 + 2ab\mu + b^2$$

and $\quad \mathrm{E}[(aX+b)^2] = \mathrm{E}(a^2X^2 + 2abX + b^2)$
$$= a^2\mathrm{E}(X^2) + 2ab\mu + b^2$$

Thus $\quad \mathrm{Var}(aX+b) = a^2\mathrm{E}(X^2) + 2ab\mu + b^2 - a^2\mu^2 - 2ab\mu - b^2$
$$= a^2\mathrm{E}(X^2) - a^2\mu^2$$
$$= a^2\mathrm{Var}(X)$$

Note: This calculation uses sums of random variables, which we discuss later in this section.

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Variance of a linear function of a random variable

If X is a random variable and $Y = aX + b$, where a and b are constants, then

$$\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X)$$

Although initially the absence of b in the variance may seem surprising, on reflection it makes sense that adding a constant merely changes the location of the distribution, and has no effect on its spread. Similarly, multiplying by a is in effect a scale change, and this is consistent with the result obtained.

Example 3

Suppose that X is a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 2$.

a Determine $E(2X + 1)$.

b Determine $\text{Var}(1 - 3X)$.

Solution

a $E(2X + 1) = 2E(X) + 1$
$= 2 \times 10 + 1 = 21$

b $\text{Var}(1 - 3X) = (-3)^2\text{Var}(X)$
$= 9 \times 2 = 18$

Linear combinations of independent random variables

From Mathematical Methods, you are familiar with the idea of independent events, that is, events A and B such that

$$P(A \cap B) = P(A) \cdot P(B)$$

The term independent can also be applied to random variables. While a formal definition of independent random variables is beyond the scope of this course, we say that two random variables are **independent** if their joint probability function is a product of their individual probability functions.

Consider, for example, the numbers observed when two dice are rolled. Let X_1 be the number observed when the first die is rolled, and X_2 be the number observed when the second die is rolled. The two random variables X_1 and X_2 are independent and have identical distributions.

What can we say about the distribution of $X_1 + X_2$?

Since the rolling of these two dice can be considered as independent events, we can determine probabilities associated with the sum by multiplying probabilities associated with each individual random variable. For example:

$$\begin{aligned} P(X_1 + X_2 = 2) &= P(X_1 = 1,\ X_2 = 1) \\ &= P(X_1 = 1) \cdot P(X_2 = 1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \end{aligned}$$

Example 4

Suppose that X_1 is the number observed when one die is rolled, and X_2 is the number observed when another die is rolled. Determine $P(X_1 + X_2 = 4)$.

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Solution

If $X_1 + X_2 = 4$, then the possible outcomes are:

- $X_1 = 1,\ X_2 = 3$
- $X_1 = 2,\ X_2 = 2$
- $X_1 = 3,\ X_2 = 1$

Thus $P(X_1 + X_2 = 4)$

$$= P(X_1 = 1,\ X_2 = 3) + P(X_1 = 2,\ X_2 = 2) + P(X_1 = 3,\ X_2 = 1)$$
$$= P(X_1 = 1) \cdot P(X_2 = 3) + P(X_1 = 2) \cdot P(X_2 = 2) + P(X_1 = 3) \cdot P(X_2 = 1)$$
$$= \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{12}$$

The mean and variance of the sum of two random variables

We next consider the mean and variance of the sum of two independent random variables.

Example 5

Suppose again that X_1 is the number observed when one die is rolled, and X_2 is the number observed when another die is rolled. Determine:

a $\mathrm{E}(X_1 + X_2)$

b $\mathrm{Var}(X_1 + X_2)$

Solution

We can readily determine the probability distribution of $X = X_1 + X_2$.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

a $\mathrm{E}(X) = \sum_x x \cdot P(X = x)$

$$= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36}$$
$$= \frac{252}{36} = 7$$

b $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

$$\mathrm{E}(X^2) = \sum_x x^2 \cdot P(X = x)$$
$$= \frac{4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144}{36} = \frac{1974}{36}$$

$$\therefore \quad \mathrm{Var}(X) = \frac{1974}{36} - 49 = \frac{35}{6}$$

How do these values compare with the mean and variance of X_1 and X_2?

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We can calculate $\mathrm{E}(X_1) = \mathrm{E}(X_2) = 3.5$, and we know that $\mathrm{E}(X_1 + X_2) = 7$. Thus we have

$$\mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2)$$

This result holds for any two random variables X_1 and X_2.

Similarly, we can calculate $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \dfrac{35}{12}$, and we know that $\mathrm{Var}(X_1 + X_2) = \dfrac{35}{6}$. Thus we have

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$$

This result holds for any two *independent* random variables X_1 and X_2.

Note: In general, for two independent random variables X_1 and X_2 that are identically distributed, we have $\mathrm{Var}(X_1 + X_2) \neq \mathrm{Var}(2X_1)$, since $\mathrm{Var}(X_1 + X_2) = 2\mathrm{Var}(X_1)$, but $\mathrm{Var}(2X_1) = 2^2\mathrm{Var}(X_1) = 4\mathrm{Var}(X_1)$.

The mean and variance of a linear combination of two random variables

Now consider a linear combination of two random variables X and Y. We have

$$\begin{aligned} \mathrm{E}(aX + bY) &= \mathrm{E}(aX) + \mathrm{E}(bY) \\ &= a\mathrm{E}(X) + b\mathrm{E}(Y) && \text{since } \mathrm{E}(aX) = a\mathrm{E}(X) \end{aligned}$$

If X and Y are independent, then

$$\begin{aligned} \mathrm{Var}(aX + bY) &= \mathrm{Var}(aX) + \mathrm{Var}(bY) && \text{since } X \text{ and } Y \text{ are independent} \\ &= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) && \text{since } \mathrm{Var}(aX) = a^2\mathrm{Var}(X) \end{aligned}$$

A linear combination of two random variables

For random variables X and Y and constants a and b:

- $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
- $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ if X and Y are independent

Example 6

A manufacturing process involves two stages. The time taken to complete the first stage, X hours, is a continuous random variable with mean $\mu = 4$ and standard deviation $\sigma = 1.5$. The time taken to complete the second stage, Y hours, is a continuous random variable with mean $\mu = 7$ and standard deviation $\sigma = 1$. Determine the mean and standard deviation of the total processing time, if the times taken at each stage are independent.

Solution

The total processing time is given by $X + Y$. The mean of the total processing time is

$$\begin{aligned} \mathrm{E}(X + Y) &= \mathrm{E}(X) + \mathrm{E}(Y) \\ &= 4 + 7 = 11 \end{aligned}$$

Since X and Y are independent, we have

$$\begin{aligned}\mathrm{Var}(X + Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y)\\ &= (1.5)^2 + (1)^2 = 3.25\end{aligned}$$

Hence the standard deviation of the total processing time is

$$\mathrm{sd}(X + Y) = \sqrt{3.25} = 1.803$$

We can extend this result to the sum of n independent identically distributed random variables as follows:

A linear combination of n independent random variables

For independent random variables $X_1, X_2, \dots, X_n$ and constants $a_1, a_2, \dots, a_n$:

- $\mathrm{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \cdots + a_n\mathrm{E}(X_n)$
- $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \cdots + a_n^2\mathrm{Var}(X_n)$

Summary 15A

- If X is a random variable and $Y = aX + b$, where a and b are constants, then
  - $\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
- For random variables X and Y and constants a and b:
  - $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
  - $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ if X and Y are independent
- For independent random variables $X_1, X_2, \dots, X_n$ and constants $a_1, a_2, \dots, a_n$:
  - $\mathrm{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \cdots + a_n\mathrm{E}(X_n)$
  - $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \cdots + a_n^2\mathrm{Var}(X_n)$

Skill-sheet

Exercise 15A

Example 1

SF

1 The number of chocolate bars produced by a manufacturer in any week has the following distribution.

x	1000	1500	2000	2500	3000	4000
$P(X = x)$	0.05	0.15	0.35	0.25	0.15	0.05

It costs the manufacturer \$450 per week, plus an additional 50 cents per chocolate bar, to produce the bars.

a Express C, the manufacturer's weekly cost of production, as a linear function of X.

b Determine the probability distribution of C.

c Determine the probability that the cost is more than \$2000 in any given week.

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SF

2 Sam plays a game with his sister Annabelle. He tosses a coin three times, and counts the number of times that the coin comes up heads. Annabelle charges him \$5 to play, and gives him \$2.50 for each head that he tosses.

a Express W, the net amount he wins, in terms of X, the number of heads observed in the three tosses.

b Determine the probability distribution of W.

c Evaluate the probability that the net amount he wins in a game is more than \$2.

Example 2

3 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Determine $P(X < 0.3)$.

b Let $Y = X + 1$. Determine $P(Y \le 1.5)$.

4 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{4}\right) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a Determine $P(X < 0.5)$.

b Let $Y = 3X - 1$. Determine $P(Y > 2)$.

5 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x+2}{16} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a Determine $P(X < 2.5)$.

b Let $Y = 4X + 2$. Determine $P(Y > 2)$.

Example 3

6 Suppose that X is a random variable with mean $\mu = 25$ and variance $\sigma^2 = 9$.

a Let $Y = 3X + 2$. Determine $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

b Let $U = 5 - 2X$. Determine $\mathrm{E}(U)$ and $\mathrm{sd}(U)$.

c Let $V = 4 - 0.5X$. Determine $\mathrm{E}(V)$ and $\mathrm{Var}(V)$.

7 A random variable X has density function f given by

$$f(x) = \begin{cases} 0.2 & \text{if } -1 \le x \le 0 \\ 0.2 + 1.2x & \text{if } 0 < x \le 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Determine

a $\mathrm{E}(X)$

b $\mathrm{Var}(X)$

c $\mathrm{E}(4X + 2)$ and $\mathrm{sd}(4X + 2)$.

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SF

Example 4 **8** The independent random variables X and Y have probability distributions as shown.

x	1	2	3
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

y	2	4
$P(Y = y)$	$\frac{1}{3}$	$\frac{2}{3}$

Let $S = X + Y$.

a Complete a table to show the probability distribution of S.

b Determine $\mathrm{E}(S)$

c Determine $P(S \le 5)$.

Example 5 **9** Suppose that X_1 is the number observed when a five-sided die is rolled, and X_2 is the number observed when another five-sided die is rolled. Determine from first principles:

a $\mathrm{E}(X_1)$ **b** $\mathrm{Var}(X_1)$

c $\mathrm{E}(X_1 - X_2)$ **d** $\mathrm{Var}(X_1 - X_2)$

10 The random variables X_1 and X_2 are independent and identically distributed, with means $\mu_{X_1} = \mu_{X_2} = 18$ and variances $\sigma^2_{X_1} = \sigma^2_{X_2} = 4$. Determine:

a $\mathrm{E}(2X_1 + 3)$ **b** $\mathrm{Var}(2X_1 + 3)$ **c** $\mathrm{E}(X_1 + X_2)$

d $\mathrm{Var}(2X_1)$ **e** $\mathrm{Var}(X_1 + X_2)$

Example 6 **11** To get to school, Jasmine rides her bike to the station and then catches the train. The time taken for her to ride to the station and catch the train, X minutes, is a continuous random variable with mean $\mu = 17$ and standard deviation $\sigma = 4.9$. The time taken for the train journey, Y minutes, is a continuous random variable with mean $\mu = 32$ and standard deviation $\sigma = 7$. Determine the mean and standard deviation of the total time taken for her to get to school, if the times taken for each part of the journey are independent.

12 Mikki buys three bags of bananas and two bags of apples from the greengrocer. If bags of bananas have a mean weight of 750 g, with a variance of 25, and bags of apples have a mean weight of 1000 g, with a variance of 50, all assumed to be independent random variables, determine the mean and standard deviation of the total weight of her purchases.

CF

13 Pippi is going on a school family picnic. Family groups will sit together on tables in the park. The number of children in a family, X, follows the distribution shown.

x	1	2	3	4
$P(X = x)$	0.5	0.3	0.15	0.05

The number of children in a family is independent of the number of children in any other family. Determine the probability that, if two families sit at the one table, there will be more than three children in the combined group.

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15B The distribution of sample means

Learning intentions

- To understand the sample mean, $\bar{X}$, as a random variable.
- To determine the mean and standard deviation of the sample mean $\bar{X}$ from the mean and standard deviation of X.
- To understand that the distribution of the sample mean $\bar{X}$ is normally distributed when X is normally distributed.
- To understand that the distribution sample mean $\bar{X}$ is normally distributed for any random variable X when the sample size is sufficiently large (the central limit theorem).

Suppose that in a particular country the mean IQ for the population is $\mu = 100$ and the standard deviation is $\sigma = 15$, and that we select a random sample of 10 people from this population and determine the mean IQ for the sample, $\bar{x}$.

A random sample of 10 people had IQ scores as follows:

105, 109, 104, 86, 118, 100, 81, 94, 70, 88

Here the sample mean is:

$$\bar{x} = \frac{105 + 109 + 104 + 86 + 118 + 100 + 81 + 94 + 70 + 88}{10} = 95.5$$

A second sample of 10 people had the following IQ scores:

114, 124, 128, 133, 95, 107, 117, 91, 115, 104

with sample mean:

$$\bar{x} = \frac{114 + 124 + 128 + 133 + 95 + 107 + 117 + 91 + 115 + 104}{10} = 112.8$$

Clearly these two sample means are different from both each other, and the population mean $\mu = 100$. If we take a number of samples of size 10 from the same population and determine the mean IQ for each of these samples, we obtain a 'distribution' of sample means. The means of these samples, $\bar{x}$, are the values of the random variable $\bar{X}$.

Since $\bar{x}$ varies according to the contents of the random samples, we can consider the sample means $\bar{x}$ as being the values of a random variable, which we denote by $\bar{X}$.

Let X be a random variable, with mean μ and standard deviation σ,

Samples of size n from the population can be described by n independent random variables, $X_1, X_2, \dots, X_n$, which have identical distributions to X.

The **sample mean** is defined to be

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

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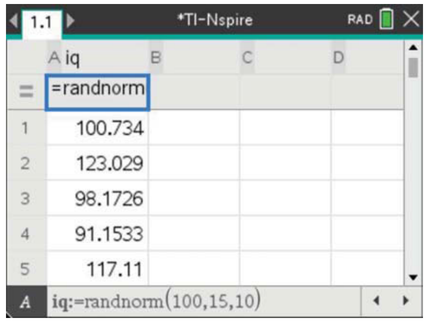
Since the distribution of IQ scores can be assumed to be normal, we can use a calculator to simulate drawing a random sample of size 10 from this population as follows:

Using the TI-Nspire CX non-CAS

To generate a random sample of size 10 from a normal population with mean 100 and standard deviation 15. On **Lists & Spreadsheet** page:

- Name the list 'iq' in Column A.
- In the formula cell of Column A, enter the formula using menu > **Data** > **Random** > **Normal** and complete as:
  = randnorm(100, 15, 10)

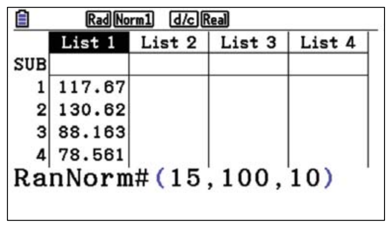


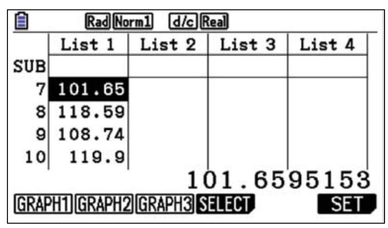
Note: The syntax is: randnorm(*mean*, *standard deviation*, *sample size*)

Using the Casio

To generate a random sample of size 10 from a normal population with mean 100 and standard deviation 15:

- Press MENU 2 to select **Statistics** mode.
- Move the cursor to the top of List1.
- Go to the **Probability** menu OPTN F5 and select **Rand** F4, **Norm** F3.
- Complete by entering:
  15, 100, 10)
- Use the cursor keys to scroll through the list of random numbers.

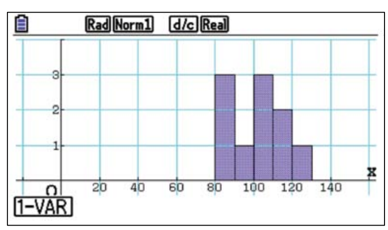




Note: The syntax is: RanNorm#(*standard deviation*, *mean*, *sample size*)

To plot a histogram of the sample:

- Press EXIT three times and select **Graph**.
  Use 0 for the start value and 10 for the width.



Since $\bar{X}$ is a linear combination of independent identically distributed random variables we use our findings from 15A to determine the mean and standard deviation of $\bar{X}$.

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The mean of the sample mean $\bar{X}$ is found as follows:

$$\begin{aligned}
\mathrm{E}(\bar{X}) &= \mathrm{E}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= \frac{1}{n}(\mathrm{E}(X_1) + \mathrm{E}(X_2) + \cdots + \mathrm{E}(X_n)) && \text{since } \mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y) \\
&= \frac{1}{n} \times n\mu && \text{since } \mathrm{E}(aX + b) = a\mathrm{E}(X) + b \\
&= \mu
\end{aligned}$$

Similarly, we can determine the variance of the sample mean $\bar{X}$:

$$\begin{aligned}
\mathrm{Var}(\bar{X}) &= \mathrm{Var}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= \frac{1}{n^2}\mathrm{Var}(X_1 + X_2 + \cdots + X_n) && \text{as } \mathrm{Var}(aX) = a^2\mathrm{Var}(X) \\
&= \frac{1}{n^2}(\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n)) && \text{as } \mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \\
& && \text{for } X \text{ and } Y \text{ independent} \\
&= \frac{1}{n^2} \times n\sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Hence, the standard deviation of $\bar{X}$ is:

$$\mathrm{sd}(\bar{X}) = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$$

The mean and standard deviation of the sample mean

Let X be a random variable with mean μ and standard deviation σ, and $\bar{X}$ be the sample mean determined from a sample of n. Then:

- The mean of $\bar{X}$, $\mathrm{E}(\bar{X}) = \mu$
- The standard deviation of $\bar{X}$, $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$

Example 7

The heights of a certain population of women have a mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. Determine the mean and standard deviation of the sample mean determined from a sample of size 16.

Solution

Let $\bar{X}$ the mean height of a sample of 16 women from this population.

$$\mathrm{E}(\bar{X}) = \mu = 160 \qquad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{16}} = 2.$$

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The distribution of the sample mean for a normal random variable

We now know how to determine the mean and standard deviation of the sample mean from the population mean. What else can we say about the distribution of $\bar{X}$? We can use technology to simulate values of the sample mean, in order to to examine this distribution further.

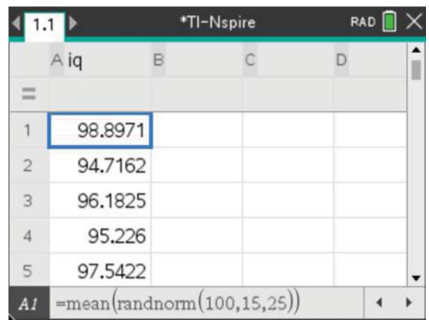
We will start by examining the distribution of the sample mean for a normal random variable. Consider again the IQ random variable, X, which is normally distributed with a mean $\mu = 100$ and standard deviation $\sigma = 15$. We can use technology to simulate the value of the sample mean from samples of size 25.

TI-84

Using the TI-Nspire CX non-CAS

To generate the sample means for 10 random samples of size 25 from a normal population with mean 100 and standard deviation 15. On a **Lists & Spreadsheet** page:

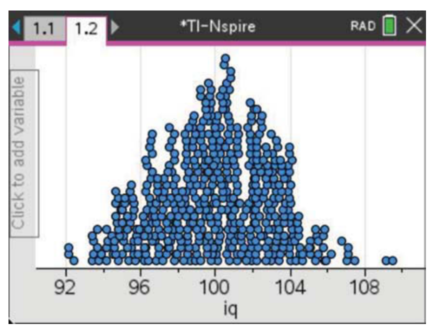
- Name the list 'iq' in Column A.
- In cell A1, enter the formula using (menu) > **Data** > **List Math** > **Mean** > (menu) > **Random** > **Normal** and complete as follows: = mean(randnorm(100, 15, 25))
- Fill down to obtain the 10 sample means.



For a large number of simulations, an alternative method is easier. To generate the sample means for 500 random samples of size 25, enter the following formula in the formula cell of Column A:

$$= \text{seq}(\text{mean}(\text{randnorm}(100, 15, 25)), k, 1, 500)$$

The dotplot on the right was created on a **Data & Statistics** page.

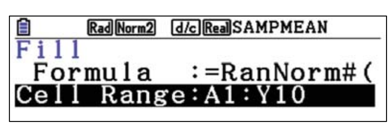


Using the Casio

Method 1: Using Spreadsheet mode

To generate 10 random samples of size 25 from a normal population with mean 100 and standard deviation 15:

- Press (MENU) (4) to select **Spreadsheet** mode.
- Move the cursor to cell A1.
- Select **Edit** (F2) and then **Fill** (F6) (F1).



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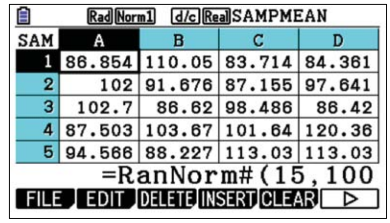
- Enter the formula:

  = RanNorm#(15, 100)

Note: To obtain the **RanNorm#** command, select **Probability** (OPTN) (F5), **Rand** (F4), **Norm** (F3).

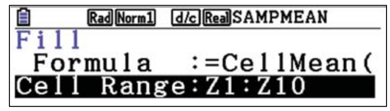


- Enter the cell range A1:Y10. (The first 10 rows will each contain a random sample of size 25.)

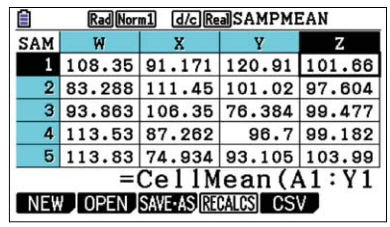
To determine the means of the 10 random samples:

- Select **Edit** (F2) and then **Fill** (F6) (F1).
- Enter the formula:

  = CellMean(A1:Y1)



Note: To obtain the **CellMean** command, select **Cell** (F4), **Mean** (F3).

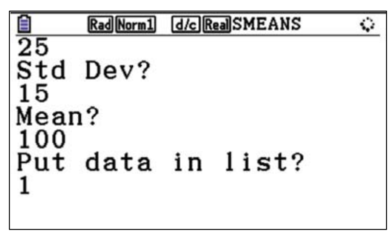


- Enter the cell range Z1:Z10. (Column Z will contain the mean of each random sample.)
- You can generate new random samples by selecting **File** (F1), **Recalcs** (F4).

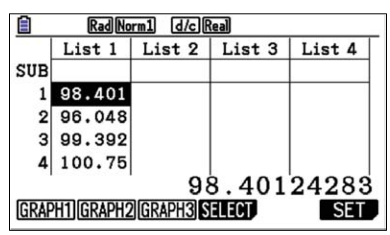
Method 2: Using a program

The program **SMEANS.g3m** can be used to generate a list of sample means for random samples from a normal distribution. Download this program from the Casio Education Australia website (www.casio.edu.shriro.com.au/app/view_module.php?id=339).

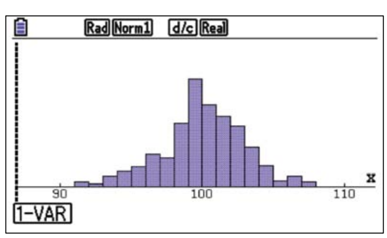
The program will prompt you to enter statistical parameters to generate a list of sample means in **Statistics** mode.

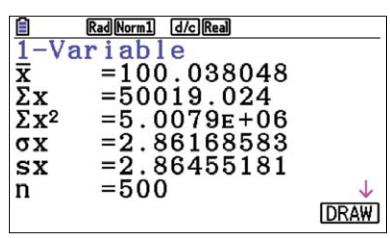


Here the program has been used to generate the sample means for 500 random samples of size 25 from a normal distribution with mean 100 and standard deviation 15.



The sample means are displayed in the histogram below. The statistics for the sample means are obtained by selecting **1-Var**.





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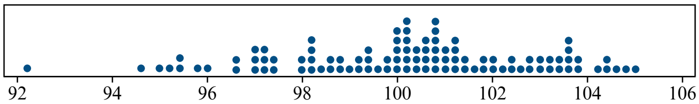
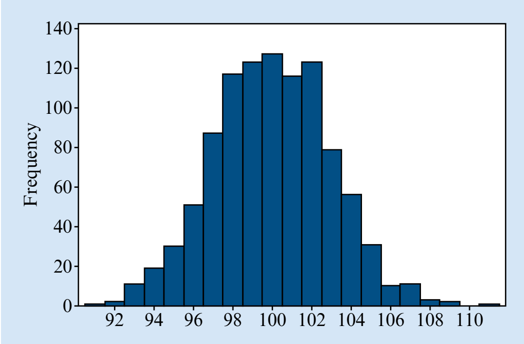
The following dotplot summarises the values of $\bar{x}$ observed for 100 samples (each of size 25).



We can see from the dot plot that the values of the sample mean are clustered around the values of the population mean (100) but we need more values of $\bar{x}$ to obtain a clear picture of the shape of the distribution.

This histogram shows the distribution of the sample mean when 1000 samples (each of size 25) were selected from a population with mean 100 and standard deviation 15.

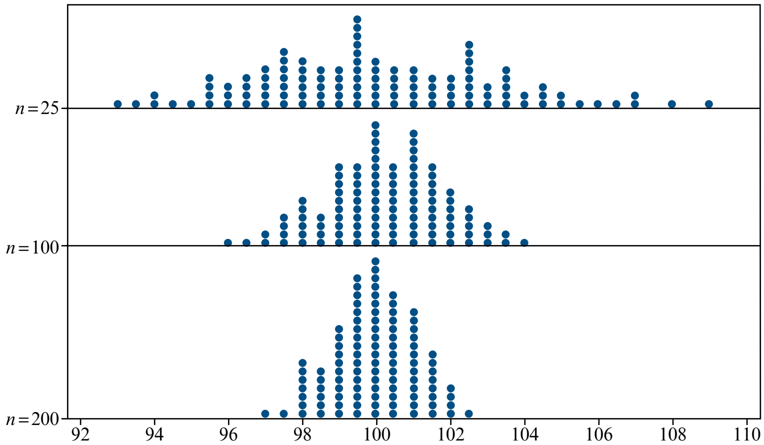


We see now see clearly from this plot that the distribution of sample means is symmetric and bell-shaped, suggesting that the sampling distribution of the sample mean may also be described by the normal distribution.

The effect of sample size on the distribution of the sample mean

We can also use simulation to demonstrate empirically the relationship between the standard deviation of the sample mean, and the size of the sample from which is was determined.

The following dotplots show the sample means $\bar{x}$ obtained when 200 samples of size 25, then size 100 and then size 200 were chosen from a population.



Each symbol represents up to 2 observations.

We can see from the dotplots that all three sampling distributions appear to be centred at 100, the value of the population mean μ. Furthermore, as the sample size increases, the values of the sample mean $\bar{x}$ are more tightly clustered around that value.

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These observations are confirmed in the following table, which gives the mean and standard deviation for each of the three simulated sampling distributions shown in the dotplots. The theoretical values of the mean and standard deviation of the sample mean are also included for comparison.

Sample size	25	100	200
Population mean μ	100	100	100
Mean of the values of $\bar{x}$	99.24	100.24	100.03
Standard deviation of the values of $\bar{x}$	3.05	1.59	1.06
$\mathrm{E}(\bar{X}) = \mu$	100	100	100
$\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$	3	1.5	1.061

Example 8

The sizes of kindergarten classes in a certain city are normally distributed, with a mean size of $\mu = 24$ children and a standard deviation of $\sigma = 2$.

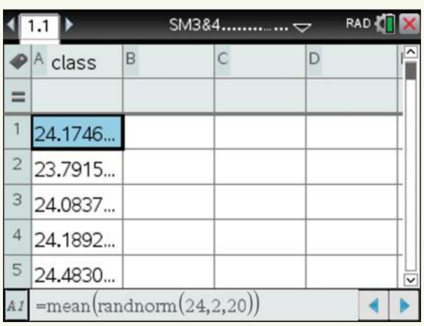
a Use your calculator to generate the sample means for 100 samples, each of size 20. Calculate the mean and standard deviation of these values of the sample mean.

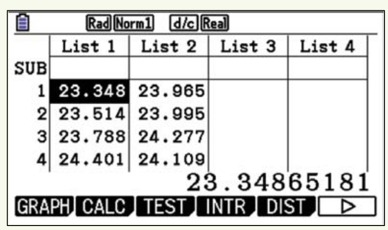
b Use your calculator to generate the sample means for 100 samples, each of size 50. Calculate the mean and standard deviation of these values of the sample mean.

c Compare the values of the mean and standard deviation calculated in **a** and **b** to each other.

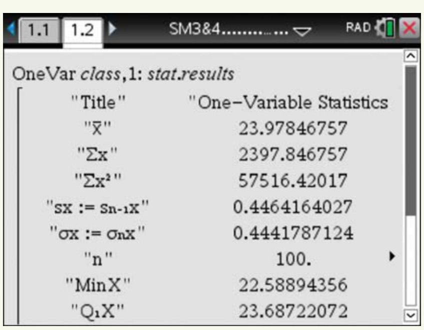
Solution

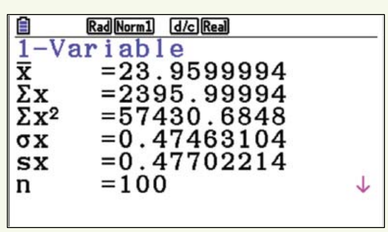
a

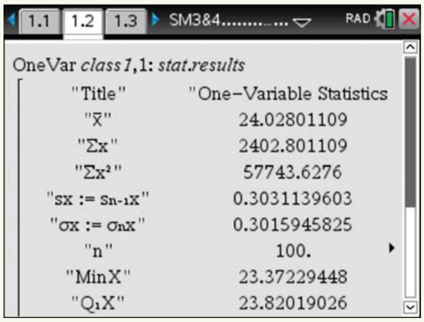


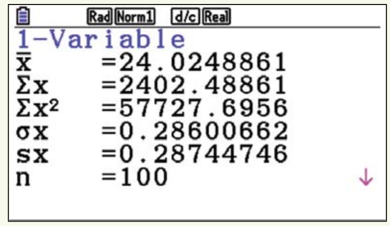


Note: Sample means for part a in List1 and part b in List2.





b 



c The means determined from the simulations are very similar, and close to the population mean of 24, as expected. The standard deviation for the samples of size 50 is much smaller than the standard deviation for the samples of size 20.

The distribution of the sample mean determined from a normal population

If X is a normally distributed random variable with mean μ and standard deviation σ, and $\bar{X}$ is the sample mean determined from a sample of n, then $\bar{X}$ is also normally distributed and with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

We can use this knowledge about the distribution of the sample mean to make predictions about its behaviour.

Example 9

Consider the population described in Example 7. Determine the probability that:

a a woman chosen at random has a height greater than 168 cm

b a sample of four women chosen at random has an average height greater than 168 cm.

Solution

a $P(X > 168) = P\left(Z > \dfrac{168 - 160}{8}\right) = P(Z > 1) = 0.1587$

b The distribution of the sample mean $\bar{X}$ is normal with mean $\mu_{\bar{X}} = \mu = 160$ and standard deviation $\sigma_{\bar{X}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{8}{\sqrt{4}} = 4$.

Thus

$$P(\bar{X} > 168) = P\left(Z > \frac{168 - 160}{4}\right) = P(Z > 2) = 0.0228$$

The distribution of the sample mean for non-normally distributed random variables

The sampling distribution of the sample mean $\bar{X}$ is normal if the distribution of X is normal. What can we say if X is not normally distributed? Using simulation, we can investigate empirically the sampling distribution of the sample mean for a variety of distributions.

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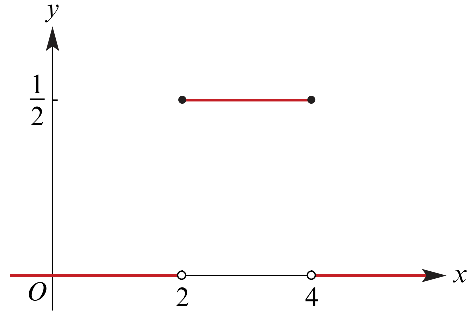
Consider, for example, the random variable X with probability density function

$$f(x) = \begin{cases} 0.5 & \text{if } 2 \le x \le 4 \\ 0 & \text{if } x < 2 \text{ or } x > 4 \end{cases}$$

The graph of this probability density function (shown on the right) is clearly not normal.

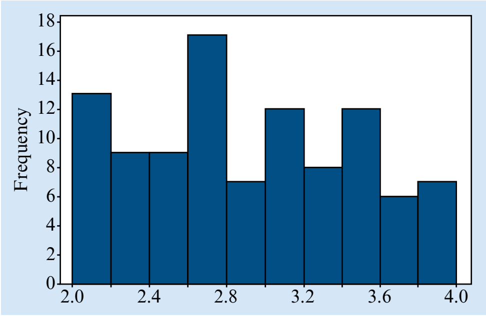
It can be readily verified that X has mean $\mu = 3$ and standard deviation $\sigma = \dfrac{1}{\sqrt{3}}$.



Simulating the distribution of data values

Suppose that we select a sample of size 100 from this distribution. The data arising from simulating one such sample are summarised in the following histogram.

From the theoretical probability distribution, we would expect the sample values to be reasonably evenly distributed between 2 and 4. That is, we might expect all of the columns in the histogram to be about the same height.

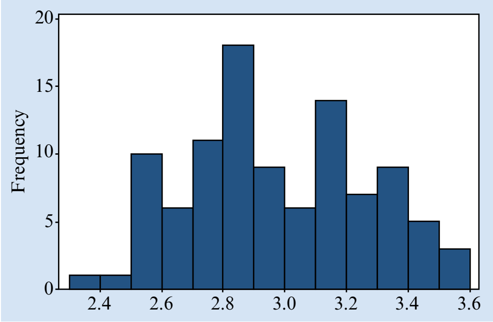


The actual histogram of the data shows a reasonable amount of variation in the individual values. The mean of the sample shown, $\bar{x}$, is 2.9 and the sample standard deviation, s, is 0.56.

Simulating the distribution of sample means

Consider now what the histogram might look like if each value represented was not an individual data value, but the mean of five data values.

To investigate the distribution of the sample mean, we select 100 samples, each of size 5. The distribution of sample means $\bar{x}$ is shown in the histogram on the right.



We can see that now the histogram does not show values evenly spread across the whole range. Instead, even with quite small samples, the sample means are clustering around the population mean $\mu = 3$.

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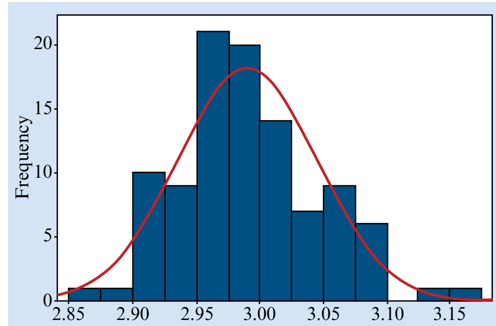
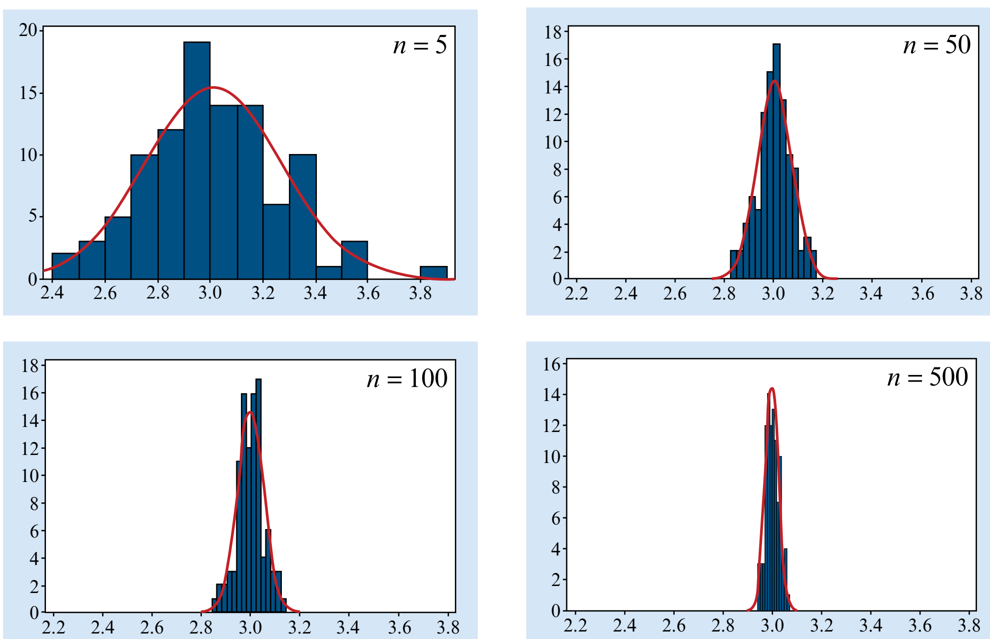
Increasing the sample size

What would be the effect of increasing the sample size from 5 to 100? To investigate this, we now select 100 samples, each of size 100. We can see from this histogram that these sample means are distributed quite symmetrically around the population mean $\mu = 3$ and that the sampling distribution can be quite well described as approximately normal.



So, while the distribution of X is clearly not normal, the sampling distribution of $\bar{X}$ is quite well approximated by a normal distribution.

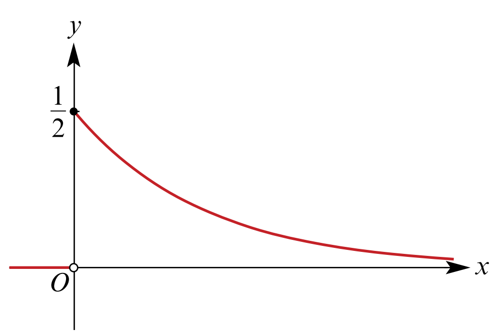
The following plots show how the sampling distribution of the sample mean becomes increasingly normal and less variable as the sample size increases.



Another example of the distribution of sample means

Let us consider another random variable X, with probability density function

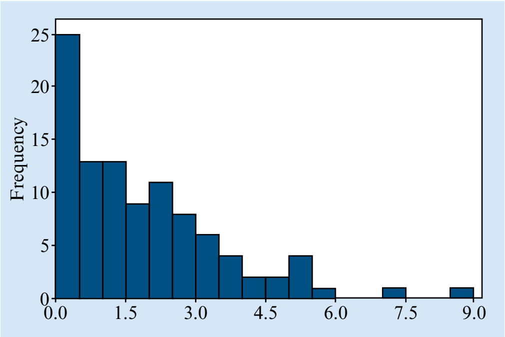
$$f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



Thus X is the exponential random variable with parameter $\lambda = \frac{1}{2}$, and so we know that X has mean $\mu = 2$ and standard deviation $\sigma = 2$.

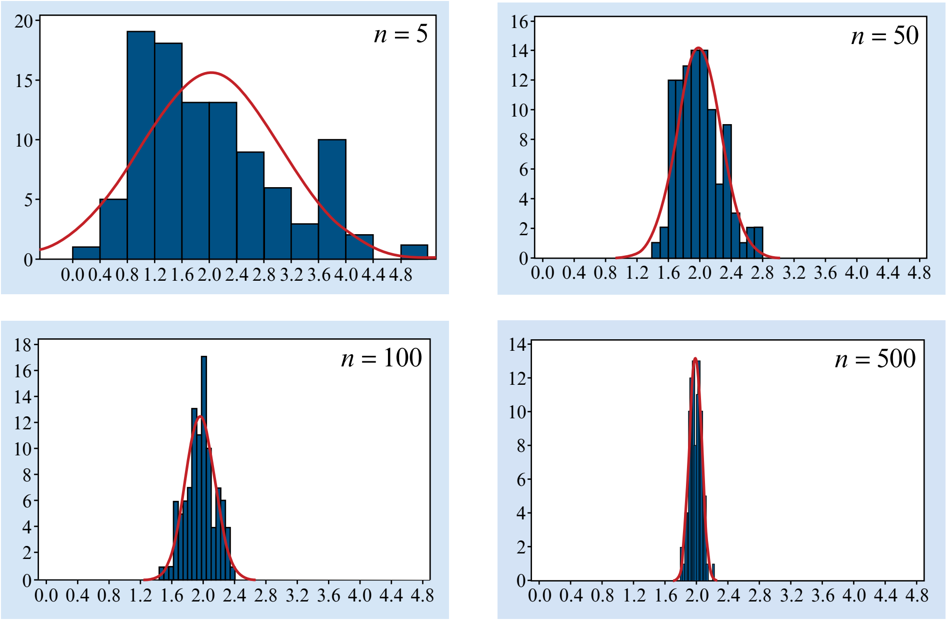
Suppose that we select a sample of size 100 from this distribution. The data arising from simulating one such sample are summarised in the histogram on the right.

The distribution is quite similar to the theoretical distribution, as we would expect. The mean of the sample shown, $\bar{x}$, is 1.9 and the sample standard deviation, s, is 1.7.



We now investigate the distribution of the sample mean by selecting 100 samples of size 5, then size 50, then size 100 and then size 500. The distributions of sample means $\bar{x}$ obtained are shown in the following histograms.

We see that the sampling distribution of the sample mean becomes increasingly normal and less variable as the sample size increases. Since the distribution is quite skewed to start with, a larger sample size is required before the sampling distribution of the sample mean begins to look normal.



Again, the distribution of X is clearly not normal, but the sampling distribution of $\bar{X}$ is quite well approximated by a normal distribution when the sample size is large enough.

The central limit theorem

From these two examples we have found that, for different underlying distributions, the sampling distribution of the sample mean is approximately normal, provided the sample size n is large enough. Furthermore, the approximation to the normal distribution improves as the sample size increases. This fact is known as the **central limit theorem**.

Central limit theorem

Let X be any random variable, with mean μ and standard deviation σ. Then, provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean $E(\bar{X}) = \mu$ and standard deviation $sd(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Note: For most distributions, a sample size of 30 is sufficient.

The central limit theorem may be used to solve problems associated with sample means, as illustrated in the following example.

Example 10

The amount of coffee, X mL, dispensed by a machine has a distribution with probability density function f defined by

$$f(x) = \begin{cases} \dfrac{1}{20} & \text{if } 160 \le x \le 180 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability that the average amount of coffee contained in 36 randomly chosen cups will be more than 173 mL.

Solution

The central limit theorem tells us that the distribution of the sample mean is approximately normal. To find the mean and standard deviation of the distribution, we first find the mean and standard deviation of X:

$$E(X) = \int_{160}^{180} \frac{x}{20}\, dx = \left[\frac{x^2}{40}\right]_{160}^{180} = 170$$

and $$E(X^2) = \int_{160}^{180} \frac{x^2}{20}\, dx = \left[\frac{x^3}{60}\right]_{160}^{180} = 28\,933.33$$

So $$sd(X) = \sqrt{28\,933.33 - 170^2} = 5.773$$

By the central limit theorem, the sample mean $\bar{X}$ is (approximately) normally distributed with

$$E(\bar{X}) = E(X) = 170 \qquad \text{and} \qquad sd(\bar{X}) = \frac{sd(X)}{\sqrt{n}} = \frac{5.77}{6} = 0.962$$

Therefore

$$P(\bar{X} > 173) = P\left(Z > \frac{173 - 170}{0.962}\right) = P(Z > 2.61) = 0.0009$$

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Summary 15B

- If X is a normally distributed random variable with mean μ and standard deviation σ, and $\bar{X}$ is the sample mean determined from a sample of n then $\bar{X}$ is also normally distributed and with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.
- If X is any random variable, with mean μ and standard deviation σ then, provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Exercise 15B

SF

Example 7

1 The distribution of final marks in an examination has a mean of 74 and a standard deviation of 8. A random sample of three students is selected and their mean mark calculated. Determine the mean and standard deviation of this sample mean.

2 A machine produces nails which have an intended diameter of $\mu = 25.025$ mm, with a standard deviation of $\sigma = 0.003$ mm. A sample of five nails is selected for inspection each hour and their average diameter calculated. Determine the mean and standard deviation of this average diameter.

Example 8

3 The lengths of a species of fish are normally distributed with mean length $\mu = 40$ cm and standard deviation $\sigma = 4$ cm.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 50 drawn from this population of fish.

b Summarise the values obtained in part **a** in a dotplot.

c Calculate the mean and standard deviation of these values of the sample mean.

4 The marks in a statistics examination at a certain university are normally distributed with a mean of $\mu = 48$ marks and a standard deviation of $\sigma = 15$ marks.

a Use your calculator to simulate 100 values of the sample mean calculated from a sample of size 20 drawn from the students at this university.

b Summarise the values obtained in part **a** in a dotplot.

c Determine the mean and standard deviation of these values of the sample mean.

5 Fuel consumption for a certain model of car is normally distributed with a mean of $\mu = 15$ litres per 100 km and a standard deviation of $\sigma = 0.75$ litres per 100 km.

a **i** Use your calculator to simulate 50 values of the sample mean calculated from a sample of size 25 drawn from this model of car.

ii Summarise these values of the sample mean in a dot plot, and describe the distribution.

iii Determine the mean and standard deviation of these values of the sample mean.

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SF

b Repeat part **a** using samples of size 50.

c Compare the means and standard deviations determined in parts **a** and **b** with each other and with the population mean μ and population standard deviation σ.

Example 9

6 The distribution of final marks in a statistics course is normal with a mean of 70 and a standard deviation of 6.

a Determine the probability that a randomly selected student has a final mark above 80.

b Determine the probability that the mean final mark for two randomly selected students is above 80.

c Compare the answers to parts **a** and **b**.

7 Suppose that IQ in a certain population is a normally distributed random variable, X, with mean $\mu = 100$ and standard deviation $\sigma = 15$.

a Determine the probability that a randomly selected individual has an IQ greater than 120.

b Determine the probability that the mean IQ of three randomly selected individuals is greater than 120.

c Compare the answers to parts **a** and **b**.

8 Gestation time for pregnancies without problems in humans is approximately normally distributed, with a mean of $\mu = 266$ days and a standard deviation of $\sigma = 16$ days. In the maternity ward of a large hospital, a random sample of seven women who had just given birth after pregnancies without problems was selected. Determine the probability that the average gestation period for these seven pregnancies exceeded 280 days.

9 Yearly income for those in the 18–25 age group living in a certain state is normally distributed with mean $\mu = \$32\,500$ and standard deviation $\sigma = \$6000$. Determine the probability that 10 randomly chosen individuals in this age group have an average income of less than \$28 000.

10 The IQ scores of adults are known to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. Determine the probability that a randomly chosen group of 25 adults will have an average IQ of more than 105.

11 The actual weight of sugar in a 1 kg package produced by a food-processing company is normally distributed with mean $\mu = 1.00$ kg and standard deviation $\sigma = 0.03$ kg. If the probability that the average weight of a randomly chosen sample of 20 packages is more than k is 5%, determine the value of k correct to three decimal places.

12 The tar content of a certain brand of cigarettes is known to be normally distributed with mean $\mu = 10$ mg and standard deviation $\sigma = 0.5$ mg. A random sample of 50 cigarettes is chosen and the average tar content determined. Determine the probability that this average is between 9.95 and 10.05 mg.

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Example 10

13 The lengths of blocks of cheese, X cm, produced by a machine have a distribution with probability density function

$$f(x) = \begin{cases} 5 & \text{if } 10.0 \le x \le 10.2 \\ 0 & \text{otherwise} \end{cases}$$

a Determine the probability that a randomly selected block is more than 10.1 cm long.

b Determine the probability that the average length of 30 randomly selected blocks is more than 10.12 cm.

SF

14 The mean number of accidents per week at an intersection is 3.2 and the standard deviation is 1.6. The distribution is discrete, and so is not normal. Determine the probability that the average number of accidents per week at the intersection over a year is less than 2.5.

15 The amount of pollutant emitted from a smokestack in a day, X kg, has probability density function f defined by

CF

$$f(x) = \begin{cases} \frac{4}{9}x(5 - x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

a Determine the probability that the amount of pollutant emitted on any one day is more than 0.5 kg.

b Determine the probability that the average amount of pollutant emitted on a random sample of 30 days is more than 0.5 kg.

16 The length of time, T minutes, that customers wait to be served at a coffee shop is an exponential random variable with the probability density function

$$f(t) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}t} & \text{if } t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine the probability that the average waiting time for a random sample of 30 customers is less than 1.9 minutes.

17 The number of cars, Y, sold in a week by a car salesperson has the following probability distribution.

y	0	1	2	3	4	5	6	7	8
$P(Y = y)$	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.002

Determine the probability that, over one year, the average number of cars sold per week by this salesperson is more than 2.

18 The time for a customer to be served at a fast-food outlet is normally distributed with a mean of 3.5 minutes and a standard deviation of 1.0 minutes. Determine the probability that 20 customers can be served in less than one hour.

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CU

19 The incubation period for a certain disease is between 5 and 11 days after contact. The probability of showing the first symptoms at various times during the incubation period is described by the probability density function

$$f(t) = \begin{cases} \frac{1}{36}(t-5)(11-t) & \text{if } 5 \le t \le 11 \\ 0 & \text{otherwise} \end{cases}$$

a Determine the probability that the average time for the appearance of symptoms for a random sample of 40 people with the disease was less than 7 days.

b Given that the probability that the average time for the appearance of symptoms for a random sample of n people was between 7 and 9 days is 95%, determine the value of n.

20 The life-time of a particular brand of electric light bulb has mean of μ hours and a standard deviation of 200 hours. The manufacture claims that they last on average for 1200 hours. Given that there is 10% probability that the mean life-time of 64 randomly selected bulbs is less than the claimed life time, determine the value of μ.

15C Confidence intervals for the population mean

Learning intentions

- To use the sample mean as a **point estimate** of the population mean.
- To understand the concept of an interval estimate for a parameter.
- To understand and be able to calculate a confidence level for any given level of confidence.
- To understand the effect on the confidence interval of varying the level of confidence.

In practice, the reason we analyse samples is to further our understanding of the population from which they are drawn. That is, we know what is in the sample, and from that knowledge we would like to infer something about the population.

Point estimates

Suppose, for example, we are interested in the mean IQ score of all Year 12 mathematics students in Australia. The value of the population mean μ is unknown. Collecting information about the whole population is not feasible, and so we rely on a random sample to help us.

What information can be obtained from a single sample? Certainly, a single value of the sample mean $\bar{x}$ gives some indication of the value of the population mean μ, and can be used when we have no other information.

The value of the sample mean $\bar{x}$ can be used to estimate the population mean μ. Since this is a single-valued estimate, it is called a **point estimate** of μ.

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Thus, if we select a random sample of 100 Year 12 mathematics students and determine that their mean IQ is 108.6, then the value $\bar{x} = 108.6$ serves as an estimate of the population mean μ.

Interval estimates

The value of the sample mean $\bar{x}$ obtained from a single sample is going to change from sample to sample, and while sometimes the value will be close to the population mean μ, at other times it will not. To use a single value to estimate μ can be rather risky. What is required is an interval that we are reasonably sure contains the parameter value μ.

An **interval estimate** for the population mean μ is called a **confidence interval** for μ.

Calculating confidence intervals

We have seen in the previous section that, whatever the underlying distribution of the random variable X, if the sample size n is large, then the sampling distribution of $\bar{X}$ is approximately normal with

$$\mathrm{E}(\bar{X}) = \mu \qquad \text{and} \qquad \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

By standardising, we can say that the distribution of the random variable

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

is approximated by that of the standard normal random variable Z.

For the standard normal random variable Z, we have

$$P(-1.9600 < Z < 1.9600) = 0.95$$

So we can state that, for large n:

$$P\left(-1.9600 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.9600\right) \approx 0.95$$

Multiplying through gives

$$P\left(-1.9600\,\frac{\sigma}{\sqrt{n}} < \bar{X} - \mu < 1.9600\,\frac{\sigma}{\sqrt{n}}\right) \approx 0.95$$

Further simplifying, we obtain

$$P\left(\bar{X} - 1.9600\,\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1.9600\,\frac{\sigma}{\sqrt{n}}\right) \approx 0.95$$

This final expression gives us an interval which, with 95% probability, will contain the value of the population mean μ (which we do not know).

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An approximate **95% confidence interval** for μ is given by

$$\left(\bar{x} - 1.9600\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\,\frac{\sigma}{\sqrt{n}}\right)$$

where:

- μ is the population mean (unknown)
- $\bar{x}$ is a value of the sample mean
- σ is the value of the population standard deviation
- n is the size of the sample from which $\bar{x}$ was calculated.

Note: Often when determining a confidence interval for the population mean, the population standard deviation σ is unknown. If the sample size is large (say $n \geq 30$), then we can use the sample standard deviation s in this formula as an approximation to the population standard deviation σ.

Example 11

Determine a 95% confidence interval for the mean IQ of Year 12 mathematics students in Australia, if we select a random sample of 100 students and determine the sample mean $\bar{x}$ to be 108.6. Assume that the standard deviation for this population is 15.

Solution

The interval is found by substituting $\bar{x} = 108.6$, $n = 100$ and $\sigma = 15$ into the expression for a 95% confidence interval:

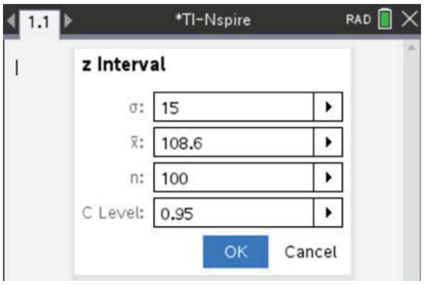
$$\left(\bar{x} - 1.9600\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\,\frac{\sigma}{\sqrt{n}}\right) = \left(108.6 - 1.9600 \times \frac{15}{\sqrt{100}},\ 108.6 + 1.9600 \times \frac{15}{\sqrt{100}}\right)$$

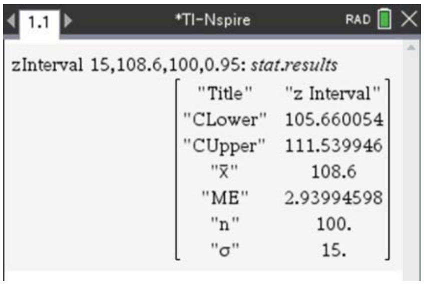
$$= (105.66, 111.54)$$

TI-84

Using the TI-Nspire CX non-CAS

- On a **Calculator** page, use menu > **Statistics** > **Confidence Intervals** > **z Interval**.
- If necessary, change the **Data Input Method** to **Stats**.
- Enter the given values and the confidence level as shown.
- The 'CLower' and 'CUpper' values give the 95% confidence interval (105.66, 111.54).

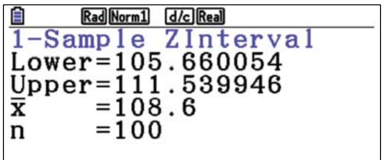




Note: 'ME' stands for margin of error, which is covered later in this section.

Using the Casio

- Press (MENU) (2) to select **Statistics** mode.
- For the Confidence Interval menu, select **Intr** (F4).
- Select **Z** (F1) and then **1-Sample** (F1).
- Ensure that the Data setting is Variable and the C-Level setting is 0.95.
- Enter the values $\sigma = 15$, $\bar{x} = 108.6$ and $n = 100$.
- The 'Lower' and 'Upper' values give the 95% confidence interval (105.66, 111.54).

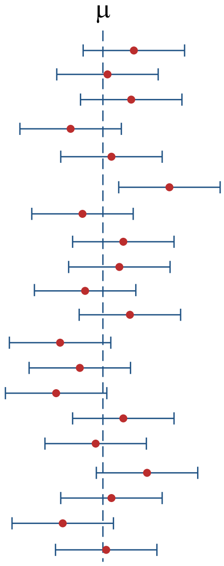


Interpretation of confidence intervals

The confidence interval found in Example 11 should not be interpreted as meaning that $P(105.66 < \mu < 111.54) = 0.95$. Since μ is a constant, the value either does or does not lie in the stated interval.

The particular confidence interval found is just one of any number of confidence intervals which could be found for the population mean μ, each one depending on the particular value of the sample mean $\bar{x}$.

The correct interpretation of the confidence interval is that we expect approximately 95% of such intervals to contain the population mean μ. Whether or not the particular confidence interval obtained contains the population mean μ is generally not known.



If we were to repeat the process of taking a sample and calculating a confidence interval many times, the result would be something like that indicated in the diagram.

The diagram shows the confidence intervals obtained when 20 different samples were drawn from the same population. The round dot indicates the value of the sample estimate in each case. The intervals vary, because the samples themselves vary. The value of the population mean μ is indicated by the vertical line.

It is quite easy to see from the diagram that none of the values of the sample estimate is exactly the same as the population mean, but that all the intervals except one (19 out of 20, or 95%) have captured the value of the population mean, as would be expected in the case of a 95% confidence interval.

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Varing the level of confidence

So far we have only considered 95% confidence intervals, but in fact we can choose any level of confidence for a confidence interval.

Consider again a 95% confidence interval:

$$\left(\bar{x} - 1.9600\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.9600\,\frac{\sigma}{\sqrt{n}}\right)$$

From our knowledge of the normal distribution, we can say that a 99% confidence interval will be given by

$$\left(\bar{x} - 2.5758\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + 2.57588\,\frac{\sigma}{\sqrt{n}}\right)$$

In general, an approximate ***C*% confidence interval** for μ is given by

$$\left(\bar{x} - z\,\frac{\sigma}{\sqrt{n}},\ \bar{x} + z\,\frac{\sigma}{\sqrt{n}}\right)$$

where the value of z is chosen so that $P(-z < Z < z) = C\%$, where Z is the standard normal random variable .

Note: The values of z (to four decimal places) for commonly used confidence intervals are:

- **90%** $z = 1.6449$
- **95%** $z = 1.9600$
- **99%** $z = 2.5758$

Example 12

Calculate and compare 90%, 95% and 99% confidence intervals for the mean IQ of Year 12 mathematics students in Australia, if we select a random sample of 100 students and determine the sample mean $\bar{x}$ to be 108.6. (Assume that $\sigma = 15$.)

Solution

From Example 11, we know that the 95% confidence interval is (105.66, 111.54).

The 90% confidence interval is

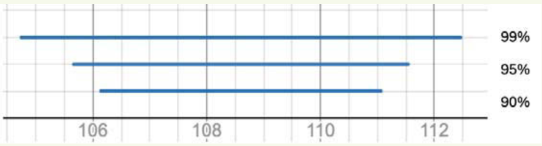
$$\left(108.6 - \frac{1.6449 \times 15}{10},\ 108.6 + \frac{1.6449 \times 15}{10}\right) = (106.13, 111.07)$$

The 99% confidence interval is

$$\left(108.6 - \frac{2.5758 \times 15}{10},\ 108.6 + \frac{2.5758 \times 15}{10}\right) = (104.74, 112.46)$$

It is helpful to use a diagram to compare these confidence intervals.

From the diagram, it can be clearly seen that the effect of being more confident that the confidence interval captures the true value of the population proportion means that a wider interval is required.



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Summary 15C

- The value of the sample mean $\bar{x}$ calculated from a sample can be used as a **point estimate** of μ, the population mean.
- An **interval estimate** for the population mean μ is called a **confidence interval**.
- An approximate ***C*% confidence interval** for μ is given by

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}},\ \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$$

  where:
  - μ is the population mean (unknown)
  - $\bar{x}$ is a value of the sample mean
  - n is the size of the sample from which $\bar{x}$ was calculated
  - z is such that $P(-z < Z < z) = C\%$, where Z is the standard normal random variable.
- When the sample size is large enough (say $n \geq 30$), and σ is unknown, then the sample standard deviation s can be used in the formula of the approximate ***C*% confidence interval** for μ. That is given by

$$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$$

- Values of z (to four decimal places) for commonly used confidence intervals:
  - **90%** $z = 1.6449$
  - **95%** $z = 1.9600$
  - **99%** $z = 2.5758$

Skill-sheet

Exercise 15C

Example 11

SF

1 A university lecturer selects a sample of 40 of her first-year students to determine how many hours per week they spend on study outside class time. She determines that their average study time is 7.4 hours. If the standard deviation of study time, σ, is known to be 1.8 hours, determine a 95% confidence interval for the mean study time for the population of first-year students.

2 The lengths of time (in seconds) for which each of a randomly selected sample of 12-year-old girls could hold their breath are as follows.

14	43	16	25	25	35	14	42	23	33	20	60
39	68	18	20	25	30	20	32	54	35	45	48

If breath-holding time is known to be normally distributed, with a standard deviation of 15 seconds, determine a 95% confidence interval for the mean time for which a 12-year-old girl can hold her breath.

3 A random sample of 49 of a certain brand of batteries was found to last an average of 14.6 hours. If the standard deviation of battery life is known to be 20 minutes, determine a 95% confidence interval for the mean time that the batteries will last.

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Example 12

SF

4 Calculate and compare 90%, 95% and 99% confidence intervals for the mean battery life for a certain brand of batteries, if the mean life of 35 batteries was found to be 35.7 hours. (Assume that $\sigma = 15$.)

5 Resting pulse rates were measured for a group of 90 randomly chosen 7-year-old children from a city, giving a sample mean of $\bar{x} = 82.6$ beats per minute and a standard deviation of $s = 10.3$ beats per minute. Calculate and compare 90%, 95% and 99% confidence intervals for the mean resting pulse rate of all 7-year-old children in the city.

6 In an investigation of physical fitness of students, resting heart rates were recorded for a sample of 30 female students. The sample had a mean of 71.1 beats per minute. The investigator knows from experience that resting heart rates are normally distributed and have a standard deviation of 6.4 beats per minute. Determine a 98% confidence interval for the mean resting heart rate of the relevant population of female students.

7 Fifty plots are planted with a new variety of corn. The average yield for these plots is 130 bushels per acre. Assuming that the standard deviation is equal to 10, determine a 99% confidence interval for the mean yield, μ, of this variety of corn.

8 An investigation was conducted into the total distance travelled by cars currently in use in a city. A random sample of 500 cars were stopped and the distances they had travelled were recorded. The sample mean was 46 724 km and the sample standard deviation was 15 172 km. Calculate a 98% confidence interval for the average distance travelled by cars in this city.

9 Twenty-two air samples taken at the same place over a period of six months showed the following amounts of suspended matter (in micrograms per cubic metre of air).

68	22	36	32	42	24	28	38	39	26	21
79	45	57	59	34	43	57	30	31	28	30

Assuming these measurements to be a random sample from a normally distributed population with standard deviation 10, determine a 97% confidence interval for the mean amount of suspended matter during that time period.

10 It is known that IQ scores in the general population are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

- **a** Use your calculator to generate 10 values of the sample mean $\bar{x}$ for random samples of size 20 drawn from this population.
- **b** Determine a 90% confidence interval for the population mean μ from each of these values of the sample mean $\bar{x}$.
- **c** How many of these intervals contain the value of the population mean μ?
- **d** How many of these intervals would you expect to contain the value of the population mean μ?
- **e** Determine the probability that they all contain the value of the population mean.

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SF

11 Suppose that marks on a mathematics test are normally distributed with mean $\mu = 30$ and standard deviation $\sigma = 7$.

a Use your calculator to generate 10 values of the sample mean $\bar{x}$ for random samples of size 30 drawn from this population.

b Use your calculator to determine an 80% confidence interval for the population mean μ from each of these values of the sample mean $\bar{x}$.

c How many of these intervals contain the value of the population mean μ?

d How many of these intervals would you expect to contain the value of the population mean μ?

e Determine the probability that they will all contain the value of the population mean.

12 Each hour the supervisor of a production line packing bags of potato crisps selects a random sample of 30 bags of crisps, weighs each bag, calculates the sample mean weight, and then determines a 95% confidence interval for the mean weight of the bags being packed. The production line operates for 20 hours each day, 7 days per week.

a How many many of the confidence intervals determined each week would be expected to contain μ?

b Determine the probability that at least one of the confidence intervals determined one week does not contain μ.

CF

13 The mean lifetime of a certain brand of light bulb is known to be 120 hours. In a new complex of rooms, 130 new light bulbs were installed, and the mean length of time that the light bulbs lasted was $\bar{x} = 118.6$ hours, with a standard deviation of $s = 7.5$ hours. Determine the smallest confidence level that could be used to produce a confidence interval, centred on the sample mean, that contains μ, based on this sample data. Give your answer as a percentage correct to the nearest whole number.

14 The mean height of a population of women is known to be 160 cm. A random sample of 50 women was chosen and their mean height was $\bar{x} = 162.4$ cm, with a standard deviation of $s = 8.9$ cm. Determine the smallest confidence level that could be used to produce a confidence interval, centred on the sample mean, that contains μ, based on this sample data. Give your answer as a percentage correct to the nearest whole number.

CU

15 The volume of soft drink in a 1 litre bottle is normally distributed with a mean of 1.012 litres and a standard deviation of 2 ml. After the machine filling the bottles was serviced, a random sample of 40 bottles was selected to see if the mean volume produced by the machine had changed. The sample mean was found to be 1.008 litres.

a Determine a 95% confidence interval for the mean volume of soft drink in the 1 litres bottles produced after the machine was serviced, assuming the standard deviation is unchanged.

b Use your answer to part **a** to comment on whether or not the mean volume of soft drink produced by the machine has changed after servicing.

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15D Margin of error

Learning intentions

- To understand and use the approximate margin of error.
- To understand and use the relationship between margin of error, level of confidence and sample size.

The width of a confidence interval is an important consideration when we are designing a research study. If the confidence interval is too wide, then our results may be unhelpful. Often we discuss the confidence interval in terms of its width or, more formally, in terms of the distance between the sample estimate and the endpoints of the confidence interval.

We know that the general rule for a confidence interval for the population mean when σ is known is

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}},\ \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$$

where the value of z is chosen so that $P(-z < Z < z) = C\%$.

The distance between the sample mean $\bar{x}$ and the confidence interval endpoints is called the **margin of error**, and denoted $\boldsymbol{E}$.

The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (E). For a C% confidence interval the approximate margin of error, E, is given by

$$E = z\frac{\sigma}{\sqrt{n}} \quad \text{when } \sigma \text{ is known}$$

or

$$E = z\frac{s}{\sqrt{n}} \quad \text{when } \sigma \text{ is unknown and } n \geq 30$$

and where the value of z is chosen so that $P(-z < Z < z) = C\%$, where Z is the standard normal random variable.

Example 13

Determine the margin of error in a 95% confidence interval for the mean IQ of Year 12 mathematics students in Australia, if we select a random sample of 400 students and determine the sample mean $\bar{x}$ to be 108.6. Assume that the standard deviation for this population is 15.

Solution

The margin of error is found by substituting $n = 400$ and $\sigma = 15$ into the expression for the margin of error in a 95% confidence interval:

$$E = 1.9600 \times \frac{15}{\sqrt{400}} = 1.47$$

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In the previous section we noted that changing the level of confidence changes the width of the confidence interval. More specifically, we saw that effect of being more confident that the confidence interval captures the true value of the population proportion meant that a wider interval was required.

Example 14

Calculate and compare the margins of error in 90%, 95% and 99% confidence intervals for the number of hours each week which Year 12 students spend doing homework if we select a random sample of 100 students and determine the sample mean $\bar{x}$ to be 8.65 hours and the sample standard deviation s to be 2.9 hours.

Solution

The margin of error in the 90% confidence interval is:

$$E = 1.6449 \times \frac{2.9}{\sqrt{100}} = 0.477$$

The margin of error in the 95% confidence interval is:

$$E = 1.9600 \times \frac{2.9}{\sqrt{100}} = 0.568$$

The margin of error in the 99% confidence interval is:

$$E = 2.5758 \times \frac{2.9}{\sqrt{100}} = 0.747$$

Comparing the margins of error we see that, as expected, the margin of error in the 90% confidence interval is less than the margin of error in the 95% confidence interval, which is in turn less than the margin of error in the 99% confidence interval.

Reducing the level of confidence in order to reduce the margin of error is not advisable, as it may mean we are estimating with an unacceptable level of confidence. In practice, we rarely estimate a population parameter with a level of confidence less than 95%.

Looking at the rule for margin of error we can see that E is is a function of the sample size n, and that increasing the sample size will also reduce the margin of error.

Example 15

Calculate and compare the margins of error in the 95% confidence intervals for the number of hours each week which Year 11 students spend doing homework when the sample mean ($\bar{x}$ =5.65 hours) and the sample standard deviation (s = 1.6 hours) are determined from samples of size n = 25, n = 100 or n = 400 respectively.

Solution

The margin of error in the 95% confidence interval when n = 25 is

$$E = 1.9600 \times \frac{1.6}{\sqrt{25}} = 0.627$$

The margin of error in the 95% confidence interval when n = 100 is

$$E = 1.9600 \times \frac{1.6}{\sqrt{100}} = 0.314$$

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The margin of error in the 95% confidence interval when $n = 400$ is

$$E = 1.9600 \times \frac{1.6}{\sqrt{400}} = 0.157$$

Comparing the margins of error we see that the margin of error when $n = 100$ is half the margin of error when $n = 25$, and that the margin of error when $n = 400$ is half the margin of error when $n = 100$.

Thus, we can see that, as expected, the margin of error in the 95% confidence interval decreases as the sample size increases, and that this decrease is proportional to the square root of the increase in the sample size.

The best way to ensure that the margin of error is acceptably small is to determine the size of the sample required for a specified margin of error before the data is collected. This calculation is illustrated in the next example.

Example 16

Consider again the problem of estimating the average IQ of Year 12 mathematics students in Australia. What size sample is required to ensure a margin of error of 1.5 points or less at the 95% confidence level? (Assume that $\sigma = 15$.)

Solution

In a 95% confidence interval, when σ is known:

$$E = 1.9600 \frac{\sigma}{\sqrt{n}}$$

Substituting $E = 1.5$ and $\sigma = 15$ gives

$$1.5 = 1.9600 \times \frac{15}{\sqrt{n}}$$

Solving for n:

$$1.5 \times \sqrt{n} = 1.9600 \times 15$$

$$\therefore \quad n = \left(\frac{1.9600 \times 15}{1.5}\right)^2 = 384.16$$

Thus a minimum sample of 385 students is needed to achieve a margin of error of at most 1.5 points in a 95% confidence interval for the population mean.

A C% confidence interval for a population mean μ will have margin of error equal to a specified value of E when the sample size is

$$n = \left(\frac{z\sigma}{E}\right)^2 \quad \text{when } \sigma \text{ is known}$$

or

$$n = \left(\frac{zs}{E}\right)^2 \quad \text{when } \sigma \text{ is unknown and } n \geq 30$$

and where the value of z is chosen so that $P(-z < Z < z) = C\%$, where Z is the standard normal random variable.

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Summary 15D

- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (E).
- For a C% confidence interval the approximate margin of error, E is given by

$$E = z\frac{\sigma}{\sqrt{n}} \quad \text{when } \sigma \text{ is known}$$

or

$$E = z\frac{s}{\sqrt{n}} \quad \text{when } \sigma \text{ is unknown and } n \geq 30$$

- A C% confidence interval for a population mean μ will have margin of error equal to a specified value of E when the sample size is

$$n = \left(\frac{z\sigma}{E}\right)^2 \quad \text{when } \sigma \text{ is known}$$

or

$$n = \left(\frac{zs}{E}\right)^2 \quad \text{when } \sigma \text{ is unknown and } n \geq 30$$

- In these formulae:
  - n is the size of the sample from which the mean was calculated
  - σ is the population standard deviation
  - s is the sample standard deviation
  - z is chosen so that $P(-z < Z < z) = C\%$, where Z is the standard normal random variable.

Skill-sheet

Exercise 15D

Example 13

SF

1 A random sample of 100 males who had married were asked the age at which they married. The average age given was 29.5 years, with a standard deviation of 10 years. Use this information to determine the margin of error in a 95% confidence interval for the mean age of marriage for males.

2 The mean time spent in physical exercise by a sample of 40 Year 12 students was 6.2 hours with a standard deviation of 2.3 hours. Use this information to determine the margin of error in a 99% confidence interval for the population mean.

3 The following is a list of scores on a manual-dexterity test for children with a particular learning disability.

20	30	19	21	33	20	21	17	25	25	32
26	31	22	23	26	26	23	25	17	27	21
23	27	24	28	21	33	22	23	17	26	24

Determine the margin of error in a 95% confidence interval for the mean score on this test for children with this learning disability.

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SF

Example 14

4 The weights of a certain species of cats are normally distributed. A random sample of 50 cats was found to have mean weight of 4.8 kg with a standard deviation of 0.9 kg.

a Determine the margin of error in a confidence interval for the mean weight of this species of cat when the level of confidence is:

i 90% **ii** 95% **iii** 99%

b Compare your answers to parts **ai**, **aii** and **aiii** and comment on the relationship between the margin of error and the level of confidence.

5 In order to estimate the mean height of adult males in a certain population a random sample of 60 adult males was chosen. The sample mean was determined to be 176.4 cm and the sample standard deviation was 7.3 cm.

a Determine the margin of error in a confidence interval for the mean height of adults males in this population when the level of confidence is:

i 90% **ii** 95% **iii** 99%

b Compare your answers to parts **ai**, **aii** and **aiii** and comment on the relationship between the margin of error and the level of confidence.

6 The volume of soft drink in a 1-litre bottle is normally distributed. A random sample of 30 bottles was found to have a mean volume of 1.05 litres and a standard deviation of 0.05 litres.

a Determine the margin of error in a confidence interval for the mean volume of soft drink in the bottles when the level of confidence is:

i 90% **ii** 95% **iii** 99%

b Compare your answers to parts **ai**, **aii** and **aiii** and comment on the relationship between the margin of error and the level of confidence.

Example 15

7 A quality-control engineer in a factory needs to estimate the mean weight, μ grams, of bags of potato chips that are packed by a machine. The engineer knows by experience that $\sigma = 2.0$ grams for this machine.

a The engineer takes a random sample of 36 bags and calculates the sample mean to be 25.4 grams. Calculate the margin of error in a 95% confidence interval for μ.

b Suppose the mean of 25.4 grams was calculated from a sample of 100 bags. Calculate the margin of error in a 95% confidence interval for μ.

c Compare your answers in parts **a** and **b**.

8 Calculate and compare the margins of error in the 99% confidence intervals for the useful life (in hours) of a fluorescent tube designed for indoor gardening, when the sample mean ($\bar{x} = 605$ hours) and the sample standard deviation ($s = 4$ hours) are determined from samples of size $n = 50$, $n = 100$ or $n = 200$ respectively.

Example 16

9 For a population with a standard deviation of 100, how large a random sample is needed in order to ensure that the margin of error in a 95% confidence interval is less than 20?

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SF

10 A quality-control engineer in a factory needs to estimate the mean weight, μ grams, of bags of potato chips that are packed by a machine. The engineer knows by experience that $\sigma = 2.0$ grams for this machine. Determine the size of the sample that should be used to ensure that we can be 95% confident that the estimate will be within 0.5 g of μ.

11 The number of customers per day at a fast-food outlet is known to have a standard deviation of 50. Determine the size of the sample required so that the owner can be 99% confident that the difference between the sample mean and the population mean is not more than 10.

12 A manufacturer knows that the standard deviation of the lifetimes of their light bulbs is 150 hours. Determine the size of the sample required so that the manufacturer can be 90% confident that the sample mean, $\bar{x}$, will be within 20 hours of the population mean.

13 Consider once again the problem of estimating the average IQ score, μ, of Year 12 mathematics students. (Assume that $\sigma = 15$.)

a Determine the size of the sample required to ensure with 95% confidence that the estimated mean IQ will be within 2 points of μ.

b Determine the size of the sample to ensure with 95% confidence that the estimated mean IQ will be within 1 point of μ.

c In general, what is the effect on the sample size of halving the margin of error?

CF

14 A 95% confidence interval for the mean weight of tomatoes (in kg) produced by a certain variety of tomato plants based on a sample of n plants is $(4.898, 5.502)$.

a What is the margin of error in this confidence interval?

b If the standard deviation of the weight of tomatoes produced by this sample of plants is 1.0 kg, determine the value of n.

15 A confidence interval for the mean score on an aptitude test determined from a sample of 45 people is $(67.7, 72.3)$.

a What is the margin of error in this confidence interval?

b If the standard deviation of the score from this sample is 8.4, determine the level of confidence used to calculate the confidence interval. Give the answer as a percentage correct to one decimal place.

16 A 95% confidence interval for μ is (a, b). Determine an expression for a 99% confidence interval for μ in terms of a and b based on the same sample data.

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Chapter summary

Linear combinations of random variables

- For a random variable X and constants a and b:
  - $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
- For random variables X and Y and constants a and b:
  - $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
  - $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ if X and Y are independent
- For independent random variables $X_1, X_2, \dots, X_n$ and constants $a_1, a_2, \dots, a_n$:
  - $\mathrm{E}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \cdots + a_n\mathrm{E}(X_n)$
  - $\mathrm{Var}(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \cdots + a_n^2\mathrm{Var}(X_n)$
- Let X and Y be independent normal random variables and let a and b be constants. Then $aX + bY$ is also a normal random variable.

Distribution of sample means

- The **population mean** μ is the mean of all values of a measure in a population. The **sample mean** $\bar{x}$ is the mean of these values in a particular sample.
- The sample mean $\bar{X}$ can be viewed as a random variable, and its distribution is called a **sampling distribution**.
- If X is a normally distributed random variable with mean μ and standard deviation σ, and $\bar{X}$ is the sample mean determined from a sample of n then $\bar{X}$ is also normally distributed and with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

  Central limit theorem
- If X is any random variable, with mean μ and standard deviation σ then, provided that the sample size n is large enough, the distribution of the sample mean $\bar{X}$ is approximately normal with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Confidence intervals for the population mean

- The value of the sample mean $\bar{x}$ calculated from a sample can be used as a **point estimate** of μ, the population mean.
- An **interval estimate** for the population mean μ is called a **confidence interval**.
- An approximate ***C*% confidence interval** for μ is given by

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}},\ \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$$

  where:
  - μ is the population mean (unknown)
  - $\bar{x}$ is a value of the sample mean
  - n is the size of the sample from which $\bar{x}$ was calculated
  - z is such that $P(-z < Z < z) = C\%$, where Z is the standard normal random variable.

Review

- When the sample size is large enough (say $n \geq 30$), and σ is unknown, then the sample standard deviation s can be used in the formula of the approximate ***C*% confidence interval** for μ. That is given by

$$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$$

- Values of z (to four decimal places) for commonly used confidence intervals:
  - **90%** $z = 1.6449$
  - **95%** $z = 1.9600$
  - **99%** $z = 2.5758$
- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (E).
- For a C% confidence interval the approximate margin of error, E is given by

$$E = z\frac{\sigma}{\sqrt{n}} \quad \text{when } \sigma \text{ is known}$$

or

$$E = z\frac{s}{\sqrt{n}} \quad \text{when } \sigma \text{ is unknown and } n \geq 30$$

- A C% confidence interval for a population mean μ will have margin of error equal to a specified value of E when the sample size is

$$n = \left(\frac{z\sigma}{E}\right)^2 \quad \text{when } \sigma \text{ is known}$$

or

$$n = \left(\frac{zs}{E}\right)^2 \quad \text{when } \sigma \text{ is unknown and } n \geq 30$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

15A **1** I can understand and use the distribution of a linear function of a discrete random variable. ☐

See Example 1 and Question 1

15A **2** I can understand and use the distribution of a linear function of a continuous random variable. ☐

See Example 2 and Question 3

15A **3** I can determine the mean and variance of a linear function of a random variable. ☐

See Example 3 and Question 6

15A **4** I can determine the probability distribution of the sum of two discrete random variables. ☐

See Example 4 and Question 8

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Review

15A **5** I can determine the mean and variance of the linear combination of two random variables. ☐

See Example 5, Example 6 and Questions 9 and 11

15B **6** I can determine the mean and variance of the sample mean. ☐

See Example 7 and Question 1

15B **7** I can use simulation to investigate the distribution of the sample mean from a normal population. ☐

See Example 8 and Question 3

15B **8** I can calculate probabilities associated with the sample mean from a normal population. ☐

See Example 9 and Question 6

15B **9** I can calculate probabilities associated with the sample mean from a a non-normal population. ☐

See Example 10 and Question 13

15C **10** I can calculate and compare confidence intervals for the population mean for varying levels of confidence. ☐

See Example 11, Example 12 and Questions 1 and 4

15D **11** I can calculate and compare the margin of error in a confidence interval for varying levels of confidence. ☐

See Example 13, Example 14 and Questions 1 and 4

15D **12** I can calculate and compare the margin of error in a confidence interval for varying sample sizes. ☐

See Example 15 and Question 7

15D **13** I can determine the sample size required for a given margin of error. ☐

See Example 16 and Question 16

Short - response questions

Technology-free short-response questions

SF

1 X is a random variable with mean $\mu = 15$ and variance $\sigma^2 = 25$, and $X_1, X_2, \ldots, X_n$ are independent random variables with the same distribution as X. For each of the following, determine $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$:

a $\bar{X} = \dfrac{X_1 + X_2}{2}$

b $\bar{X} = \dfrac{X_1 + X_2 + X_3}{3}$

c $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

SF

2 The final marks in a mathematics examination are normally distributed with mean 65 and standard deviation 7. A random sample of 10 students is selected and the mean mark calculated. Determine the mean and standard deviation of this sample mean.

3 The number of customers per day at a fast-food outlet is known to be normally distributed with a standard deviation of 50. In a sample of 25 randomly chosen days, an average of 155 customers were served.

a Give a point estimate for μ, the mean number of customers served per day.

b Write down an expression for a 95% confidence interval for μ.

4 Suppose that 60 independent random samples are taken from a large population and a 95% confidence interval for the population mean is computed from each of them.

a How many of the 95% confidence intervals would you expect to contain the population mean μ?

b Write down an expression for the probability that all 60 confidence intervals contain the population mean μ.

CF

5 The tolerance in the diameter of bolts, X mm, produced by a machine has a distribution with probability density function

$$f(x) = \begin{cases} 0.5 & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

A random sample of 100 bolts is selected. Determine the mean and variance of the sample mean.

6 A newspaper determined a 95% confidence interval for the number of people (in millions) in Australia who regularly read the news online was (1.32, 1.44).

a Determine the value of $\bar{x}$ that was used to calculate this confidence interval.

b Calculate the margin of error.

c Explain how the newspaper could reduce the margin of error in their study.

Technology-active short-response questions

SF

7 Fifty plots are planted with a new variety of corn. The average yield for these plots is 130 bushels per acre. Assuming that the standard deviation is equal to 10, determine a 90% confidence interval for the mean yield per plot for this variety of corn.

8 A manufacturer knows that the lifetimes of their light bulbs are normally distributed with a standard deviation of 150 hours. Determine the minimum size sample required to ensure a margin of error of no more than $E = 20$ hours at the 95% confidence level.

9 A quality-control engineer in a factory needs to estimate the mean weight, μ grams, of bags of potato chips that are packed by a machine. The engineer knows by experience that $\sigma = 2.0$ grams for this machine. What size sample is required to ensure that we can be 99% confident that the estimate will be within 0.5 g of μ?

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Review

CF

10 The weight, Y kg, of nails sold each day by a hardware shop is a continuous random variable with probability density function f given by

$$f(y) = \begin{cases} \dfrac{2(y-1)}{25} & 1 \le y \le 6 \\ 0 & y < 1 \text{ or } y > 6 \end{cases}$$

Determine the probability that the average weight of nails sold over a a random sample of 30 days is more than 4.8 kg.

11 In a given manufacturing process, a batch of components is rejected if the mean length of a sample of 30 of these components is greater than 60.28 mm or less than 58.85 mm. It is found that 1% are rejected as the mean is too large and 5% are rejected as the mean is too small. Determine the mean and standard deviation of the distribution of the length, correct to one decimal place.

12 A 90% confidence interval for the mean weight of potatoes (in kg) produced by a certain variety of potato plants based on a sample of n plants is (5.31, 5.69).

a What is the margin of error in this confidence interval?

b If the standard deviation of the weight of potatoes produced by this sample of plants is 0.75 kg, determine the value of n.

13 A confidence interval for the mean score on a psychological test determined from a sample of 60 people is (37.878, 41.122).

a What is the margin of error in this confidence interval?

b If the standard deviation of the score from this sample is 5.4, determine the level of confidence used to calculate the confidence interval. Give the answer as a percentage correct to one decimal place.

CU

14 Jan uses the lift in her multi-storey office building each day. She has noted that, when she goes to her office each morning, the time she waits for the lift is normally distributed with a mean of 60 seconds and a standard deviation of 20 seconds.

a What is the probability that Jan will wait less than 54 seconds on a particular day?

b Determine a and b such that the probability that Jan waits between a seconds and b seconds is 0.95.

c During a five-day working week, determine the probability that:

i Jan's average waiting time is less than 54 seconds

ii Jan's total waiting time is less than 270 seconds

iii she waits for less than 54 seconds on more than two days in the week.

d Determine c and d such that there is a probability of 0.95 that her average waiting time over a five-day period is between c seconds and d seconds.

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Review

CU

15 The daily rainfall in Brisbourne is normally distributed with mean μ mm and standard deviation σ mm. The rainfall on one day is independent of the rainfall on any other day. On a randomly selected day, there is a 5% chance that the rainfall is more than 10.2 mm. In a randomly selected seven-day week, there is a probability of 0.025 that the mean daily rainfall is less than 6.1 mm. Determine the values of μ and σ.

16 **a** Researchers have established that the time it takes for a certain drug to cure a headache is normally distributed, with a mean of 14.5 minutes and a standard deviation of 2.4 minutes. Determine the probability that:

i in a random sample of 20 patients, the mean time for the headache to be cured is between 12 and 15 minutes

ii in a random sample of 50 patients, the mean time for the headache to be cured is between 12 and 15 minutes.

b The researchers modify the formula for the drug, and carry out some trials to determine the new mean time for a headache to be cured.

i Determine a 95% confidence interval for the mean time for a headache to be cured, if the average time it took for the headache to be cured in a random sample of 20 subjects was 12.5 minutes. (Assume that $\sigma = 2.4$.)

ii Determine a 95% confidence interval for the mean time for a headache to be cured, if the average time it took for the headache to be cured in a random sample of 50 subjects was 13.5 minutes. (Assume that $\sigma = 2.4$.)

iii Determine a 95% confidence interval for the mean time for a headache to be cured based on the combined data from the two studies in **i** and **ii**.

iv In order to ensure a margin of error of 0.5 minutes at the 95% confidence level, what size sample should the researchers use to determine the mean time to cure a headache for the new drug?

17 An aeroplane is licensed to carry 100 passengers.

a If the weights of passengers are normally distributed with a mean of 80 kg and a standard deviation of 20 kg, determine the probability that the combined weight of 100 passengers will exceed 8500 kg.

b The weight of the luggage that passengers check in before they travel is normally distributed, with a mean of 27 kg and a standard deviation of 4 kg. Determine the probability that the combined weight of the checked luggage of 100 passengers is more than 2850 kg.

c Passengers are also allowed to take hand luggage on the plane. The weight of the hand luggage that they carry is normally distributed, with a mean of 8 kg and a standard deviation of 2.5 kg. Determine the probability that the combined weight of the hand luggage for 100 passengers is more than 900 kg.

d What is the probability that the combined weight of the 100 passengers, their checked luggage and their hand luggage is more than 12 000 kg?

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18 A sociologist asked randomly selected workers in two different industries to fill out a questionnaire on job satisfaction. The answers were scored from 1 to 20, with higher scores indicating greater job satisfaction.

- The scores on the questionnaire for industry A are known to be normally distributed with a standard deviation of 2.2.
- The scores on the questionnaire for industry B are known to be normally distributed with a standard deviation of 3.1.

This information, together with the sample sizes used and the means obtained from the samples, is given in the following table.

Industry	n	σ	Sample mean
A	30	2.2	15.3
B	35	3.1	12.1

a **i** Determine a 95% confidence interval for μ_A, the mean satisfaction score in industry A.

ii Determine a 95% confidence interval for μ_B, the mean satisfaction score in industry B.

iii Compare the two confidence intervals. Do they seem to indicate that there is a difference in job satisfaction in the two industries?

b To properly compare the two industries, we should determine a confidence interval for the difference in the means between the two industries, that is, for $\mu_A - \mu_B$.

i What is a point estimate of $\mu_A - \mu_B$?

ii Determine the standard deviation of $\bar{X}_A - \bar{X}_B$, the difference between the mean score from industry A and the mean score from industry B.

iii Use this information to construct a 95% confidence interval for $\mu_A - \mu_B$.

iv Interpret this interval in the context of the random variables in this situation.

Multiple-choice questions

Technology-free multiple-choice questions

1 Which of the following statements is true?

A We use sample statistics to estimate population parameters.
B We use sample parameters to estimate population statistics.
C We use population parameters to estimate sample statistics.
D We use population statistics to estimate sample parameters.

2 Suppose that X is a random variable with mean $\mu = 3.6$ and variance $\sigma^2 = 1.44$. If $\bar{X}$ is the mean of a sample of size 9, then

A $\mathrm{E}(\bar{X}) = 0.4,\ \mathrm{sd}(\bar{X}) = 0.16$
B $\mathrm{E}(\bar{X}) = 0.4,\ \mathrm{sd}(\bar{X}) = 0.13$
C $\mathrm{E}(\bar{X}) = 3.6,\ \mathrm{sd}(\bar{X}) = 0.4$
D $\mathrm{E}(\bar{X}) = 3.6,\ \mathrm{sd}(\bar{X}) = 0.13$

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Review

3 The monthly mortgage payments for recent home buyers are normally distributed with mean $\mu = \$1732$ and standard deviation $\sigma = \$554$. A random sample of 100 recent home buyers is selected. The distribution of the mean of this sample is

A normal with mean \$17.32 and standard deviation \$5.54

B normal with mean \$1732 and standard deviation \$55.40

C normal with mean \$1732 and standard deviation \$5.54

D normal with mean \$173.20 and standard deviation \$55.40

4 Which of the following is a statement of the central limit theorem?

A If the sample size is large, then the distribution of the sample can be closely approximated by a normal curve.

B If the sample size is large and the population is normal, then the variance of the sample mean must be small.

C If the sample size is large, then the sampling distribution of the sample mean can be closely approximated by a normal curve.

D If the sample size is large and the population is normal, then the sampling distribution of the sample mean can be closely approximated by a normal curve.

5 The central limit theorem tells us that the sampling distribution of the sample mean is approximately normal. Which of the following conditions are necessary for the theorem to be valid?

A We have to be sampling from a normal population.

B The sample size has to be sufficiently large.

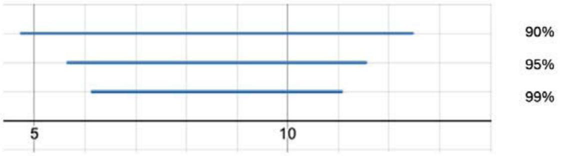
C The population distribution has to be symmetric.

D The population variance has to be small.

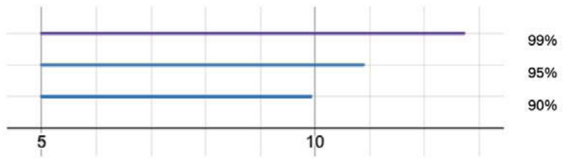
6 The weekly error (in seconds) of a brand of watch is known to be normally distributed with a mean of 10 seconds. A random sample of 10 of these watches is found to have a mean weekly error of 8.2 seconds. Which option could represent 90%, 95% and 99% confidence interval for μ based on this sample?

A

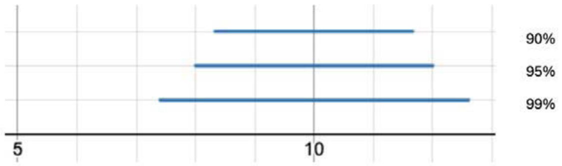
B



C



D



7 The sampling distribution of the sample mean refers to

A the distribution of the various sample sizes which might be used in a given study
B the distribution of the different possible values of the sample mean together with their respective probabilities of occurrence
C the distribution of the values of the random variable in the population
D the distribution of the values of the random variable in a given sample

8 If 50 random samples are chosen from a population and a 90% confidence interval for the population mean μ is computed from each sample, then on average we would expect the number of intervals which contain μ to be

A 50 **B** 48
C 45 **D** 40

9 If the sample size and standard deviation remain unchanged, then an increase in the level of confidence will lead to a confidence interval which is

A narrower **B** wider
C unchanged **D** asymmetric

10 Which of the following statements is true?

I Increasing the level of confidence increases the margin of error.
II Decreasing the level of confidence increases the margin of error.
III Increasing the sample size increases the margin of error.
IV Decreasing the sample size increases the margin of error.

A I and IV **B** II and III **C** I and III **D** II and IV

11 If a researcher increases her sample size by a factor of 4, then the width of a 95% confidence interval would

A increase by a factor of 2 **B** increase by a factor of 4
C decrease by a factor of 2 **D** decrease by a factor of 4

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Technology-active multiple-choice questions

12 Suppose that people's weights are normally distributed with a mean of 80 kg and a standard deviation of 10 kg. The probability that the average weight of 100 people will exceed 100 kg is

A 0.0228 **B** 0.0022 **C** 0 **D** 0.9772

13 The time required to assemble an electronic component is normally distributed, with a mean of 10 minutes and a standard deviation of 1.5 minutes. These components are packaged in boxes of 12. The probability that the average assembly time for the components in a box is greater than 11 minutes is

A 0 **B** 0.0105 **C** 0.2525 **D** 0.7575

14 The amount of money that customers spend at the supermarket each week in a certain town is known to be normally distributed with a standard deviation of $84. If the average amount spent by a random sample of 50 customers is $162, then a 98% confidence interval for the population mean is

A ($138.72, $185.28) **B** ($142.46, $181.54)
C ($151.31, $172.69) **D** ($134.36, $189.64)

15 A random sample of 100 observations is taken from a population known to be normally distributed with a standard deviation of 25. If the sample mean is 45, then the margin of error in a 95% confidence interval calculated from these data would be

A 4.9 **B** 0.49 **C** 0.98 **D** 40.1

16 In order to be 99% confident that the sample mean is within 1.4 of the population mean when a random sample is drawn from a population with a standard deviation of 6.7, the size of the sample should be

A 13 **B** 124 **C** 152 **D** 289

17 The length of time taken to complete a board game is normally distributed with a mean of μ minutes and a standard deviation of 6 minutes. There is a 5% probability that the average of 25 games will take less than 100 minutes. The value of μ is closest to

A 98.8 **B** 99.0 **C** 101.0 **D** 102.0

18 The volume of liquid in a 1-litre bottle of mineral water is a normally distributed random variable with a mean of 1.005 litres. A random sample of 50 bottles of mineral water was selected. The mean volume for this sample was 1.003 litres with a standard deviation of 0.01 litres. The largest confidence level that could be used for a confidence interval, centred on the sample mean, which does not contain μ is

A 84% **B** 92% **C** 96% **D** 98%

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16

Revision of Unit 4

Revision

16A Short-response questions

Technology-free short-response questions

Unit 4 Topic 1: Integration techniques

SF

1 Determine an anti-derivative of each of the following:

a $2xe^{-2x}$ **b** $x\sin(3x)$ **c** $\arccos(2x)$ **d** $4x^3\ln(3x)$

2 Evaluate:

a $\int_0^1 e^{2x}\cos(e^{2x})\,dx$ **b** $\int_1^2 (x-1)\sqrt{2-x}\,dx$ **c** $\int_0^1 \dfrac{x-2}{x^2-7x+12}\,dx$

3 Let $f(x) = \arcsin(4x^2-3)$. Determine the maximal domain of f.

4 Evaluate each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} x\cos\left(\dfrac{x}{2}\right) dx$ **b** $\int_0^2 x^2e^{2x+2}\,dx$ **c** $\int_1^2 \ln\left(\dfrac{x}{2}\right) dx$

5 Let a be a real number. Determine an expression for b given that

$$\int_a^{a+b} e^x\,dx = \frac{1}{2}\int_a^{a+1} e^x\,dx$$

6 Determine each of the following anti-derivatives:

a $\int \left(x-\dfrac{\pi}{4}\right)\sin\left(x-\dfrac{\pi}{4}\right) dx$ **b** $\int (x+1)\ln(x+1)\,dx$

7 **a** Determine the derivative of $f(x) = x\sin^{-1}(2x)$.

b Hence, or otherwise, determine $\int \sin^{-1}(2x)\,dx$.

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Revision

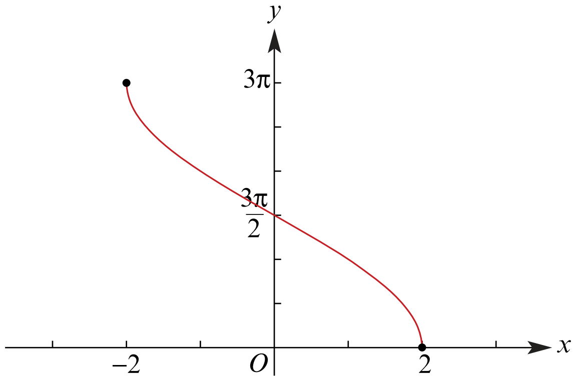
Unit 4 Topic 2: Applications of integration

CF

8 The graph of $y = 3\arccos\left(\frac{x}{2}\right)$ is shown opposite.

a Determine the area bounded by the graph, the x-axis and the line $x = -2$.

b Determine the volume of the solid of revolution formed when the graph is rotated about the y-axis.



9 Determine the volume of the solid formed when the region bounded by the x-axis and the curve with equation $y = a - \frac{x^2}{16a^3}$, where $a > 0$, is rotated about the y-axis.

10 Let $f(x) = e^x \sin x$ for $0 \le x \le \pi$. Determine the volume of the solid of revolution formed when the graph of $y = f(x)$ is rotated about the x-axis.

11 Assume that, for a certain kind of cable, the distance between consecutive flaws is exponentially distributed. Given that the average number of flaws per metre is 0.02, determine the probability that the distance between consecutive flaws is more than 100 m. (Give your answer in exact form.)

12 Consider the line $y = \frac{1}{3}(x+2)$ and the curve $y = \sqrt{x}$.

a Determine the coordinates of the points of intersection of these two graphs by solving a quadratic equation in x.

b Hence determine the area of the region enclosed by the two graphs.

13 The lifetime, X hours, of a particular type of light bulb is a random variable with the cumulative distribution function

$$P(X \le x) = 1 - e^{-\frac{x}{10}} \quad \text{for } x \ge 0$$

a Determine the median lifetime of a light bulb.

b Determine the probability density function f of X.

c Determine the mean and variance of X.

Unit 4 Topic 3: Rates of change and differential equations

SF

14 Consider the relation $5x^2 + 2xy + y^2 = 13$.

a Determine the gradient of each of the tangents to the graph at the points where $x = 1$.

b Determine the equation of the normal to the graph at the point in the first quadrant where $x = 1$.

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SF

15 For the curve defined by the parametric equations

$$x = 2\sin t + 1 \quad \text{and} \quad y = 2\cos t - 3$$

determine $\frac{dy}{dx}$ and its value at $t = \frac{\pi}{4}$.

CF

16 Determine y as a function of x given that $\frac{dy}{dx} = e^{2y}\sin(2x)$ and that $y = 0$ when $x = 0$.

17 Determine the solution of the differential equation $(1 + x^2)\frac{dy}{dx} = 2xy$, given $y = 2$ when $x = 0$.

18 A curve has equation $ye^x + \ln(x + 1) - y^2 + 2 = 0$.

a Determine $\frac{dy}{dx}$.

b Determine the equation of the tangent to the curve at the point $(0, 2)$.

19 Solve each of the following differential equations:

a $\frac{dy}{dx} = \frac{y}{(x + 1)^2}, \quad y(0) = 1$

b $\frac{dy}{dx} = y(y - 1), \quad y(0) = 2$

20 Determine the general solution of the differential equation

$$\frac{dy}{dx} = xy^2$$

using separation of variables. First consider what happens if $y = 0$ as a separate case.

21 Determine the general solution of the differential equation

$$\frac{dy}{dx} = y\sin x - \sin x$$

using separation of variables. First consider what happens if $y = 1$ as a separate case.

Unit 4 Topic 4: Modelling motion

CF

22 A particle is moving in a straight line and is subject to a deceleration of $1 + v^2$ m/s^2, where v m/s is the speed of the particle at time t seconds. The initial speed is u m/s. Determine an expression for the distance travelled, in metres, for the particle to come to rest.

23 A particle falls vertically from rest such that the acceleration, a m/s^2, is given by $a = g - 0.4v$, where v m/s is the speed at time t seconds. Determine an expression for v in terms of t in the form $v = A(1 - e^{-Bt})$, where A and B are positive constants. Hence state the values of A and B.

CU

24 A train, when braking, has an acceleration, a m/s^2, given by $a = -\left(1 + \frac{v}{100}\right)$, where v m/s is the velocity. The brakes are applied when the train is moving at 20 m/s and it travels x metres after the brakes are applied. Determine the distance that the train travels to come to rest in the form $x = A\ln(B) + C$, where A, B and C are positive constants.

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Revision

CU

25 **a** A 50 kg person stands in a lift which accelerates downwards at 1 m/s^2. Determine the reaction of the lift floor on the person.

b Determine the reaction of the lift floor on the person when the lift accelerates upwards at 1 m/s^2.

26 A body of mass 10 kg, on a horizontal plane, is initially at rest and is acted upon by a resultant force of $v - 5$ newtons, where v m/s is the speed of the body. The body will move in a straight line.

a Determine the acceleration of the body at time t in terms of v.

b Determine v in terms of t.

27 A body of mass 5 kg is held in place on a smooth plane inclined at 30° to the horizontal by a string with a tension force T N, acting parallel to the plane. Determine the value of T.

28 The acceleration of an object is inversely proportional to its velocity at any time t seconds. The object is travelling at 1 m/s when its acceleration is 2 m/s^2. The velocity of the particle when $t = 0$ was 2 m/s to the left. Determine its velocity at time t seconds.

29 Two forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ act in the directions of the vectors $4\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{i} - 2\boldsymbol{j}$ respectively and the magnitude of $\boldsymbol{P}$ is 25 newtons. If the magnitude of the resultant of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is also 25 newtons, determine the magnitude of $\boldsymbol{Q}$.

Unit 4 Topic 5: Statistical inference

SF

30 A random variable X has mean $\mu = 100$ and standard deviation $\sigma = 10$. A sample of size 64 is selected. Determine the mean and standard deviation of the sample mean $\bar{X}$.

31 A random sample of 36 fish was removed from a large tank. The average weight of these fish was 84.0 grams, with a standard deviation of 12.0 grams. Determine a 95% confidence interval for the mean weight of all the fish in the tank (use $z = 2$).

32 Suppose that 30 independent random samples are taken from a large population and a 90% confidence interval for the population mean μ is computed from each of them.

a How many of the 90% confidence intervals would you expect to contain the population mean μ?

b Write down an expression for the probability that all 30 confidence intervals contain the population mean μ.

CF

33 A random variable X has probability density function f given by

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $\bar{X}$ is the mean of a random sample of size 100 from this distribution, determine the mean and variance of $\bar{X}$.

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Technology-active short-response questions

Unit 4 Topic 1: Integration techniques

CF

34 Determine the value of each of the following integrals:

a $\int_0^{\frac{1}{4}} \tan^2(\pi x)\sec^2(\pi x)\, dx$ **b** $\int_0^2 x\sqrt{x+2}\, dx$

35 **a** Differentiate $f(x) = e^{-x}x^n$ and hence prove that

$$\int e^{-x}x^n\, dx = n\int e^{-x}x^{n-1}\, dx - e^{-x}x^n$$

b Define the function g with rule $g(n) = \int_0^{\infty} e^{-x}x^n\, dx$ for all $n \in [0, \infty)$.

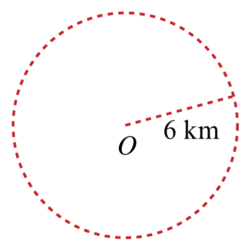
Note: $\int_a^{\infty} f(x)\, dx = \lim\limits_{b\to\infty} \int_a^b f(x)\, dx$

i Show that $g(0) = 1$.

ii Using the answer to **a**, show that $g(n) = ng(n-1)$.

iii Using your answers to **b i** and **b ii**, show that $g(n) = n!$, for $n = 0, 1, 2, 3, \ldots$.

36 Point O is the centre of a city with a population of 600 000. All of the population lives within 6 km of the city centre. The number of people who live within r km $(0 \le r \le 6)$ of the city centre is given by $\int_0^r 2\pi k(6-x)^{\frac{1}{2}}x^2\, dx$.



a Determine the value of k, correct to three significant figures.

b Determine the number of people who live within 3 km of the city centre, correct to three significant figures.

37 **a** Evaluate $\int_0^{\frac{\pi}{4}} \tan^4\theta\sec^2\theta\, d\theta$.

b Hence show that $\int_0^{\frac{\pi}{4}} \tan^6\theta\, d\theta = \frac{1}{5} - \int_0^{\frac{\pi}{4}} \tan^4\theta\, d\theta$.

c Deduce that $\int_0^{\frac{\pi}{4}} \tan^6\theta\, d\theta = \frac{13}{15} - \frac{\pi}{4}$.

Unit 4 Topic 2: Applications of integration

SF

38 Use Simpson's rule with strips of width $w = 0.1$ to approximate $\int_0^1 \sqrt{x}\sin(\pi x)\, dx$. (Give your answer to four significant figures.)

39 The lifetime of a particular type of electronic component (measured in thousands of hours) is exponentially distributed with a mean of 2. What is the probability that such a component will last more than 1500 hours?

40 **a** Determine the equation of the line segment joining the points $(-1, 4)$ and $(2, -2)$.

b Determine the volume of the solid of revolution that is formed when this line segment is rotated about the x-axis.

41 Suppose that the number of large plastic bags sold each day at a particular supermarket can be modelled by an exponential random variable. If the average number sold each day is 1000, what is the probability that 1500 or more are sold in one day?

Revision

42 A bowl can be described as the solid of revolution formed by rotating the graph of $y = \frac{1}{4}x^2$ around the y-axis for $0 \le y \le 25$. CU

a Determine the volume of the bowl.

b The bowl is filled with water and then, at time $t = 0$, the water begins to run out of a small hole in the base. The rate at which the water runs out is proportional to the depth, h, of the water at time t. Let V denote the volume of water at time t.

i Show that $\dfrac{dh}{dt} = \dfrac{-k}{4\pi}$, where $k > 0$.

ii Given that the bowl is empty after 30 seconds, determine the value of k.

iii Determine h in terms of t.

iv Determine V in terms of t.

c Sketch the graph of:

i V against h **ii** V against t

43 Suppose that the time, T minutes, between successive trains at an inner-city platform is exponentially distributed with the probability density function

$$f(t) = \begin{cases} \frac{1}{6}e^{-\frac{t}{6}} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

a If you arrive at the platform just as a train is leaving, what is the probability that you will have to wait at least 5 minutes for the next train? That is, determine $P(T \ge 5)$.

b If you have waited at the platform for 5 minutes and no train has arrived, what is the probability that you will have to wait at least 5 minutes more for a train? That is, determine $P(T \ge 10 \mid T \ge 5)$.

c In general, let X be an exponential random variable with parameter λ. Show that

$$P(X \ge a + b \mid X \ge a) = P(X \ge b)$$

for all $a, b > 0$.

Unit 4 Topic 3: Rates of change and differential equations

44 The rate of change of a given population P is proportional to $\sqrt{P}$. Initially the population was 10 000, and after five years the population was 11 025. CF

a Determine an expression for the population P after t years.

b Hence determine how long it takes for the population to reach 15 000. (Give your answer in years, correct to two decimal places.)

45 A straight ladder AB has length 5 m. One end of the ladder, A, is leaning on a vertical wall. The other end, B, is resting on horizontal ground. When point A is at a height of 4 m, it is sliding down the wall at the rate of 1 m/s. At this time, what is the rate at which point B is sliding across the ground? CU

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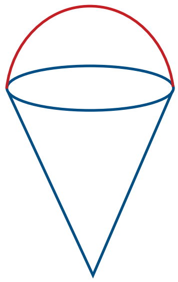
46 A large weather balloon is in the shape of a hemisphere on a cone, as shown in this diagram. When inflated, the height of the cone is twice the radius length of the hemisphere. The shapes and conditions are true as long as the radius of the hemisphere is at least 2 metres.



At time t minutes, the radius length of the hemisphere is r metres and the volume of the balloon is V m^3, for $r \geq 2$.

The balloon has been inflated so that the radius length is 10 m and it is ready to be released, when a leak develops. The gas leaks out at the rate of t^2 m^3 per minute.

a Determine the relationship between V and r.

b Construct a differential equation of the form $f(r)\dfrac{dr}{dt} = g(t)$.

c Solve the differential equation with respect to t, given that the initial radius length is 10 m.

d Determine how long it will take for the radius length to reduce to 2 metres.

47 Two boats leave the same point at the same time. One boat moves north at 30 km/h and the other moves east at 40 km/h. How fast will the distance between the two boats be changing after 2 hours?

48 An advertising campaign is started for a new type of deodorant. The proportion, $N(t)$, of the people in the target demographic who have seen the new product after t weeks can be modelled by the differential equation

$$\frac{dN}{dt} = \frac{N(1-N)}{t} \qquad \text{for } t > 0 \text{ and } 0 < N < 1$$

a Determine the general solution of this differential equation.

b For each of the following, determine the particular solution:

i $N(1) = 0.5$ **ii** $N(1) = 0.25$

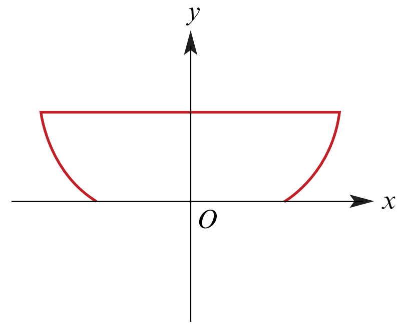
c Sketch the graph of N against t for each of the solutions from part **b**.

49 A searchlight is located at ground level vertically below the path of an approaching aircraft, which is flying with a constant velocity of 400 m/s at a height of 10 000 m. If the light is continuously directed at the aircraft, determine the rate at which the searchlight is turning at the instant when the aircraft is at a horizontal distance of 5000 m from the searchlight. (Give your answer in degrees per second, correct to three decimal places.)

Revision

CU

50 The vertical cross-section of a bucket is shown in this diagram. The sides are arcs of a parabola with the y-axis as the central axis and the horizontal cross-sections are circular. The depth is 36 cm, the radius length of the base is 10 cm and the radius length of the top is 20 cm.



a Prove that the parabolic sides are arcs of the parabola $y = 0.12x^2 - 12$.

b Prove that the bucket holds 9π litres when full.

Water starts leaking from the bucket, initially full, at the rate given by $\dfrac{dv}{dt} = \dfrac{-\sqrt{h}}{A}$, where at time t seconds the depth is h cm, the surface area is A cm^2 and the volume is v cm^3.

c Prove that $\dfrac{dv}{dt} = \dfrac{-3\sqrt{h}}{25\pi(h+12)}$.

d Show that $v = \pi \displaystyle\int_0^h \left(\frac{25y}{3} + 100\right) dy$.

e Hence construct a differential equation expressing:

i $\dfrac{dv}{dh}$ as a function of h **ii** $\dfrac{dh}{dt}$ as a function of h

f Hence determine the time taken for the bucket to empty.

51 A hemispherical bowl can be described as the solid of revolution generated by rotating $x^2 + y^2 = a^2$ about the y-axis for $-a \le y \le 0$. The bowl is filled with water. At time $t = 0$, water starts running out of a small hole in the bottom of the bowl, so that the depth of water in the bowl at time t is h cm. The rate at which the volume is decreasing is proportional to h. (All length units are centimetres.)

a **i** Show that, when the depth of water is h cm, the volume, V cm^3, of water remaining is $V = \pi(ah^2 - \frac{1}{3}h^3)$, where $0 < h \le a$.

ii If $a = 10$, determine the depth of water in the hemisphere if the volume is 1 litre.

b Show that $\pi(2ah - h^2)\dfrac{dh}{dt} = -kh$, for a positive constant k.

c Given that the bowl is empty after time T, show that $k = \dfrac{3\pi a^2}{2T}$.

d If $a = 10$ and $T = 30$, determine k (correct to three significant figures).

e Sketch the graph of:

i $\dfrac{dV}{dt}$ against h for $0 \le h \le a$ **ii** $\dfrac{dh}{dt}$ against h for $0 \le h \le a$

f Determine the rate of change of the depth with respect to time when:

i $h = \dfrac{a}{2}$ **ii** $h = \dfrac{a}{4}$

g If $a = 10$ and $T = 30$, determine the rate of change of depth with respect to time when there is 1 litre of water in the hemisphere.

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52 A disease spreads through a population. Let p denote the proportion of the population who have the disease at time t. The rate of change of p is proportional to the product of p and the proportion $1 - p$ who do not have the disease.

When $t = 0$, $p = \frac{1}{10}$ and when $t = 2$, $p = \frac{1}{5}$.

a **i** Show that $t = \frac{1}{k}\ln\left(\frac{9p}{1-p}\right)$, where $k = \ln\left(\frac{3}{2}\right)$.

ii Hence show that $\frac{9p}{1-p} = \left(\frac{3}{2}\right)^t$.

b Determine p when $t = 4$.

c Determine p in terms of t.

d Determine the values of t for which $p > \frac{1}{2}$.

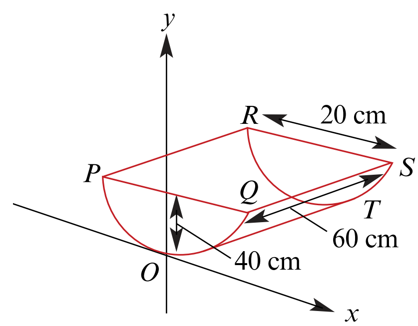
e Sketch the graph of p against t.

53 A car moves along a straight level road. Its speed, v, is related to its displacement, x, by the differential equation $v\frac{dv}{dx} = \frac{p}{v} - kv^2$, where p and k are positive constants.

a Given that $v = 0$ when $x = 0$, show that $v^3 = \frac{1}{k}(p - pe^{-3kx})$.

b Determine $\lim_{x\to\infty} v$.

54 This container has an open rectangular horizontal top, $PQSR$, and parallel vertical ends, PQO and RST. The ends are parabolic in shape. The x-axis and y-axis intersect at O, with the x-axis horizontal and the y-axis the line of symmetry of the end PQO. The dimensions are shown on the diagram.



a Determine the equation of the parabolic arc QOP.

b If water is poured into the container to a depth of y cm, with a volume of V cm^3, determine the relationship between V and y.

c Calculate the depth, to the nearest mm, when the container is half full.

d Water is poured into the empty container so that the depth is y cm at time t seconds. If the water is poured in at the rate of 60 cm^3/s, construct a differential equation expressing $\frac{dy}{dt}$ as a function of y and solve it.

e Calculate, to the nearest second:

i how long it will take the water to reach a depth of 20 cm

ii how much longer it will take for the container to be completely full.

55 The rate of change of a population, y, is given by

$$\frac{dy}{dt} = \frac{2y(N-y)}{N}$$

where N is a positive constant. When $t = 0$, $y = \frac{N}{4}$.

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a Determine y in terms of t and determine $\frac{dy}{dt}$ in terms of t.
b What limiting value does the population size approach for large values of t?
c Explain why the population is always increasing.
d What is the population when the population is increasing most rapidly?
e For $N = 10^6$:
  i Sketch the graph of $\frac{dy}{dt}$ against y.
  ii At what time is the population increasing most rapidly?

Unit 4 Topic 4: Modelling motion

CF

56 A particle of mass 10 kg is acted on by a variable force of $(100 - v)$ N, where v m/s is the velocity of the particle at time t seconds.
a Given that the particle was initially at rest, determine an expression for v in terms of t.
b What is the limiting speed of the particle as $t \to \infty$?
c Determine the time taken for the particle's speed to reach 90% of this limiting speed.

57 A particle of mass 6 kg starts at the point $(3, -1)$ on a Cartesian plane, where the unit of distance is metres. The particle is initially stationary and is acted on by two constant forces of $(4\boldsymbol{i} - 2\boldsymbol{j})$ N and $(\boldsymbol{i} + 5\boldsymbol{j})$ N simultaneously. Determine the position of the particle after 5 seconds.

58 A particle is moving in a straight line such that its position, x cm, at time t seconds is given by

$$x = 4 - 3\sin\left(\frac{\pi}{6}(t+1)\right), \quad t \geq 0$$

a Show that this particle is in simple harmonic motion, and state the centre and amplitude of the motion.
b Determine the particle's initial position and velocity.
c Determine the first time $t > 0$ at which the particle's velocity is equal to its initial velocity.

CU

59 A particle is moving in a straight line such that its position, x metres, at time t seconds is given by

$$x = a\cos(\omega t) + b\sin(\omega t)$$

for positive constants a, b and ω.
a Show that the motion of the particle is simple harmonic.
b If $a = 3$, $b = 4$ and $\omega = 2$, determine:
  i the period of the motion
  ii the amplitude of the motion
  iii the maximum speed of the particle.

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60 An object projected vertically upwards from the surface of the Earth experiences an acceleration of a m/s^2 at a point x m from the centre of the Earth (neglecting air resistance). This acceleration is given by

$$a = \frac{-gR^2}{x^2}$$

where g m/s^2 is the acceleration due to gravity and R m is the radius length of the Earth.

- **a** Given that $g = 9.8$, $R = 6.4 \times 10^6$ and the object has an upwards velocity of u m/s at the Earth's surface:
  - **i** Express v^2 in terms of x, using $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$.
  - **ii** Use the result of part **i** to determine the position of the object when it has zero velocity.
  - **iii** For what values of u does the result in part **ii** not exist?
- **b** The minimum value of u for which the object does not fall back to Earth is called the escape velocity. Determine the escape velocity in km/h.

61 A lift that has mass 1000 kg when empty is carrying a man of mass 80 kg. The lift is descending with a downwards acceleration of 1 m/s^2.

- **a**
  - **i** Calculate the tension in the lift cable.
  - **ii** Calculate the vertical force exerted on the man by the floor of the lift.
- **b** The man drops a coin from a height of 2 m. Calculate the time taken for it to hit the floor of the lift.
- **c** The lift is designed so that during any journey the magnitude of the acceleration reaches but does not exceed 1 m/s^2. Safety regulations do not allow the lift cable to bear a tension greater than 20 000 N. Making reasonable assumptions, suggest the number of people that the lift should be licensed to carry. (**Hint:** The maximum tension in the lift cable occurs when the lift is accelerating upwards.)

62 A particle is oscillating between the points P and Q with simple harmonic motion. The midpoint of PQ is O. The particle's maximum acceleration is 16 m/s^2 and its speed at the midpoint of OP is $4\sqrt{3}$ m/s. Determine:

- **a** the length of PQ
- **b** the period of the motion
- **c** the time taken for the particle to travel directly from the midpoint of OP to the midpoint of OQ.

63 Moving in the same direction along parallel tracks, objects A and B pass the point O simultaneously with speeds of 20 m/s and 10 m/s respectively.

From then on, the deceleration of A is $\dfrac{v^3}{400}$ m/s^2 and the deceleration of B is $\dfrac{v^2}{100}$ m/s^2, when the speeds are v m/s.

- **a** Determine the speeds of A and B at time t seconds after passing O.
- **b** Determine the positions of A and B at time t seconds after passing O.
- **c** Use a calculator to plot the graphs of the positions of objects A and B.
- **d** Use a calculator to determine, to the nearest second, when the objects pass.

64 A stone, initially at rest, is released and falls vertically. Its velocity, v m/s, at time t s after release is determined by the differential equation

$$5\frac{dv}{dt} + v = 50$$

a Determine an expression for v in terms of t.

b Determine v when $t = 47.5$.

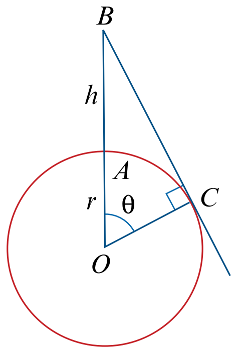
c Sketch the graph of v against t.

d **i** Let x be the displacement from the point of release at time t. Determine an expression for x in terms of t.

ii Determine x when $t = 6$.

65 The diagram shows a plane circular section through O, the centre of the Earth (which is assumed to be stationary for the purpose of this problem).

From the point A on the surface, a rocket is launched vertically upwards. After t hours, the rocket is at B, which is h km above A. Point C is on the horizon as seen from B, and the length of the chord AC is y km. The angle AOC is θ radians. The radius of the Earth is r km.



a **i** Express y in terms of r and θ.

ii Express $\cos\theta$ in terms of r and h.

b Suppose that after t hours the vertical velocity of the rocket is $\dfrac{dh}{dt} = r\sin t,\ t \in [0, \pi)$. Assume that $r = 6000$.

i Determine $\dfrac{dy}{d\theta}$ and $\dfrac{dy}{dt}$. **ii** How high is the rocket when $t = \dfrac{\pi}{2}$?

iii Determine $\dfrac{dy}{dt}$ when $t = \dfrac{\pi}{2}$.

66 A particle moves in a straight line, starting from point A. Its motion is assumed to be with constant deceleration. During the first, second and third seconds of its motion, it covers distances of 70 m, 60 m and 50 m respectively, measured in the same sense.

a **i** Verify that these distances are consistent with the assumption that the particle is moving with constant deceleration.

ii Determine the deceleration and an expression for the displacement of the particle.

b If the particle comes instantaneously to rest at B, determine distance AB.

c At the same instant that the first particle leaves A, a second particle leaves B with an initial velocity of 75 m/s and travels with constant acceleration towards A. It meets the first particle at a point C, $1\frac{1}{2}$ seconds after leaving B.

i Determine distance BC.

ii Show that the acceleration of the second particle is 60 ms^{-2}.

Unit 4 Topic 5: Statistical inference

SF

67 The marks in a mathematics examination are normally distributed with a mean of 65 and a standard deviation of 5.

a Determine the probability that a randomly selected student receives a mark above 70.

b Determine the probability that the average mark for five randomly selected students is above 70.

68 A random sample of 30 newborn babies in a certain country was found to have an average weight of 3.6 kg with a standard deviation of 0.55 kg. Determine a 95% confidence interval for the mean birth weight of babies in that country.

69 A manufacturer knows that the voltage (V) of a certain type of battery is normally distributed with a variance of 0.09. Determine the minimum size sample required to ensure a margin of error of no more than $E = 0.1$ V at the 99% confidence level.

CF

70 A 90% confidence interval for the mean height (in m) of Xmas trees sold by a certain company based on a sample of 60 trees is (2.38, 2.62). Determine the values of the sample mean and the sample standard deviation used in the calculation of this confidence interval.

71 A confidence interval for the mean hourly pay rate for hospitality staff determined from a sample of 90 people is (\$29.35, \$31.65).

a Determine the margin of error in this confidence interval.

b If the standard deviation of the mean hourly pay rate is \$6.45, determine the level of confidence used to calculate the confidence interval. Give the answer as a percentage correct to one decimal place.

72 A 95% confidence interval for the mean score on a mathematics examination is (61.804, 65.796). Determine a 99% confidence interval for the mean score using the same data.

73 Batteries are sold in packs of six. The lifetime of these batteries can be assumed to be normally distributed with a mean of 32 hours and a standard deviation of σ hours. Determine the value of σ such that, for one in 100 packs, the average life of the batteries in the pack is less than 30 hours.

74 The volume of liquid in a 600 mL bottle is normally distributed with a mean of μ mL and a standard deviation of σ mL.

- For a randomly selected six-pack of bottles, there is a probability of 0.110 that the mean volume of liquid is more than 602 mL.
- For a randomly selected 12-pack of bottles, there is a probability of 0.193 that the mean volume of liquid is more than 601 mL.

Determine the values of μ and σ, correct to two decimal places.

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Revision

CF

75 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = \frac{\pi}{20}\sin\left(\frac{\pi(x-7)}{10}\right) \quad 7 \le x \le 17$$

Determine the probability that the average mass of a random sample of 30 2-year-old children is more than 12.5 kg.

76 The volume of liquid in a 1 litre bottle is normally distributed with a mean of μ mL and a standard deviation of σ mL. For a randomly selected bottle, there is a probability of 0.057 that the volume of liquid is more than 1020 mL. For a randomly selected six-pack of bottles, there is a probability of 0.033 that the mean volume of liquid is more than 1010 mL. Determine the values of μ and σ.

77 Let $X_1, X_2, \dots, X_{30}$ be independent random variables, each having a probability distribution given by

$$P(X = x) = 0.4^{x-1} \times 0.6 \quad \text{for } x = 1, 2, 3, \dots$$

with $\mathrm{E}(X) = \dfrac{5}{3}$ and $\mathrm{Var}(X) = \dfrac{10}{9}$.

Given that $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_{30}}{30}$ determine $P(\bar{X} > 2)$, correct to two decimal places.

CU

78 A continuous random variable X has a normal distribution with a mean of $\mu = 20$ and a standard deviation of $\sigma = 3$. Let $\bar{X}$ be the mean of a random sample of size n. Given that $0.01 < P(\bar{X} > 21) < 0.05$, determine the possible values of n.

79 Suppose that people's weights are normally distributed, with a mean of 78 kg and a standard deviation of 12.5 kg. Due to changes in diet, researchers believe that the mean weight of 78 kg may have changed. To investigate this possibility, they take a random sample of 40 people and determine a sample mean of 80 kg. Does this sample data support the researchers' belief that the mean has changed?

80 This table gives the marks on two exams, X and Y, for a random sample of 18 students. You may assume that the marks on each exam are normally distributed

X	80	79	70	79	81	69	80	82	80	74	76	58	65	80	81	82	80	73
Y	74	73	72	80	74	67	72	74	75	76	67	59	61	76	74	76	74	79
D	6																	

a Use the data in the table to determine a 95% confidence interval for:

i the average mark on exam X **ii** the average mark on exam Y.

b Consider the random variable $D = X - Y$. Complete the third row of the table by entering the values for D. Hence determine a 95% confidence interval for the mean of D, the difference in marks on exams X and Y.

c Does the confidence interval from part **b** support the suggestion that, in general, students did better on exam X than on exam Y? Use mathematical reasoning to explain your conclusion.

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16B Multiple-choice questions

Technology-free multiple-choice questions

1 One solution to the differential equation $\dfrac{d^2y}{dx^2} = 2\cos x + 1$ is

A $y = -4 + \cos x + x$

B $y = 2\sin x + x + 1$

C $y = -\dfrac{1}{4}\cos(2x) + \dfrac{x^2}{2} + x$

D $y = -2\cos x + \dfrac{x^2}{2} + x$

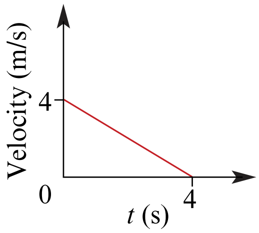
2 The graph shows the motion of an object which, in 4 seconds, covers a distance of

A 1 m

B 8 m

C 16 m

D −8 m



3 At a certain instant, a sphere is of radius 10 cm and the radius is increasing at a rate of 2 cm/s. The rate of increase (in cm^3/s) of the volume of the sphere is

A 80π

B $\dfrac{800\pi}{3}$

C 400π

D 800π

4 A curve passes through the point $(2, 3)$ and is such that the tangent to the curve at each point (a, b) is perpendicular to the tangent to $y = 2x^3$ at $(a, 2a^3)$. The equation of the curve can be found by using the differential equation

A $\dfrac{dy}{dx} = 2x^3$

B $\dfrac{dy}{dx} = -\dfrac{1}{6x^2}$

C $\dfrac{dy}{dx} = -6x^2$

D $\dfrac{dy}{dx} = \dfrac{2}{x} + c$

5 A curve passes through the point $(1, 1)$ and is such that the gradient at any point is twice the reciprocal of the x-coordinate. The equation of this curve can be found by solving the differential equation with the given boundary condition

A $x\dfrac{dy}{dx} = 2, \quad y(1) = 1$

B $\dfrac{d^2y}{dx^2} = \dfrac{x}{2}, \quad y(1) = 1$

C $y\dfrac{dy}{dx} = 2, \quad y(1) = 1$

D $\dfrac{dy}{dx} = x, \quad y(1) = 1$

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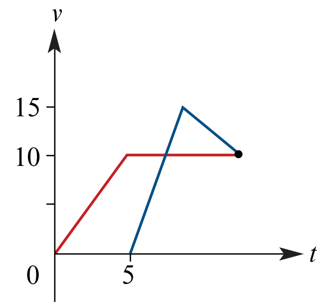
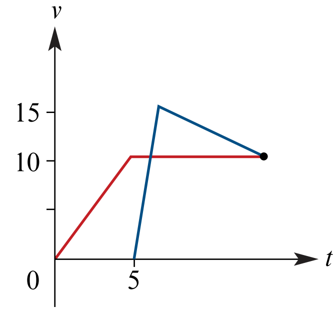
6 Car P leaves a garage, accelerates at a constant rate to a speed of 10 m/s and continues at that speed. Car Q leaves the garage 5 seconds later, accelerates at the same rate as car P to a speed of 15 m/s and continues at that speed until it hits the back of car P. Which one of the following pairs of graphs represents the motion of these cars?

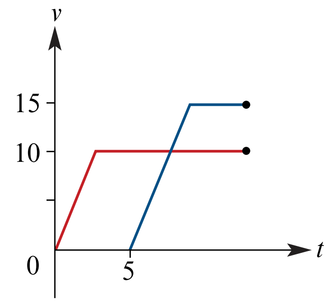
A



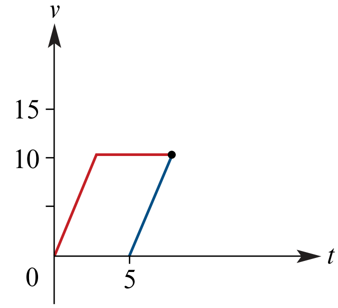
B



C



D



7 $\dfrac{d}{d\theta}\big(\ln(\sec\theta + \tan\theta)\big)$ equals

A $\sec\theta$ **B** $\sec^2\theta$ **C** $\sec\theta\tan\theta$ **D** $\cot\theta - \tan\theta$

8 A particle is moving along the x-axis such that $x = 3\cos(2t)$ at time t. When $t = \dfrac{\pi}{2}$, the acceleration of the particle in the positive x-direction is

A 12 **B** -6 **C** 0 **D** 6

9 A container initially holds 20 litres (L) of water. A salt solution of concentration 3 g/L is poured into the container at a rate of 2 L/min. The mixture is kept uniform by stirring and flows out at a rate of 2 L/min. If Q g is the amount of salt in the container t minutes after pouring begins, then Q satisfies the equation

A $\dfrac{dQ}{dt} = \dfrac{Q}{10}$ **B** $\dfrac{dQ}{dt} = Q$

C $\dfrac{dQ}{dt} = 6 - \dfrac{Q}{10}$ **D** $\dfrac{dQ}{dt} = 6 - \dfrac{Q}{10 + t}$

10 The equation of the particular member of the family of curves defined by $\dfrac{dy}{dx} = 3x^2 + 1$ that passes through the point $(1, 3)$ is

A $y = 6x$ **B** $y = x^3 + x^2 + 1$

C $y = x^3 + x + 1$ **D** $y = x^3 + x + 3$

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11 One solution of the differential equation $\frac{d^2y}{dx^2} = e^{3x}$ is

A $y = 3e^{3x}$ **B** $y = \frac{1}{9}e^{3x} + x$ **C** $y = \frac{1}{3}e^{3x}$ **D** $y = \frac{1}{3}e^{3x} + x$

12 A body initially travelling at 12 m/s is subject to a constant deceleration of 4 m/s^2. The time taken to come to rest (t seconds) and the distance travelled before it comes to rest (s metres) are

A $t = 3,\ s = 24$ **B** $t = 3,\ s = 18$

C $t = 3,\ s = 8$ **D** $t = 4,\ s = 18$

13 If $y = 1 - \sin(\cos^{-1} x)$, then $\frac{dy}{dx}$ equals

A $\frac{x}{\sqrt{1 - x^2}}$ **B** $-x$

C $\cos(\sqrt{1 - x^2})$ **D** $-\cos(\sqrt{1 - x^2})$

14 If $x = 2\sin^2(y)$, then $\frac{dy}{dx}$ equals

A $4\sin(y)$ **B** $\frac{1}{2}\operatorname{cosec}(2y)$ **C** $4\sqrt{\frac{x}{2}}$ **D** $2\sqrt{2x}$

15 $\int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x \, dx$ equals

A $\sqrt{3}$ **B** 3 **C** $\frac{\pi^3}{81}$ **D** $\frac{\pi^2}{9}$

16 Assume that $\ddot{y} = e^x + e^{-2x}$. If $y = 0$ and $\dot{y} = \frac{1}{2}$ when $x = 0$, then

A $y = e^x + \frac{1}{4}e^{-2x} - \frac{5}{4}$ **B** $y = e^x + e^{-2x} - \frac{1}{2}$

C $y = e^x + e^{-2x}$ **D** $y = e^x + e^{-2x} + \frac{1}{2}$

17 If $\frac{dy}{dx} = 2y + 1$ and $y = 3$ when $x = 0$, then

A $y = \frac{7e^{2x} - 1}{2}$ **B** $y = \frac{1}{2}\ln(2x + 1)$

C $y = y^2 + y + 1$ **D** $y = e^{2x}$

18 A rock falls from the top of a cliff 45 metres high ($g = -10$ m/s^2). The rock's speed (in m/s) just before it hits the ground is

A 5 **B** 10 **C** 20 **D** 30

19 The velocity, v m/s, of a particle at time t seconds is given by $v = t - t^2$, $t \geq 0$. The acceleration (in m/s^2) at time $t = 5$ is

A -20 **B** -9 **C** 11 **D** 1

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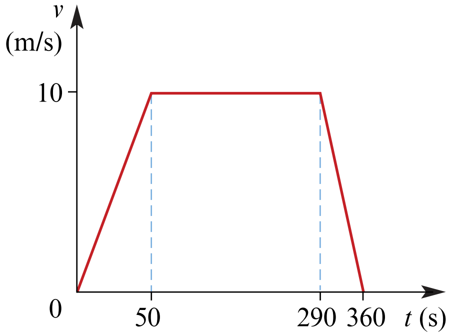
20 If $y = x\tan^{-1}(x)$, then $\dfrac{dy}{dx} = \dfrac{x}{1+x^2} + \tan^{-1}(x)$. It follows that an anti-derivative of $\tan^{-1}(x)$ is

A $x\tan^{-1}(x)$
B $x\tan^{-1}(x) - \dfrac{x}{1+x^2}$
C $x\tan^{-1}(x) - \ln(\sqrt{1+x^2})$
D $\dfrac{1}{1+x^2} + \dfrac{1}{x}\tan^{-1}(x)$

21 The velocity–time graph shows the motion of a train between two stations. The distance between the stations, in metres, is

A 2500
B 2900
C 3000
D 3400



22 If $\dfrac{dy}{dx} = x^2 + x$ and $x = -3$ when $y = -\frac{1}{2}$, then

A $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 4$
B $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4$
C $y = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - 4$
D $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4$

23 The equation of the particular member of the family of curves defined by $\dfrac{dy}{dx} = 1 - e^{-x}$ that passes through the point $(0, 6)$ is

A $y = x - e^{-x} + 5$
B $y = x + e^{-x} + 5$
C $y = x + e^{-x} + 7$
D $y = x + e^{-x} + 6$

24 If $y = \sin^{-1}(\sqrt{1-x})$, then $\dfrac{dy}{dx}$ equals

A $\cos^{-1}(\sqrt{1+x})$
B $\dfrac{1}{\sqrt{x}}$
C $-\dfrac{1}{2\sqrt{x(1-x)}}$
D $\sqrt{\dfrac{1-x}{x}}$

25 The values of m for which $y = e^{mx}$ satisfies the differential equation $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0$ are

A $m = 1,\ m = 2$
B $m = 3,\ m = -1$
C $m = -2,\ m = 3$
D $m = \pm 1$

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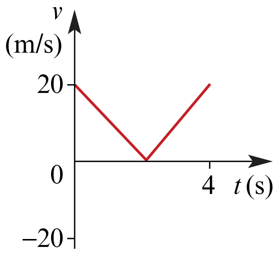
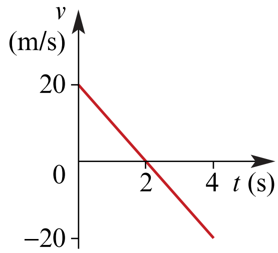
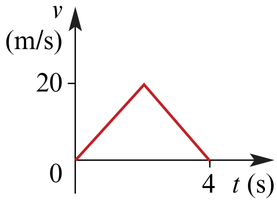
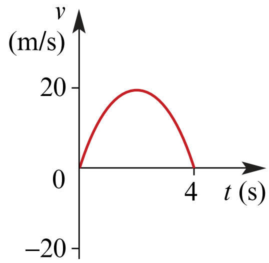
26 A particle is projected vertically upwards from ground level with a velocity of 20 m/s and returns to the point of projection. The velocity–time graph illustrating this could be

A


B


C


D


27 A particle has initial velocity 3 m/s and its acceleration t seconds later is given by $(6t^2 + 5t - 3)$ m/s^2. After 2 seconds, its velocity in m/s is

A 15 **B** 23 **C** 18 **D** 21

28 A particle starts from rest at a point O and moves in a straight line so that after t seconds its velocity, v, is given by $v = 4\sin(2t)$. Its displacement from O is given by

A $s = 2 - 2\cos(2t)$ **B** $s = 8\cos(2t)$
C $s = 2\cos(2t)$ **D** $s = -2\cos(2t)$

29 Which one of the following differential equations is satisfied by $y = e^{3x}$ for all values of x?

A $\dfrac{d^2y}{dx^2} + 9y = 0$ **B** $\dfrac{d^2y}{dx^2} - 9y = 0$
C $\dfrac{d^2y}{dx^2} + \dfrac{y}{9} = 0$ **D** $\dfrac{d^2y}{dx^2} - 27y = 0$

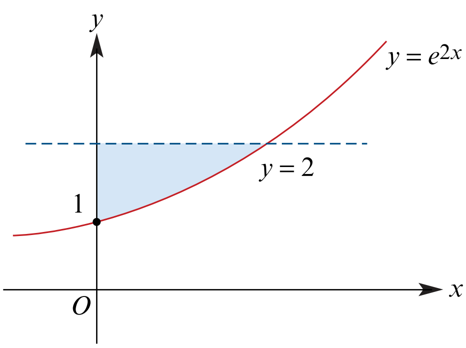
30 The volume of the solid of revolution when the shaded region of the diagram is rotated about the y-axis is given by

A $\pi \displaystyle\int_1^2 \frac{1}{4}(\ln y)^2 \, dy$

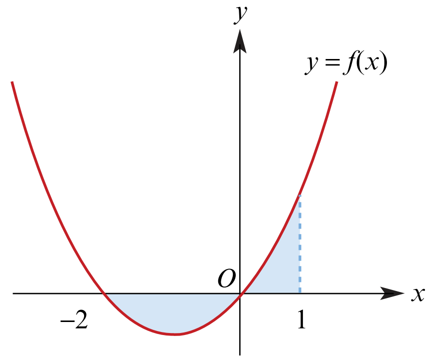
B $\pi \displaystyle\int_0^{\frac{1}{2}\ln 2} e^{2x} \, dx$

C $\pi \displaystyle\int_0^2 \frac{1}{2} \ln y \, dy$

D $\pi \left(\ln 2 - \displaystyle\int_0^{\frac{1}{2}\ln 2} e^{2x} \, dx\right)$



31 The area of the shaded region in the graph is



A $\int_0^1 f(x)\,dx + \int_0^{-2} f(x)\,dx$

B $\int_{-2}^1 f(x)\,dx$

C $\int_{-2}^0 f(x)\,dx + \int_0^1 f(x)\,dx$

D $-\int_1^0 f(x)\,dx + \int_0^{-2} f(x)\,dx$

32 An arrangement of the integrals

$$P = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx, \quad Q = \int_0^{\frac{\pi}{4}} \cos^2 x\,dx, \quad R = \int_0^{\frac{\pi}{4}} \sin^2 x\,dx$$

in ascending order of magnitude is

A $P,\ R,\ Q$ **B** $Q,\ P,\ R$ **C** $R,\ Q,\ P$ **D** $R,\ P,\ Q$

33 $\int_{-a}^{a} \tan x\,dx$ can be evaluated if a equals

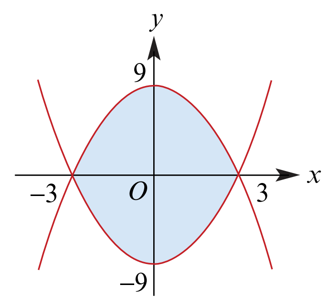
A $\dfrac{\pi}{2}$ **B** $\dfrac{3\pi}{2}$ **C** $\dfrac{\pi}{4}$ **D** π

34 The value of $\int_0^1 \dfrac{e^{2x}}{e^{2x}+1}\,dx$ is

A $\dfrac{1}{2}(e^2+1)$ **B** $\dfrac{1}{2}\ln(e^2-1)$

C $\dfrac{1}{2}\ln\left(\dfrac{e^2+1}{2}\right)$ **D** $\ln\left(\dfrac{e^2+1}{2}\right)$

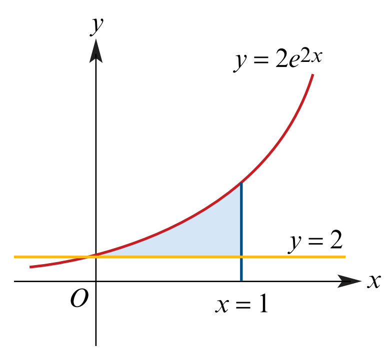
35 In the diagram on the right, the area of the region enclosed between the graphs with equations $y = x^2 - 9$ and $y = 9 - x^2$ is given by



A $\int_{-3}^{3} 2x^2 - 18\,dx$ **B** $\int_{-3}^{3} 18 - 2x^2\,dx$

C 0 **D** $\int_{-9}^{9} 2x^2 - 18\,dx$

36 The volume of the solid of revolution when the shaded region of this graph is rotated about the x-axis is given by



A $\pi\int_0^1 4e^{4x} - 4\,dx$

B $\pi\int_0^1 e^{2x} - 4\,dx$

C $\pi\int_0^1 (2e^{2x} - 2)^2\,dx$

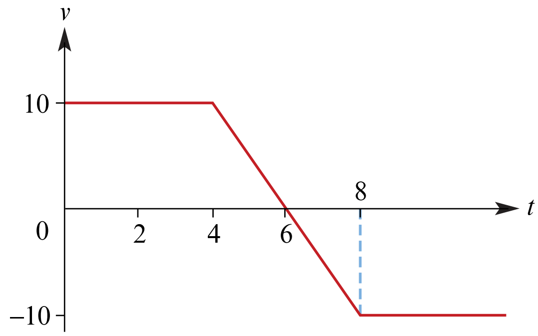
D $\pi\int_2^{2e} 1\,dy$

37 A body moves in a straight line so that its acceleration (in m/s^2) at time t seconds is given by $\frac{d^2x}{dt^2} = 4 - e^{-t}$. If the body's initial velocity is 3 m/s, then when $t = 2$ its velocity (in m/s) is

A e^{-2} **B** $2 + e^{-2}$ **C** $8 + e^{-2}$ **D** $10 + e^{-2}$

38 A particle moves with velocity v m/s. The distance travelled, in metres, by the particle in the first 8 seconds is

A 40 **B** 50
C 60 **D** 70



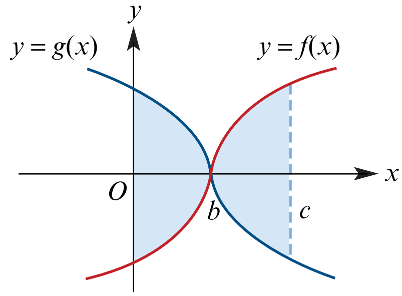
39 The area of the region shaded in the graph is equal to

A $\int_0^c f(x) - g(x)\, dx$

B $\int_b^c f(x) - g(x)\, dx + \int_0^b f(x) - g(x)\, dx$

C $\int_b^c f(x) - g(x)\, dx + \int_b^0 f(x) - g(x)\, dx$

D $\int_b^c f(x)\, dx + \int_0^b g(x)\, dx$



40 $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$ is equal to

A $\int_0^1 u\, du$ **B** $\int_0^{\frac{\pi}{4}} u^2\, du$

C $-\int_0^1 u^2\, du$ **D** $\int_0^1 \sqrt{1 - u^2}\, du$

41 An anti-derivative of $\frac{\sin x}{\cos^2 x}$ is

A $\sec x$ **B** $\tan x \cos x$ **C** $\tan^2 x$ **D** $\cot x \sec x$

42 A partial fraction expansion of $\frac{1}{(2x+6)(x-4)}$ shows that it has an anti-derivative $\frac{a}{2}\ln(2x+6) + b\ln(x-4)$, where

A $a = -\frac{1}{7},\ b = \frac{1}{14}$ **B** $a = 1,\ b = 1$

C $a = \frac{1}{2},\ b = \frac{1}{2}$ **D** $a = -1,\ b = -1$

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Revision

43 $\int_0^1 x\sqrt{2x+1}\,dx$ is equal to

A $\frac{1}{2}\int_0^1 (u-1)\sqrt{u}\,du$ **B** $\frac{1}{4}\int_1^3 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

C $\int_0^1 u\sqrt{u}\,du$ **D** $\frac{1}{4}\int_1^3 \sqrt{u}\,du$

44 If $\int_0^{\frac{\pi}{6}} \sin^n x \cos x\,dx = \frac{1}{64}$, then n equals

A 6 **B** 5 **C** 4 **D** 3

45 Of the integrals

$$\int_0^{\pi} \sin^3\theta\cos^3\theta\,d\theta, \quad \int_0^2 t^3(4-t^2)^2\,dt, \quad \int_0^{\pi} x^2\cos x\,dx$$

one is negative, one is positive and one is zero. Without evaluating them, determine which is the correct order of signs.

A 0 + − **B** − 0 + **C** + − 0 **D** + 0 −

46 $\int_0^{\frac{\pi}{4}} \cos(2x)\,dx$ is equal to

A $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2x)\,dx$ **B** $\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2x)\,dx$

C $\int_{-\frac{\pi}{4}}^{0} \sin(2x)\,dx$ **D** $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(4x)\,dx$

47 An anti-derivative of $\dfrac{2x}{\sqrt{x^2-1}}$ is

A $2\sqrt{x^2-1}$ **B** $\dfrac{x^2}{\sqrt{x^2-1}}$ **C** $2x\sqrt{x^2-1}$ **D** $\dfrac{2}{\sqrt{x^2-1}}$

48 If $\dfrac{3}{(x-1)(2x+1)} = \dfrac{A}{x-1} + \dfrac{B}{2x+1}$, for all $x \in \mathbb{R} \setminus \{1, -\frac{1}{2}\}$, then

A $A = 4,\ B = 3$ **B** $A = 1,\ B = 4$

C $A = 1,\ B = -2$ **D** $A = 3,\ B = 3$

49 $\int \tan x\,dx$ is equal to

A $\sec^2 x + c$ **B** $\ln(\cos x) + c$ **C** $\ln(\sec x) + c$ **D** $\ln(\sin x) + c$

50 The volume of the solid of revolution formed by rotating the region bounded by the curve $y = 2\sin x - 1$ and the lines with equations $x = 0$, $x = \frac{\pi}{4}$ and $y = 0$ about the x-axis is given by

A $\int_0^{\frac{\pi}{2}} \pi^2(2\sin x - 1)^2\,dx$ **B** $\int_0^{\frac{\pi}{4}} \pi(4\sin^2 x - 1)\,dx$

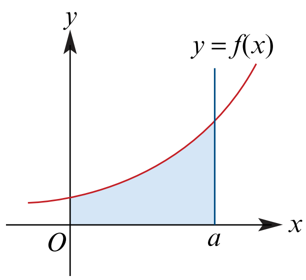
C $\int_0^{\frac{\pi}{4}} \pi(1 - 2\sin x)^2\,dx$ **D** $\int_0^{\frac{\pi}{4}} (2\sin x - 1)^2\,dx$

51 The area of the region bounded by the graphs of $f(x) = \sin x$ and $g(x) = \sin(2x)$ for $0 \le x \le \frac{\pi}{2}$ is

A $\int_0^{\frac{\pi}{2}} \sin x - x\sin(2x)\,dx$ **B** $\int_0^{\frac{\pi}{3}} \sin(2x) - \sin x\,dx$

C $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(2x) - \sin x\,dx$ **D** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \sin(2x)\,dx$

52 The shaded region is bounded by the curve $y = f(x)$, the coordinate axes and the line $x = a$. Which one of the following statements is false?



A The area of the shaded region is $\int_0^a f(x)\,dx$.

B The volume of the solid of revolution formed by rotating the region about the x-axis is $\int_0^a \pi(f(x))^2\,dx$.

C The volume of the solid of revolution formed by rotating the region about the y-axis is $\int_{f(0)}^{f(a)} \pi x^2\,dy$.

D The area of the shaded region is greater than $af(0)$.

53 $\int \frac{1}{\sqrt{9-4x^2}}\,dx$ equals

A $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$ **B** $\frac{1}{3}\sin^{-1}\left(\frac{2x}{3}\right) + c$

C $\frac{1}{2}\sin^{-1}\left(\frac{3x}{2}\right) + c$ **D** $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$

54 $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1-x)^2}\,dx$ equals

A $\frac{4}{3}$ **B** $-\frac{4}{3}$ **C** 1 **D** $\ln 3$

55 $\int \frac{1}{\sqrt{\frac{1}{9} - x^2}}\,dx$ equals

A $\sin^{-1}\left(\frac{x}{3}\right) + c$ **B** $\sin^{-1}(3x) + c$

C $\sin^{-1}\left(\frac{3}{x}\right) + c$ **D** $\frac{3}{2}\ln\left(\frac{1}{9} - x^2\right) + c$

56 $\int \frac{1}{9+4x^2}\,dx$ is

A $\frac{1}{9}\tan^{-1}\left(\frac{2x}{9}\right) + c$ **B** $\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right) + c$

C $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c$ **D** $9\tan^{-1}\left(\frac{2x}{9}\right) + c$

57 $\frac{d}{d\theta}(\sec^3\theta)$ is

A $3\sec^3\theta\tan\theta$ **B** $3\sec^2\theta$ **C** $3\sec^2\theta\tan\theta$ **D** $3\sec^2\theta\tan^2\theta$

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58 If $\int \sin^2(4x)\cos(4x)\,dx = k\sin^3(4x) + c$, then k is

A $\dfrac{1}{12}$ **B** $\dfrac{1}{4}$ **C** $\dfrac{1}{3}$ **D** $-\dfrac{1}{4}$

59 $\dfrac{x+7}{x^2-x-6}$ written as partial fractions is

A $\dfrac{2}{x-3} - \dfrac{1}{x+2}$

B $\dfrac{2}{x+2} - \dfrac{1}{x-3}$

C $\dfrac{9}{5(x-2)} - \dfrac{9}{5(x+3)}$

D $\dfrac{4}{5(x-2)} - \dfrac{9}{5(x+3)}$

60 If $y = \sin^{-1}(3x)$, then $\dfrac{dy}{dx}$ equals

A $-\dfrac{3\cos(3x)}{\sin^2(3x)}$ **B** $3\cos^{-1}(3x)$ **C** $\dfrac{1}{\sqrt{1-9x^2}}$ **D** $\dfrac{3}{\sqrt{1-9x^2}}$

61 $\dfrac{d}{dx}\big(\ln(\tan x)\big)$ equals

A $\ln(\sec^2 x)$ **B** $\cot x$ **C** $\dfrac{2}{\sin(2x)}$ **D** $\dfrac{1}{\sin(2x)}$

62 The general solution of the differential equation $\dfrac{dy}{dx} + y = 1$ (with P being an arbitrary constant) is

A $2x + (1-y)^2 = P$

B $2x - (1-y)^2 = P$

C $y = 1 + Pe^x$

D $y = 1 + Pe^{-x}$

63 $\displaystyle\int \frac{x^2}{(x^3+1)^{\frac{1}{2}}}\,dx$ equals

A $\dfrac{1}{3}\ln\Big((x^3+1)^{\frac{1}{2}}\Big) + c$

B $\dfrac{2}{3}\ln\Big((x^3+1)^{\frac{1}{2}}\Big) + c$

C $\dfrac{2}{3}(x^3+1)^{\frac{1}{2}} + c$

D $\dfrac{1}{6}(x^3+1)^{\frac{1}{2}} + c$

64 Air leaks from a spherical balloon at a constant rate of 2 m^3/s. When the radius of the balloon is 5 m, the rate (in m^2/s) at which the surface area is decreasing is

A $\dfrac{4}{5}$ **B** $\dfrac{8}{5}$ **C** $\dfrac{1}{50}\pi$ **D** $\dfrac{1}{100}\pi$

65 $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$ equals

A $\dfrac{1}{4}$ **B** $\dfrac{1}{2}$ **C** 1 **D** $\dfrac{\pi}{3}$

66 For $-1 < x < 1$, the integral $\displaystyle\int \frac{1}{1-x^2}\,dx$ can be written as

A $\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) + c$

B $\dfrac{1}{2}\ln\left(\dfrac{1-x}{1+x}\right) + c$

C $\ln\left(\dfrac{1+x}{1-x}\right) + c$

D $\dfrac{1}{2}\ln((1-x)(1+x)) + c$

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67 Which one of the following is not an equation of simple harmonic motion, where x is the position of a particle at time t?

A $x = 5\sin(t) + 1$ **B** $x = 2\sin(-3t)$
C $x = 2\cos(3t) + 1$ **D** $x = \sin(2t) + \sin(3t)$

68 A particle is in simple harmonic motion with $\ddot{x} = -16x + 16$, where x m is the position of the particle relative to the origin O at time t s. The centre of the motion, $x = c$, and the period of the motion, T s, are given by

A $c = 1, \quad T = \dfrac{\pi}{8}$ **B** $c = -1, \quad T = \dfrac{\pi}{2}$
C $c = 1, \quad T = \dfrac{\pi}{2}$ **D** $c = 16, \quad T = \dfrac{\pi}{2}$

69 The velocity, V, of a body is given by $V = (x-2)^2$, where x is the position of the body at time t. The acceleration of the body at time t is given by

A $2(x-2)$ **B** $\dfrac{(x-2)^2}{t}$ **C** $2(x-2)^3$ **D** $x^2 - 4x + 4$

70 A particle, P, of unit mass moves under a resisting force $-kv$, where k is a positive constant and v is the velocity of P. No other forces act on P, which has velocity V at time $t = 0$. At time t, the velocity of the particle is

A Ve^{kt} **B** $\left(\dfrac{V}{k}\right)e^{kt}$ **C** Ve^{-kt} **D** $\left(\dfrac{V}{k}\right)e^{-kt}$

71 A particle of mass m lies on a horizontal platform that is being accelerated upwards with an acceleration f. The force exerted by the platform on the particle is

A $m(f-g)$ **B** $m(g+f)$ **C** $m(g-f)$ **D** $\dfrac{mf}{g}$

72 A particle of mass 10 kg is subject to forces of $3\boldsymbol{i}$ newtons and $4\boldsymbol{j}$ newtons. The acceleration of the particle is described by the vector

A $5\boldsymbol{i}$ **B** $0.3\boldsymbol{i} + 0.4\boldsymbol{j}$ **C** $\dfrac{5}{\sqrt{2}}(\boldsymbol{i}+\boldsymbol{j})$ **D** $5\boldsymbol{j}$

73 A body falls, under gravity, against a resistance of kv^2 per unit mass, where v is the speed and k is a constant. After time t, the body has fallen a distance s. Which of the following equations describes the motion?

A $v\dfrac{dv}{ds} = g - kv^2$ **B** $v\dfrac{dv}{dt} = g + kv^2$
C $\dfrac{d^2s}{dt^2} = g + kv^2$ **D** $v\dfrac{dv}{ds} = -(g + kv^2)$

74 A particle starts at rest at a point O and moves in a straight line so that, after t seconds, its velocity, v, is given by $v = 4\sin(2t)$. At this time the displacement, s, from O is given by

A $s = 2 - 2\cos(2t)$ **B** $s = 8\cos(2t)$
C $s = 2\cos(2t)$ **D** $s = -2\cos(2t)$

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75 A boy of mass 60 kg slides down a frictionless slope that is inclined at $\theta°$ to the horizontal, where $\sin\theta° = \frac{4}{5}$. The boy's acceleration down the slide (in m/s^2) is

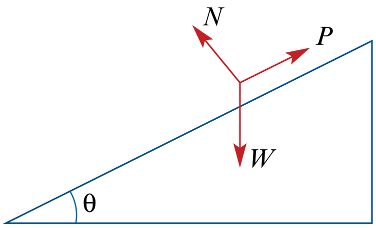
A $\frac{3}{5}g$

B $\frac{4}{5}g$

C $36g$

D $48g$

76 The diagram shows a particle of weight W on an inclined plane. The normal force exerted by the plane is N, and the friction force is F. The force P pulls the particle up the plane at a constant speed. Which one of the following is true?



A $P = W\sin\theta - F$

B $P = F + W\sin\theta$

C $P = F$

D $N = W\sin\theta$

77 A particle of mass 5 kg is subjected to forces of $3\boldsymbol{i}$ newtons and $4\boldsymbol{j}$ newtons. The magnitude of the particle's acceleration is equal to

A 1 ms^{-2} **B** 7 ms^{-2} **C** 1.2 ms^{-2} **D** -1.2 ms^{-2}

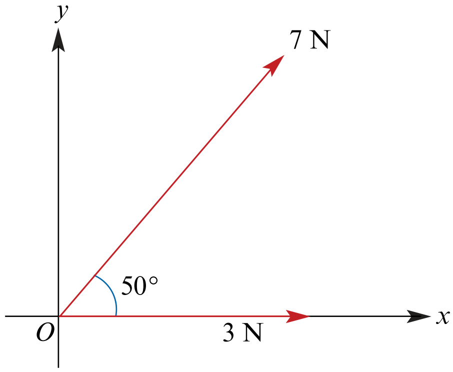
78 A particle of mass m kg is moving with constant velocity down a plane inclined at $\theta°$ to the horizontal. The frictional force in newtons is

A $m\cos\theta$ **B** $mg\sin\theta$ **C** $mg\tan\theta$ **D** mg

79 A particle of mass 8 kg, travelling at a constant velocity of 20 m/s, is acted upon by a force of 5 N. The magnitude of the resulting acceleration is

A 32 ms^{-2} **B** $\dfrac{5}{8g}\text{ ms}^{-2}$ **C** $\dfrac{5}{8}\text{ ms}^{-2}$ **D** 1.6 ms^{-2}

80 Two forces of 7 N and 3 N act at a point as shown in the diagram. The magnitude of the resultant force (in newtons) is



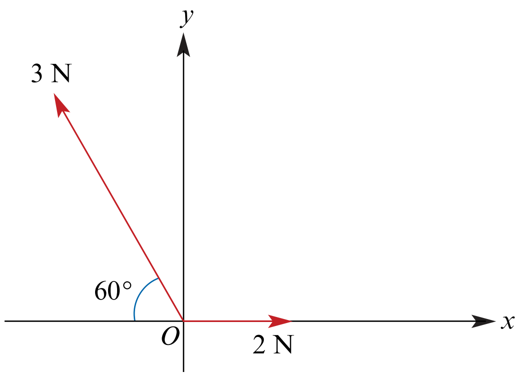
A $7 + 3\cos 50°$

B $3 + 7\cos 50°$

C $10\cos 25°$

D none of these

81 Two forces of 3 N and 2 N act at a point as shown in the diagram. The resultant of these forces makes an angle θ with the positive direction of the x-axis. Which one of the following is true?



A $\cos\theta = \dfrac{2}{3}$ **B** $\tan\theta = 3\sqrt{3}$

C $\theta = 90°$ **D** $\sin\theta = \dfrac{6\sqrt{3}}{\sqrt{107}}$

82 A particle of mass 5 kg has its momentum defined by the vector $30\boldsymbol{i} - 15\boldsymbol{j} + 10\boldsymbol{k}$ kg m/s. The magnitude of the velocity of the particle is

A 25 m/s **B** 5 m/s **C** 7 m/s **D** $\sqrt{31}$ m/s

83 Which of the following is a population parameter?

A $\bar{x}$ **B** s **C** σ **D** z

84 Suppose that X is a random variable with mean $\mu = 48$ and variance $\sigma^2 = 144$. If $\bar{X}$ is the mean of a sample of size 64, then

A $\mathrm{E}(\bar{X}) = 48,\ \mathrm{sd}(\bar{X}) = 1.5$ **B** $\mathrm{E}(\bar{X}) = 6,\ \mathrm{sd}(\bar{X}) = 1.5$

C $\mathrm{E}(\bar{X}) = 48,\ \mathrm{sd}(\bar{X}) = 2.25$ **D** $\mathrm{E}(\bar{X}) = 48,\ \mathrm{sd}(\bar{X}) = 18$

85 The Central Limit Theorem states that the sampling distribution of the sample mean is approximately normal under certain conditions. Which of the following is a necessary condition for the Central Limit Theorem to be used?

A The sample size must be large (at least 30).

B The population size must be large (at least 30).

C The population from which we are sampling must be normally distributed.

D The population from which we are sampling must not be normally distributed.

86 Which of the following statements about the sampling distribution of the sample mean is incorrect?

A The sampling distribution is generated by repeatedly taking samples of size n and computing the sample means.

B The standard deviation of the sampling distribution is σ .

C The sampling distribution is approximately normal whenever the sample size is sufficiently large.

D The mean of the sampling distribution is μ.

87 If 80 random samples are chosen from a population and a 90% confidence interval for the population mean μ is computed from each sample, then on average we would expect the number of intervals which do not contain μ to be

A 0 **B** 8 **C** 56 **D** 72

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88 If a researcher increases her sample size by a factor of 16, then the margin of error in a 99% confidence interval would

A increase by a factor of 4
B increase by a factor of 16
C decrease by a factor of 4
D decrease by a factor of 16

Technology-active multiple-choice questions

89 $\int_0^{\sqrt{3}} \frac{2x+3}{9+x^2}\,dx$ is closest to

A 0.7 **B** 0.8 **C** 0.9 **D** 1.0

90 A particle of mass 5 kg is acted upon by two forces of 0.3 kg wt and 0.4 kg wt at right angles to each other. The magnitude of the acceleration of the particle is

A 0.1 ms^{-2} **B** 10 ms^{-2} **C** 0.98 ms^{-2} **D** $\frac{50}{7} \text{ ms}^{-2}$

91 The rate of decay of a radioactive substance is proportional to the amount, x, of the substance present. This is described by the differential equation $\frac{dx}{dt} = -kx$, where k is a positive constant. Given that initially $x = 20$ and that $x = 5$ when $t = 20$, the time at which $x = 2$ is closest to

A 22.33 **B** 10.98 **C** 50 **D** 33.22

92 The area of the region enclosed by the graphs of $f(x) = \sqrt{4 - x^2}$ and $g(x) = e^{x^2} - 2$ is closest to

A 2.03 **B** 3.57 **C** 5.05 **D** 4.82

93 Let $f(x) = \sqrt[3]{x^2 - 4x}$ and $G(x)$ an antiderivative of $f(x)$ and $G(4) = 5.67$ then the value of $G(1)$ is closest to

A 1.46 **B** 1.58 **C** 9.88 **D** 10.82

94 Consider the differential equation $\frac{dy}{dx} = -\frac{1}{3}\sqrt{y}$ with $y = 17$ when $x = 0$. When $y = 0$ the value of x is closest to

A 4.12 **B** 24.74 **C** 5.54 **D** 25.87

95 The adult weight of a certain breed of cat is normally distributed with mean 5.2 kg and standard deviation 0.55 kg. The probability that the average weight of 20 of these cats, randomly selected, is more than 5.3 kg is closest to

A 0.208 **B** 0.273 **C** 0.428 **D** 0.572

96 A random variable X is has an unknown mean, μ, and a known variance 0.04. A random sample of size 50 was selected, and the average of this sample was determined to be 5.30. A 90% confidence interval for μ is approximately

A $(5.29, 5.31)$ **B** $(5.24, 5.36)$ **C** $(5.23, 5.37)$ **D** $(5.25, 5.35)$

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97 For the margin of error in a 99% confidence interval to be ±0.3 units when the standard deviation is equal to 1.365, the minimum sample size should be

A 56 **B** 80 **C** 113 **D** 138

98 The area of the region enclosed by the graph of $y = x^2 - x - 1$ and the x-axis is (correct to 2 decimal places)

A 1.86 **B** 1.68 **C** 1.74 **D** 1.98

99 The region enclosed by the graphs of $y = x^2$, $y = 2 - x$ and the x-axis is rotated about the y-axis to form a volume of revolution.
The volume of revolution is

A 2π **B** $\dfrac{7\pi}{3}$ **C** $\dfrac{11\pi}{6}$ **D** $\dfrac{32\pi}{15}$

100 The time T that a customer has to wait to order a coffee is exponentially distributed with a mean of $\frac{1}{\lambda}$ minutes.

If $P(T \leq 2) = 0.4$, then the value of $\frac{1}{\lambda}$ correct to 2 decimal places is

A 3.92 **B** 4.05 **C** 3.78 **D** 3.85

101 Values of a continuous function $f(x)$ are given in the table below

x	0	0.5	1	1.5	2
$f(x)$	1	1.8	1.5	0.6	−0.2

An approximation to $\int_0^2 f(x)\,dx$ using Simpson's rule is

A 2.25 **B** 2.20 **C** 2.23 **D** 2.30

102 Given a graph represented by the equation $xy^2 - x^2 = 0$, $x > 0$, then at the point where $y = 2.5$, the gradient is (correct to 2 decimal places)

A 0.20 **B** −0.20 **C** 1.25 **D** −1.25

103 The position x of a particle moving in a straight line is given by $x = \frac{5}{2}\sin 2t - 4,\ t \geq 0$.
The value of t correct to 2 decimal places when $\dot{x} = \ddot{x}$ for the first time is

A 0.23 **B** 1.34 **C** 0.39 **D** 1.80

104 Forces of 10 N and 6 N act on a body and the angle between the two forces is 110°. The magnitude of the resultant force, correct to 2 decimal places is

A 9.74 **B** 12.24 **C** 11.48 **D** 10.33

105 A differential equation is $\dfrac{dy}{dx} = y + 1, x \geq 0$.
Given that $y(1) = 3$, then correct to 2 decimal places, $y(2)$ is equal to

A 9.87 **B** 10.87 **C** 11.87 **D** 12.87

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106 A body is projected up vertically from ground level with a velocity of 30 ms^{-1}. The time in seconds taken by the body to reach maximum height is (correct to 2 decimal places, $g = 9.8\ \mathrm{ms}^{-1}$)

A 3.00 **B** 3.06 **C** 3.10 **D** 3.01

107 A body weighing 3 kg is projected along a horizontal straight track with an initial speed of 30 ms^{-1}. It is subject to a resisting force of $\frac{v}{2}$ N where v is the velocity of the body at time t. The total distance travelled in metres before the body stops is

A 90 **B** 120 **C** 180 **D** 240

108 A particle of mass 2 kg moving in a straight line is subject to an acceleration of $(t^2 - t + 1)\ \mathrm{ms}^{-2}$. If the particle started at rest, its momentum when $t = 3$ is

A 15 **B** 14 **C** 7.5 **D** 7

109 A help-desk receives on average 12 queries per hour. The time between queries is exponentially distributed.
Correct to 2 decimal places, the probability that the next query will occur between 5 and 8 minutes from the last query is

A 0.15 **B** 0.18 **C** 0.17 **D** 0.16

110 The position of a particle moving along a straight line is given by $x = A\sin(\omega t), t \geq 0$. The simple harmonic motion of the particle has a period of 5 seconds and the maximum speed is 10 ms^{-1}.
The value of A correct to 2 decimal places is

A 7.54 **B** 8.22 **C** 6.92 **D** 7.96

111 Two forces of P N and 10 N act on a body. The angle between the two forces is 125°. If the resultant force is perpendicular to the P N force, the value of P correct to 2 decimal places is

A 5.74 **B** 6.26 **C** 7.50 **D** 6.01

112 The change in a particular variable x is given by the differential equation

$$\frac{dx}{dt} = \frac{2(x+1)}{\sqrt{t+1}}, \quad t > 0 \text{ and } x > 0$$

When $t = 0, x = 0$.
Correct to 2 decimal places, the value of x when $t = 3$ is

A 6.39 **B** 55.60 **C** 53.60 **D** 8.39

113 The velocity v of a particle moving in a straight line is given by

$$v = t\sqrt{t^2 + 1}, t \geq 0$$

The displacement of the particle in the first 2 seconds is (correct to 2 decimal places)

A 5.57 **B** 8.35 **C** 3.39 **D** 3.44

114 The velocity v at time t of a body moving in simple harmonic motion is given by $v^2 = 8x - 3x^2$ where x is the position of the body at time t
The period of the motion (correct to 2 decimal places) is

A 9.42 **B** 1.05 **C** 3.63 **D** 4.44

115 The area of the region bounded by the graphs of $f(x) = \sin x$ and $g(x) = 2\cos x$ and the y-axis, $x \geq 0$ is (correct to 2 decimal places)

A 1.00 **B** 1.24 **C** 1.31 **D** 1.17

116 A body moves in a straight line and its velocity is given by $v = \sin t$.
The distance travelled in the first 4 seconds (correct to 2 decimal places) is

A 3.65 **B** 2.35 **C** 1.65 **D** 2.21

117 From a sample of 36 loaves of bread a 95% confidence interval for the mean cost of a loaf was determined to be (\$2.55, \$3.13). The mean and standard deviation of this sample are

A $\bar{x} = \$2.84$, $s = \$0.22$ **B** $\bar{x} = \$2.55$, $s = \$0.15$
C $\bar{x} = \$2.84$, $s = \$0.89$ **D** $\bar{x} = \$2.84$, $s = \$0.30$

118 A 98% confidence interval for the mean amount spent per person in a restaurant μ, based on a sample of 35 people, was found to be from \$32.00 to \$45.00. The standard deviation of the amount spent is closest to:

A \$4.30 **B** \$16.53 **C** \$9.89 **D** \$38.50

119 A confidence interval is to be used to estimate the population mean μ based on a sample mean $\bar{x}$. To decrease the width of a confidence interval by 75%, the sample size must be multiplied by a factor of

A $\dfrac{3}{4}$ **B** $\dfrac{1}{4}$ **C** 4 **D** 16

120 The weight of a bag of 10 mandarins is normally distributed with a mean of μ kg and a standard deviation of 0.08 kg. There is a 5% probability that the average of 25 bags will weigh more than 1 kg . The value of μ is closest to

A 0.973 kg **B** 0.969 kg **C** 1.026 kg **D** 1.031 kg

121 The volume of liquid in a 600 mL bottle of mineral water is a normally distributed random variable with a mean of 602 mL. A random sample of 60 bottles of mineral water was selected. The mean volume for this sample was 601 mL with a standard deviation of 5 mL. To ensure that the confidence interval based on this sample contains μ, the smallest confidence level that could be used for is

A 90% **B** 95% **C** 88% **D** 92%

122 A random variable is normally distributed with mean μ. A 95% confidence interval for μ from a sample from this distribution is $(77.55, 91.65)$. A confidence interval for μ, based on the same sample, using a confidence level greater than 95% could be

A (78.68, 90.52) **B** (77.55, 101.65) **C** (75.34, 93.85) **D** (75.55, 89.55)

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Revision

17

Revision of Units 3 & 4

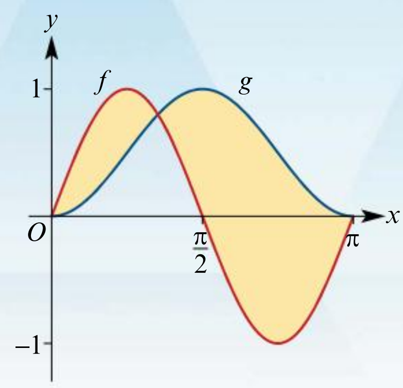
17A Short-response questions

Technology-free short-response questions

SF

1 The diagram shows the graphs of $f(x) = \sin(2x)$ and $g(x) = \sin^2 x$ for $0 \le x \le \pi$.

Determine the total area of the shaded regions.



2 **a** Show that $\sin x + \cos x = \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$.

b Hence show that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$.

3 Let $\mathbf{M} = \begin{bmatrix} k+1 & 3 \\ 2 & k \end{bmatrix}$.

a Determine the two values of k for which the matrix $\mathbf{M}$ does not have an inverse.

b Determine $\mathbf{M}^{-1}$, given that k does not take either of the values from part **a**.

Now consider the two lines with equations $(k+1)x + 3y = 8$ and $2x + ky = 6$.

c Use $\mathbf{M}^{-1}$ to determine the point of intersection of these two lines when $k = 4$.

d Explain what happens geometrically if k takes either of the values from part **a**.

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4 A curve has equation $x^2 + y^2 + 2x + 4y = 24$.

a Determine the gradient of this curve at the point $(1, 3)$.

b A particle moves along the curve. When the particle is at the point $(1, 3)$, its y-coordinate is increasing by 2 units per second. Determine the corresponding rate of change in its x-coordinate.

5 Use mathematical induction to prove that

$$\sum_{r=1}^{n} r(r^2 + 1) = \frac{1}{4}n(n+1)(n^2 + n + 2)$$

for every natural number n.

6 A curve is specified by the vector equation $\boldsymbol{r} = (2t^2 + t)\hat{\boldsymbol{i}} + (3t - 1)\hat{\boldsymbol{j}}$ for $t \in \mathbb{R}$.

a Determine a Cartesian equation of the curve.

b Determine the equation of the tangent to the curve at the point where $t = 1$.

7 Prove by induction that $3^{2n-1} + 8^{2n-1}$ is divisible by 11, for each positive integer n.

8 Points A, B and C have coordinates $(2, -1, 0)$, $(1, 1, 1)$ and $(-3, 0, 2)$ respectively.

a Determine the magnitude of $\angle ABC$.

b Determine the coordinates of the point D such that $ABCD$ is a parallelogram.

9 Let $z_1 = 1 - i$ and $z_2 = 2\operatorname{cis}\left(\frac{2\pi}{3}\right)$.

a Express each of the following in polar form:

i $(z_1)^3$ **ii** $(z_2)^2$ **iii** $(z_1)^3 \times (z_2)^2$

b Express $(z_1)^3 + (z_2)^2$ in Cartesian form.

10 Determine each of the following anti-derivatives:

a $\displaystyle\int \frac{5}{\sqrt{4 - x^2}}\,dx$ **b** $\displaystyle\int \frac{5}{4 - x^2}\,dx$ **c** $\displaystyle\int \frac{5}{4 + x^2}\,dx$

11 Determine each of the following anti-derivatives:

a $\displaystyle\int \frac{x}{9 - 4x}\,dx$ **b** $\displaystyle\int \frac{x}{9 - 4x^2}\,dx$ **c** $\displaystyle\int \frac{x}{\sqrt{9 - 4x^2}}\,dx$

12 **a** Determine the derivative of $f(x) = x\ln(x + 2)$.

b Hence evaluate $\int_0^2 \ln(x + 2)\,dx$.

13 Let ℓ be the line with vector equation $\boldsymbol{r} = -8\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$, $t \in \mathbb{R}$, and let Π be the plane with Cartesian equation $12x - 2y - z = 17$. Show that the line ℓ and the plane Π do not intersect.

14 The line given by $\boldsymbol{r} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + 9\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$, $t \in \mathbb{R}$, crosses the x–z plane and the y–z plane at the points A and B respectively. What is the length of the line segment AB?

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Revision

SF

15 Determine a vector equation that represents the line of intersection of the planes defined by the equations $3x - y + 2z = 100$ and $x + 3y = 45$.

16 Determine the distance between the two parallel lines $5x + 5y - 11 = 0$ and $x + y - 1 = 0$ in the Cartesian plane.

17 Consider the following system of linear equations, where m is a constant:

$$\begin{aligned} x + 5y - 6z &= 2 \\ mx + y - z &= 0 \\ 5x - my + 3z &= 7 \end{aligned}$$

For what values of m does this system have a unique solution?

18 Determine the coordinates of the point where the line through $A(3, 4, 1)$ and $B(5, 1, 6)$ crosses the x–y plane.

19 The line ℓ passes through the points $A(-1, -3, -3)$ and $B(5, 0, 6)$. Determine a vector equation of the line ℓ, and determine the distance from the origin to the line.

20 Prove by induction that $2^{n+2} + 3^{2n+1}$ is divisible by 7, for each positive integer n.

21 Determine the gradient of the curve $2y^2 - xy^3 = 8$ at the point where $y = -1$.

22 Let $f(x) = 4\arccos(2x - 1)$. Determine:

a the maximal domain

b the range

c the value of $f(\frac{1}{2})$

d the value of a such that $f(a) = 3\pi$

e the equation of the tangent to the graph at the point where $x = \frac{1}{2}$.

23 For the graph of $f(x) = \dfrac{x+3}{x^2+3}$, determine:

a the equations of any asymptotes

b the coordinates of any stationary points

c the area bounded by the x-axis, the y-axis, the line $x = 3$ and the graph of $y = f(x)$.

24 **a** Determine:

i $(5 + i)(4 + i)$

ii $(\sqrt{3} + i)(-2\sqrt{3} + i)$

iii $\left(\dfrac{1}{2} + i\right)\left(-\dfrac{3}{4} + i\right)$

iv $(1.2 - i)(0.4 + i)$

b Let $z = a + i$ and $w = b + i$, where both a and b are integers.

i Determine zw, in terms of a and b.

ii If $\mathrm{Re}(zw) = \mathrm{Im}(zw)$, express b in terms of a.

iii Hence sketch the graph of b against a.

25 Suppose that X is a random variable with mean μ and variance σ^2, and $\bar{X}$ is the mean of a sample of size 25. Given that $\mathrm{E}(\bar{X}) = 10$ and $\mathrm{Var}(\bar{X}) = 5$, determine the values of μ and σ.

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26 Let $P(z) = z^4 + 3z^2 - 6z + 10$.

a Show that $P(1 + i) = 0$.

b Factorise $P(z)$ into linear factors.

27 Determine an anti-derivative of each of the following:

a $(2x - 6)e^x$ **b** $x\ln(2x)$ **c** $x\sec^2(3x)$ **d** $x\tan^2(x)$

28 Evaluate each of the following definite integrals:

a $\int_0^1 xe^{3x}\,dx$ **b** $\int_0^{\frac{\pi}{2}} x\sin(3x)\,dx$ **c** $\int_0^{\pi} (2x - 2)\cos\left(\frac{x}{3}\right)dx$

29 Calculate the area of the triangle XYZ, where the three vertices have coordinates $X(1, 3, 2)$, $Y(2, -1, 0)$ and $Z(1, 10, 6)$.

30 The two planes with equations $2x - 2y + z = 5$ and $x - 3y + 2z = 2$ intersect in a line. Describe this line using:

a parametric equations **b** a vector equation **c** Cartesian equations.

31 **a** Determine the solutions of the equation $z^4 + 36 = 0$ in both polar form and Cartesian form.

b Hence write $z^4 + 36$ as a product of two quadratic factors with real coefficients.

c Illustrate the solutions of the equation $z^4 + 36 = 0$ on an Argand diagram.

32 Determine the values of A and B such that the function

$$y = e^{-x}(A\sin(\pi x) + B\cos(\pi x))$$

is a solution of the differential equation $\frac{dy}{dx} = e^{-x}\sin(\pi x)$.

33 The region bounded by the graph of $y = e^{-x} - 2$ and the two axes is rotated about the x-axis. Determine the volume of this solid of revolution.

34 A tank originally holds 40 litres of water, in which 10 grams of a chemical is dissolved. Pure water is poured into the tank at 4 litres per minute. The mixture is well stirred and flows out at 6 litres per minute until the tank is empty.

a State how long it takes the tank to empty.

b Set up a differential equation for the mass, m grams, of chemical in the tank at time t minutes, including the initial condition.

c Express m in terms of t.

d Hence determine how long it takes for the concentration of the solution to reach 0.2 grams per litre.

35 **a** Solve the differential equation $\frac{dy}{dx} = e^{x+y}$, $y(1) = 1$, expressing y as a function of x.

b State the maximal domain of this function.

c Determine the equation of the tangent to the curve at $x = 0$.

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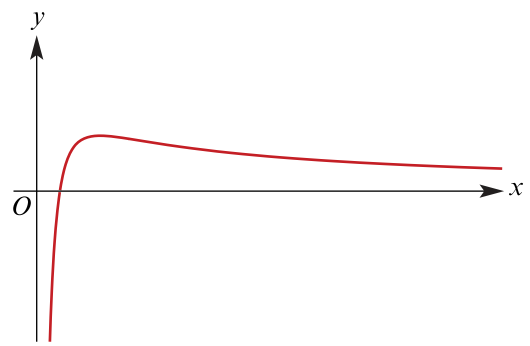
Revision

CF

36 The graph of $y = \dfrac{\ln x}{x}$ is shown.

Point P is the stationary point, and Q is the point of intersection of the graph with the x-axis.



a Determine the coordinates of P and Q.

b Determine the area of the region bounded by the x-axis, the curve and the line $x = e$.

37 **a** Solve $\dfrac{dy}{dx} = x(4 + y^2)$, where $y(0) = 2$, expressing y as a function of x.

b State the maximal domain of this function.

c Determine the equation of the normal to the curve at $x = \dfrac{1}{2}\sqrt{\dfrac{\pi}{3}}$.

38 **a** Show that $\dfrac{x}{\sqrt{x-1}} = \sqrt{x-1} + \dfrac{1}{\sqrt{x-1}}$.

b The graph of $f(x) = \dfrac{x}{\sqrt{x-1}}$, for $x \in [2, a]$, is rotated about the x-axis to form a solid of revolution. Determine the volume of this solid in terms of a.

39 Let P be a point on the line $x + y = 1$ and write $\overrightarrow{OP} = m\hat{\boldsymbol{i}} + n\hat{\boldsymbol{j}}$, where O is the origin and $m, n \in \mathbb{R}$.

a Determine the unit vectors parallel to the line $x + y = 1$.

b Determine a relation between m and n, and hence express $\overrightarrow{OP}$ in terms of m only.

c Determine the two values of m such that $\overrightarrow{OP}$ makes an angle of 60° with the line $x + y = 1$.

40 Points A, B and C are represented by position vectors $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$, $2\hat{\boldsymbol{i}} + m\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ respectively.

a The position vector $\boldsymbol{r} = \overrightarrow{OA} + t\overrightarrow{AC}$, $t \in \mathbb{R}$, can be used to represent any point on the line AC. Determine the value of t for which $\boldsymbol{r}$ is perpendicular to $\overrightarrow{AC}$.

b Determine the value of m such that $\angle BAC$ is a right angle.

41 Let $f(x) = \dfrac{4x^2 + 16x}{(x-2)^2(x^2+4)}$.

a Given that $f(x) = \dfrac{a}{x-2} + \dfrac{6}{(x-2)^2} - \dfrac{bx+4}{x^2+4}$, determine a and b.

b Given that $\int_{-2}^{0} f(x)\,dx = \dfrac{c - \pi - \ln d}{2}$, determine c and d.

42 Lines ℓ_1 and ℓ_2 are defined by the following two vector equations:

$$\ell_1: \quad \boldsymbol{r}_1 = (4 + \lambda)\hat{\boldsymbol{i}} + (1 + \lambda)\hat{\boldsymbol{j}} + 4\lambda\hat{\boldsymbol{k}} \qquad \text{for } \lambda \in \mathbb{R}$$

$$\ell_2: \quad \boldsymbol{r}_2 = (4 + \mu)\hat{\boldsymbol{i}} + (4 + 4\mu)\hat{\boldsymbol{j}} + (m + \mu)\hat{\boldsymbol{k}} \qquad \text{for } \mu \in \mathbb{R}$$

Given that lines ℓ_1 and ℓ_2 intersect at a point, determine the value of m.

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CF

43 A particle is moving in a straight line such that its position, x metres, at time t seconds is given by $x = 3 - 2\cos^2 t$.

a Show that the motion of the particle is simple harmonic.

b **i** Determine the period of the motion.

ii Determine the amplitude of the motion.

iii Determine the maximum speed of the particle.

44 Let $z = 3\operatorname{cis}\left(\frac{\pi}{4}\right)$.

a Write the complex numbers z, z^2 and iz in both polar form and Cartesian form.

b On an Argand diagram, plot and clearly label the points corresponding to z, z^2 and iz.

45 Consider the polynomial $P(z) = z^2 - (m + 2i)z + n(1 + i)$, where m and n are real numbers such that $P(1 + 3i) = 0$.

a Determine the values of m and n.

b Solve the equation $P(z) = 0$ for z.

46 A market researcher determined, from a sample of size n, a 95% confidence interval for the mean age of people who buy a particular brand of sunscreen to be $(23.6, 26.4)$.

a Determine the value of $\bar{x}$ that was used to calculate this confidence interval.

b Calculate the margin of error.

c Determine the confidence interval that would have resulted from the same values of $\bar{x}$ and s if the sample had been of size $\frac{n}{4}$.

CU

47 Let z be a non-zero complex number such that

$$z + \frac{4}{z} = k \quad \text{for some } k \in \mathbb{R}$$

and let z be written in Cartesian form as $z = x + yi$, where $x, y \in \mathbb{R}$.

a Prove that $y = 0$ or $x^2 + y^2 = 4$.

b Prove that if $y = 0$, then $|k| \geq 4$.

c Prove that if $x^2 + y^2 = 4$, then $|k| \leq 4$.

48 Variables x and y are related by $2(x + y) = (x - y)^2$.

a Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

b If $\frac{dy}{dx} = 0$, determine the values of x and y.

c If $\frac{dy}{dx} = 0$, determine the value of $\frac{d^2y}{dx^2}$.

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Revision

Technology-active short-response questions

SF

49 Use Simpson's rule with $n = 2$ to determine an approximate value for $\int_{\frac{1}{2}}^{\frac{3}{2}} x^{-1}\, dx$

50 Determine the acute angle, correct to the nearest degree, between the planes given by $x - y + 3z = 2$ and $3x + y - z = -5$.

51 Solve the following simultaneous equations:

$$-3x + y + 2z = 5$$
$$2x + 5y - 3z = 3$$
$$3x + 6y - 5z = 0$$

52 Lines ℓ_1 and ℓ_2 are defined by vector functions as follows:

$$\ell_1: \quad \boldsymbol{r}_1(s) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \ell_2: \quad \boldsymbol{r}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a Determine the coordinates of the point of intersection of the two lines.

b Determine the acute angle between the two lines.

c Determine the distance, in terms of the parameter t, from the point $(1, -2)$ to the point with position vector $\boldsymbol{r}_2(t)$ on line ℓ_2.

d Hence determine the shortest distance from the point $(1, -2)$ to the line ℓ_2.

53 Consider the system of linear equations

$$2x - 5y + 7z = 4$$
$$3x + y - 12z = -8$$
$$5x + 2y - 4z = 3$$

a Write this system as a matrix equation $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix and both $\mathbf{X}$ and $\mathbf{B}$ are 3×1 matrices.

b Solve the matrix equation for $\mathbf{X}$.

54 The weights of bags of oranges are normally distributed with a mean of 1.5 kg and a standard deviation of 50 g. Lacey buys four bags of oranges. Determine the probability (correct to three decimal places) that the mean weight of the bags is less than 1.45 kg.

55 A random sample of 100 students is selected from those studying Year 12 mathematics, and their mean IQ is determined to be 115.6 with a standard deviation of 11.5. Determine a 90% confidence interval for the mean IQ of all students studying Year 12 mathematics.

56 Suppose that the durations of phone conversations are exponentially distributed with a mean of 3 minutes. Determine the probability that a phone conversation takes:

a less than 2 minutes

b less than 2 minutes, given that it takes less than 2.5 minutes

c less than 2 minutes, given that it takes more than 1 minute.

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57 Consider the points $A(3, 2, 4)$, $B(-2, 3, 5)$ and $C(1, 0, 2)$ in three-dimensional space.

a Express the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ in component form.

b Determine $\overrightarrow{CA} \times \overrightarrow{CB}$.

c Determine the area of the triangle ABC.

d Determine the area of the parallelogram spanned by the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

58 A certain population of female marsupials is divided into six age groups, each spanning 3 months. The population can then be modelled by the following Leslie matrix, $\mathbf{L}$, and initial population matrix, $\mathbf{P}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & 0.3 & 0.8 & 0.7 & 0.4 & 0 \\ 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{bmatrix} 24 \\ 16 \\ 24 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

a Determine the population matrices $\mathbf{P}_1$, $\mathbf{P}_{40}$ and $\mathbf{P}_{41}$.

b Using $\mathbf{P}_{40}$ and $\mathbf{P}_{41}$, show that the long-term growth rate of the population is approximately 1.03.

Now assume that the initial population is 2202 and the sizes of the six age groups (from youngest to oldest) are 715, 416, 363, 317, 247 and 144.

c Write down $\mathbf{P}_0$ and determine $\mathbf{P}_1$.

d Using $\mathbf{P}_0$ and $\mathbf{P}_1$, show that the growth rate of the population over the first 3 months is approximately 1.03.

59 A round-robin debating tournament was held between five schools, P, Q, R, S and T.

- P won against R, and lost to Q, S and T.
- Q won against P, S and T, and lost to R.
- R won against Q, S and T, and lost to P.
- S won against P, and lost to Q, R and T.

Using this information:

a Represent all the results of the tournament as a dominance matrix $\mathbf{M}$.

b Use $\mathbf{M} + \mathbf{M}^2$ to determine a ranking of the five schools.

60 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = \begin{cases} \dfrac{\pi}{20} \sin\left(\dfrac{\pi(x-7)}{10}\right) & \text{if } 7 \le x \le 17 \\ 0 & \text{otherwise} \end{cases}$$

a Determine the mean and standard deviation of X.

b Determine the mean and standard deviation of $\bar{X}$, the average weight of the 2-year-old children in a random sample of size 30.

c Using the central limit theorem, determine an approximate value for $P(\bar{X} > 12.8)$.

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Revision

CF

61 A particle moves in a line such that the velocity, v m/s, at time t seconds ($t \geq 0$) satisfies the differential equation $\dfrac{dv}{dt} = \dfrac{-v}{50}(1 + v^2)$. The particle starts from O with an initial velocity of 10 m/s.

a **i** Express as an integral the time taken for the particle's velocity to decrease from 10 m/s to 5 m/s.

ii Hence calculate the time taken for this to occur.

b **i** Show that, for $v \geq 0$, the motion of this particle is described by the differential equation $\dfrac{dv}{dx} = \dfrac{-(1 + v^2)}{50}$, where x metres is the displacement from O.

ii Given that $v = 10$ when $x = 0$, solve this differential equation, expressing x in terms of v.

iii Hence show that $v = \dfrac{10 - \tan\left(\dfrac{x}{50}\right)}{1 + 10\tan\left(\dfrac{x}{50}\right)}$.

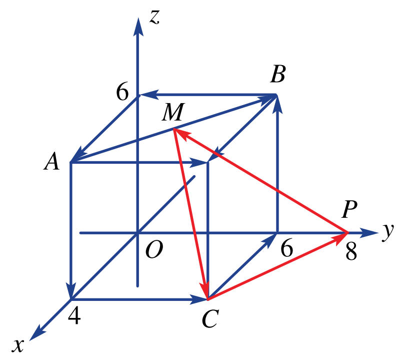
iv Hence determine the displacement of the particle from O, to the nearest metre, when it first comes to rest.

62 A, B, C and O are vertices of a cuboid with edges of length as indicated in the diagram. M is the midpoint of AB and P is the point (0, 8, 0).

a Determine the vector $\overrightarrow{OM}$.

b Determine vectors $\overrightarrow{MC}$ and $\overrightarrow{MP}$.

c Use the cross product to find the area of triangle CMP.



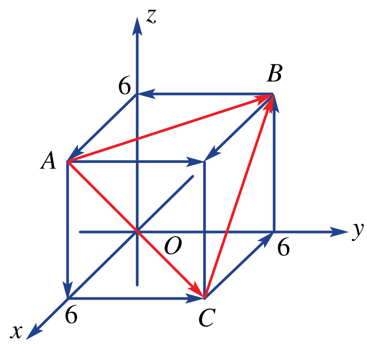
63 The diagram shows a cube set on Cartesian axes.

a Determine vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

b M is the midpoint of AC. Determine the vector $\overrightarrow{OM}$.

c N is a point on BM such that $\overrightarrow{BN} = \dfrac{2}{3}\overrightarrow{BM}$. Determine $\overrightarrow{ON}$.

d Show that $\overrightarrow{ON}$ is perpendicular to the plane defined by points A, B and C.



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64 A stone is dropped from a great height. The speed, v m/s, of the stone after t seconds can be modelled by the differential equation

$$\frac{dv}{dt} = 10 - 0.1v^2$$

The initial speed is 0 m/s.

a Determine t in terms of v.

b Determine v in terms of t.

c Determine the exact time taken for the stone to reach a speed of 2 m/s.

d Sketch the graph of v against t.

65 A curve in the plane is defined by the vector equation

$$\boldsymbol{r}(t) = \sin(2t)\,\hat{\boldsymbol{i}} + \tan(t)\,\hat{\boldsymbol{j}} \quad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}$$

a Show that the Cartesian relation corresponding to the curve has equation $x = \dfrac{2y}{1+y^2}$, and state the domain and range of this relation.

b Determine the coordinates of the points where the curve intersects the line with equation $y = x$, and give the values of the parameter t corresponding to these points.

c Determine the equation of the tangent to the curve at the point where:

i $t = \dfrac{\pi}{6}$ **ii** $t = \dfrac{\pi}{3}$

d Determine the coordinates of the point of intersection of these two tangents.

e Determine the area of the region enclosed by the curve, the x-axis and the line $x = 1$.

66 The population, P, of goats on an island grows at a rate proportional to $P - 0.5P_0$, where P_0 is the initial population and time t is measured in years.

a Write down a differential equation that represents this situation.

b Solve the differential equation, given that the initial population was 1000 and the population increased to 1100 after 1 year.

c Determine the increase in population in the third year.

d Determine the time taken for the population to reach 2000. (Answer in years, correct to two decimal places.)

67 Riley's factory has two machines, A and B, for making nails.
The nails produced by Machine A have a mean diameter of 3 mm, with a standard deviation of 0.03 mm.
The nails produced by Machine B have a mean diameter of 3.01 mm, with a standard deviation of 0.02 mm.

a A random sample of 30 nails is collected from Machine A. Determine the approximate probability that the mean diameter of the nails in this sample is less than 2.99 mm or greater than 3.01 mm.

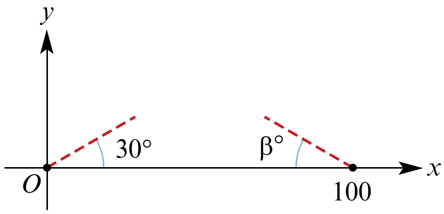
b A random sample of 30 nails is collected from Machine B. Determine the approximate probability that the mean diameter of the nails in this sample is less than 2.99 mm or greater than 3.01 mm.

c Morgan brings Riley another random sample of 30 nails, but cannot remember which of the two machines they were collected from. Riley determines that this sample of nails has a mean diameter of 3.0 mm, with a standard deviation of 0.025 mm.

i Use this sample to determine a 95% confidence interval for the mean diameter of the population of nails.

ii Which machine do you think this sample came from, and why?

68 In the diagram, the x-axis represents horizontal ground, the y-axis is vertical and the unit of distance is metres.
Particle A is projected with speed 60 m/s from the origin at an angle of 30° to the positive x-direction. At the same time, particle B is projected with speed 50 m/s from the point $(100, 0)$ at an angle of $\beta°$ to the negative x-direction.



a Give expressions for $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$, the position vectors of particles A and B after t seconds.

b Given that the two particles collide, determine the value of β.

c Determine the time of collision (in seconds, correct to two decimal places).

d Determine the coordinates of the point of collision (correct to two decimal places).

69 The points A, B and C have position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ respectively, with respect to an origin O.

a Determine a vector equation of the line BC.

b Determine a vector equation of the plane Π that contains the point A and is perpendicular to the line OA.

c Show that the line BC is parallel to the plane Π.

d A circle with centre O passes through A and B. Determine the length of the minor arc AB.

e Verify that $2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ is perpendicular to the plane OAB. Write down a vector perpendicular to the plane OAC. Hence, determine the acute angle between the planes OAB and OAC.

70 The plane Π_1 is given by the Cartesian equation $3x + 2y - z = -1$, and the line ℓ_1 is given by the vector equation $\boldsymbol{r} = (4 - t)\hat{\boldsymbol{i}} + (2t - 3)\hat{\boldsymbol{j}} + (t + 7)\hat{\boldsymbol{k}}$, $t \in \mathbb{R}$.

a Show that the line ℓ_1 lies in the plane Π_1.

The line ℓ_2 is given by the vector equation $\boldsymbol{r} = 10\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$, $t \in \mathbb{R}$, and the line ℓ_2 intersects the plane Π_1 at the point A.

b Determine the coordinates of the point A.

c Determine a Cartesian equation of the plane Π_2 that passes through the point A and is perpendicular to the line ℓ_1.

d Determine the coordinates of the point where the line ℓ_1 intersects the plane Π_2.

e Determine a vector equation of the line ℓ_3 that lies in the plane Π_1, passes through the point A and is perpendicular to the line ℓ_1.

71 A plane Π_1 in three-dimensional space has vector equation $\boldsymbol{r} \cdot (2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) = -6$.

a Determine a vector equation of the line that is normal to Π_1 and passes through $P(2, 1, 4)$.

b Determine the coordinates of Q, the foot of the perpendicular on Π_1 from the point P.

c Determine the angle between OQ and Π_1.

d Planes Π_2 and Π_3 have vector equations $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 5$ and $\boldsymbol{r} \cdot \hat{\boldsymbol{i}} = 0$ respectively. Determine the point of intersection of the three planes Π_1, Π_2 and Π_3.

72 Consider the function

$$f(x) = \begin{cases} x \ln x - 3x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a Determine the derivative for $x > 0$.

b One x-axis intercept is at $(0, 0)$. Determine the coordinates of the other x-axis intercept, A.

c Determine the equation of the tangent at A.

d Determine the ratio of the area of the region bounded by the tangent and the coordinate axes to the area of the region bounded by the graph of $y = f(x)$ and the x-axis.

73 **a** Consider $y = \dfrac{a + b \sin x}{b + a \sin x}$, where $0 < a < b$.

i Determine $\dfrac{dy}{dx}$.

ii Determine the maximum and minimum values of y.

b For the graph of $y = \dfrac{1 + 2 \sin x}{2 + \sin x}$, $-\pi \le x \le 2\pi$:

i State the coordinates of the y-axis intercept.

ii Determine the coordinates of the x-axis intercepts.

iii Determine the coordinates of the stationary points.

iv Sketch the graph of $y = f(x)$.

v Determine the area of the region bounded by the graph and the line $y = -1$.

74 Consider the function

$$f(x) = \cos x + \sqrt{3} \sin x, \quad 0 \le x \le 2\pi$$

Given that $f(x)$ can be expressed in the form $r \cos(x - a)$, where $r > 0$ and $0 < a < \dfrac{\pi}{2}$:

a Determine the values of r and a.

b Determine the range of the function.

c Determine the coordinates of the y-axis intercept.

d Determine the coordinates of the x-axis intercepts.

e Determine x, if $f(x) = \sqrt{2}$.

f If $g(x) = \dfrac{1}{f(x)}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} g(x)\,dx$.

g Determine the volume measure of the solid formed when the region bounded by the graph of $y = f(x)$, the x-axis and the y-axis is rotated about the x-axis.

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Revision

CU

75 **a** **i** Determine the derivative of $x\cos(\pi x)$.

ii Hence use calculus methods to determine an anti-derivative of $x\sin(\pi x)$.

Let $f(x) = \sin(\pi x) + px$, $x \in [0, 1]$.

b **i** Determine the value of p for which $f'(1) = 0$.

ii Hence, using this value of p, show that $f'(x) \geq 0$ for $x \in [0, 1]$.

c Sketch the graph of $y = f(x)$, $x \in [0, 1]$.

d Determine the exact value for the volume of revolution formed when the graph of $y = f(x)$, $x \in [0, 1]$, is rotated around the x-axis.

e For $g(x) = k\arcsin(x)$, $x \in [0, 1]$, determine the value of k such that $f(1) = g(1)$.

f Using this value of p, determine the area of the region enclosed by the graphs of $y = f(x)$ and $y = g(x)$, correct to three decimal places.

g If $f(x) - g(x)$ has a maximum at $x = a$, determine a, correct to three decimal places.

76 The complex number $z_1 = \sqrt{3} - 3i$ is a solution of the equation $z^3 + a = 0$.

a **i** Determine the value of a.

ii Hence determine the other solutions z_2 and z_3, where z_2 is real.

b Plot the solutions on an Argand diagram.

c A set of points on the Argand diagram is defined by the equation

$$|z - z_1| + |z - z_3| = b$$

i This set of points includes the point z_2. Show that the value of b is 12.

ii Hence determine the two complex numbers on the line through z_1 and z_3 which belong to this set of points.

iii Hence or otherwise, and using $z = x + yi$, determine the Cartesian equation of the set of points.

77 Points A and B are represented by position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$ and $\boldsymbol{b} = m(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ respectively, relative to a point O, where $m > 0$.

a Determine the value of m for which A and B are equidistant from O.

Points A and B lie on a circle with centre O. Point C is represented by the position vector $-\boldsymbol{a}$.

b **i** Give reasons why C also lies on the circle.

ii By using the scalar product, show that $\angle ABC = 90°$.

Now assume that all points on this circle can be represented by the general position vector $\boldsymbol{d} = k\boldsymbol{a} + \ell\boldsymbol{b}$, for different values of k and ℓ.

c **i** Show that the relation between k and ℓ is given by $9k^2 - 2\sqrt{3}k\ell + 9\ell^2 = 9$.

ii When $k = 1$, determine the two position vectors that represent points on the circle.

d Let P be a point on the circle such that OP bisects AB. Determine the position vectors which represent P. Do not attempt to simplify your answer.

A particle is travelling such that its position at time t seconds is given by $\boldsymbol{r} = (5 - t)\hat{\boldsymbol{i}} + (2 + t)\hat{\boldsymbol{j}} + (t - 3)\hat{\boldsymbol{k}}$.

e Determine the value of t when $\boldsymbol{r}$ can be expressed in the form $k\boldsymbol{a} + \ell\boldsymbol{b}$, and determine the corresponding values of k and ℓ.

f Hence determine whether the particle lies inside, outside or on this circle at this time.

78 A curve is defined by the vector equation

$$\boldsymbol{r} = 3\sin(t)\,\hat{\boldsymbol{i}} + (6\cos(t) - a)\,\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$$

where a is a real constant with $0 \le a < 6$.

a **i** Determine the Cartesian equation of the curve.

ii Determine the intercepts of the curve with the x-axis.

b Define the function which represents the part of the curve above the x-axis.

c Differentiate $x\sqrt{9 - x^2}$.

d **i** Show that $\dfrac{x^2}{\sqrt{9 - x^2}}$ can be expressed in the form $\dfrac{A}{\sqrt{9 - x^2}} - \sqrt{9 - x^2}$ by determining the appropriate value for A.

ii Hence show that the result in **c** can be written as $2\sqrt{9 - x^2} - \dfrac{9}{\sqrt{9 - x^2}}$.

e Use this result and calculus to determine an anti-derivative of $\sqrt{9 - x^2}$.

f Hence determine the area of the region enclosed by the curve above the x-axis.

g For $a = 0$, determine the area of the region enclosed by the curve.

h For $a = 0$, determine the volume of the solid of revolution formed when the curve is rotated about its horizontal axis.

79 The coordinates, $P(x, y)$, of points on a curve satisfy the differential equations $\dfrac{dx}{dt} = -3y$ and $\dfrac{dy}{dt} = \sin(2t)$ and when $t = 0$, $y = -\frac{1}{2}$ and $x = 0$.

a Determine x and y in terms of t.

b Determine the Cartesian equation of the curve.

c Determine the gradient of the tangent to the curve at a point $P(x, y)$ in terms of t.

d Determine the axis intercepts of the tangent in terms of t.

e Let the x- and y-axis intercepts of the tangent be points A and B respectively, and let O be the origin. Determine an expression for the area of triangle AOB in terms of t, and hence determine the minimum area of this triangle and the values of t for which this occurs.

f Give a pair of parametric equations in terms of t which describe the circle with centre the origin and the same x-axis intercepts as the curve.

g Determine the volume of the solid formed by rotating the region between the circle and the curve about the x-axis.

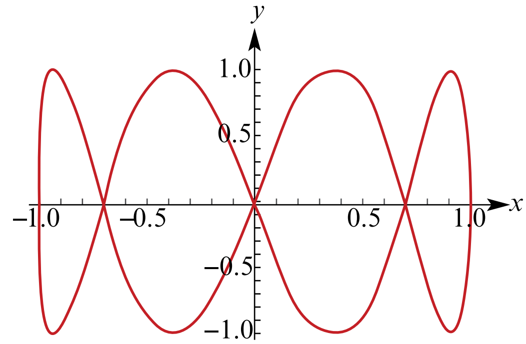
Revision

CU

80 A curve is defined by the parametric equations

$$x = \sin(t), \quad y = \sin(4t)$$

for $0 \le t \le 2\pi$. The graph is shown.



a Determine the Cartesian equation of the curve with y in terms of x.

b Determine $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ in terms of t.

c **i** Determine the values of t for which $\frac{dy}{dx} = 0$.

ii Determine the values of x for which $\frac{dy}{dx} = 0$.

iii Determine the coordinates of the stationary points of the graph.

iv Determine the gradients of the graph at $x = \frac{1}{\sqrt{2}}$, at $x = \frac{-1}{\sqrt{2}}$ and at the origin.

v Show that the gradient is undefined when $x = -1$ or $x = 1$.

d Determine the total area of the regions enclosed by the curve.

e Determine the volume of the solid of revolution formed by rotating the curve around the x-axis.

81 A curve is defined by the vector equation

$$\boldsymbol{r} = t^2\,\hat{\boldsymbol{i}} + \left(\tfrac{1}{3}t^3 - t\right)\hat{\boldsymbol{j}}, \quad t \in \mathbb{R}$$

a The curve can be described by a Cartesian equation of the form $y^2 = g(x)$. Determine $g(x)$.

b Determine the coordinates of the stationary points of the curve.

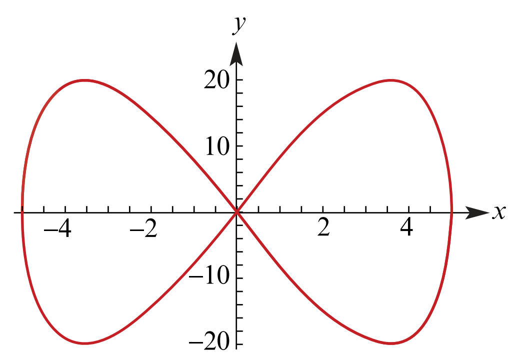
c Determine the area of the region enclosed by the curve.

d Determine the volume of the solid formed by rotating this region around the x-axis.

82 The position of a particle at time t is given by

$$\boldsymbol{r}(t) = 5\sin\left(\frac{\pi t}{30}\right)\hat{\boldsymbol{i}} + 20\sin\left(\frac{\pi t}{15}\right)\hat{\boldsymbol{j}}$$

for $t \ge 0$.



a Determine the Cartesian equation of the path of the particle. The curve is shown.

b Determine the gradients of the curve when:

i $x = 0$ **ii** $x = 3$

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c **i** Determine the velocity of the particle when $t = 7.5$.

ii Determine the speed of the particle when $t = 7.5$.

d Determine, using the method of substitution, the area of the regions enclosed by the curve.

e Determine the greatest distance from the origin reached by the particle.

f Determine the volume of the solid of revolution formed by rotating the curve around the x-axis.

83 Linh rides her bike to work each day. She knows that the time it takes is normally distributed with a mean of 55 minutes and a standard deviation of 5 minutes.

a What is the probability that Linh will ride to work in less than 48 minutes on a particular day?

b Determine k_1 and k_2 such that the probability that Linh takes between k_1 and k_2 minutes to ride to work is 0.95.

c During a five-day working week, Linh makes the ride 10 times. Determine the probability that, in a randomly chosen week:

i Linh's average riding time is less than 50 minutes

ii Linh's total riding time is more than 580 minutes

iii the ride takes less than 50 minutes more than three times during the week.

d Determine c_1 and c_2 such that there is a probability of 0.95 that her average riding time over a five-day period is between c_1 and c_2 minutes.

84 A continuous random variable X has the probability density function

$$f(x) = \begin{cases} \dfrac{1}{b} & \text{if } 0 \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where b is a positive constant.

a Determine the mean and standard deviation of X in terms of b.

b Determine the mean and standard deviation of $\bar{X}$ in terms of b and n, where $\bar{X}$ is the mean of a random sample of size n.

c For a particular random sample of size 50, the sample mean was 2.4. Give an expression in terms of b for a 90% confidence interval for the mean of X.

d What does this confidence interval tell us about the value of b?

85 Let $f(x) = x \arcsin(x)$, $x \in [-1, 1]$, and let $g(x) = \arcsin(x)$, $x \in [-1, 1]$.

a Determine $f'(x)$ and the coordinates of any turning points for $x \in (-1, 1)$.

b Determine $f''(x)$ and show that there are no points of inflection for $x \in (-1, 1)$.

c Prove that $f(x) \ge 0$ for all $x \in [-1, 1]$.

d Determine the values of x for which $f(x) = g(x)$.

e Sketch the graphs of f and g on the one set of axes.

f Determine the area of the region enclosed by the graphs of f and g.

Revision

17B Multiple-choice questions

Technology-free multiple-choice questions

1 The simultaneous equations

$$(m-2)x + 8y = 7$$
$$4x + (m+2)y = m$$

have a unique solution for

A $m \in \mathbb{R} \setminus \{6, -6\}$ **B** $m \in \mathbb{R} \setminus \{0\}$
C $m \in \mathbb{R} \setminus \{2, -2\}$ **D** $m = 6$

2 One solution of $z^3 - 6z^2 + 21z - 26 = 0$ is $z = 2$. Which equation could be used to determine the remaining solutions?

A $z^2 + 4z + 13 = 0$ **B** $z^2 + 4z - 13 = 0$
C $z^2 - 4z + 13 = 0$ **D** $z^2 - 2z - 13 = 0$

3 Point $A(3, 0, 2)$ is the centre of a sphere and the origin $O(0, 0, 0)$ lies on its surface. The equation of the sphere is

A $x^2 - 6x + y^2 + z^2 - 8z = 0$ **B** $x^2 + 6x + y^2 + z^2 - 4z = 0$
C $x^2 - 6x + y^2 + z^2 - 4z = 0$ **D** $x^2 + 6x + y^2 + z^2 + 21z = 0$

4 The equation of the plane through $(2, 1, 3)$ that is parallel to the plane $x - 9y + 13z = 8$ is

A $x + 9y + 13z = 8$ **B** $x - 9y + 13z = 8$
C $x - 9y + 13z = 32$ **D** $x + 9y - 13z = 13$

5 The radius of the sphere with equation $x^2 + y^2 + z^2 = x + y + z$ is

A 2 **B** $\dfrac{\sqrt{3}}{2}$ **C** $\dfrac{1}{2}$ **D** $\dfrac{\sqrt{5}}{2}$

6 A system of linear equations has an augmented matrix in row-echelon form as shown.

$$x - 2y + 3z = 1$$
$$x + ky + 2z = 2$$
$$-2x + k^2y - 4z = 3k - 4$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & k+2 & -1 & 1 \\ 0 & 0 & k & 2k \end{array}\right]$$

This system has a unique solution for

A $k \in \mathbb{R}$ **B** $k \in \mathbb{R} \setminus \{-2, 0\}$
C $k = -2$ **D** $k = -2$ or $k = 0$

7 A normal vector to the plane with Cartesian equation $8x + 6y - 3z = -12$ is

A $8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ **B** $-8\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$
C $6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}}$ **D** $-12\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

8 The gradient of the curve with equation $x^3 + y^3 + 3xy = 1$ at the point $(2, -1)$ is

A 0 **B** -1 **C** 1 **D** -2

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9 $(1-\sqrt{3}i)^3 \div (1+i)$ equals

A $-4+4i$ **B** -8 **C** $4-4i$ **D** $4\sqrt{2}\,\text{cis}\left(\frac{5\pi}{4}\right)$

10 The solution of the equation $\frac{z-2i}{z-(3-2i)} = 2$, where $z \in \mathbb{C}$, is $z =$

A $6+2i$ **B** $6-2i$ **C** $-6-6i$ **D** $6-6i$

11 The polynomial $z^3 - 2z + 4$ can be factorised as

A $(z+4)(z-1)(z+1)$ **B** $(z+2)(z-1+i)(z-1-i)$

C $(z-4)(z-i)(z+1)$ **D** $(z-1)(z+1+i)(z+1-i)$

12 The solution of the equation $\frac{\tan(2x)}{1+\sec(2x)} = -\sqrt{3}$, for $0 \le x \le \pi$, is

A $\frac{\pi}{3}$ **B** $\frac{2\pi}{3}$ **C** $\frac{5\pi}{6}$ **D** $\frac{\pi}{4}$

13 The solution of the inequality $\cot\left(\frac{\theta}{2}\right) \ge \sqrt{3}$, for $-\pi \le \theta \le \pi$, is

A $\left(-\pi, \frac{\pi}{3}\right)$ **B** $\left[-\pi, \frac{\pi}{3}\right)$ **C** $\left[0, \frac{\pi}{3}\right]$ **D** $\left(0, \frac{\pi}{3}\right]$

14 The graph of $y = -\sec(ax+b)$ is identical to the graph of $y = \text{cosec}\left(x + \frac{\pi}{3}\right)$. The values of a and b could be

A $a = 1$ and $b = \frac{\pi}{6}$ **B** $a = -1$ and $b = \frac{\pi}{6}$

C $a = 1$ and $b = \frac{2\pi}{3}$ **D** $a = -1$ and $b = \frac{7\pi}{6}$

15 $\frac{d}{dx}(x\ln y) - \frac{x}{y}\frac{dy}{dx} =$

A 0 **B** $\ln y$

C $x\ln y$ **D** $\ln y - \frac{x}{y}\frac{dy}{dx}$

16 The graph with parametric equations $x = 2 + 3\sec(t)$ and $y = 1 + 2\tan(t)$, where $t \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, has

A two asymptotes, $y = \frac{2x}{3} - \frac{1}{3}$ and $y = -\frac{2x}{3} + \frac{7}{3}$

B two asymptotes, $y = \frac{2}{3}(x-1)$ and $y = -\frac{2}{3}(x-1)$

C two asymptotes, $y - 1 = \frac{3}{2}(x-2)$ and $y - 1 = -\frac{3}{2}(x-2)$

D one asymptote, $y = \frac{2x}{3} - \frac{1}{3}$

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Revision

17 $\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \cos^2(2x)\,dx$ is not equal to

A $\left[\frac{1}{6}\cos^3(2x)\right]_{\frac{\pi}{6}}^{\frac{2\pi}{3}}$

B $\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \cos^2(x)\,dx$

C $\frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} \sin^2(2x)\,dx$

D $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} 1 + \cos(4x)\,dx$

18 A hyperbola has asymptotes given by the equations $y = \pm\frac{3}{2}(x+1)+3$ and passes through the origin. The equation of the hyperbola is

A $\frac{4(y-3)^2}{27} - \frac{(x+1)^2}{3} = 1$

B $\frac{(y-3)^2}{9} - \frac{(x+1)^2}{4} = 1$

C $\frac{(x+1)^2}{4} - \frac{(y-3)^2}{9} = 1$

D $\frac{3(x+1)^2}{16} - \frac{(y-3)^2}{12} = 1$

19 An ellipse has a horizontal semi-axis length of 4 units and a vertical semi-axis length of 2 units. The centre of the ellipse is $(-2, 1)$. The pair of parametric equations which cannot represent this ellipse is

A $x = -2 + 4\cos(t)$ and $y = 1 - 2\sin(t)$

B $x = -2 + 4\cos(t)$ and $y = 1 + 2\sin(t)$

C $x = -2 + 2\cos(2t)$ and $y = 1 + \sin(2t)$

D $x = -2 - 4\sin(2t)$ and $y = 1 - 2\cos(2t)$

20 Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1+i) = 2 - 2i$, then the Cartesian form of one value of z could be

A $\sqrt{2}i$

B $-\sqrt{2}i$

C $-1 - i$

D $-1 + i$

21 A block of ice is pulled from rest along a smooth horizontal surface by a constant horizontal force of F newtons. The ice block initially has a mass of m kg, but gradually melts as it is pulled, losing mass at the rate of c kg per second. Let x m and v m/s represent the position and velocity of the ice block after t seconds. An appropriate differential equation for the motion of the ice block after t seconds could be

A $\frac{dv}{dt} = \frac{F}{m - ct}$

B $\frac{dv}{dt} = \frac{F}{m}$

C $\frac{dv}{dx} = \frac{F}{m - c}$

D $\frac{dv}{dt} = \frac{F}{v(m - ct)}$

22 Let $\boldsymbol{a} = p\hat{\boldsymbol{i}} + q\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$. If the scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\frac{2}{3}$ and the scalar resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is 2, then the values of p and q are

A $p = 0$ and $q = 0$

B $p = 2 - \sqrt{7}$ and $q = \sqrt{7}$

C $p = \frac{8+\sqrt{10}}{5}$ and $q = \frac{4\sqrt{10}}{5}$

D $p = 1$ and $q = 0.5$

23 The gradient of the tangent to the graph of $y = e^{xy}$ at the point where $x = 0$ is

A 0 **B** 1 **C** 2 **D** $\ln 2$

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24 The position of a particle at time t seconds is defined by $\boldsymbol{r} = \dfrac{a}{t+1}\hat{\boldsymbol{i}} + (1+t^2)\hat{\boldsymbol{j}}$, $t \geq 0$, where $a > 0$. The Cartesian equation which represents the path of the particle is

A $y = \dfrac{a^2}{x^2}$, for $x \in [0, \infty)$ **B** $y = \dfrac{a^2 - 2ax + 2x^2}{x^2}$, for $x \in [a, \infty)$

C $y = \left(\dfrac{a}{x} - 1\right)^2 + 1$, for $x \in (0, a]$ **D** $y = \dfrac{x^2 - 2ax + 2a^2}{a^2}$, for $x \in \mathbb{R} \setminus \{-1\}$

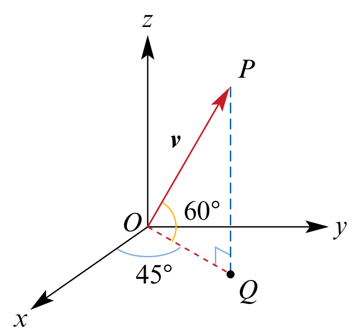
25 Using an appropriate substitution, the integral $\int_1^2 x(2-x)(x^3 - 3x^2 + 4)\,dx$ can be expressed as

A $-\dfrac{1}{3}\int_2^0 u\,du$ **B** $3\int_1^2 u\,du$ **C** $\dfrac{1}{3}\int_2^1 u\,du$ **D** $\dfrac{1}{6}\int_2^0 u^2\,du$

26 A particle is projected at an angle of $\arctan\left(\dfrac{3}{4}\right)$ to the horizontal with a speed of 40 m/s. After 2 seconds, the particle is moving in a direction at an angle of θ to the horizontal. If the acceleration due to gravity is taken as $g = 10$ m/s^2, then $\tan\theta$ is equal to

A $\dfrac{1}{8}$ **B** $\dfrac{1}{2}$ **C** 2 **D** 4

27 The diagram shows a vector $\boldsymbol{v} = \overrightarrow{OP}$ of magnitude 12, where Q is the projection of P in the x–y plane and $\angle POQ = 60°$. Writing $\boldsymbol{v}$ as a column vector gives



A $\boldsymbol{v} = \begin{bmatrix} 12 \\ 45° \\ 60° \end{bmatrix}$ **B** $\boldsymbol{v} = \begin{bmatrix} 6 \\ 6 \\ 6\sqrt{3} \end{bmatrix}$

C $\boldsymbol{v} = \begin{bmatrix} 3\sqrt{2} \\ 3\sqrt{2} \\ 6\sqrt{3} \end{bmatrix}$ **D** $\boldsymbol{v} = \begin{bmatrix} 6\sqrt{2} \\ 6\sqrt{2} \\ 6\sqrt{3} \end{bmatrix}$

28 A particle is oscillating between the two positions $x = 2$ m and $x = 8$ m with simple harmonic motion. If the period of the motion is $\dfrac{\pi}{2}$ seconds, then the maximum speed of the particle is

A 12 m/s **B** 3 m/s **C** 4 m/s **D** 6 m/s

29 Which one of the following points is equidistant from the two planes given by the equations $-2x + y - 2z + 3 = 0$ and $-2x + y - 2z + 21 = 0$?

A $(4, 3, 4)$ **B** $(2, 4, 6)$ **C** $(3, -5, 5)$ **D** $(1, 3, 5)$

30 A plane is represented by the equation $4x - 2y + 6z = 8$. A vector normal to this plane is

A $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix}$ **D** $\begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}$

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31 Suppose that X is a random variable with mean $\mu = 20$ and standard deviation $\sigma = 5$. If $\bar{X}$ is the mean of a sample of size 25, then

A $E(\bar{X}) = 0.8,\ sd(\bar{X}) = 0.2$ **B** $E(\bar{X}) = 20,\ sd(\bar{X}) = 0.2$

C $E(\bar{X}) = 20,\ sd(\bar{X}) = 1$ **D** $E(\bar{X}) = 20,\ sd(\bar{X}) = \dfrac{5}{\sqrt{5}}$

32 If 20 random samples are chosen from a population and a 90% confidence interval for the population mean μ is computed from each sample, then the expected number of confidence intervals which will contain μ is

A 2 **B** 9 **C** 18 **D** 19

33 If the sample size and standard deviation remain unchanged, then a decrease in the level of confidence will lead to a margin of error which is

A smaller **B** larger **C** unchanged **D** asymmetric

Technology-active multiple-choice questions

34 Given that $1 + i$ is a root of $z^3 - pz^2 - q$ where $p, q \in \mathbb{R}$, determine the values of p and q.

A $p = 2, q = -1$ **B** $p = -1, q = 2$ **C** $p = 1, q = -2$ **D** $p = 1, q = 1$

35 A plane contains the origin and the points $(4, -5, 9)$ and $(3, -2, 6)$. A vector normal to the plane is

A $\begin{bmatrix} -12 \\ 3 \\ 7 \end{bmatrix}$ **B** $\begin{bmatrix} -12 \\ 3 \\ -7 \end{bmatrix}$ **C** $\begin{bmatrix} 12 \\ -3 \\ 7 \end{bmatrix}$ **D** $\begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$

36 The position of a particle at time t can be modelled by the vector equation

$$\boldsymbol{r} = 9\cos(2t)\hat{\boldsymbol{i}} + 4\sin(2t)\hat{\boldsymbol{j}}$$

The Cartesian equation of the path is

A $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ **B** $\dfrac{x^2}{16} - \dfrac{y^2}{81} = 1$ **C** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **D** $\dfrac{x^2}{81} + \dfrac{y^2}{16} = 1$

37 Given that $f(x) = \sin^{-1}(3x)$, determine $f'(0.2)$.

A 2.86 **B** 3.75 **C** 3.20 **D** 4.28

38 If $\int_0^k xe^{-x}\,dx = 0.5$ and $k > 0$, then k is closest to

A 0.7 **B** 1.7 **C** 2.7 **D** 3.7

39 If $\dfrac{dy}{dx} = x \ln x$ with $y(2) = 2$, then $y(3)$ is closest to

A 4.31 **B** 2.3 **C** -1.7 **D** 0

40 In the interval $(-\pi, \pi)$, the number of points of intersection of the graphs of $f(x) = \sec x$ and $g(x) = \operatorname{cosec}(2x)$ is

A 0 **B** 1 **C** 2 **D** 3

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41 The velocity, v m/s, of a particle at time t seconds is given by $v = \dfrac{4t}{1+t^2}$, $t \geq 0$.

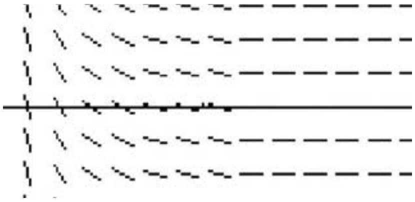
The distance, in metres, travelled by the particle in the first 10 seconds is closest to

A 9.23 **B** 533.33 **C** 1 **D** 2

42 A small rocket is fired vertically upwards. The initial speed of the rocket is 200 m/s. The acceleration of the rocket, a m/s^2, is given by $a = -\dfrac{20+v^2}{50}$, where v m/s is the velocity of the rocket at time t seconds. The time that the rocket takes to reach the highest point, in seconds, is closest to

A 5 **B** 8 **C** 12 **D** 17

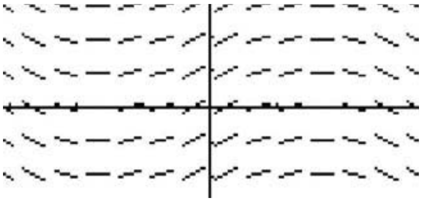
43 This is the slope field for a differential equation, produced by a calculator, with $0 \leq x \leq 2$ and $-3 \leq y \leq 3$.



A solution for the differential equation could be

A $y = -\dfrac{1}{x^2}$ **B** $y = -\dfrac{1}{x^3}$ **C** $y = \dfrac{1}{x}$ **D** $y = e^x$

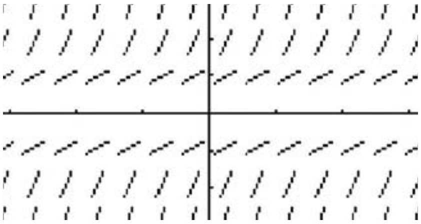
44 This is the slope field for a differential equation, produced by a calculator, with $-\pi \leq x \leq \pi$ and $-3 \leq y \leq 3$.



The differential equation could be

A $\dfrac{dy}{dx} = \sin x$ **B** $\dfrac{dy}{dx} = -\cos x$ **C** $\dfrac{dy}{dx} = \tan x$ **D** $\dfrac{dy}{dx} = \cos x$

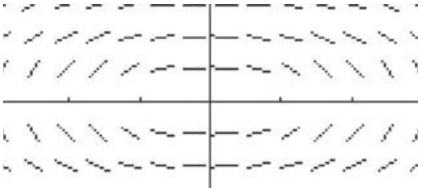
45 This is the slope field for a differential equation, produced by a calculator, with $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.



A solution for the differential equation could be

A $y = \dfrac{1}{x}$ **B** $x = y^3$ **C** $y = \dfrac{1}{x^2}$ **D** $x = -\dfrac{1}{y}$

46 This is the slope field for a differential equation, produced by a calculator, with $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.



The differential equation could be

A $\dfrac{dy}{dx} = x^2$ **B** $\dfrac{dy}{dx} = -\dfrac{x}{y}$ **C** $\dfrac{dy}{dx} = -\dfrac{y}{x}$ **D** $\dfrac{dy}{dx} = \dfrac{y}{x}$

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Revision

47 The weight of a certain type of large dog is normally distributed with mean 42 kg and standard deviation 4.5 kg. The probability that the average weight of 20 randomly selected dogs is between 38 kg and 43 kg is closest to

A 0.8398 **B** 0.4009
C 0.7564 **D** 0.6862

48 Based on a random sample of size 45 selected from a population, a sample mean of 9.1 and a known $\sigma = 3$, the 99% confidence interval for the population mean is

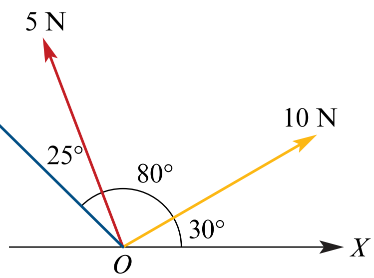
A $(8.224, 9.977)$ **B** $(7.896, 10.304)$
C $(8.199, 10.001)$ **D** $(7.948, 10.252)$

49 The area, correct to two decimal places, of the parallelogram with vertices $A(2, 1, 1)$, $B(4, 1, 3)$, $C(3, -2, 1)$ and $D(1, -2, -1)$ is

A 6.48 **B** 8.72
C 9.54 **D** 3.87

50 Three forces act concurrently on a 2 kg object placed at the origin. The magnitude of the acceleration is closest to

A $6.21\ \text{ms}^{-2}$ **B** $12.42\ \text{ms}^{-2}$
C $4.84\ \text{ms}^{-2}$ **D** $5.03\ \text{ms}^{-2}$



51 Determine the solution of the differential equation $\dfrac{dy}{dx} = \dfrac{\cos(3x) + 2}{\sin(3x) + 6x}$ given that $y = 0$ when $x = \pi$. Give the value of the constant correct to two decimal places.

A $y = \dfrac{1}{3}\ln|\sin(3x) + 6x| - 0.98$ **B** $y = \ln|\cos(3x) + 6| - 1.61$
C $y = 3\ln|\sin(3x) + 6x| - 1.36$ **D** $y = \dfrac{1}{6}\ln|\sin(3x) + 6| - 1.72$

52 In order to estimate the mean price of strawberries per punnet within ±30 cents in a 99% confidence interval the size of the sample should be (assume $\sigma = \$1.00$)

A 7 **B** 9 **C** 43 **D** 74

53 The length of time taken to complete a puzzle is normally distributed with a mean of 25 minutes and a standard deviation of σ minutes. There is a 5% probability that the average of the time taken by a sample of 30 people will take more than 28 minutes. The value of σ is closest to

A 8.4 **B** 10.0 **C** 1.8 **D** 18.4

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Appendix A

The problem-solving and modelling task

Joel Speranza

Chapter contents

A1 About the problem-solving and modelling task

Mathematisation is the process of taking a real-world problem, translating it to a mathematically purposeful representation, and solving that problem. In Specialist Mathematics you will be assessed on your ability to do this through an assessment item called a problem-solving and modelling task (PSMT).

This chapter outlines how to plan, solve and present your PSMT at a high level and provides real examples of high-level student work. Each section of this chapter is accompanied by a video lesson with additional advice, accessible via the included QR codes.

How is the PSMT marked?

Before you begin any assessment, you should consider how teachers will make judgements of your work. Your teacher will use the Instrument-specific Marking Guide (ISMG) from the syllabus to determine your mark. The ISMG is broken into four criteria (Formulate, Solve, Evaluate and Communicate) and each criterion is assessed using three to five descriptors. Throughout this chapter, we will be focusing on the top descriptor in each criterion. These top descriptors will be displayed in this appendix where needed. The full ISMG with all descriptors can accessed through the Interactive Textbook.

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A2 A content guide for a PSMT report

A suggested set of headings for writing the PSMT report has been provided in a downloadable Word document in the Interactive Textbook. The rest of this appendix provides notes on what to include under these headings, which are reproduced here in black ('Introduction to the task', 'Formulating a solution', 'Developing a solution', and so on). Extracts from the ISMG display the criteria and descriptors covered by each heading of the report.

Video A1

You also need to adhere to the required word limit and conventions of the mathematical report genre. You may have seen another report genre, the scientific report, and the two have similarities.

Word length guides of each section of the report are indicative only and will be dependent on your specific PSMT. Setting out your PSMT in this way is not compulsory but does help you to achieve this highlighted descriptor from the ISMG:

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

1. Introduction to the task (200 words)

These are the criteria and descriptors to be covered by the introduction:

Video A2

Formulate	Marks
The student response has the following characteristics:	
• justified statements of important assumptions • justified statements of important observations • justified mathematical translation of important aspects of the task	3–4

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

Note that, as long as you understand the task, and have written down what you need to do, you do not have to finalise the introduction at the beginning of the process. You may find it better to write the introduction alongside writing the conclusion, ensuring that the conclusion addresses the goal of the task which was stated in the introduction.

The purpose of the introduction is twofold:

1 To introduce a report which 'can be read independently' of the task sheet. This means that a person who has never seen the task sheet will be able to understand what the task is,

simply by reading your introduction. No reference to the task sheet should be made here. You should also begin with providing some context on what the problem is to be solved, and why solving the problem is important.

2 To show 'justified mathematical translation of important aspects of the task'. Mathematical translation is the process of taking a real-world problem and moving it into the mathematical world – a process known as **mathematisation**. In this part of an introduction, you should give a brief description of the mathematical techniques you will use to solve the problem. This description can include:
- Applicable mathematical or statistical principles
- Mathematical concepts and techniques
- Technology that you will use throughout the task

The student sample below shows:

- how the task can be read independently of the task sheet
- justified mathematical translation of important aspects of the task.

Introduction

As the population of Australia continues to grow, it is crucial that we plan ahead. Important aspects in life such as resources, housing, and medical products and services need to account for this growing population before the demand for necessary requirements of life have such a demand that cannot be catered for (Sommerfeld, 2018). By knowing predictions for a future population count, we can plan ahead and make choices that will benefit the future now.

The purpose of this task is to determine a prediction for the rate of population growth in Australia for 2061 by exploring multiple mathematical models. A multitude of mathematical techniques will be utilised throughout this task, including; modelling of exponential, logarithmic, logistic, and sinusoidal functions, as well as deriving to find a rate of change, determining percentage error, and using technology such as GeoGebra, a TI-84 Plus CE Graphics Calculator and Excel to model equations and manage data.

Student sample taken from the 2020 Mathematical Methods subject report

Using mathematical language?

In the introduction and throughout the PSMT, you must demonstrate correct use of mathematical language.

Video A3

These are the criteria and descriptors to be covered:

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

The syllabus objectives elaborate on mathematical language as ‘terminology, symbols, conventions and representations’. What follows is a non-exhaustive list of ways to demonstrate each aspect of mathematical language.

Terminology

- Procedural mathematical language (mathematical verbs) e.g. determine, solve, verify, calculate integrate, derive.
- Technical mathematical language (mathematical nouns) e.g. search the unit outline for terminology specific to the content being assessed.

Conventions

- Equations have a left and right hand side
- Equal signs are aligned
- Define variables before using them

Symbols

- Use mathematical symbols correctly
- If typing mathematics, use equation editor

Representations: Graphs

- An appropriate title
- Axes labelled with appropriate units
- An appropriate scale for each axis
- A legend if appropriate

Representations: Diagrams

- An appropriate title
- Labelled
- Drawn to scale or a label indicating otherwise
- Vertices of shapes labelled with capital letters
- Angles labelled with Greek letters

2. Formulating a solution (400 words)

In the section titled ‘Formulating a solution’, you are aiming to demonstrate the descriptors of the ISMG shown below.

Formulate	Marks
The student response has the following characteristics:	
• justified statements of important assumptions • justified statements of important observations • justified mathematical translation of important aspects of the task	3–4

It is vital that your assumptions are both important and expressed as justified statements. This is the most important section of the PSMT, as many sections of the PSMT cannot be completed to a high-standard without them. This is demonstrated in the following flowchart.

Observations are considered and assumptions made in order to 'mathematise' the problem.

↓

If the solution does not rely on these, then they are not 'important' and therefore do not meet the criteria. If they are not backed by justified statements, then the evidence for your solution being valid is low.

↓

Once a solution is found, 'justified statements about the reasonableness of the solution by considering observations and assumptions' must be made. If these observations and assumptions are poorly justified or are not important, then it will be difficult to do this.

↓

At the end of the PMST, 'justified statements of relevant strengths and limitations of the solution' are made. These are often found by examining the assumptions and observations that have been made. If assumptions and observations are not important, this section becomes more difficult to complete.

Observations vs assumptions

Students can often be confused about the difference between an observation and an assumption. Both are vital to creating a mathematical model but serve different purposes. The table below outlines the key differences between them.

Aspect	Observations	Assumptions
Definition	Data or information required to solve a mathematical problem and/or develop a mathematical model.	Conditions that are stated to be true when beginning to solve a mathematical problem and/or develop a mathematical model.
Nature and verifiability	Factual, based on actual data or information, and can be empirically verified.	While not directly verifiable, are rational and necessary for solving the problem. They are based on plausible reasoning or existing knowledge.
Role in modelling	Provides data and empirical evidence for creation of the model.	Simplifies complexity and fills data gaps. Dictates the strengths, limitations, and applicability of the model.
Flexibility	Generally rigid; they are facts that can't be altered. However, the interpretation of observations can evolve with new data.	More flexible and can be adjusted or replaced as new information becomes available or as the model evolves.

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Aspect	Observations	Assumptions
Examples	**1** Measuring the temperature and humidity levels in various regions over a year. **2** Recording the frequency and intensity of rainfall in a specific area. **3** The observable fact that warm air rises and cool air sinks, affecting weather patterns.	**1** Assuming that future weather patterns will reflect past trends due to climate consistency. **2** Assuming a certain level of accuracy in satellite data used for cloud cover analysis. **3** Assuming that ocean currents will remain relatively stable over the short-term forecasting period.
Role in validation	Used for validating the model by comparing its predictions with actual observations.	Validated indirectly through the model's performance and its ability to make accurate predictions within the defined framework of these assumptions.

2.1 Observations

To ensure that your observations reach the level of 'justified statements' they should include:

- a discussion of how the observations affect the mathematical model/solution
- in-text referencing to a reputable source (while it is possible to justify statements without this, consider the inclusion of a reference the 'gold-standard').

Video A4

The following student sample demonstrates justified statements of observations. Each observation contains:

- a statement of what the observation is
- a reference that provides justification for the observation being made
- justification for how this observation impacts the mathematical model/solution.

Structuring your observations in this way provide a high likelihood of them being considered justified statements on the ISMG.

Observations

It was observed that the male-to-female ratio in Western Australia is 102 males for every 100 females (McCrindle, 2014). This observation directly impacts the mathematical model as the initial total population must divide by this ratio only to consider the female Western Australian population. This is relevant as males do not reproduce any offspring and cannot be factored into the Leslie matrix.

It was observed that Western Australia takes 30% of immigrants into Australia each year (Australian Bureau of Statistic, 2021). Further, 12,706 to 18,200 immigrants settled in Australia during 2018 and 2020. (Lawrence, 2018). These observations impacted the mathematical model as an increase in immigrants will impact the projected populations, causing an increase or decrease in total population growth rate.

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Humans live to approximately 100 years (Vaupel, 2010); therefore, a 21 × 21 matrix is reasonable. From this, the age classes for the Leslie matrix were split into 5-year categories. Hence, each new generation produced by the Leslie matrix represents a 5-year gap. This observation is relevant as it impacts the scope mathematical model by investigating only eight new generations.

Student sample taken from the 2022 Specialist Mathematics subject report

2.2 Assumptions

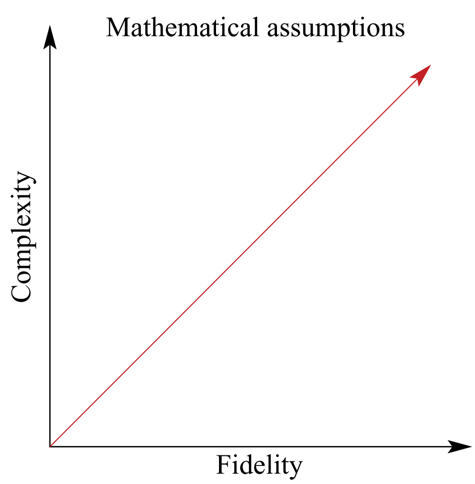
Video A5

As with observations, to justify the assumptions you make you should include discussion of how the assumptions affect the mathematical model/solution; and in-text referencing to a reputable source. While it is possible to justify statements without this, it is considered best practice to include one.

Referencing should be used to justify an assumption and it can be done in two different ways:

- use a source that brings your mathematical model closer to representing the real world
- use a source that takes your mathematical model further from representing the real world but is required to reduce the complexity of the model.

To illustrate this further, each assumption can be thought of as existing on the graph shown below, with mathematical complexity increasing as fidelity (how closely the assumption models the real world) increases. Therefore, each assumption can be thought of as a trade-off between complexity and fidelity. Adopting this approach from the beginning makes future sections of the PSMT (evaluating the reasonableness of the solution and strengths and limitations) easier to complete.



The student sample which follows demonstrates justified statements of assumptions. Each assumption contains:

- a statement of what the assumption is
- a reference that provides justification for the assumption being made
- justification for how the assumption impacts the mathematical model/solution.

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Assumptions

It was assumed that birth rates would only impact women aged 15–44 as in Australia. The average woman's reproductive years are between ages 15 and 44 (Watson, 2018). This assumption restricts birth rates to only six of twenty-one age classes. The assumption was made to reduce the anomalies to develop clean data.

It was assumed that the new immigrant population introduced into Western Australia was equally divided into the age classes from 18 to 34. Most migrants to Australia are young adults, with 61.2% aged between 18 and 34 years (abs.gov.au, 2018). Further, 12 706 to 18 200 immigrants settled in Australia during 2018 and 2020. (Lawrence, 2018). This assumption was made to create a realistic data spread that included the possible impact immigrants' survival and or birth rate would have on the total population.

It was assumed that the investigation started during 2016 as the female population data collected was from 2016 (abs.gov.au, 2016); therefore, a 21×21 matrix is reasonable. This assumption impacts the investigation as the potential increasing or decreasing growth rate and total population can be compared to secondary data to determine the model's validity.

Student sample taken from the 2022 Specialist Mathematics subject report.

3. Developing a Solution (700 words)

Video A6

In this section of the PSMT, you are attempting to demonstrate almost all the descriptors of the ISMG, as shown in the following example. This is also a section where students can get a little confused about how to set things out. Below is a flowchart you can use throughout this section to ensure that you demonstrate the full range of ISMG descriptors.

STEP 1 Justify the 'mathematical decision' you have made or the 'mathematical translation' you have performed.

STEP 2 Do some mathematics using:

- Equations
- Graphs
- Tables
- Technology
- Diagrams

STEP 3 Make a 'justified statement' of one of the following:

- Observation
- Assumption
- The reasonableness of the solution

OR

- Verify your result.

The sample following is a simple example of how this flowchart can be put into practice. It is from a PSMT in which the task is to develop a flying fox. The mathematics has been simplified to allow us to focus on the structure of the response, rather than the mathematics. To guide you, the example has been annotated with:

- in the left margin, the numbered steps from the flowchart above
- in the right margin, the matching ISMG descriptors from the syllabus.

Use them to guide your solution, but do not annotate your own report with them.

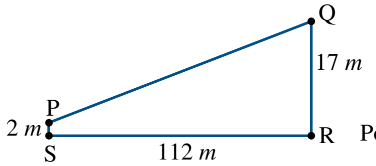
3.1 Calculating the length of the cable

STEP 1

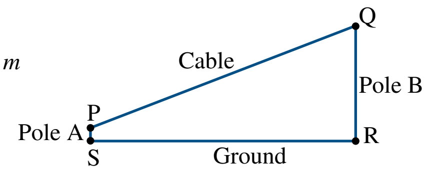
The flying fox cable, two supporting poles and the ground can be modelled as the quadrilateral, *PQRS*, as shown below.

Formulate

- Justified mathematical translation of important aspects of the task



STEP 2



Communicate

- Correct use of appropriate mathematical language

STEP 3

It is assumed that the cable is perfectly taut and has no sag. While the cable in a real flying fox will have a sag (Evans, 2017), this assumption allows the cable's length to be calculated.

Formulate

- Justified statements of important assumptions

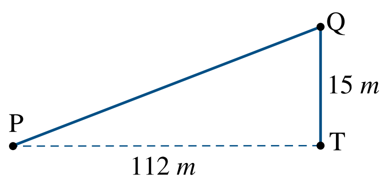
STEP 1

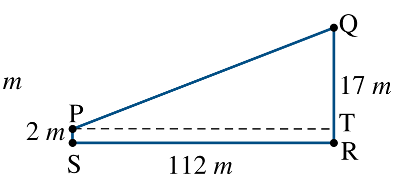
A horizontal line drawn through point *P* creates a right triangle *PQT*. Creating a right triangle will allow the length of the cable to be calculated using Pythagoras' theorem.

Communicate

- Justification of decisions using mathematical reasoning

STEP 2





$$\begin{aligned}\text{Length of cable} &= \sqrt{112^2 + 15^2}\\ &= \sqrt{12544 + 225}\\ &= \sqrt{12769}\\ &= 113\end{aligned}$$

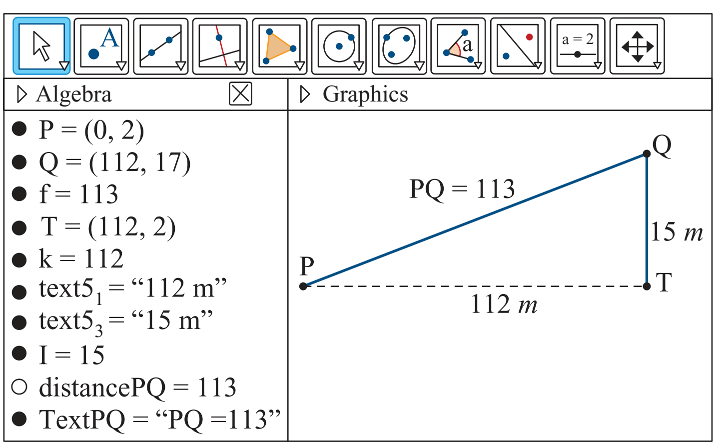
Solve

- Accurate use of mathematical knowledge for important aspects of the task

STEP 3

The length of the cable is calculated to be 113 metres.

This result can be verified using a scale diagram drawn in Geogebra, as shown in this screenshot.



Solve

- Efficient use of technology

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3.2 Calculating the total cost of the cable

STEP 1

The total cost of cable can now be calculated. In construction, it is common to assume that you require an additional 10% of materials to allow for wastage (Jones, 2017). 8 mm aircraft-grade galvanised cable is perfect for ziplines up to 150 metres in length and has a current cost of \$8 per metre. (cable-ride.com, n.d)

Formulate

- Justified statements of important assumptions
- Justified statements of important observations

STEP 2

$$\text{Total cost} = 113 \times 8 \times 1.1$$
$$= \$994.40$$

Solve

- Accurate use of mathematical knowledge for important aspects of the task

Adding an additional 10% to materials is a reasonable solution when considering the assumption made that there is no sag in the cable. In reality there will be sag, and this will increase the amount of the cable required.

Evaluate

- Justified statements about the reasonableness of the solution by considering the assumptions

STEP 3

The \$994.40 total cost for the cable can be verified by comparing it to a 90 metre zipline kit available online for \$1387. (cable-ride.com, n.d). While this is \$392.60 more expensive than our cable, it is for a complete kit rather than just a cable. These two prices are close enough to support our result.

Evaluate

- Verified results

Efficient use of technology

Video A7

Solve

The student response has the following characteristics:

- Efficient use of technology

Throughout your assignment, you should be looking for opportunities to demonstrate the efficient use of technology, as required by the ISMG descriptor shown. The key descriptor here is 'efficient', which can be understood as using technology in a way that minimises wasted effort and/or time. An example of efficiency would be using Excel formulas to perform repeated calculations, rather than performing each calculation by hand or on a calculator.

It is often the case that while students use quite a bit of technology throughout their PSMT, they often don't provide the evidence that they have used it. Here are some types of technology students use and the evidence that they can provide to demonstrate that they have used it.

Types of technology	Evidence provided
Spreadsheet software (e.g. Microsoft Excel)	■ Screenshots of graphs or tables ■ Samples of spreadsheet formulas used
Calculator	■ Photos or screenshots of the calculator screen ■ Descriptions of how the calculator was used
Graphing software (e.g. Desmos or Geogebra)	■ Screenshots ■ Descriptions of how the software was used
Logging software (e.g. data collectors)	■ Sample of the data collected ■ Screenshots

The following examples which evidence efficient use of technology are taken from the 2022 Mathematics subject reports.

To verify this a larger Excel spreadsheet was created to solve the parametric equations for a variety of t values (also extending to ten years):

t	Xleft	Xright	Difference	Intersect?	Yleft	Yright	Difference	Intersect?	Zleft	Zright	Difference	Intersect?
0.5	0.999963	1.999997	1.0000337	FALSE	0.0086	0.001805	0.0067944	FALSE	0	-2	1.999997	FALSE
1	0.999852	1.999987	1.0001349	FALSE	0.017199	0.003611	0.0135882	FALSE	0	-1.99999	1.999987	FALSE

Figure 12: Spreadsheet

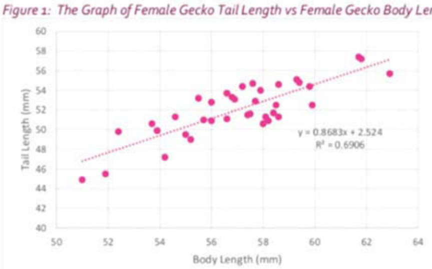
The Excel equation inserted in the 'Intersect?' column (Figure 13), calculated if the difference between the two sides of the parametric equation was less than 4.2587×10^{-5}, the conversion of Earth's radius to AU, to ensure constant units.

```
=IF(D2<(0.0000425875),"YES")
```

Student sample taken from the 2022 Specialist Mathematics subject report

The data without outliers was utilised for further analysis and extrapolation.

Figure 1: The Graph of Female Gecko Tail Length vs Female Gecko Body Length without Outlier



Student sample taken from the 2022 General Mathematics subject report

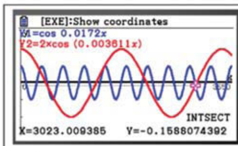


Figure 9:

$x: \cos(0.0172t) = 2 \times \cos(0.003611t)$

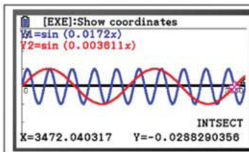


Figure 10:

$y: \sin(0.0172t) = \sin(0.003611t)$

Student sample taken from the 2022 Specialist Mathematics subject report

4. Evaluating and verifying the solution (600 words)

Video A8

4.1 Verifying results

Evaluate
The student response has the following characteristics:
• Verified results

In this section you are aiming to demonstrate the descriptor shown here, verifying the overall solution. If you have been following the advice from the previous section, you will have been verifying some of your results in the process of coming to your solution. Below, we look at four techniques students can use for verifying results.

1. **Verifying through estimation:** This method simplifies the problem before calculating an approximate solution. For instance, if you have calculated the area of a composite shape, you can verify your solution by simplifying it into a basic rectangle and recalculating the area. If the estimated area is close to your calculated area, it helps confirm the accuracy of your solution.
2. **Verifying through research:** This method involves comparing results with reliable sources. This method is particularly useful for tasks related to historical data or correlation studies. Findings can be matched against data from textbooks, academic journals, or credible online sources. For example, in a task exploring the correlation between car weight and fuel consumption, you can verify the solution against existing research on this topic.

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3 **Verifying through technology:** Using technology to redo algebraic calculations can aid in verifying results. For example, when calculating the area under a curve, employing graphing calculators or software for an approximation and comparing it to manual calculations can serve as a verification method.

4 **Verifying through an alternative method:** This method is effective in problems with multiple solutions and involves employing various techniques to solve the same issue. For example, you may initially solve an algebraic equation by factoring, then recheck the solution using the quadratic formula. If both methods produce identical results, it strongly suggests that the solution is correct.

4.2 Evaluating reasonableness of the solution

In this section you are aiming to demonstrate the content descriptor below:

Video A9

Evaluate	Marks
The student response has the following characteristics:	
• verified results • justified statements about the reasonableness of the solution by considering the assumptions • justified statements about the reasonableness of the solution by considering the observations • justified statements of relevant strengths of the solution • justified statements of relevant limitations of the solution	4–5

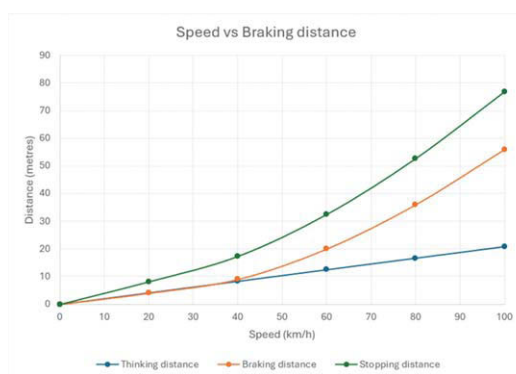
When evaluating the reasonableness of your solution by considering assumptions and observations you should:

- consider the solution found
- consider how it is affected by the observation or assumption
- consider how the solution might be different if the observation or assumption was altered (often with some mathematical working included).

Below is an example of evaluating the reasonableness of a solution by considering an assumption from a PSMT investigating the braking distance of a car dependent on the speed at which it is travelling.

Example

The distance calculated for a car to come to a complete stop is underestimated because of the assumption that driver's response times were instantaneous. If driver response time were factored in, the braking distance would be greater than the distances calculated in this report.



Assuming average driver response time of 1.5 seconds (Muttart, 2004), we can see how the solution would change if this time was taken into account in the graph pictured. At the top speed of 100 km/h, braking distance is increased from the initial solution of 56 m to 77 m, an increase of 37.5%.

Note: Students often make the mistake of evaluating the reasonableness of their assumption, rather than evaluating the reasonableness of their solution by considering their assumption. The distinction is subtle but important. The justified statements made must refer to the solution and how it is affected by the assumption, not just the assumption itself.
Correct: The solution is reasonable because...

Incorrect: The assumption is reasonable because...

A subtle but important difference.

4.3 Strengths and limitations of the solution

In this section you are aiming to demonstrate the content descriptor below:

Video A10

Evaluate	Marks
The student response has the following characteristics:	
• verified results • justified statements about the reasonableness of the solution by considering the assumptions • justified statements about the reasonableness of the solution by considering the observations • justified statements of relevant strengths of the solution • justified statements of relevant limitations of the solution	4–5

Strengths and limitations of the solution can be thought of in the following way.

- Strengths – aspects of the model that make it useful
- Weaknesses – aspects of the model that limit its usefulness

A series of questions to help identify these strengths and limitations is below. Do not aim to answer all these questions but use them as prompts to generate ideas.

Strengths	Limitations
What assumptions were made that closely align to the real world?	What assumptions were made that do not align closely with the real world?
What aspects of the real world does the solution consider?	What aspects of the real world does the solution not consider?
Could the method used to create this solution be easily adapted and used to solve other, related problems?	Are there other, related problems that the method used could not be easily adapted to solve?
What aspects of the solution can be verified using other observational data?	What aspects of the solution cannot be verified using observational data?
What are the potential, positive consequences of using this solution in the real world?	What are the potential, negative consequences of using this solution in the real world?
What aspects of the solution will continue to be accurate into the future?	What aspects of the solution will cease to be accurate into the future?

To make justified statements of strengths/limitations:

- state the strength/limitation
- justify why it is a strength/limitation.

The student sample of limitations below provides examples of this.

The following limitations were observed

There was a limited amount of data points that were used as a sample.This means the findings were less reliable as it may not be an accurate representation of all rugby games.

Another limitation is that when using extrapolation with regards to the regression line, it may not be accurate to predict further outcomes because the prediction is outside the sample data range.

One final limitation is that the R2 value found is not considerably strong, therefore a smaller percentage of the points scored per game can be attributed to the line breaks achieved per game, decreasing the reliability of the study.

Student sample taken from the 2022 General Mathematics subject report

5. Conclusion (100 words)

The conclusion is another opportunity to show logical organisation of your response. In the conclusion you should:

- restate the purpose of the mathematical report
- provide a summary of your solution, stating an appropriate answer to the task.

Conclusion

The purpose of this report was to use functions and derivatives to create a reasonable prediction for the rate of change for the Australian population in 2061. It was found that the most reasonable model was solution three as it produces a reasonable population and somewhat reasonable rates of change. Therefore, using this logistic function, it is predicted that the population will reach approximately 39.6 million in 2061, with a percentage rate of change of 0.95% and an instantaneous rate of change of approximately 370 000 addition people per year.

Student sample taken from the 2022 Mathematical Methods subject report

6. Reference List (Not included in page or word count)

Use a standard referencing style. Ask your teacher for guidance on this if you need it.

7. Appendix (Not included in page or word count)

An appendix is for supporting material such as data, diagrams, calculations and screenshots or print-outs from technology, that don't form a direct part of the solution or evaluation. The appendix is not marked so don't include important items that you want a mark for. If you haven't already included your use of technology in the report, you could put a small sample into the body of your assignment to get marks for efficient use of technology.

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Glossary

A

Absolute value [p. 455] The absolute value (or *modulus*) of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Acceleration [pp. 309, 615] the rate of change of velocity with respect to time

Acceleration, average

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{change in time}}$$

Acceleration, instantaneous [pp. 615, 630]

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Addition of complex numbers [p. 82]
If $z_1 = a + bi$ and $z_2 = c + di$, then
$z_1 + z_2 = (a + c) + (b + d)i$.

Addition of vectors [p. 157]
If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$.

Amplitude of trigonometric functions [p. 5]
The distance between the mean position and the maximum position is called the amplitude.
The graph of $y = a \sin x$ has an amplitude of $|a|$.

Angle between two lines [p. 250] Let θ be the angle between two vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ that are parallel to the two lines. The angle between the lines is θ or $180^\circ - \theta$, whichever is in $[0^\circ, 90^\circ]$.

Angle between two planes [p. 266] Let θ be the angle between two vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ that are normal to the two planes. The angle between the planes is θ or $180^\circ - \theta$, whichever is in $[0^\circ, 90^\circ]$.

Angle between two vectors [p. 185] can be found using the scalar product:
$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$
where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Angle sum and difference identities [p. 22]

- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Angular velocity, ω [p. 323] the rate of change of angle with respect to time

Anti-derivative [p. 295] To find the general anti-derivative of $f(x)$: If $F'(x) = f(x)$, then
$\int f(x)\,dx = F(x) + c$
where c is an arbitrary real number.

Anti-derivative of a vector function [p. 307]
If $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$, then
$\int \boldsymbol{r}(t)\,dt = X(t)\boldsymbol{i} + Y(t)\boldsymbol{j} + Z(t)\boldsymbol{k} + \boldsymbol{c}$

where $\dfrac{dX}{dt} = x(t)$, $\dfrac{dY}{dt} = y(t)$, $\dfrac{dZ}{dt} = z(t)$
and $\boldsymbol{c}$ is a constant vector.

Arccosine function (inverse cosine) [p. 27]
$\cos^{-1} x = y$ if $\cos y = x$,
for $x \in [-1, 1]$ and $y \in [0, \pi]$

Arcsine function (inverse sine) [p. 26]
$\sin^{-1} x = y$ if $\sin y = x$,
for $x \in [-1, 1]$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

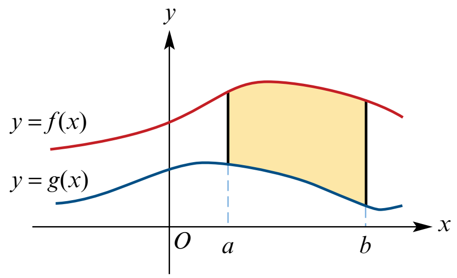
Arctangent function (inverse tangent) [p. 27]
$\tan^{-1} x = y$ if $\tan y = x$,
for $x \in \mathbb{R}$ and $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Area of a parallelogram [p. 252] The area of the parallelogram spanned by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $|\boldsymbol{a} \times \boldsymbol{b}|$.

Area of a region between two curves [p. 502]

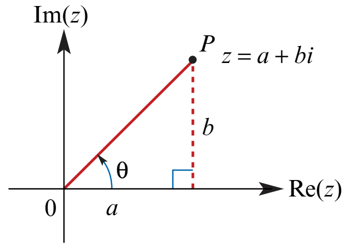
$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

where $f(x) \geq g(x)$ for all $x \in [a, b]$



Area of image [SM1&2] If a linear transformation (with matrix $\mathbf{B}$) is applied to a region of the plane, then
Area of image $= |\det(\mathbf{B})| \times$ Area of region.

Argand diagram [p. 84] a geometric representation of the set of complex numbers



Argument of a complex number [pp. 93, 94]

- An argument of a non-zero complex number z is an angle θ from the positive direction of the x-axis to the line joining the origin to z.
- The *principal value* of the argument, denoted by $\operatorname{Arg} z$, is the angle in the interval $(-\pi, \pi]$.

Argument, properties [pp. 99, 100]

- $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1
- $\operatorname{Arg}\left(\dfrac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1
- $\operatorname{Arg}\left(\dfrac{1}{z}\right) = -\operatorname{Arg}(z)$, provided z is not a negative real number

Augmented matrix [p. 370] represents a system of linear equations. For example:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \qquad \begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Augmented matrix, row leader [p. 371] the first non-zero entry of a row

Augmented matrix, row operations [p. 370] operations that produce an equivalent system of equations (i.e. the solution set is the same):

- Interchange two rows.
- Multiply a row by a non-zero number.
- Add a multiple of one row to another row.

Augmented matrix, row-echelon form [p. 371] Each successive row leader is further to the right, and so each row leader has only 0s below.

C

$\mathbb{C}$ [p. 80] the set of complex numbers: $\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$

Cartesian equation An equation in variables x and y describes a curve in the plane by giving the relationship between the x- and y-coordinates of the points on the curve; e.g. $y = x^2 + 1$.
An equation in x, y and z describes a surface in three-dimensional space; e.g. $x^2 + y^2 + z^2 = 1$.

Cartesian form of a complex number [p. 84] A complex number is expressed in Cartesian form as $z = a + bi$, where a is the real part of z and b is the imaginary part of z.

Central limit theorem [p. 698] Let X be any random variable, with mean μ and standard deviation σ. Then, if the sample size n is large enough ($n \geq 30$), the distribution of the sample mean $\bar{X}$ is approximately normal with mean $\mathrm{E}(\bar{X}) = \mu$ and standard deviation $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$.

Chain rule [p. 292]

- If $q(x) = f(g(x))$, then $q'(x) = f'(g(x))\, g'(x)$.
- If $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

Change of variable rule *see* integration by substitution

Circle, general Cartesian equation [p. 55] The circle with radius r and centre (h, k) has equation $(x - h)^2 + (y - k)^2 = r^2$.

cis θ [p. 93] $\cos\theta + i\sin\theta$

Collinear points [p. 192] Three or more points are collinear if they all lie on a single line.

Column-vector notation [pp. 156, 160] writing a vector as a column of numbers. For example, the position vector of the point $P(a, b, c)$ is written

$$\overrightarrow{OP} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Complex conjugate, $\bar{z}$ [pp. 89, 97]

- If $z = a + bi$, then $\bar{z} = a - bi$.
- If $z = r \operatorname{cis} \theta$, then $\bar{z} = r \operatorname{cis}(-\theta)$.

Complex conjugate, properties [p. 89]

- $z + \bar{z} = 2\operatorname{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$

Complex number [p. 80] an expression of the form $a + bi$, where a and b are real numbers

Complex plane *see* Argand diagram

Component form of a vector [p. 167]

- In two dimensions, each vector $\boldsymbol{v}$ can be written in the form $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j}$.
- In three dimensions, each vector $\boldsymbol{v}$ can be written in the form $\boldsymbol{v} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

Concurrent lines [p. 249] Three or more lines are concurrent if they all pass through a single point.

Conditional statement [p. 139] a statement of the form 'If P is true, then Q is true', which can be abbreviated to $P \Rightarrow Q$

Confidence interval [p. 704] an interval estimate for a population parameter (eg μ) based on the value of a sample statistic (eg $\bar{x}$).

Conjugate root theorem [p. 112]
If a polynomial has real coefficients, then the non-real roots occur in conjugate pairs.

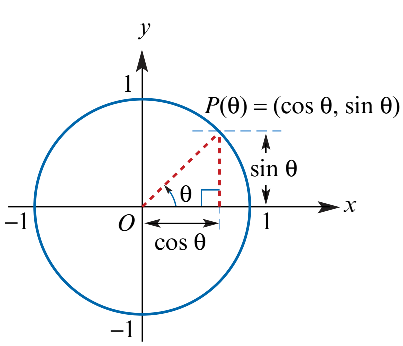
Constant acceleration formulas [p. 619]

- $v = u + at$
- $s = ut + \frac{1}{2}at^2$
- $v^2 = u^2 + 2as$
- $s = \frac{1}{2}(u + v)t$

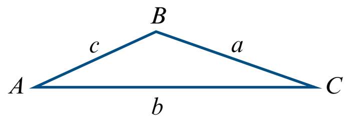
Converse [p. 139] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Cosecant function [p. 15] $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$ for $\sin \theta \neq 0$

Cosine function [p. 2] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Cosine rule For triangle ABC:
$c^2 = a^2 + b^2 - 2ab \cos C$



Cotangent function [p. 16] $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$ for $\sin \theta \neq 0$

Counterexample [p. 140] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

Cross product *see* vector product

D

De Moivre's theorem [p. 102]
$(r \operatorname{cis} \theta)^n = r^n \operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$

Definite integral [pp. 454, 496] $\int_a^b f(x)\,dx$ denotes the signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

Derivative function [p. 292] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Derivative of a vector function [p. 303]

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

$$\dot{\boldsymbol{r}}(t) = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

$$\ddot{\boldsymbol{r}}(t) = \frac{d^2x}{dt^2}\boldsymbol{i} + \frac{d^2y}{dt^2}\boldsymbol{j} + \frac{d^2z}{dt^2}\boldsymbol{k}$$

Derivatives, basic [pp. 292, 293, 456]

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^{ax}	ae^{ax}
$\ln \lvert ax \rvert$	$\frac{1}{x}$

$f(x)$	$f'(x)$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$

Derivatives, inverse trigonometric [p. 458]

$f(x)$	$f'(x)$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2 + x^2}$

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Determinant of a matrix [pp. 348, 360]
Associated with each square matrix $\mathbf{A}$, there is a real number called the determinant of $\mathbf{A}$, which is denoted by $\det(\mathbf{A})$. A square matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Differential equation [p. 551] an equation involving derivatives of a particular function or variable; e.g.

$$\frac{dy}{dx} = \cos x, \qquad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = \frac{y}{y+1}$$

Differential equation, general solution [p. 551]
$y = \sin x + c$ is the general solution of the differential equation $\dfrac{dy}{dx} = \cos x$.

Differential equation, particular solution [p. 551] $y = \sin x$ is the particular solution of the differential equation $\dfrac{dy}{dx} = \cos x$, given $y(0) = 0$.

Dilation [SM1&2] A dilation scales the x- or y-coordinate of each point in the plane.

- Dilation from the x-axis: $(x, y) \to (x, cy)$
- Dilation from the y-axis: $(x, y) \to (cx, y)$

Direct proof [p. 139] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Displacement [p. 614] the change in position. If a particle moves from point A to point B, then its displacement is described by the vector $\overrightarrow{AB}$.

Distance from a point P to a line [p. 242] given by $|\overrightarrow{PQ}|$, where Q is the point on the line such that PQ is perpendicular to the line

Distance from a point P to a plane [p. 263] given by $|\overrightarrow{PQ} \cdot \hat{\boldsymbol{n}}|$, where $\hat{\boldsymbol{n}}$ is a unit vector normal to the plane and Q is any point on the plane

Divisible [p. 137] For two integers a and b, we say that a is divisible by b if there exists an integer k such that $a = bk$.

Division of complex numbers [pp. 90, 100]

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Divisor [p. 137] For two integers a and b, we say that b is a divisor of a if there exists an integer k such that $a = bk$.

Dominance matrix [p. 393] an $n \times n$ matrix, $\mathbf{D}$, that represents a competition between the members of a group of size n. Entry d_{ij} in row i and column j of $\mathbf{D}$ is given by

$$d_{ij} = \begin{cases} 1 & \text{if member } i \text{ defeats member } j \\ 0 & \text{otherwise} \end{cases}$$

The members of the group can be ranked using the dominance scores from the matrix $\mathbf{D} + \mathbf{D}^2$.

Dot product *see* scalar product

Double-angle identities [p. 23]

- $\cos(2A) = \cos^2 A - \sin^2 A$
  $= 1 - 2\sin^2 A$
  $= 2\cos^2 A - 1$
- $\sin(2A) = 2\sin A \cos A$

E

Ellipse [p. 57] The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

Equality of complex numbers [p. 82]
$a + bi = c + di$ if and only if $a = c$ and $b = d$

Equilibrium A particle is in equilibrium if the resultant force acting on it is zero; the particle will remain at rest or continue moving with constant velocity.

Equivalence of vectors [p. 167]
Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$.
If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Equivalent statements [p. 140]
Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.
For equivalent statements P and Q, we also say 'P is true *if and only if* Q is true'.

Existence statement [p. 140] a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier 'there exists'.

Expected value of a random variable, E(X) [p. 680] also called the mean, μ.

- For a discrete random variable X:
  $$\mathrm{E}(X) = \sum_x x \cdot P(X = x) = \sum_i p_i \cdot x_i$$
  where p_i is the probability of outcome x_i occurring.
- For a continuous random variable X:
  $$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

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Exponential distribution [p. 522] For $\lambda > 0$, an exponential random variable X with parameter λ has a probability density function given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The mean and standard deviation of X are given by $\mathrm{E}(X) = \dfrac{1}{\lambda}$ and $\mathrm{sd}(X) = \dfrac{1}{\lambda}$.

F

Factor theorem [p. 111] Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.

Factorise [p. 114] In the complex number system, every non-constant polynomial can be expressed as a product of linear factors.

Force [p. 641] causes a change in motion; e.g. gravitational force, tension force, normal reaction force. Force is a vector quantity.

Frictional force [p. 655] When an object moves across a rough surface, there is a resistance force due to friction. The frictional force acts in the opposite direction to the velocity of the object.

Fundamental theorem of algebra [p. 114] Every non-constant polynomial with complex coefficients has at least one linear factor in the complex number system.

Fundamental theorem of calculus [p. 496] If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any anti-derivative of f and $\int_a^b f(x)\,dx$ is the definite integral from a to b.

G

g [p. 650] the acceleration of a particle due to gravity. Close to the Earth's surface, the value of g is approximately $9.8\ \text{m/s}^2$.

Gaussian elimination [p. 370] a systematic method for solving simultaneous linear equations

Gradient function *see* derivative function

H

Hyperbola [p. 60] The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k); the asymptotes are given by

$$y - k = \pm\frac{b}{a}(x - h)$$

I

Imaginary number i [p. 80] $i^2 = -1$

Imaginary part of a complex number [p. 80] If $z = a + bi$, then $\mathrm{Im}(z) = b$.

Implication *see* conditional statement

Implicit differentiation [p. 546] used to find the gradient at a point on a curve such as $x^2 + y^2 = 1$, which is not defined by a rule of the form $y = f(x)$

Integers [p. 137] the elements of $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Integrals, standard [pp. 296, 456, 462]

$f(x)$	$\int f(x)\,dx$
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + c$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert + c$
e^{ax+b}	$\dfrac{1}{a}e^{ax+b} + c$
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right) + c$
$\dfrac{-1}{\sqrt{a^2-x^2}}$	$\cos^{-1}\left(\dfrac{x}{a}\right) + c$
$\dfrac{a}{a^2+x^2}$	$\tan^{-1}\left(\dfrac{x}{a}\right) + c$

Integration by parts [p. 479]

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Integration by substitution [p. 465]

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

Inverse cosine function (arccosine) [p. 27] $\cos^{-1} x = y$ if $\cos y = x$, for $x \in [-1, 1]$ and $y \in [0, \pi]$

Inverse sine function (arcsine) [p. 26] $\sin^{-1} x = y$ if $\sin y = x$, for $x \in [-1, 1]$ and $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Inverse tangent function (arctangent) [p. 27] $\tan^{-1} x = y$ if $\tan y = x$, for $x \in \mathbb{R}$ and $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Glossary F → I

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K

Kilogram weight, kg wt [p. 641] a unit of force. If an object on the surface of the Earth has a mass of 1 kg, then the gravitational force acting on this object is 1 kg wt.

L

Leslie matrix [p. 399] an $n \times n$ matrix, $\mathbf{L}$, used to model the change over time in a population that has been divided into n age groups. The size of each age group after the kth time period is given by the population matrix $\mathbf{P}_k$, where

$\mathbf{P}_k = \mathbf{L}\mathbf{P}_{k-1} = \mathbf{L}^k\mathbf{P}_0$

Limits of integration [p. 454] In the expression $\int_a^b f(x)\,dx$, the number a is called the *lower limit* of integration and b the *upper limit* of integration.

Line in three dimensions [pp. 238, 240] can be described as follows, where $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ is the position vector of a point A on the line, and $\boldsymbol{d} = d_1\boldsymbol{i} + d_2\boldsymbol{j} + d_3\boldsymbol{k}$ is parallel to the line:

Vector equation	$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d},\ t \in \mathbb{R}$
Parametric equations	$x = a_1 + d_1t$ $y = a_2 + d_2t$ $z = a_3 + d_3t$
Cartesian form	$\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Linear combination of independent normal random variables [p. 684] If X and Y are independent normal random variables, then $aX + bY$ is also a normal random variable (provided a and b are not both zero).

Linear combination of random variables [p. 684]

- $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$
- $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$ if X and Y are independent

Linear equation [p. 339] an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $x_1, x_2, \dots, x_n$ are variables and $a_1, a_2, \dots, a_n, b$ are constants.

- **Two variables** An equation $ax + by = c$ represents a line in two-dimensional space (provided a and b are not both zero).
- **Three variables** An equation $ax + by + cz = d$ represents a plane in three-dimensional space (provided a, b and c are not all zero).

Linear function of a random variable [p. 679]

- $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
- $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Linear transformation [SM1&2] a transformation of the plane with a rule of the form

$(x, y) \to (ax + by, cx + dy)$

Each linear transformation can be represented by a 2×2 matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Linear transformation, inverse [SM1&2] If $\mathbf{A}$ is the matrix of a linear transformation and $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.

Linear transformations, composition [SM1&2] If $\mathbf{A}$ and $\mathbf{B}$ are the matrices of two different linear transformations, then the product $\mathbf{BA}$ is the matrix of the transformation $\mathbf{A}$ followed by $\mathbf{B}$.

Locus [p. 121] a set of points described by a geometric condition; e.g. the locus of the equation $|z - 1 - i| = 2$ is the circle with centre $1 + i$ and radius 2

Logistic differential equation [p. 581]

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad 0 < P < K$$

This differential equation can be used to model a population P at time t, where:

- the constant r is called the *growth parameter*
- the constant K is called the *carrying capacity*.

M

Magnitude of a vector [p. 156] the length of a directed line segment corresponding to the vector.

- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Margin of error, E [p. 711] the distance between the sample estimate and the endpoints of the confidence interval

Mass [p. 641] The mass of an object is the amount of matter it contains, and can be measured in kilograms. Mass is not the same as weight.

Mathematical induction [p. 143] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

Matrices, addition [p. 341] Addition is defined for two matrices of the same size (same number of rows and same number of columns). The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

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Matrices, equal [p. 341] Two matrices **A** and **B** are equal, and we write **A** = **B**, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

Matrices, multiplication [p. 344] The product of two matrices **A** and **B** is only defined if the number of columns of **A** is the same as the number of rows of **B**. If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Matrix [p. 340] a rectangular array of numbers

Matrix, dimensions [p. 340] A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, identity [pp. 347, 358] For square matrices of a given size (e.g. 2×2), a multiplicative identity **I** exists.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ for each 2×2 matrix **A**.

Matrix, inverse [pp. 347, 359] If **A** is a square matrix and there exists a matrix **B** such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **B** is called the inverse of **A**. When it exists, the inverse of a square matrix **A** is unique and is denoted by $\mathbf{A}^{-1}$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

provided $ad - bc \neq 0$.

Matrix, invertible [p. 347] A square matrix is said to be invertible if its inverse exists.

Matrix, multiplication by a scalar [p. 341] If **A** is an $m \times n$ matrix and k is a real number, then k**A** is an $m \times n$ matrix whose entries are k times the corresponding entries of **A**. For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Matrix, non-invertible [p. 347] A square matrix is said to be non-invertible if it does not have an inverse.

Matrix, size [p. 340] A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, square [p. 347] A matrix with the same number of rows and columns is called a square matrix; e.g. a 2×2 matrix.

Matrix, zero [p. 342] The $m \times n$ matrix with all entries equal to zero is called the zero matrix and is usually denoted by **O**.

Matrix algebra [pp. 340–222] Some properties of arithmetic operations on $n \times n$ matrices:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ commutative law
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ associative law
- $\mathbf{A} + \mathbf{O} = \mathbf{A}$ zero matrix
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$ additive inverse
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ associative law
- $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ identity matrix
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ distributive law
- $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ distributive law

Note: Matrix multiplication is not commutative.

Mean of a random variable, μ *see* expected value of a random variable, $\mathrm{E}(X)$

Midpoint of a line segment [p. 162] If M is the midpoint of line segment AB, then

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

Modulus function [p. 455] The modulus (or *absolute value*) of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Modulus of a complex number, $|z|$ [pp. 88, 93] the distance of the complex number from the origin. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Modulus, properties [p. 88] For complex numbers z_1 and z_2:

- $|z_1 z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)

Modulus–argument form of a complex number *see* polar form of a complex number

Momentum [p. 651] The momentum of a particle is the product of its mass and velocity: $\boldsymbol{P} = m\boldsymbol{v}$. Momentum can be considered as the fundamental quantity of motion.

Multiplication of a complex number by a real number [pp. 83, 98]

- If $z = a + bi$ and $k \in \mathbb{R}$, then $kz = ka + kbi$.
- If $z = r \operatorname{cis} \theta$ and $k > 0$, then $kz = kr \operatorname{cis} \theta$.
- If $z = r \operatorname{cis} \theta$ and $k < 0$, then $kz = |k| r \operatorname{cis}(\theta + \pi)$.

Multiplication of a complex number by i [pp. 86, 99] corresponds to a rotation about the origin by 90° anticlockwise. If $z = a + bi$, then $iz = i(a + bi) = -b + ai$.

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Multiplication of a vector by a scalar [p. 157] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $m \in \mathbb{R}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$.

Multiplication of complex numbers [pp. 86, 99] If $z_1 = a + bi$ and $z_2 = c + di$, then
$z_1z_2 = (ac - bd) + (ad + bc)i$
If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then
$z_1z_2 = r_1r_2 \operatorname{cis}(\theta_1 + \theta_2)$

N

Natural numbers [p. 137] the elements of $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Newton, N [p. 641] the standard unit of force. $1 \text{ N} = 1 \text{ kg m/s}^2$

Newton's first law of motion [p. 652] If the resultant force on a particle is zero, then the particle will remain stationary or in uniform straight-line motion.

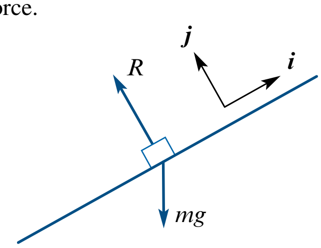
Newton's law of cooling [p. 568] The rate at which a body cools is proportional to the difference between its temperature and the temperature of its immediate surroundings.

Newton's second law of motion [p. 652]
$\boldsymbol{F} = m\boldsymbol{a}$
The rate of change of momentum of a particle at any instant is proportional to the resultant force on the particle.

Newton's third law of motion [p. 652] If an object A exerts a force on another object B (*action*), then B exerts a force on A of equal magnitude but opposite direction (*reaction*).

Normal distribution a symmetric, bell-shaped distribution that often occurs for a measure in a population (e.g. height, weight, IQ); its centre is determined by the mean, μ, and its width by the standard deviation, σ.

Normal reaction force [p. 654] A mass placed on a surface (horizontal or inclined) experiences a force perpendicular to the surface, called the normal reaction force.



Normal vector to a plane [p. 257] a vector that is perpendicular to the plane

P

Parametric equations [p. 64] A pair of equations of the form $x = f(t)$ and $y = g(t)$ describes a curve in the plane, where t is called the *parameter* of the curve. For example:
- **Circle** $x = a\cos t$ and $y = a\sin t$
- **Ellipse** $x = a\cos t$ and $y = b\sin t$
- **Hyperbola** $x = a\sec t$ and $y = b\tan t$

Similarly, equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ describe a curve in three-dimensional space.

Partial fractions [p. 475] Some rational functions may be expressed as a sum of partial fractions; e.g.
$$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2} + \frac{Dx+E}{ex^2+fx+g}$$

Particle model [p. 641] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

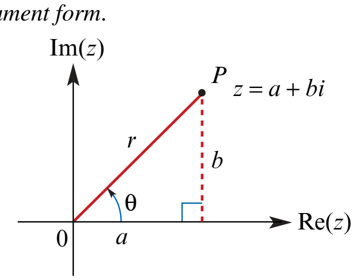
Period of a function [p. 5] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f.
- Sine and cosine have period 2π.
- Tangent has period π.
- A function of the form $y = a\cos(nx + \varepsilon) + b$ or $y = a\sin(nx + \varepsilon) + b$ has period $\dfrac{2\pi}{n}$.

Plane in three dimensions [p. 257] can be described as follows, where $\boldsymbol{a}$ is the position vector of a point A on the plane, $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ is normal to the plane, and $k = \boldsymbol{a} \cdot \boldsymbol{n}$:

Vector equation	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation	$n_1x + n_2y + n_3z = k$

Point estimate [p. 703] If the value of the sample mean $\bar{x}$ is used as an estimate of the population mean μ, then it is called a point estimate of μ.

Polar form of a complex number [p. 93] A complex number is expressed in polar form as $z = r \operatorname{cis} \theta$, where r is the modulus of z and θ is an argument of z. This is also called *modulus–argument form*.



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Polar form of a vector [p. 178]

- In two dimensions, each vector $\boldsymbol{v}$ can be written in the form $\boldsymbol{v} = [r, \theta]$.
- In three dimensions, each vector $\boldsymbol{v}$ can be written in the form $\boldsymbol{v} = [r, \theta, \varphi]$.

Population [p. 688] the set of all eligible members of a group which we intend to study

Population mean, μ [p. 688] the mean of all values in the entire population

Population parameter [p. 688] a statistical measure that is based on the whole population; the value is constant for a given population

Position [p. 614] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Position vector [p. 159] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point P relative to the origin O.

Principle of mathematical induction [p. 143] To prove that a statement $P(n)$ is true for every natural number n:

- Show that $P(1)$ is true.
- Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

Product rule [p. 292]

- If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.
- If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Product-to-sum identities [p. 39]

- $2\cos A \cos B = \cos(A - B) + \cos(A + B)$
- $2\sin A \sin B = \cos(A - B) - \cos(A + B)$
- $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

Projection [SM1&2] A projection maps each point in the plane onto an axis.

- Projection onto the x-axis: $(x, y) \to (x, 0)$
- Projection onto the y-axis: $(x, y) \to (0, y)$

Proof by contradiction [p. 138] a proof that begins by assuming the negation of what is to be proved

Proof by induction [p. 143] a proof that uses the *principle of mathematical induction*

Pythagorean identity [pp. 6, 19]

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Q

Quadratic formula [p. 108] An equation of the form $az^2 + bz + c = 0$, with $a \neq 0$, may be solved using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quantifier [p. 140] ‘for all’, ‘there exists’

Quotient rule [p. 293]

- If $f(x) = \dfrac{u(x)}{v(x)}$, then
$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$
- If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

R

Radian [p. 4] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Radioactive decay [p. 568] The rate at which a radioactive substance decays is proportional to the mass of the substance remaining.

Random sample [p. 688] A sample of size n is called a *simple random sample* if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

Rational function [p. 475] a function of the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials

Real part of a complex number [p. 80] If $z = a + bi$, then $\operatorname{Re}(z) = a$.

Reciprocal trigonometric functions [p. 15] the cosecant, secant and cotangent functions

Reflection [SM1&2] A reflection in a line ℓ maps each point in the plane to its mirror image on the other side of the line.

- Reflection in the x-axis: $(x, y) \to (x, -y)$
- Reflection in the y-axis: $(x, y) \to (-x, y)$
- Reflection in the line $y = x$: $(x, y) \to (y, x)$
- Reflection in the line $y = -x$: $(x, y) \to (-y, -x)$

Reflection matrix [SM1&2] A reflection in the line $y = mx = x\tan\theta$ is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Glossary Q → R

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Remainder theorem [p. 110] Let $\alpha \in \mathbb{C}$. When a polynomial $P(z)$ is divided by $z - \alpha$, the remainder is $P(\alpha)$.

Resultant force [p. 642] the vector sum of the forces acting at a point

Roots of a complex number [p. 119] The nth roots of a complex number a are the solutions of the equation $z^n = a$. If $a = 1$, then the solutions are called the *nth roots of unity*.

Rotation matrix [SM1&2] A rotation about the origin by angle θ anticlockwise is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

S

Sample [SM1&2] a subset of the population which we select in order to make inferences about the whole population

Sample mean, $\bar{x}$ [p. 688] the mean of all values of a measure in a particular sample. The values $\bar{x}$ are the values of a random variable $\bar{X}$.

Sample statistic [SM1&2] a statistical measure that is based on a sample from the population; the value varies from sample to sample

Sampling distribution [p. 688] the distribution of a statistic which is calculated from a sample

Scalar product [p. 183] The scalar product of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is given by

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

Scalar product, properties [p. 184]

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Scalar quantity [p. 614] a quantity determined only by its magnitude; e.g. distance, time, mass

Scalar resolute [p. 189] The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

Secant function [p. 15] $\sec\theta = \dfrac{1}{\cos\theta}$ for $\cos\theta \neq 0$

Second derivative [p. 294]

- The second derivative of a function f with rule $f(x)$ is denoted by f'' and has rule $f''(x)$.
- The second derivative of y with respect to x is denoted by $\dfrac{d^2y}{dx^2}$.

Separation of variables [p. 561] If $\dfrac{dy}{dx} = f(x)\,g(y)$, then $\displaystyle\int f(x)\,dx = \int \frac{1}{g(y)}\,dy$.

Shear [SM1&2] A shear moves each point in the plane by an amount proportional to its distance from an axis.

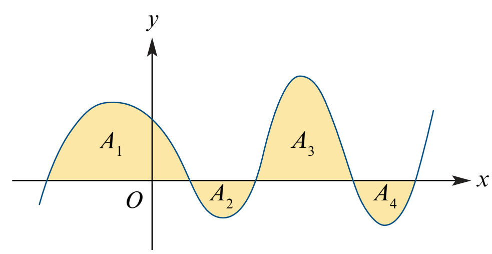
- Shear parallel to the x-axis: $(x, y) \to (x + cy, y)$
- Shear parallel to the y-axis: $(x, y) \to (x, cx + y)$

Sigma notation [p. 147] *see* summation notation

Signed area [p. 496]

- Regions *above* the x-axis are defined to have *positive* signed area.
- Regions *below* the x-axis are defined to have *negative* signed area.

For example, the signed area of the shaded region in the following graph is $A_1 - A_2 + A_3 - A_4$.



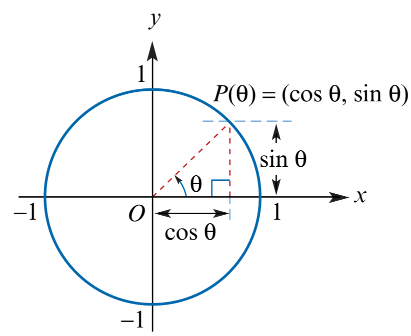
Simple harmonic motion [p. 635] motion in a straight line such that $\ddot{x} = -\omega^2(x - c)$, for constants ω and c with $\omega > 0$. The particle oscillates about the centre point $x = c$.

Simpson's rule [p. 529] a method for estimating a definite integral $\int_a^b f(x)\,dx$. The interval $[a, b]$ on the x-axis is divided into an even number n of equal subintervals, each of width w:

$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \dots,\ [x_{n-1}, x_n]$

Simulation [p. 688] using technology (calculators or computers) to repeat a random process many times; e.g. random sampling

Sine function [p. 2] $\sin\theta$ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



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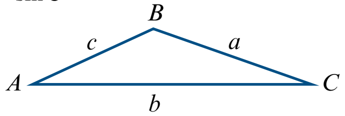
Sine rule For triangle ABC:

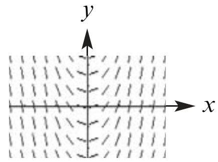
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Skew lines [p. 247] In three-dimensional space, two lines are skew if they do not intersect and are not parallel.

Slope field [p. 597] The slope field of a differential equation

$$\frac{dy}{dx} = f(x, y)$$



assigns to each point $P(x, y)$ in the plane the number $f(x, y)$, which is the gradient of the solution curve through P.

Solid of revolution [p. 516] the solid formed by rotating a region about a line

Speed [pp. 309, 615] the magnitude of velocity

Speed, average [p. 615]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Sphere, general Cartesian equation [p. 270] The sphere with radius a and centre (h, k, ℓ) has equation

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = a^2$$

Sphere, general vector equation [p. 270] The sphere with radius a and centre C has vector equation

$$|\boldsymbol{r} - \overrightarrow{OC}| = a$$

Standard deviation of a random variable, σ a measure of the spread or variability, given by

$$\text{sd}(X) = \sqrt{\text{Var}(X)}$$

Standard deviation of a sample, s a measure of the spread or variability of a sample about the sample mean $\bar{x}$, given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Subtraction of complex numbers [p. 82] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = (a - c) + (b - d)i$.

Subtraction of vectors [p. 158] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$.

Summation notation [p. 147] used for writing sums concisely. For example:

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

Sum-to-product identities [p. 40]

- $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

System of linear equations [pp. 351, 362, 355, 366] a finite set of linear equations that are to be solved simultaneously

T

Tangent function [p. 2] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$

Tension force the pulling force exerted by a string that connects two objects. The forces at each end of the string have equal magnitude.

Transformation [SM1&2] A transformation of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the *image* of (x, y).

Translation [SM1&2] a transformation that moves each point in the plane in the same direction and over the same distance: $(x, y) \to (x + a, y + b)$. A translation is *not* a linear transformation.

Trigonometric functions [p. 2] the sine, cosine and tangent functions

U

Unit vector [p. 167] a vector of magnitude 1. The unit vectors in the positive directions of the x-, y- and z-axes are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. The unit vector in the direction of $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Universal statement [p. 140] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier 'for all'.

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V

Variance of a random variable, σ^2 [p. 681] a measure of the spread or variability, defined by

$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$

An alternative (computational) formula is

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Vector [p. 156] a set of equivalent directed line segments

Vector function [p. 227] If $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, then we say that $\boldsymbol{r}$ is a vector function of t.

Vector product, formula [p. 254] For $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is given by

$(a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$

Vector product, geometric properties [p. 252]

- **Magnitude** $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin\theta$, where θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
- **Direction** $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ (if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel non-zero vectors)

Vector product, properties [pp. 252, 254]

- $k(\boldsymbol{a} \times \boldsymbol{b}) = (k\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (k\boldsymbol{b})$
- $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$
- $\boldsymbol{b} \times \boldsymbol{a} = -(\boldsymbol{a} \times \boldsymbol{b})$
- $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{0} = \boldsymbol{0}$

Vector quantity [p. 614] a quantity determined by its magnitude and direction; e.g. force, velocity

Vector resolute [p. 188] The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\,\boldsymbol{b} = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Vectors, parallel [p. 159] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = k\boldsymbol{b}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Vectors, perpendicular [p. 184] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Vectors, properties [p. 160]

$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$	commutative law
$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$	associative law
$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$	zero vector
$\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$	additive inverse
$m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$	distributive law

Vectors, resolution [p. 188] A vector $\boldsymbol{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

Velocity [pp. 309, 615] the rate of change of position with respect to time

Velocity, average [p. 615]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

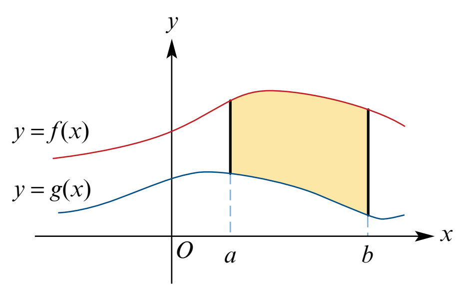
Velocity, instantaneous [p. 615] $v = \dfrac{dx}{dt}$

Velocity–time graph [p. 617]

- Acceleration is given by the gradient.
- Displacement is given by the signed area bounded by the graph and the t-axis.
- Distance travelled is given by the total area bounded by the graph and the t-axis.

Volume of a solid of revolution [p. 516]

- **Rotation about the x-axis**
  If the region is bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis, then
  $V = \int_a^b \pi y^2\, dx = \pi \int_a^b (f(x))^2\, dx$
- **Rotation about the y-axis**
  If the region is bounded by the curve $x = f(y)$, the lines $y = a$ and $y = b$ and the y-axis, then
  $V = \int_a^b \pi x^2\, dy = \pi \int_a^b (f(y))^2\, dy$
- **Region not bounded by the x-axis**
  If the shaded region is rotated about the x-axis, then the volume V is given by
  $V = \pi \int_a^b (f(x))^2 - (g(x))^2\, dx$



W

Weight [p. 650] On the Earth's surface, a mass of m kg has a force of m kg wt (or mg newtons) acting on it; this force is known as the weight.

Z

Zero vector, $\boldsymbol{0}$ [p. 158] a line segment of zero length with no direction

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Answers

Chapter 1

Exercise 1A

1 a **i** 4π **ii** 3π **iii** $-\dfrac{5\pi}{2}$

iv $\dfrac{\pi}{12}$ **v** $-\dfrac{\pi}{18}$ **vi** $-\dfrac{7\pi}{4}$

b **i** $225°$ **ii** $-120°$ **iii** $105°$

iv $-330°$ **v** $260°$ **vi** $-165°$

2 a **i** 0.12^c **ii** -1.75^c

iii -0.44^c **iv** 0.89^c

v 3.60^c **vi** -7.16^c

b **i** $97.40°$ **ii** $-49.85°$

iii $160.43°$ **iv** $5.73°$

v $-171.89°$ **vi** $-509.93°$

3 a $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{1}{2}$ **c** $\dfrac{\sqrt{3}}{2}$

d $-\dfrac{1}{2}$ **e** $\dfrac{1}{\sqrt{2}}$ **f** $\dfrac{\sqrt{3}}{2}$

4 a $\dfrac{\sqrt{3}}{2}$ **b** $-\dfrac{1}{\sqrt{2}}$ **c** $\dfrac{1}{2}$

d $-\dfrac{1}{\sqrt{2}}$ **e** $\dfrac{1}{\sqrt{2}}$ **f** $-\dfrac{\sqrt{3}}{2}$

g $-\dfrac{\sqrt{3}}{2}$ **h** $-\dfrac{\sqrt{3}}{2}$ **i** $\dfrac{1}{2}$

5 a $-\dfrac{\sqrt{3}}{2}$ **b** $-\dfrac{1}{\sqrt{3}}$

6 a $-\dfrac{\sqrt{51}}{10}$ **b** $\dfrac{\sqrt{51}}{7}$

7 a $-\dfrac{\sqrt{3}}{2}$ **b** $\dfrac{1}{\sqrt{3}}$

8 a $\dfrac{\sqrt{91}}{10}$ **b** $\dfrac{-3\sqrt{91}}{91}$

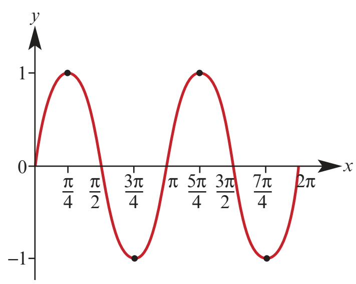
9 $2\pi - a,\ 2\pi - b,\ 2\pi - c,\ 2\pi - d$

10 a $\dfrac{4\pi}{3}, \dfrac{5\pi}{3}$ **b** $\dfrac{2\pi}{3}, \dfrac{5\pi}{6}, \dfrac{5\pi}{3}, \dfrac{11\pi}{6}$

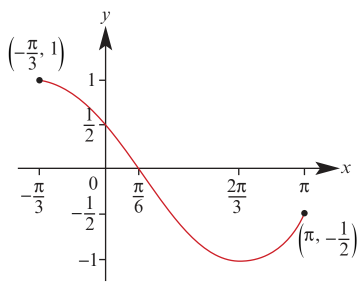
c $\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$ **d** $\dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

e $0, \dfrac{\pi}{3}, \pi, \dfrac{4\pi}{3}, 2\pi$ **f** $\dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{2}, \dfrac{5\pi}{3}$

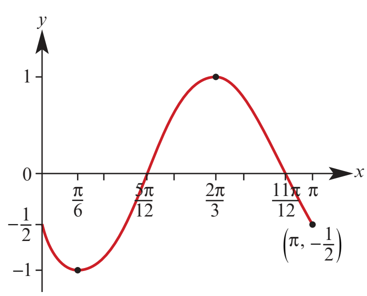
11 a



b



c



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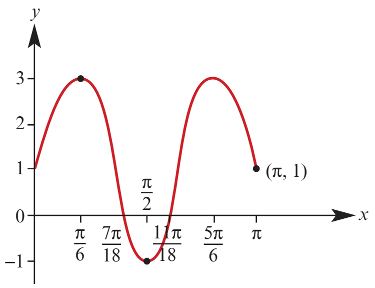
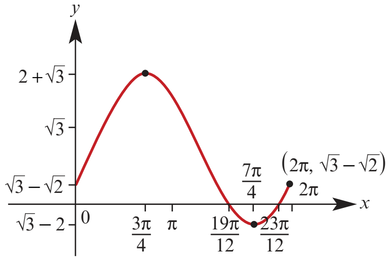
d



e



12 a 1 **b** $\sqrt{3}$ **c** $\frac{1}{\sqrt{3}}$ **d** $\sqrt{3}$

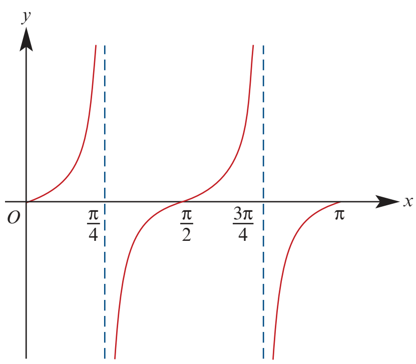
13 a $\frac{-\sqrt{17}}{17}$ **b** $\frac{-4\sqrt{17}}{17}$ **c** $\frac{-1}{4}$ **d** $\frac{-1}{4}$

14 a $\frac{\sqrt{21}}{7}$ **b** $\frac{-2\sqrt{7}}{7}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{-\sqrt{3}}{2}$

15 a $\frac{2\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

c $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

16 a



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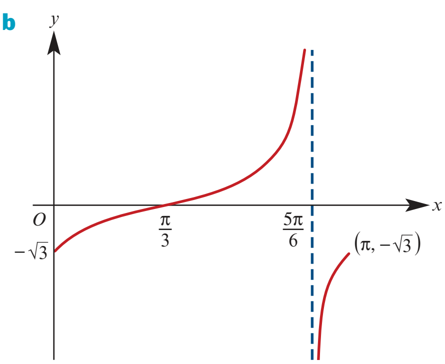

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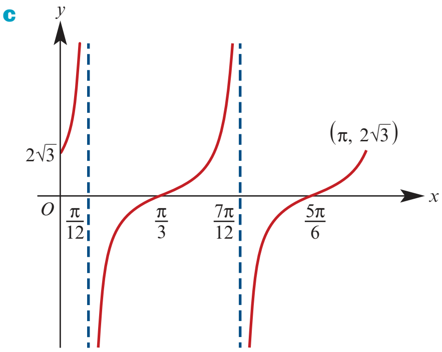

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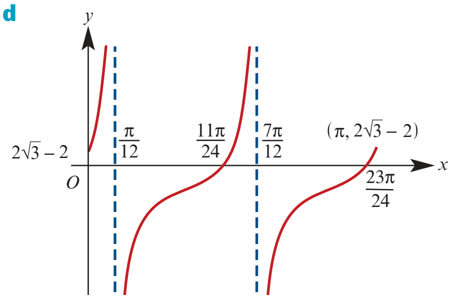
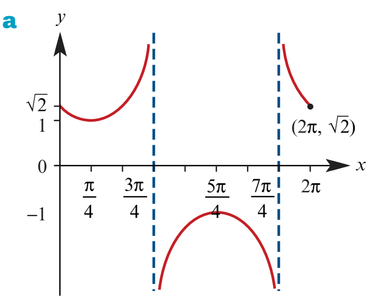
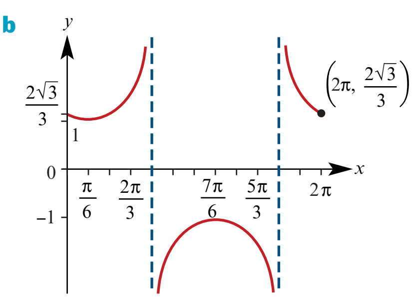


Exercise 1B

1 a



b



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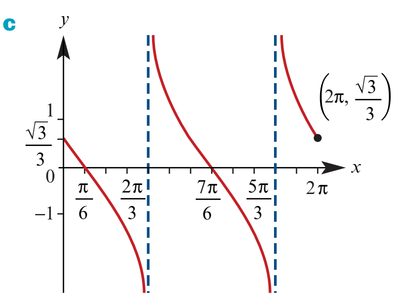

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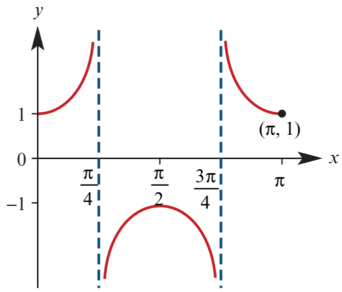
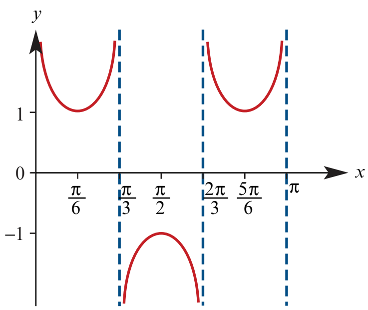
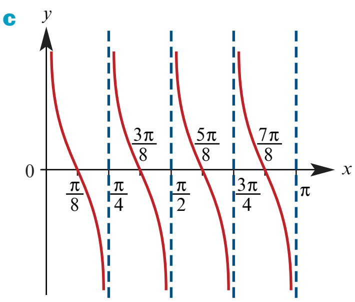
2 a



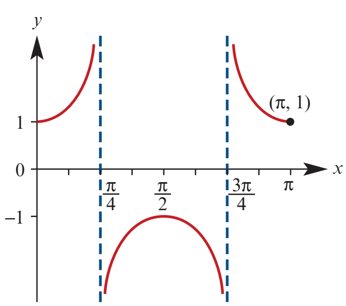
b



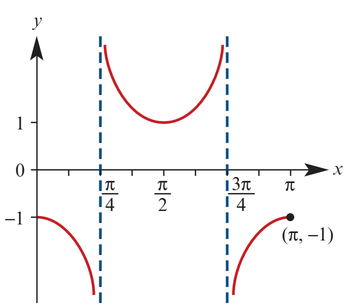
c



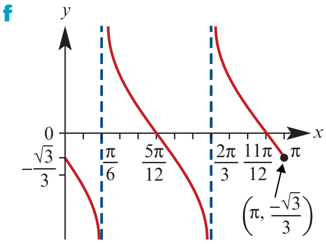
d



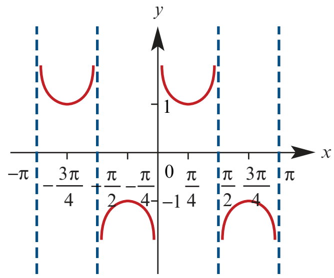
e



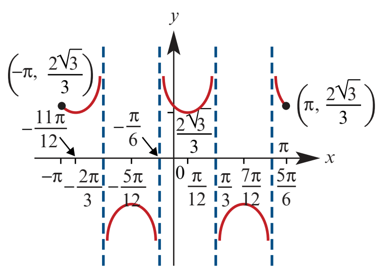
f



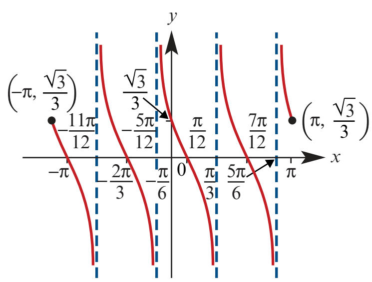
3 a



b



c



4 a $\cot x = \frac{5}{8}$, $\sec x = \frac{\sqrt{89}}{5}$, $\operatorname{cosec} x = \frac{\sqrt{89}}{8}$

b $\cot x = \frac{2\sqrt{6}}{5}$, $\sec x = \frac{7\sqrt{6}}{12}$, $\operatorname{cosec} x = \frac{7}{5}$

c $\cot x = \frac{7\sqrt{2}}{8}$, $\sec x = \frac{9}{7}$, $\operatorname{cosec} x = \frac{9\sqrt{2}}{8}$

5 a 2 **b** $\sqrt{2}$ **c** $-\sqrt{3}$ **d** 2
e $\sqrt{2}$ **f** 1 **g** $-\sqrt{2}$ **h** 2

6 a 1 **b** -1
c $\operatorname{cosec}^2 x$ **d** $\sec x$
e $\sin^2 x - \cos^2 x = -\cos(2x)$
f $\tan x \sec^2 x$

7 a $\sqrt{17}$ **b** $\frac{\sqrt{17}}{17}$ **c** $-\frac{\sqrt{17}}{4}$

8 a $-\sqrt{10}$ **b** $-\frac{\sqrt{10}}{10}$ **c** $-\frac{\sqrt{10}}{3}$

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9 a $-3\sqrt{11}$ **b** $\dfrac{-3\sqrt{11}}{10}$

10 a $-\sqrt{35}$ **b** $\dfrac{\sqrt{35}}{6}$

11 a $\dfrac{-\sqrt{3}}{2}$ **b** $-\sqrt{3}$ **c** 2

12 a $-\dfrac{1}{3}$ **b** $-\dfrac{2\sqrt{2}}{3}$ **c** $-\dfrac{3\sqrt{2}}{4}$

13 a $\dfrac{\sqrt{51}}{10}$ **b** $-\dfrac{\sqrt{51}}{7}$ **c** $-\dfrac{7\sqrt{51}}{51}$

14 a 0.2 **b** $-\dfrac{2\sqrt{6}}{5}$ **c** $-\dfrac{\sqrt{6}}{12}$

15 a 0 **b** $\dfrac{1}{2}\sin(2\theta)$ **c** 1 **d** 1

16 $x - \dfrac{1}{x} = -2\tan\theta$

Exercise 1C

1 a $\dfrac{\sqrt{2}}{4}(\sqrt{3}-1)$ **b** $\dfrac{\sqrt{2}}{4}(1-\sqrt{3})$

2 a $\sin(2x)\cos(5y) - \cos(2x)\sin(5y)$
b $\cos(x^2)\cos(y) - \sin(x^2)\sin(y)$

3 a $\sin(x-2y)$ **b** $\cos x$
c $\sin(2A)$ **d** $\cos y$

4 a $\sin x\cos(2x) + \cos x\sin(2x)$
b $3\sin x - 4\sin^3 x$

5 a $\cos x\cos(2x) - \sin x\sin(2x)$
b $4\cos^3 x - 3\cos x$

6 a -0.8 **b** 2.6 **c** $\dfrac{5}{13}$ **d** $\dfrac{12}{13}$
e $\dfrac{16}{65}$ **f** $\dfrac{63}{65}$

7 a -0.71 **b** 0.92 **c** -0.93 **d** -0.36

8 a $\dfrac{1}{4}\sin(2x)$ **b** $-\cos(2x)$ **c** -1
d $-2\tan x$ **e** $\sin(2x)$

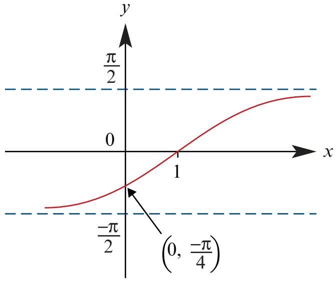
9 a 0.96 **b** -0.28 **c** $-\dfrac{24}{7}$

10 a -0.66 **b** 0.91

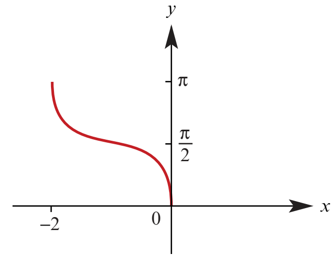
11 0.97

Exercise 1D

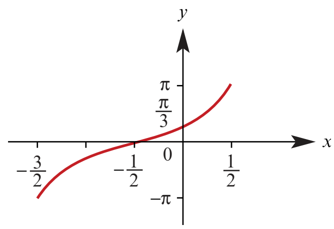
1 a dom $= \mathbb{R}$, ran $= \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$



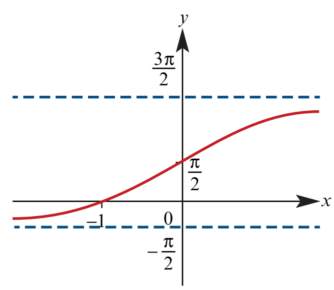
b dom $= [-2, 0]$, ran $= [0, \pi]$



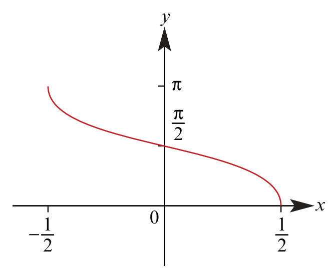
c dom $= \left[-\dfrac{3}{2}, \dfrac{1}{2}\right]$, ran $= [-\pi, \pi]$



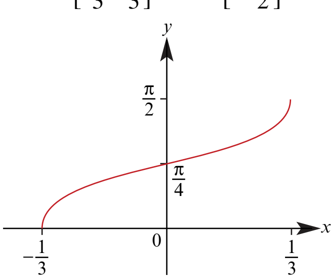
d dom $= \mathbb{R}$, ran $= \left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$



e dom $= \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$, ran $= [0, \pi]$



f dom $= \left[\dfrac{-1}{3}, \dfrac{1}{3}\right]$, ran $= \left[0, \dfrac{\pi}{2}\right]$



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2 a $\frac{\pi}{2}$ b $-\frac{\pi}{4}$ c $\frac{\pi}{6}$ d $\frac{5\pi}{6}$ e $\frac{\pi}{3}$
f $\frac{\pi}{4}$ g $-\frac{\pi}{3}$ h $\frac{\pi}{6}$ i π

3 a $\frac{\sqrt{3}}{2}$ b $-\frac{\pi}{3}$ c -1 d $\frac{\sqrt{2}}{2}$ e $\frac{\pi}{4}$
f $\sqrt{3}$ g $\frac{\pi}{3}$ h $-\frac{\pi}{3}$ i $-\frac{\pi}{4}$ j $\frac{5\pi}{6}$
k π l $-\frac{\pi}{4}$

4 a $[1,3], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ b $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right], [-1,1]$
c $\left[-\frac{5}{2}, -\frac{3}{2}\right], \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ d $\left[-\frac{\pi}{18}, \frac{5\pi}{18}\right], [-1,1]$
e $\left[\frac{\pi}{6}, \frac{7\pi}{6}\right], [-1,1]$ f $[-2,0], [0,\pi]$
g $[-1,1], \left[0, \frac{\pi}{2}\right]$ h $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right], [-1,1]$
i $\mathbb{R}, \left[0, \frac{\pi}{2}\right)$ j $\left(0, \frac{\pi}{2}\right), \mathbb{R}$
k $\mathbb{R}, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
l $\left(-\frac{\sqrt{2\pi}}{2}, \frac{\sqrt{2\pi}}{2}\right), [0,\infty)$

5 a $\frac{3}{5}$ b $\frac{12}{5}$ c $\frac{24}{25}$ d $\frac{40}{9}$ e $\sqrt{3}$
f $\frac{\sqrt{5}}{3}$ g $\frac{-2\sqrt{5}}{5}$ h $\frac{2\sqrt{10}}{7}$ i $\frac{7\sqrt{149}}{149}$

6 a i $\frac{4}{5}$
ii $\frac{12}{13}$

Exercise 1E

1 a $\frac{7\pi}{6}, \frac{11\pi}{6}$ b $\frac{\pi}{12}, \frac{17\pi}{12}$ c $\frac{\pi}{6}, \frac{11\pi}{6}$
d $\frac{\pi}{4}, \frac{5\pi}{4}$ e $\frac{5\pi}{6}, \frac{11\pi}{6}$
f $\frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$

2 a $\frac{\pi}{6}, \frac{5\pi}{6}$ b $\frac{5\pi}{6}, \frac{7\pi}{6}$ c $\frac{\pi}{3}, \frac{4\pi}{3}$
d $\frac{3\pi}{4}, \frac{7\pi}{4}$ e $\frac{2\pi}{3}, \frac{4\pi}{3}$ f $\frac{5\pi}{4}, \frac{7\pi}{4}$

3 a $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{3\pi}{4} + 2n\pi, n \in \mathbb{Z}$
b $x = 2n\pi, n \in \mathbb{Z}$
c $x = \frac{\pi}{6} + n\pi, n \in \mathbb{Z}$
d $x = \frac{(12n-5)\pi}{12}$ or $x = \frac{(4n+1)\pi}{4}, n \in \mathbb{Z}$
e $x = \frac{(2n-1)\pi}{3}$ or $x = \frac{2(3n+1)\pi}{9}, n \in \mathbb{Z}$
f $x = \frac{2n\pi}{3}$ or $x = \frac{(6n+1)\pi}{9}, n \in \mathbb{Z}$
g $x = \frac{(3n-2)\pi}{6}, n \in \mathbb{Z}$
h $x = \frac{n\pi}{2}, n \in \mathbb{Z}$
i $x = \frac{(8n-5)\pi}{8}, n \in \mathbb{Z}$

4 a ± 1.16 b $-0.20, -2.94$
c $1.03, -2.11$

5 a $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$ b $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$
c $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$
d $\frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{5\pi}{8}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{9\pi}{8}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{13\pi}{8}, \frac{41\pi}{24}, \frac{15\pi}{8}$
e $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ f $0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$
g $\frac{3\pi}{4}, \frac{7\pi}{4}$ h $\frac{\pi}{3}, \frac{5\pi}{3}$ i $0, \frac{\pi}{2}, 2\pi$

6 a max = 3, min = 1 b max = 1, min = $\frac{1}{3}$
c max = 5, min = 4 d max = $\frac{1}{4}$, min = $\frac{1}{5}$
e max = 3, min = −1 f max = 9, min = 5

7 a $(-1.14, -2.28), (0,0), (1.14, 2.28)$
b $(-1.24, -1.24), (0,0), (1.24, 1.24)$
c $(3.79, -0.79)$ d $(0,0), (4.49, 4.49)$

8 $2\pi - q$

9 a $\pi + \alpha, 2\pi - \alpha$
b $\frac{\pi}{2} - \alpha, \frac{3\pi}{2} + \alpha$

10 a $\pi - \beta, \beta - \pi$ b $\frac{\pi}{2} - \beta, \beta - \frac{3\pi}{2}$

11 a $2\pi - \gamma, 3\pi - \gamma$ b $\frac{3\pi}{2} - \gamma, \frac{5\pi}{2} - \gamma$

12 b 45.07

13 b 1.113

14 When $t = 0$, $x_A = x_B = 0$;
when $t = 1.29$, $x_A = x_B = 0.48$

15 b 0.94

Exercise 1F

1 a $\sin(11\pi t) - \sin(3\pi t)$
b $\frac{1}{2}(\sin 60° + \sin 40°)$
c $\frac{3}{2}\left(\sin(\pi x) + \sin\left(\frac{\pi x}{3}\right)\right)$
d $\sin(A) + \sin(B + C)$

2 $\cos(3\theta) - \cos(5\theta)$

3 $\sin A - \sin B$

5 a $2 \sin 50° \cos 16°$ b $2 \cos 50° \cos 16°$
c $2 \sin 16° \cos 50°$ d $-2 \sin 50° \sin 16°$

6 a $2 \sin(5A) \cos(3A)$ b $2 \cos\left(\frac{5x}{2}\right) \cos\left(\frac{3x}{2}\right)$
c $2 \sin(x) \cos(5x)$ d $-2 \sin(4A) \sin(A)$

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11 a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$

d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

12 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$

d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

Chapter 1 review

Short-response questions

1 a $\frac{7}{\sqrt{113}}$

b $\frac{9}{2}$

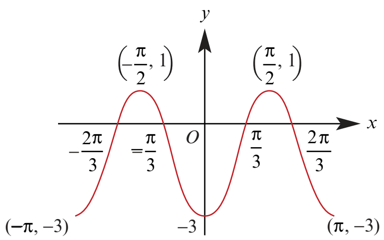
2 a $\frac{1}{\sqrt{2}}$ **b** $-\frac{4}{5}$ **c** $210°$ is a possible answer

3 $\tan^{-1}(3\sqrt{2})$

4 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{\pi}{4}, \frac{5\pi}{4}$

5 a $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$ or $\frac{2\pi}{3}$

b



c $x \in \left[-\pi, -\frac{2\pi}{3}\right) \cup \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right]$

6 a $\frac{7}{25}$ **b** $\frac{24}{25}$ **c** $\frac{5}{3}$ **d** $\frac{4}{3}$

7 a $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ **b** $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

c $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$

d $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ **e** $\frac{\pi}{2}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, -\frac{5\pi}{6}$

f $0, 2\pi, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$

8 a $\frac{7\pi}{6}, \frac{11\pi}{6}, \sin^{-1}\left(\frac{1}{3}\right), \pi - \sin^{-1}\left(\frac{1}{3}\right)$

b $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{\pi}{4}, \frac{5\pi}{4}$

d $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

9 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{2\sqrt{3}}{3}$ **c** 2

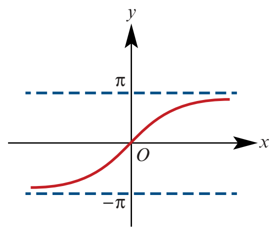
d 2 **e** 1 **f** $-\sqrt{3}$

10 a $-p$ **b** $-p$ **c** $\frac{1}{p}$ **d** $-\frac{1}{p}$ **e** $-p$

11 a $\frac{\pi}{3}$ **b** $\frac{1}{2}$ **c** $\frac{2\pi}{3}$

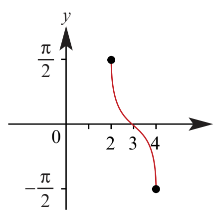
d $\frac{2\pi}{3}$ **e** $\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{2}}{2}$

12 a



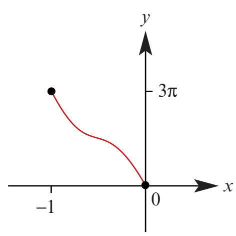
Domain $= \mathbb{R}$
Range $= (-\pi, \pi)$

b



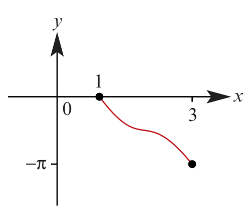
Domain $= [2, 4]$
Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

c



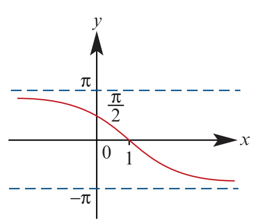
Domain $= [-1, 0]$
Range $= [0, 3\pi]$

d



Domain $= [1, 3]$
Range $= [-\pi, 0]$

e



Domain $= \mathbb{R}$
Range $= (-\pi, \pi)$

13 $0, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \pi$

15 a i x

ii $\sqrt{1 - x^2}$

iii $\frac{x}{\sqrt{1 - x^2}}$

iv $2x$

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v $\sqrt{1-4x^2}$

vi $\dfrac{2x}{\sqrt{1-4x^2}}$

b **i** $2x\sqrt{1-x^2} - x\sqrt{1-4x^2}$

ii $\sqrt{(1-4x^2)(1-x^2)} + 2x^2$

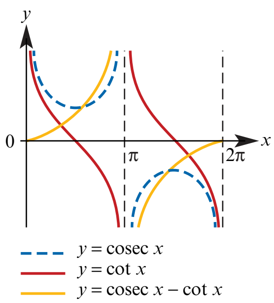
iii $\dfrac{2x\sqrt{1-x^2} - x\sqrt{1-4x^2}}{\sqrt{(1-4x^2)(1-x^2)} + 2x^2}$

iv $2x\sqrt{1-x^2}$ **v** $1-2x^2$

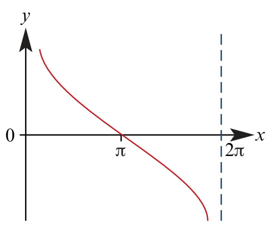
vi $\dfrac{2x\sqrt{1-x^2}}{1-2x^2}$

c $\angle B_2AB_1 = 0.34$, $2\alpha = 0.61$

16 **a**



c $y = \cot\left(\dfrac{x}{2}\right)$, $y = \text{cosec}(x) + \cot(x)$



$y = \text{cosec}(x) + \cot(x)$ not defined when $x = \pi$

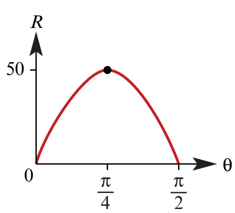
d **ii** $\cot\left(\dfrac{\pi}{8}\right) = 1 + \sqrt{2}$, $\cot\left(\dfrac{\pi}{12}\right) = 2 + \sqrt{3}$

iii $\dfrac{1}{\sqrt{4 + 2\sqrt{2}}}$

e $\cot\left(\dfrac{\theta}{2}\right) - \cot(4\theta)$

17 **a** **i** $100 \sin\theta \cos\theta$

ii



iii 50

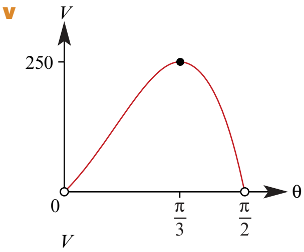
iv $\dfrac{\pi}{4}$

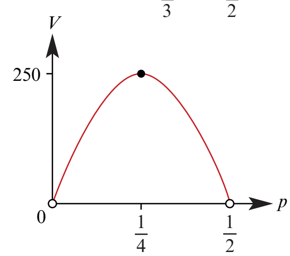
b **ii** $a = 2000$, $b = -4000$

iii $V = 2000p - 4000p^2$

iv $0 < p < \dfrac{1}{2}$

v

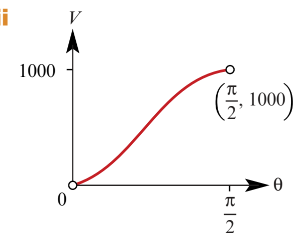




vi Max volume $= 250$ when $p = \dfrac{1}{4}$, $\theta = \dfrac{\pi}{3}$

c **i** $V = 1000 \sin^2\theta$, for $0 < \theta < \dfrac{\pi}{2}$

ii



iii V is an increasing function: as the angle θ gets larger, so does the volume of the cuboid

18 **a** Each triangle has a right angle, and angle CAD is common to both triangles

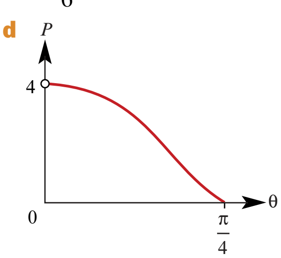
b $(\cos(2\theta), \sin(2\theta))$

c **i** $2\cos\theta$ **ii** $2\sin\theta$

19 **b** $p = 8\cos^3\theta - 4\cos\theta$

c **iii** $\dfrac{\pi}{6}$ **iv** 1

d



e $\dfrac{\pi}{4}$

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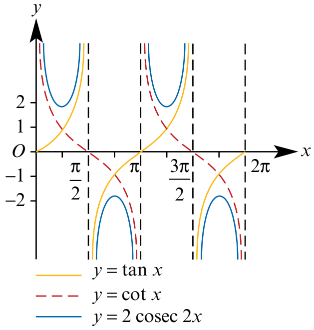
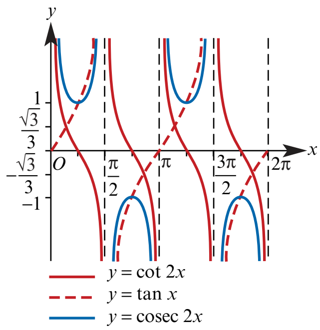
20 a ii $x = \pm\dfrac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$

iii



b ii $x = n\pi \pm \dfrac{\pi}{6},\ n \in \mathbb{Z}$

iii

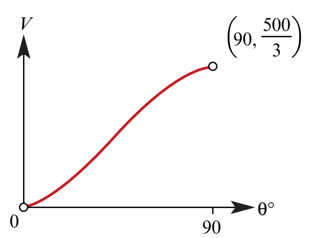


21 a i $\angle BAE = 72°$, $\angle AEC = 72°$, $\angle ACE = 72°$

ii $36°$

e $\dfrac{\sqrt{5}-1}{4}$

22 a ii



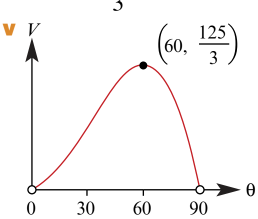
iii V is an increasing function: as the angle θ gets larger, so does the volume of the pyramid

b ii $\theta \in (0, 90)$

iii $V = -\dfrac{2000}{3}a^2 + \dfrac{1000}{3}a$

iv $V_{max} = \dfrac{125}{3}$ when $\theta = 60$

v



23 a i $V = \dfrac{500}{3}\cos(\theta°)\sin^2(\theta°)$

ii $V_{max} = 64.15$ when $\theta = 54.74$

b ii $\theta \in (0, 90)$

c $V_{max} = 24.69$ when $a = 0.67$, $\theta = 48.19$

Multiple-choice questions

1 C	**2** A
3 C	**4** C
5 C	**6** D
7 D	**8** A
9 A	**10** D
11 D	**12** B
13 B	**14** B
15 D	**16** B
17 D	**18** C
19 D	**20** C

Chapter 2

Exercise 2A

1 a $(x-2)^2 + (y-3)^2 = 1$

b $(x+3)^2 + (y-4)^2 = 25$

c $x^2 + (y+5)^2 = 25$

d $(x-3)^2 + y^2 = 2$

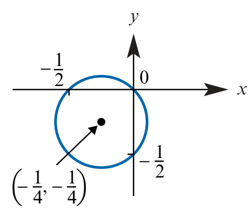
2 a Centre $(-2, 3)$; radius 1

b Centre $(1, 2)$; radius 2

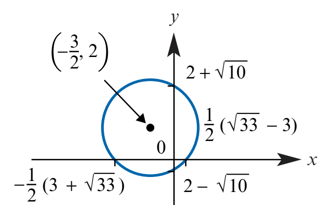
c Centre $\left(\dfrac{3}{2}, 0\right)$; radius $\dfrac{3}{2}$

d Centre $(-2, 5)$; radius 2

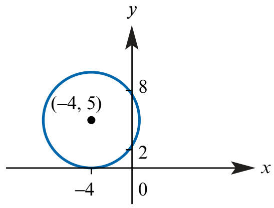
3 a $\left(x + \dfrac{1}{4}\right)^2 + \left(y + \dfrac{1}{4}\right)^2 = \dfrac{1}{8}$



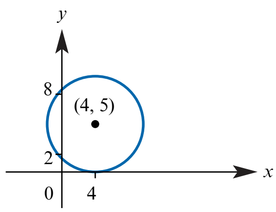
b $\left(x + \dfrac{3}{2}\right)^2 + (y-2)^2 = \dfrac{49}{4}$



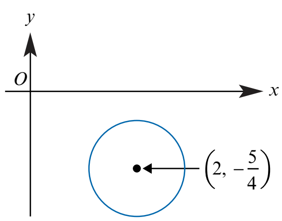
c $(x+4)^2+(y-5)^2=25$



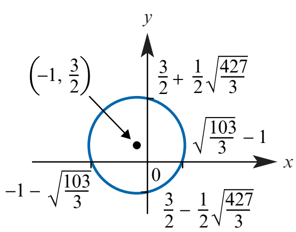
d $(x-4)^2+(y-5)^2=25$



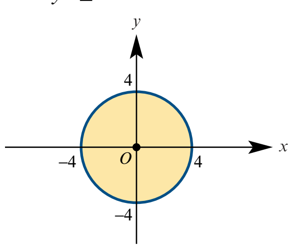
e $(x-2)^2+\left(y+\dfrac{5}{4}\right)^2=\dfrac{9}{16}$



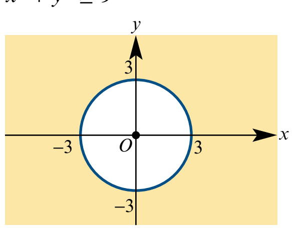
f $(x+1)^2+\left(y-\dfrac{3}{2}\right)^2=\dfrac{439}{12}$



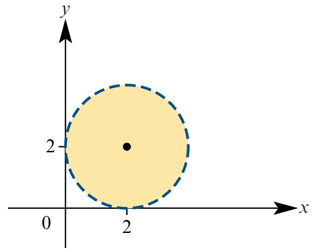
4 a $x^2+y^2\le 16$



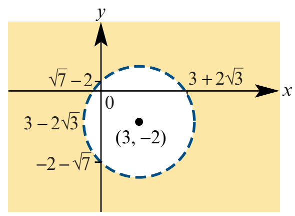
b $x^2+y^2\ge 9$



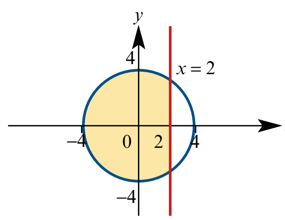
c $(x-2)^2+(y-2)^2<4$



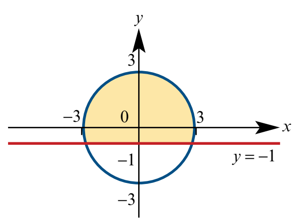
d $(x-3)^2+(y+2)^2>16$



e $x^2+y^2\le 16$ and $x\le 2$



f $x^2+y^2\le 9$ and $y\ge -1$



5 Centre $(5, 3)$; radius $\sqrt{10}$

6 $(x-2)^2+(y+3)^2=9$

7 $(x-5)^2+(y-4)^2=13$

8 a First circle: centre $\left(\dfrac{15}{2}, \dfrac{19}{2}\right)$; radius $\dfrac{5\sqrt{2}}{2}$

Second circle: centre $(5, 7)$; radius 5

b $(5, 12)$, $(10, 7)$

9 a $\left(\dfrac{5\sqrt{2}}{2}, \dfrac{5\sqrt{2}}{2}\right), \left(\dfrac{-5\sqrt{2}}{2}, \dfrac{-5\sqrt{2}}{2}\right)$

b $(\sqrt{5}, 2\sqrt{5}), (-\sqrt{5}, -2\sqrt{5})$

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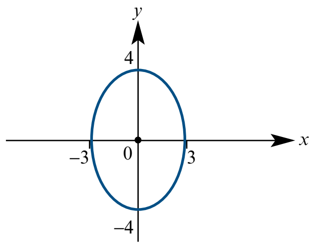
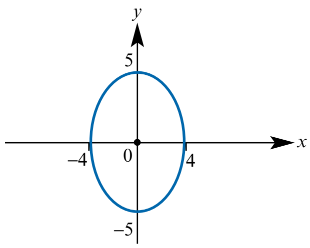
Exercise 2B

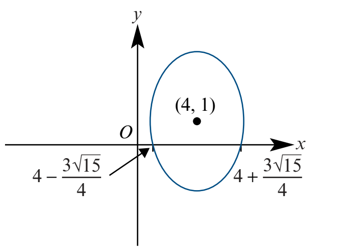
1 a $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, centre $(0, 0)$



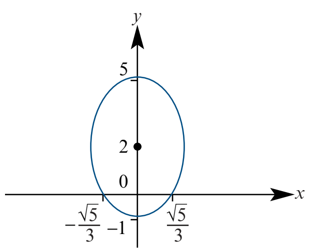
b $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$, centre $(0, 0)$



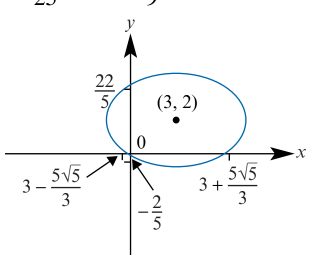
c $\dfrac{(x-4)^2}{9} + \dfrac{(y-1)^2}{16} = 1$, centre $(4, 1)$



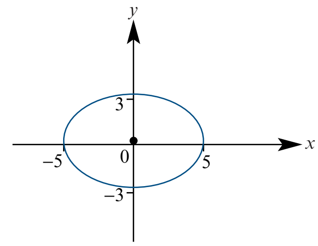
d $x^2 + \dfrac{(y-2)^2}{9} = 1$, centre $(0, 2)$



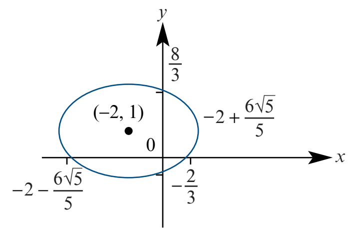
e $\dfrac{(x-3)^2}{25} + \dfrac{(y-2)^2}{9} = 1$, centre $(3, 2)$



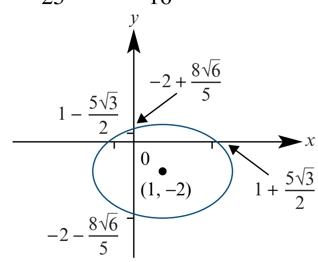
f $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, centre $(0, 0)$



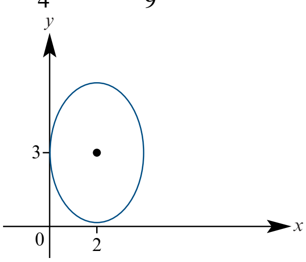
g $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{5} = 1$, centre $(-2, 1)$



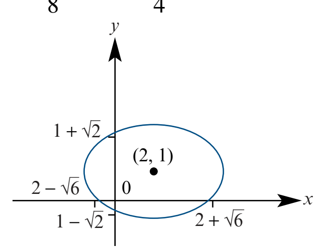
h $\dfrac{(x-1)^2}{25} + \dfrac{(y+2)^2}{16} = 1$, centre $(1, -2)$



i $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$, centre $(2, 3)$



j $\dfrac{(x-2)^2}{8} + \dfrac{(y-1)^2}{4} = 1$, centre $(2, 1)$



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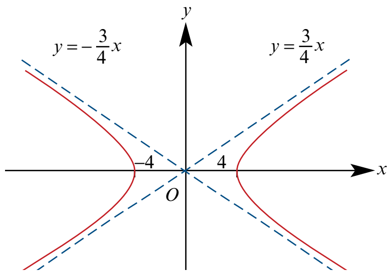
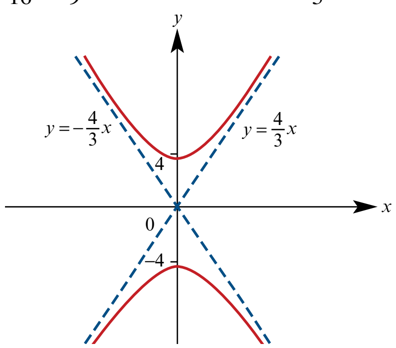
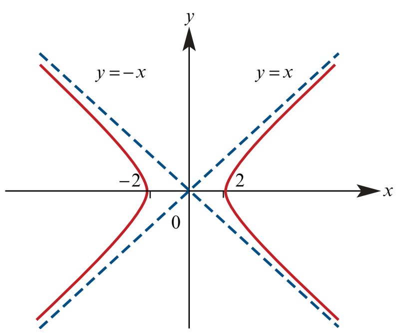
2 a $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, asymptotes $y = \pm\dfrac{3}{4}x$



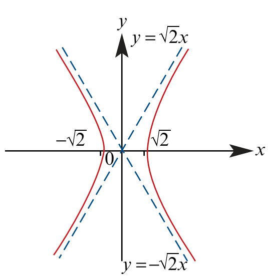
b $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$, asymptotes $y = \pm\dfrac{4}{3}x$



c $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$, asymptotes $y = \pm x$

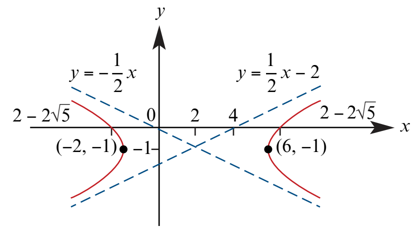


d $\dfrac{x^2}{2} - \dfrac{y^2}{4} = 1$, asymptotes $y = \pm\sqrt{2}x$



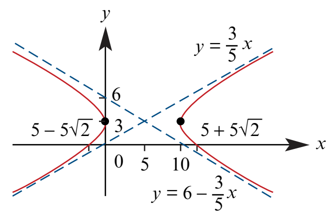
e $\dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{4} = 1$,

asymptotes $y = \dfrac{1}{2}x - 2$, $y = -\dfrac{1}{2}x$



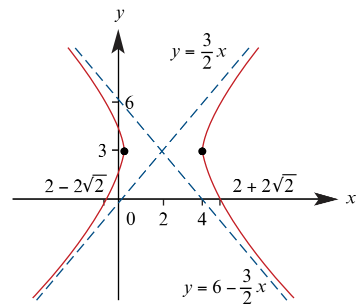
f $\dfrac{(x-5)^2}{25} - \dfrac{(y-3)^2}{9} = 1$,

asymptotes $y = \dfrac{3}{5}x$, $y = 6 - \dfrac{3}{5}x$



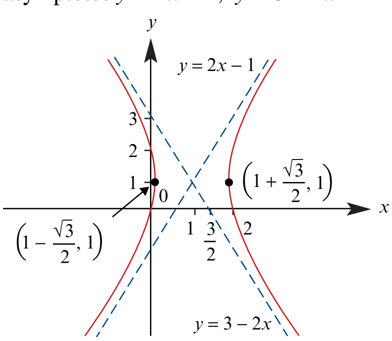
g $\dfrac{(x-2)^2}{4} - \dfrac{(y-3)^2}{9} = 1$,

asymptotes $y = \dfrac{3}{2}x$, $y = 6 - \dfrac{3}{2}x$



h $\dfrac{4(x-1)^2}{3} - \dfrac{(y-1)^2}{3} = 1$,

asymptotes $y = 2x - 1$, $y = 3 - 2x$



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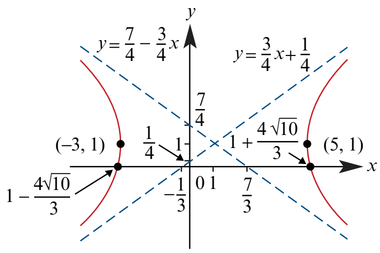
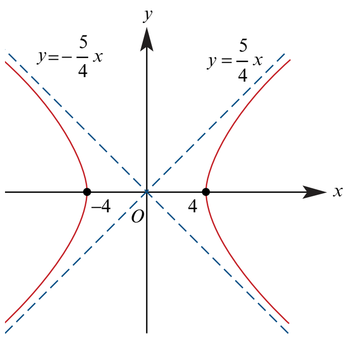
i $\dfrac{(x-1)^2}{16} - \dfrac{(y-1)^2}{9} = 1$,

asymptotes $y = \dfrac{3}{4}x + \dfrac{1}{4}$, $y = \dfrac{7}{4} - \dfrac{3}{4}x$



j $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$, asymptotes $y = \pm\dfrac{5}{4}x$



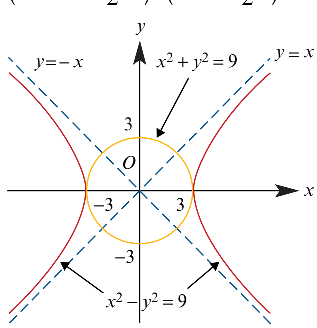
3 a $\left(\dfrac{2\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right), \left(\dfrac{-2\sqrt{3}}{3}, \dfrac{-\sqrt{3}}{3}\right)$

b $\left(\sqrt{2}, \dfrac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \dfrac{-\sqrt{2}}{2}\right)$

5 $\left(\dfrac{-6\sqrt{13}}{13}, \dfrac{-6\sqrt{13}}{13}\right), \left(\dfrac{6\sqrt{13}}{13}, \dfrac{6\sqrt{13}}{13}\right),$
$\left(\dfrac{-6\sqrt{13}}{13}, \dfrac{6\sqrt{13}}{13}\right), \left(\dfrac{6\sqrt{13}}{13}, \dfrac{-6\sqrt{13}}{13}\right)$

6 $\left(-2\sqrt{2}, \dfrac{-5\sqrt{2}}{2}\right), \left(2\sqrt{2}, \dfrac{5\sqrt{2}}{2}\right)$

7



8 a $h = 1, k = 5$ **b** $a = 1, b = 2$

Exercise 2C

1 $x^2 + y^2 = 4$, dom $= [-2, 2]$, ran $= [-2, 2]$

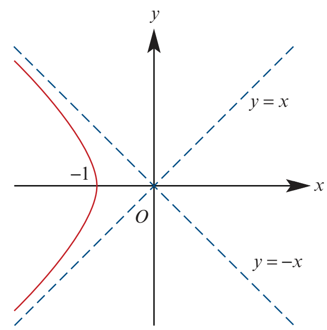
2 a $y^2 = 16x$
b $x = 4$
c $32\sqrt{2}$

3 $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{4} = 1$;
ellipse with centre $(2, 3)$

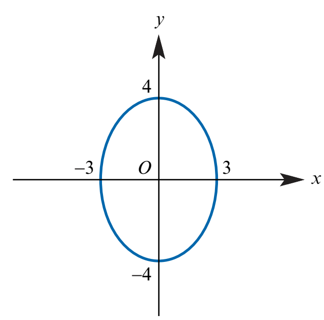
4 $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$, $x \le -2$;
left branch of hyperbola with centre $(0, 0)$ and x-axis intercept $(-2, 0)$; asymptotes $y = \pm\dfrac{3x}{2}$

5 a $x^2 + y^2 = 16$ **b** $x^2 + y^2 = 4$
c $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ **d** $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$
e $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$ **f** $y = x^2 - 2x - 3$
g $y = \dfrac{1}{x-2}$ **h** $y = x + 2$, $x \ge -1$
i $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$

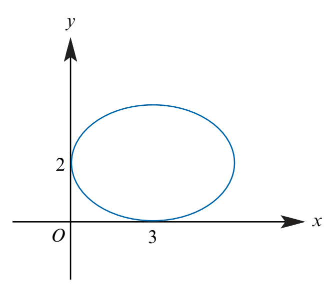
6 a $x^2 - y^2 = 1$, $x \in (-\infty, -1]$



b $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$



c $\dfrac{(x-3)^2}{9} + \dfrac{(y-2)^2}{4} = 1$



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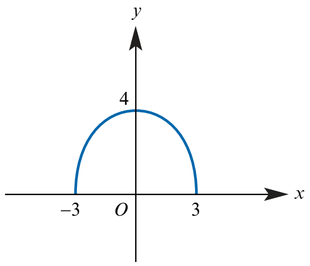
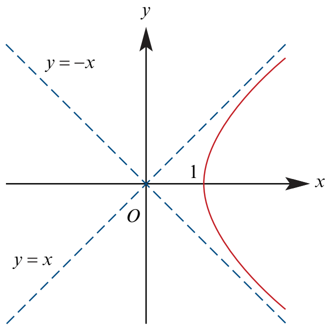
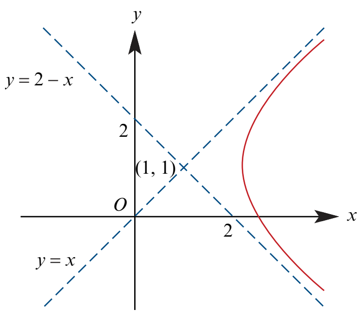
d $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1,\ x \in [-3, 3],\ y \in [0, 4]$



e $x^2 - y^2 = 1,\ x \in [1, \infty)$



f $(x-1)^2 - (y-1)^2 = 1,\ x \in [2, \infty)$



7 a $P = (-1, -\sqrt{3})$
b $\sqrt{3}x + 3y = -4\sqrt{3}$

8 a $x = 4\cos t \qquad y = 4\sin t$
b $x = 3\sec t \qquad y = 2\tan t$
c $x = 3\cos t + 1 \qquad y = 3\sin t - 2$
d $x = 9\cos t + 1 \qquad y = 6\sin t - 3$
Other answers are possible.

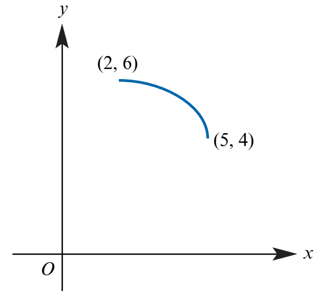
9 $a = 1,\ b = 2,\ c = 3,\ d = 2$

10 $x = 4\cos t,\ y = 3\sin t$

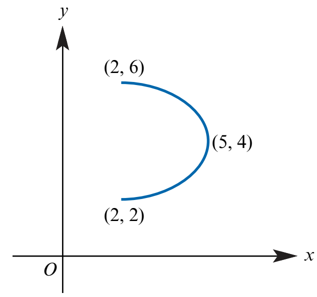
11 a $x = 2\cos t,\ y = 6\sin t$
b $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$

12 a $x = -2\cos\left(\dfrac{t}{2}\right),\ y = 2 + 3\sin\left(\dfrac{t}{2}\right)$
b $\dfrac{x^2}{4} + \dfrac{(y-2)^2}{9} = 1$

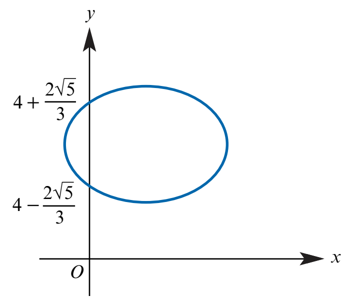
13 a dom $= [2, 5]$, ran $= [4, 6]$



b dom $= [2, 5]$, ran $= [2, 6]$



c dom $= [-1, 5]$, ran $= [2, 6]$



Chapter 2 review

Short-response questions

1 $\dfrac{(x+2)^2}{4} + \dfrac{(y-3)^2}{16} = 1$
2 $y = 3x + 2,\ y = -3x + 2$
3 $\dfrac{(x-4)^2}{9} + (y+6)^2 = 1$
4 $x^2 + (y-2)^2 = 4$
5 $a = 1,\ c = 2,\ b = d = 3$
6 Centre $(-4, 6)$, radius 7
7 $(\pm 9, 0), (0, \pm 3)$
8 a i $[-\sqrt{2}, \sqrt{2}]$ **ii** $[-3 - \sqrt{5}, -3 + \sqrt{5}]$
iii $(0, -3)$
b 2, 3, 1, 2 **c** $\left(\dfrac{37}{13}, \dfrac{11}{13}\right)$ **d** $\left(0, \dfrac{48}{13}\right)$
e $\left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{35}{26}\right)^2 = \dfrac{1945}{338}$
9 d Centre $(2, 2)$ and radius 2; centre $(10, 10)$ and radius 10

e Gradient undefined; gradient $\frac{3}{4}$

f $y = 4$; $y = \frac{-4}{3}x + \frac{20}{3}$

10 a $y = (\tan t)x$

b $(-a\cos t, -a\sin t)$

c $y - a\sin t = -\frac{\cos t}{\sin t}(x - a\cos t)$

d $A\left(\frac{a}{\cos t}, 0\right)$, $B\left(0, \frac{a}{\sin t}\right)$

e Area $= \frac{a^2}{2\sin t\cos t} = \frac{a^2}{\sin(2t)}$;

Minimum when $t = \frac{\pi}{4}$

Multiple-choice questions

1 D **2** D **3** D **4** A **5** B
6 B **7** B **8** D **9** C **10** B
11 C **12** B **13** C **14** A **15** B
16 D

Chapter 3

Exercise 3A

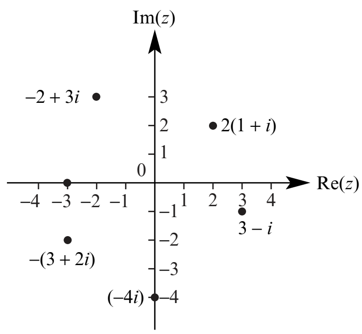
1 a 6 **b** -7 **c** 13

2 a $5i$ **b** $3\sqrt{3}i$ **c** $-5i$
d $13i$ **e** $5\sqrt{2}i$ **f** $-2\sqrt{3}$
g $-1 + 2i$ **h** 4 **i** 0

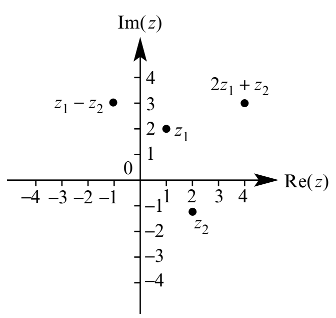
3 a $x = 5, y = 0$ **b** $x = 0, y = 2$
c $x = 0, y = 0$ **d** $x = 9, y = -4$
e $x = -2, y = -2$ **f** $x = 13, y = 6$

4 a $5 + i$ **b** $4 + 4i$ **c** $5 - 5i$
d $4 - 3i$ **e** $-1 + i$ **f** 2
g 2 **h** 1 **i** $3 - 2i$

5



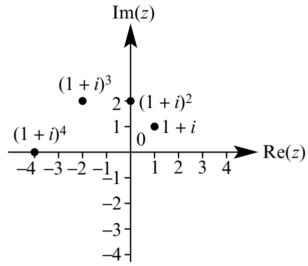
6



7 a $11 + 3i$ **b** $-23 + 41i$ **c** 13
d $-8 + 6i$ **e** $3 - 4i$ **f** $-2 + 2i$
g 1 **h** $5 - 6i$ **i** -1

8 a $x = 4, y = -3$ **b** $x = -2, y = 5$
c $x = -3$ **d** $x = 3, y = -3$ or $x = -3, y = 3$
e $x = 3, y = 2$

9 a



b Anticlockwise turn by $\frac{\pi}{4}$ about the origin; distance from origin increases by factor $\sqrt{2}$

10 a $\overrightarrow{PQ} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \overrightarrow{OR}$ **b** $|\overrightarrow{PQ}| = \sqrt{10}$

Exercise 3B

1 a $\sqrt{3}, \sqrt{3}$ **b** $-8i$, 8 **c** $4 + 3i$, 5
d $-1 + 2i$, $\sqrt{5}$ **e** $4 - 2i$, $2\sqrt{5}$
f $-3 + 2i$, $\sqrt{13}$

2 a i **b** $\frac{3}{10} - \frac{1}{10}i$ **c** $-3 + 4i$
d $\frac{17}{5} + \frac{1}{5}i$ **e** $\frac{-1-\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i$
f $4 + i$

4 a $5 - 5i$ **b** $6 + i$ **c** $2 + 3i$
d $\frac{2 - i}{5}$ **e** $-8i$ **f** $8 + 6i$

5 a $a^2 + b^2$ **b** $\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}i$
c $2a$ **d** $2bi$ **e** $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$
f $\frac{a^2 - b^2}{a^2 + b^2} - \frac{2ab}{a^2 + b^2}i$

6 $\frac{3}{4}$

7 $0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$

Exercise 3C

1 a $3; \pi$ **b** $5; \frac{\pi}{2}$ **c** $\sqrt{2}; \frac{3\pi}{4}$
d $2; \frac{\pi}{6}$ **e** $4; -\frac{\pi}{3}$ **f** $16; -\frac{2\pi}{3}$

2 a 1.18 **b** 2.06 **c** -2.50
d -0.96 **e** 0.89 **f** -1.98

3 a $\frac{5\pi}{3}$ **b** $\frac{3\pi}{2}$ **c** $\frac{5\pi}{6}$
d $\frac{\pi}{4}$ **e** $-\frac{11\pi}{6}$ **f** $-\frac{3\pi}{2}$

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4 a $-\frac{3\pi}{4}$ **b** $\frac{5\pi}{6}$ **c** $\frac{\pi}{8}$ **d** $-\frac{\pi}{2}$

5 a $\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **b** $\operatorname{cis}\left(-\frac{\pi}{3}\right)$

c $\sqrt{6}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **d** $\frac{2}{3}\operatorname{cis}\left(\frac{\pi}{6}\right)$

e $2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{6}\right)$ **f** $4\operatorname{cis}\left(\frac{5\pi}{6}\right)$

6 a $-\sqrt{2}+\sqrt{2}i$ **b** $\frac{5}{2}-\frac{5\sqrt{3}}{2}i$ **c** $2+2i$

d $-\frac{3\sqrt{3}}{2}-\frac{3}{2}i$ **e** $6i$ **f** -4

8 a $2\operatorname{cis}\left(-\frac{3\pi}{4}\right)$ **b** $7\operatorname{cis}\left(\frac{2\pi}{3}\right)$

c $3\operatorname{cis}\left(\frac{\pi}{3}\right)$ **d** $5\operatorname{cis}\left(\frac{\pi}{4}\right)$

Exercise 3D

1 $(2\sqrt{3}-3)+(3\sqrt{3}+2)i$

2 a $12\operatorname{cis}\left(-\frac{7\pi}{12}\right)$ **b** $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$

c $\frac{7}{6}\operatorname{cis}\left(-\frac{\pi}{15}\right)$ **d** $8\operatorname{cis}\left(-\frac{19\pi}{20}\right)$ **e** $-\frac{1}{8}$

3 a $\operatorname{Arg}(z_1z_2)=\frac{7\pi}{12}$; $\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)=\frac{7\pi}{12}$;
$\operatorname{Arg}(z_1z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$

b $\operatorname{Arg}(z_1z_2)=\frac{7\pi}{12}$; $\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)=\frac{-17\pi}{12}$;
$\operatorname{Arg}(z_1z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)+2\pi$

c $\operatorname{Arg}(z_1z_2)=\frac{-5\pi}{6}$; $\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)=\frac{7\pi}{6}$;
$\operatorname{Arg}(z_1z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)-2\pi$

5 a $\frac{\pi}{4}$ **b** $\frac{-3\pi}{4}$ **c** $\frac{-\pi}{4}$

6 a i $\sec\theta\operatorname{cis}\theta$

ii $\operatorname{cosec}\theta\operatorname{cis}\left(\frac{\pi}{2}-\theta\right)$

iii $\frac{1}{\sin\theta\cos\theta}\operatorname{cis}\theta=\operatorname{cosec}\theta\sec\theta\operatorname{cis}\theta$

b i $\sec^2\theta\operatorname{cis}(2\theta)$

ii $\sin^3\theta\operatorname{cis}\left(3\theta-\frac{3\pi}{2}\right)$

iii $\operatorname{cosec}\theta\sec\theta\operatorname{cis}(-\theta)$

7 a $(\sqrt{6}-\sqrt{2})+(\sqrt{6}+\sqrt{2})i$

b $u=2\operatorname{cis}\frac{\pi}{4}$ and $v=2\operatorname{cis}\frac{\pi}{6}$

c $4\cos\left(\frac{5\pi}{12}\right)+4i\sin\left(\frac{5\pi}{12}\right)$

d $\cos\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$

and $\sin\left(\frac{5\pi}{12}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$

Exercise 3E

1 a $8\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\frac{8}{27}\operatorname{cis}\left(\frac{\pi}{8}\right)$

c $27\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $-32i$ **e** -216

f $1024\operatorname{cis}\left(-\frac{\pi}{12}\right)$ **g** $\frac{27}{4}\operatorname{cis}\left(-\frac{\pi}{20}\right)$

2 a $64\operatorname{cis}0=64$ **b** $\frac{\sqrt{2}}{8}\operatorname{cis}\left(\frac{-3\pi}{4}\right)$

c $128\operatorname{cis}\left(\frac{-2\pi}{3}\right)$ **d** $\frac{\sqrt{3}}{72}\operatorname{cis}\left(\frac{-\pi}{2}\right)=-\frac{\sqrt{3}}{72}i$

e $\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$ **f** $\frac{64\sqrt{3}}{3}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

g $\frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{\pi}{2}\right)=\frac{\sqrt{2}}{2}i$ **h** $\frac{1}{4}\operatorname{cis}\left(-\frac{2\pi}{15}\right)$

i $8\sqrt{2}\operatorname{cis}\left(\frac{11\pi}{12}\right)$

3 b i $\operatorname{cis}\left(\frac{3\pi}{2}-7\theta\right)$ **ii** i

iii $\operatorname{cis}(4\theta)$ **iv** $\operatorname{cis}(\pi-\theta-\varphi)$

4 b i $\operatorname{cis}(-5\theta)$ **ii** $\operatorname{cis}(3\theta)$ **iii** 1

iv $\operatorname{cis}\left(\frac{\pi}{2}-2\theta\right)$

5 b i $\operatorname{cis}(6\theta-3\pi)$ **ii** $\operatorname{cis}(\pi-2\theta)$

iii $\operatorname{cis}(\theta-\pi)$ **iv** $-i$

7 a $(\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta)$
$+i(4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta)$

Exercise 3F

1 a $(z+4i)(z-4i)$ **b** $(z+\sqrt{5}i)(z-\sqrt{5}i)$

c $(z+1+2i)(z+1-2i)$

d $\left(z-\frac{3}{2}+\frac{\sqrt{7}}{2}i\right)\left(z-\frac{3}{2}-\frac{\sqrt{7}}{2}i\right)$

e $2\left(z-2+\frac{\sqrt{2}}{2}i\right)\left(z-2-\frac{\sqrt{2}}{2}i\right)$

f $3\left(z+1+\frac{\sqrt{3}}{3}i\right)\left(z+1-\frac{\sqrt{3}}{3}i\right)$

g $3\left(z+\frac{1}{3}+\frac{\sqrt{5}}{3}i\right)\left(z+\frac{1}{3}-\frac{\sqrt{5}}{3}i\right)$

h $2\left(z-\frac{1}{4}+\frac{\sqrt{23}}{4}i\right)\left(z-\frac{1}{4}-\frac{\sqrt{23}}{4}i\right)$

2 a $5i,\ -5i$ **b** $2\sqrt{2}i,\ -2\sqrt{2}i$

c $2+i,\ 2-i$

d $-\frac{7}{6}+\frac{\sqrt{11}}{6}i,\ -\frac{7}{6}-\frac{\sqrt{11}}{6}i$

e $1-\sqrt{2}i,\ 1+\sqrt{2}i$

f $\frac{3}{10}+\frac{\sqrt{11}}{10}i,\ \frac{3}{10}-\frac{\sqrt{11}}{10}i$

g $-i,\ -1-i$ **h** $i,\ -1-i$

Exercise 3G

1 a $(z-5)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

b $(z+2)\left(z-\frac{3}{2}+\frac{\sqrt{11}}{2}i\right)\left(z-\frac{3}{2}-\frac{\sqrt{11}}{2}i\right)$

c $3(z-4)\left(z-\frac{1}{6}+\frac{\sqrt{11}}{6}i\right)\left(z-\frac{1}{6}-\frac{\sqrt{11}}{6}i\right)$

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d $2(z+3)\left(z-\frac{3}{4}+\frac{\sqrt{31}}{4}i\right)\left(z-\frac{3}{4}-\frac{\sqrt{31}}{4}i\right)$

e $(z+i)(z-i)(z-2+i)$

2 b $z-1+i$ **c** $(z+6)(z-1+i)(z-1-i)$

3 b $z+2+i$ **c** $(2z+1)(z+2+i)(z+2-i)$

4 b $z-1-3i$

c $(z-1+3i)(z-1-3i)(z+1+i)(z+1-i)$

5 a $(z+3)(z-3)(z+3i)(z-3i)$

b $(z+2)(z-2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)$
$(z+1+\sqrt{3}i)(z+1-\sqrt{3}i)$

6 a $(z-i)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

b $(z+i)(z-1+\sqrt{2})(z-1-\sqrt{2})$

c $(z-2i)(z-3)(z+1)$

d $2(z-i)\left(z+\frac{1}{4}+\frac{\sqrt{41}}{4}\right)\left(z+\frac{1}{4}-\frac{\sqrt{41}}{4}\right)$

7 a 8 **b** -4 **c** -6

8 a $3,\ -2\pm\sqrt{2}i$ **b** $5,\ \frac{1\pm\sqrt{23}i}{2}$

c $-1,\ \frac{5\pm\sqrt{7}i}{2}$ **d** $-2,\ 3,\ \frac{1\pm\sqrt{23}i}{2}$

9 a $a=0, b=4$ **b** $a=-6, b=13$

c $a=2, b=10$

10 a $1-3i,\ \frac{1}{3}$ **b** $-2+i,\ 2\pm\sqrt{2}i$

11 $P(x)=-2x^3+10x^2-18x+10$;
$x=1$ or $x=2\pm i$

12 $a=6, b=-8$

13 a $z^2-4z+5,\ a=-7,\ b=6$

b $z=2\pm i$ or $z=-\frac{1}{2}$

14 a $P(1+i)=(-4a+d-2)+2(a-1)i$

b $a=1, d=6$

c $z=1\pm i$ or $z=-1\pm\sqrt{2}i$

15 $p=-(5+4i), q=1+7i$

16 $z=1+i$ or $z=2$

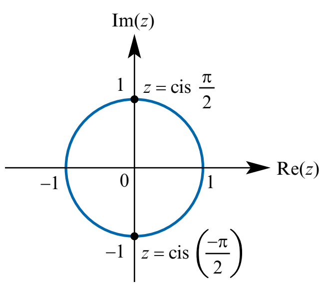
17 a $3+i$ **b** $2i,\ \pm\sqrt{6}$

c $1,\ \pm\sqrt{6}i$ **d** $2,\ -\frac{1}{2}\pm\frac{\sqrt{15}}{2}i$

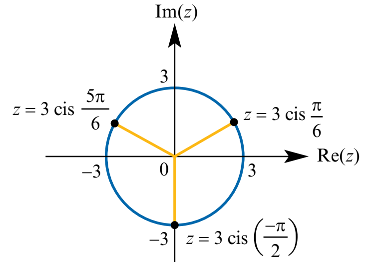
e $\frac{\sqrt{2}}{4}\pm\frac{\sqrt{14}}{4}i$ **f** $0,\ -1\pm 2\sqrt{2}i$

Exercise 3H

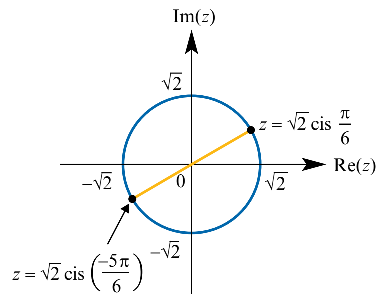
1 a $z=i$ or $z=-i$



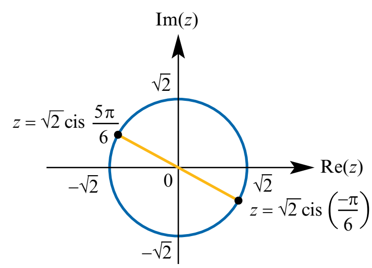
b $z=3\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$, $z=3\left(\frac{-\sqrt{3}}{2}+\frac{1}{2}i\right)$ or
$z=-3i$



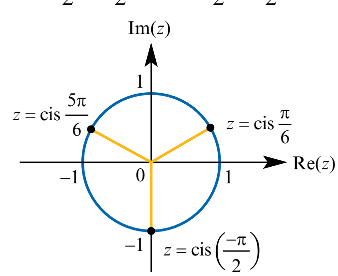
c $z=\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$ or $z=\sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$



d $z=\sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)$ or $z=\sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$



e $z=\frac{\sqrt{3}}{2}+\frac{1}{2}i$, $z=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ or $z=-i$



f $z=\frac{\sqrt{3}}{2}-\frac{1}{2}i$, $z=i$ or $z=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$

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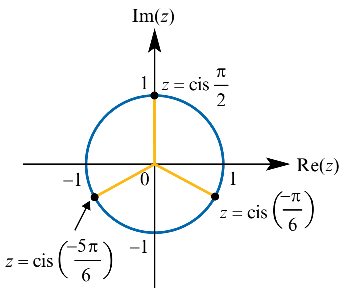


2 a $2\operatorname{cis}\left(\frac{-\pi}{12}\right), 2\operatorname{cis}\left(\frac{7\pi}{12}\right), 2\operatorname{cis}\left(\frac{-3\pi}{4}\right)$

b $2\operatorname{cis}\left(\frac{\pi}{4}\right), 2\operatorname{cis}\left(\frac{11\pi}{12}\right), 2\operatorname{cis}\left(\frac{-5\pi}{12}\right)$

c $2\operatorname{cis}\left(\frac{-5\pi}{18}\right), 2\operatorname{cis}\left(\frac{7\pi}{18}\right), 2\operatorname{cis}\left(\frac{-17\pi}{18}\right)$

d $2\operatorname{cis}\left(\frac{-\pi}{18}\right), 2\operatorname{cis}\left(\frac{11\pi}{18}\right), 2\operatorname{cis}\left(\frac{-13\pi}{18}\right)$

e $5\operatorname{cis}\left(\frac{-\pi}{6}\right), 5\operatorname{cis}\left(\frac{\pi}{2}\right), 5\operatorname{cis}\left(\frac{-5\pi}{6}\right)$

f $2^{\frac{1}{6}}\operatorname{cis}\left(\frac{\pi}{4}\right), 2^{\frac{1}{6}}\operatorname{cis}\left(\frac{11\pi}{12}\right), 2^{\frac{1}{6}}\operatorname{cis}\left(\frac{-5\pi}{12}\right)$

3 a $a^2 - b^2 = 3, 2ab = 4$

b $a = \pm 2, b = \pm 1$;
square roots of $3 + 4i$ are $\pm(2 + i)$

4 a $\pm(1 - 4i)$

b $\pm\frac{\sqrt{2}}{2}(7 + i)$

c $\pm(1 + 2i)$

d $\pm(3 + 4i)$

5 $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{6}\right), \sqrt{2}\operatorname{cis}\left(\frac{-5\pi}{6}\right), \sqrt{2}\operatorname{cis}\left(\frac{-\pi}{6}\right),$
$\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{6}\right)$

6 $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ or $z = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$;
$z^2 - i = \left(z - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\left(z + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

7 $z = \operatorname{cis}\frac{\pi}{8}, \operatorname{cis}\frac{3\pi}{8}, \operatorname{cis}\frac{5\pi}{8}, \operatorname{cis}\frac{7\pi}{8}, \operatorname{cis}\frac{9\pi}{8},$
$\operatorname{cis}\frac{11\pi}{8}, \operatorname{cis}\frac{13\pi}{8}$ or $\operatorname{cis}\frac{15\pi}{8}$;

$$z^8 + 1 = \left(z - \operatorname{cis}\frac{\pi}{8}\right)\left(z - \operatorname{cis}\frac{3\pi}{8}\right)\left(z - \operatorname{cis}\frac{5\pi}{8}\right)\left(z - \operatorname{cis}\frac{7\pi}{8}\right)\left(z - \operatorname{cis}\frac{9\pi}{8}\right)\left(z - \operatorname{cis}\frac{11\pi}{8}\right)\left(z - \operatorname{cis}\frac{13\pi}{8}\right)\left(z - \operatorname{cis}\frac{15\pi}{8}\right)$$

8 a i $\pm\left(\sqrt{\frac{1 + \sqrt{2}}{2}} + \sqrt{\frac{\sqrt{2} - 1}{2}}\,i\right)$

ii $2^{\frac{1}{4}}\operatorname{cis}\left(\frac{\pi}{8}\right), 2^{\frac{1}{4}}\operatorname{cis}\left(\frac{-7\pi}{8}\right)$

b $\cos\left(\frac{\pi}{8}\right) = \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2}, \sin\left(\frac{\pi}{8}\right) = \frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2}$

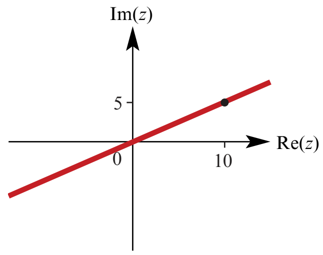
9 a $2 - 2i, 2 - 2i, 2 + 2i, -2 + 2i$

b $a = 4, b = 8$

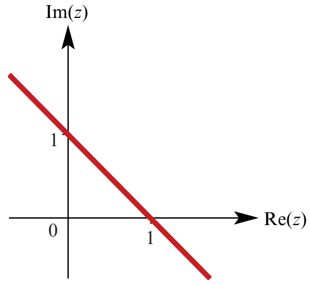
10 d i -1 **ii** 1 **iii** -1

Exercise 3I

1 a



b



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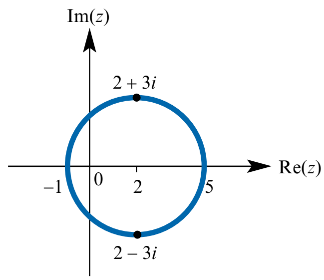
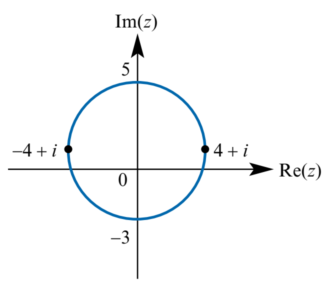
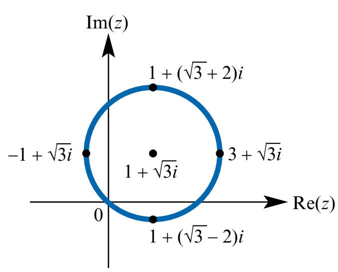


d

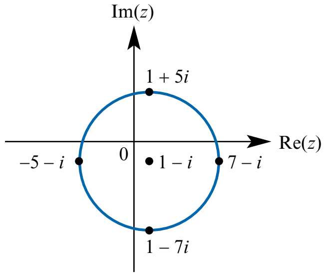


e



Answers 3I

f



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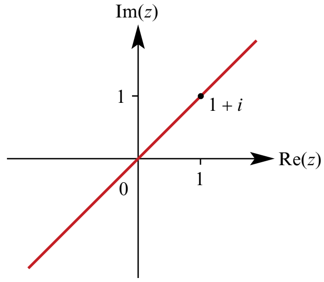
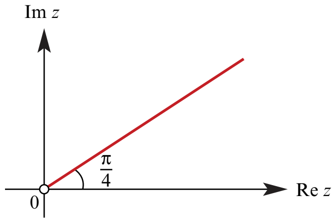


3 The imaginary axis, i.e. $\{z : \mathrm{Re}(z) = 0\}$

4 a



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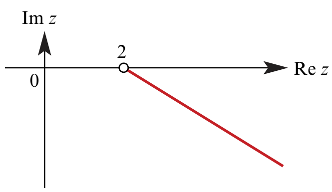

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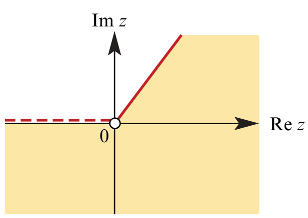

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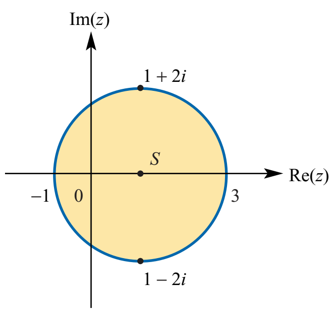

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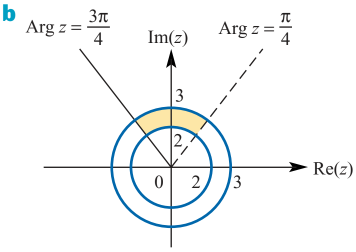

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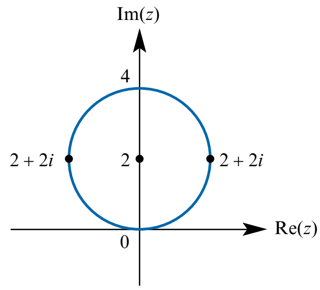

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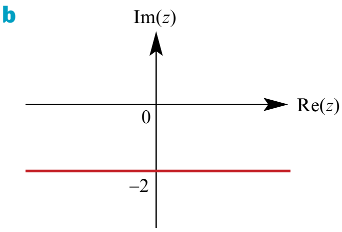

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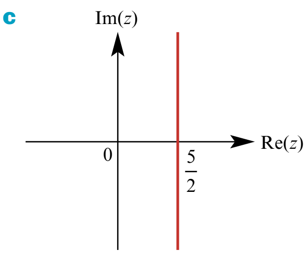

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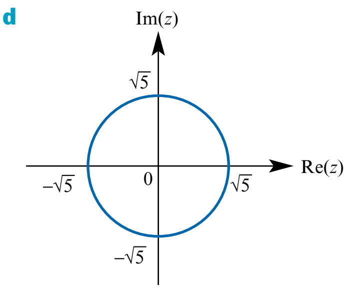

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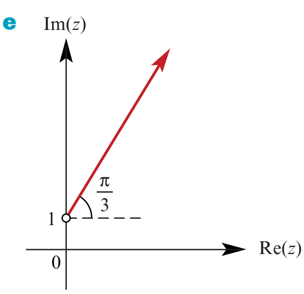

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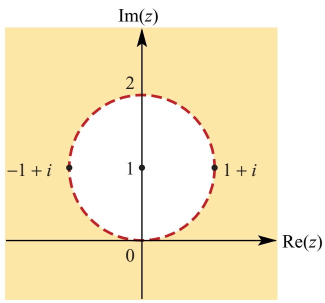
7 a



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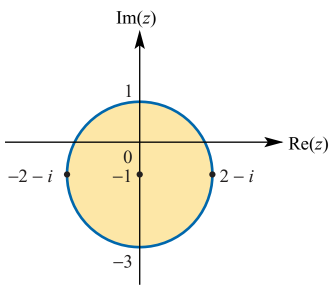

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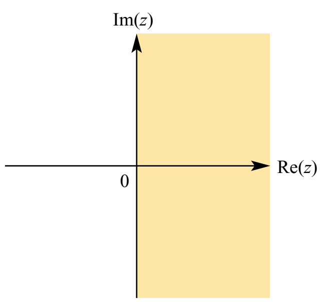

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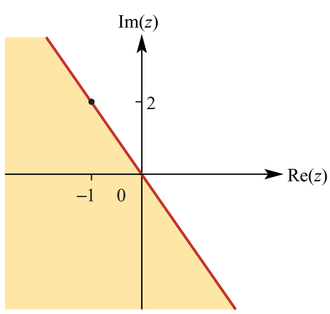

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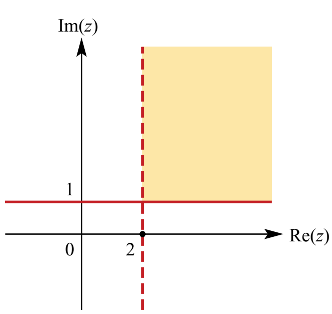

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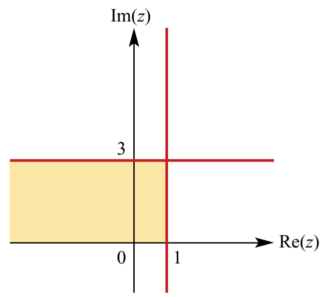

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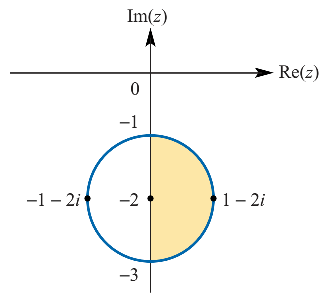

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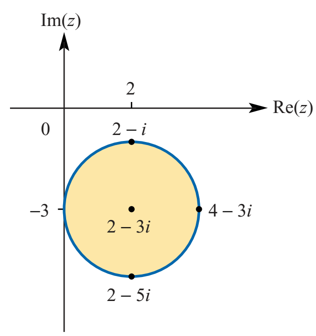
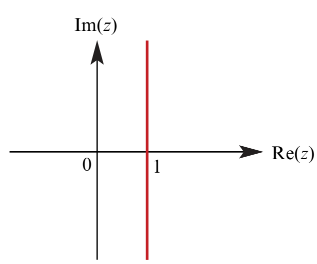


11 a



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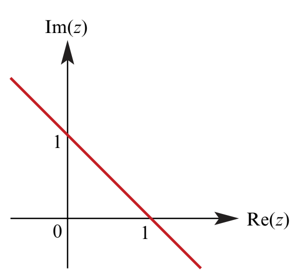

12 $x^2 + y^2 = 1$

13 $|z|^2 : 1$

14 a Circle with centre $(1, 1)$ and radius 1

b $y = -x$

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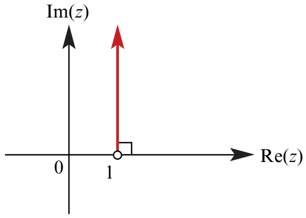
c



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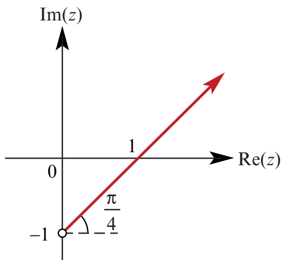
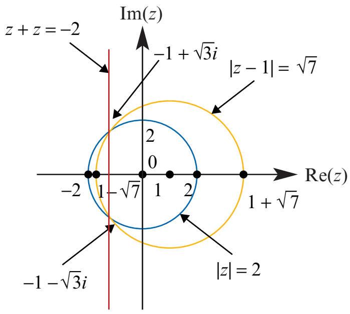


15 Circle with centre $2 + 4i$ and radius 6

16 a $z = -1 \pm \sqrt{3}i$

c



Chapter 3 review

Short-response questions

1 a $8 - 5i$
b $-i$
c $29 + 11i$
d 13
e $\dfrac{6}{13} + \dfrac{4}{13}i$
f $\dfrac{9}{5} - \dfrac{7}{5}i$
g $\dfrac{3}{5} + \dfrac{6}{5}i$
h $-8 - 6i$
i $\dfrac{43}{10} + \dfrac{81}{10}i$

2 a $2 \pm 3i$
b $-6 + 2i$
c $-3 \pm \sqrt{3}i$
d $\dfrac{3}{\sqrt{2}}(1 \pm i), \dfrac{3}{\sqrt{2}}(-1 \pm i)$
e $3, \dfrac{3}{2}(-1 \pm \sqrt{3}i)$ or $3\,\mathrm{cis}\left(\pm\dfrac{2\pi}{3}\right)$
f $-\dfrac{3}{2}, \dfrac{3}{4}(1 \pm \sqrt{3}i)$ or $\dfrac{3}{2}\,\mathrm{cis}\left(\pm\dfrac{\pi}{3}\right)$

3 a $2 - i, 2 + i, -2$
b $3 - 2i, 3 + 2i, -1$
c $1 + i, 1 - i, 2$

4 a $2\left(x + \dfrac{3}{4} + \dfrac{\sqrt{7}}{4}i\right)\left(x + \dfrac{3}{4} - \dfrac{\sqrt{7}}{4}i\right)$
b $(x - 1)(x + i)(x - i)$
c $(x + 2)^2(x - 2)$

5 2 and −1; −2 and 1

6 a iv
b ii
c i
d iii

7 −1 and 5; 1 and −5

8 $a = 2, b = 5$

9 $\dfrac{1}{2}\,\mathrm{cis}\left(-\dfrac{\pi}{3}\right)$

10 $a = \dfrac{3}{2} - \dfrac{\sqrt{3}}{2}, b = \dfrac{1}{2} + \dfrac{3\sqrt{3}}{2}$

11 a $2 + 2i$
b $\dfrac{1}{2}(1 + i)$
c $8\sqrt{2}$
d $\dfrac{\pi}{4}$

12 a i $\sqrt{2}$ **ii** 2 **iii** $\dfrac{\pi}{4}$ **iv** $-\dfrac{\pi}{3}$
b $\dfrac{\sqrt{2}}{2}, -\dfrac{\pi}{12}$

13 $2\,\mathrm{cis}\left(\dfrac{\pi}{6}\right), -64\sqrt{3} - 64i$

14 $\pm 3; \pm 3i, 1 \pm i$

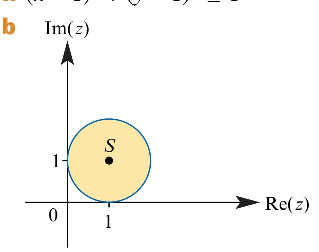
15 $16 - 16i$

16 $-2i, i, -2, k = -2$ or 1

17 a $(z + 2)(z - 1 + i)(z - 1 - i)$
b 25

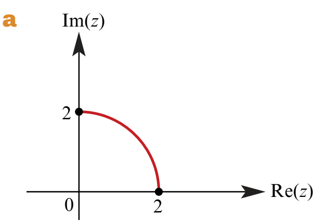
18 $-1 + 2i, \ -1 - \dfrac{1}{2}i$

19 a $(x - 1)^2 + (y - 1)^2 \le 1$
b



20 The real axis, i.e. $\{ z : \mathrm{Im}(z) = 0 \}$

21 a



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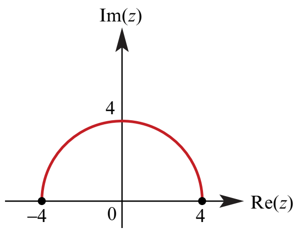
b



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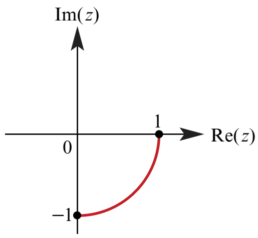


22 $\left(\frac{5}{6}, \frac{-7}{6}\right)$

23 a $4 - 3i$

b $c = 12 + 3i, d = 9 - i$ or
$c = 4 + 9i, d = 1 + 5i$

24 a $2\operatorname{cis}\left(\frac{\pi}{3}\right),\ 2\operatorname{cis}\pi,\ 2\operatorname{cis}\left(-\frac{\pi}{3}\right)$

b $2\operatorname{cis}\left(\frac{\pi}{6}\right),\ 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

25 a $x^6 - 1 = (x + 1)(x - 1)(x^2 - x + 1)(x^2 + x + 1)$

b $x^6 - 1 = (x + 1)(x - 1)$

$$\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

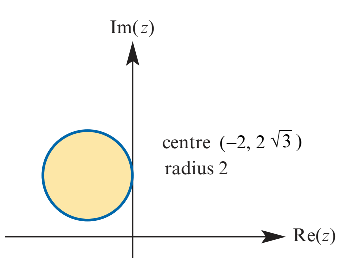
c $-1,\ 1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

26 a 1 **b** 1 **c** 0

27 $\frac{-\pi}{4}$

28 a $-2 + 2\sqrt{3}i$ **b** $-3 - 6i$

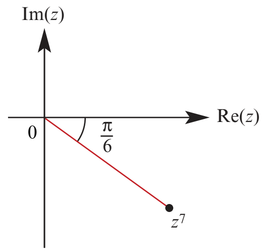
29 a



b i 2 **ii** $\frac{5\pi}{6}$

30 a $|z^7| = 16\,384;\quad \operatorname{Arg}(z^7) = \frac{-\pi}{6}$

b



c $2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$

d $z = -2\sqrt{3} + 2i,\ w = 1 + i,$

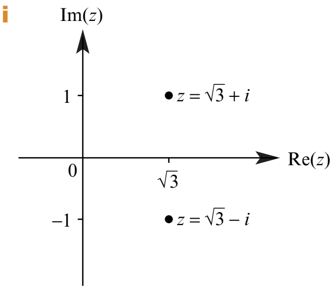
$\frac{z}{w} = (1 - \sqrt{3}) + (1 + \sqrt{3})i$

e $-2 - \sqrt{3}$

31 b $3,\ 2 - i$

32 a $z = \sqrt{3} \pm i$

b i

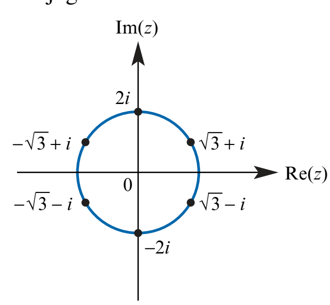


ii $x^2 + y^2 = 4$

iii $a = 2$

iv $P(z) = z^2 + 2\sqrt{3}z + 4$

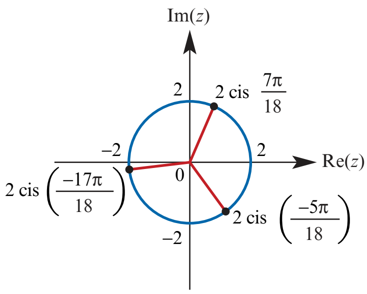
The solutions to the equation $z^6 + 64 = 0$ are equally spaced around the circle $x^2 + y^2 = 4$, and represent the sixth roots of -64. Three of the solutions are the conjugates of the other three solutions.



33 a $8\operatorname{cis}\left(\frac{-5\pi}{6}\right)$

b $2\operatorname{cis}\left(\frac{-5\pi}{18}\right),\ 2\operatorname{cis}\left(\frac{7\pi}{18}\right), 2\operatorname{cis}\left(\frac{-17\pi}{18}\right)$

c



d **i** $(z - \sqrt{3}i)^3 = -4\sqrt{3} - 4i$

ii $2\cos\left(\frac{-5\pi}{18}\right) + \left(2\sin\left(\frac{-5\pi}{18}\right) + \sqrt{3}\right)i$,

$2\cos\left(\frac{7\pi}{18}\right) + \left(2\sin\left(\frac{7\pi}{18}\right) + \sqrt{3}\right)i$,

$2\cos\left(\frac{-17\pi}{18}\right) + \left(2\sin\left(\frac{-17\pi}{18}\right) + \sqrt{3}\right)i$

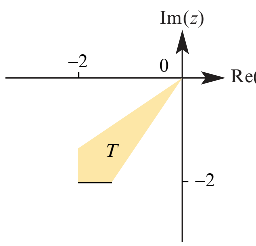
34 **a** $\overrightarrow{XY} = \sqrt{3}\hat{i} - \hat{j}, \quad \overrightarrow{XZ} = 2\sqrt{3}\hat{i} + 2\hat{j}$

b $z_3 = 1 + \sqrt{3}i$

c $z_3 = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$; W corresponds to $6\sqrt{3}$

d $(4\sqrt{3}, 0)$

35 **a**

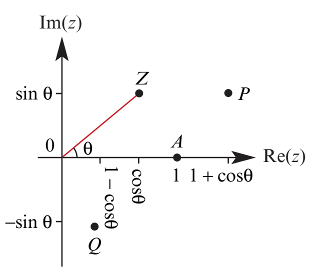


b $T = \{z : \operatorname{Re}(z) > -2\} \cap \{z : \operatorname{Im}(z) \geq -2\}$

$\cap \left\{z : \frac{-5\pi}{6} < \operatorname{Arg}(z) < \frac{-2\pi}{3}\right\}$

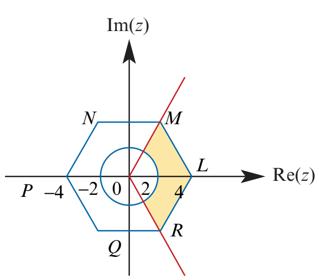
36 **a** $k > -\frac{5}{4}$ **b** $k = -\frac{5}{4}$ **c** $-2 < k < -\frac{5}{4}$

37 **b**



c $\operatorname{cosec}\theta + \cot\theta = \cot\left(\frac{\theta}{2}\right)$

38 **a**



b $|z - 4| = 4$

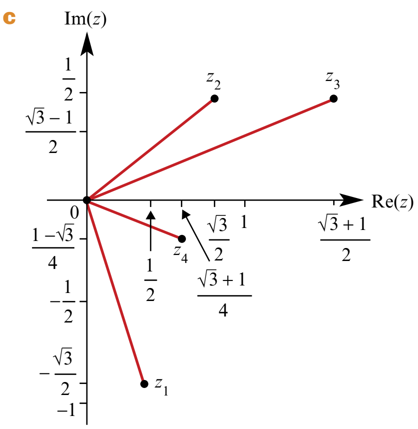
c N is $4\operatorname{cis}\left(\frac{2\pi}{3}\right)$; Q is $4\operatorname{cis}\left(\frac{-2\pi}{3}\right)$

d New position of N is $4\operatorname{cis}\left(\frac{5\pi}{12}\right)$;

new position of Q is $4\operatorname{cis}\left(\frac{-11\pi}{12}\right)$

39 **b** $z_3 = \sqrt{2}\operatorname{cis}(\tan^{-1}(2 - \sqrt{3})) = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)$

c



40 **a** **ii** $q = 2k^3$

b $b = -1 - i, \ c = 2 + 2i$

41 **a** **i** $6\sqrt{2}$

ii 6

b **ii** Isosceles right-angled

42 **a** **i** 13

ii $157.38° = 2.75^c$

b **i** $\cos\alpha = \frac{-12}{13}, \sin\alpha = \frac{5}{13}$

ii $r = \sqrt{13}, \cos(2\theta) = \frac{-12}{13}, \sin(2\theta) = \frac{5}{13}$

iii $\sin\theta = \pm\frac{5\sqrt{26}}{26}, \cos\theta = \pm\frac{\sqrt{26}}{26}$

iv $w = \pm\frac{\sqrt{2}}{2}(1 + 5i)$

d $\pm\frac{\sqrt{2}}{2}(5 + i)$; a reflection of the square roots of $-12 + 5i$ in the line $\operatorname{Re}(z) = \operatorname{Im}(z)$

43 **a** $\left(x + \frac{3}{2}\right)^2 + y^2 = \frac{29}{4}$

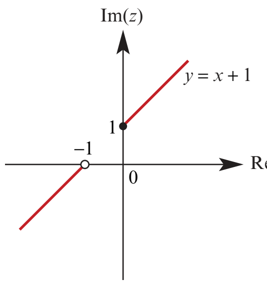
b $\left(x + \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{15}{2}$

c $\left(x + \frac{\beta}{\alpha}\right)^2 + y^2 = \frac{\beta^2 - \alpha\gamma}{\alpha^2}$

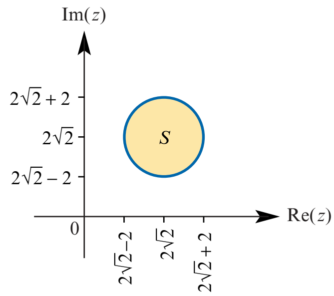
d $\left(x + \frac{a}{\alpha}\right)^2 + \left(y - \frac{b}{\alpha}\right)^2 = \frac{a^2 + b^2 - \alpha\gamma}{\alpha^2}$, where $\beta = a + bi$

44 **a** $(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)i$

45 a



b i



ii max = 6; min = 2

iii max = 75° = $\frac{5\pi}{12}$; min = 15° = $\frac{\pi}{12}$

46 a $\frac{(\sqrt{6}+\sqrt{2})+(\sqrt{6}-\sqrt{2})i}{4}$

b $u = \text{cis}\left(\frac{\pi}{3}\right)$ and $v = \text{cis}\left(\frac{\pi}{4}\right)$

c $\text{cis}\left(\frac{\pi}{12}\right)$

d $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

and $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$.

Multiple-choice questions

1 D	**2** D	**3** A	**4** C	**5** B
6 B	**7** D	**8** B	**9** A	**10** D
11 C	**12** B	**13** A	**14** C	**15** D
16 A	**17** C	**18** C	**19** A	**20** B

Chapter 4

See worked solutions in the Interactive Textbook

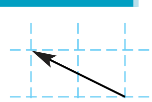
Multiple-choice questions

1 D	**2** C	**3** A	**4** D
5 A	**6** C	**7** B	

Chapter 5

Exercise 5A

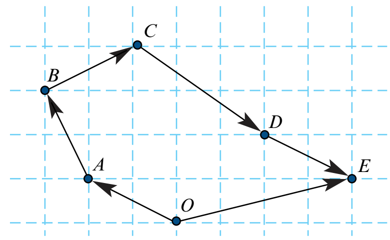
1



Magnitude = $\sqrt{5}$

2 $a = 3,\ b = 2$

3



4 a i $2\boldsymbol{b}$ **ii** $4\boldsymbol{a}$ **iii** $2\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}$

iv $\frac{1}{2}\boldsymbol{b} - 2\boldsymbol{a}$ **v** $2\boldsymbol{a} - \frac{3}{2}\boldsymbol{b}$

b i 4 **ii** 4 **iii** $\sqrt{13}$

5 a 6 **b** $\frac{9}{2}$ **c** $\frac{3}{2}$

6 a $\frac{21}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b} - 21\boldsymbol{c}$ **b** $\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$

7 a i $\frac{1}{4}\boldsymbol{a}$ **ii** $\frac{1}{4}\boldsymbol{b}$ **iii** $\frac{1}{4}(\boldsymbol{b} - \boldsymbol{a})$

iv $\boldsymbol{b} - \boldsymbol{a}$

b i $\frac{1}{2}\boldsymbol{a}$ **ii** $\frac{1}{2}\boldsymbol{b}$ **iii** $\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

8 a $\boldsymbol{a} + \boldsymbol{b}$

b $-(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$

c $-(\boldsymbol{b} + \boldsymbol{c})$

9 a $\boldsymbol{b} - \boldsymbol{a}$ **b** $\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$ **c** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

10 a $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

11 a $\boldsymbol{a} + \boldsymbol{c} - \boldsymbol{b}$ **b** $\boldsymbol{a} + \boldsymbol{c} - 2\boldsymbol{b}$

12 a $\begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$ **c** $\begin{bmatrix} -14 \\ 1 \\ -4 \end{bmatrix}$ **d** $\begin{bmatrix} 4 \\ -4 \\ -8 \end{bmatrix}$

e 3 **f** 4

13 a $x = 0, y = 1$ **b** $x = -1, y = \frac{7}{3}$

c $x = -\frac{5}{2}, y = 0$

14 a $-\boldsymbol{c}$ **b** $\boldsymbol{c}$ **c** $-\frac{1}{2}\boldsymbol{a}$

d $\boldsymbol{c} + \boldsymbol{g} + \frac{1}{2}\boldsymbol{a}$ **e** $\boldsymbol{c} + \boldsymbol{g} - \frac{1}{2}\boldsymbol{a}$

15 a i $\boldsymbol{b} - \boldsymbol{a}$ **ii** $\boldsymbol{c} - \boldsymbol{d}$ **iii** $\boldsymbol{b} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{d}$

b i $\boldsymbol{c} - \boldsymbol{b}$ **ii** $-\frac{1}{2}\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

16 a $k = 3, \ell = \frac{1}{2}$ **b** $k = \frac{55}{2}, \ell = -10$

17 a i $k(2\boldsymbol{a} - \boldsymbol{b})$ **ii** $(2m + 1)\boldsymbol{a} + (4 - 3m)\boldsymbol{b}$

b $k = \frac{11}{4}, m = \frac{9}{4}$

18 a i $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ **ii** $\frac{4}{5}(\boldsymbol{a} + \boldsymbol{b})$

iii $\frac{1}{5}(4\boldsymbol{b} - \boldsymbol{a})$ **iv** $\frac{4}{5}(4\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{RP} = 4\overrightarrow{AR}$, 1 : 4 **c** 4

19 a $\frac{AX}{AB} = k$ **c** $\frac{AX}{XB} = \frac{k}{1-k}$ **d** $k = \frac{m}{1+m}$

Exercise 5B

1 a i $3\hat{i} + \hat{j}$ **ii** $-2\hat{i} + 3\hat{j}$ **iii** $-3\hat{i} - 2\hat{j}$
iv $4\hat{i} - 3\hat{j}$
b i $-5\hat{i} + 2\hat{j}$ **ii** $7\hat{i} - \hat{j}$ **iii** $-\hat{i} + 4\hat{j}$
c i $\sqrt{10}$ **ii** $\sqrt{29}$ **iii** $\sqrt{17}$

2 a $\hat{i} + 4\hat{j}$ **b** $4\hat{i} + 4\hat{j} + 2\hat{k}$
c $6\hat{j} - 3\hat{k}$ **d** $-8\hat{i} - 8\hat{j} + 8\hat{k}$
e $\sqrt{6}$ **f** 4

3 a i $-5\hat{i}$ **ii** $3\hat{k}$
iii $2\hat{j}$ **iv** $5\hat{i} + 3\hat{k}$
v $5\hat{i} + 2\hat{j} + 3\hat{k}$ **vi** $5\hat{i} + 2\hat{j}$
vii $-5\hat{i} - 3\hat{k}$ **viii** $2\hat{j} - 3\hat{k}$
ix $-5\hat{i} + 2\hat{j} - 3\hat{k}$ **x** $-5\hat{i} - 2\hat{j} + 3\hat{k}$
xi $5\hat{i} + 2\hat{j} - 3\hat{k}$ **xii** $5\hat{i} - 2\hat{j} - 3\hat{k}$
b i $\sqrt{34}$ **ii** $\sqrt{38}$ **iii** $\sqrt{29}$
c i $\frac{5}{2}\hat{i}$ **ii** $\frac{5}{2}\hat{i} + 2\hat{j}$
iii $\frac{-5}{2}\hat{i} + 2\hat{j} - 3\hat{k}$
d i $\frac{-4}{3}\hat{j}$ **ii** $\frac{2}{3}\hat{j}$ **iii** $\frac{2}{3}\hat{j} + 3\hat{k}$
iv $5\hat{i} - \frac{2}{3}\hat{j} - 3\hat{k}$ **v** $\frac{5}{2}\hat{i} + \frac{4}{3}\hat{j} - 3\hat{k}$
e i $\frac{\sqrt{613}}{6}$
ii $\frac{\sqrt{77}}{2}$
iii $\frac{\sqrt{310}}{3}$

4 a $x = 3, y = -\frac{1}{3}$ **b** $x = 4, y = \frac{2}{5}$
c $x = -\frac{3}{2}, y = 7$

5 a i $-2\hat{i} + 4\hat{j}$ **ii** $3\hat{i} + 2\hat{j}$
iii $-2\hat{i} - 12\hat{j}$
b $-\hat{i} + 2\hat{j}$ **c** $-8\hat{i} - 32\hat{j}$

6 $\left(3, -\frac{7}{2}, 8\right)$

7 a i $4\hat{i} - 2\hat{j} - 4\hat{k}$ **ii** $-5\hat{i} + 4\hat{j} + 9\hat{k}$
iii $2\hat{i} - \hat{j} - 2\hat{k}$ **iv** $-\hat{i} - \hat{j} - 3\hat{k}$
b i $\sqrt{30}$
ii $\sqrt{67}$
c $\overrightarrow{AB}, \overrightarrow{CD}$

8 a i $2\hat{i} - 3\hat{j} + 4\hat{k}$ **ii** $\frac{4}{5}(2\hat{i} - 3\hat{j} + 4\hat{k})$
iii $\frac{1}{5}(13\hat{i} - 7\hat{j} - 9\hat{k})$
b $\left(\frac{13}{5}, -\frac{7}{5}, -\frac{9}{5}\right)$

9 a i $\overrightarrow{OA} = 2\hat{i} + \hat{j}$ **ii** $\overrightarrow{AB} = -\hat{i} - 4\hat{j}$
iii $\overrightarrow{BC} = -6\hat{i} + 5\hat{j}$ **iv** $\overrightarrow{BD} = 2\hat{i} + 8\hat{j}$
b $\overrightarrow{BD} = -2\overrightarrow{AB}$
c Points A, B and D are collinear

10 a i $\overrightarrow{OB} = 2\hat{i} + 3\hat{j} + \hat{k}$
ii $\overrightarrow{AC} = -\hat{i} - 5\hat{j} + 8\hat{k}$
iii $\overrightarrow{BD} = 2\hat{i} + 2\hat{j} + 5\hat{k}$
iv $\overrightarrow{CD} = 4\hat{i} + 6\hat{j} + 2\hat{k}$
b $\overrightarrow{CD} = 2(2\hat{i} + 3\hat{j} + \hat{k}) = 2\overrightarrow{OB}$

11 a i $\overrightarrow{AB} = 2\hat{i} - \hat{j} + 2\hat{k}$
ii $\overrightarrow{BC} = -\hat{i} + 2\hat{j} + 3\hat{k}$
iii $\overrightarrow{CD} = -2\hat{i} + \hat{j} - 2\hat{k}$
iv $\overrightarrow{DA} = \hat{i} - 2\hat{j} - 3\hat{k}$
b Parallelogram

12 a $(-6, 3)$ **b** $(6, 5)$ **c** $\left(\frac{3}{2}, -\frac{3}{2}\right)$

13 a i $\overrightarrow{BC} = 6\hat{i} + 3\hat{j}$
ii $\overrightarrow{AD} = (x - 2)\hat{i} + (y - 1)\hat{j}$
b $(8, 4)$

14 a $(1.5, 1.5, 4)$
b $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

15 $\left(\frac{17}{5}, \frac{8}{5}, -3\right)$ **16** $\left(\frac{17}{2}, 3\right)$

17 $\left(-11, -\frac{11}{3}\right)$

19 a i $\hat{i} + \hat{j}$ **ii** $-\hat{i} - 6\hat{j}$ **iii** $-\hat{i} - 15\hat{j}$
b $k = \frac{19}{8},\ \ell = -\frac{1}{4}$

20 a i $2\hat{i} + 4\hat{j} - 9\hat{k}$ **ii** $14\hat{i} - 8\hat{j} + 3\hat{k}$
iii $5.7\hat{i} - 0.3\hat{j} - 1.6\hat{k}$
b There are no values for k and ℓ such that $k\boldsymbol{a} + \ell\boldsymbol{b} = \boldsymbol{c}$

21 a i $\sqrt{29}$ **ii** $\sqrt{13}$ **iii** $\sqrt{97}$ **iv** $\sqrt{19}$
b i 21.80° anticlockwise
ii 23.96° clockwise **iii** 46.51°

22 a $-3.42\hat{i} + 9.40\hat{j}$ **b** $-2.91\hat{i} - 7.99\hat{j}$
c $4.60\hat{i} + 3.86\hat{j}$ **d** $2.50\hat{i} - 4.33\hat{j}$

23 a $-6.43\hat{i} + 1.74\hat{j} + 7.46\hat{k}$
b $5.14\hat{i} + 4.64\hat{j} - 4\hat{k}$
c $6.13\hat{i} - 2.39\hat{j} - 2.39\hat{k}$
d $-6.26\hat{i} + 9.77\hat{j} + 3.07\hat{k}$

24 c $\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$

25 a $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 3$ **b** $\overrightarrow{OM} = -\hat{i} + 3\hat{j} + 4\hat{k}$
c $\overrightarrow{AM} = \hat{i} + 2\hat{j} - \hat{k}$ **d** $3\sqrt{2}$

26 a $5\hat{i} + 5\hat{j}$ **b** $\frac{1}{2}(5\hat{i} + 5\hat{j})$
c $\frac{5}{2}\hat{i} + \frac{5}{2}\hat{j} + 3\hat{k}$ **d** $-\frac{5}{2}\hat{i} - \frac{5}{2}\hat{j} + 3\hat{k}$
e $\frac{\sqrt{86}}{2}$

27 a $\overrightarrow{MN} = \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$
b $\overrightarrow{MN} \parallel \overrightarrow{AB},\ MN = \frac{1}{2}AB$

28 a $\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}$ **b** $\frac{3\sqrt{3}}{2}\hat{i} - \frac{3}{2}\hat{j}$
c $\frac{3\sqrt{3}}{2}\hat{i} + \frac{7}{2}\hat{j}$ **d** $\sqrt{19}$ km

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29 a $\overrightarrow{OA} = 50\hat{\boldsymbol{k}}$
b i $-80\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}} - 10\hat{\boldsymbol{k}}$ **ii** $10\sqrt{69}$ m
c $-80\hat{\boldsymbol{i}} + 620\hat{\boldsymbol{j}} + 100\hat{\boldsymbol{k}}$

30 a 2.66 km
b i $-0.5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 0.1\hat{\boldsymbol{k}}$ **ii** 1.12 km
c $-0.6\hat{\boldsymbol{i}} - 0.8\hat{\boldsymbol{j}}$

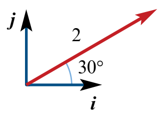
31 a $-100\sqrt{2}\hat{\boldsymbol{i}} + 100\sqrt{2}\hat{\boldsymbol{j}}$ **b** $50\hat{\boldsymbol{j}}$
c $-100\sqrt{2}\hat{\boldsymbol{i}} + (50 + 100\sqrt{2})\hat{\boldsymbol{j}}$ **d** $30\hat{\boldsymbol{k}}$
e $-100\sqrt{2}\hat{\boldsymbol{i}} + (50 + 100\sqrt{2})\hat{\boldsymbol{j}} + 30\hat{\boldsymbol{k}}$

32 a $\overrightarrow{OP} = 50\sqrt{2}\hat{\boldsymbol{i}} + 50\sqrt{2}\hat{\boldsymbol{j}}$
b i $(50\sqrt{2} - 100)\hat{\boldsymbol{i}} + 50\sqrt{2}\hat{\boldsymbol{j}}$ **ii** 337.5°

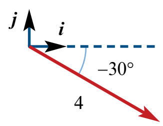
33 a $\boldsymbol{c} = (3m + 1)\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + (1 - 3m)\hat{\boldsymbol{k}}$ **b** $p = -5$

Exercise 5C

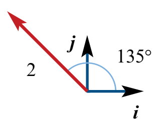
1 a $\sqrt{3}\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$



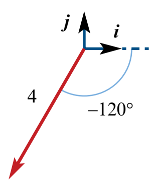
b $2\sqrt{3}\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$



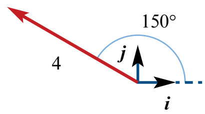
c $-\sqrt{2}\hat{\boldsymbol{i}} + \sqrt{2}\hat{\boldsymbol{j}}$



d $-2\hat{\boldsymbol{i}} - 2\sqrt{3}\hat{\boldsymbol{j}}$



e $-2\sqrt{3}\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$



2 a $4\hat{\boldsymbol{i}} + 4\sqrt{3}\hat{\boldsymbol{j}}$ **b** $5\sqrt{3}\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$
c $-3\sqrt{3}\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **d** $4\sqrt{2}\hat{\boldsymbol{i}} - 4\sqrt{2}\hat{\boldsymbol{j}}$
e $-6\sqrt{3}\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$

3 a $[4\sqrt{2}, -45°]$ **b** $[3\sqrt{2}, 135°]$
c $[4, 150°]$ **d** $[5, 53.13°]$
e $[13, -112.62°]$

4 a $[2, 30°]$ **b** $[\sqrt{5}, 63.43°]$
c $[\sqrt{2}, -135°]$ **d** $[2, -30°]$
e $[\sqrt{2}, 135°]$

5 a $5.36\hat{\boldsymbol{i}} + 4.50\hat{\boldsymbol{j}}$ **b** $8.19\hat{\boldsymbol{i}} + 5.74\hat{\boldsymbol{j}}$
c $-9.97\hat{\boldsymbol{i}} + 4.65\hat{\boldsymbol{j}}$ **d** $7.37\hat{\boldsymbol{i}} - 5.16\hat{\boldsymbol{j}}$
e $-6.88\hat{\boldsymbol{i}} - 9.83\hat{\boldsymbol{j}}$

6 a $[20, 60°]$ **b** $[1, -165°]$
c $[18, 160°]$

7 a $(6 + 4\sqrt{3})\hat{\boldsymbol{i}} + (4 + 6\sqrt{3})\hat{\boldsymbol{j}}$
b $5\sqrt{2}\hat{\boldsymbol{i}} + (5\sqrt{2} - 10)\hat{\boldsymbol{j}}$
c $4.80\hat{\boldsymbol{i}} + 7.28\hat{\boldsymbol{j}}$

8 a $3\sqrt{3}\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 6\sqrt{3}\hat{\boldsymbol{k}}$ **b** $15\hat{\boldsymbol{i}} + 15\hat{\boldsymbol{j}} + 15\sqrt{2}\hat{\boldsymbol{k}}$
c $-12\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}} - 12\sqrt{2}\hat{\boldsymbol{k}}$
d $-15\sqrt{3}\hat{\boldsymbol{i}} - 45\hat{\boldsymbol{j}} + 30\hat{\boldsymbol{k}}$
e $100\hat{\boldsymbol{j}} - 100\sqrt{3}\hat{\boldsymbol{k}}$ **f** $-200\hat{\boldsymbol{k}}$

9 a $\left[\sqrt{5}, \dfrac{\pi}{6}, \sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)\right] \approx [\sqrt{5}, 30°, 26.57°]$
b $\left[\sqrt{11}, \tan^{-1}\left(\dfrac{2}{\sqrt{3}}\right), \sin^{-1}\left(\dfrac{2}{\sqrt{11}}\right)\right] \approx$
$[\sqrt{11}, 49.11°, 37.09°]$
c $\left[\sqrt{3}, \dfrac{\pi}{4}, \sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\right] \approx [\sqrt{3}, 45°, 35.25°]$

10 a $[\sqrt{33}, -45°, 10.02°]$ **b** $[\sqrt{19}, 135°, -13.26°]$
c $[2\sqrt{5}, 150°, 26.57°]$
d $[\sqrt{41}, 53.13°, -38.66°]$
e $[\sqrt{173}, -112.62°, 8.75°]$
f $[2\sqrt{13}, -90°, 33.69°]$

Exercise 5D

1 a 66 **b** 22 **c** 6 **d** 11 **e** 25
f 86 **g** −43

2 a 14 **b** 13 **c** 0 **d** −8 **e** 14

3 a 21 **b** −21

4 a $\boldsymbol{a} \cdot \boldsymbol{a} + 4(\boldsymbol{a} \cdot \boldsymbol{b}) + 4(\boldsymbol{b} \cdot \boldsymbol{b})$ **b** $4(\boldsymbol{a} \cdot \boldsymbol{b})$
c $\boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{b} \cdot \boldsymbol{b}$ **d** $|\boldsymbol{a}|$

5 a −4 **b** 5 **c** 5 **d** −6 or 1

6 a $\overrightarrow{AB} = -2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ **b** $|\overrightarrow{AB}| = 3$
c 105.8°

7 $\sqrt{66}$

8 a i $\boldsymbol{c}$ **ii** $\boldsymbol{a} + \boldsymbol{c}$ **iii** $\boldsymbol{c} - \boldsymbol{a}$
b 0

9 $\boldsymbol{d}$ and $\boldsymbol{f}$; $\boldsymbol{a}$ and $\boldsymbol{e}$; $\boldsymbol{b}$ and $\boldsymbol{c}$

10 b 109.47°

11 a $\overrightarrow{AP} = -\boldsymbol{a} + q\boldsymbol{b}$
b $q = \dfrac{13}{15}$
c $\left(\dfrac{26}{15}, \dfrac{13}{3}, -\dfrac{13}{15}\right)$

12 $x = 1,\ y = -3$

13 a 2.45 **b** 1.11 **c** 0.580 **d** 2.01

15 a $\overrightarrow{OM} = \dfrac{3}{2}\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ **b** 36.81° **c** 111.85°

16 a i $-\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **ii** $3\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$
b 37.87° **c** 31.00°

17 a i $\dfrac{1}{2}(4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}})$ **ii** $\dfrac{1}{2}(2\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{k}})$
b 80.12° **c** 99.88°

18 69.71°

Exercise 5E

1 a $\dfrac{\sqrt{11}}{11}(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ **b** $\dfrac{1}{3}(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$
c $\dfrac{\sqrt{10}}{10}(-\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})$

2 a i $\dfrac{\sqrt{26}}{26}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ **ii** $\dfrac{\sqrt{3}}{3}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
b $\dfrac{\sqrt{78}}{26}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$

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3 a i $\frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$ **ii** $\frac{1}{5}(3\hat{i} + 4\hat{k})$

b $\frac{\sqrt{510}}{510}(19\hat{i} - 10\hat{j} + 7\hat{k})$

4 a $\frac{-11}{18}(\hat{i} - 4\hat{j} + \hat{k})$ **b** $\frac{-1}{9}(\hat{i} - 4\hat{j} + \hat{k})$

c $\frac{13}{17}(4\hat{i} - \hat{k})$

5 a 2 **b** $\frac{\sqrt{5}}{5}$ **c** $\frac{2\sqrt{21}}{7}$ **d** $\frac{-(1+4\sqrt{5})\sqrt{17}}{17}$

6 a $\frac{9}{26}(5\hat{i} - \hat{k})$, $\frac{1}{26}(7\hat{i} + 26\hat{j} + 35\hat{k})$

b $\frac{3}{2}(\hat{i} + \hat{k})$, $\frac{3}{2}\hat{i} + \hat{j} - \frac{3}{2}\hat{k}$

c $-\frac{1}{9}(2\hat{i} + 2\hat{j} - \hat{k})$, $\frac{-7}{9}\hat{i} + \frac{11}{9}\hat{j} + \frac{8}{9}\hat{k}$

7 a $\hat{j} + \hat{k}$ **b** $\frac{1}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$

8 a $\hat{i} - \hat{j} - \hat{k}$ **b** $3\hat{i} + 2\hat{j} + \hat{k}$ **c** $\sqrt{14}$

9 a i $\hat{i} - \hat{j} - 2\hat{k}$ **ii** $\hat{i} - 5\hat{j}$

b $\frac{3}{13}(\hat{i} - 5\hat{j})$ **c** $\frac{2}{13}\sqrt{195}$ **d** $\sqrt{30}$

10 b i $\frac{2}{7}(\hat{i} - 3\hat{j} - 2\hat{k})$ **ii** $\frac{1}{3}(5\hat{i} + \hat{j} + \hat{k})$

c $\frac{1}{21}(\hat{i} + 11\hat{j} - 16\hat{k})$

Exercise 5F

1 a $\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$ **b** $\frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{b}$

2 a $\frac{5}{2}\hat{i} - \hat{j} + \frac{5}{2}\hat{k}$ **b** $\frac{5}{3}\hat{i} - \frac{8}{3}\hat{j}$

c $\frac{10}{3}\hat{i} + \frac{2}{3}\hat{j} + 5\hat{k}$

3 b 2 : 1

4 a $\frac{a+x}{2}\hat{i} + \frac{y}{2}\hat{j}$ **b** $x^2 + y^2 = a^2$

5 b 1 : 5

6 a $\overrightarrow{OB} = -\hat{i} + 7\hat{j}$ **b** $\overrightarrow{OD} = -2\hat{i} + \frac{17}{3}\hat{j}$

c $\lambda = \frac{2}{5}$

7 b i $\overrightarrow{OP} = 2\hat{i} + \hat{j} + \hat{k}$

ii $\overrightarrow{OP} = \frac{18}{11}\hat{i} + \frac{15}{11}\hat{j} - \frac{1}{11}\hat{k}$

iii $\overrightarrow{OP} = \frac{7}{4}\hat{i} + \frac{5}{4}\hat{j} + \frac{1}{4}\hat{k}$

Exercise 5G

1 a $-\hat{i} - 11\hat{j}$ **b** $5\hat{i} - 6\hat{j}$ **c** $\hat{i} + 5\hat{j}$
d $-11\hat{j} + 4\hat{k}$ **e** $4\hat{i} + 12\hat{k}$
f $6\hat{i} + 11\hat{j} - 13\hat{k}$

2 a $\sqrt{41}$ ms^{-1} **b** 5 m/s **c** $\sqrt{17}$ ms^{-1}
d $2\sqrt{14}$ ms^{-1} **e** $\sqrt{173}$ ms^{-1} **f** $\sqrt{195}$ ms^{-1}

3 a $24\hat{i} + 62\hat{j} + 17\hat{k}$
b $(5t-1)\hat{i} + (12t+2)\hat{j} + (3t+2)\hat{k}$

4 $-4\hat{i} + 4\hat{j} + \hat{k}$ ms^{-1}

5 a i $26\hat{i} + 99\hat{j} + 10\hat{k}$
ii $(7t-2)\hat{i} + (24t+3)\hat{j} + (3t-2)\hat{k}$
b i 102.84 m
ii $\sqrt{(7t-2)^2 + (24t+3)^2 + (3t-2)^2}$ m

6 a $-\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{10}\hat{k}$ ms^{-1}

b $\frac{3\sqrt{14}}{10}$ ms^{-1}

7 After $\frac{2\sqrt{6}+12}{5}$ seconds;

position vector $\frac{2\sqrt{6}+12}{5}(\hat{i} + 2\hat{j} + \hat{k})$

8 a $8\hat{i} + 4\hat{j}$ **b** $2\hat{i} - 4\hat{j}$ ms^{-1}

9 a $20\hat{i} + 10\hat{j}$ **b** $\hat{j}$ ms^{-1}

10 a On a bearing of 143.13°
b 5 km/h

11 100.08 km/h on a bearing of 357.71°

12 a 20 km/h west **b** 180 km/h west

13 47 ms^{-1} north

14 10 ms^{-1}

15 a 20 km/h north **b** 20 km/h south

16 252.98 km/h on a bearing of 018.43°

17 100 km/h on a bearing of 053.13°

18 42.5 km/h on a bearing of 41.73°

19 a $\hat{i} - 4\hat{j} + 7\hat{k}$ ms^{-1} **b** 8.12 ms^{-1}

20 10.36 ms^{-1}

21 196.83 km/h on a bearing of 345.44°

22 a Bearing 210.67°
b 243.28 km/h

Exercise 5H

12 a i $\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$ **ii** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

b $\frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b})$

13 c 3 : 1

14 a i $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ **ii** $\boldsymbol{a} + 2\boldsymbol{b}$ **iii** $2\boldsymbol{b}$

15 a $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{t}$

b $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{s})$, $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

16 b $\overrightarrow{AB} = \hat{i} - 3\hat{j}$, $\overrightarrow{DC} = \hat{i} - \hat{j}$ **c** $4\hat{i} + 2\hat{j}$
e $4\hat{j}$

18 $\frac{2}{3}\boldsymbol{b} - \frac{5}{12}\boldsymbol{a}$

19 b $\lambda = \frac{k+2}{2}$, $\mu = \frac{k+2}{2}$

c $\lambda = \frac{3}{2}$, $\mu = \frac{3}{2}$

20 b $12r^2$

22 a $\overrightarrow{OX} = \dfrac{c^2}{a^2+c^2+d^2}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})$

b $\overrightarrow{OY} = \dfrac{a^2+c^2}{a^2+c^2+d^2}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})$

c i $\overrightarrow{OX} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}),\ \overrightarrow{OY} = \frac{2}{3}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d})$

ii $120°$ **iii** $120°$

25 a $\overrightarrow{OG} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b})$

29 a $\lambda = \frac{1}{2}$ **c** $\arccos\left(\frac{1}{3}\right)$

30 a $\overrightarrow{OG} = \boldsymbol{b}+\boldsymbol{d}+\boldsymbol{e},\ \overrightarrow{DF} = \boldsymbol{b}-\boldsymbol{d}+\boldsymbol{e},$
$\overrightarrow{BH} = -\boldsymbol{b}+\boldsymbol{d}+\boldsymbol{e},\ \overrightarrow{CE} = -\boldsymbol{b}-\boldsymbol{d}+\boldsymbol{e}$

b $|\overrightarrow{OG}|^2 = |\boldsymbol{b}|^2+|\boldsymbol{d}|^2+|\boldsymbol{e}|^2 + 2(\boldsymbol{b}\cdot\boldsymbol{d}+\boldsymbol{b}\cdot\boldsymbol{e}+\boldsymbol{d}\cdot\boldsymbol{e})$

$|\overrightarrow{DF}|^2 = |\boldsymbol{b}|^2+|\boldsymbol{d}|^2+|\boldsymbol{e}|^2 + 2(-\boldsymbol{b}\cdot\boldsymbol{d}+\boldsymbol{b}\cdot\boldsymbol{e}-\boldsymbol{d}\cdot\boldsymbol{e})$

$|\overrightarrow{BH}|^2 = |\boldsymbol{b}|^2+|\boldsymbol{d}|^2+|\boldsymbol{e}|^2 + 2(-\boldsymbol{b}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{e}+\boldsymbol{d}\cdot\boldsymbol{e})$

$|\overrightarrow{CE}|^2 = |\boldsymbol{b}|^2+|\boldsymbol{d}|^2+|\boldsymbol{e}|^2 + 2(\boldsymbol{b}\cdot\boldsymbol{d}-\boldsymbol{b}\cdot\boldsymbol{e}-\boldsymbol{d}\cdot\boldsymbol{e})$

Chapter 5 review

Short-response questions

1 a $2\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ **b** $\frac{\sqrt{2}}{3}$

2 a i $\frac{3}{7}(-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+6\hat{\boldsymbol{k}})$ **ii** $\frac{1}{7}(6\hat{\boldsymbol{i}}-11\hat{\boldsymbol{j}}-12\hat{\boldsymbol{k}})$

3 a $x = 5$ **b** $y = 2.8,\ z = -4.4$

4 a $\cos\theta = \frac{1}{3}$ **b** 6

5 a $\frac{1}{9}(43\hat{\boldsymbol{i}}-46\hat{\boldsymbol{j}}+20\hat{\boldsymbol{k}})$

b $\frac{485}{549}(3\hat{\boldsymbol{i}}-6\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}})$

6 a i $(2-3t)\hat{\boldsymbol{j}}+(-3-2t)\hat{\boldsymbol{k}}$

ii $(-2-3t)\hat{\boldsymbol{j}}+(3-2t)\hat{\boldsymbol{k}}$

b ± 1

7 a i $2\sqrt{17}$ **ii** $4\sqrt{3}$ **iii** -40

b $\cos^{-1}\left(\frac{5\sqrt{51}}{51}\right)$

8 $5\sqrt{3}\hat{\boldsymbol{i}}-15\hat{\boldsymbol{j}}+10\hat{\boldsymbol{k}}$

9 $[20, 30°, 60°]$

10 a $3\hat{\boldsymbol{i}}-\frac{3}{2}\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ **b** $\hat{\boldsymbol{i}}-\frac{1}{2}\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$ **c** $\frac{8\sqrt{5}}{21}$

11 a $34-4p$ **b** 8.5 **c** $\frac{5}{13}$

12 -6.5

13 $\lambda = \frac{3}{2},\ \mu = -\frac{3}{2}$

14 $AB \parallel DC,\ AB : CD = 1 : 2$

15 $\frac{\sqrt{19}}{5}$

16 a $(-1, 10)$ **b** $h = 3,\ k = -2$

17 a $2\boldsymbol{c},\ 2\boldsymbol{c}-\boldsymbol{a}$ **b** $\frac{1}{2}\boldsymbol{a}+\boldsymbol{c}$ **c** 1.5

18 $h = \frac{2}{3},\ k = \frac{3}{4}$

19 $3(\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}})$

20 a $\boldsymbol{c}-\boldsymbol{a}$

21 a i $\frac{1}{3}\boldsymbol{c}$ **ii** $\frac{2}{3}\boldsymbol{a}+\frac{1}{3}\boldsymbol{b}$ **iii** $\frac{2}{3}\boldsymbol{a}+\frac{1}{3}\boldsymbol{b}-\frac{1}{3}\boldsymbol{c}$

22 a $\frac{1}{4}\boldsymbol{a}+\frac{3}{4}\boldsymbol{b}$

b i $\frac{\lambda}{4}\boldsymbol{a}+\left(\frac{3\lambda}{4}-1\right)\boldsymbol{b}$ **ii** $\frac{4}{3}$

23 a i $\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ **ii** $\sqrt{3}$

b i $(\lambda-0.5)\hat{\boldsymbol{i}}+(\lambda-1)\hat{\boldsymbol{j}}+(\lambda-0.5)\hat{\boldsymbol{k}}$

ii $\lambda = \frac{2}{3},\ \overrightarrow{OQ} = \frac{1}{3}(8\hat{\boldsymbol{i}}+11\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}})$

c $5\hat{\boldsymbol{i}}+6\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$

24 a i $|\overrightarrow{OA}| = \sqrt{14},\ |\overrightarrow{OB}| = \sqrt{14}$ **ii** $\hat{\boldsymbol{i}}-5\hat{\boldsymbol{j}}$

b i $\frac{1}{2}(5\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}})$

c $5\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$

e i $5\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}-13\hat{\boldsymbol{k}}$ or $-5\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+13\hat{\boldsymbol{k}}$

iii The vector is perpendicular to the plane containing $OACB$

25 a $\overrightarrow{OX} = 7\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}},\ \overrightarrow{OY} = 2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}},$
$\overrightarrow{OZ} = 6\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}},\ \overrightarrow{OD} = 6\hat{\boldsymbol{i}}+3\hat{\boldsymbol{k}},\ |\overrightarrow{OD}| = 3\sqrt{5},$
$|\overrightarrow{OY}| = \sqrt{29}$

b $48.27°$

c i $\left(\frac{5\lambda}{\lambda+1}+1\right)\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}$ **ii** $-\frac{1}{6}$

26 a i $\boldsymbol{b}-\boldsymbol{a}$ **ii** $\boldsymbol{c}-\boldsymbol{b}$ **iii** $\boldsymbol{a}-\boldsymbol{c}$

iv $\frac{1}{2}(\boldsymbol{b}+\boldsymbol{c})$ **v** $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{c})$ **vi** $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$

27 a $\frac{1}{3}\boldsymbol{b}+\frac{2}{3}\boldsymbol{c}$

c ii $5 : 1$

d $1 : 3$

28 a i $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$ **ii** $-\frac{1}{2}\boldsymbol{a}+\left(\lambda-\frac{1}{2}\right)\boldsymbol{b}$

29 a i $12(1-a)$ **ii** 1

b i $x-4y+2 = 0$ **ii** $x = -2,\ y = 0$

c i $\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$ **ii** $\hat{\boldsymbol{i}}-12\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}}$

iii $3\hat{\boldsymbol{i}}-11\hat{\boldsymbol{j}}+7\hat{\boldsymbol{k}}$

d X has height 5 units; Y has height 7 units

30 a i $\frac{3}{4}\boldsymbol{c}$ **ii** $\frac{2}{5}\boldsymbol{a}+\frac{3}{5}\boldsymbol{c}$ **iii** $-\boldsymbol{a}+\frac{3}{4}\boldsymbol{c}$

b $\mu = \frac{5}{6},\ \lambda = \frac{2}{3}$

31 a $\boldsymbol{b} = q\hat{\boldsymbol{i}}-p\hat{\boldsymbol{j}},\ \boldsymbol{c} = -q\hat{\boldsymbol{i}}+p\hat{\boldsymbol{j}}$

b i $\overrightarrow{AB} = -(x+1)\hat{\boldsymbol{i}}-y\hat{\boldsymbol{j}},\ \overrightarrow{AC} = (1-x)\hat{\boldsymbol{i}}-y\hat{\boldsymbol{j}}$

ii $\overrightarrow{AE} = y\hat{\boldsymbol{i}}+(1-x)\hat{\boldsymbol{j}},\ \overrightarrow{AF} = -y\hat{\boldsymbol{i}}+(x+1)\hat{\boldsymbol{j}}$

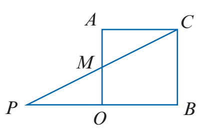
32 a i $\overrightarrow{BC} = m\boldsymbol{v},\ \overrightarrow{BE} = n\boldsymbol{v},\ \overrightarrow{CA} = m\boldsymbol{w},\ \overrightarrow{CF} = n\boldsymbol{w}$

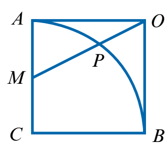
ii $|\overrightarrow{AE}| = \sqrt{m^2-mn+n^2},$
$|\overrightarrow{FB}| = \sqrt{m^2-mn+n^2}$

33 a $\overrightarrow{CF} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{c}$, $\overrightarrow{OE} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$
b ii $60°$
c ii HX is parallel to EX; KX is parallel to FX; HK is parallel to EF

34 a $\overrightarrow{OA} = -2(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$, $\overrightarrow{OB} = 2(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$, $\overrightarrow{OC} = 2(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$, $\overrightarrow{OD} = -2(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$
b $\overrightarrow{PM} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + h\hat{\boldsymbol{k}}$, $\overrightarrow{QN} = -3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + h\hat{\boldsymbol{k}}$
c $\overrightarrow{OX} = \frac{1}{2}\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}} + \frac{h}{2}\hat{\boldsymbol{k}}$
d i $\sqrt{2}$ **ii** $71°$
e ii $\sqrt{6}$

35 a i $\overrightarrow{OM} = \frac{a}{2}\hat{\boldsymbol{j}}$ **ii** $\overrightarrow{MC} = a\hat{\boldsymbol{i}} + \frac{a}{2}\hat{\boldsymbol{j}}$
b $\overrightarrow{MP} = a\lambda\hat{\boldsymbol{i}} + \frac{a\lambda}{2}\hat{\boldsymbol{j}}$,
$\overrightarrow{BP} = a(\lambda - 1)\hat{\boldsymbol{i}} + \frac{a}{2}(\lambda + 1)\hat{\boldsymbol{j}}$,
$\overrightarrow{OP} = a\lambda\hat{\boldsymbol{i}} + \frac{a}{2}(\lambda + 1)\hat{\boldsymbol{j}}$
c i $\lambda = \frac{3}{5}$, $|\overrightarrow{BP}| = \frac{2\sqrt{5}a}{5}$, $|\overrightarrow{OP}| = a$, $|\overrightarrow{OB}| = a$
ii $\frac{\sqrt{5}}{5}$
d $\lambda = -1$ and $\lambda = \frac{3}{5}$





e $\overrightarrow{OY} = \frac{14}{15}a\hat{\boldsymbol{i}} + \frac{29}{30}a\hat{\boldsymbol{j}} + \frac{1}{6}a\hat{\boldsymbol{k}}$

Multiple-choice questions

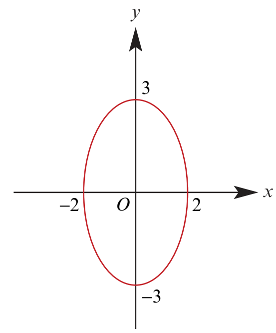
1 C	**2** D	**3** B	**4** B	**5** C
6 D	**7** B	**8** D	**9** B	**10** C
11 C	**12** B	**13** A	**14** C	**15** D
16 A	**17** D	**18** B	**19** C	**20** D

Chapter 6

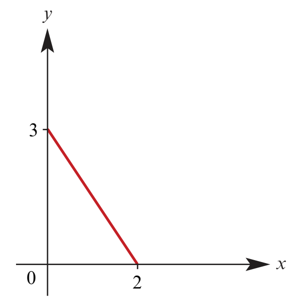
Exercise 6A

1 a $y = 2x$; dom $= \mathbb{R}$; ran $= \mathbb{R}$
b $x = 2$; dom $= \{2\}$; ran $= \mathbb{R}$
c $y = 7$; dom $= \mathbb{R}$; ran $= \{7\}$
d $y = 9 - x$; dom $= \mathbb{R}$; ran $= \mathbb{R}$
e $x = \frac{1}{9}(2 - y)^2$; dom $= [0, \infty)$; ran $= \mathbb{R}$
f $y = (x + 3)^3 + 1$; dom $= \mathbb{R}$; ran $= \mathbb{R}$
g $y = 3^{\left(\frac{x-1}{2}\right)}$; dom $= \mathbb{R}$; ran $= (0, \infty)$
h $y = \cos(2x + \pi) = -\cos(2x)$; dom $= \mathbb{R}$; ran $= [-1, 1]$
i $y = \left(\frac{1}{x} - 4\right)^2 + 1$; dom $= \mathbb{R} \setminus \{0\}$; ran $= [1, \infty)$
j $y = \frac{x}{1 + x}$; dom $= \mathbb{R} \setminus \{-1, 0\}$; ran $= \mathbb{R} \setminus \{0, 1\}$

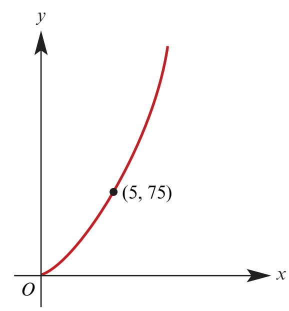
2 a $\frac{x^2}{4} + \frac{y^2}{9} = 1$
dom $= [-2, 2]$
ran $= [-3, 3]$



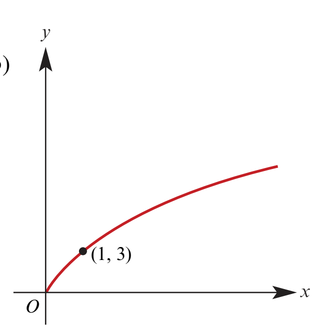
b $3x + 2y = 6$
dom $= [0, 2]$
ran $= [0, 3]$



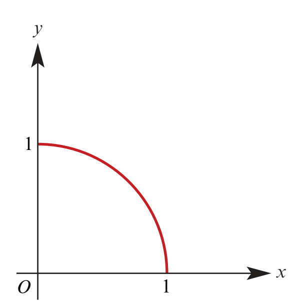
c $y = 3x^2$
dom $= [0, \infty)$
ran $= [0, \infty)$



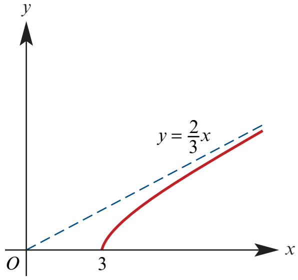
d $y = 3x^{\frac{2}{3}}$
dom $= [0, \infty)$
ran $= [0, \infty)$



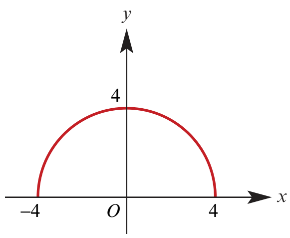
e $x^2 + y^2 = 1$
dom $= [0, 1]$
ran $= [0, 1]$



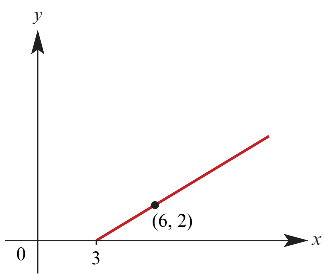
f $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$; dom $= (3, \infty)$; ran $= (0, \infty)$



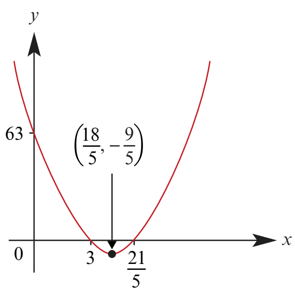
g $x^2 + y^2 = 16$; dom $= [-4, 4]$; ran $= [0, 4]$



h $3y = 2x - 6$; dom $= [3, \infty)$; ran $= [0, \infty)$



i $y = 5x^2 - 36x + 63$;
dom $= \mathbb{R}$; ran $= [-\frac{9}{5}, \infty)$

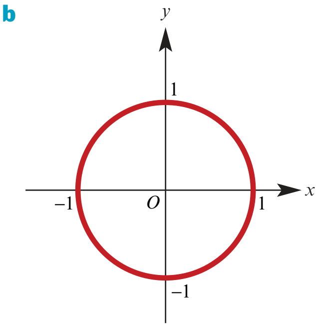


3 a $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + (3 - 2t)\hat{\boldsymbol{j}},\ t \in \mathbb{R}$
b $\boldsymbol{r}(t) = 2\cos t\,\hat{\boldsymbol{i}} + 2\sin t\,\hat{\boldsymbol{j}},\ t \in \mathbb{R}$
c $\boldsymbol{r}(t) = (2\cos t + 1)\hat{\boldsymbol{i}} + 2\sin t\,\hat{\boldsymbol{j}},\ t \in \mathbb{R}$
d $\boldsymbol{r}(t) = 2\sec t\,\hat{\boldsymbol{i}} + 2\tan t\,\hat{\boldsymbol{j}},\ t \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
e $\boldsymbol{r}(t) = t\hat{\boldsymbol{i}} + ((t - 3)^2 + 2(t - 3))\hat{\boldsymbol{j}},\ t \in \mathbb{R}$
f $\boldsymbol{r}(t) = \sqrt{6}\cos t\,\hat{\boldsymbol{i}} + 2\sin t\,\hat{\boldsymbol{j}},\ t \in \mathbb{R}$

4 a $\boldsymbol{r}(\theta) = (2 + 5\cos\theta)\hat{\boldsymbol{i}} + (6 + 5\sin\theta)\hat{\boldsymbol{j}}$
b $(x - 2)^2 + (y - 6)^2 = 25$

Exercise 6B

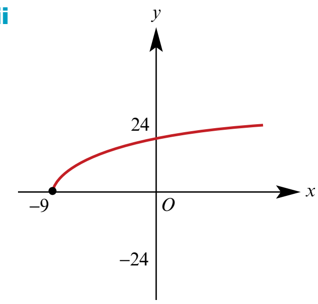
1 a $x^2 + y^2 = 1$
b



c $t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$

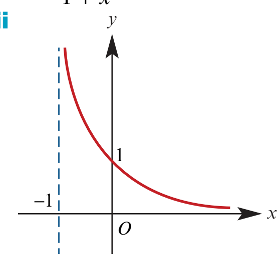
i.e. $t = \dfrac{(2n - 1)\pi}{2},\ n \in \mathbb{N}$

2 a i $x = \dfrac{y^2}{64} - 9$
ii



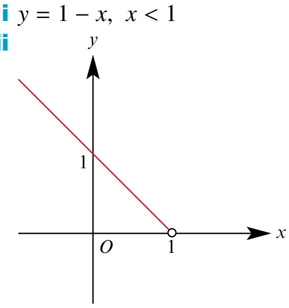
iii $t = 3$

b i $y = \dfrac{1}{1 + x},\ x > -1$
ii



iii $t = -1$

c i $y = 1 - x,\ x < 1$
ii



iii $t = 1$

3 a Position vector $\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$; Coordinates $(1, 4)$
b $(1, 4)$ and $(7, -8)$ **c** $\sqrt{65}$

4 a $\frac{9}{2}\hat{\boldsymbol{i}} - \frac{3}{2}\hat{\boldsymbol{j}}$, $\left(\frac{9}{2}, \frac{-3}{2}\right)$

b $(6, -1)$ and $\left(\frac{9}{2}, \frac{-3}{2}\right)$

c $5\sqrt{2}$

5 a $t = 9$ **b** $22\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 15\hat{\boldsymbol{k}}$

6 a $t = \frac{1}{2}$, $-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{k}}$

b $t = 2$, $4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

c $t = 2$, $4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

7 a $\sqrt{137}$

b $t = \frac{-2}{5}$ and $t = -1$

8 a $3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

b $3\sqrt{6}$

c $4\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

d $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

9 a $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

b $\sqrt{14}$

10 $a = \frac{2}{3}$, $b = 7$

11 a $\frac{x^2}{9} + \frac{y^2}{4} = 1$

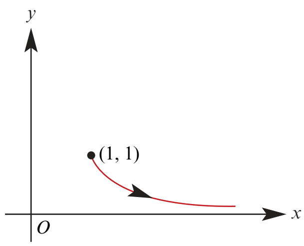
b $3\hat{\boldsymbol{i}}$

c i 303.69° **ii** 285.44°

12 a $y = \frac{1}{x}$, for $x \geq 1$

b $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

c



13 a $\boldsymbol{r}(0) = 2\hat{\boldsymbol{i}}$

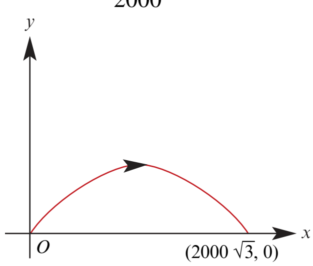
b $\frac{5}{2}\hat{\boldsymbol{i}} + \frac{3}{2}\hat{\boldsymbol{j}}$

c $x^2 - y^2 = 4$, $x \geq 2, y]ge0$

14 a $\boldsymbol{r}(0) = \boldsymbol{0}$, $\boldsymbol{r}(20\sqrt{3}) = 2000\sqrt{3}\hat{\boldsymbol{i}}$

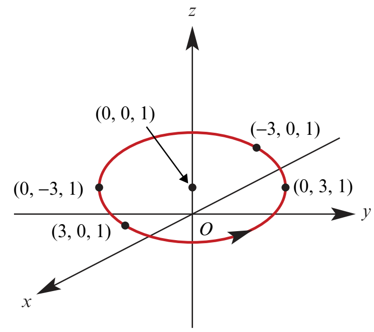
b $y = \sqrt{3}x - \frac{x^2}{2000}$, $0 \leq x \leq 2000\sqrt{3}$

c

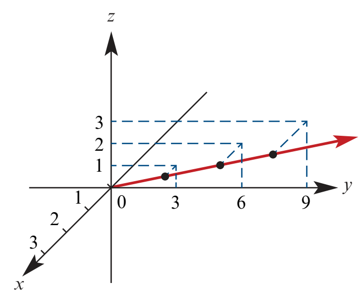


15 Collide when $t = \frac{3}{2}$; $\boldsymbol{r}\left(\frac{3}{2}\right) = \frac{27}{2}\hat{\boldsymbol{i}} - \frac{81}{4}\hat{\boldsymbol{j}}$

16 Particle is moving along a circular path, with centre $(0, 0, 1)$ and radius 3, starting at $(3, 0, 1)$ and moving anticlockwise; always a distance of 1 above the x–y plane. It takes 2π units of time to complete one circle.



17 Particle is moving along a straight line, starting at $(0, 0, 0)$, and moving 'forward 1', 'across 3' and 'up 1' at each step.



18 a $\frac{(x-1)^2}{4} + \frac{(y-3)^2}{25} = 1$

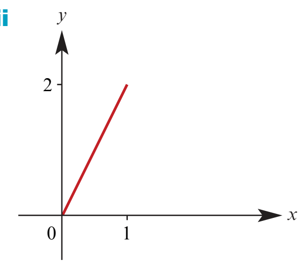
b i $(-1, 3)$ **ii** $(1, -2)$ **iii** $(3, 3)$

c π units of time

d Anticlockwise

19 a i $y = 2x$, $0 \leq x \leq 1$

ii



iii Particle starts at $(1, 2)$ and moves along a linear path towards the origin. When it reaches $(0, 0)$, it reverses direction and heads towards $(1, 2)$. It continues in this pattern, taking $\frac{1}{3}$ units of time to complete each cycle.

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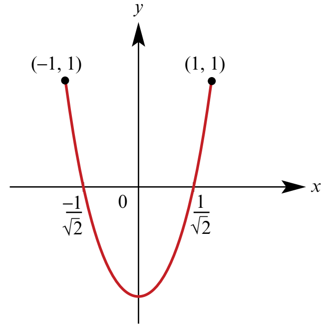
b **i** $y = 2x^2 - 1, \; -1 \le x \le 1$

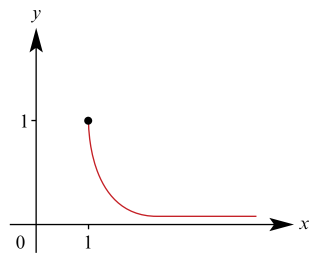
ii



iii Particle is moving along a parabolic path, starting at (1, 1) and reversing direction at (−1, 1). It takes 1 unit of time for each cycle.

c **i** $y = \dfrac{1}{x^2}, \; x \ge 1$

ii



iii Particle is moving along a 'truncus' path, starting at (1, 1) and moving to the 'right' indefinitely.

20 **a** $y = x^2 - 4x, \; y = 4x - 7$

b $(1, -3), \; (7, 21)$ **c** $t = \dfrac{7}{2}, \; (7, 21)$

Exercise 6C

1 **a** Yes **b** Yes **c** No

2 **a** $\boldsymbol{r} = \hat{\boldsymbol{i}} + (1 + 2t)\hat{\boldsymbol{j}}$
b $\boldsymbol{r} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$
c $\boldsymbol{r} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + t(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
d $\boldsymbol{r} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$

3 **a** $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(-5\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$
b $\boldsymbol{r} = -\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} + t(3\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}})$
c $\boldsymbol{r} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}})$
d $\boldsymbol{r} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

4 **a** **i** $x = 3 - 5t, \; y = 1 + t$
ii $y = \dfrac{1}{5}(8 - x)$
b **i** $x = -1 + 3t, \; y = 5 - 6t$
ii $y = -2x + 3$
c **i** $x = 1 + t, \; y = 2 - 2t, \; z = 3 - 4t$
ii $x - 1 = \dfrac{2 - y}{2} = \dfrac{3 - z}{4}$
d **i** $x = 1 + t, \; y = -4 + 7t, \; z = t$
ii $x - 1 = \dfrac{y + 4}{7} = z$

5 **a** $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + t(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}})$
b $\boldsymbol{r} = 4\hat{\boldsymbol{j}} + t(-9\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}})$
c $\boldsymbol{r} = 6\hat{\boldsymbol{i}} + t(-6\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$

6 **a** $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$
b $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$

7 **a** $\boldsymbol{r} = t(2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$ **b** $\boldsymbol{r} = t(\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

8 **a** $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(-3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$
b **i** No **ii** No **iii** Yes

9 **a** $\boldsymbol{r} = \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
c $m = -\dfrac{5}{3}, \; n = -\dfrac{4}{3}$

10 **a** $4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **b** $\boldsymbol{r} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + t(4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$
c $\left(-\dfrac{7}{3}, 0\right), \left(0, \dfrac{7}{4}\right)$

11 **a** $x = 2 - 3t, \; y = 5 + t, \; z = 4 - 2t;$
$\dfrac{2 - x}{3} = y - 5 = \dfrac{4 - z}{2}$
b $x = 2t, \; y = 2 + t, \; z = -1 + 4t;$
$\dfrac{x}{2} = y - 2 = \dfrac{z + 1}{4}$

12 **a** $(2t + 1)\hat{\boldsymbol{i}} + (2 - 3t)\hat{\boldsymbol{j}} + 5t\hat{\boldsymbol{k}}$
b $\dfrac{4\sqrt{133}}{19}$

13 **a** $\dfrac{11\sqrt{114}}{38}$ **b** $\dfrac{\sqrt{23\,123}}{19}$ **c** $\sqrt{17}$
d 3

14 **c** $t \in [-3, 2]$

15 $\left(\dfrac{7}{3}, \dfrac{2}{3}, \dfrac{8}{3}\right)$

16 $\boldsymbol{r} = (t - 2)\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

17 $\dfrac{\sqrt{165}}{3}$

18 **a** $(-1, -1, 3), (-5, 1, 7)$
b $(2, 6, -4), (5, 0, 2)$

19 **a** $(1, 1, 2)$ **c** $t = 2$

20 **a** $(1 + t)\hat{\boldsymbol{i}} + (-4 + 2t)\hat{\boldsymbol{j}} + (1 - t)\hat{\boldsymbol{k}}$
b $\sqrt{18 - 16t + 6t^2}$ **c** $\dfrac{\sqrt{66}}{3}$ **d** $\dfrac{\sqrt{786}}{6}$

Exercise 6D

1 $\dfrac{17}{2}\hat{\boldsymbol{i}} + \dfrac{9}{4}\hat{\boldsymbol{j}}$

2 $(-1, 2, 3)$

4 **a** **i** No **ii** No **iii** No **iv** $(-7, -6)$
b **i** No **ii** Yes **iii** No **iv** $(-1, 1)$
c **i** Yes **ii** No **iii** Yes
d **i** Yes **ii** No **iii** No **iv** None
e **i** No **ii** Yes **iii** No **iv** $(3, 1, -2)$
f **i** No **ii** No **iii** No **iv** None
g **i** No **ii** No **iii** No **iv** $(3, 0, -1)$
h **i** Yes **ii** No **iii** Yes
i **i** No **ii** No **iii** No **iv** $(0, 1, -2)$
j **i** Yes **ii** No **iii** Yes

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5 a $(1, 2, -1)$ **b** None **c** None **d** None

6 a $25.21°$ **b** $0°$

7 a $30°$

8 a $(3, 3, 1)$ **b** $\frac{1}{\sqrt{15}}$

9
- Lines ℓ_1 and ℓ_2 do not intersect
- Lines ℓ_1 and ℓ_3 intersect at $(2, 3, -1)$
- Lines ℓ_2 and ℓ_3 intersect at $(4, -5, -1)$

Exercise 6E

1 a $-3\hat{i} + 4\hat{j} + 19\hat{k}$ **b** $\hat{i} - 7\hat{j} - 4\hat{k}$
c $\hat{i} - \hat{j}$ **d** $\hat{i} + 2\hat{k}$
e $-9\hat{i} - 26\hat{j} - 12\hat{k}$ **f** $2\hat{j} + \hat{k}$
g $2\hat{j} + \hat{k}$ **h** $\hat{i} - 2\hat{k}$

2 a $\boldsymbol{a} \times \boldsymbol{b}$ **b** $\boldsymbol{0}$ **c** $2(\boldsymbol{a} \times \boldsymbol{b})$
d $(\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$ **e** 0 **f** 0

3 $\frac{\sqrt{10}}{6}(4\hat{i} - 5\hat{j} - 7\hat{k})$

4 $\hat{i} + \hat{j}$ is a possible vector.

5 1

6 $\frac{\sqrt{374}}{2}$

Exercise 6F

1 a $\boldsymbol{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3,\ x + y + z = 3$
b $\boldsymbol{r} \cdot (\hat{i} - 2\hat{k}) = 3,\ x - 2z = 3$
c $\boldsymbol{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0,\ 2x + 3y - z = 0$
d $\boldsymbol{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = -8,\ x + 3y - z = -8$

2 $\frac{1}{\sqrt{170}}(12\hat{i} + 5\hat{j} + \hat{k}),\ \boldsymbol{r} \cdot (12\hat{i} + 5\hat{j} + \hat{k}) = 28$

3 $\boldsymbol{r} \cdot (\hat{i} - \hat{j} - 3\hat{k}) = -1,\ x - y - 3z = -1$

4 a $5\hat{i} + 4\hat{j} + 13\hat{k}$
b $\boldsymbol{r} = -3\hat{i} + \hat{j} + \hat{k} + t(5\hat{i} + 4\hat{j} + 13\hat{k})$

5 $\frac{1}{\sqrt{77}}(-6\hat{i} + 5\hat{j} + 4\hat{k}),\ \boldsymbol{r} \cdot (-6\hat{i} + 5\hat{j} + 4\hat{k}) = 11$

8 a $x = 0$ **b** $x = 6$ **c** $x = 3$ **d** $x = 4$

9 $6x + 2y + z = 10$ **10** $x - 2y + 8z = 7$

11 $5x - 3y + 2z = 27$ **12** $13x + 7y + 9z = 61$

13 $-3x + 8y + 7z = 41$

Exercise 6G

1 a 2 **b** $\frac{22}{9}$

2 $\frac{8}{3}$

3 a $80.41°$ **b** $\boldsymbol{r} = 22\hat{j} + 14\hat{k} + t(\hat{i} - 5\hat{j} - 3\hat{k})$

4 a $(-1, -9, 7)$ **b** $7.82°$

5 a $7\hat{i} + \hat{j} + 5\hat{k}$ **b** $\hat{i} + 3\hat{j} - 5\hat{k}$
c $72.98°$

6 a $(2, -2, -1),\ 29.50°$
b $\left(\frac{7}{2}, -\frac{3}{2}, -\frac{5}{2}\right),\ 32.98°$
c $\left(\frac{1}{2}, \frac{3}{2}, -\frac{7}{2}\right),\ 79.98°$ **d** $(-7, 4, -3),\ 7.45°$

7 a $\boldsymbol{r} \cdot (\hat{i} - 2\hat{j} + 6\hat{k}) = -9$ **b** $\frac{9}{\sqrt{41}}$

8 a $\frac{7}{3}$ **b** $\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$ **c** 1 **d** $\frac{4}{3}$

9 $\frac{5}{3}$

10 a $(5, -1, -1)$ **b** $25.7°$

11 a $x + y + z = 4$ **b** $2\sqrt{3}$
c $\frac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

12 a $88.18°$
b $\boldsymbol{r} = -\frac{5}{2}\hat{j} - 9\hat{k} + t\left(\hat{i} + \frac{19}{2}\hat{j} + 30\hat{k}\right)$

13 a $\hat{i} - 5\hat{j} - 3\hat{k}$ **b** $2\hat{i} + 3\hat{j} + 7\hat{k}$
c $43.12°$

14 a $-6\hat{i} - 4\hat{j} + \hat{k}$
b $\boldsymbol{r} = 2\hat{i} + \hat{j} - 2\hat{k} + t(-6\hat{i} - 4\hat{j} + \hat{k})$

Exercise 6H

1 a $(x + 1)^2 + (y - 3)^2 + (z - 2)^2 = 4$
b $|\boldsymbol{r} - (-\hat{i} + 3\hat{j} + 2\hat{k})| = 2$

2 a $(x + 1)^2 + (y + 3)^2 + (z - 1)^2 = 16$
b $|\boldsymbol{r} - (-\hat{i} - 3\hat{j} + \hat{k})| = 4$

3 $\left(\frac{8}{\sqrt{17}}, \frac{12}{\sqrt{17}}, -\frac{8}{\sqrt{17}}\right)$

4 $\left(1 + \frac{\sqrt{22}}{2}, 1 + \frac{\sqrt{22}}{2}, 1 - \sqrt{22}\right)$,
$\left(1 - \frac{\sqrt{22}}{2}, 1 - \frac{\sqrt{22}}{2}, 1 + \sqrt{22}\right)$

5 $(2 + 3\sqrt{2}, 3 + 3\sqrt{2}, 4),\ (2 - 3\sqrt{2}, 3 - 3\sqrt{2}, 4)$

6 a $(0, 0, 3),\ 3\sqrt{3}$ **b** $(3, 0, 0),\ 3\sqrt{3}$
c $(0, 0, 0),\ 6$

7 $(1, 2, -4),\ 2$

8 a $(x - 1)^2 + y^2 + (z + 1)^2 = 16$
b $(x - 1)^2 + (y + 3)^2 + (z - 2)^2 = 14$
c $(x - 3)^2 + (y + 2)^2 + (z - 4)^2 = 33$
d $x^2 + y^2 + z^2 = 9$

9 a $(0, 0, 0),\ (4, 0, 0),\ (0, 6, 0),\ (0, 0, 8)$
b $\boldsymbol{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + t(2\hat{i} - 3\hat{j} - 4\hat{k})$
c $2x - 3y - 4z = 8$

Exercise 6I

1 a $\boldsymbol{r} = \hat{i} + \hat{j} + \hat{k} - 2s\hat{j} - 2t\hat{k}$
b $\boldsymbol{r} = 2\hat{i} - \hat{k} + s(-\hat{i} - \hat{j} + 3\hat{k}) + t(\hat{i} + \hat{j})$
c $\boldsymbol{r} = -\hat{i} + \hat{j} + s(3\hat{i} - 2\hat{j} + 2\hat{k}) + t(\hat{i} - \hat{k})$
d $\boldsymbol{r} = \hat{i} + \hat{k} + s(-2\hat{i} - 2\hat{j} + 2\hat{k}) + t(\hat{i} + \hat{i} - 2\hat{k})$

2 a $s = 4, t = 3$

3 a $\frac{5}{4}$ **b** -4 **c** $-\frac{9}{7}$

5 $3x + 4y + 2z = 7$

6 a $\boldsymbol{r} = (\hat{i} + 2\hat{j} + \hat{k}) + s(\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} - \hat{j} + \hat{k})$
b $\boldsymbol{r} \cdot (\hat{i} + 3\hat{j} + \hat{k}) = 8$ **c** $x + 3y + z = 8$
d $x + 3y + z = 6$ **e** $\frac{2}{\sqrt{11}}$

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7 a $\mathbf{r} \cdot (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 4$
b $\mathbf{r} = 4\hat{\mathbf{i}} + s(-4\hat{\mathbf{i}} - 2\hat{\mathbf{j}} +) + t(-4\hat{\mathbf{i}} + 4\hat{\mathbf{k}})$
c $x = 4 - 4s - 4t,\ y = -2s,\ z = 4t$

8 a i $x + y + z = 5$
ii $\mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$
b i $11x + 5y + 9z = 1$
ii $\mathbf{r} \cdot (11\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 9\hat{\mathbf{k}}) = 1$
c i $3x + 4y + 5z = 7$
ii $\mathbf{r} \cdot (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) = 7$

9 a $\overrightarrow{OA} = 4\hat{\mathbf{i}} + \hat{\mathbf{k}}$
b $18°$

Chapter 6 review

Short-response questions

1 a $7\hat{\mathbf{i}} + \hat{\mathbf{j}}$ **b** $t = 1$

2 a $(7, 21)$ **b** $(7, 21),\ \left(\frac{61}{3}, \frac{469}{9}\right)$ **c** $2\sqrt{10}$

3 a $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **b** $y = \frac{3x^2}{4}$ for $x \geq 6$
c $x = \frac{2y^2}{3} - 3$ for $-3 \leq y \leq 3$
d $(x - 5)^2 + \frac{y^2}{16} = 1$

4 a Domain $= [-3, 3]$; range $= [-4, 4]$
b Domain $= [6, \infty)$; range $= [27, \infty)$
c Domain $= [-3, 3]$; range $= [-3, 3]$
d Domain $= [4, 6]$; range $= [-4, 4]$

5 $4\hat{\mathbf{i}} - \hat{\mathbf{k}}$

7 $4x + 5y + 6z = 32$

8 $\mathbf{r} = (t - 2)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

9 $\left(\frac{7}{3}, \frac{2}{3}, \frac{8}{3}\right)$ **10** $\frac{\sqrt{165}}{3}$

11 $(-1, -1, 4)$ **12** $2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

13 $\mathbf{r} = 5\hat{\mathbf{i}} + 6\hat{\mathbf{k}} + t(6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 9\hat{\mathbf{k}})$; $(1, -2, 0)$

14 $\frac{\sqrt{3}}{2}$ **15** $2x - 8y + 5z = 18$

16 $12x + 8y + 20z = 16$

17 a $\mathbf{r} \cdot (\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = 19$
b $x + 10y + 6z = 19$

18 a $x - 2y + z = 0$ **b** $\frac{\sqrt{6}}{2}$ **c** $\left(0, \frac{4}{3}, \frac{8}{3}\right)$

19 $(1, -2, 1)$ or $(1, 1, -2)$

22 a $(3, -1, 2)$ **b** $\frac{\sqrt{30}}{3}$

23 c $\sqrt{30}$ **d** $47.73°$ **e** $k = 2$ or $k = 80$

24 c $\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} + t(5\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + \hat{\mathbf{k}})$
d $(13, -12, 3),\ 10\sqrt{3}$

26 a $\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$
b $\sqrt{19}$
c $8x - 11y - 9z = 0$
d $29.9°$

27 a $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
b $\mathbf{r}(t) = 2\hat{\mathbf{i}} + 1\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) + t(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
c $\frac{5}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} - \frac{7}{3}\hat{\mathbf{k}}$

28 a $\lambda = -3$
b $-\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$
c $-x + 5y - 3z = 5$

29 a $-x + 5y + 3z = 0$
b $-x + 5y + 3z = -10$
c $\frac{2\sqrt{35}}{7}$

30 a $(7, 0, 0), (-5, 0, 0)$
c $\left(\frac{14}{3}, -\frac{2}{3}, \frac{7}{3}\right)$

31 a $x + y + 2z = 3$
b $\mathbf{v} = (1 - \mu)\hat{\mathbf{i}} + (2\lambda - \mu)\hat{\mathbf{j}} + (1 - \lambda + \mu)\hat{\mathbf{k}}$.

32 a $x = 3 - 2t,\ y = 2 + t,\ z = t$
b $\mathbf{r}(t) = (2 - 2t)\hat{\mathbf{i}} + (2 + t)\hat{\mathbf{j}} + t\hat{\mathbf{k}}$
c $(1, 3, 1)$

33 d $(x - 2y + 6)^2 + (x - 8z - 10)^2 = 0$
e $(x + y - 5)^2 + (5x - z - 10)^2 = 0$

34 a $5\hat{\mathbf{i}}$
b i $(5 - 3t_1)\hat{\mathbf{i}} + 2t_1\hat{\mathbf{j}} + t_1\hat{\mathbf{k}}$,
$(5 - 3t_2)\hat{\mathbf{i}} + 2t_2\hat{\mathbf{j}} + t_2\hat{\mathbf{k}}$
ii $-3(t_2 - t_1)\hat{\mathbf{i}} + 2(t_2 - t_1)\hat{\mathbf{j}} + (t_2 - t_1)\hat{\mathbf{k}}$
c $-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$
d i $36.70°$
ii 13.42

35 a $y = 5 - 2x,\ x \leq 2$
b i $\mathbf{r}_1(t) = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + t(-\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$
ii $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is the starting position; $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ is the velocity
c i $-13\hat{\mathbf{i}} + 6\hat{\mathbf{j}}$ **ii** $5\sqrt{10}$

36 a $13\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$
b $\frac{\sqrt{14}}{14}(-3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}),\ \frac{\sqrt{6}}{6}(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$
c $40.20°$ **d** $7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$
e $13\hat{\mathbf{i}} - \hat{\mathbf{j}} - 8\hat{\mathbf{k}} + t(-5\hat{\mathbf{i}} + 3\hat{\mathbf{k}})$ **f** $\frac{\sqrt{1190}}{34}$

37 a $\sqrt{2}$ **b** $\frac{\sqrt{3}}{2}$
c $x + y + z = 1,\ x - y - z = -1,\ \cos^{-1}\left(\frac{1}{3}\right)$

38 a $\sqrt{2}$ **b** $\frac{\sqrt{3}}{2}$
d $x + y + z = 1,\ x - y - z = 1,\ \cos^{-1}\left(-\frac{1}{3}\right)$

Multiple-choice questions

1 B	**2** A	**3** C
4 D	**5** D	**6** D
7 D	**8** C	**9** A
10 D	**11** D	**12** C
13 C	**14** A	**15** B
16 B	**17** C	**18** A
19 C	**20** B	

Chapter 7

Exercise 7A

1 a $x^4(5\sin x + x\cos x)$ **b** $\sqrt{x}\left(\dfrac{\cos x}{2x} - \sin x\right)$

c $e^x(\cos x - \sin x)$ **d** $x^2e^x(3 + x)$

e $\cos^2 x - \sin^2 x = \cos(2x)$

2 a $e^x(\tan x + \sec^2 x)$ **b** $x^3(4\tan x + x\sec^2 x)$

c $\sec^2 x \ln x + \dfrac{\tan x}{x}$ **d** $\sin x\,(1 + \sec^2 x)$

e $\sqrt{x}\left(\dfrac{\tan x}{2x} + \sec^2 x\right)$

3 a $\dfrac{\ln x - 1}{(\ln x)^2}$

b $\sqrt{x}\left(\dfrac{\cot x}{2x} - \operatorname{cosec}^2 x\right)$

c $e^x(\cot x - \operatorname{cosec}^2 x)$ **d** $\dfrac{\sec^2 x}{\ln x} - \dfrac{\tan x}{x(\ln x)^2}$

e $\dfrac{\cos x}{x^2} - \dfrac{2\sin x}{x^3}$

f $\sec x\,(\sec^2 x + \tan^2 x)$

g $\dfrac{-(\sin x + \cos x)}{e^x}$ **h** $-\operatorname{cosec}^2 x$

4 a $2x\sec^2(x^2 + 1)$ **b** $\sin(2x)$

c $e^{\tan x}\sec^2 x$ **d** $5\tan^4 x\sec^2 x$

e $\dfrac{\sqrt{x}\cos(\sqrt{x})}{2x}$ **f** $\dfrac{1}{2}\sec^2 x\sqrt{\cot x}$

g $x^{-2}\sin\left(\dfrac{1}{x}\right)$ **h** $2\tan x\sec^2 x$

i $\dfrac{1}{4}\sec^2\left(\dfrac{x}{4}\right)$ **j** $-\operatorname{cosec}^2 x$

5 a $k\sec^2(kx)$ **b** $2\sec^2(2x)\,e^{\tan(2x)}$

c $6\tan(3x)\sec^2(3x)$ **d** $e^{\sin x}\left(\dfrac{1}{x} + \ln x\cos x\right)$

e $6x\sin^2(x^2)\cos(x^2)$

f $e^{3x+1}\sec^2 x\,(3\cos x + \sin x)$

g $e^{3x}(3\tan(2x) + 2\sec^2(2x))$

h $\dfrac{\sqrt{x}\tan(\sqrt{x})}{2x} + \dfrac{\sec^2(\sqrt{x})}{2}$

i $\dfrac{2(x+1)\tan x\sec^2 x - 3\tan^2 x}{(x+1)^4}$

j $20x\sec^3(5x^2)\sin(5x^2)$

6 a $5(x-1)^4$ **b** $\dfrac{1}{x}$

c $e^x(3\sec^2(3x) + \tan(3x))$

d $-\sin x\,e^{\cos x}$ **e** $-12\cos^2(4x)\sin(4x)$

f $4\cos x\,(\sin x + 1)^3$

g $-\sin x\sin(2x) + 2\cos(2x)\cos x$ **h** $1 - \dfrac{1}{x^2}$

i $\dfrac{x^2(3\sin x - x\cos x)}{\sin^2 x}$ **j** $\dfrac{-(1 + \ln x)}{(x\ln x)^2}$

7 a $224(2x+5)^6$

b $-4\sin(2x)$

c $-\dfrac{1}{9}\cos\left(\dfrac{x}{3}\right)$

d $2\sin x\sec^3 x$

e $16e^{-4x}$

f $\dfrac{-1}{x^2}$

g $-\operatorname{cosec}^2 x$

h $18\sin(1-3x)\sec^3(1-3x)$

i $\dfrac{1}{9}\sec\left(\dfrac{x}{3}\right)\left(2\tan^2\left(\dfrac{x}{3}\right) + 1\right)$

j $\dfrac{1 + \cos^2\left(\dfrac{x}{4}\right)}{16\sin^3\left(\dfrac{x}{4}\right)}$

8 a $-\dfrac{1}{2}\cos\left(2x + \dfrac{\pi}{4}\right)$ **b** $\dfrac{1}{\pi}\sin(\pi x)$

c $-\dfrac{3}{2\pi}\cos\left(\dfrac{2\pi x}{3}\right)$ **d** $\dfrac{1}{3}e^{3x+1}$

e $\dfrac{1}{5}e^{5(x+4)}$ **f** $-\dfrac{3}{2x}$

g $\dfrac{3}{2}x^4 - \dfrac{2}{3}x^3 + 2x^2 + x$

h $\dfrac{2}{3}\sin\left(\dfrac{3x}{2}\right)$

9 a $\dfrac{(3x+2)^6}{18}$ **b** $\dfrac{1}{3}\ln(3x-2)$

c $\dfrac{2}{9}(3x+2)^{\frac{3}{2}}$ **d** $-\dfrac{1}{3(3x+2)}$

e $\dfrac{3}{20}(5x-1)^{\frac{4}{3}}$ **f** $3x - 2\ln(x+1)$

g $2x - 5\ln(x+3)$

10 a $\dfrac{x^3}{3} + 6$

b $3 - \cos x$

c $\dfrac{x^3}{3} + x^2 - 3x + \dfrac{17}{3}$

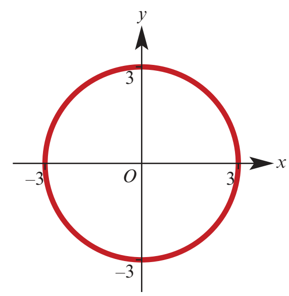
11 a $-4.9x^2 + 20$

b $\dfrac{x^3}{3} + 4x + \dfrac{2}{3}$

c $4x - \cos(x)$

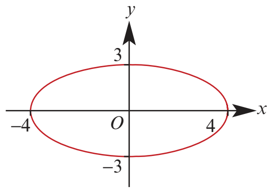
Exercise 7B

1 a i $x^2 + y^2 = 9$,
dom $= [-3, 3]$, ran $= [-3, 3]$



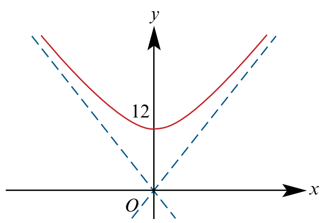
ii Particle starts at $(3, 0)$ and moves clockwise with period 6

b **i** $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$,
dom $= [-4, 4]$, ran $= [-3, 3]$



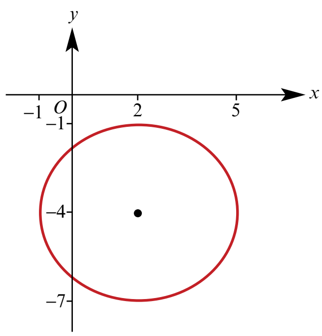
ii Particle starts at $(-4, 0)$ and moves anticlockwise with period 6

c **i** $\dfrac{y^2}{144} - \dfrac{x^2}{25} = 1$, dom $= \mathbb{R}$, ran $= [12, \infty)$



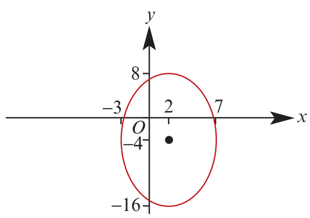
ii Particle moves from right to left

d **i** $(x-2)^2 + (y+4)^2 = 9$,
dom $= [-1, 5]$, ran $= [-7, -1]$



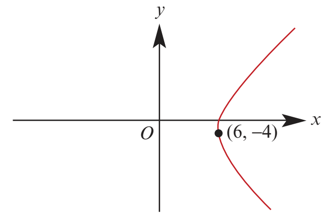
ii Particle starts at $(-1, -4)$ and moves anticlockwise with period 2π

e **i** $\dfrac{(x-2)^2}{25} + \dfrac{(y+4)^2}{144} = 1$,
dom $= [-3, 7]$, ran $= [-16, 8]$



ii Particle starts at $(-3, -4)$ and moves anticlockwise with period 2π

f **i** $\dfrac{(x-2)^2}{16} - \dfrac{(y+4)^2}{144} = 1$,
dom $= [6, \infty)$, ran $= \mathbb{R}$

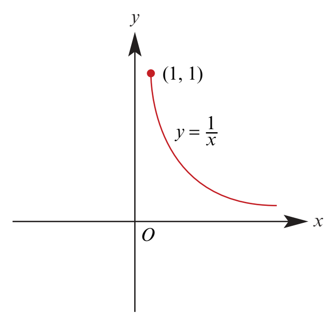


ii Particle moves downwards

Exercise 7C

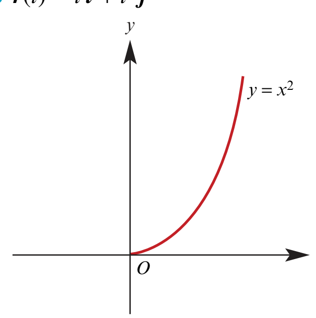
1 **a** $\dot{\boldsymbol{r}}(t) = e^t\hat{\boldsymbol{i}} - e^{-t}\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(t) = e^t\hat{\boldsymbol{i}} + e^{-t}\hat{\boldsymbol{j}}$
b $\dot{\boldsymbol{r}}(t) = \hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(t) = 2\hat{\boldsymbol{j}}$
c $\dot{\boldsymbol{r}}(t) = \frac{1}{2}\hat{\boldsymbol{i}} + 2t\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(t) = 2\hat{\boldsymbol{j}}$
d $\dot{\boldsymbol{r}}(t) = 16\hat{\boldsymbol{i}} - 32(4t-1)\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(t) = -128\hat{\boldsymbol{j}}$
e $\dot{\boldsymbol{r}}(t) = \cos t\,\hat{\boldsymbol{i}} - \sin t\,\hat{\boldsymbol{j}}$,
$\ddot{\boldsymbol{r}}(t) = -\sin t\,\hat{\boldsymbol{i}} - \cos t\,\hat{\boldsymbol{j}}$
f $\dot{\boldsymbol{r}}(t) = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(t) = \boldsymbol{0}$
g $\dot{\boldsymbol{r}}(t) = 100\hat{\boldsymbol{i}} + (100\sqrt{3} - 9.8t)\hat{\boldsymbol{j}}$,
$\ddot{\boldsymbol{r}}(t) = -9.8\hat{\boldsymbol{j}}$
h $\dot{\boldsymbol{r}}(t) = \sec^2 t\,\hat{\boldsymbol{i}} - \sin(2t)\,\hat{\boldsymbol{j}}$,
$\ddot{\boldsymbol{r}}(t) = (2\sec^2 t\,\tan t)\hat{\boldsymbol{i}} - 2\cos(2t)\hat{\boldsymbol{j}}$

2 **a** $\boldsymbol{r}(t) = e^t\hat{\boldsymbol{i}} + e^{-t}\hat{\boldsymbol{j}}$



$\boldsymbol{r}(0) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\dot{\boldsymbol{r}}(0) = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(0) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

b $\boldsymbol{r}(t) = t\,\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$



$\boldsymbol{r}(1) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\dot{\boldsymbol{r}}(1) = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(1) = 2\hat{\boldsymbol{j}}$

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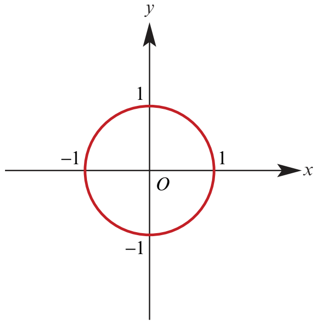
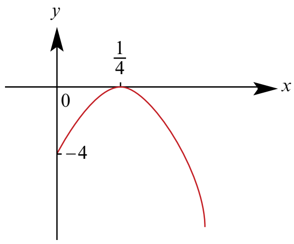
c $\boldsymbol{r}(t) = \sin t\,\hat{\boldsymbol{i}} + \cos t\,\hat{\boldsymbol{j}}$



$\boldsymbol{r}\left(\frac{\pi}{6}\right) = \frac{1}{2}\hat{\boldsymbol{i}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{j}}, \quad \dot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\hat{\boldsymbol{i}} - \frac{1}{2}\hat{\boldsymbol{j}},$

$\ddot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = -\frac{1}{2}\hat{\boldsymbol{i}} - \frac{\sqrt{3}}{2}\hat{\boldsymbol{j}}$

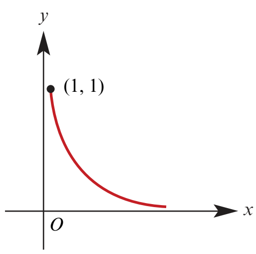
d $\boldsymbol{r}(t) = 16t\,\hat{\boldsymbol{i}} - 4(4t-1)^2\hat{\boldsymbol{j}}$



$\boldsymbol{r}(1) = 16\hat{\boldsymbol{i}} - 36\hat{\boldsymbol{j}}, \quad \dot{\boldsymbol{r}}(1) = 16\hat{\boldsymbol{i}} - 96\hat{\boldsymbol{j}},$

$\ddot{\boldsymbol{r}}(1) = -128\hat{\boldsymbol{j}}$

e $\boldsymbol{r}(t) = \frac{1}{t+1}\hat{\boldsymbol{i}} + (t+1)^2\hat{\boldsymbol{j}}$



$\boldsymbol{r}(1) = \frac{1}{2}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}, \quad \dot{\boldsymbol{r}}(1) = -\frac{1}{4}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}},$

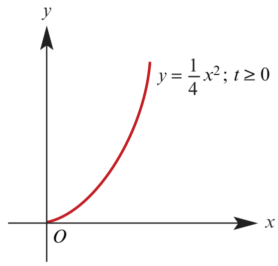
$\ddot{\boldsymbol{r}}(1) = \frac{1}{4}\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

3 **a** -1 **b** Undefined **c** $-2e^{-3}$
d $\frac{1}{2}$ **e** 4 **f** $2\sqrt{2}$

4 **a** $\boldsymbol{r}(t) = (4t+1)\hat{\boldsymbol{i}} + (3t-1)\hat{\boldsymbol{j}}$
b $\boldsymbol{r}(t) = (t^2+1)\hat{\boldsymbol{i}} + (2t-1)\hat{\boldsymbol{j}} - t^3\hat{\boldsymbol{k}}$
c $\boldsymbol{r}(t) = \frac{1}{2}e^{2t}\hat{\boldsymbol{i}} + 4(e^{0.5t}-1)\hat{\boldsymbol{j}}$
d $\boldsymbol{r}(t) = \left(\frac{t^2+2t}{2}\right)\hat{\boldsymbol{i}} + \frac{1}{3}t^3\hat{\boldsymbol{j}}$
e $\boldsymbol{r}(t) = -\frac{1}{4}\sin(2t)\,\hat{\boldsymbol{i}} + 4\cos\left(\frac{1}{2}t\right)\hat{\boldsymbol{j}}$

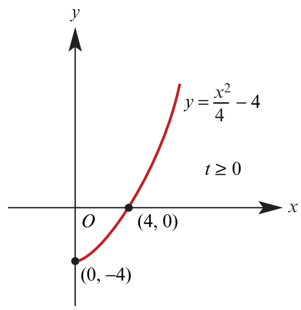
6 **a** $t = 0, 2$
b $\dot{\boldsymbol{r}}(0) = 2\hat{\boldsymbol{i}}$ and $\ddot{\boldsymbol{r}}(0) = 96\hat{\boldsymbol{j}}$;
$\dot{\boldsymbol{r}}(2) = 2\hat{\boldsymbol{i}}$ and $\ddot{\boldsymbol{r}}(2) = -96\hat{\boldsymbol{j}}$

7 **a**



b $t = \frac{2}{a}$

8 **a**



b 45° **c** $t = \sqrt{3}$

9 **a** $\dot{\boldsymbol{r}} = 3\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}} + 3t^2\hat{\boldsymbol{k}}$ **b** $|\dot{\boldsymbol{r}}| = \sqrt{9+10t^4}$
c $\ddot{\boldsymbol{r}} = 2t\hat{\boldsymbol{j}} + 6t\hat{\boldsymbol{k}}$ **d** $|\ddot{\boldsymbol{r}}| = 2\sqrt{10}\,t$
e $t = \frac{4\sqrt{10}}{5}$

10 **a** $\dot{\boldsymbol{r}} = V\cos\alpha\,\hat{\boldsymbol{i}} + (V\sin\alpha - gt)\hat{\boldsymbol{j}}$ **b** $\ddot{\boldsymbol{r}} = -g\hat{\boldsymbol{j}}$
c $t = \frac{V\sin\alpha}{g}$
d $\boldsymbol{r} = \frac{V^2\sin(2\alpha)}{2g}\hat{\boldsymbol{i}} + \frac{V^2\sin^2\alpha}{2g}\hat{\boldsymbol{j}}$

Exercise 7D

1 **a** $2t\,\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ **b** $2\hat{\boldsymbol{i}}$ **c** $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

2 **a** $2\hat{\boldsymbol{i}} + (6 - 9.8t)\hat{\boldsymbol{j}}$
b $2t\,\hat{\boldsymbol{i}} + (6t - 4.9t^2 + 6)\hat{\boldsymbol{j}}$

3 **a** $2\hat{\boldsymbol{j}} - 4\hat{\boldsymbol{k}}$
b $3t\,\hat{\boldsymbol{i}} + (t^2+1)\hat{\boldsymbol{j}} + (t - 2t^2 + 1)\hat{\boldsymbol{k}}$
c $\sqrt{20t^2 - 8t + 10}$
d **i** $\frac{1}{5}$ seconds **ii** $\frac{1}{5}\sqrt{230}$ m/s

4 **a** $(10t+20)\hat{\boldsymbol{i}} - 20\hat{\boldsymbol{j}} + (40 - 9.8t)\hat{\boldsymbol{k}}$
b $(5t^2 + 20t)\hat{\boldsymbol{i}} - 20t\hat{\boldsymbol{j}} + (40t - 4.9t^2)\hat{\boldsymbol{k}}$

5 Speed = $10t$

6 45°

7 Min speed $3\sqrt{2}$ m/s; position $24\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$

8 **a** $t = 61\frac{11}{49}$ s **b** 50 m/s **c** $\frac{2250}{49}$ m
d 50 m/s **e** $\theta = 36.87°$

9 **a** $\boldsymbol{r}(t) = (\frac{1}{3}\sin(3t) - 3)\hat{\boldsymbol{i}} + (\frac{1}{3}\cos(3t) + \frac{8}{3})\hat{\boldsymbol{j}}$
b $(x+3)^2 + (y - \frac{8}{3})^2 = \frac{1}{9}$; centre $(-3, \frac{8}{3})$

10 Max speed $2\sqrt{5}$ m/s; min speed $2\sqrt{2}$ m/s

11 a Magnitude $\dfrac{\sqrt{11\,667}}{9}$ m/s^2;

direction $\dfrac{1}{\sqrt{11\,667}}(108\hat{\boldsymbol{i}} - \sqrt{3}\hat{\boldsymbol{j}})$

b $\boldsymbol{r}(t) = (\frac{4}{3}t^3 + 2t^2 + t)\hat{\boldsymbol{i}} + (\sqrt{2t+1} - 1)\hat{\boldsymbol{j}}$

12 a $t = 6$ **b** $7\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$

13 a $-16\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ **b** $-80\hat{\boldsymbol{i}} + 60\hat{\boldsymbol{j}}$

14 a $8\cos(2t)\,\hat{\boldsymbol{i}} - 8\sin(2t)\,\hat{\boldsymbol{j}},\ t \geq 0$

b 8 **c** $-4\boldsymbol{r}$

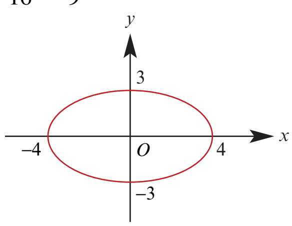
15 a $(t^2 - 5t - 2)\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ **b** $-\dfrac{33}{4}\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

c $y = 2$ for $x \geq -8.25$

16 a $\dfrac{x^2}{36} - \dfrac{y^2}{16} = 1$

b $6\tan t\,\sec t\,\hat{\boldsymbol{i}} + 4\sec^2 t\,\hat{\boldsymbol{j}},\ t \geq 0$

17 a $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$



b **i** $t = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

ii $\boldsymbol{r}(0) = 4\hat{\boldsymbol{i}},\quad \boldsymbol{r}\left(\dfrac{\pi}{2}\right) = 3\hat{\boldsymbol{j}},\quad \boldsymbol{r}(\pi) = -4\hat{\boldsymbol{i}},$

$\boldsymbol{r}\left(\dfrac{3\pi}{2}\right) = -3\hat{\boldsymbol{j}},\quad \boldsymbol{r}(2\pi) = 4\hat{\boldsymbol{i}}$

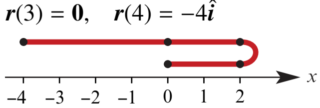
c **i** $\sqrt{9 + 7\sin^2 t}$ **ii** $\sqrt{16 - 7\cos^2 t}$

iii Max speed 4 m/s; min speed 3 m/s

Exercise 7E

1 a $\boldsymbol{r}(0) = \boldsymbol{0},\quad \boldsymbol{r}(1) = 2\hat{\boldsymbol{i}},\quad \boldsymbol{r}(2) = 2\hat{\boldsymbol{i}},$

$\boldsymbol{r}(3) = \boldsymbol{0},\quad \boldsymbol{r}(4) = -4\hat{\boldsymbol{i}}$



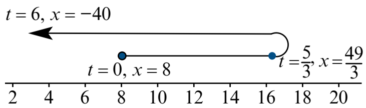
b $-6\hat{\boldsymbol{i}}$ cm **c** $-\hat{\boldsymbol{i}}$ cm/s

d $\dot{\boldsymbol{r}}(t) = (3 - 2t)\hat{\boldsymbol{i}}$ **e** $-2\hat{\boldsymbol{i}}$ cm/s

f $t = \dfrac{3}{2};\ \boldsymbol{r}\left(\dfrac{3}{2}\right) = \dfrac{9}{4}\hat{\boldsymbol{i}}$ **g** $\dfrac{17}{2}$ cm

2 a $\dot{\boldsymbol{r}}(t) = (-6t + 10)\hat{\boldsymbol{i}}$ **b** $\ddot{\boldsymbol{r}}(t) = -6\hat{\boldsymbol{i}}$

c



d $-5\hat{\boldsymbol{i}}$ cm **e** $\dfrac{41}{3}$ cm

3 a $t = 2, 4$ **b** $12\hat{\boldsymbol{i}}$ cm/s^2

c $10\hat{\boldsymbol{i}}$ cm/s

4 a $\dot{\boldsymbol{r}}(t) = (-gt + 20)\hat{\boldsymbol{j}}$

b $\boldsymbol{r}(t) = (-\frac{1}{2}gt^2 + 20t + 10)\hat{\boldsymbol{j}}$

c Approx 30.41 m **d** $t \approx 4.53$

5 a $|\dot{\boldsymbol{r}}_A| = \sqrt{13}$ m/s, $|\dot{\boldsymbol{r}}_B| = \sqrt{157}$ m/s

b $2\sqrt{5}\,|2t - 1|$ m **c** $t = \dfrac{1}{2},\ 5\hat{\boldsymbol{i}} + \dfrac{5}{2}\hat{\boldsymbol{j}}$

6 a $|\dot{\boldsymbol{r}}_A| = \sqrt{30}$ m/s, $|\dot{\boldsymbol{r}}_B| = \sqrt{26}$ m/s

b $2\sqrt{2}\,|t - 3|$ m

c $t = 3,\ 10\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 15\hat{\boldsymbol{k}}$

7 a 3 m/s **b** $\boldsymbol{r}(t) = \dfrac{1}{2}\sin(2t)\,(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

c $\boldsymbol{a}(t) = -2\sin(2t)\,(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$

d $\pm\dfrac{1}{2}(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$ **e** 6 m/s^2 **f** $\boldsymbol{0}$

Exercise 7F

1 a $\dot{\boldsymbol{r}}(0) = 49\sqrt{3}\,\hat{\boldsymbol{i}} + 49\hat{\boldsymbol{j}}$

b $\dot{\boldsymbol{r}}(t) = 49\sqrt{3}\,\hat{\boldsymbol{i}} + (49 - 9.8t)\hat{\boldsymbol{j}}$

c $\boldsymbol{r}(t) = 49\sqrt{3}t\,\hat{\boldsymbol{i}} + (49t - 4.9t^2)\hat{\boldsymbol{j}}$

d $y = \dfrac{\sqrt{3}}{3}x - \dfrac{1}{1470}x^2$

2 a $y = -\dfrac{1}{150g}x^2 + \dfrac{\sqrt{3}}{3}x + 50$

b $25\sqrt{3}\left(\sqrt{g^2 + 2g} + g\right)$

3 $\dfrac{25g}{8}$ m

4 a $\boldsymbol{r}(t) = 40\cos(20°)t\,\hat{\boldsymbol{i}} - \left(40\sin(20°)t + \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

b 31.7 m **c** 30.3°

5 16.3° or 87.7°

6 a $\dot{\boldsymbol{r}} = u\cos\alpha\,\hat{\boldsymbol{i}} + (u\sin\alpha - gt)\hat{\boldsymbol{j}}$

b $T = \dfrac{u}{g\sin\alpha}$

7 23.40°

8 $13\sqrt{\dfrac{g}{5}}$ m/s

9 a 4.9 m

b 37.5 m/s

10 a $\boldsymbol{r} = 16t\,\hat{\boldsymbol{i}} + \left(30t - \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

b $t = 2.5;\ \boldsymbol{r}(2.5) = 40\hat{\boldsymbol{i}} + \left(75 - \dfrac{25g}{8}\right)\hat{\boldsymbol{j}}$

11 a $\boldsymbol{v} = \boldsymbol{u} + tg\hat{\boldsymbol{j}}$ **b** $\boldsymbol{r} = t\boldsymbol{u} + \dfrac{1}{2}gt^2\hat{\boldsymbol{j}}$

Exercise 7G

1 a 2 radians per second **b** $2.5\hat{\boldsymbol{i}}$

c $5\hat{\boldsymbol{j}}$ **d** $-10\hat{\boldsymbol{i}}$

2 a 1 radian per second **b** $2\hat{\boldsymbol{j}}$

c $-2\hat{\boldsymbol{i}}$ **d** $-2\hat{\boldsymbol{j}}$

3 a $\dfrac{35\pi}{3}$ radians per second **b** $\dfrac{7\pi}{6}$ m/s

4 a $\boldsymbol{r} = 25\left(\cos\left(\dfrac{2t}{5}\right)\hat{\boldsymbol{i}} + \sin\left(\dfrac{2t}{5}\right)\hat{\boldsymbol{j}}\right)$

b $\dot{\boldsymbol{r}} = 10\left(-\sin\left(\dfrac{2t}{5}\right)\hat{\boldsymbol{i}} + \cos\left(\dfrac{2t}{5}\right)\hat{\boldsymbol{j}}\right)$

c $\ddot{\boldsymbol{r}} = 4\left(-\cos\left(\dfrac{2t}{5}\right)\hat{\boldsymbol{i}} - \sin\left(\dfrac{2t}{5}\right)\hat{\boldsymbol{j}}\right)$

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5 $\frac{100}{3}$ radians per second

6 **a** 3π radians per second **b** 6π m/s
c $18\pi^2$ m/s^2 **d** $\frac{2}{3}$ s

7 **a** 12π m/s **b** $48\pi^2$ m/s^2
c $\boldsymbol{r}(\frac{1}{2}) = -3\hat{\boldsymbol{j}}$; $\dot{\boldsymbol{r}}(\frac{1}{2}) = 12\pi\hat{\boldsymbol{i}}$; $\ddot{\boldsymbol{r}}(\frac{1}{2}) = 48\pi^2\hat{\boldsymbol{j}}$
d 4π radians per second

8 **a** 4π radians per second **b** 1.4π m/s
c 1583.36 m

9 34.29 radians per second

10 **a** Circle with centre $(0, 0)$ and radius 4
b $\dot{\boldsymbol{r}} = -8t\sin(t^2)\,\hat{\boldsymbol{i}} + 8t\cos(t^2)\,\hat{\boldsymbol{j}}$
c $\ddot{\boldsymbol{r}} = -4t^2\boldsymbol{r} + \frac{1}{t}\dot{\boldsymbol{r}}$

11 **a** 12π m/s
b $48\pi^2$ m/s^2
c $\boldsymbol{r}(1) = 7\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$; $\dot{\boldsymbol{r}}(1) = -12\pi\hat{\boldsymbol{j}}$;
$\ddot{\boldsymbol{r}}(1) = -48\pi^2\hat{\boldsymbol{i}}$
d 4π radians per second
e $(x-4)^2 + (y-2)^2 = 9$

Chapter 7 review

Short-response questions

1 **a** $2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}, 2\hat{\boldsymbol{j}}$ **b** $4y = x^2 - 16$

2 **a** $\dot{\boldsymbol{r}}(t) = 4t\,\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, $\ddot{\boldsymbol{r}}(t) = 4\hat{\boldsymbol{i}}$
b $\dot{\boldsymbol{r}}(t) = 4\cos t\,\hat{\boldsymbol{i}} - 4\sin t\,\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{k}}$,
$\ddot{\boldsymbol{r}}(t) = -4\sin t\,\hat{\boldsymbol{i}} - 4\cos t\,\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$

3 $0.6\hat{\boldsymbol{i}} + 0.8\hat{\boldsymbol{j}}$

4 **a** $5\sqrt{3}\hat{\boldsymbol{i}} + \frac{5}{2}\hat{\boldsymbol{j}}$ **b** $\frac{2\sqrt{7}}{7}$

5 $\cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}}$

6 **a** $5(-\sin t\,\hat{\boldsymbol{i}} + \cos t\,\hat{\boldsymbol{j}})$ **b** 5
c $-5(\cos t\,\hat{\boldsymbol{i}} + \sin t\,\hat{\boldsymbol{j}})$
d 0, acceleration perpendicular to velocity

7 $\frac{3\pi}{4}$

8 **a** $|\dot{\boldsymbol{r}}| = 1, |\ddot{\boldsymbol{r}}| = 1$
b $(x-1)^2 + (y-1)^2 = 1$
c $\frac{3\pi}{4}$

9 $-2\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$

10 **a** $\boldsymbol{r} = \left(\frac{t^2}{2} + 1\right)\hat{\boldsymbol{i}} + (t-2)\hat{\boldsymbol{j}}$ **b** $(13.5, 3)$
c 12.5 s

11 **a** $\dot{\boldsymbol{r}} = t\,\hat{\boldsymbol{i}} + (2t-5)\hat{\boldsymbol{j}}$
b $\boldsymbol{r} = \left(\frac{t^2}{2} - 1\right)\hat{\boldsymbol{i}} + (t^2 - 5t + 6)\hat{\boldsymbol{j}}$
c $-\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$, $-5\hat{\boldsymbol{j}}$

12 **a** **i** $\dot{\boldsymbol{r}}_2(t) = (2t-4)\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$
ii $\dot{\boldsymbol{r}}_1(t) = t\,\hat{\boldsymbol{i}} + (k-t)\hat{\boldsymbol{j}}$
b **i** 4 **ii** 8 **iii** $4(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$

13 **b** **i** $\dot{\boldsymbol{r}}(t) = e^t\hat{\boldsymbol{i}} + 8e^{2t}\hat{\boldsymbol{j}}$ **ii** $\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ **iii** $\ln 1.5$

14 **b** **i** $x = 2$ for $y \geq -3.5$ **ii** $(2, -3.5)$

15 **a** 6π m/s **b** $12\pi^2$ m/s^2
c $\boldsymbol{r}(1) = 3\hat{\boldsymbol{j}}$; $\dot{\boldsymbol{r}}(1) = 6\pi\hat{\boldsymbol{i}}$; $\ddot{\boldsymbol{r}}(1) = -12\pi^2\hat{\boldsymbol{j}}$
d 2π radians per second

16 $a = 20\sqrt{3}$

17 **a** 12π radians per second **b** 6π m/s

18 **a** $a = 2,\ n = \frac{8\pi}{3}$
b $\frac{16\pi}{3}$ m/s
c $\frac{128\pi^2}{9}$ m/s^2

19 $\sqrt{a^2 + \left(b + \frac{g}{2}\right)^2}$

20 **a** $(4t - \frac{1}{2}t^2 + 8)\hat{\boldsymbol{i}} + 3t\hat{\boldsymbol{j}} + (20 - \frac{3}{2}t^2)\hat{\boldsymbol{k}}$
b $\sqrt{10t^2 - 8t + 25}$
c $t = 0.4$ $\boldsymbol{x} = 9.52\hat{\boldsymbol{i}} + 1.2\hat{\boldsymbol{j}} + 19.76\hat{\boldsymbol{k}}$

21 **a** $\boldsymbol{v}(t) = 20\cos\theta\hat{\boldsymbol{i}} + (20\sin\theta - gt)\hat{\boldsymbol{j}}$
b $\boldsymbol{x}(t) = 20\cos\theta t\hat{\boldsymbol{i}} + (20\sin\theta t - \frac{g}{2}t^2)\hat{\boldsymbol{j}}$
c $t_1 = \frac{1}{2\cos\theta}, t_1 = \frac{40\sin\theta}{g}$
d $\theta = \frac{1}{2}\sin^{-1}\frac{g}{40}$ and $\frac{\pi}{2} - \frac{1}{2}\sin^{-1}\frac{g}{40}$

22 **a** $\boldsymbol{x} = 10\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} + t(-3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}})$ **b** $t = \frac{7}{2}$
c $\frac{\sqrt{26}}{2}$

23 **a** $-2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ **b** $t = 3$ **c** $\lambda = 4$
d $(1, 2, 8)$

24 **a** $\boldsymbol{v} = \cos t\hat{\boldsymbol{i}} + 2\cos 2t\hat{\boldsymbol{j}}$
b $t = \frac{(2n+1)\pi}{2}, n \in \boldsymbol{Z}$, and $t = \frac{(2n+1)\pi}{4} n \in \boldsymbol{Z}$
c ≈ 0.9117

25 **a** $\overrightarrow{BF} = -3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$ **b** 9 m
c 3 m/s **d** $(-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$ m/s
e 2 seconds, $2\sqrt{26}$ metres

26 **a** Speed of P is $3\sqrt{13}$ m/s;
speed of Q is $\sqrt{41}$ m/s
b **i** Position of P is $60\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$;
position of Q is $80\hat{\boldsymbol{i}} + 80\hat{\boldsymbol{j}}$
ii $\overrightarrow{PQ} = (20 - 4t)\hat{\boldsymbol{i}} + (60 - 2t)\hat{\boldsymbol{j}}$
c 10 seconds, $20\sqrt{5}$ metres

27 **a** $\overrightarrow{AB} = ((v+3)t - 56)\hat{\boldsymbol{i}} + ((7v - 29)t + 8)\hat{\boldsymbol{j}}$
b 4
c **i** $\overrightarrow{AB} = (6t - 56)\hat{\boldsymbol{i}} + (8 - 8t)\hat{\boldsymbol{j}}$
ii 4 seconds

28 **a** **i** 200 s **ii** $\frac{1}{2}$ **iii** 5 m/s **iv** $(1200, 0)$
b 8 seconds, 720 metres

29 **a** **i** $\overrightarrow{OA} = (6t - 1)\hat{\boldsymbol{i}} + (3t + 2)\hat{\boldsymbol{j}}$
ii $\overrightarrow{BA} = (6t - 3)\hat{\boldsymbol{i}} + (3t + 1)\hat{\boldsymbol{j}}$
b 1 second
c **i** $\boldsymbol{c} = \frac{1}{5}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$ **ii** $\boldsymbol{d} = \frac{1}{5}(4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}})$
iii $6\boldsymbol{c} + 3\boldsymbol{d}$

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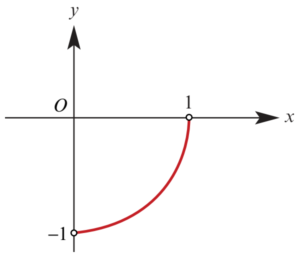
30 a



b **i** $a = 16$ **ii** $b = -16$ **iii** $n = 2$

iv $\boldsymbol{v}(t) = -32\sin(2t)\,\hat{\boldsymbol{i}} - 32\cos(2t)\,\hat{\boldsymbol{j}}$

$\boldsymbol{a}(t) = -64\cos(2t)\,\hat{\boldsymbol{i}} + 64\sin(2t)\,\hat{\boldsymbol{j}}$

c **i** $\overrightarrow{PQ} = 8((\sin t - 2\cos(2t))\hat{\boldsymbol{i}} + (\cos t + 2\sin(2t))\hat{\boldsymbol{j}})$

ii $|\overrightarrow{PQ}|^2 = 64(5 + 4\sin t)$

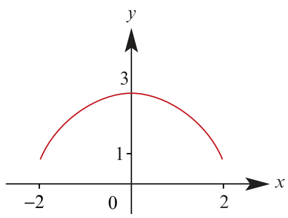
d 8 cm

31 a $2\sin t\,\hat{\boldsymbol{i}} + (\cos(2t) + 2)\hat{\boldsymbol{j}},\ t \ge 0$

b $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

c **i** $y = 3 - \dfrac{x^2}{2},\ -2 \le x \le 2$

ii



d $|\boldsymbol{v}|^2 = -16\cos^4 t + 20\cos^2 t$,

max speed is $\dfrac{5}{2}$

e $\dfrac{3\pi}{2}$

f **ii** $t = \dfrac{(2k-1)\pi}{2},\ k \in \mathbb{N}$

32 a $a\hat{\boldsymbol{i}} + (b + 2t)\hat{\boldsymbol{j}} + (20 - 10t)\hat{\boldsymbol{k}}$

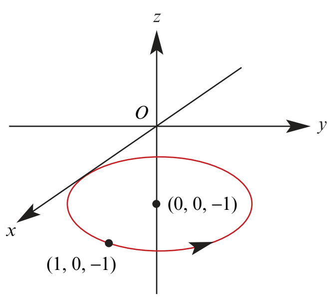
b $at\hat{\boldsymbol{i}} + (bt + t^2)\hat{\boldsymbol{j}} + (20t - 5t^2)\hat{\boldsymbol{k}}$ **c** 4 s

d $a = 25,\ b = -4$ **e** 38.3°

33 a $-5.25\hat{\boldsymbol{j}}$ **b** $5.25\hat{\boldsymbol{i}}$ **c** $5.25\hat{\boldsymbol{j}}$ **d** $-5.25\hat{\boldsymbol{i}}$

e $-3\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ **f** $-3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$

34 a **i** Particle P is moving on a circular path, with centre $(0, 0, -1)$ and radius 1, starting at $(1, 0, -1)$ and moving 'anticlockwise' a distance of 1 'below' the x–y plane. The particle finishes at $(1, 0, -1)$ after one revolution.



ii $\sqrt{2}$

iii $\dot{\boldsymbol{p}}(t) = -\sin t\,\hat{\boldsymbol{i}} + \cos t\,\hat{\boldsymbol{j}},\ 0 \le t \le 2\pi$

v $\ddot{\boldsymbol{p}}(t) = -\cos t\,\hat{\boldsymbol{i}} - \sin t\,\hat{\boldsymbol{j}},\ 0 \le t \le 2\pi$

b **i** $\overrightarrow{PQ} = (\cos(2t) - \cos t)\hat{\boldsymbol{i}} + (-\sin t - \sin(2t))\hat{\boldsymbol{j}} + \frac{3}{2}\hat{\boldsymbol{k}}$

iii $\dfrac{5}{2}$

iv $\dfrac{\pi}{3},\ \pi,\ \dfrac{5\pi}{3}$

v $\dfrac{3}{2}$

vi $0,\ \dfrac{2\pi}{3},\ \dfrac{4\pi}{3},\ 2\pi$

c **ii** $\dfrac{\sqrt{10}}{5}\left(\cos(3t) - \dfrac{1}{2}\right)$

iii 162°

35 a $\boldsymbol{r}(t) = 35t\,\hat{\boldsymbol{i}} + 5t\,\hat{\boldsymbol{j}} + (24.5t - 4.9t^2)\hat{\boldsymbol{k}}$

b 5 s

c 35 m

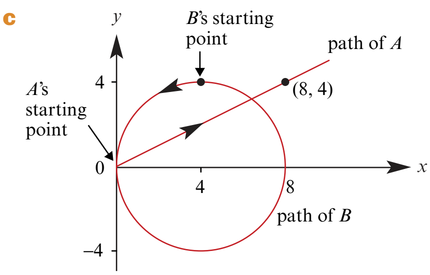
d 43.0 m/s

36 a 4α

b A: $y = \dfrac{x}{2},\ x \ge 0$;

B: $(x - 4)^2 + y^2 = 16$

c



d $(0, 0),\ \left(\dfrac{32}{5}, \dfrac{16}{5}\right)$

e 1.76

37 a **i** $-9.8\hat{\boldsymbol{j}}$

ii $2\hat{\boldsymbol{i}} - 9.8t\hat{\boldsymbol{j}}$

iii $2t\,\hat{\boldsymbol{i}} - 4.9t^2\hat{\boldsymbol{j}}$

b **i** $\dfrac{2\sqrt{2}}{7}$ seconds

ii $\dfrac{4\sqrt{2}}{7}$ metres

38 a **i** $6\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ **ii** $\dfrac{\sqrt{5}}{5}(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$

b $4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}},\ (4, -2)$

c **i** $\overrightarrow{LP} = (1 - \frac{7}{2}t)\hat{\boldsymbol{i}} + (7 - 2t)\hat{\boldsymbol{j}}$

ii 1:05 p.m.

iii $\dfrac{9\sqrt{65}}{13}$ km

Multiple-choice questions

1 D	**2** D	**3** B	**4** D	**5** C
6 C	**7** C	**8** D	**9** C	**10** D
11 D	**12** D	**13** A	**14** A	**15** C
16 C	**17** C	**18** C	**19** D	**20** B

Chapter 8

Exercise 8A

1 $\begin{bmatrix} 220 & 150 \\ 180 & 125 \\ 135 & 102 \\ 112 & 91 \\ 86 & 83 \\ 48 & 53 \end{bmatrix}$ **2** $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$

3 $x = 0, \ y = 2$

4 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $2\mathbf{X} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$

$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$

$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$

5 $\mathbf{X} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} \frac{-9}{2} & \frac{-23}{2} \\ \frac{-1}{2} & 11 \end{bmatrix}$

6 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 310 & 180 & 220 & 90 \\ 200 & 0 & 125 & 0 \end{bmatrix}$
represents the total production at two factories in two successive weeks

7 $\mathbf{AX} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\mathbf{AY} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ $\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$ $\mathbf{A}(\mathbf{CA}) = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$

8 b $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9 One possible answer is

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$,

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$, $\mathbf{AB} + \mathbf{AC} = \begin{bmatrix} -1 & 11 \\ -4 & 24 \end{bmatrix}$,

$(\mathbf{B} + \mathbf{C})\mathbf{A} = \begin{bmatrix} 11 & 7 \\ 16 & 12 \end{bmatrix}$

10 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

11 a $\begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

b $\begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$,
John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

Exercise 8B

1 a 1

b $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

c 2

d $\frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

2 a $\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$ **d** $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

4 a $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$, $(\mathbf{AB})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

c $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$,

$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

5 a $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$

6 a $\begin{bmatrix} \frac{-3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{-11}{16} & \frac{17}{16} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$

7 $m \in \mathbb{R} \setminus \{-4, 2\}$

8 $m = 0, m = \frac{3}{2}$ or $m = 2$

9 $\begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$

11 $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix},$
$\begin{bmatrix}1&0\\k&-1\end{bmatrix}, \begin{bmatrix}-1&0\\k&1\end{bmatrix}, \begin{bmatrix}1&k\\0&-1\end{bmatrix}, \begin{bmatrix}-1&k\\0&1\end{bmatrix}, k \in \mathbb{R},$
$\begin{bmatrix}a&b\\\frac{1-a^2}{b}&-a\end{bmatrix}, b \neq 0$

Exercise 8C

1 a $x = 4,\ y = -3$ **b** $x = -\frac{3}{2},\ y = \frac{1}{2}$
c $x = \frac{51}{38},\ y = -\frac{31}{38}$ **d** $x = \frac{37}{10},\ y = \frac{7}{5}$

2 a One solution
b Infinitely many solutions
c No solution

3 Their graphs are parallel straight lines that do not coincide

4 $x = 6 + \lambda,\ y = \lambda$ for $\lambda \in \mathbb{R}$

5 a $m = -5$ **b** $m = 3$

6 $m = 9$

7 a i $m = -2$ **ii** $m = 4$
b $x = \frac{4}{m+2},\ y = \frac{2(m+4)}{m+2}$

8 a $x = 2,\ y = 0,\ k \neq -\frac{3}{2}$ **b** $k = -\frac{3}{2}$

9 a $b \neq 10$ **b** $b = 10,\ c = 8$
c $b = 10,\ c \neq 8$

10 a ii Infinitely many solutions for $b = 8$
iii No solutions for $b \neq 8$
iv If $b = 8$, then the solutions are $x = 4 - \lambda$, $y = \lambda$ for $\lambda \in \mathbb{R}$
b i Unique solution for all $b \in \mathbb{R}$
iv Solution is $x = b - 4,\ y = 8 - b$
c i Unique solution for $b \neq 1$
iii No solution for $b = 1$
iv If $b \neq 1$, then $x = \frac{4}{b-1},\ y = \frac{4b-8}{b-1}$

Exercise 8D

1 a $\begin{bmatrix}3\\10\end{bmatrix}$ **b** $\begin{bmatrix}5\\17\end{bmatrix}$

2 a $x = -\frac{1}{7},\ y = \frac{10}{7}$ **b** $x = 4,\ y = 1.5$
c $x = 3,\ y = 4$ **d** $x = \frac{11}{2},\ y = 2$

3 $(2, -1)$

4 a $\begin{bmatrix}\sqrt{3}&-1\\1&\sqrt{3}\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}8\\12\end{bmatrix}$
b $\frac{1}{4}\begin{bmatrix}\sqrt{3}&1\\-1&\sqrt{3}\end{bmatrix}$
c $a = 2\sqrt{3} + 3,\ b = 3\sqrt{3} - 2$

5 Book \$12, DVD \$18

6 $m = -2$ or $m = 4$

7 a $\begin{bmatrix}2&-3\\4&-6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$
b Non-invertible
c System has solutions (not a unique solution)
d Solution set contains infinitely many pairs

Exercise 8E

2 $\mathbf{AB} = 7\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{7}\mathbf{B}$

3 $\mathbf{A}^2 = 9\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{9}\mathbf{A}$

4 $\mathbf{A}^2 = 4\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{4}\mathbf{A}$

5 a $\frac{1}{90}\begin{bmatrix}20&10&-10\\-10&4&23\\-10&-14&32\end{bmatrix}$
b $\frac{1}{51}\begin{bmatrix}-16&-19&14\\2&-4&11\\15&21&6\end{bmatrix}$
c $\frac{1}{51}\begin{bmatrix}6&6&-9&-21\\-13&-30&62&54\\8&42&-46&-45\\-4&-21&23&48\end{bmatrix}$
d $\begin{bmatrix}2&-5&4&-4\\7&-18&15&-15\\-8&21&-17&17\\-17&43&-35&36\end{bmatrix}$

6 a i -2 **ii** -2
b i -4 **ii** -16

Exercise 8F

1 a $x = 2, y = 3, z = 1$ **b** $x = -3, y = 5, z = 2$
c $x = 5, y = 0, z = 7$ **d** $x = 6, y = 5, z = 1$

2 a $y = 4z - 2$
b $x = 8 - 5\lambda,\ y = 4\lambda - 2,\ z = \lambda$

3 a $x = \lambda - 1,\ y = \lambda,\ z = 5$
b $x = \lambda + 3,\ y = 3\lambda,\ z = \lambda$
c $x = \frac{14 - 3\lambda}{6},\ y = \frac{10 - 3\lambda}{6},\ z = \lambda$

4 a $-y + 5z = 15,\ -y + 5z = 15$
b The two equations are the same
c $y = 5\lambda - 15$ **d** $x = 43 - 13\lambda$

5 $w = \frac{\lambda - 2}{2}, x = \frac{26 - 3\lambda}{4}, y = \frac{-3(\lambda + 2)}{4}, z = \lambda$ for $\lambda \in \mathbb{R}$; $w = 6, x = -4, y = -12, z = 14$

6 a $x = 1,\ y = -2,\ z = 3$
b $x = -\frac{5}{3},\ y = \frac{3\lambda - 5}{3},\ z = \lambda$
c $x = \frac{2 - 3\lambda}{2},\ y = -2(\lambda - 1),\ z = \lambda$

Exercise 8G

1 a $x = \frac{13}{2},\ y = \frac{16}{3},\ z = \frac{7}{6}$
b $x = -3,\ y = 5,\ z = 2$

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c $x = -15,\ y = 16,\ z = 11$
d $x = 6,\ y = 5,\ z = 1$
e $x = 5,\ y = 2,\ z = 4,\ w = -1$
f $x = 2,\ y = -3,\ z = 1,\ w = -1$

2 a $\mathbf{AB} = 24\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{24}\mathbf{B}$
b $x = -\frac{37}{6},\ y = \frac{16}{3},\ z = \frac{29}{6}$

3 a $\mathbf{AB} = 3\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{3}\mathbf{B}$
b $x = 18,\ y = 44,\ z = -49$

4

Spray	P	Q	R
Barrels	$\frac{8}{13}$	$\frac{46}{39}$	$\frac{12}{13}$

Exercise 8H

1 a $x = -\frac{7}{5},\ y = \frac{11}{5},\ z = -2$
b $x = -\frac{1}{2},\ y = 2,\ z = -\frac{3}{2}$
c $x = \frac{115}{13},\ y = -\frac{29}{13},\ z = \frac{44}{13}$

2 a $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$x = 3 - \lambda,\ y = 2 + 3\lambda,\ z = \lambda$ for $\lambda \in \mathbb{R}$

b $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$x = 4 + 2\lambda,\ y = 3 - \lambda,\ z = \lambda$ for $\lambda \in \mathbb{R}$

c $\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$
$x = 1 - \frac{3}{2}\lambda,\ y = 2 - 2\lambda,\ z = \lambda$ for $\lambda \in \mathbb{R}$

3 $\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
No solutions, as row 3 represents the equation $0x + 0y + 0z = 1$

4 a $x = 1,\ y = 2,\ z = 4$
b $x = 3,\ y = 0,\ z = -2$
c $x = -\frac{100}{317},\ y = -\frac{248}{317},\ z = \frac{335}{317}$
d No solutions

5 a
- Unique solution for $a \in \mathbb{R} \setminus \{-4, 4\}$
- Infinitely many solutions for $a = 4$
- No solutions for $a = -4$

b
- If $a \in \mathbb{R} \setminus \{-4, 4\}$, then the solution is $x = \frac{25 + 8a}{28 + 7a},\ y = \frac{2(27 + 5a)}{7(4 + a)},\ z = \frac{1}{4 + a}$
- If $a = 4$, then the solutions are $x = \frac{8 - 7\lambda}{7},\ y = \frac{10 + 14\lambda}{7},\ z = \lambda,\ \lambda \in \mathbb{R}$

6 a
- Unique solution for $a \in \mathbb{R} \setminus \{-2, 2\}$
- Infinitely many solutions for $a = 2$
- No solutions for $a = -2$

b
- If $a \in \mathbb{R} \setminus \{-2, 2\}$, then the solution is $x = 2,\ y = \frac{3 + 2a}{2 + a},\ z = \frac{1}{2 + a}$
- If $a = 2$, then the solutions are $x = 2,\ y = 2 - \lambda,\ z = \lambda$ for $\lambda \in \mathbb{R}$

7 a $\boldsymbol{r} = (6 - \lambda)\hat{\boldsymbol{i}} - \frac{1}{3}(2 + \lambda)\hat{\boldsymbol{j}} + \lambda\hat{\boldsymbol{k}}$
b $\boldsymbol{r} = \frac{1}{11}(26 - 2\lambda)\hat{\boldsymbol{i}} + \frac{1}{11}(17 - 3\lambda)\hat{\boldsymbol{j}} + \lambda\hat{\boldsymbol{k}}$

8 a Infinitely many solutions for $a = 6$
b The three planes intersect along a line
c The first two planes intersect along a line in the plane $3x - 2y + 2z = 6$. (Add the first two equations.) The third plane is parallel to this plane, and coincides only if $a = 6$.

9 $\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$
No solutions, as row 3 represents the equation $0x + 0y + 0z = 1$

10 a $x = -\lambda,\ y = \lambda,\ z = 0,\ w = \lambda$
b $x = -\frac{11\lambda}{39},\ y = -\frac{4\lambda}{39},\ z = -\frac{5\lambda}{39},\ w = \lambda$
c $x = -\frac{\lambda}{5},\ y = \lambda,\ z = 0,\ w = 0$

11 $9x + y - 5z - 16 = 0$ **12** $x + 9y - 5z - 26 = 0$

Chapter 8 review

Short-response questions

1 a $\begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 \\ 9 & 3 \end{bmatrix}$

2 a $\begin{bmatrix} m & 4 \\ 2 & m+2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ -1 \end{bmatrix}$
b $m \in \mathbb{R} \setminus \{-4, 2\}$
c $m = -4$ and $a = 2$; $m = 2$ and $a = -1$

3 a $m = -4$ **b** $m \in \mathbb{R} \setminus \{-4, -3\}$

4 $\mathbf{A} = \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$

5 $\mathbf{A}^2 = 25\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{25}\mathbf{A}$

6 a $\det(\mathbf{A}) = 6,\ \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{2} & \frac{2}{3} \end{bmatrix}$
b $x = -2,\ y = 6$

7 $k \in \mathbb{R} \setminus \{-\frac{3}{2}\}$

8 $a \in \mathbb{R} \setminus \{6\}$

9 a $a = -1$ **b** $a = 2$ **c** $a \in \mathbb{R} \setminus \{-1, 2\}$

10 $y = 3x^2 - 8x + 12$

12 a $a \in \mathbb{R} \setminus \{-2, 2\}$ **b** $a = -2$ or $a = 2$

13 a $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \lambda(\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$
b $(2, -1, -2),\ (2, 2, 1)$

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14 a $\mathbf{AB} = 8\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{8}\mathbf{B}$

b $x = -10,\ y = 45,\ z = 7$

15 a i $\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

ii $\det(\mathbf{A}) = -11,\ \mathbf{A}^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$

iii $x = \frac{32}{11},\ y = -\frac{4}{11}$

iv Lines intersect at the point $\left(\frac{32}{11}, -\frac{4}{11}\right)$

b i $\begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$ **ii** $\det(\mathbf{A}) = 0$

c Distinct parallel lines

16 a $\lambda = -\frac{17}{7}$ **b** $\lambda = -1$

17 a $\lambda = 4$ **b** $\mu = 2$

c $\boldsymbol{r} = 8\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + t((5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

18 a $b = -2, 4$ **b** $(3, -2, 2)$

19 a $5x + 6y + 10z = 310$

$3x + 4y + 5z = 175$

$2x + 6y + 5z = 175$

b $\left[\begin{array}{ccc|c} 5 & 6 & 10 & 310 \\ 3 & 4 & 5 & 175 \\ 2 & 6 & 5 & 175 \end{array}\right]$

c Brad 20; Flynn 10; Lina 15

20 a $4a + 2b + c = 0,\ a + b + c = 1$

b $a = \frac{c-2}{2},\ b = \frac{4-3c}{2}$

c $a = \frac{1}{6},\ b = -\frac{3}{2},\ c = \frac{7}{3}$

21 a $a + b + c + d = 1,\ a - b + c - d = 4,$

$4a + 3b + 2c + d = 0$

b $a = \frac{4d-1}{4},\ b = \frac{-2d-3}{2},\ c = \frac{11-4d}{4}$

c $d = \frac{38}{67}$

22 a $x = -\lambda,\ y = \lambda + 1,\ z = \lambda$ for $\lambda \in \mathbb{R}$

b i $p = 2$ or $p = -2$ **ii** $p \in \mathbb{R} \setminus \{-2, 2\}$

iii No value of p

23 a $\boldsymbol{r} = \frac{1}{2}(3 + \lambda)\hat{\boldsymbol{i}} + \frac{1}{2}(1 - \lambda)\hat{\boldsymbol{j}} + \lambda\hat{\boldsymbol{k}}$

b i $p \in \mathbb{R} \setminus \{-1, 1\};\ \left(\frac{2p+1}{p+1}, \frac{1}{p+1}, \frac{p-1}{p+1}\right)$

ii $p = 1$ **iii** $p = 0$

c $(0,0,0),\ (2,0,0),\ (0,2,0),\ (0,0,-2)$

d $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}, -1 - \sqrt{2}\right)$,

$\left(1 + \frac{\sqrt{2}}{2}, 1 - \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$

24 a $\mathbf{AB} = 20\mathbf{I},\ \mathbf{A}^{-1} = \frac{1}{20}\mathbf{B}$

b

Product	P	Q	R
Number per day	13.5	0.5	13

25 Bananas \$20, Apples \$5, Peaches \$10

26 component x: \$15; component y: \$6.60; component z: \$1.20

27 Bank account \$9000; shares \$9000; bonds \$7000

28 a $a = 1, b = 5, c = 3$

b No

30 c $\mathbf{E} = \begin{bmatrix} 2\lambda + 1 & -2\lambda & -4\lambda \\ -\lambda & 3\lambda + 1 & 4\lambda \\ \lambda & -2\lambda & -3\lambda + 1 \end{bmatrix}$

d $\mu = \frac{-\lambda}{\lambda + 1}$

e $\mathbf{E} = \begin{bmatrix} 7 & -6 & -12 \\ -3 & 10 & 12 \\ 3 & -6 & -8 \end{bmatrix},$

$\mathbf{F} = \frac{1}{4}\begin{bmatrix} -2 & 6 & 12 \\ 3 & -5 & -12 \\ -3 & 6 & 13 \end{bmatrix}$

f $x = 9,\ y = -\frac{11}{2},\ z = \frac{15}{2}$

Multiple-choice questions

1 A	**2** D	**3** A	**4** A	**5** D
6 D	**7** A	**8** B	**9** C	**10** D
11 A	**12** D	**13** A	**14** A	**15** D
16 D	**17** B	**18** D	**19** D	**20** D
21 D	**22** A	**23** A		

Chapter 9

Exercise 9A

1 a 10

b
- Ash defeats Carl and Dot
- Ben defeats Ash, Carl and Dot
- Carl defeats Dot
- Dot defeats Elle
- Elle defeats Ash, Ben and Carl

c Ben = Elle, Ash, Carl = Dot

2 a $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$ **b** $E,\ B,\ A = C,\ D$

3 a $\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 1 & 1 & 0 \end{array}$

b $\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 0 \end{array}$

c $\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 0 \end{array}$

d $\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 0 & 0 & 1 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 0 & 0 & 0 \end{array}$

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4 a $\begin{bmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ **b** A

5 a $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **b** A, B, D, C

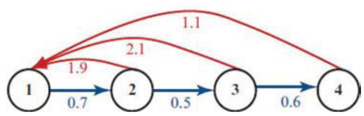
6 A, B, D, C

7 $E, B, A = C, D$

Exercise 9B

1 a i 1.9 **ii** 0.6

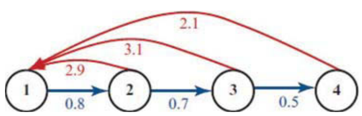
b



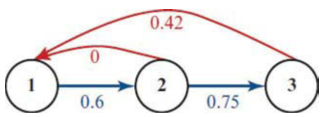
c i $\begin{bmatrix} 510 \\ 70 \\ 50 \\ 60 \end{bmatrix}$ **ii** $\begin{bmatrix} 784.8 \\ 212.8 \\ 178.5 \\ 21 \end{bmatrix}$ **iii** $\begin{bmatrix} 208\,276 \\ 103\,876 \\ 36\,984 \\ 15\,615.8 \end{bmatrix}$

d $\begin{bmatrix} 3483 \\ 1829 \\ 600 \\ 291 \end{bmatrix}$

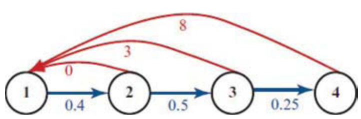
2 a



b



c

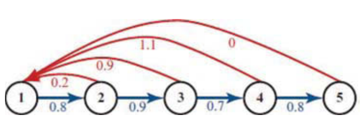


3 a $\begin{bmatrix} 0 & 1.3 & 2.4 \\ 0.7 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 2.3 & 3 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1.4 & 2.6 & 0.6 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.05 & 0 \end{bmatrix}$

4 a $\begin{bmatrix} 15 \\ 20 \\ 30 \\ 15 \\ 10 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0.2 & 0.9 & 1.1 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$

c



d i $\begin{bmatrix} 48 \\ 12 \\ 18 \\ 21 \\ 12 \end{bmatrix}$ **ii** $\begin{bmatrix} 58 \\ 37 \\ 22 \\ 21 \\ 19 \end{bmatrix}$

e 37 **f** 90

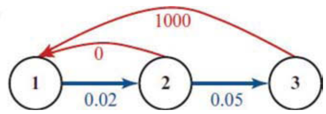
g i 111 **ii** 158 **iii** 235

h i 99 **ii** 145 **iii** 233

It appears that the population rate of increase approaches 10% per year. Further investigation confirms this.

5 a $\begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}$

c



d i $\begin{bmatrix} 50\,000 \\ 0 \\ 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$

iii $\begin{bmatrix} 50\,000 \\ 0 \\ 0 \end{bmatrix}$

e i $\begin{bmatrix} 50\,000 \\ 1 \\ 5 \end{bmatrix}$ **ii** $\begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix}$

iii $\begin{bmatrix} 50\,000 \\ 1 \\ 5 \end{bmatrix}$

6 a i $\begin{bmatrix} 269 \\ 123 \\ 30 \end{bmatrix}$ **ii** $\begin{bmatrix} 356 \\ 168 \\ 40 \end{bmatrix}$ **iii** $\begin{bmatrix} 622 \\ 294 \\ 70 \end{bmatrix}$

i 427 **ii** 564 **iii** 986

b At this stage the rate of increase is approximately 5.7%.

7 a i $\begin{bmatrix} 1400 \\ 240 \\ 100 \end{bmatrix}$ **ii** $\begin{bmatrix} 700 \\ 840 \\ 60 \end{bmatrix}$ **iii** $\begin{bmatrix} 2160 \\ 420 \\ 210 \end{bmatrix}$

b i $\begin{bmatrix} 976 \\ 460 \\ 91 \end{bmatrix}$ **ii** $\begin{bmatrix} 1241 \\ 586 \\ 115 \end{bmatrix}$ **iii** $\begin{bmatrix} 1579 \\ 745 \\ 146 \end{bmatrix}$

c i $\begin{bmatrix} 974 \\ 460 \\ 90 \end{bmatrix}$ **ii** $\begin{bmatrix} 1237 \\ 584 \\ 115 \end{bmatrix}$ **iii** $\begin{bmatrix} 1571 \\ 742 \\ 145 \end{bmatrix}$

The population appears to be iincreasing at a rate of 27%.

8 a i $\begin{bmatrix} 0 \\ 300 \\ 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$ **iii** $\begin{bmatrix} 1200 \\ 0 \\ 0 \end{bmatrix}$

b Population cycles through three states

c **i** Population decreases by 50% every three time periods

ii Population increases by 25% every three time periods

9 **a** $\begin{bmatrix} 2800 \\ 200 \\ 200 \\ 40 \end{bmatrix}$ **b** $\begin{bmatrix} 1080 \\ 1400 \\ 100 \\ 20 \end{bmatrix}$ **c** $\begin{bmatrix} 4440 \\ 540 \\ 700 \\ 10 \end{bmatrix}$

10 **a** **i** Every 3 years, the population returns to 1000 newborns

ii Every 3 years, the population increases by 50% and returns to only newborns

iii Every 3 years, the population decreases by 40% and returns to only newborns

b Long-term growth rate 1.26

Chapter 9 review

Short-response questions

1 **a** $\mathbf{S}_1 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}, \mathbf{S}_2 = \begin{bmatrix} 50 \\ 50 \\ 50 \\ 50 \end{bmatrix}$

b 50%

c 4 time periods

2 **a**
- A defeats B and C
- B defeats C, D and E
- C defeats E
- D defeats A, C and E
- E defeats A

b $B,\ D,\ A,\ E,\ C$

3 **a** $\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$

b $A,\ B = E,\ C,\ D$

4 **a** $c = \dfrac{a^2}{mb},\ d = \dfrac{ab}{m^2}$

b **i** $\dfrac{1}{4}$ **ii** $\mathbf{S}_0 = \begin{bmatrix} 4000 \\ 2000 \\ 1000 \end{bmatrix}$

iii $\mathbf{S}_1 = \begin{bmatrix} 1000 \\ 500 \\ 250 \end{bmatrix}, \mathbf{S}_2 = \begin{bmatrix} 250 \\ 125 \\ 62.5 \end{bmatrix}$

iv After 5 months

c **i** 1 **ii** $\mathbf{S}_0 = \begin{bmatrix} 6400 \\ 800 \\ 100 \end{bmatrix}$

iii $\mathbf{S}_1 = \mathbf{S}_2 = \mathbf{S}_3 = \mathbf{S}_4 = \begin{bmatrix} 6400 \\ 800 \\ 100 \end{bmatrix}$

d Long-term growth rate 2 (i.e. after a certain stage, the population doubles each month)

5 **a** Preferences:
- DogDelite over Doggy+ and EasyFeed
- CanineCandy over DogDelite, Doggy+ and SuperDog
- Doggy+ over SuperDog and EasyFeed
- SuperDog over DogDelite and EasyFeed
- EasyFeed over CanineCandy

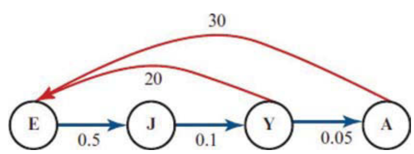
b 1st: CanineCandy; Equal 2nd: DogDelite, Doggy+, SuperDog; 5th: EasyFeed

6 **a** $\begin{bmatrix} 0 & 0.1 & 0.9 & 0.2 & 0 & 0 & 0 & 0 \\ 0.98 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.95 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.95 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \end{bmatrix}$

b $\mathbf{S}_2 = \begin{bmatrix} 104.5 \\ 107.8 \\ 0 \\ 90.25 \\ 85.5 \\ 31.5 \\ 0 \\ 0 \end{bmatrix}, \mathbf{S}_3 = \begin{bmatrix} 28.83 \\ 102.41 \\ 102.41 \\ 0 \\ 81.225 \\ 59.85 \\ 15.75 \\ 0 \end{bmatrix}$

c 1.035

7 **a**



b 920

c 5 weeks

d 5%

e **i** 1607

ii 1706

f 2%

Multiple-choice questions

1 C **2** B **3** A **4** A **5** D
6 D **7** A **8** A

Chapter 10

Short-response questions

1 $z = \pm 2,\ z = \pm\sqrt{3}i$

2 $\dfrac{\pi}{12}$

3 $z = 1, 2, -2 + i, -2 - i$

4 **a** $z = 1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

b $\operatorname{cis}(0), \operatorname{cis}\left(\dfrac{\pi}{3}\right), \operatorname{cis}\left(-\dfrac{\pi}{3}\right)$

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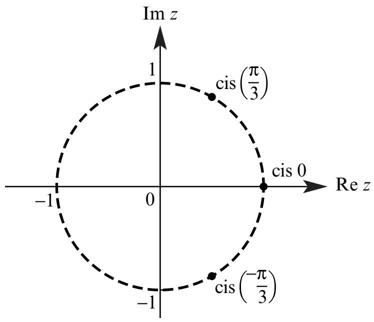
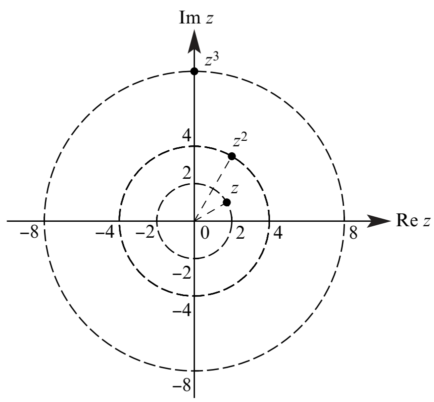
c



5 $(-2\sqrt{2}, \sqrt{2})$

6 $z = 2\operatorname{cis}\left(\frac{\pi}{6}\right),\ z^2 = 4\operatorname{cis}\left(\frac{\pi}{3}\right),\ z^3 = 8\operatorname{cis}\left(\frac{\pi}{2}\right) = 8i$



7 $\cos\theta - (\sin\theta)i$

8 b $(z-1-i)(z-2+3i)(z-2-3i)$

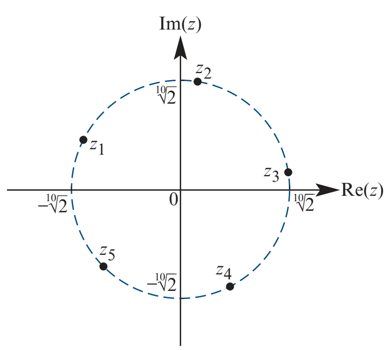
9 b i $(-1 \pm \sqrt{2})i$ **ii** i **iii** $\pm 1 - i$

10 a $a = 3, b = 4, c = 2$ **b** $-\sqrt{3} + i$

11 $z_1 = \sqrt[10]{2}\operatorname{cis}\left(\frac{17\pi}{20}\right),\quad z_2 = \sqrt[10]{2}\operatorname{cis}\left(\frac{9\pi}{20}\right),$

$z_3 = \sqrt[10]{2}\operatorname{cis}\left(\frac{\pi}{20}\right),\quad z_4 = \sqrt[10]{2}\operatorname{cis}\left(-\frac{7\pi}{20}\right),$

$z_5 = \sqrt[10]{2}\operatorname{cis}\left(-\frac{15\pi}{20}\right)$



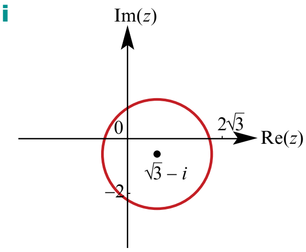
12 $c = 1, r = \sqrt{2}$

13 $a = 3, k = -30$

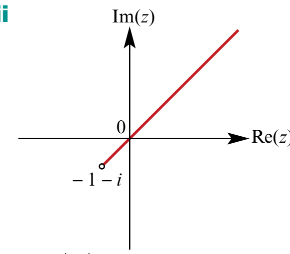
14 a $z_1 = 2\operatorname{cis}\left(-\frac{\pi}{6}\right),\ z_2 = \sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

b $\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$ **c** $\sqrt{2}\operatorname{cis}\left(-\frac{7\pi}{12}\right)$

d i



ii



15 a i $2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **ii** $2^n\operatorname{cis}\left(\frac{n\pi}{3}\right)$

iii $n = 3k, k \in \mathbb{Z}$

iv no such values exist

b $z_1^2 = -2 + 2\sqrt{3}i,\ z_1^3 = -8$

c $a = 1,\ b = -2$

d $z = -\frac{5}{2}, 1 + \sqrt{3}i, 1 - \sqrt{3}i$

16 a $C = 1 - 2i,\ D = 3 + 2i$ **b** Centre i

31 $(9, 4, -5), (-1, 6, 1), (-5, -2, 9)$

32 $\frac{2}{3}(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}),\ \frac{1}{3}(5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}})$

33 a $-\hat{\boldsymbol{i}}$ **b** $\frac{7\sqrt{6}}{18}$ **c** $\frac{\sqrt{5}}{2}$

34 a $m = \pm 5$ **b** $m = -\frac{5}{4}$

c $4\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 7\hat{\boldsymbol{k}}$ **d** $m = -\frac{7}{2}$

35 b $\lambda = \frac{2}{7}$

36 a $\frac{\sqrt{13}}{13}(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$

b i $-\frac{10}{13}(3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$ **ii** $\frac{10\sqrt{13}}{13}$

37 a $p = \frac{1}{3}(4 + 2\sqrt{2}i),\ q = \frac{1}{3}(2 + 4\sqrt{2}i)$

b i $\boldsymbol{b} - \boldsymbol{a}$ **ii** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ **iii** $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$

iv $\frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$ **v** $\frac{1}{3}(2\boldsymbol{b} - \boldsymbol{a})$

38 a $\boldsymbol{r} = \lambda(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{k}}),\ \lambda \in \mathbb{R}$

b $\boldsymbol{r} = 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + \lambda(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}),\ \lambda \in \mathbb{R}$

c $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + \lambda(-3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}),\ \lambda \in \mathbb{R}$

39 a $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) = 0$ **b** $\boldsymbol{r} \cdot (-2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{k}}) = 6$

c $\boldsymbol{r} \cdot (4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}) = -6$

40 a $\frac{4}{3}$ **b** $\frac{\sqrt{14}}{7}$ **c** 0 **d** $\frac{2\sqrt{3}}{3}$

41 $(3, -1, -3)$

42 a $(0, 1, 0)$ **b** $(2, -1, 2)$ **c** $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

43 a $m = \frac{1}{7}$

b i $\frac{19}{7}$ **ii** $\frac{4}{7}(2\hat{j} - 2\hat{k})$

44 3

45 a $a = -3$

b $r = \frac{1}{11}(25 - 14\lambda)\hat{i} + \frac{27}{11}(\lambda - 1)\hat{j} + \lambda\hat{k}$

46 a $\frac{\sqrt{21}}{14}$ **b** $r = \hat{j} + t(-\hat{i} - \hat{j} + \hat{k})$

47 c $r = 8\hat{j} + 5\hat{k} + t(\hat{i} - 2\hat{j} + 2\hat{k})$

48 a $\frac{1}{\sqrt{51}}$ **b** $\frac{2}{29}$

49 a $r = 3\hat{i} + 5\hat{j} + 9\hat{k} + t(-2\hat{i} + 4\hat{j} + \hat{k})$

b $\frac{x-3}{-2} = \frac{y-5}{4} = z - 9$

c $x = 3 - 2t,\ y = 5 + 4t,\ z = 9 + t$

50 a $-\hat{i}$ **b** $\frac{7\sqrt{6}}{18}$ **c** $\frac{\sqrt{5}}{2}$

51 c $3 : 1$

52 a $(3, 3, 0)$ **b** $-\frac{3\sqrt{35}}{35}$

c $6\sqrt{26}$

53 a $\overrightarrow{OA} = \hat{i} + 6\hat{j} + 7\hat{k}$, $\overrightarrow{OB} = 10\hat{i} + 10\hat{k}$, $\overrightarrow{OC} = 9\hat{i} - 6\hat{j} + 3\hat{k}$

b i $\overrightarrow{OD} = 4\hat{i} + 4\hat{j} + 8\hat{k}$ **ii** $\overrightarrow{ON} = 6\hat{i} + 6\hat{k}$

54 3

55 a $\sqrt{3}$ **b** $\sqrt{2}$

56 $a = \pm\sqrt{2}$

57 a $v(t) = \cos t\,\hat{i} + \cos(2t)\hat{j}$

b $a(t) = -\sin t\,\hat{i} - 2\sin(2t)\hat{j}$

c $d(t) = |\sin t|\sqrt{2 - \sin^2 t}$

d $s(t) = \sqrt{2 - 5\sin^2 t + 4\sin^4 t}$

e $y^2 = x^2(1 - x^2)$

58 a $x^2 + y^2 = 9$; circle with centre $(0, 0)$ and radius 3; particle starts at $(0, -3)$ and moves anticlockwise

b $v = 6\cos(2t)\hat{i} + 6\sin(2t)\hat{j}$,
$a = -12\sin(2t)\hat{i} + 12\cos(2t)\hat{j}$

c 6 m/s

59 a $\frac{x^2}{4} - 4y^2 = 1,\ x \geq 2,\ y \geq 0$

b $v(t) = 2\tan t\,\sec t\,\hat{i} + 0.5\sec^2 t\,\hat{j}$

c $2\sqrt{13}$ m/s

60 $\frac{5}{2}\hat{i} + \hat{j} - \frac{19}{8}\hat{k}$

61 b $y = \sqrt{3}x - \frac{g}{200}x^2$

62 a $r(t) = (\cos(2t) + 1)\hat{i} + (\sin(2t) - 1)\hat{j}$

b $(x - 1)^2 + (y + 1)^2 = 1$

c $t = \frac{\pi}{4}, \frac{5\pi}{4}$

63 a $\frac{28}{g}$ seconds **b** $y = \frac{\sqrt{3}}{3}x - \frac{g}{1176}x^2$

c $\frac{98}{g} = 10$ metres

64 c i $-(\hat{i} + \hat{j}), \mathbf{0}$ **iii** $-0.43\hat{i} - 0.68\hat{j}$

66 a $\frac{6}{5}(4\hat{i} + 3\hat{j})$

b i $\frac{1}{5}(-11\hat{i} + 28\hat{j})$ **ii** $\frac{1}{5}(13\hat{i} + 46\hat{j})$

iii $-7\hat{i} + 2\hat{j} + \frac{6}{5}t(4\hat{i} + 3\hat{j})$

c i $\frac{1}{5}(29\hat{i} + 58\hat{j})$ **ii** $\frac{8}{3}$ hours

iii $\frac{1}{5}\sqrt{(15 + 11t)^2 + (27t - 15)^2}$

iv 3.91 km

67 $\begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$; $x = \frac{4}{19},\ y = \frac{9}{19}$

68 $x = -\frac{1}{4},\ y = 0,\ z = \frac{1}{2}$

69 a $\det(\mathbf{A}) = 2k^2 - 4k - 6$

b $k \in \mathbb{R} \setminus \{-1, 3\}$

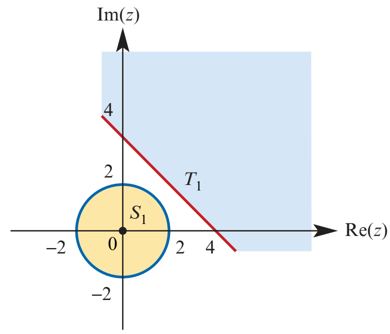
70 $a = -2$

71 $a = \frac{2}{5},\ b = -18;$

$x = \frac{5(\lambda - 7)}{6},\ y = \frac{19 - 7\lambda}{3},\ z = \lambda$

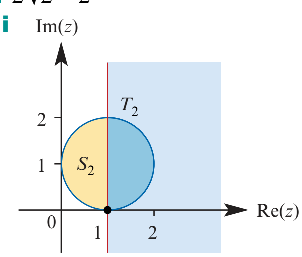
72 $a = \pm b$

73 a i



ii $2\sqrt{2} - 2$

b i



ii Maximum $\sqrt{2} + 1$; minimum 1

74 a $z^2 - 2z + 4$

b i $2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ **ii** $4\operatorname{cis}\left(-\frac{2\pi}{3}\right), -8$

iii $1 \pm \sqrt{3}i,\ -1$

c i $\sqrt{7},\ \sqrt{7}$ **ii** Isosceles

75 a $(z + 2i)(z - 2i)$ **b** $(z^2 + 2i)(z^2 - 2i)$

d $(z - 1 - i)(z + 1 + i)(z - 1 + i)(z + 1 - i)$

e $(z^2 - 2z + 2)(z^2 + 2z + 2)$

76 a $2 + 11i$

b i $\frac{2\sqrt{5}}{25}$ **ii** $\frac{11\sqrt{5}}{25}$

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77 **c** **i** 1 **ii** −1
d **i** $z^2 - 3z + 3 = 0$ **ii** $z^2 + 2z + 13 = 0$
e 0, 3

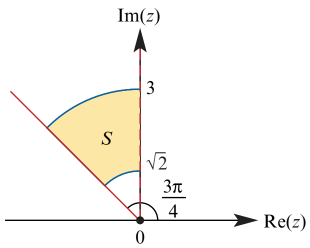
78 **a** $z^4 + z^3 + z^2 + z + 1$ **c** $\operatorname{cis}\left(-\frac{2\pi}{5}\right)$
d $\operatorname{cis}\left(\pm\frac{2\pi}{5}\right)$, $\operatorname{cis}\left(\pm\frac{4\pi}{5}\right)$, 1
e $\left(z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$

79 **a** 4, 9, −4 **b** 5

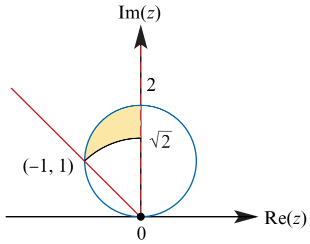
80 **b** $\cos(5\theta) = \cos^5\theta\,(1 - 10\tan^2\theta + 5\tan^4\theta)$,
$\sin(5\theta) = \cos^5\theta\,(5\tan\theta - 10\tan^3\theta + \tan^5\theta)$

81 **a** $\operatorname{cis}(\pm\theta)$

82 **a**



b $-1 + i$, $-1 + 2i$, $-2 + 2i$
c



83 **a** 103.63° **b** $2\sqrt{17}$

84 **a** **i** $\frac{3}{2}(\boldsymbol{b} - \boldsymbol{a})$ **ii** $\frac{1}{2}(3\boldsymbol{b} - \boldsymbol{a})$
b **i** $\overrightarrow{AB} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\overrightarrow{BC} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ **iv** $3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$
c $x = 4, y = 5, z = 2$

85 **a** **i** $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ **iii** $\frac{2}{3}(\boldsymbol{a} - \boldsymbol{b})$
b $\overrightarrow{DA} = 2\overrightarrow{BD}$

86 **a** **i** 151° **ii** $\frac{1}{9}(34\hat{\boldsymbol{i}} + 40\hat{\boldsymbol{j}} + 23\hat{\boldsymbol{k}})$
iii $x = 3, y = -2, z = 16$
b **i** $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\overrightarrow{OA} = 2\overrightarrow{BQ}$

87 **b** 4 : 1 : 3 **c** $4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$ **e** $s = 3, t = -2$

89 **c** 8 : 1

90 **a** **i** $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ **ii** $\frac{1}{6}(2\boldsymbol{b} - 5\boldsymbol{a})$
b **i** 2 : 3 **ii** 6 : 1

91 **a** **i** $2\boldsymbol{c} - \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ **iii** $\frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{c})$

92 **a** Since $\boldsymbol{a} \times (\boldsymbol{b} - 3\boldsymbol{c}) = \boldsymbol{0}$ and $\boldsymbol{a} \neq \boldsymbol{0}$, we must have $\boldsymbol{b} - 3\boldsymbol{c} = k\boldsymbol{a}$ for some $k \in \mathbb{R}$
b **i** 1 **ii** $2\sqrt{3}$ **iii** $\pm 2\sqrt{3}$
c $\frac{1}{\sqrt{3}}$

93 **b** **i** $\boldsymbol{r} = \lambda(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) + \mu(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) + \nu(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$, where $\lambda + \mu + \nu = 1$
ii $\boldsymbol{r} = \lambda(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) + \mu(-\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}) + \nu(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}})$, where $\lambda + \mu + \nu = 1$
c $\boldsymbol{r} = t\hat{\boldsymbol{i}} + (4t - 2)\hat{\boldsymbol{j}} + (6 - 9t)\hat{\boldsymbol{k}}$

94 **b** **i** $\boldsymbol{p} + \frac{(k - \boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}$
ii $\left|\frac{(k - \boldsymbol{p}\cdot\boldsymbol{n})\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}\right| = \frac{|k - \boldsymbol{p}\cdot\boldsymbol{n}|}{|\boldsymbol{n}|}$

95 **a** $\overrightarrow{AB} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\overrightarrow{AC} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{k}}$
b $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$ **c** $x - y + 2z = 5$
d $\boldsymbol{r} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}})$
e $\frac{5}{3}\hat{\boldsymbol{i}} + \frac{4}{3}\hat{\boldsymbol{j}} + \frac{7}{3}\hat{\boldsymbol{k}}$ **f** $\frac{2\sqrt{6}}{3}$

96 **a** $\overrightarrow{OA} = \hat{\boldsymbol{i}} + \sqrt{\lambda}\hat{\boldsymbol{k}}$, $\overrightarrow{CA} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + \sqrt{\lambda}\hat{\boldsymbol{k}}$
b 56°
c $13 + 8\sqrt{3}$ since $\lambda > 0$

97 **a** $\overrightarrow{OX} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$, $\overrightarrow{OY} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$,
$\overrightarrow{OZ} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d})$, $\overrightarrow{OW} = \frac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
b $\overrightarrow{DX} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}) - \boldsymbol{d}$, $\overrightarrow{BY} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}) - \boldsymbol{b}$,
$\overrightarrow{CZ} = \frac{1}{3}(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{d}) - \boldsymbol{c}$, $\overrightarrow{AW} = \frac{1}{3}(\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) - \boldsymbol{a}$
c $\overrightarrow{OP} = \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
d $\overrightarrow{OQ} = \overrightarrow{OR} = \overrightarrow{OS} = \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
e $Q = R = S = P$, which is the centre of the sphere that circumscribes the tetrahedron

98 12.58°

99 93.79 m

100 **a** **i** $\frac{3}{2}\sin(2t)\,\hat{\boldsymbol{i}} - 2\cos(2t)\,\hat{\boldsymbol{j}}$
ii $-6\sin(2t)\,\hat{\boldsymbol{i}} + 8\cos(2t)\,\hat{\boldsymbol{j}}$
iii $t = \frac{n\pi}{4}$, $n \in \mathbb{N} \cup \{0\}$
iv $16x^2 + 9y^2 = 36$
b $a = \frac{(2n+1)\pi}{4}$, $n \in \mathbb{N} \cup \{0\}$

101 $(x - 1)^2 + (y - 2)^2 = 1$

102 **a** $(3, 0)$, $(0, 4)$, $(-3, 0)$, $(0, -4)$
b Anticlockwise **c** $\frac{x^2}{9} + \frac{y^2}{16} = 1$

103 **a** $\boldsymbol{r} = (\cos(4t) - 1)\hat{\boldsymbol{i}} + (\sin(4t) + 1)\hat{\boldsymbol{j}}$
b $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ **c** $\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = 0$

104 **a** $\boldsymbol{r}_A(t) = 10\sqrt{3}t\,\hat{\boldsymbol{i}} + \left(10t - \frac{g}{2}t^2\right)\hat{\boldsymbol{j}}$,
$\boldsymbol{r}_B(t) = 10t\,\hat{\boldsymbol{i}} + \left(10\sqrt{3}t - \frac{g}{2}t^2\right)\hat{\boldsymbol{j}}$
b $\boldsymbol{r}_A\left(\frac{10}{g}\right) = \frac{100\sqrt{3}}{g}\hat{\boldsymbol{i}} + \frac{50}{g}\hat{\boldsymbol{j}}$,
$\boldsymbol{r}_B\left(\frac{10\sqrt{3}}{g}\right) = \frac{100\sqrt{3}}{g}\hat{\boldsymbol{i}} + \frac{150}{g}\hat{\boldsymbol{j}}$
c 10.35 m

105 **a** $|\dot{\boldsymbol{r}}_A| = 3\sqrt{13}$ m/s, $|\dot{\boldsymbol{r}}_B| = \sqrt{41}$ m/s
b $\boldsymbol{r}_A(t) = (9t + 2)\hat{\boldsymbol{i}} + (6t + 2)\hat{\boldsymbol{j}}$,
$\boldsymbol{r}_B(t) = (5t + 3)\hat{\boldsymbol{i}} + (4t + 3)\hat{\boldsymbol{j}}$

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c $\overrightarrow{AB} = (-4t + 1)\hat{\boldsymbol{i}} + (-2t + 1)\hat{\boldsymbol{j}}$

d $|\overrightarrow{AB}| = \sqrt{20t^2 - 12t + 2}$

e $t = \dfrac{3}{10}$

106 a $2\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}}$ m/s **b** $\dot{\boldsymbol{r}}_1(t) = 2\hat{\boldsymbol{i}} - 2t\hat{\boldsymbol{j}}$

c $\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ **d** $t = 0$ **e** $t = 5$

f Yes; $t = 2$

107 a 6π s

b i $-(3\sqrt{3}\hat{\boldsymbol{i}} + 2.25\hat{\boldsymbol{j}})$ **ii** $\hat{\boldsymbol{i}} - \dfrac{3\sqrt{3}}{4}\hat{\boldsymbol{j}}$

c i $1.5\sqrt{9 + 7\sin^2\left(\dfrac{t}{3}\right)}$

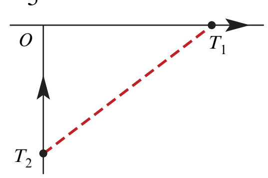
ii $t = 3\left(\dfrac{\pi}{2} + n\pi\right), n \in \mathbb{N} \cup \{0\}$

d $\ddot{\boldsymbol{r}} = -\dfrac{1}{9}\boldsymbol{r}, t = 3n\pi, n \in \mathbb{N} \cup \{0\}$

108 a i T_1 **ii** t_0

b ii $\dfrac{2\sqrt{5}}{5}Vt_0$

iii



109 a $\sqrt{g(2h + d)}$ m/s

b $\sqrt{\dfrac{d}{2(2h + d)}}$

110 a i $h\hat{\boldsymbol{j}}$, for $0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$ at the base of the cliff

ii $V\cos\alpha\,\hat{\boldsymbol{i}} + V\sin\alpha\,\hat{\boldsymbol{j}}$

b i $V\cos\alpha\,\hat{\boldsymbol{i}} + (V\sin\alpha - gt)\hat{\boldsymbol{j}}$

ii $Vt\cos\alpha\,\hat{\boldsymbol{i}} + \left(h + Vt\sin\alpha - \dfrac{gt^2}{2}\right)\hat{\boldsymbol{j}}$

c $\dfrac{V\sin\alpha}{g}$

111 b i $\boldsymbol{r}_2 = (0.2t - 1.2)\hat{\boldsymbol{i}} + (-0.2t + 3.2)\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$

ii $t = 16$ at $2\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$

112

Day	Sat	Sun	Wed	Thu	Total
Calories	1380	2200	1540	940	6060

113 a $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

b Amina, Carl, Dylan, Bessie

114 a $\mathbf{A}^{-1} = \dfrac{1}{12}\begin{bmatrix} 0 & -6 & 9 \\ 0 & 6 & -3 \\ 4 & -8 & 0 \end{bmatrix}$

b $x = 15, y = 3, z = -12$

115 a $x = \dfrac{ak - 2a - k - 2}{a - 5}$, $y = \dfrac{2(3 - k)}{a - 5}$,

$z = \dfrac{-ak + 4a - k - 2}{a - 5}$

b $k \neq 3$ **c** $k = 3$

116 a $c = a + b$

b $\boldsymbol{r} = (b - \lambda)\hat{\boldsymbol{i}} + (a - b - \lambda)\hat{\boldsymbol{j}} + \lambda\hat{\boldsymbol{k}},\ \lambda \in \mathbb{R}$

117 a i $3a - b + c = -10,\ -a - 2b + c = -5,$
$4a - 5b + c = -41$

ii $\left[\begin{array}{ccc|c} 3 & -1 & 1 & -10 \\ -1 & -2 & 1 & -5 \\ 4 & -5 & 1 & -41 \end{array}\right]$

iii Centre $\left(\dfrac{3}{2}, -\dfrac{7}{2}\right)$; radius $\dfrac{\sqrt{34}}{2}$

b i $3a - b + c = -10,\ -a - 2b + c = -5,$
$kb + c = -k^2$

ii $\left[\begin{array}{ccc|c} 3 & -1 & 1 & -10 \\ -1 & -2 & 1 & -5 \\ 0 & k & 1 & -k^2 \end{array}\right]$

iii $k \neq -\dfrac{7}{4}$

c $k \neq -\dfrac{3}{2}$

118 a $\mathbf{L} = \begin{bmatrix} 0 & 0.9 & 0.7 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$

b $\mathbf{P}_0 = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$

c $\mathbf{P}_1 = \begin{bmatrix} 90 \\ 0 \\ 80 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} 56 \\ 81 \\ 0 \end{bmatrix}, \mathbf{P}_3 = \begin{bmatrix} 73 \\ 50 \\ 65 \end{bmatrix}, \mathbf{P}_4 = \begin{bmatrix} 91 \\ 66 \\ 40 \end{bmatrix}$;
number of cases is increasing and spreading through the three stages

d $\mathbf{P}_{40} = \begin{bmatrix} 5309 \\ 4258 \\ 3036 \end{bmatrix}, \mathbf{P}_{41} = \begin{bmatrix} 5957 \\ 4778 \\ 3406 \end{bmatrix}$, 1.122

e Not sufficient to eradicate disease; growth rate after 40 weeks is approximately 1.027

f Sufficient to eradicate disease; growth rate after 40 weeks is approximately 0.94

Multiple-choice questions

1 D	**2** D	**3** C	**4** D	**5** B	**6** D
7 D	**8** D	**9** B	**10** C	**11** D	**12** B
13 A	**14** D.	**15** C	**16** D	**17** A	**18** C
19 C	**20** B	**21** C	**22** C	**23** D	**24** C
25 C	**26** B	**27** B	**28** D	**29** D	**30** D
31 D	**32** C	**33** D	**34** C	**35** B	**36** D
37 B	**38** A	**39** C	**40** B	**41** A	**42** D
43 A	**44** B	**45** D	**46** D	**47** C	**48** C
49 A	**50** C	**51** B	**52** C	**53** C	**54** D
55 B	**56** C	**57** B	**58** D	**59** C	**60** D
61 D	**62** C	**63** D	**64** A	**65** A	**66** A
67 D	**68** C	**69** A	**70** B	**71** C	**72** B
73 C	**74** D	**75** C	**76** D	**77** A	**78** D
79 A	**80** C	**81** D	**82** C	**83** A	**84** C
85 A	**86** C	**87** D	**88** A	**89** B	**90** D
91 B	**92** B	**93** D	**94** A	**95** B	**96** A
97 A	**98** C	**99** D	**100** A	**101** A	**102** C
103 C	**104** D	**105** C	**106** D	**107** A	**108** B
109 D	**110** D				

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Chapter 11

Exercise 11A

1 a 0 **b** 20 **c** 1
d $\frac{5}{24}$ **e** $\frac{1}{\sqrt{2}}+\frac{\pi^2}{16}$ **f** $\frac{e^3}{3}+\frac{1}{6}$
g 0 **h** 0 **i** 1

2 a 8 **b** 8 **c** 2 **d** -2 **e** -2 **f** 4

3 a $\frac{2}{2x+1}$
b $\frac{2}{2x-1}$
c $\cot x$
d $\sec x$
e $\frac{\sin^2 x-\cos^3 x}{\sin x\cos x(\cos x+\sin^2 x)}$
f $\operatorname{cosec} x$
g $\operatorname{cosec} x$
h $\frac{1}{\sqrt{x^2-4}}, x \neq \pm 2$
i $\frac{1}{\sqrt{x^2+4}}$

4 a $\frac{1}{2}\ln|2x-5|$ **b** $\frac{1}{2}\ln\left(\frac{3}{5}\right)$ **c** $\frac{1}{2}\ln\left(\frac{7}{9}\right)$

5 a $\frac{1}{3}\ln\left(\frac{5}{2}\right)$ **b** $\frac{1}{3}\ln\left(\frac{5}{11}\right)$ **c** $\frac{1}{3}\ln\left(\frac{7}{4}\right)$
d $\ln\left(\frac{4}{5}\right)$ **e** $\frac{1}{2}\ln\left(\frac{5}{3}\right)$ **f** $\frac{1}{3}\ln 3$
g $3-\ln 2$ **h** $3+2\ln 2$ **i** $2+7\ln\left(\frac{3}{4}\right)$

Exercise 11B

1 a $\frac{1}{\sqrt{4-x^2}}, x\in(-2,2)$
b $\frac{-1}{\sqrt{16-x^2}}, x\in(-4,4)$
c $\frac{3}{9+x^2}$
d $\frac{3}{\sqrt{1-9x^2}}, x\in\left(-\frac{1}{3},\frac{1}{3}\right)$
e $\frac{-2}{\sqrt{1-4x^2}}, x\in\left(-\frac{1}{2},\frac{1}{2}\right)$
f $\frac{5}{1+25x^2}$
g $\frac{3}{\sqrt{16-9x^2}}, x\in\left(-\frac{4}{3},\frac{4}{3}\right)$
h $\frac{-3}{\sqrt{4-9x^2}}, x\in\left(-\frac{2}{3},\frac{2}{3}\right)$
i $\frac{10}{25+4x^2}$
j $\frac{1}{\sqrt{25-x^2}}, x\in(-5,5)$

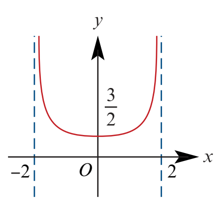
2 a $\frac{1}{\sqrt{-x(x+2)}}, x\in(-2,0)$
b $\frac{-1}{\sqrt{-x(x+1)}}, x\in(-1,0)$
c $\frac{1}{x^2+4x+5}$
d $\frac{3}{\sqrt{6x-9x^2}}, x\in\left(0,\frac{2}{3}\right)$
e $\frac{6}{\sqrt{-3(3x^2+2x-1)}}, x\in\left(-1,\frac{1}{3}\right)$
f $\frac{20}{\sqrt{-5(5x^2-6x+1)}}, x\in\left(\frac{1}{5},1\right)$
g $\frac{-10}{x^2-2x+5}$
h $\frac{-2x}{\sqrt{1-x^4}}, x\in(-1,1)$

3 a $\frac{3}{x\sqrt{x^2-9}}$ **b** $\frac{-5}{x\sqrt{x^2-25}}$ **c** $\frac{3}{x\sqrt{4x^2-9}}$

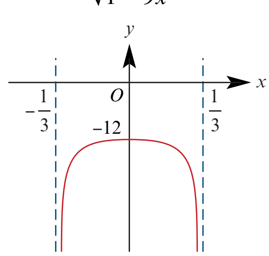
4 a $\frac{a}{\sqrt{1-a^2x^2}}, x\in\left(-\frac{1}{a},\frac{1}{a}\right)$
b $\frac{-a}{\sqrt{1-a^2x^2}}, x\in\left(-\frac{1}{a},\frac{1}{a}\right)$
c $\frac{a}{1+a^2x^2}$

5 a $\frac{4x}{\sqrt{(1-x^2)^3}}$ **b** $\frac{-2x}{(1+x^2)^2}$
c $\frac{3x}{\sqrt{(16-x^2)^3}}$ **d** $\frac{-27x}{\sqrt{(1-9x^2)^3}}$
e $\frac{-96x}{(9+4x^2)^2}$

6 a i $[-2,2]$ **ii** $\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$
b $f'(x)=\frac{3}{\sqrt{4-x^2}}, x\in(-2,2)$
c



7 a i $\left[-\frac{1}{3},\frac{1}{3}\right]$ **ii** $[0,4\pi]$
b $f'(x)=\frac{-12}{\sqrt{1-9x^2}}, x\in\left(-\frac{1}{3},\frac{1}{3}\right)$
c



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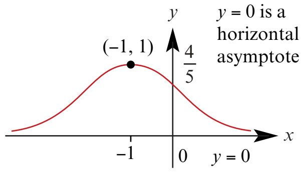
8 a i $\mathbb{R}$ **ii** $(-\pi, \pi)$

b $f'(x) = \dfrac{4}{x^2 + 2x + 5}$

c



9 a $f'(x) = \dfrac{2\sin^{-1} x}{\sqrt{1 - x^2}}, x \in (-1, 1)$

b $f'(x) = 0, x \in (-1, 1)$

c $f'(x) = \dfrac{-x}{\sqrt{1 - x^2}}, x \in (-1, 1)$

d $f'(x) = \dfrac{-x}{\sqrt{1 - x^2}}, x \in (-1, 1)$

e $f'(x) = \dfrac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}, x \in (-1, 1)$

f $f'(x) = \dfrac{e^x}{1 + e^{2x}}$

10 a 0.35 **b** −6.29 **c** $\dfrac{3}{5}$

11 a $\pm\dfrac{\sqrt{3}}{2}$ **b** $\pm\dfrac{\sqrt{391}}{10}$ **c** $\pm\dfrac{\sqrt{5}}{3}$

d $-1 \pm \dfrac{\sqrt{1599}}{20}$ **e** $\pm\dfrac{\sqrt{35}}{4}$

f $\dfrac{1}{2}(1 \pm \sqrt{7})$

12 a $y = \dfrac{4\sqrt{3}}{3}x - \dfrac{\sqrt{3}}{3} + \dfrac{\pi}{6}$

b $y = x - \dfrac{1}{2} + \dfrac{\pi}{4}$

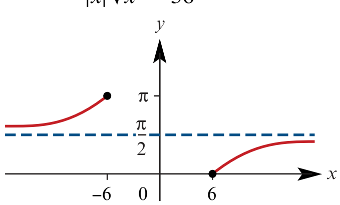
c $y = -2\sqrt{3}x + \dfrac{\sqrt{3} + \pi}{3}$

d $y = -6x + \sqrt{3} + \dfrac{\pi}{6}$

13 a $(-\infty, -6] \cup [6, \infty)$

b $f'(x) = \dfrac{6}{|x|\sqrt{x^2 - 36}}, x < -6 \text{ or } x > 6$

c



Exercise 11C

1 a $\sin^{-1}\left(\dfrac{x}{3}\right) + c$ **b** $\dfrac{1}{\sqrt{5}}\tan^{-1}\left(\dfrac{x}{\sqrt{5}}\right) + c$

c $\tan^{-1}(t) + c$ **d** $5\sin^{-1}\left(\dfrac{x}{\sqrt{5}}\right) + c$

e $\dfrac{3}{4}\tan^{-1}\left(\dfrac{x}{4}\right) + c$ **f** $\dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{2}\right) + c$

g $10\sin^{-1}\left(\dfrac{t}{\sqrt{10}}\right) + c$ **h** $\dfrac{1}{12}\tan^{-1}\left(\dfrac{4t}{3}\right) + c$

i $\dfrac{\sqrt{2}}{2}\sin^{-1}\left(\dfrac{x\sqrt{10}}{5}\right) + c$

j $\dfrac{7}{\sqrt{3}}\tan^{-1}\left(\dfrac{y}{\sqrt{3}}\right) + c$

2 a $\dfrac{\pi}{2}$ **b** $\dfrac{\pi}{2}$ **c** $\dfrac{5\pi}{6}$ **d** $\dfrac{3\pi}{10}$

e $\dfrac{\pi}{8}$ **f** $\dfrac{\pi}{16}$ **g** $\dfrac{\pi}{6}$ **h** $\dfrac{\pi}{8}$

i $\dfrac{\pi}{2}$ **j** $\dfrac{1}{\sqrt{3}}\tan^{-1}(2\sqrt{3})$

Exercise 11D

1 a $\dfrac{(x^2 + 1)^4}{4} + c$ **b** $-\dfrac{1}{2(x^2 + 1)} + c$

c $\dfrac{1}{4}\sin^4 x + c$ **d** $-\dfrac{1}{\sin x} + c$

e $\dfrac{1}{12}(2x + 1)^6 + c$ **f** $\dfrac{5}{3}(9 + x^2)^{\frac{3}{2}} + c$

g $\dfrac{1}{12}(x^2 - 3)^6 + c$ **h** $-\dfrac{1}{4(x^2 + 2x)^2} + c$

i $-\dfrac{1}{3(3x + 1)^2} + c$ **j** $2\sqrt{1 + x} + c$

k $\dfrac{1}{15}(x^3 - 3x^2 + 1)^5 + c$

l $\dfrac{3}{2}\ln(x^2 + 1) + c$ **m** $-\dfrac{3}{2}\ln|2 - x^2| + c$

2 a $\tan^{-1}(x + 1) + c$

b $\dfrac{2\sqrt{3}}{3}\tan^{-1}\left(\dfrac{\sqrt{3}(2x - 1)}{3}\right) + c$

c $\sin^{-1}\left(\dfrac{x + 2}{5}\right) + c$ **d** $\sin^{-1}(x - 5) + c$

e $\sin^{-1}\left(\dfrac{x + 3}{7}\right) + c$

f $\dfrac{\sqrt{3}}{6}\tan^{-1}\left(\dfrac{\sqrt{3}(x + 1)}{2}\right) + c$

3 a $-\dfrac{1}{2}(2x + 3)^{\frac{3}{2}} + \dfrac{1}{10}(2x + 3)^{\frac{5}{2}} + c$

b $\dfrac{2(1 - x)^{\frac{5}{2}}}{5} - \dfrac{2(1 - x)^{\frac{3}{2}}}{3} + c$

c $\dfrac{4}{9}(3x - 7)^{\frac{3}{2}} + \dfrac{28}{3}(3x - 7)^{\frac{1}{2}} + c$

d $\dfrac{4}{45}(3x - 1)^{\frac{5}{2}} + \dfrac{10}{27}(3x - 1)^{\frac{3}{2}} + c$

e $2\ln|x - 1| - \dfrac{1}{x - 1} + c$

f $\dfrac{2}{45}(3x + 1)^{\frac{5}{2}} + \dfrac{16}{27}(3x + 1)^{\frac{3}{2}} + c$

g $\dfrac{3}{7}(x + 3)^{\frac{7}{3}} - \dfrac{3(x + 3)^{\frac{4}{3}}}{4} + c$

h $\dfrac{5}{4}\ln|2x + 1| + \dfrac{7}{4(2x + 1)} + c$

i $\dfrac{2}{105}(x - 1)^{\frac{3}{2}}(15x^2 + 12x + 8) + c$

j $\dfrac{2\sqrt{x - 1}}{15}(3x^2 + 4x + 8) + c$

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Exercise 11E

1 a $\frac{61}{3}$ **b** $\frac{1}{16}$ **c** $\frac{1}{3}$ **d** $\frac{25}{114}$
e $\frac{4}{15}$ **f** $\ln 2$ **g** $\frac{4}{3}$ **h** 1
i $\frac{1}{2}$ **j** $\ln 2$ **k** $\ln\left(\frac{\sqrt{6}}{2}\right)$ **l** $\ln\left(\frac{15}{8}\right)$
m $\ln\left(\frac{e+1}{e}\right) = \ln(e+1) - 1$

Exercise 11F

1 a $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$
b $\frac{1}{32}\sin(4x) - \frac{1}{4}\sin(2x) + \frac{3}{8}x + c$
c $2\tan x - 2x + c$
d $-\frac{1}{6}\cos(6x) + c$
e $\frac{1}{2}x - \frac{1}{8}\sin(4x) + c$
f $\frac{1}{2}\tan(2x) - x + c$
g $\frac{1}{8}x - \frac{1}{32}\sin(4x) + c$
h $\frac{1}{2}\sin(2x) + c$
i $-\cot x - x + c$
j $\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + c$

2 a $\tan x + c$ **b** $\frac{1}{2}\tan(2x) + c$
c $2\tan\left(\frac{1}{2}x\right) + c$ **d** $\frac{1}{k}\tan(kx) + c$
e $\frac{1}{3}\tan(3x) - x + c$ **f** $2x - \tan x + c$
g $-x + c$ **h** $\tan x + c$

3 a $\frac{\pi}{4}$ **b** $\frac{1}{2} + \ln\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} - \frac{1}{2}\ln 2$
c $\frac{1}{3}$ **d** $\frac{1}{4} + \frac{3\pi}{32}$ **e** $\frac{4}{3}$ **f** $\frac{\pi}{4}$
g $\frac{\pi}{24} + \frac{\sqrt{3}}{64}$ **h** 1

4 a $\sin x - \frac{\sin^3 x}{3} + c$
b $\frac{4}{3}\cos^3\left(\frac{x}{4}\right) - 4\cos\left(\frac{x}{4}\right) + c$
c $\frac{1}{2}x + \frac{1}{16\pi}\sin(8\pi x) + c$
d $7\sin t\left(\cos^2 t + \frac{3}{5}\sin^4 t - \frac{1}{7}\sin^6 t\right) + c$
e $\frac{1}{5}\sin(5x) - \frac{1}{15}\sin^3(5x) + c$
f $3x - 2\sin(2x) + \frac{1}{4}\sin(4x) + c$
g $\frac{1}{48}\sin^3(2x) - \frac{1}{64}\sin(4x) + \frac{x}{16} + c$
h $\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$

Exercise 11G

1 a $\frac{2}{x-1} + \frac{3}{x+2}$ **b** $\frac{1}{x+1} - \frac{2}{2x+1}$
c $\frac{2}{x+2} + \frac{1}{x-2}$ **d** $\frac{1}{x+3} + \frac{3}{x-2}$
e $\frac{3}{5(x-4)} - \frac{8}{5(x+1)}$

2 $3 + \frac{3}{x-1} + \frac{2}{x-2}$

3 $\frac{1}{x-10} - \frac{1}{x-1}$; $\ln\left|\frac{x-10}{x-1}\right| + c$

4 a $\ln\left|\frac{x-2}{x+5}\right| + c$ **b** $\ln\left|\frac{(x-2)^5}{(x-1)^4}\right| + c$
c $\frac{1}{2}\ln|(x+1)(x-1)^3| + c$
d $2x + \ln\left|\frac{x-1}{x+1}\right| + c$
e $\frac{5}{8}\ln(|x-2|) + \frac{11}{8}\ln(|x+6|)$
f $\ln|(x-2)(x+4)^3| + c$

5 a $\ln\left|\frac{(x-3)^3}{x-2}\right| + c$
b $\ln|(x-1)^2(x+2)^3| + c$
c $\frac{x^2}{2} - 2x + \ln\left|(x+2)^{\frac{1}{4}}(x-2)^{\frac{3}{4}}\right| + c$
d $\ln((x+1)^2(x+4)^2) + c$
e $\frac{x^3}{3} - \frac{x^2}{2} - x + 5\ln|x+2| + c$
f $\frac{x^2}{2} + x + \ln\left|\frac{(x-1)^4}{x^3}\right| + c$

6 a $\ln\left(\frac{4}{3}\right)$ **b** $\ln\left(\frac{4}{3}\right)$ **c** $\frac{1}{3}\ln\left(\frac{625}{512}\right)$
d $1 + \ln\left(\frac{32}{81}\right)$ **e** $\ln\left(\frac{10}{3}\right)$ **f** $\frac{1}{2}\ln\left(\frac{7}{4}\right)$
g $\ln\left(\frac{2}{3}\right)$ **h** $\frac{1}{4}\ln\left(\frac{1}{3}\right)$
i $5\ln\left(\frac{3}{4}\right) - \ln 2$

Exercise 11H

1 a $(-x-1)e^{-x}$ **b** $x\ln x - x$
c $\sin x - x\cos x$ **d** $x\arccos(x) - \sqrt{1-x^2}$
e $\frac{1}{9}\cos(3x) + \frac{1}{3}x\sin(3x)$
f $\ln|\cos x| + x\tan x$
g $-\frac{1}{2}x^2 + x\tan x + \ln|\cos x|$
h $x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2}$
i $x\arctan x - \frac{1}{2}\ln(1+x^2)$ **j** $(-x-2)e^{-x}$
k $\frac{1}{2}(-x + \arctan x + x^2\arctan x)$
l $\frac{1}{4}x^2(2\ln x - 1)$ **m** $\frac{1}{9}x^3(3\ln x - 1)$

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n $2\sqrt{x}(\ln x - 2)$ **o** $(x+2)e^x$

p $\frac{1}{36}x^6(6\ln x - 1)$ **q** $\frac{1}{4}(2x-1)e^{2x+1}$

r $\frac{1}{4}x^2(2\ln(2x) - 1)$

2 a $-(x^2+2x+2)e^{-x}$
b $(2-x^2)\cos x + 2x\sin x$

3 a $\frac{1}{2}e^x(\sin x - \cos x)$

b $\frac{1}{13}e^{2x}(3\sin(3x) + 2\cos(3x))$

c $-\frac{1}{10}e^{3x}(\cos x - 3\sin x)$

d $-\frac{2}{5}e^x\left(\cos\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)\right)$

4 a $\frac{1}{4}(1+3e^4)$ **b** $-\frac{\pi}{2}$ **c** $-\frac{1}{8}$ **d** $\frac{2}{9}(1+2e^3)$

e $-12 + 38\sqrt{2} - 8\sqrt{2}\pi$ **f** $\frac{-2+5e^3}{27e}$

g $\ln(12) - 1$ **h** $\frac{1}{4}(5e^4 - 1)$ **i** $3\ln(27) - \frac{26}{9}$

Exercise 11I

1 $p = \frac{4}{3}$ **2** $\frac{1}{24}$

3 $e - 1 - \ln\left(\frac{1+e}{2}\right)$ **4** $\frac{9}{64}$

5 $\frac{1}{3}\ln 5$ **6** $c = \frac{3}{2}$

7 $-\frac{1}{18}\cos^6(3x) + c$ **8** $p = \left(\frac{3}{2}\right)^{\frac{1}{2}}$

9 $p = \frac{8}{5}$

10 a $-\frac{1}{2\sin^2 x} + c$ **b** $\frac{1}{20}(4x^2+1)^{\frac{5}{2}} + c$

c $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

d $\frac{1}{1-e^x} + c$

11 1

12 a $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$ **b** $\frac{1}{3}\sin^{-1}(3x) + c$

c $\frac{1}{2}\sin^{-1}(2x) + c$ **d** $\frac{1}{6}\tan^{-1}\left(\frac{2x+1}{3}\right) + c$

13 a $-\frac{1}{2x\sqrt{x-1}}$ **b** $\frac{\pi}{6}$

14 a $\frac{1}{3}(f(x))^3 + c$ **b** $-\frac{1}{f(x)} + c$

c $\ln(f(x)) + c$ **d** $-\cos(f(x)) + c$

15 $\frac{dy}{dx} = \frac{8-3x}{2\sqrt{4-x}}$; $4\sqrt{2}$

16 $a = 2, b = -3, c = -1$; $x^2 - 3x + \frac{1}{x-2} + c$

17 a $\frac{\pi}{8}$ **b** 42 **c** 0 **d** $\ln 2$

e $1 - \frac{\pi}{4}$ **f** $\ln\left(\frac{3}{2}\right)$

18 a $\frac{1}{2}\sin^2 x + c$ **b** $-\frac{1}{4}\cos(2x) + c$

19 a $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$;
$\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln|x + \sqrt{x^2+1}| + c$

b $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$

20 a $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$ **b** $\frac{1}{4}\ln\left|\frac{x+2}{2-x}\right| + c$

c $4\ln|x| + \frac{1}{2}x^2 + c$ **d** $\frac{1}{2}\ln(4+x^2) + c$

e $x - 2\tan^{-1}\left(\frac{x}{2}\right) + c$ **f** $\frac{1}{2}\tan^{-1}(2x) + c$

g $\frac{1}{3}(4+x^2)^{\frac{3}{2}} + c$

h $\frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}} + c$

i $-2\sqrt{4-x} + c$ **j** $\sin^{-1}\left(\frac{x}{2}\right) + c$

k $-8\sqrt{4-x} + \frac{2}{3}(4-x)^{\frac{3}{2}} + c$

l $-\sqrt{4-x^2} + c$

21 a $\frac{1}{4}(3e^4 - e^2)$ **b** $\frac{1}{4}(e^2+1)$ **c** $-\pi$

22 $c = \frac{5}{2}, d = \frac{3}{2}$

23 a $f'(x) = -(n-1)\sin^2 x\cos^{n-2} x + \cos^n x$

c i $\frac{3\pi}{16}$ **ii** $\frac{5\pi}{32}$ **iii** $\frac{\pi}{32}$ **iv** $\frac{4}{3}$

24 a $\frac{1}{2-n}(x+1)^{2-n} - \frac{1}{1-n}(x+1)^{1-n} + c$

b $\frac{1}{n+2} + \frac{1}{n+1}$

25 a $\frac{1}{3}a^2 + a + 1$ **b** $-\frac{3}{2}$

26 a $\frac{a^2+b^2}{(a\cos x + b\sin x)^2}$ **b** $\frac{1}{ab}$

27 a $U_n + U_{n-2} = \frac{1}{n-1}$

28 a 1 **c** $\frac{\pi}{4}$

Chapter 11 review

Short-response questions

1 a $\frac{1}{2}\ln\left(\frac{5}{7}\right)$ **b** $\frac{1}{3}\ln\left(\frac{5}{2}\right)$ **c** $4 - 7\ln 3$

2 a $\frac{1}{\sqrt{x-x^2}}$ **b** $\frac{-1}{\sqrt{-x-x^2}}$

c $\frac{1}{x^2-2x+2}$ **d** $\frac{-x}{\sqrt{1-x^2}}$

e $\frac{\sec^2(\tan^{-1} x)}{1+x^2} = 1$ **f** $\frac{e^x}{\sqrt{1-e^{2x}}}$

3 $-\frac{1}{2}$

Answers 11I → 11 review

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4 a $\frac{1}{6}\sin(2x)\,(3-\sin^2(2x))$

b $\frac{1}{4}(\ln(4x^2+1)+6\tan^{-1}(2x))$ **c** $\frac{1}{4}\ln\left|\frac{1+2x}{1-2x}\right|$

d $-\frac{1}{4}\sqrt{1-4x^2}$

e $-\frac{1}{4}x+\frac{1}{16}\ln\left|\frac{1+2x}{1-2x}\right|$

f $-\frac{1}{6}(1-2x^2)^{\frac{3}{2}}$

g $\frac{1}{2}x-\frac{1}{4}\sin\left(2x-\frac{2\pi}{3}\right)$ **h** $(x^2-2)^{\frac{1}{2}}$

i $\frac{1}{2}x-\frac{1}{12}\sin(6x)$

j $\frac{1}{6}\cos(2x)\,(\cos^2(2x)-3)$

k $2(x+1)^{\frac{3}{2}}\left(\frac{1}{5}(x+1)-\frac{1}{3}\right)$ **l** $\frac{1}{2}\tan x$

m $\frac{x}{e}-\frac{1}{3e^{3x+1}}$ **n** $\frac{1}{2}\ln|x^2-1|$ **o** $\frac{x}{8}-\frac{\sin 4x}{32}$

p $\frac{1}{2}x^2-x+\ln|1+x|$

5 a $\frac{1}{3}-\frac{\sqrt{3}}{8}$ **b** $\frac{1}{2}\ln 3$

c $\frac{1}{3}\left(\frac{5\sqrt{5}}{8}-1\right)$ **d** $\frac{1}{6}\ln\left(\frac{7}{4}\right)$

e $2+\ln\left(\frac{32}{81}\right)$ **f** $\frac{2}{3}$

g $\frac{\pi}{6}$ **h** $\frac{\pi}{4}$

i $\frac{\pi}{4}$ **j** $\frac{\pi}{16}$

k $\ln\left(\frac{3\sqrt{2}}{2}\right)$ **l** 6

6 b $\frac{1}{2}\ln|x^2+2x+3|-\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}(x+1)}{2}\right)+c$

7 a $\frac{1}{2\sqrt{x(1-x)}}$; $2\sin^{-1}(\sqrt{x})+c$

b $\frac{2x}{\sqrt{1-x^4}}$; $\sin^{-1}(x^2)+c$

8 a $\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$; $x\sin^{-1}x+\sqrt{1-x^2}+c$

b $\ln|x|+1$; $x\ln|x|-x+c$

c $\tan^{-1}x+\frac{x}{1+x^2}$; $x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)+c$

9 a $-\frac{1}{8}\cos(4x)$ **b** $\frac{1}{9}(x^3+1)^3$

c $\frac{-1}{2(3+2\sin\theta)}$ **d** $-\frac{1}{2}e^{1-x^2}$

e $\tan(x+3)-x$ **f** $\sqrt{6+2x^2}$

g $\frac{1}{3}\tan^3 x$ **h** $\frac{1}{3\cos^3 x}$

i $\frac{1}{3}\tan(3x)-x$

10 a $\frac{8}{15}$ **b** $-\frac{39}{4}$ **c** $\frac{1}{2}$

d $\frac{2}{3}(2\sqrt{2}-1)$ **e** $\frac{\pi}{2}$ **f** $1-\frac{11\ln(2)}{3}$

11 $\frac{1}{2}\left(x^2+\frac{1}{x}\right)^{-\frac{1}{2}}(2x-x^{-2})$; $\sqrt{2}$

12 a 1, 1 **b** 3, 2

13 a $\frac{1}{4}e^{-2x}(\sin(2x+3)-\cos(2x+3))$

b $x\tan x+\ln(\cos x)$

c $\frac{2}{37}e^{3x}\left(\sin\left(\frac{x}{2}\right)+6\cos\left(\frac{x}{2}\right)\right)$

14 a $\frac{1}{9}(8\ln 8-7)$

b $\frac{1}{2}(\ln 2)^2$

c $\frac{1}{4}(1-3e^{-2})$

15 a 0.7854 **b** 1.4627 **c** 1.0000 **d** 1.1184
e 0.5032 **f** 0.2527 **g** 1.6858 **h** 0.8491

16 89.85

17 0.71

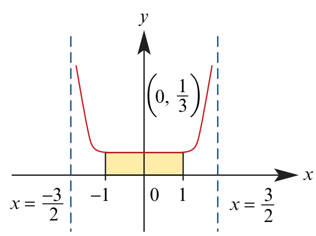
Multiple-choice questions

1 D	**2** A	**3** A	**4** D	**5** A
6 C	**7** C	**8** D	**9** A	**10** D
11 C	**12** A	**13** D	**14** B	**15** D
16 C	**17** B	**18** D	**19** A	**20** C
21 B	**22** C			

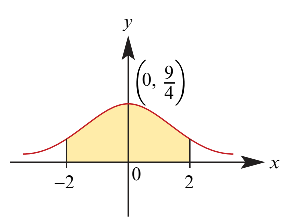
Chapter 12

Exercise 12A

1 Area $=\sin^{-1}\left(\frac{2}{3}\right)$ square units

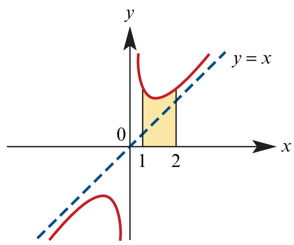


2 Area $=\frac{9\pi}{4}$ square units

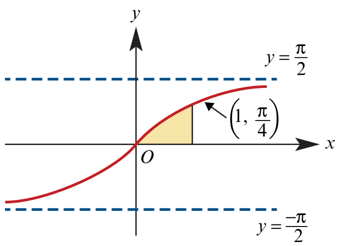


3 Area $=2\frac{2}{3}$ square units

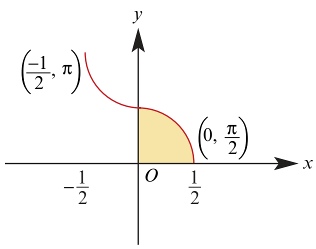
4 Area $= \frac{3}{2} + 2\ln 2$ square units



5 a Area $= \frac{\pi}{4} - \ln\sqrt{2}$ square units

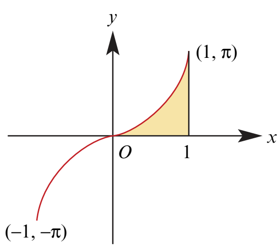


b Area $= \frac{1}{2}$ square units

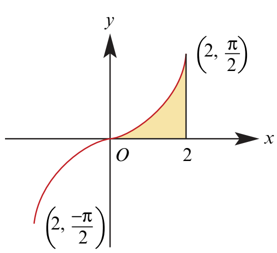


c Area $= \frac{\pi}{2}$ square units

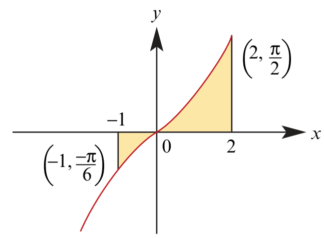
d Area $= \pi - 2$ square units



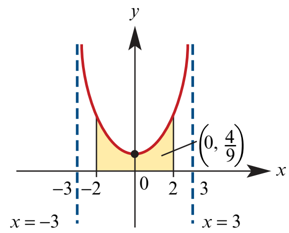
e Area $= \pi - 2$ square units



f Integral $= \frac{5\pi}{6} - \sqrt{3}$



6 Area $= \frac{4}{3}\ln 5$ square units



7 a $(0, 1)$

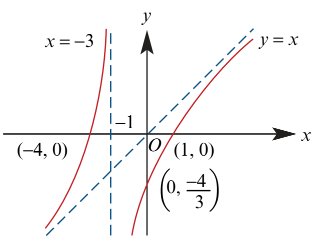
b $y = -1$

c $\pi - 2$ square units

8 a $\left(0, -\frac{4}{3}\right)$, $(-4, 0)$, $(1, 0)$

b $y = x$, $x = -3$

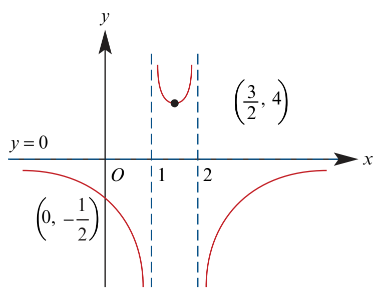
c



d Area $= 31\frac{1}{2} + 4\ln\frac{4}{11}$ square units

9 a $\mathbb{R} \setminus \{1, 2\}$

b



c $\mathbb{R}^- \cup [4, \infty)$

d Area $= -\ln\left(\frac{3}{4}\right) = \ln\left(\frac{4}{3}\right)$ square units

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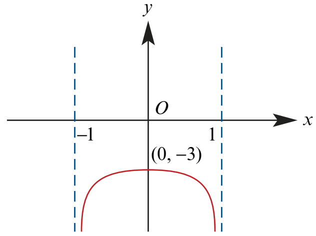
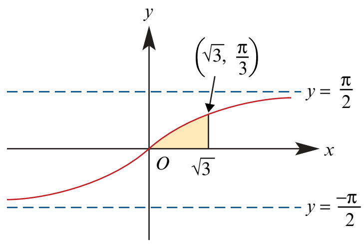
10 $\int_0^{\frac{1}{2}} \frac{-3}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}$



11 $\frac{\pi}{12}$ square units

12 Area $= \frac{\pi\sqrt{3}}{3} - \ln 2$ square units

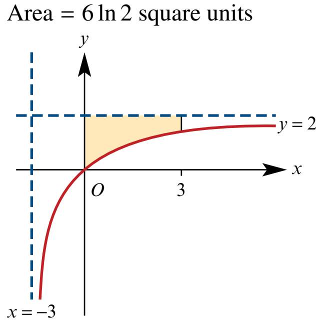


13 1 square unit

14 $\frac{2}{3}$ square units

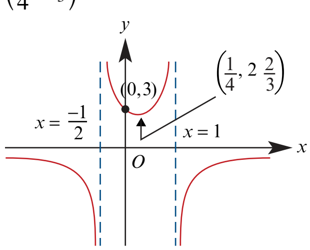
15 $\frac{1}{3}$ square units

16 Area $= 6 \ln 2$ square units



17 b $\left(\frac{1}{4}, 2\frac{2}{3}\right)$ local minimum

c



d $\frac{3}{2} - \ln 4$ square units

Exercise 12B

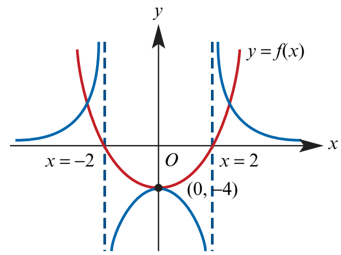
1 (3, 3), (2, 0); $\frac{1}{3}$ square units

2 $\frac{1}{3}$ square units

3 a $\frac{17}{24}$ square units

b $\frac{5}{6}$ square units

4 Area $= 8 \ln 3 - \frac{22}{3}$ square units



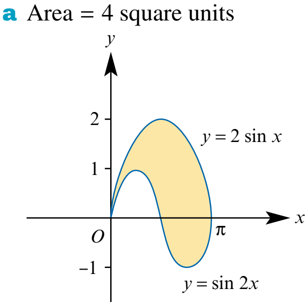
5 $a = e^2$

6 a $4\frac{1}{2}$ square units

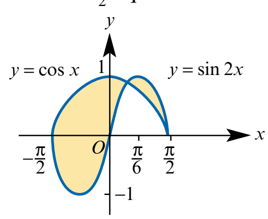
b $\frac{11}{6}$ square units

c $\frac{11}{6}$ square units

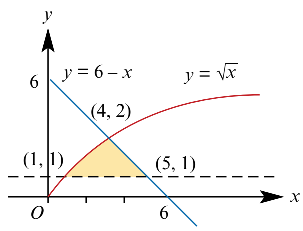
7 a Area = 4 square units



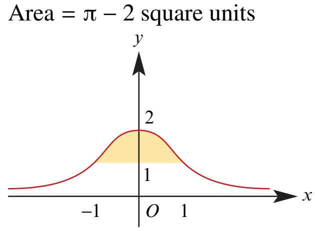
b Area $= 2\frac{1}{2}$ square units



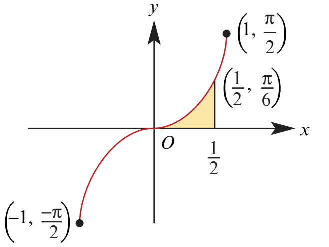
c Area $= 2\frac{1}{6}$ square units



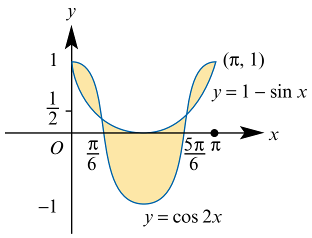
d Area $= \pi - 2$ square units



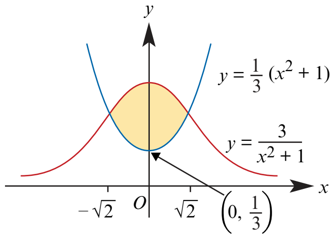
e Area $= \dfrac{\pi}{12} - 1 + \dfrac{\sqrt{3}}{2}$ square units



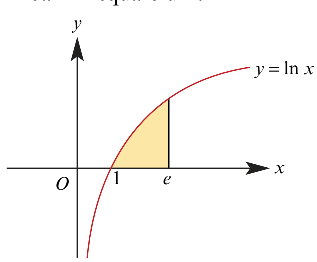
f Area $= 2 + \dfrac{\pi}{3} - \sqrt{3}$ square units



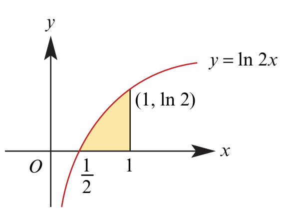
g Area ≈ 4.161 square units



8 a Area $= 1$ square unit

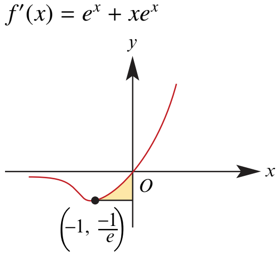


b Area $= \ln 2 - \dfrac{1}{2}$ square units



9 a $f'(x) = e^x + xe^x$ **b** $x = -1$

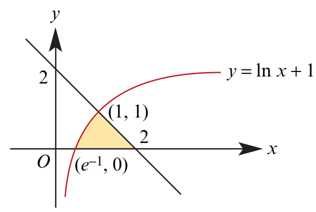
c



d $y = -\dfrac{1}{e}$ **e** Area $= \dfrac{3}{e} - 1$ square units

Note: As $f'(x) = e^x + xe^x$, $\int xe^x \, dx = xe^x - e^x + c$

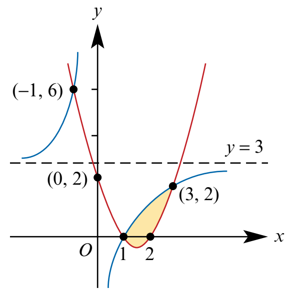
10



a $y = 2 - x$

b Area $= \dfrac{1}{2} + \dfrac{1}{e}$ square units

11



Area $= \dfrac{16}{3} - 3\ln 3$ square units

13 a $(-2\sqrt{2}, 1), (2\sqrt{2}, 1)$

b 33.36

14 $\dfrac{9}{2}$

15 3.772

16 a $a = 4, b = 2\sqrt{5}$

b 5.06

17 4

Exercise 12C

1 a 4.24 **b** 3.14 **c** 1.03

d 0.67 **e** 1.95 **f** 0.66

g 0.64 **h** 0.88 **i** 1.09

j 0.83

2 a 1.359

b 1.419

3 a $\ln x$

b $-\ln x$

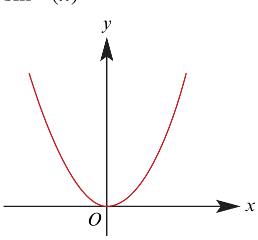
c $e^x - 1$

d $1 - \cos x$

e $\tan^{-1}(x) + \dfrac{\pi}{4}$

f $\sin^{-1}(x)$

4 a



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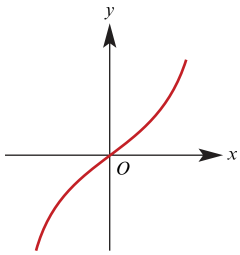
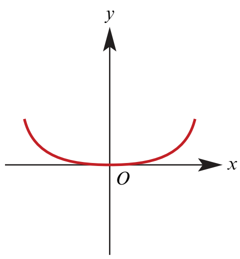
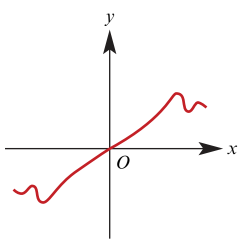
b



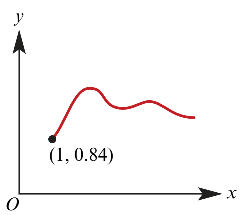
c



d



e



Exercise 12D

1 Area $= \dfrac{32}{3}$ square units;
Volume $= 8\pi$ cubic units

2 a 8π cubic units **b** $\dfrac{364\pi}{3}$ cubic units

c $\dfrac{343\pi}{6}$ cubic units **d** $\dfrac{\pi^2}{4}$ cubic units

e $\dfrac{\pi}{2}(e^4 - 1)$ cubic units

f 36π cubic units

3 $\dfrac{2\pi}{3}$ cubic units

4 a $\dfrac{3\pi}{4}$ cubic units **b** $\dfrac{28\pi}{15}$ cubic units

c 2π cubic units **d** $\dfrac{4\pi a^3}{3}$ cubic units

e 36π cubic units **f** 18π cubic units

5 $\dfrac{1088\pi}{15}$ cubic units

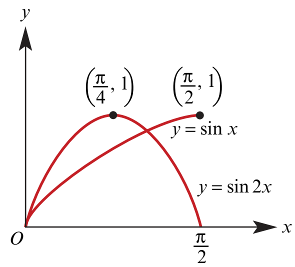
6 $\dfrac{\pi}{2}$ cubic units

7 $\dfrac{21\pi}{4}$ cubic units

8 $\dfrac{3\pi}{10}$ cubic units

9 $\dfrac{32\pi}{3}$ cubic units

11



12 $b = \dfrac{4}{13}$

13 $\dfrac{7\pi}{6}$ cubic units

14 a $\dfrac{16\pi}{3}$ **b** $\pi\left(\dfrac{e^4}{2} - 4e^2 + \dfrac{23}{2}\right)$

15 a $\dfrac{e}{2} - 1$ **b** $\dfrac{\pi}{6}(e^2 - 3)$

16 $\dfrac{16\pi}{15}$ cubic units

17 $\dfrac{\pi^2}{2}$ cubic units

18 $\dfrac{7\pi}{10}$ cubic units

19 $\dfrac{19\pi}{6}$ cubic units

20 $\pi\left(\ln 2 - \dfrac{1}{2}\right)$ cubic units

22 $2\pi(4 - \pi)$ cubic units

24 a $\dfrac{\pi}{3}\tan^{-1}\left(\dfrac{4}{3}\right)$ **b** 4π

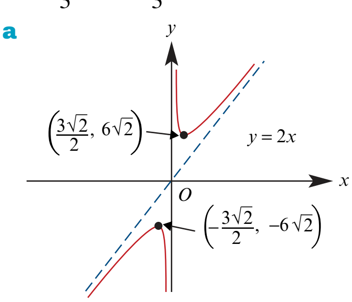
25 176 779 cm^3

26 a $\dfrac{4\pi ab^2}{3}$ **b** $\dfrac{4\pi a^2 b}{3}$

27 a $x + y = 8$

b i $\dfrac{64\pi}{3}$ **ii** $\dfrac{64\pi}{3}$

28 a



b $\dfrac{482\pi}{3}$ cubic units

29 2.642 cubic units

30 $4\pi\left(\dfrac{4\pi}{3} - \sqrt{3}\right)$ cubic units

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Exercise 12E

1 a 0.6065 **b** 0.6321

2 a 0.6514 **b** 0.2325

3 a $E(X) = 10$, $sd(X) = 10$ **b** 0.3679

4 a $f(t) = \frac{1}{6}e^{-\frac{t}{6}}$ for $t \geq 0$ **b** 0.3935
c 0.4346

5 a 0.7769 **b** 0.3679 **c** 0.1447

6 0.5507

7 a 0.2636 **b** 0.5134

8 a 5 minutes **b** 5 minutes **c** 0.3935

9 a 0.0183 **b** $Var(X) = 0.25$
c 0.0159

10 a $F(x) = 1 - e^{-\frac{x}{4}}$ for $x \geq 0$ **b** $m = 2.773$

11 0.4866

12 a 0.5488 **b** $E(X) = 500$ hours
c $m = 346.57$ hours

13 $m = 2.77$ minutes

14 a 0.2231 **b** 0.6065

Exercise 12F

1 a 12
b 10

2 0.842

3 a 10
b 19.667
c 29.667

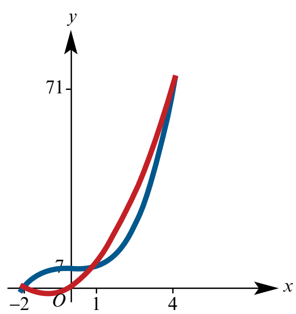
4 a 0.2083
b 0.2005

5 a 0.69444 **b** 1.50201 **c** 0.45751

6 a 2.05979 **b** −0.35229 **c** 0.76355

7 a 0.8556 **b** 0.3413 **c** 0.6827

9 a $a = 3$, $b = 6$, $c = -1$
b 102, 102
c



10 b 0.78539 **c** 3.14157

11 8.5 m

12 a $\pi \int_0^6 \left(e^{\frac{x}{12}} + 4\right)^2 dx$ **b** 529.631
c 529.647 **d** 0.003%

13 a 0.5236
b 0.5236

Chapter 12 review

Short-response questions

1 $\frac{1}{3}$

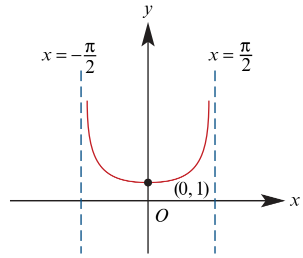
2 a $\frac{\pi}{2} - 1$ **b** 1

3 a π
b $\frac{\pi}{8}(\pi - 2)$
c $\frac{\pi}{8}(\pi + 2)$
d $\frac{2048\pi}{15}$
e 40π

4 $\frac{119\pi}{6}$

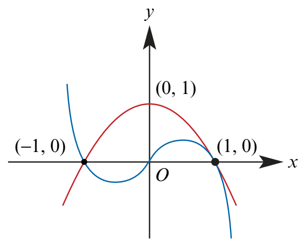
5 a 12π
b $\frac{20\sqrt{10}\pi}{3} - \frac{2\pi}{3}$

6 Volume $= 2\pi$



7 a (0, 0), (2, 4)
b $\frac{16\pi}{3}$

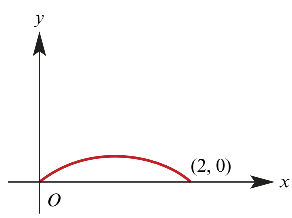
8 a



b $\frac{4}{3}$

9 a $A = (-1, 1)$, $B = (1, 1)$, $C = (0, \sqrt{2})$
b $\frac{44\pi}{15}$

10 a



b $\frac{4}{3}$ **c** $\frac{16\pi}{15}$

11 a i $\frac{\pi b^5}{5}$ **ii** $\frac{\pi b^4}{2}$
b $b = 2.5$ or $b = 0$

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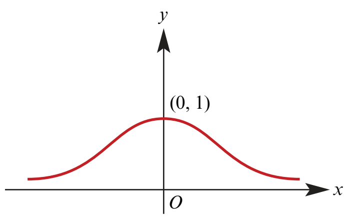
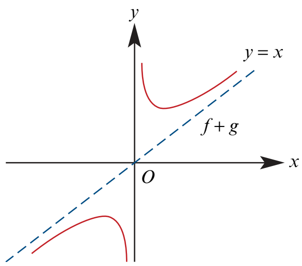
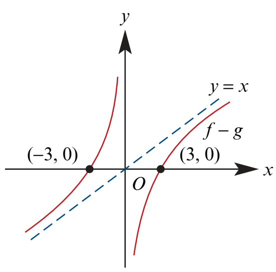
12 a



b $\dfrac{dy}{dx} = \dfrac{-8x}{(4x^2+1)^2}$, $x + y = 1$ **c** $\dfrac{\pi - 3}{8}$

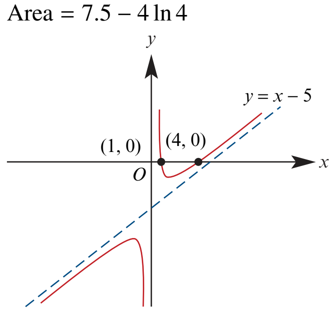
13 a



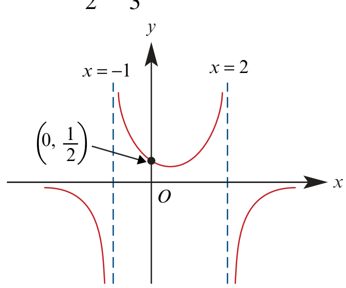


b $18 \ln 3$

14 Area $= 7.5 - 4\ln 4$



15 Area $= \dfrac{1}{2} - \dfrac{1}{3}\ln 4$



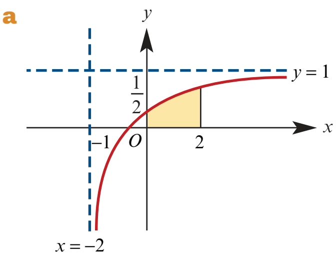
16 e^{-2}

17 a $F(x) = 1 - e^{-\frac{x}{8}}$ for $x \ge 0$
b $m = 8\ln 2$

18 a 13 **b** $\dfrac{31}{3}$

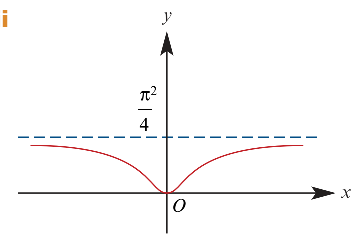
19 $\dfrac{113}{3}$

20 a



b $2 - \ln 2$ square units
c $2\pi\left(\dfrac{9}{8} - \ln 2\right)$ cubic units

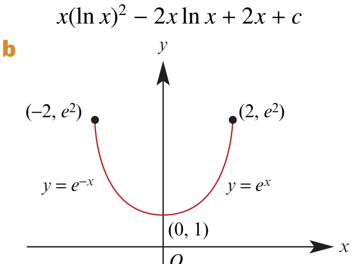
21 a $f'(x) = \dfrac{x}{1+x^2} + \tan^{-1} x$
b $\dfrac{\pi}{4} - \dfrac{1}{2}\ln 2$
c $\dfrac{1}{2}\ln 2$ square units
d i $g'(x) = \dfrac{2\tan^{-1} x}{1+x^2}$
iii



e $\pi\left(\dfrac{\pi}{2} - 1\right)$ cubic units

22 a i $\ln x + 1$; $\quad x\ln x - x + c$
ii $(\ln x)^2 + 2\ln x$;
$x(\ln x)^2 - 2x\ln x + 2x + c$

b



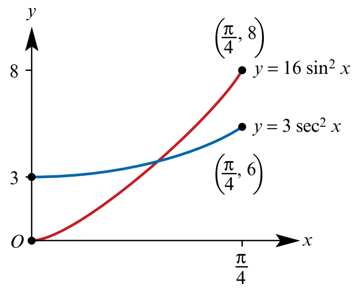
c $V = 2\pi(e^2 - 1)\text{ cm}^3 \approx 40\text{ cm}^3$

23 a 0.1353 **b** 0.0900 **c** 0.9502
d $L = \ln(\frac{5}{4}) \approx 0.2231$ years **e** 0.6723

24 a $\dfrac{\pi}{2}$ cubic units
b i $\dfrac{\pi}{8}$ cubic units
ii $\dfrac{\sqrt{2}}{2}$ units

25 b i $a = 1$
ii $\dfrac{2\sqrt{2}}{3}$
c $\dfrac{\pi a}{2(a^2+1)}$ cubic units

26 a



b $\left(\frac{\pi}{6}, 4\right)$ **c** $3\sqrt{3} - \frac{4\pi}{3}$

28 a Area $= \pi(r^2 - y^2)$

29 a $\frac{4\pi ab^2}{3}$ **b** $4\sqrt{3}\pi a^2 b$

30 b $\frac{\pi}{6} - \frac{3}{16}$

c $\frac{\pi}{2}\left(\frac{-3}{16} + \ln 3\right) = \pi\left(\frac{-3}{32} + \ln(\sqrt{3})\right)$

31 a $\frac{\pi}{3}$ **b** $k = \frac{\sqrt{3}}{3}$; $\frac{2\pi\sqrt{3}}{27}$ cubic units

32 a i
$$d = 0$$
$$125a + 25b + 5c = 1$$
$$1000a + 100b + 10c = 2.5$$
$$27\,000a + 900b + 30c = 10$$

ii $a = \frac{-7}{30\,000}, b = \frac{27}{2000}, c = \frac{83}{600}, d = 0$

b $\frac{273}{2}$

c i $V = \frac{\pi}{900\,000\,000} \times \int_0^{30} (-7x^3 + 405x^2 + 4150x)^2\, dx$

ii $\frac{362\,083\pi}{400} \approx 2843.79$

d i $w = 16.729\,335$

ii $\frac{197\,881\,099\pi}{250\,000} \approx 2487$

e $\left(\frac{135}{7}, \frac{1179}{196}\right)$

33 a $\frac{\pi H}{3}(a^2 + ab + b^2)$ cm^3

b $\frac{\pi H}{24}(7a^2 + 4ab + b^2)$ cm^3

c $V = \frac{\pi H(r^3 - a^3)}{3(b - a)}$ **d** $\frac{dV}{dr} = \frac{\pi H r^2}{b - a}$

e $h = \frac{H(r - a)}{b - a}$

34 a $k + \ell + m = 2w$ **b** $-k + m = 0$

c $k + m = \frac{2w}{3}$

Multiple-choice questions

1 B	**2** C	**3** D	**4** B	**5** A
6 D	**7** C	**8** C	**9** D	**10** B
11 D	**12** C	**13** D	**14** C	**15** B
16 A	**17** D	**18** B	**19** D	**20** D

Chapter 13

Exercise 13A

1 a x **b** $-\frac{2y}{x}$ **c** $\frac{-x^2}{y^2}$ **d** $\frac{2x}{3y^2}$

e $2\sqrt{y}$ **f** $\frac{2 - y}{x + 3}$ **g** $\frac{2a}{y}$ **h** $\frac{2}{1 - y}$

2 a $\frac{x + 2}{y}$ **b** $\frac{-y^2}{x^2}$

c $\frac{2(x + y)}{1 - 2(x + y)}$ **d** $\frac{y - 2x}{2y - x}$

e $\frac{2xe^y}{1 - x^2e^y}$ **f** $\frac{-\sin(2x)}{\cos y}$

g $\frac{\cos x - \cos(x - y)}{\cos y - \cos(x - y)}$ **h** $\frac{\sin y}{5y^4 - x\cos y + 6y}$

3 a $x + y = -2$ **b** $5x - 12y = 9$

c $16x - 15y = 8$ **d** $y = -3$

4 $\frac{dy}{dx} = \frac{y}{x}$ **5** $\frac{-1}{4}$

6 -1 **7** $\frac{-2}{5}$

8 $\frac{-7}{5}$

9 a $\frac{dy}{dx} = \frac{-x^2}{y^2}$ **c** $\frac{-1}{9}$

10 $y = -1, y = 1$

11 a $\frac{dy}{dx} = \frac{-(3x^2 + y)}{x + 6y^2}$

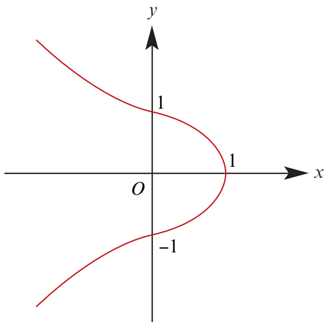
d $k = -220$ or $k = -212$

12 a $\frac{dy}{dx} = \frac{y - x}{2y - x}$ **b** $(-2, -2), (2, 2)$

13 a $\frac{dy}{dx} = \frac{-3x^2}{2y}$ **b** $(0, -1), (0, 1)$

d $y = \pm\sqrt{1 - x^3}$ **e** $(0, -1), (0, 1)$

f



Exercise 13B

1 a $y = 4e^{2t} - 2$ **b** $y = x\ln|x| - x + 4$

c $y = \sqrt{2x + 79}$ **d** $y - \ln|y + 1| = x - 3$

e $y = \frac{1}{2}x^4 - \frac{1}{2}x + 2$ **f** $y = \frac{11}{5}e^{2x} + \frac{4}{5}e^{-2x}$

g $x = 3\sin(3t) + 2\cos(3t) + 2$

3 $-2, 5$

4 $a = 0, b = -1, c = 1$

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5 $a = 0, b = \frac{1}{2}$

6 $a = 1, b = -6, c = 18, d = -24$

Exercise 13C

1 a $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c$

b $y = \frac{1}{2}x^2 + 3x - \ln|x| + c$

c $y = 2x^4 + 4x^3 + 3x^2 + x + c$

d $y = 2\sqrt{x} + c$

e $y = \frac{1}{2}\ln|2t - 1| + c$

f $y = -\frac{1}{3}\cos(3t - 2) + c$

g $y = -\frac{1}{2}\ln|\cos(2t)| + c$ **h** $x = -\frac{1}{3}e^{-3y} + c$

i $x = \sin^{-1}\left(\frac{y}{2}\right) + c$ **j** $x = \frac{1}{y - 1} + c$

2 a $y = \frac{1}{4}x^5 + cx + d$

b $y = \frac{4}{15}(1 - x)^{\frac{5}{2}} + cx + d$

c $y = -\frac{1}{4}\sin\left(2x + \frac{\pi}{4}\right) + cx + d$

d $y = 4e^{\frac{x}{2}} + cx + d$

e $y = -\ln|\cos x| + cx + d$

f $y = -\ln|x + 1| + cx + d$

3 a $y = \frac{x - 1}{x}$ **b** $y = 1 - e^{-x}$

c $y = \frac{1}{2}x^2 - 4\ln x + 1$ **d** $y = \frac{1}{2}\ln|x^2 - 4|$

e $y = \frac{1}{3}(x^2 - 4)^{\frac{3}{2}} - \frac{95\sqrt{3}}{12}$

f $y = \sin^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{6}$ **g** $y = \frac{1}{4}\ln\left|\frac{2 + x}{2 - x}\right| + 2$

h $y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{4}$

i $y = \frac{2}{5}(4 - x)^{\frac{5}{2}} - \frac{8}{3}(4 - x)^{\frac{3}{2}} + 8$

j $y = \ln\left(\frac{e^x + 1}{2}\right)$

4 a $y = e^{-x} - e^x + 2x$ **b** $y = x^2 - 2x^3$

c $y = x^2 + \frac{1}{4}\sin(2x) - 1$

d $y = \frac{1}{2}x^2 - 2x + \ln|x| + 3$

e $y = x - \tan^{-1}x + \frac{\pi}{4}$ **f** $y = 8x^3 + 12x^2 + 6x$

g $y = \sin^{-1}\left(\frac{x}{2}\right)$

5 a $y = \frac{3}{2}x^2 + 4x + c$

b $y = -\frac{1}{3}x^3 + cx + d$

c $y = \ln|x - 3| + c$

6 a $y = 2x + e^{-x}$

b $y = \frac{1}{2}x^2 - \frac{1}{2}\cos(2x) + \frac{9}{2}$

c $y = 2 - \ln|2 - x|$

Exercise 13D

1 a $y = Ae^{\frac{x^2}{2}}$ **b** $y^2 = x^2 + c$

c $y = Ae^{\frac{x^3}{12}}$ **d** $y^2 = 2\ln|x| + c$

2 a $y = \frac{1}{3}(Ae^{3x} + 5)$ **b** $y = \frac{1}{2}(Ae^{-2x} + 1)$

c $y = \frac{1}{2} - \frac{1}{2}\ln|2c - 2x|$

d $y = \tan^{-1}(x - c)$ **e** $y = \cos^{-1}(e^{c-x})$

f $y = \frac{1 - Ae^{2x}}{1 + Ae^{2x}}$ **g** $y = \tan(x - c)$

h $x = \frac{5}{3}y^3 + y^2 + c$ **i** $y = \frac{1}{4}(x - c)^2$

3 a $y = e^{x+1}$ **b** $y = e^{x-4} - 1$

c $y = e^{2x-2}$ **d** $y = -\frac{1}{2}(e^{2x} + 1)$

e $x = y - e^{-y} + 1$

f $y = 3\cos x, -\pi < x < 0$

g $y = \frac{3(e^{6x-7} - 1)}{e^{6x-7} + 1}$

h $y = \frac{1}{3}\tan(3x), -\frac{\pi}{6} < x < \frac{\pi}{6}$

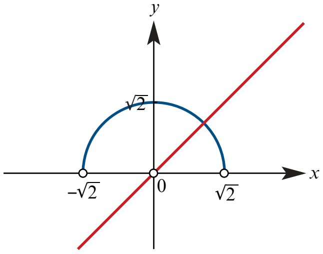
i $y = \frac{4}{e^{-x} - 2}$

4 a $y = (3(x - c))^{\frac{1}{3}}$ **b** $y = \frac{1}{2}(Ae^{2x} + 1)$

5 a $y^2 + x^2 = 2, y > 0$ or $y = \sqrt{2 - x^2}$

b $y = x$

c



6 $y = \frac{1}{2}(x^2 + 1)^2$

7 $y^2 - x^2 = 5, y \geq \sqrt{5}$

8 Circles centre $(-1, 3)$

9 $y^3 = c - \frac{3}{2x^2}$

10 $y = \frac{-2x^2}{2Ax^2 - 2x + 3}$

11 a $y = Ae^{e^x + x}$

b $y = Ae^{3x^3}$

c $y^2 = -\frac{2}{\ln(x) + c}$

d $y^2 = \frac{2}{d - \ln|x|}$

e $\ln|y| = \frac{1}{2}e^{x^2} + c$ or $y = 0$

f $y = \dfrac{3}{2(1-x^2)^{\frac{3}{2}} + d}$

12 a $y = \sqrt{\dfrac{2x^3}{3} + 2x + 1}$

b $\tan y = 2 - \dfrac{1}{x}$

13 $\dfrac{y^3}{3} - \dfrac{y^2}{2} = \dfrac{x^3}{3} - \dfrac{x^2}{2} + c$

14 b $x = A(t-25)^2$

c $\dfrac{9}{25}$

15 b $\dfrac{13}{25}e^{\frac{72}{5}}N_0$

16 a $y = 0$ or $y = -\dfrac{2}{x^2 + c}$

b $y = Ae^{-\cos x} + 1$

c $y = 1 - \dfrac{1}{x^2 + c}$ or $y = 1$

17 $y = 2xe^{\frac{x^2}{2}}$

18 $y = \dfrac{-3}{\sin^3 x - 1} - 1$

Exercise 13E

1 a $\dfrac{dx}{dt} = 2t + 1, \quad x = t^2 + t + 3$

b $\dfrac{dx}{dt} = 3t - 1, \quad x = \dfrac{3}{2}t^2 - t + \dfrac{1}{2}$

c $\dfrac{dx}{dt} = -2t + 8, \quad x = -t^2 + 8t - 15$

2 a $\dfrac{dy}{dx} = \dfrac{1}{y}, \ y \neq 0$ **b** $\dfrac{dy}{dx} = \dfrac{1}{y^2}, \ y \neq 0$

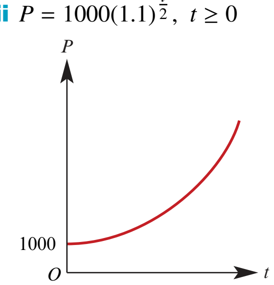
c $\dfrac{dN}{dt} = \dfrac{k}{N^2}, \ N \neq 0, k > 0$

d $\dfrac{dx}{dt} = \dfrac{k}{x}, \ x \neq 0, k \neq 0$

e $\dfrac{dm}{dt} = km, \ k < 0$ **f** $\dfrac{dy}{dx} = \dfrac{-x}{3y}, \ y \neq 0$

3 a i $\dfrac{dP}{dt} = kP$ **ii** $t = \dfrac{1}{k}\ln P + c, \ P > 0$

b i 1269 **ii** $P = 1000(1.1)^{\frac{t}{2}}, \ t \geq 0$

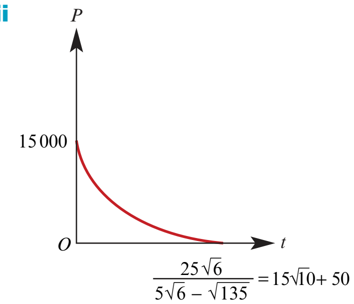


4 a i $\dfrac{dP}{dt} = k\sqrt{P}, \ k < 0, P > 0$

ii $t = \dfrac{2\sqrt{P}}{k} + c, \ k < 0$

b i 12 079

ii

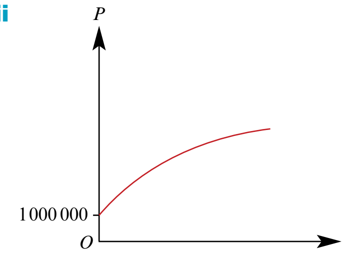


5 a i $\dfrac{dP}{dt} = \dfrac{k}{P}, \ k > 0, P > 0$

ii $t = \dfrac{1}{2k}P^2 + c$

b i $P = 50\,000\sqrt{21t + 400}, \ t \geq 0$

ii



6 $y = 10e^{\frac{x}{10}}$

7 $\dfrac{140}{3}$ °C

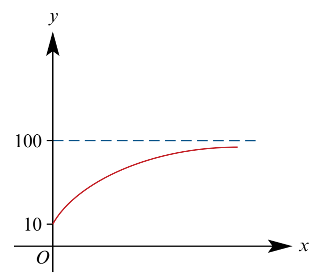
8 $\theta = 331.55$ K

9 23.22

11 a $x = \dfrac{1}{3}\left(20 - 14e^{-\frac{1}{10}t}\right)$

b 19 minutes

12 $y = 100 - 90e^{\frac{-x}{10}}$



13 13 500

14 a 14 400

b 13 711

c 14 182

15 a $\dfrac{dV}{dt} = 0.3 - 0.2\sqrt{V}, \ V > 0$

b $\dfrac{dm}{dt} = 50 - \dfrac{6m}{100 - t}, \ 0 \leq t < 100$

c $\dfrac{dx}{dt} = \dfrac{-5x}{200 + t}, \ t \geq 0$

16 a $\dfrac{dm}{dt} = \dfrac{1}{4}(1 - 4m)$ **b** $\dfrac{1}{4}(1 - e^{-t})$

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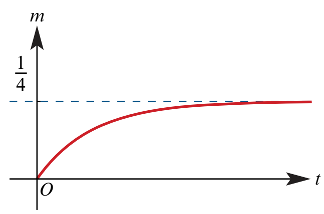
c

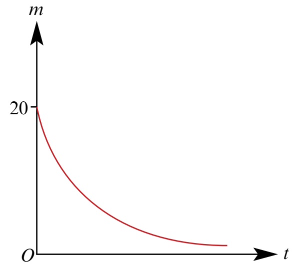


d $\frac{1}{4}(1 - e^{-2})$ kg

17 a $\frac{m}{100}$ kg/min **b** $\frac{dm}{dt} = \frac{-m}{100}$

c $m = 20e^{\frac{-t}{100}},\ t \geq 0$

d



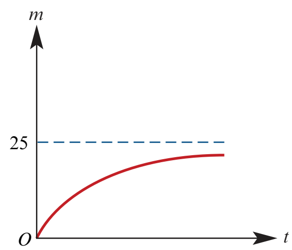
18 a 0.25 kg/min **b** $\frac{m}{100}$ kg/min

c $\frac{dm}{dt} = 0.25 - \frac{m}{100}$

d $m = 25\left(1 - e^{\frac{-t}{100}}\right),\ t \geq 0$

e 51 minutes

f



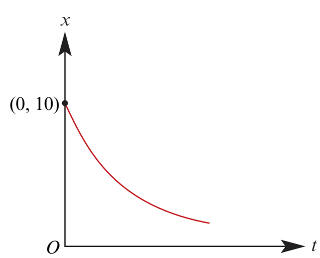
19 a $\frac{dx}{dt} = \frac{10 - x}{50}$ **b** 11.16 minutes

20 a $\frac{dx}{dt} = \frac{80 - x}{200},\quad x = 80 - 70e^{\frac{-t}{200}}$

b $\frac{dx}{dt} = 0.4 - \frac{x}{400 + t}$

21 a $\frac{dx}{dt} = -\frac{x}{10}$ **b** $x = 10e^{\frac{-t}{10}}$

c



d $10 \ln 2 \approx 6.93$ mins

22 a $N = 50\,000\left(99e^{\frac{t}{10}} + 1\right),\ t \geq 0$

b At the end of 2016

23 a **i** \$10 609.00 **ii** \$10 613.64

iii \$10 615.99 **iv** \$10 617.57

v \$10 618.34

b \$10 618.37

c Compounding interest more often gives a higher return. Result for daily compounded interest is very close to that for continuously compounded interest.

24 a $A = A_0 e^{\frac{rt}{100}},\ t \geq 0$ **b** 2.23% p.a.

c \$70 469

25 a 3.25% p.a. **b** \$90 709.20

c 3.41% p.a.

26 b $A = 2500(5 - e^{0.08t})$ **c** After 20.12 years

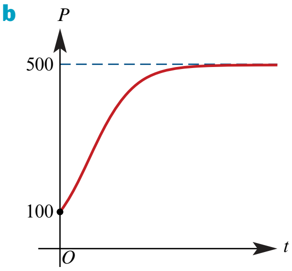
d \$20 118

Exercise 13F

1 $P = \frac{2e^t}{2e^t - 1}$

2 a $P = \frac{500e^{0.02t}}{4 + e^{0.02t}}$

b

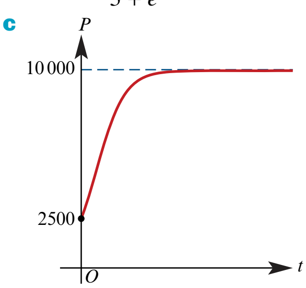


c 250

3 a $P'(t) = 0.3P\left(1 - \frac{P}{10\,000}\right)$

b $P(t) = \frac{10\,000e^{0.3t}}{3 + e^{0.3t}}$

c



d 5990

e $\frac{10}{3} \ln 3 \approx 3.66$ years

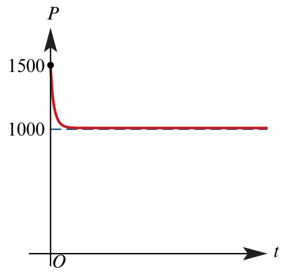
4 12.5 wasps per month

5 $P = \frac{3000e^{0.05t}}{7 + 3e^{0.05t}}$

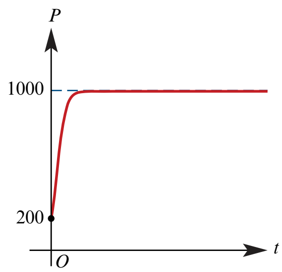
6 a 5 **b** 400 **c** $t = \frac{5}{4}\ln(79)$

d 80 cases per week **e** 60 cases per week

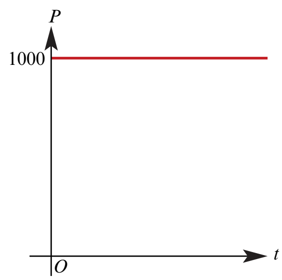
7 a $P = \dfrac{3000e^{0.1t}}{3e^{0.1t} - 1}$



b $P = \dfrac{1000e^{0.1t}}{e^{0.1t} + 4}$



c $P = 1000$

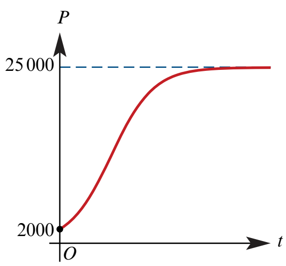


8 a $P = \dfrac{50\,000e^{0.1t}}{23 + 2e^{0.1t}}$

b i 3419 **ii** 24 307

c 24 months **d** 38 months

e



9 a $y = \dfrac{30 - 10e^{-0.1x}}{3 - 2e^{-0.1x}}, \ x \geq 0$

b $y = \dfrac{30 + 10e^{-0.1x}}{3 + 2e^{-0.1x}}, \ x \geq 0$

c $y = \dfrac{20 - 35e^{-0.1x}}{2 - 7e^{-0.1x}}, \ 0 \leq x \leq 10\ln\left(\dfrac{7}{2}\right)$

Exercise 13G

1 a $\dfrac{dr}{dt} \approx 0.00127$ m/min

b $\dfrac{dA}{dt} = 0.08$ m^2/min

2 $\dfrac{dx}{dt} \approx 0.56$ cm/s

3 $\dfrac{dy}{dt} = 39$ units/s

4 $\dfrac{dx}{dt} = \dfrac{3}{20\pi} \approx 0.048$ cm/s

5 $\dfrac{dv}{dt} = -\dfrac{5}{6}$ units/min

6 $\dfrac{dA}{dt} = 0.08\pi \approx 0.25$ cm^2/h

7 $\dfrac{dc}{dt} = \dfrac{1}{2}$ cm/s

8 a $\dfrac{dy}{dt} = \dfrac{1 - t^2}{(1 + t^2)^2}, \quad \dfrac{dx}{dt} = \dfrac{-2t}{(1 + t^2)^2}$

b $\dfrac{dy}{dx} = \dfrac{t^2 - 1}{2t}$

9 $\dfrac{dy}{dx} = \dfrac{-\sin(2t)}{1 + \cos(2t)} = -\tan t$

10 $y = \dfrac{\sqrt{3}}{3}x - \dfrac{\pi\sqrt{3}}{18} + 1$

11 a $\dfrac{dy}{dt} = 12$ cm/s

b $\dfrac{dy}{dt} = \pm 16$ cm/s

12 2.4

13 a $\dfrac{-5\sqrt{6}}{2}$ cm/s **b** $-4\sqrt{3}$ cm/s

14 72π cm^3/s

15 a 4 cm **b** 2 cm/s

16 $\dfrac{7}{12\pi}$ cm/s

17 $\dfrac{dV}{dt} = A\dfrac{dh}{dt}$

18 a $\dfrac{dh}{dt} = -\dfrac{\sqrt{h}}{4\pi}$

b i $\dfrac{dV}{dt} = -\dfrac{\sqrt{10}}{2}$ m^3/h **ii** $\dfrac{dh}{dt} = -\dfrac{\sqrt{10}}{8\pi}$ m/h

19 a $y = -\dfrac{1}{2}x + \sqrt{2}$ **b** $y = \dfrac{-\cos t}{2\sin t}x + \dfrac{1}{\sin t}$

20 a $y = \dfrac{\sqrt{2}}{2}x - 1$ **b** $y = -\sqrt{2}x + 5$

c $y = \dfrac{1}{2\sin\theta}x - \dfrac{\cos\theta}{\sin\theta}$

21 a $2\operatorname{cosec} t$ **b** $y = 2\sqrt{2}x + 6\sqrt{2} - 2$

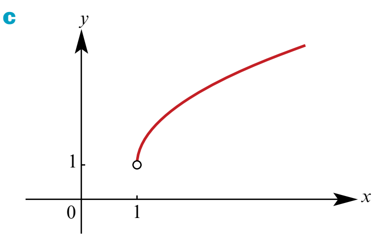
22 a $y = -\sin(t)x + 2\tan(t)$

b $\dfrac{2\sin t}{\cos^2 t}$

c $\dfrac{\pi}{3}$

23 a e^{-t} **b** $(1, \infty)$

c



d $4x - 2y = 1$

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Exercise 13H

1 a $\dfrac{dh}{dt} = \dfrac{-2000}{\pi h^2},\ h > 0$

b $\dfrac{dh}{dt} = \dfrac{1}{A}(Q - c\sqrt{h}),\ h > 0$

c $\dfrac{dh}{dt} = \dfrac{3 - 2\sqrt{V}}{60\pi},\ V > 0$

d $\dfrac{dh}{dt} = \dfrac{-4\sqrt{h}}{9\pi},\ h > 0$

2 a $\dfrac{dy}{dt} = 5\sin t$ **b** $y = -5\cos t + c$

3 a $t = -\dfrac{2\pi}{25}h^{\frac{5}{2}} + 250\pi$ **b** 13 hrs 5 mins

4 a $\dfrac{dx}{dt} = -\dfrac{1}{480\sqrt{4 - x}}$ **b** $t = 320(4 - x)^{\frac{3}{2}}$

c 42 hrs 40 mins

5 a $\dfrac{dr}{dt} = -8\pi r^2$ **b** $r = \dfrac{2}{16\pi t + 1}$

6 a $\dfrac{dh}{dt} = \dfrac{1000}{A}(Q - kh),\ h > 0$

b $t = \dfrac{A}{1000k}\ln\left(\dfrac{Q - kh_0}{Q - kh}\right),\ Q > kh_0$

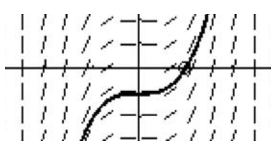
c $\dfrac{A\ln 2}{1000k}$ minutes

Exercise 13I

1 a 1.7443 **b** 1.8309 **c** 4 **d** 3.2556

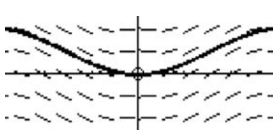
Exercise 13J

1 a



$y = x^3 - 1$

b

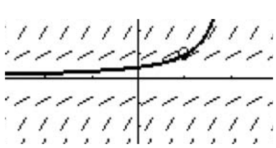


$y = 1 - \cos x$

c

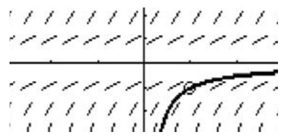
$y = \dfrac{1}{2}(3 - e^{-2x})$

d



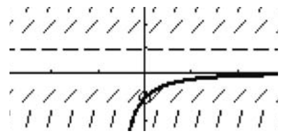
$y = \dfrac{1}{2 - x},\ x < 2$

e



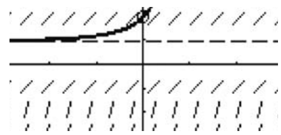
$y = -\dfrac{1}{x},\ x > 0$

f



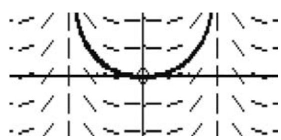
$y = \dfrac{1}{1 - 2e^x},\ x > -\ln 2$

g



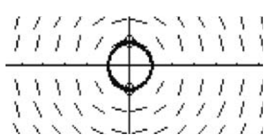
$y = \dfrac{2}{2 - e^x},\ x < \ln 2$

h

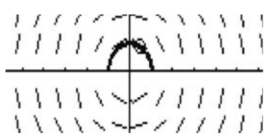


$y = -\ln(\cos x),\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

2 a



b



Chapter 13 review

Short-response questions

1 a $y = x - \dfrac{1}{x} + c$ **b** $y = e^{10x+c}$

c $y = -\dfrac{1}{2}\left(\dfrac{\sin(3t)}{9} + \dfrac{\cos(2t)}{4}\right) + at + b$

d $y = \dfrac{e^{-3x}}{9} + e^{-x} + ax + b$

e $y = 3 - e^{-\frac{x}{2}+c}$ **f** $y = \dfrac{3x}{2} - \dfrac{1}{4}x^2 + c$

2 a $y = \dfrac{1}{2}\sin(2\pi x) - 1$ **b** $y = \dfrac{1}{2}\ln|\sin(2x)|$

c $y = \ln|x| + \dfrac{1}{2}x^2 - \dfrac{1}{2}$ **d** $y = \dfrac{1}{2}\ln(1 + x^2) + 1$

e $y = e^{-\frac{x}{2}}$ **f** $x = 64 + 4t - 5t^2$

3 a $k = 2,\ m = -2$

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4 a $\frac{2}{\sqrt{3}} - 1$ **b** $\frac{8}{3}$

5 $n = -3, 5$

6 a -1

b $\frac{-(x+1)}{y+3}$

c $\frac{-2y^2}{x^2}$

d $\frac{-(x+1)}{y-3}$

7 a 324 cm/s **b** 36 cm/s

8 b $y = 2\tan(2x - \arctan(\frac{1}{2}) - 4)$

9 a $k = \frac{1}{10}\ln\left(\frac{5}{4}\right)$

b 78.67°C

10 $y = 43 - \frac{2(25 - x^2)^{\frac{3}{2}}}{3}$

11 $k = -1$

12 $\frac{dx}{dt} = \frac{3}{\pi x(12 - x)}$

13 $\frac{dC}{dt} = \frac{8\pi}{C}$

14 $100 \ln 2 \approx 69$ days

15 $\frac{dS}{dt} = -\frac{S}{25}, \quad S = 3e^{\frac{-t}{25}}$

16 a $\theta = 30 - 20e^{\frac{-t}{20}}$

b 29°C

c 14 mins

17 a $\frac{dA}{dt} = 0.02A$ **b** $0.5e^{0.2}$ ha **c** $89\frac{1}{2}$ h

18 $x = \frac{2L}{3}$; maximum deflection $= \frac{L^3}{216}$

19 $\frac{dh}{dt} = \frac{8 - 0.2\sqrt{h}}{\pi h^2}$

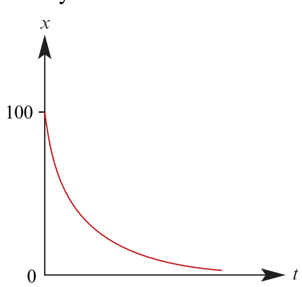
20 0.1 km/s

21 a i $\frac{dx}{dt} = -kx, \ k > 0$

ii $x = 100e^{\frac{-t\ln 2}{5760}} = 100 \cdot 2^{\frac{-t}{5760}}, \ t \geq 0$

b 6617 years

c



22 a $\frac{dx}{dt} = \frac{3k}{16}(8 - x)(4 - x)$

b $t = \frac{1}{\ln(\frac{7}{6})}\ln\left(\frac{8 - x}{8 - 2x}\right)$

c 2 min 38 sec **d** $\frac{52}{31}$ kg

23 a $\frac{dT}{dt} = k(T - T_S), \ k < 0$

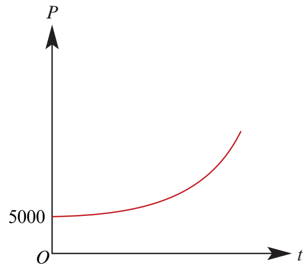
b i 19.2 mins

ii 42.2°C

24 b $t = \frac{1}{k}\ln\left(\frac{kp - 1000}{5000k - 1000}\right), \ kp > 1000$

c ii 0.22

d $p = \frac{1}{k}(e^{kt}(5000k - 1000) + 1000)$

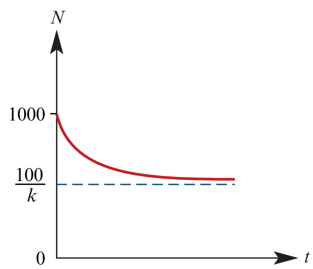


25 a $\frac{dN}{dt} = 100 - kN, \ k > 0$

b $t = \frac{1}{k}\ln\left(\frac{100 - 1000k}{100 - kN}\right)$

c 0.16

d $N = \frac{1}{k}(100 - e^{-kt}(100 - 1000k))$



e $\frac{100}{k}$

26 a $\frac{2L}{3}$

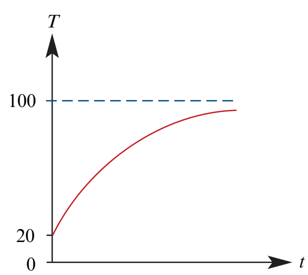
b $\frac{L}{60}$

27 a $\frac{dT}{dt} = \frac{100 - T}{40}$

b $T = 100 - 80e^{\frac{-t}{40}}$

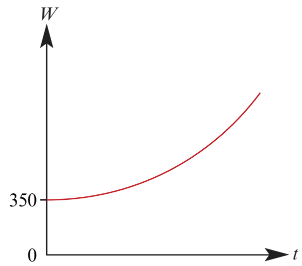
c 62.2°C

d



28 a i $t = 25\ln\left(\frac{W}{350}\right), \ W > 0$

ii $W = 350e^{\frac{t}{25}}$

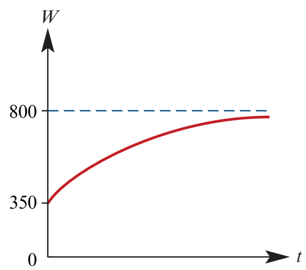


iii 2586

b 0

c **i** $t = 25\ln\left(\dfrac{9W}{7(800 - W)}\right),\ 0 < W < 800$

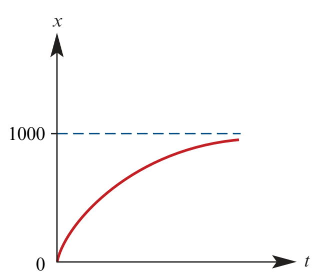
ii $W = \dfrac{5600e^{\frac{t}{25}}}{9 + 7e^{\frac{t}{25}}}$



iii 681

29 a **ii** $x = \dfrac{R}{k}(1 - e^{-kt})$

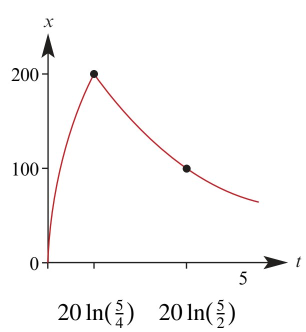
b **i**



ii 4.46 hours

c **i** 13.86 hours after drip is disconnected

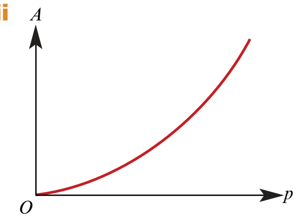
ii $x = \begin{cases} 1000\left(1 - e^{\frac{-t}{20}}\right) & 0 \le t \le 20\ln\left(\dfrac{5}{4}\right) \\ 250e^{-\frac{t}{20}} & t > 20\ln\left(\dfrac{5}{4}\right) \end{cases}$



30 a $A = \dfrac{p\sqrt{p^2 + 4}}{2} - p$

b **i** $\dfrac{dA}{dp} = \dfrac{p^2}{2\sqrt{p^2 + 4}} + \dfrac{\sqrt{p^2 + 4}}{2} - 1$

ii



iii 10.95

c **i** 0.315 sq. units/s **ii** 0.605 sq. units/s
iii 9.800 sq. units/s **iv** 15.800 sq. units/s

31 b **i** $\left(3, \dfrac{2 - 2\cos\theta}{\sin\theta}\right)$

c **i** $M = \left(\dfrac{3}{2\cos\theta}, \dfrac{1}{\sin\theta}\right)$

ii $\dfrac{9}{4x^2} + \dfrac{1}{y^2} = 1$

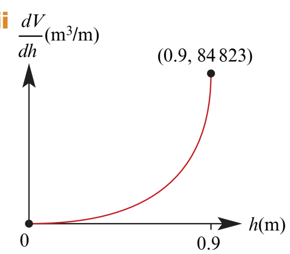
d **i** $y = \dfrac{2\sin\theta}{3\cos\theta}x + \dfrac{2}{\cos\theta}$

ii $Z = (3(\cos\theta - \sin\theta), 2(\cos\theta + \sin\theta))$

iii $(2x + 3y)^2 + (3y - 2x)^2 = 144$

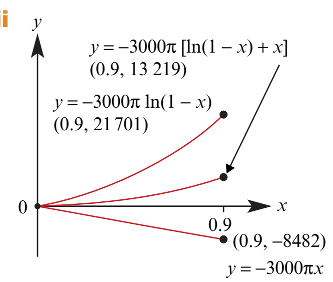
32 a **i** $\dfrac{dV}{dh} = \dfrac{3000\pi h}{1 - h}$

ii



b **i** 13 219 litres

ii



c 0.0064 m/min

33 a $\dfrac{x^2}{4} + y^2 = 1$

Multiple-choice questions

1 D	**2** B	**3** D	**4** D	**5** C
6 C	**7** A	**8** C	**9** D	**10** A
11 A	**12** D	**13** A	**14** C	**15** B
16 A	**17** D	**18** B	**19** A	**20** C

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Chapter 14

Exercise 14A

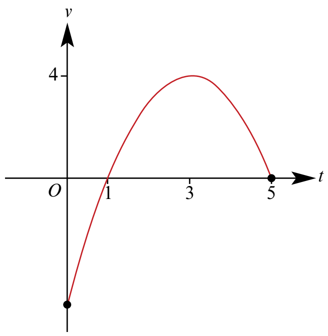
1 a -3 m **b** -2 m/s **c** $v = 2 - 2t$
d -4 m/s **e** $t = 1,\ x = 2$ **f** 10 m
g 2.5 m/s

2 a -3 m/s **b** 1, 3 **c** 12 m/s^2

3 $0, \frac{4}{3}$

4 a 42 m/s **b** 6 s **c** 198 m

5 a



b $8\frac{1}{3}$ m **c** 13 m

6 a 32 m **b** 16 m/s

7 a 60 m **b** 20 m **c** 30 m **d** 55 m
e 44 m **f** $\frac{70}{3}$ m **g** $\frac{165}{2}$ m **h** $\frac{49}{2}$ m

8 a -30 m/s^2 **b** 4, 6 **c** 4 m **d** 120 m

9 $x = \frac{1}{6}t - \frac{1}{4}\ln\left(\frac{2t+3}{3}\right)$

10 $\left(\frac{3\sqrt{3}}{2} - \frac{\pi}{3}\right)$ m

11 a 0 m/s **b** $\frac{1}{2}$ m/s **c** $\frac{1}{2}\ln 2$ m
d $x = \frac{1}{2}\ln(1 + t^2)$ **e** $\ddot{x} = \frac{1 - t^2}{(1 + t^2)^2}$
f -0.1 m/s^2 **g** $-\frac{1}{8}$ m/s^2

12 3 m/s^2

13 a 12 960 km/h^2 **b** 1 m/s^2

14 a 3 m/s^2 **b** $\frac{175}{2}$ m **c** $\frac{10(\sqrt{7} - 1)}{3}$ s

15 -5 m/s^2

16 a 12 m **b** 14 m/s **c** 2.5 s **d** 37 m

17 a i 22.4 m **ii** 22.5 m
b i 5 s **ii** -28 m/s

18 a $\frac{10}{7}$ s **b** 10 m **c** $\frac{20}{7}$ s

19 a 200 s **b** 2 km

20 a $\frac{10\sqrt{10}}{7}$ s **b** $14\sqrt{10}$ m/s

21 a 4.37 s **b** $-6\sqrt{30}$ m/s

22 -0.64 m/s^2

23 a 4 s **b** $\frac{1}{2}$ m/s^2

Exercise 14B

1 a 7 m **b** 4 m

2 a $x = \frac{1}{2}\ln(2e^{2t} - 1)$ **b** $-\frac{100}{2401}$ m/s^2

3 a $v = 3(e^t - 1)$ **b** $a = 3e^t$
c $x = 3(e^t - t - 1)$

4 a $v = \frac{g}{k}(1 - e^{-kt})$ **b** $\frac{g}{k}$

5 a $v = \tan\left(\frac{\pi}{3} - \frac{3t}{10}\right)$
b $x = \frac{10}{3}\ln\left(2\cos\left(\frac{\pi}{3} - \frac{3t}{10}\right)\right)$

6 $v = 450\left(1 - e^{\frac{-t}{50}}\right)$

7 $v = 15\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \frac{2t}{5}\right)$

8 a $x = 5e^{\frac{2t}{5}}$ **b** 273 m

9 a $t = 50\ln\left(\frac{500}{500 - v}\right)$ **b** $v = 500\left(1 - e^{\frac{-t}{50}}\right)$

10 $\frac{1}{k}\ln 2$

11 $v = 8e^{\frac{-t}{5}}$; 3.59 m/s

12 a $v = \frac{90}{2t + 3}$ **b** 91.66 m

Exercise 14C

1 -2 m/s^2

2 a $v = \pm 4$ **b** $t = -\ln 2$
c $x = 2(1 - \ln 2)$

3 a $v = \frac{1}{x + 1}$
b i $x = e^t - 1$ **ii** $a = e^t$ **iii** $a = v$

4 $x = \frac{-5}{2}\ln\left(\frac{g + 0.2v^2}{g + 2000}\right)$; $x_{\max} = \frac{5}{2}\ln\left(\frac{g + 2000}{g}\right)$

5 a $x = \cos(2t)$ **b** $a = -4x$

6 a $v = \ln(1 + t)$ **b** $v^2 = 2\ln(1 + x)$
c $v = \sqrt{2t + 1} - 1$

7 $v^2 = \frac{x}{2 + x}$

8 a 4 **b** $2\ln 2 - 1$

9 a 9.83 m **b** 1.01 s

Exercise 14D

1 a 0.5 m **b** 1 s **c** π m/s **d** $2\pi^2$ m/s^2

2 a $T = \frac{2\pi}{5}$; $A = 4$
b $x = 4\cos(5t)$, $\dot{x} = -20\sin(5t)$,
$\ddot{x} = -100\cos(5t)$

3 a $\frac{2}{3\pi}$ m/s **b** $\frac{1}{45\pi^2}$ m **c** $\frac{4}{3\pi^2}$ m/s

4 0.347 m; 5.9186 m/s

5 16 m

6 $x = 2\sin(2t + \alpha)$, where the initial position is $x = 2\sin\alpha$

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7 a 0.2 m **b** 0.5 s

8 $\frac{\pi}{5}$ m/s^2

9 96 cm/s

10 a 2 cm **b** 12 s **c** $\frac{\pi}{3}$ cm/s

d $\frac{\pi^2}{18}$ cm/s^2

11 Four times: $t = 4.18, 13.82, 40.18, 49.82$

12 a $\frac{2\pi}{3}$ s **b** 15 m/s **c** $\dot{x} = -15\sin(3t)$

d $x = 5\cos(3t)$

13 a $\dot{x} = -\frac{5\pi}{6}\sin\left(\frac{\pi t}{12}\right)$ **b** $\ddot{x} = -\frac{5\pi^2}{72}\cos\left(\frac{\pi t}{12}\right)$

c 24 s **d** $\frac{5\pi}{6}$ cm/s **e** $\frac{5\pi^2}{72}$ cm/s^2

14 a i $t = 2.5$ **ii** $t = 0,\ t = 5$

iii None **iv** $t = \frac{5}{6},\ t = \frac{25}{6}$

b $t = 0,\ t = 5$

c i $t = \frac{5}{3}$ **ii** $t = \frac{10}{3},\ t = \frac{20}{3}$

15 a $x = 5\cos(3t)$ **b** $x = \frac{10}{3}\sin(3t)$

c $x = \frac{5\sqrt{10}}{3}\sin(3t + \alpha)$,

where $\tan\alpha = 3, 0 < \alpha < \frac{\pi}{2}$

16 a $v^2 = 2\left(\frac{81}{4} - \left(x + \frac{3}{2}\right)^2\right)$; $\ddot{x} = -2\left(x + \frac{3}{2}\right)$

b i $\pi\sqrt{2}$ **ii** $\frac{9}{2}$ **iii** $\frac{9\sqrt{2}}{2}$

17 a $\dot{x} = -\frac{5\pi}{4}\sin\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$;

$\ddot{x} = -\frac{5\pi^2}{16}\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

b $\dot{x}^2 = \frac{\pi^2}{16}(25 - x^2)$; $\ddot{x} = -\frac{\pi^2}{16}x$

c i 8 s **ii** 5 cm **iii** $\frac{5\pi}{8}\sqrt{3}$ cm/s

iv $-\frac{5\pi^2}{32}$ cm/s^2

18 a $x = 3 - 2\cos(2t)$; $\ddot{x} = 8\cos(2t) = -4(x - 3)$

b i 3 **ii** 2 **iii** π

19 a $x = 1 + \sqrt{2}\sin\left(3t - \frac{\pi}{4}\right)$

b $T = \frac{2\pi}{3}$; $A = \sqrt{2}$

20 a $\ddot{x} = -32(x - 1)$

b i $x = 1$ **ii** 2 **iii** $\frac{\pi\sqrt{2}}{4}$

Exercise 14E

1 a i $5\hat{i} + 5\hat{j}$ **ii** $5\sqrt{2} \approx 7.07$ N; 45°

b i $-4\hat{i} - 4\hat{j}$ **ii** $4\sqrt{2} \approx 5.66$ N; 225°

c i $-\hat{i} - 5\hat{j}$ **ii** $\sqrt{26} \approx 5.10$ N; 258.7°

d i $3\hat{i} + 10\hat{j}$ **ii** $\sqrt{109} \approx 10.44$ N; 73.3°

e i $-4\hat{j}$ **ii** 4 N; 270°

f i $10\hat{i}$ **ii** 10 N

2 $\boldsymbol{R} = (11\hat{i} - 3\hat{j})$ N

3 25.43 N

4 $\frac{\sqrt{781} - 9}{2} \approx 9.5$ N

5 $\boldsymbol{F}_3 = -2\hat{i} + \hat{k}$

6 386 N

7 a i $6.06\hat{i} + 2.57\hat{j}$ **ii** 6.59 N; 22.98°

b i $19.41\hat{i} + 7.44\hat{j}$ **ii** 20.79 N; 20.96°

c i $1.38\hat{i} + 5.39\hat{j}$ **ii** 5.57 N; 75.63°

d i $2.19\hat{i} - 2.19\hat{j}$ **ii** 3.09 N; 315°

e i $18.13\hat{i}$ **ii** 18.13 N; 0°

f i $-2.15\hat{i} - 1.01\hat{j}$ **ii** 2.37 N; 205.28°

9 a $5\hat{j}$ **b** 5 N; 90°

10 a 11.28 N

b 6.34 N

c 0 N

d −9.01 N

11 a 17.72 N **b** 14.88 N

12 a $\frac{11}{5}(2\hat{i} - \hat{j})$

b $\frac{-6}{25}(3\hat{i} + 4\hat{j})$

13 a −1.97 N

b 5.35 N

c −0.48 N

14 −3.20 N

15 a 32.15 N **b** 33.23 N

16 a 4.55 N; 19.7°

b 12.42 N; 63.5°

17 15.46 N

18 a 6.93 N **b** 14 N

19 1.15 N

Exercise 14F

1 a 10 kg m/s **b** 0.009 kg m/s

c $8333\frac{1}{3}$ kg m/s **d** 60 kg m/s

e $41\,666\frac{2}{3}$ kg m/s

2 a $10(\hat{i} + \hat{j})$ kg m/s

b i $10(5\hat{i} + 12\hat{j})$ kg m/s **ii** 130 kg m/s

3 a −30 kg m/s **b** 40 kg m/s **c** 90 kg m/s

4 a $5g \approx 49$ N **b** $3000g \approx 29\,400$ N

c $0.06g \approx 0.588$ N

5 a 32 N **b** $\frac{1}{2}$ m/s^2

6 a 4 **b** 7

7 $\frac{96}{1.2 + g} \approx 8.73$ kg **8** 660 N

9 2.076 kg wt **10** 5.4×10^{-14} N

11 $\boldsymbol{a} = \hat{i} + 5\hat{j}$ m/s^2 **12** $\boldsymbol{a} = \hat{i} - \frac{2}{5}\hat{j}$ m/s^2

13 a 2.78 kg wt **b** 3.35 kg wt

14 $-34\,722\frac{2}{9}$ N

15 $\boldsymbol{F}_3 = 19.6\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

16 113 N

17 5 N

18 $\boldsymbol{a} = \frac{7}{2}\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ m/s^2

19 663 N

20 a 2 m/s^2 **b** 20 m/s

21 $\frac{1}{2}$ m/s

22 42.517 s

23 Pushing force = 62.5 N; Resistance = 25 N

24 60 000 N; -0.1 m/s^2

25 5 m/s

26 a $200g \approx 1960$ N **b** 2060 N

Exercise 14G

1 7.3 m/s^2; 18.4°

2 a $\sqrt{3}$ m/s^2
b 1.124 m/s^2

3 $g\cos 45° \approx 6.93$ m/s^2

4 3.9 m/s^2; 84.9 N

5 29.223 N

6 a $\frac{3\sqrt{3}}{4}$ m/s^2 **b** 181 N

7 $\boldsymbol{a} = -\frac{g}{2}\hat{\boldsymbol{i}}$

8 14.9 m, -10 m/s

9 6.76 m/s

10 $a = \frac{P}{m} - kg\cos\theta - g\sin\theta$

11 a $a = \frac{F}{M}(\cos\theta + k\sin\theta) - kg$
b $a = \frac{F}{M}(\cos\theta - k\sin\theta) - kg$

12 a 490 N **b** 1980 N

13 $8 + 4\sqrt{3} \approx 14.93$ N

14 $\sqrt{2}g$ N

15 a 15.99 N
b 1.13 m/s^2

Exercise 14H

1 $33\frac{1}{3}$ m/s; 250 m

2 a $x = 6t - 2\sin t$ **b** $\pm 4\sqrt{3}$ m/s
c $x = \frac{1}{4}(2t^2 - \cos(2t) + 1)$

3 $\frac{110}{9}$ m/s; $\left(\frac{400}{3} - \frac{50}{3}\ln 3\right)$ m

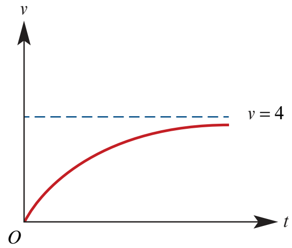
4 $x = \frac{t^2}{2} + 16\sin\left(\frac{t}{4}\right) - 4t$

5 a $\dot{x} = t - 2\sin\left(\frac{1}{2}t\right)$
b $x = \frac{t^2}{2} + 4\cos\left(\frac{1}{2}t\right) - 4$

6 10 m/s

7 $10 - \ln 11 \approx 7.6$ m/s

8 a $v = 4(1 - e^{-0.5t})$ m/s
b



c Approx 112 m

9 a 5.5 **b** $\frac{275}{6} - 10\ln 2$

10 $\frac{um}{k}\left(e^{\frac{kt}{m}} - 1\right)$ metres

11 $V - \frac{k}{m}x$

12 $\frac{b}{c}\left(1 - e^{-\frac{ct}{m}}\right)$ m/s; $\frac{b}{c}$ m/s

13 Max height = $\frac{m}{2k}\ln\left(1 + \frac{ku^2}{mg}\right)$;
speed = $u\sqrt{\frac{mg}{ku^2 + mg}}$

15 b $\frac{4375}{3}$
c $1000\ln 2 + \frac{4375}{3} \approx 2151.48$

Chapter 14 review

Short-response questions

1 a After 3.5 seconds
b 2 m/s^2
c 14.5 m
d When $t = 2.5$ s and the particle is 1.25 m to the left of O

2 $x = 215\frac{1}{3}$, $v = 73$

3 16 m

4 a 2 s
b $v = \frac{-t}{\sqrt{9 - t^2}}$, $a = \frac{-9}{(9 - t^2)^{\frac{3}{2}}}$
c 3 m
d $t = 0$

5 a i $v = 35 - 3g$ up **ii** $v = 5g - 35$ down
b $\frac{35^2}{g}$ m
c -35 m/s

6 a $80 + 0.4g$ m/s
b $\frac{80 + 0.4g}{g}$ s
c $\frac{(80 + 0.4g)^2}{2g}$ m
d $\frac{2(80 - 0.4g)}{g}$ s

7 a $v^2 = 16(9 - (x + 1)^2)$; $\ddot{x} = -16(x + 1)$
b i $\frac{\pi}{2}$ **ii** 9 **iii** 36

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8 a 885 N
b 6785 N

9 a $\dfrac{4}{(t+1)^2}$ m/s^2 **b** $\dfrac{4t}{t+1}$ m/s
c $(4t - 4\ln(t+1))$ m

10 2000 N

11 a $\dfrac{g}{4}$ m/s^2 **b** Particle lowered with $a \geq \dfrac{g}{6}$

12 4 m/s

13 a $(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$ m/s^2
b i $(t+1)(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$ m/s **ii** $\sqrt{5}(t+1)$ m/s
c $\left(\dfrac{t^2}{2} + t\right)(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$ m
d $y = 2x,\ x \geq 0$

14 $204\frac{1}{6}$ m **15** 2250 N **16** $\dfrac{10\,000}{3g}$

17 $m(g+f)$ N **18** $100\sqrt{2}$ m/s

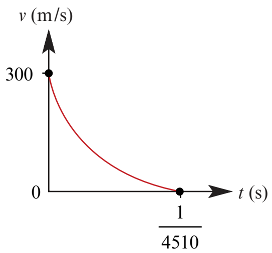
19 a 9 kg wt **b** $\dfrac{g}{9}$ m/s^2

20 a 30 minutes
b i $a = -k(\sin(\pi t) + \pi t\cos(\pi t) - 1)$
ii From 0 h to 0.18 h
c 845

21 a i $v = 4 - 10t - 3t^2$ **ii** $a = -10 - 6t$
iii 0.36 **iv** $t = 0$ or $t = 0.70$
v $t = 2.92$
b i $x = t^2 - t^3 + 2t$ **ii** $\dfrac{7}{3}$ s **iii** Yes

22 a $v = \dfrac{300(1 - 4510t)}{12\,300t + 1},\ 0 \leq t \leq \dfrac{1}{4510}$

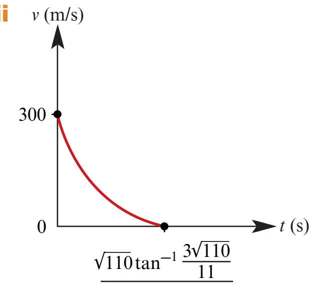
b



c i $x = -110t + \dfrac{1}{30}\ln(12\,300t + 1)$
ii $x = \dfrac{1}{30}\left(\ln\left(\dfrac{410}{v+110}\right) - \dfrac{110}{v+110} + \dfrac{11}{41}\right)$
iii 19 mm

d i $t = \dfrac{\sqrt{110}}{33\,000} \times$
$\left(\tan^{-1}\left(\dfrac{3\sqrt{110}}{11}\right) - \tan^{-1}\left(\dfrac{v\sqrt{110}}{1100}\right)\right)$
ii $v = 10\sqrt{110} \times$
$\tan\left(\tan^{-1}\left(\dfrac{3\sqrt{110}}{11}\right) - 300\sqrt{110}t\right),$
for $0 \leq t \leq \dfrac{\sqrt{110}\tan^{-1}\left(\dfrac{3\sqrt{110}}{11}\right)}{33\,000}$

iii



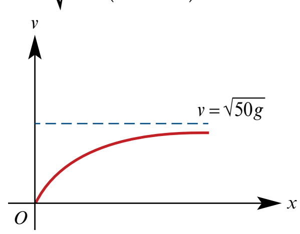
iv 20 mm

23 a 2.8 N
b 0.7 m/s^2
c i $\sqrt{\dfrac{20}{21-2g}}$ s
ii $\sqrt{5(21-2g)}$ m/s
d 0.357 metres

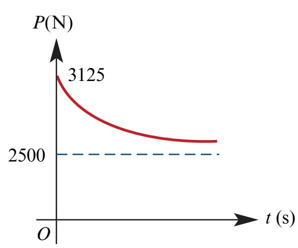
24 a i 4.85 m/s **ii** 0.49 s
b i $v = 9.8t - \dfrac{1}{2}t^2$
ii $x = 4.9t^2 - \dfrac{1}{6}t^3$
iii 0.50 s
c i $x = 1.2 - 2.45t^2$ **ii** 6 cm

25 b ii 8.96

26 a $x = 25\ln\left(\dfrac{50g}{50g - v^2}\right)$
b $v = \sqrt{50g\left(1 - e^{\frac{-x}{25}}\right)}$
c



27 a $\dfrac{25}{2}e^{-0.1t}$ m/s^2
b i $625(4 + e^{-0.1t})$ **ii** $5(625 - v)$
iii 3025 N **iv** $625(4 + e^{-3}) \approx 2531.12$ N
c



Multiple-choice questions

1 A **2** A **3** D **4** D **5** D
6 C **7** A **8** D **9** D **10** B
11 C **12** D **13** B **14** B **15** B
16 A **17** C **18** A **19** B **20** C
21 B

Chapter 15

Exercise 15A

1 a $C = 450 + 0.5X$

b

C	950	1200	1450
$P(C = c)$	0.05	0.15	0.35

C	1700	1950	2450
$P(C = c)$	0.25	0.15	0.05

c 0.05

2 a $W = 2.5X - 5$

b

W	-5	-2.5	0	2.5
$P(W = w)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

c $\frac{1}{8}$

3 a 0.027 **b** 0.125

4 a 0.3827
b 0.2929

5 a 0.5078 **b** 1

6 a $E(Y) = 77$, $Var(Y) = 81$
b $E(U) = -45$, $sd(U) = 6$
c $E(V) = -8.5$, $Var(V) = 2.25$

7 a $E(X) = 0.4$
b $Var(X) = 0.2733$
c $E(4X + 2) = 3.6$, $sd(4X + 2) = 2.0913$

8 a

S	3	4	5	6	7
$P(S = s)$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{7}{18}$	$\frac{2}{9}$	$\frac{1}{9}$

b 5 **c** $\frac{2}{3}$

9 a $E(X_1) = 3$ **b** $Var(X_1) = 2$
c $E(X_1 - X_2) = 0$ **d** $Var(X_1 - X_2) = 4$

10 a 39 **b** 16 **c** 36 **d** 16 **e** 8

11 Mean 49 mins, sd 8.5446 mins

12 Mean 4250 g, sd 13.2288 g

13 0.45

Exercise 15B

1 mean =74, sd = 4.62
2 mean = 25.025, sd = 0.0013
3 Answers will vary
4 Answers will vary
5 c Means should all be approximately equal to 15, standard deviations will be smaller for the samples of size 50 than for the samples of size 25, and both will be much smaller than the population standard deviation.
6 a 0.0478 **b** 0.0092
c Much smaller probability for the mean than for an individual student.
7 a 0.0912 **b** 0.0105
c Much smaller probability for the mean than for an individual.

8 0.0103 **9** 0.0089 **10** 0.0478
11 1.011 kg **12** 0.5206
13 a 0.5 **b** 0.0288
14 0.0008
15 a 0.7292 **b** 0.9998
16 0.3921
17 0.5041
18 0.0127
19 a 0.00001 **b** 7
20 1232

Exercise 15C

1 (6.84, 7.96) **2** (26.67, 38.67)
3 (14.51, 14.69)
4 90%: (31.53, 39.87); 95%: (30.73, 40.67); 99%: (29.17, 42.23) Increasing the level of confidence increases the width of the confidence interval.
5 90%: (80.82, 84.39); 95%: (80.47, 84.73); 99%: (79.80.85.40) Increasing the level of confidence increases the width of the confidence interval.
6 (68.38, 73.82) **7** (126.36, 133.64)
8 (45 146, 48 302) **9** (34.87, 44.13)
10 d 9 **e** 0.3487
11 d 8 **e** 0.1074
12 a 133 **b** 0.9992
13 97%
14 94%
15 a (1.0074, 1.0086)
b The sample mean does indicates that the population mean has changed after servicing, as the previous population mean ($\mu = 1.012$) is not included in the 95% confidence interval based on this sample mean.

Exercise 15D

1 1.960 **2** 0.937
3 1.496
4 a
b 0.209 kg
c 0.249 kg
d 0.328 kg
e Increasing the level of confidence increases the margin of error.
5 a i 1.550 cm
ii 1.847 cm
iii 2.428 cm
b Increasing the level of confidence increases the margin of error.
6 a i 0.015 L **ii** 0.018 L **iii** 0.024 L
b Increasing the level of confidence increases the margin of error.

7 a $E_1 = 0.653$ gm
b $E_2 = 0.392$ gm
c $E_2 = E_1 \times 0.6$

8 $n = 50, E_1 = 1.457; n = 100, E_2 = 1.030 = \dfrac{E_1}{\sqrt{2}}$

$n = 200, E_3 = 0.729 = \dfrac{E_1}{2}$

9 97 **10** 62 **11** 166
12 153

13 a 217
b 865
c Increased by a factor of 4

14 a 0.302 kg **b** 42

15 a 2.3 **b** 93.4%

16 $\left(\frac{1}{2}\left(2.314a - 0.314b\right), \left(\frac{1}{2}\left(-0.314a + 2.314b\right)\right)\right)$

Chapter 15 review

Short-response questions

1 a $\mathrm{E}(\bar{X}) = 15,\ \mathrm{Var}(\bar{X}) = \dfrac{25}{2}$

b $\mathrm{E}(\bar{X}) = 15,\ \mathrm{Var}(\bar{X}) = \dfrac{25}{3}$

c $\mathrm{E}(\bar{X}) = 15,\ \mathrm{Var}(\bar{X}) = \dfrac{25}{n}$

2 Mean 65, sd $\dfrac{7}{\sqrt{10}}$

3 a 155
b 155 ± 19.6

4 a 57
b $(0.95)^{60}$

5 $\mathrm{E}(\bar{X}) = 0,\ \mathrm{Var}(\bar{X}) = 0.0033$

6 a $\bar{x} = 1.38$ million
b $E = 0.06$ million
c The margin of error will be reduced by either reducing the level of confidence or increasing the sample size.

7 (127.67, 132.33) **8** 217
9 107 **10** 0.0151
11 $\mu = 59.44, \sigma = 1.97$

12 a 0.19 kg **b** 42

13 a 1.622 **b** 98%

14 a 0.38
b $a = 20.8, b = 99.2$
c i 0.2512 **ii** 0.2512 **iii** 0.2847
d $c = 42.47, d = 77.53$

15 $\mu = 7.37, \sigma = 1.72$

16 a i 0.8243
ii 0.9296
b i (11.45, 13.55)
ii (12.84, 14.17)
iii (12.65, 13.78)
iv 89

17 a 0.0062 **b** 0.000 088
c 0.000 032 **d** 0.0075

18 a i A: (14.51, 16.09)
ii B: (11.07, 13.13)
iii Yes, industry A seems more satisfied
b i 3.2
ii 0.6602
iii (1.91, 4.49)
iv On average, industry A workers score from 1.9 to 4.5 points higher than industry B workers

Multiple-choice questions

1 A	**2** C	**3** B	**4** C	**5** B
6 A	**7** B	**8** C	**9** B	**10** A
11 C	**12** C	**13** B	**14** D	**15** A
16 C	**17** D	**18** A		

Chapter 16

Short-response questions

1 a $-\dfrac{1}{2}(2x+1)e^{-2x}$

b $\dfrac{1}{9}\sin(3x) - \dfrac{1}{3}x\cos(3x)$

c $x\arccos(2x) - \dfrac{1}{2}\sqrt{1-4x^2}$

d $\dfrac{1}{4}x^4(4\ln(3x) - 1)$

2 a $\dfrac{1}{2}(\sin(e^2) - \sin(1))$ **b** $\dfrac{4}{15}$ **c** $\ln\left(\dfrac{27}{32}\right)$

3 $\left[-1, -\dfrac{\sqrt{2}}{2}\right] \cup \left[\dfrac{\sqrt{2}}{2}, 1\right]$

4 a $-4 + 2\sqrt{2} + \dfrac{\pi}{\sqrt{2}}$ **b** $\dfrac{1}{4}e^2(5e^4 - 1)$

c $\ln 2 - 1$

5 $\ln(e+1) - \ln 2$

6 a $\sin(x - \frac{\pi}{4}) - (x - \frac{\pi}{4})\cos(x - \frac{\pi}{4}) + c$

b $\dfrac{1}{4}(x+1)^2(2\ln(x+1) - 1) + c$

7 a $\sin^{-1}(2x) + \dfrac{2x}{\sqrt{1-4x^2}}$

b $\dfrac{1}{2}\sqrt{1-4x^2} + x\sin^{-1}(2x) + c$

8 a 6π
b $6\pi^2$

9 $8\pi a^5$

10 $\dfrac{\pi}{8}(e^{2\pi} - 1)$

11 e^{-2}

12 a $(1, 1),\ (4, 2)$ **b** $\dfrac{1}{6}$

13 a $10\ln 2$ **b** $f(x) = \dfrac{1}{10}e^{-\frac{x}{10}},\ x \geq 0$

c $\mathrm{E}(X) = 10,\ \mathrm{Var}(X) = 100$

14 a $\frac{1}{3}, -\frac{7}{3}$ **b** $3x - 7y = -11$

15 $\frac{dy}{dx} = -\tan t, \quad -1$

16 $y = -\frac{1}{2}\ln(\cos(2x))$

17 $y = 2(1 + x^2)$

18 a $\frac{(x+1)ye^x + 1}{(x+1)(2y - e^x)}$

b $y = x + 2$

19 a $y = e^{\frac{x}{x+1}}$

b $y = \frac{2}{2 - e^x}$

20 $y = 0$ or $y = -\frac{2}{x^2 + c}$

21 $y = 1 + Ae^{-\cos x}$

22 $\frac{1}{2}\ln(1 + u^2)$

23 $\frac{5g}{2}, \frac{2}{5}$

24 $10\,000\ln\left(\frac{5}{6}\right) + 2000$

25 525

a 440 N **b** 540 N

26 a $\frac{v - 5}{10}$ **b** $v = 5 - 5e^{\frac{t}{10}}$

27 $T = \frac{5g}{2}$

28 $v = -2\sqrt{t + 1}$

29 $\frac{4\sqrt{5}}{5}$ N newtons

30 $\mathrm{E}(\bar{X}) = 100, \ \mathrm{sd}(\bar{X}) = 1.25$

31 $(80, 88)$

32 a 27 **b** $(0.9)^{30}$

33 $\mathrm{E}(\bar{X}) = \frac{1}{3}, \ \mathrm{var}(\bar{X}) = \frac{1}{1800}$

34 a $\frac{1}{3\pi}$

b $\frac{32 + 16\sqrt{2}}{15}$

36 a 1180

b 129 000

37 a $\frac{1}{5}$

38 0.4375

39 0.47

40 a $y = -2x + 2, \ -1 \le x \le 2$ **b** 12π

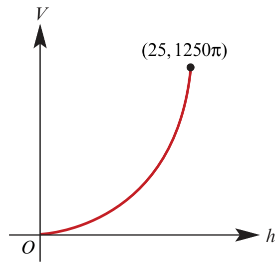
41 0.223

42 a 1250π

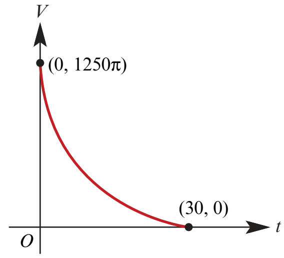
b ii $\frac{10\pi}{3}$ **iii** $h = -\frac{5t}{6} + 25$

iv $V = 2\pi\left(25 - \frac{5t}{6}\right)^2$

c i



ii



43 a $e^{-\frac{5}{6}} \approx 0.435$ **b** $e^{-\frac{5}{6}} \approx 0.435$

44 a $P = (t + 100)^2$

b 22.47 years

45 $\frac{4}{3}$ m/s

46 a $V = \frac{4}{3}\pi r^3$

b $4\pi r^2 \frac{dr}{dt} = -t^2$

c $r = \sqrt[3]{\frac{4000\pi - t^3}{4\pi}}$

d 23.2 mins

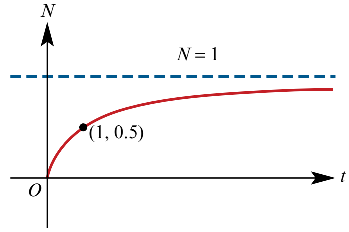
47 50 km/h

48 a $N = \frac{At}{At + 1}$, where $A > 0$

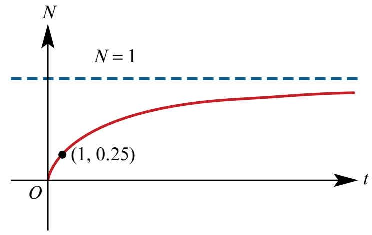
b i $N = \frac{t}{t + 1}$

ii $N = \frac{t}{t + 3}$

c i



ii



49 1.833 degrees per second

 ISBN 978-1-009-57826-4 Cambridge University Press

50 **e** **i** $\dfrac{dv}{dh} = \pi\left(\dfrac{25h}{3} + 100\right)$

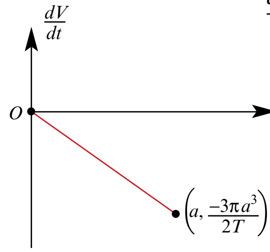
ii $\dfrac{dh}{dt} = \dfrac{-9\sqrt{h}}{625\pi^2(h+12)^2}$

f 65 days 19 hours

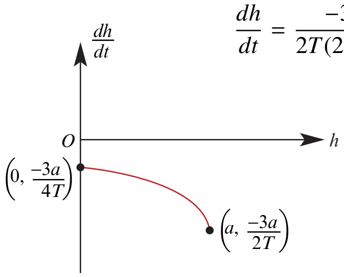
51 **a** **ii** 6.355 cm

d 15.7

e **i** $\dfrac{dV}{dt} = \dfrac{-3\pi a^2}{2T}h$



ii $\dfrac{dh}{dt} = \dfrac{-3a^2}{2T(2a-h)}$



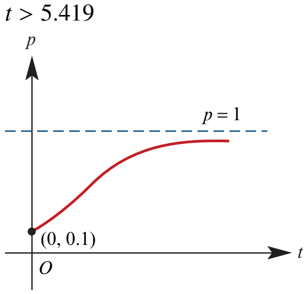
f **i** $-\dfrac{a}{T}$ **ii** $-\dfrac{6a}{7T}$

g -0.37 cms^{-1}

52 **b** $\dfrac{9}{25}$ **c** $\dfrac{1}{9\left(\frac{2}{3}\right)^t + 1}$

d $t > 5.419$

e



53 **b** $\dfrac{\sqrt[3]{k^2 p}}{k}$

54 **a** $y = \dfrac{2}{5}x^2$

b $V = 40\sqrt{10}y^{\frac{3}{2}}$

c 252 mm

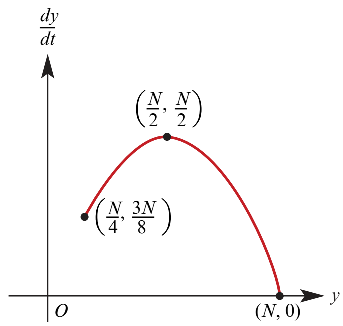
d $\dfrac{dy}{dt} = \dfrac{\sqrt{10y}}{10y}$, $t = \dfrac{2\sqrt{10}}{3}y^{\frac{3}{2}}$

e **i** 3 min 9 s **ii** 5 min 45 s

55 **a** $y = \dfrac{Ne^{2t}}{3 + e^{2t}}$, $\dfrac{dy}{dt} = \dfrac{6Ne^{2t}}{(3+e^{2t})^2}$ **b** N

c $\dfrac{dy}{dt} > 0$ for all t **d** $\dfrac{N}{2}$

e **i**



ii At $t = \dfrac{1}{2}\ln 3 \approx 0.549306$

56 **a** $v = 100\left(1 - e^{-\frac{t}{10}}\right)$

b 100 m/s

c $10\ln 10 \approx 23.03$ seconds

57 $\left(\dfrac{161}{12}, \dfrac{21}{4}\right)$

58 **a** $\ddot{x} = -\dfrac{\pi^2}{36}(x-4)$, centre $x = 4$, $A = 3$

b $x(0) = \dfrac{5}{2}$, $\dot{x}(0) = -\dfrac{\pi\sqrt{3}}{4}$

c $t = 10$

59 **a** $\ddot{x} = -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t) = -\omega^2 x$

b **i** π s

ii 5 m

iii 10 m/s

60 **a** **i** $v^2 = \dfrac{2gR^2}{x} + u^2 - 2gR$

ii $x = \dfrac{2gR^2}{2gR - u^2}$

iii $u \geq \sqrt{2gR}$

b 40 320 km/h

61 **a** **i** 9504 N **ii** 704 N

b 0.6742 s

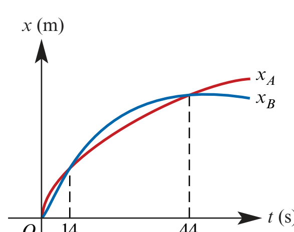
c About 10 people (852 kg)

62 **a** 8 m **b** π s **c** $\dfrac{\pi}{6}$ s

63 **a** $v_A = \dfrac{20}{\sqrt{2t+1}}$, $v_B = \dfrac{100}{t+10}$

b $x_A = 20(\sqrt{2t+1} - 1)$, $x_B = 100\ln\left(\dfrac{t+10}{10}\right)$

c



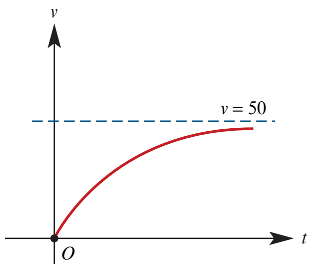
d 14 s and 44 s

64 **a** $v = 50 - 50e^{\frac{-t}{5}}$

b 49.9963

 ISBN 978-1-009-57826-4

c



d i $x = 50\left(t + 5e^{\frac{-t}{5}} - 5\right)$ **ii** 125.2986 m

65 a i $y = 2r\sin\left(\frac{1}{2}\theta\right)$ **ii** $\cos\theta = \frac{r}{r+h}$

b i $\frac{dy}{d\theta} = r\cos\left(\frac{1}{2}\theta\right)$;

$$\frac{dy}{dt} = \frac{r\cos\left(\frac{1}{2}\theta\right)\cos^2\theta\,\sin t}{\sin\theta}$$

ii 6000 km **iii** 1500 km/h

66 a ii 10 m/s^2, $75t - 5t^2$

b 281.25 m

c i 180 m

67 a 0.159 **b** 0.013

68 (3.403, 3.797)

69 60

70 $\bar{x} = 2.5$, $s = 0.565$

71 a \$1.15 **b** 90.9%

72 (61.177, 66.423)

73 2.106

74 $\mu = 600.00$, $\sigma = 4.00$

75 0.1041

76 $\mu = 1001$, $\sigma = 12$

77 0.0417

78 $25 \le n \le 48$

79 95% confidence interval for μ is (76.126, 83, 874). Since this confidence interval includes 78 kg as a possible value of the population mean, the data does not support the researchers' belief that the mean has changed.

80 a i i (73.05, 79.06) **ii ii** (69.89, 74.89)

b (1.73, 5.60)

c Yes. If the means of the two exams were the same, the confidence interval for D would include 0, which it does not. Since the values of D were calculated by subtracting the mark from exam Y from the mark from exam X, we can be 95% confident that students marks were between 1.73 and 5.60 higher on exam X than on ExamY.

Multiple-choice questions

1 D	**2** B	**3** D	**4** B	**5** A
6 C	**7** A	**8** A	**9** C	**10** C
11 B	**12** B	**13** A	**14** B	**15** A
16 A	**17** A	**18** D	**19** B	**20** C
21 C	**22** D	**23** B	**24** C	**25** B
26 B	**27** B	**28** A	**29** B	**30** A
31 A	**32** C	**33** C	**34** C	**35** B
36 A	**37** D	**38** C	**39** C	**40** A
41 A	**42** A	**43** B	**44** D	**45** A
46 A	**47** A	**48** C	**49** C	**50** C
51 B	**52** C	**53** D	**54** A	**55** B
56 C	**57** A	**58** A	**59** A	**60** D
61 C	**62** D	**63** C	**64** A	**65** B
66 A	**67** D	**68** C	**69** C	**70** C
71 B	**72** B	**73** A	**74** A	**75** B
76 B	**77** A	**78** B	**79** C	**80** D
81 B	**82** C	**83** C	**84** A	**85** A
86 B	**87** B	**88** C	**89** B	**90** C
91 D	**92** C	**93** C	**94** B	**95** A
96 D	**97** D	**98** A	**99** C	**100** A
101 C	**102** A	**103** B	**104** A	**105** A
106 B	**107** C	**108** A	**109** C	**110** D
111 A	**112** C	**113** C	**114** C	**115** B
116 B	**117** C	**118** B	**119** D	**120** A
121 C	**122** C			

Chapter 17

Short-response questions

1 $2 + \frac{\pi}{2} - \tan^{-1}(2)$

3 a $k = -3$ or $k = 2$

b $\frac{1}{(k+3)(k-2)}\begin{bmatrix} k & -3 \\ -2 & k+1 \end{bmatrix}$ **c** (1, 1)

d Parallel lines

4 a $-\frac{2}{5}$ **b** Decreasing by 5 units per second

6 a $9x = 2y^2 + 7y + 5$ **b** $y = \frac{3}{5}x + \frac{1}{5}$

8 a $\cos^{-1}\left(-\frac{\sqrt{3}}{6}\right) \approx 106.78°$ **b** $(-2, -2, 1)$

9 a i $2\sqrt{2}\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)$ **ii** $4\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)$

iii $8\sqrt{2}\,\mathrm{cis}\left(\frac{7\pi}{12}\right)$

b $-4 - (2\sqrt{3} + 2)i$

10 a $5\sin^{-1}\left(\frac{x}{2}\right) + c$ **b** $\frac{5}{4}\ln\left|\frac{2+x}{2-x}\right| + c$

c $\frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

11 a $-\frac{1}{16}(4x + 9\ln|9 - 4x|) + c$

b $-\frac{1}{8}\ln|9 - 4x^2| + c$ **c** $-\frac{1}{4}\sqrt{9 - 4x^2} + c$

12 a $\ln(x+2) + \frac{x}{x+2}$ **b** $6\ln 2 - 2$

14 $\frac{4\sqrt{91}}{9}$

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15 $\boldsymbol{r} = -\frac{3}{10}(2t - 115)\hat{\boldsymbol{i}} + \frac{1}{10}(2t + 35)\hat{\boldsymbol{j}} + t\hat{\boldsymbol{k}}$

16 $\frac{3\sqrt{2}}{5}$

17 $m \in \mathbb{R} \setminus \{\frac{2}{3}, 2\}$

18 $\left(\frac{13}{5}, \frac{13}{5}, 0\right)$

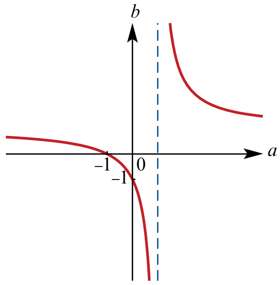
19 $\boldsymbol{r}(t) = -\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} + t(6\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}); \sqrt{5}$

21 $\frac{1}{22}$

22 a $[0, 1]$
b $[0, 4\pi]$
c 2π
d $\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$
e $y = 4 + 2\pi - 8x$

23 a $y = 0$
b $\left(-3 - 2\sqrt{3}, \frac{1}{2} - \frac{1}{\sqrt{3}}\right), \left(-3 + 2\sqrt{3}, \frac{1}{2} + \frac{1}{\sqrt{3}}\right)$
c $\frac{\pi}{\sqrt{3}} + \ln 2$

24 a i $19 + 9i$ **ii** $-7 - \sqrt{3}i$
iii $-\frac{11}{8} - \frac{1}{4}i$ **iv** $1.48 + 0.8i$
b i $(ab - 1) + (a + b)i$
ii $b = \frac{a + 1}{a - 1}$
iii



25 $\mu = 10, \ \sigma = 5\sqrt{5}$

26 b $(z - 1 - i)(z - 1 + i)(z + 1 - 2i)(z + 1 + 2i)$

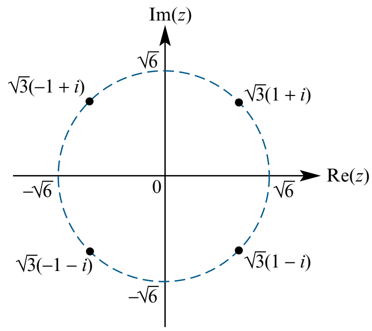
27 a $2(x - 4)e^x$ **b** $\frac{1}{4}x^2(2\ln(2x) - 1)$
c $\frac{1}{3}x\tan(3x) + \frac{1}{9}\ln(\cos(3x))$
d $-\frac{1}{2}x^2 + x\tan x + \ln(\cos x)$

28 a $\frac{1}{9}(1 + 2e^3)$ **b** $-\frac{1}{9}$
c $-3(3 + \sqrt{3}) + 3\sqrt{3}\pi$

29 $\frac{\sqrt{69}}{2}$

30 a $x = \frac{11 + t}{4}, \ y = \frac{1 + 3t}{4}, \ z = t$
b $\boldsymbol{r} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}})$
c $4x - 11 = \frac{4y - 1}{3} = z$

31 a Polar form: $\sqrt{6}\operatorname{cis}\left(\pm\frac{\pi}{4}\right), \sqrt{6}\operatorname{cis}\left(\pm\frac{3\pi}{4}\right)$;
Cartesian form: $\sqrt{3}(\pm 1 \pm i)$
b $(z^2 + 2\sqrt{3}z + 6)(z^2 - 2\sqrt{3}z + 6)$
c



32 $A = -\frac{1}{\pi^2 + 1}, \ B = -\frac{\pi}{\pi^2 + 1}$

33 $\frac{\pi}{2}(8\ln 2 - 5)$

34 a 20 mins
b $\frac{dm}{dt} = -\frac{3m}{20 - t}, m(0) = 10$
c $m = \frac{(20 - t)^3}{800}$
d $20 - 8\sqrt{5}$ mins

35 a $y = -\ln(e + e^{-1} - e^x)$
b $(-\infty, \ln(e + e^{-1}))$
c $y = \frac{x}{e + e^{-1} - 1} - \ln(e + e^{-1} - 1)$

36 a $P\left(e, \frac{1}{e}\right), Q(1, 0)$ **b** $\frac{1}{2}$

37 a $y = 2\tan\left(x^2 + \frac{\pi}{4}\right)$ **b** $\left(-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}\right)$
c $y = -\frac{x}{8}\sqrt{\frac{3}{\pi}} + 2\sqrt{3} + \frac{1}{16}$

38 b $\pi\left(\frac{a^2}{2} + a + \ln(a - 1) - 4\right)$

39 a $\pm\frac{1}{\sqrt{2}}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$
b $m + n = 1; \overrightarrow{OP} = m\hat{\boldsymbol{i}} + (1 - m)\hat{\boldsymbol{j}}$
c $m = \frac{3 \pm \sqrt{3}}{6}$

40 a $-\frac{2}{9}$ **b** -4

41 a $a = 1, b = 1$ **b** $c = 3, d = 2$

42 $m = -3$

43 a $\ddot{x} = -4(x - 2)$
b i π s **ii** 1 m **iii** 2 m/s

44 a $z = 3\operatorname{cis}\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}(1 + i)$
$z^2 = 9\operatorname{cis}\left(\frac{\pi}{2}\right) = 9i$
$iz = 3\operatorname{cis}\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}(-1 + i)$

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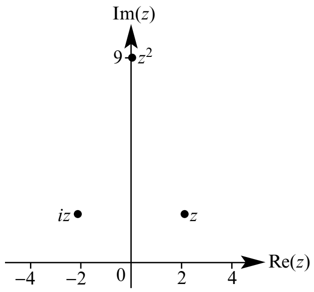
b



45 a $m = 3,\ n = 5$ **b** $1 + 3i,\ 2 - i$

46 a 25 **b** 1.4 **c** (22.2, 27.8)

48 a $\dfrac{dy}{dx} = \dfrac{x-y-1}{x-y+1},\ \dfrac{d^2y}{dx^2} = \dfrac{4}{(x-y+1)^3}$

b $x = \dfrac{3}{4},\ y = -\dfrac{1}{4}$

c $\dfrac{1}{2}$

49 $\dfrac{10}{9}$

50 85°

51 $x = 1,\ y = 2,\ z = 3$

52 a $(4, -1)$ **b** $\cos^{-1}\left(\dfrac{\sqrt{5}}{5}\right) \approx 63.43°$

c $\sqrt{2(t^2 - 2t + 5)}$ **d** $2\sqrt{2}$

53 a $\begin{bmatrix} 2 & -5 & 7 \\ 3 & 1 & -12 \\ 5 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 3 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

54 0.023

55 (113.71, 117.49)

56 a 0.4866 **b** 0.8606 **c** 0.2835

57 a $\overrightarrow{CA} = 2(\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}),\quad \overrightarrow{CB} = 3(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$

b $12(-\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ **c** $6\sqrt{2}$ **d** $12\sqrt{2}$

58 a $\mathbf{P}_1 = \begin{bmatrix} 29.6 \\ 14.4 \\ 14.4 \\ 21.6 \\ 6.4 \\ 0.0 \end{bmatrix},\ \mathbf{P}_{40} = \begin{bmatrix} 96.11 \\ 55.97 \\ 48.88 \\ 42.69 \\ 33.15 \\ 19.30 \end{bmatrix},\ \mathbf{P}_{41} = \begin{bmatrix} 99.04 \\ 57.67 \\ 50.37 \\ 43.99 \\ 34.16 \\ 19.89 \end{bmatrix}$

c $\mathbf{P}_0 = \begin{bmatrix} 715 \\ 416 \\ 363 \\ 317 \\ 247 \\ 144 \end{bmatrix},\ \mathbf{P}_1 = \begin{bmatrix} 735.9 \\ 429.0 \\ 374.4 \\ 326.7 \\ 253.6 \\ 148.2 \end{bmatrix}$

59 a $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$ **b** $R,\ Q,\ P = T,\ S$

60 a $\mathrm{E}(X) = 12,\ \mathrm{sd}(X) \approx 2.176$

b $\mathrm{E}(\bar{X}) = 12,\ \mathrm{sd}(\bar{X}) \approx 0.397$

c 0.022

61 a i $\displaystyle\int_{10}^{5} \frac{-50}{v(1+v^2)}\, dv$

ii $25\ln\left(\dfrac{104}{101}\right)$ seconds

b ii $x = 50(\tan^{-1}(10) - \tan^{-1} v)$ **iv** 74 m

62 a $\overrightarrow{OM} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}}$

b $\overrightarrow{MC} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}},\ \overrightarrow{MP} = -2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}} - 6\hat{\boldsymbol{k}}$

c $2\sqrt{61}$

63 a $\overrightarrow{OA} = 6\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{k}},\ \overrightarrow{OB} = 6\hat{\boldsymbol{j}} + 6\hat{\boldsymbol{k}},$
$\overrightarrow{OC} = 6\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$

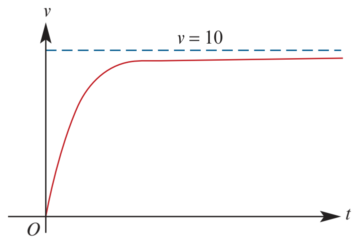
b $\overrightarrow{OM} = 6\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}}$

c $\overrightarrow{ON} = 4\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}}$

64 a $t = \dfrac{1}{2}\ln\left(\dfrac{10+v}{10-v}\right)$ **b** $v = \dfrac{10(e^{2t}-1)}{e^{2t}+1}$

c $\dfrac{1}{2}\ln\left(\dfrac{3}{2}\right)$ seconds

d



65 a dom $= (-1, 1)$, ran $= \mathbb{R}$

b $(0, 0),\ t = 0;\quad (1, 1),\ t = \dfrac{\pi}{4};$

$(-1, -1),\ t = -\dfrac{\pi}{4}$

c i $y = \dfrac{1}{3}(4x - \sqrt{3})$ **ii** $y = -4x + 3\sqrt{3}$

d $\left(\dfrac{5\sqrt{3}}{8}, \dfrac{\sqrt{3}}{2}\right)$

e $1 - \ln 2$

66 a $\dfrac{dP}{dt} = k(P - 0.5P_0)$

b $P = 500\left(1 + \left(\dfrac{6}{5}\right)^t\right)$

c 144 goats

d 6.03 years

67 a 0.0679

b 0.5

c i (2.991, 3.009)

ii Machine A, since the confidence interval contains the mean for Machine A but not Machine B

68 a $\boldsymbol{r}_A(t) = 30\sqrt{3}t\,\hat{\boldsymbol{i}} + (30t - \tfrac{1}{2}gt^2)\hat{\boldsymbol{j}}$

$\boldsymbol{r}_B(t) = (100 - 50t\cos\beta)\hat{\boldsymbol{i}} + (50t\sin\beta - \tfrac{1}{2}gt^2)\hat{\boldsymbol{j}}$

b $\beta = \sin^{-1}\left(\dfrac{3}{5}\right) \approx 36.87°$

c 1.09 seconds

d (56.50, 26.83)

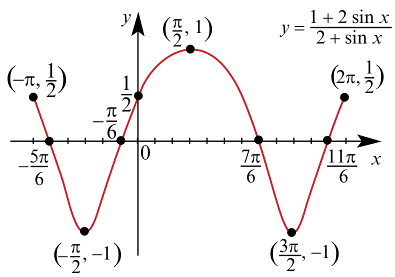
69 a $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}})$
b $\boldsymbol{r} \cdot (\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}) = 9$
d 1.43
e $2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$; $61.9°$

70 b $(-2, 4, 3)$
c $-x + 2y + z = 13$
d $\left(\frac{4}{3}, \frac{7}{3}, \frac{29}{3}\right)$
e $\boldsymbol{r} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}})$

71 a $\boldsymbol{r} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 4\hat{\boldsymbol{k}} + t(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$
b $(0, -2, 4)$ **c** $21.85°$ **d** $(0, -2, 7)$

72 a $f'(x) = \ln(x) - 2$ **b** $A(e^3, 0)$
c $y = x - e^3$ **d** $2 : 1$

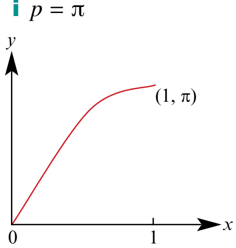
73 a i $\dfrac{dy}{dx} = \dfrac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$ **ii** $1, -1$
b i $\left(0, \frac{1}{2}\right)$
ii $\left(\frac{-5\pi}{6}, 0\right), \left(\frac{-\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$
iii $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right)$
iv



v $2\pi(3 - \sqrt{3})$

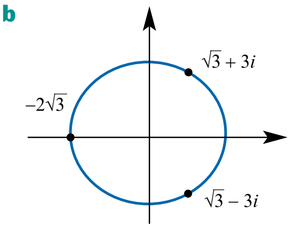
74 a $r = 2,\ a = \frac{\pi}{3}$ **b** $[-2, 2]$ **c** $(0, 1)$
d $\left(\frac{5\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$ **e** $\frac{\pi}{12}, \frac{7\pi}{12}$
f $\dfrac{\ln(21 + 12\sqrt{3})}{4}$ **g** $(10\pi + 3\sqrt{3})\dfrac{\pi}{6}$

75 a i $\cos(\pi x) - \pi x \sin(\pi x)$
ii $\dfrac{1}{\pi^2}\sin(\pi x) - \dfrac{x}{\pi}\cos(\pi x)$
b i $p = \pi$
c



d $\dfrac{(2\pi^2 + 15)\pi}{6}$ **e** $k = 2$
f 1.066 **g** 0.572

76 a i $24\sqrt{3}$
ii $-2\sqrt{3},\ \sqrt{3} + 3i$
b



c ii $\sqrt{3} \pm 6i$
iii $\dfrac{(x - \sqrt{3})^2}{27} + \dfrac{y^2}{36} = 1$

77 a $m = \sqrt{3}$
b i $\overrightarrow{OC} = -\overrightarrow{OA}$
c ii $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}},\ \frac{8}{3}\hat{\boldsymbol{i}} - \frac{1}{3}\hat{\boldsymbol{j}} + \frac{4}{3}\hat{\boldsymbol{k}}$
d $\dfrac{3}{\sqrt{18 - 2\sqrt{3}}}\left((2 + \sqrt{3})\hat{\boldsymbol{i}} + (-1 + \sqrt{3})\hat{\boldsymbol{j}} + (2 - \sqrt{3})\hat{\boldsymbol{k}}\right)$
e $t = \frac{3}{4}, k = \frac{1}{2}, \ell = \frac{13\sqrt{3}}{12}$
f Particle lies outside the circle

78 a i $\dfrac{x^2}{9} + \dfrac{(y + a)^2}{36} = 1$ **ii** $\pm\dfrac{\sqrt{36 - a^2}}{2}$
b $f(x) = 2\sqrt{9 - x^2} - a$
c $\sqrt{9 - x^2} - \dfrac{x^2}{\sqrt{9 - x^2}}$
d i $A = 9$
e $\frac{1}{2}\left(x\sqrt{9 - x^2} + 9\arcsin\left(\frac{x}{3}\right)\right)$
f $18\arcsin\left(\dfrac{\sqrt{36 - a^2}}{6}\right) - \dfrac{a}{2}\sqrt{36 - a^2}$
g 18π
h 144π

79 a $x = \frac{3}{4}\sin(2t),\ y = -\frac{1}{2}\cos(2t)$
b $\dfrac{16x^2}{9} + 4y^2 = 1$ **c** $\frac{2}{3}\tan(2t)$
d $y = -\frac{1}{2}\sec(2t),\ x = \frac{3}{4}\operatorname{cosec}(2t)$
e $\frac{3}{8}|\operatorname{cosec}(4t)|$, minimum area $= \frac{3}{8}$ when $t = \dots, -\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}, \dots$
f $x = \frac{3}{4}\sin(2t),\ y = \frac{3}{4}\cos(2t)$
(infinitely many possible answers)
g $\dfrac{5\pi}{16}$

80 a $y^2 = 16x^2(1 - x^2)(1 - 2x^2)^2$
b $\dfrac{dx}{dt} = \cos t,\ \dfrac{dy}{dt} = 4\cos(4t),\ \dfrac{dy}{dx} = \dfrac{4\cos(4t)}{\cos t}$
c i $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$
ii $-\frac{1}{2}\sqrt{2 - \sqrt{2}},\ -\frac{1}{2}\sqrt{2 + \sqrt{2}},\ \frac{1}{2}\sqrt{2 - \sqrt{2}},\ \frac{1}{2}\sqrt{2 + \sqrt{2}}$

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iii $(-\frac{1}{2}\sqrt{2-\sqrt{2}}, 1)$, $(-\frac{1}{2}\sqrt{2-\sqrt{2}}, -1)$, $(-\frac{1}{2}\sqrt{2+\sqrt{2}}, 1)$, $(-\frac{1}{2}\sqrt{2+\sqrt{2}}, -1)$, $(\frac{1}{2}\sqrt{2-\sqrt{2}}, 1)$, $(\frac{1}{2}\sqrt{2-\sqrt{2}}, -1)$, $(\frac{1}{2}\sqrt{2+\sqrt{2}}, 1)$, $(\frac{1}{2}\sqrt{2+\sqrt{2}}, -1)$

iv $\frac{dy}{dx} = \pm 4$ when $x = 0$;
$\frac{dy}{dx} = \pm 4\sqrt{2}$ when $x = \pm\frac{1}{\sqrt{2}}$

d $\frac{16}{15}(\sqrt{2}+1)$ **e** $\frac{64\pi}{63}$

81 a $y^2 = x\left(\frac{x}{3}-1\right)^2$ **b** $\left(1, \frac{2}{3}\right), \left(1, -\frac{2}{3}\right)$

c $\frac{8\sqrt{3}}{5}$ **d** $\frac{3\pi}{4}$

82 a $y^2 = \frac{64x^2(25-x^2)}{25}$

b i ± 8 **ii** $\pm\frac{14}{5}$

c i $\frac{\pi\sqrt{2}}{12}\,\boldsymbol{\hat{i}}$ **ii** $\frac{\pi\sqrt{2}}{12}$

d $\frac{800}{3}$ **e** $\frac{325}{16}$ **f** $\frac{6400\pi}{3}$

83 a 0.0808

b $k_1 = 45.2$, $k_2 = 64.8$

c i 0.0008 **ii** 0.0289 **iii** 0.0028

d $c_1 = 51.90$, $c_2 = 58.10$

84 a $\mathrm{E}(X) = \frac{b}{2}$, $\mathrm{sd}(X) = \frac{b}{\sqrt{12}}$

b $\mathrm{E}(\bar{X}) = \frac{b}{2}$, $\mathrm{sd}(\bar{X}) = \frac{b}{\sqrt{12n}}$

c $(2.4 - 0.067b, 2.4 + 0.067b)$

d $4.23 < b < 5.54$ with 90% confidence

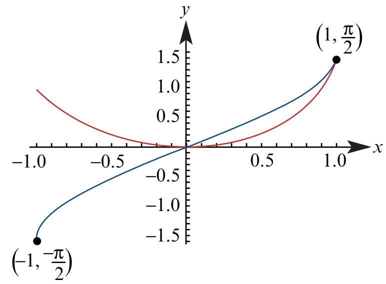
85 a $\frac{x}{\sqrt{1-x^2}} + \arcsin(x)$, $(0, 0)$ local minimum

b $\frac{x^2\sqrt{1-x^2} + 2(1-x^2)^{\frac{3}{2}}}{(x^2-1)^2} = \frac{\sqrt{1-x^2}\,(2-x^2)}{(x^2-1)^2} \geq 0$ for all $x \in (-1, 1)$

c $f(x) \geq 0$ for all x, as x and $\arcsin(x)$ have the same sign for all x

d $x = 0$ and $x = 1$

e



f $\frac{3\pi}{8} - 1$

Multiple-choice questions

1 A	**2** C	**3** C	**4** C	**5** B	**6** B
7 A	**8** B	**9** A	**10** D	**11** B	**12** B
13 D	**14** D	**15** B	**16** D	**17** A	**18** A
19 C	**20** D	**21** A	**22** A	**23** B	**24** C
25 A	**26** A	**27** C	**28** A	**29** B	**30** A
31 C	**32** C	**33** A	**34** C	**35** A	**36** D
37 B	**38** B	**39** A	**40** C	**41** A	**42** D
43 C	**44** D	**45** D	**46** B	**47** A	**48** D
49 B	**50** A	**51** A	**52** D	**53** B	

Appendix

B

Modelling with matrices

Objectives

- To use **dominance matrices** in more general situations.
- To find the **characteristic equation**, **eigenvalues** and **eigenvectors** of a 2×2 matrix and a 3×3 matrix.
- To use technology to find the eigenvalues and eigenvectors of a square matrix.
- To use matrices to solve problems involving conditional probability.
- To introduce **Markov chains** and use them to solve problems involving prediction.
- To introduce the concept of the **steady state** of a Markov chain.
- To determine the steady state of a Markov chain using eigenvalues and eigenvectors.
- To use **dominant eigenvalues** in working with **Leslie matrices**.
- To use **Leontief matrices** to model economies.
- To use matrices for encrypting and decrypting messages.
- To use a spreadsheet to perform calculations with matrices.

Matrices are part of the branch of mathematics called *linear algebra*. There are many applications of linear algebra to diverse areas such as engineering, biology and physics.

This chapter extends your study of applications of matrices in Chapter 9. In this chapter, we further investigate dominance matrices and Leslie matrices. We also introduce eigenvalues, Markov chains and Leontief matrices.

You may be able to use this chapter as a starting point to learn about a topic for a problem-solving and modelling task in Specialist Mathematics Units 3 & 4.

B1 Further approaches to dominance matrices

In Section 9A, we saw how to use a dominance matrix to determine a ranking of the teams competing in a round-robin tournament. In this section, we introduce an alternative approach to dominance matrices that can be used for other types of competitions. For example, this approach can be used when there may be draws or when some teams do not play each other.

The transpose of a matrix

We first need to introduce the concept of the transpose of a matrix.

The **transpose** of a matrix $\mathbf{A}$ is the matrix formed by interchanging the rows and columns: the ith row of $\mathbf{A}$ becomes the ith column of the transpose of $\mathbf{A}$.

The transpose of $\mathbf{A}$ is denoted by $\mathbf{A}^T$.

Note: If $\mathbf{A}$ is an $m \times n$ matrix, then its transpose $\mathbf{A}^T$ is an $n \times m$ matrix.

Example 1

Find the transpose of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution

Interchange the rows and columns of $\mathbf{A}$:

$$\mathbf{A}^T = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 4 & 6 \end{bmatrix}$$

Note: Your graphics calculator can find the transpose of a given matrix; this will be convenient for the calculations in this section.

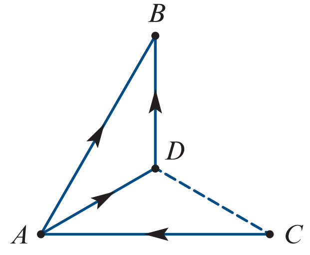
An alternative approach to dominance matrices

The approach to dominance matrices from Section 7F is commonly used to analyse competitions where there is a win/loss result for each pair of teams. In this section, we introduce an alternative approach that may be useful in other situations.

For example, suppose that a competition is held between four teams, A, B, C and D, and that the results are as follows:

- A defeats B and D
- C defeats A
- D defeats B
- C and D draw
- B and C do not play.

These results are also shown in the diagram.



One-step dominance

We represent the results of this competition as a 4×4 matrix:

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

- The first row of this matrix shows that A defeats B and D.
- The third row shows that C defeats A and draws with D.
- The fourth row shows that D defeats B and draws with C.

In general, we can represent a competition between n teams as an $n \times n$ matrix $\mathbf{M}$, where the entry m_{ij} in row i and column j of $\mathbf{M}$ is given by

$$m_{ij} = \begin{cases} 1 & \text{if team } i \text{ defeats team } j \\ \frac{1}{2} & \text{if team } i \text{ draws with team } j \\ 0 & \text{otherwise} \end{cases}$$

We now give each team a score that takes into consideration their wins and their losses.

Score based on wins To sum the wins and draws for each team, we multiply $\mathbf{M}$ by a column matrix of 1s:

$$\mathbf{M}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1.5 \\ 1.5 \end{bmatrix}\begin{array}{c} A \\ B \\ C \\ D \end{array}$$

Note that this calculation is finding the total of each row of $\mathbf{M}$.

Score based on losses To sum the losses and draws for each team, we multiply the transpose of $\mathbf{M}$ by a column matrix of 1s:

$$\mathbf{M}^T\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0.5 \\ 1.5 \end{bmatrix}\begin{array}{c} A \\ B \\ C \\ D \end{array}$$

Note that this calculation is finding the total of each column of $\mathbf{M}$.

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Score based on wins and losses To find the **first-order rank vector**, $\mathbf{r}_1$, we subtract the losses from the wins:

$$\mathbf{r}_1 = \mathbf{M}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \mathbf{M}^T\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\1.5\\1.5\end{bmatrix} - \begin{bmatrix}1\\2\\0.5\\1.5\end{bmatrix} = \begin{array}{c}\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}\end{array}\begin{array}{c}A\\B\\C\\D\end{array}$$

This gives us a ranking of the teams with A and C equal first, D third and B fourth.

Two-step dominance

As in Section 7F, we can consider two-step dominance using $\mathbf{M}^2$.

For this example, we have

$$\mathbf{M}^2 = \begin{bmatrix}0 & 1 & \frac{1}{2} & 0\\0 & 0 & 0 & 0\\0 & \frac{3}{2} & \frac{1}{4} & 1\\\frac{1}{2} & 0 & 0 & \frac{1}{4}\end{bmatrix} \quad\text{and}\quad (\mathbf{M}^2)^T = \begin{bmatrix}0 & 0 & 0 & \frac{1}{2}\\1 & 0 & \frac{3}{2} & 0\\\frac{1}{2} & 0 & \frac{1}{4} & 0\\0 & 0 & 1 & \frac{1}{4}\end{bmatrix}$$

We find the **second-order rank vector**, $\mathbf{r}_2$, using $\mathbf{M}^2$ as follows:

$$\mathbf{r}_2 = \mathbf{M}^2\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - (\mathbf{M}^2)^T\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1.5\\0\\2.75\\0.75\end{bmatrix} - \begin{bmatrix}0.5\\2.5\\0.75\\1.25\end{bmatrix} = \begin{bmatrix}1\\-2.5\\2\\-0.5\end{bmatrix}$$

Combining one- and two-step dominance

We can combine the first- and second-order rank vectors to obtain a **total rank vector**:

$$\mathbf{r}_{\text{total}} = \mathbf{r}_1 + \mathbf{r}_2 = \begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + \begin{bmatrix}1\\-2.5\\2\\-0.5\end{bmatrix} = \begin{bmatrix}2\\-4.5\\3\\-0.5\end{bmatrix}\begin{array}{c}A\\B\\C\\D\end{array}$$

This gives the ranking C, A, D, B.

When working out rankings, it may be appropriate to give less weighting to the more indirect two-step contributions. The total rank vector can be defined by

$$\mathbf{r}_{\text{total}} = \mathbf{r}_1 + \omega\mathbf{r}_2$$

where the weighting ω is a number in the interval $[0, 1]$. This is demonstrated in the next example.

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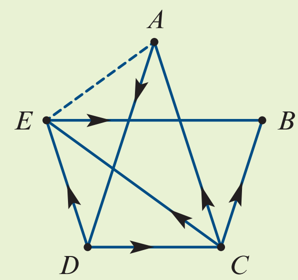
Example 2

Five friends – Alf, Bao, Cal, Dan and Ed – competed in a table-tennis competition. The results were as follows:



- Alf defeated Dan and drew with Ed
- Cal defeated Alf, Bao and Ed
- Dan defeated Cal and Ed
- Ed defeated Bao
- Bao did not play Alf or Dan.

Give a ranking of the five players using the total rank vector $\mathbf{r}_{\text{total}} = \mathbf{r}_1 + \omega\mathbf{r}_2$, where:

a $\omega = 1$ **b** $\omega = \frac{1}{2}$

Solution

Represent the results of the competition as a 5×5 matrix:

$$\mathbf{M} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- **One-step dominance**

  The first-order rank vector is

$$\mathbf{r}_1 = \mathbf{M}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \mathbf{M}^T\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \\ -1 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

- **Two-step dominance**

  The second-order rank vector is

$$\mathbf{r}_2 = \mathbf{M}^2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - (\mathbf{M}^2)^T\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -3.5 \\ 2 \\ 3 \\ -2 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

a Taking $\omega = 1$, we have

$$\mathbf{r}_{\text{total}} = \mathbf{r}_1 + \mathbf{r}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -3.5 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -5.5 \\ 4 \\ 4 \\ -3 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

This gives the ranking: Cal = Dan, Alf, Ed, Bao.

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b Taking $\omega = \frac{1}{2}$, we have

$$\mathbf{r}_{\text{total}} = \mathbf{r}_1 + \tfrac{1}{2}\mathbf{r}_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0.5 \\ -3.5 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0.25 \\ -3.75 \\ 3 \\ 2.5 \\ -2 \end{bmatrix} \end{array}\begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}$$

This gives the ranking: Cal, Dan, Alf, Ed, Bao.

Higher-order rank vectors

You can also determine the **third-order rank vector**, $\mathbf{r}_3$, the **fourth-order rank vector**, $\mathbf{r}_4$, and so on by using $\mathbf{M}^3, \mathbf{M}^4, \dots$. You can then define a total rank vector

$$\mathbf{r}_{\text{total}} = \mathbf{r}_1 + \omega_2\mathbf{r}_2 + \omega_3\mathbf{r}_3 + \cdots + \omega_n\mathbf{r}_n$$

where each weighting ω_i belongs to the interval $[0, 1]$. You can choose how far to go with the sum and what weightings to use. More is not necessarily better. You should decide which calculations are appropriate to the particular context.

Exercise B1

Example 2

1 The results of a competition between four players – A, B, C and D – are as follows:

- A defeats B
- D defeats B and C
- A and C draw
- A and D do not play
- B and C do not play.

Rank these four players using dominance matrices.

2 Consider a round-robin soccer competition between six teams. Devise a method for constructing a one-step dominance matrix to represent the results of the competition that takes into consideration the score in each game.

B2 Eigenvalues and eigenvectors

Eigenvalues and eigenvectors have many important applications in mathematics. In later sections of this chapter, we will look at applications to Markov chains, Leslie matrices and Leontief matrices.

Definition of eigenvalues and eigenvectors

We start with a simplified definition of eigenvalues and eigenvectors. We will give the more general definition at the end of this section.

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Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{x}$ be a non-zero $n \times 1$ matrix such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

for some real number λ. Then $\mathbf{x}$ is called an **eigenvector** of $\mathbf{A}$ with **eigenvalue** λ.

Note: Here we view the $n \times 1$ matrix $\mathbf{x}$ as a vector in n-dimensional space.

For example, consider the 2×2 matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$.

We can see that

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So the matrix $\mathbf{A}$ has an eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with eigenvalue $\lambda = 5$.

Note that, if $\mathbf{Ax} = \lambda\mathbf{x}$, then for any real number k we have

$$\mathbf{A}(k\mathbf{x}) = k\mathbf{Ax} = k\lambda\mathbf{x} = \lambda(k\mathbf{x})$$

Therefore any non-zero scalar multiple of an eigenvector of $\mathbf{A}$ is also an eigenvector with the same eigenvalue.

In Example 3, we will find that the family of all eigenvectors of $\mathbf{A}$ with eigenvalue $\lambda = 5$ can be described by $\mathbf{x} = \begin{bmatrix} k \\ 3k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

Finding eigenvalues and eigenvectors

Let $\mathbf{A}$ be an $n \times n$ matrix. Suppose that $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. Then $\mathbf{x}$ is a non-zero $n \times 1$ matrix such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

This equation can also be written as

$$\mathbf{Ax} = \lambda\mathbf{Ix} \qquad \text{where } \mathbf{I} \text{ is the } n \times n \text{ identity matrix}$$

Rearranging gives

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \qquad \text{where } \mathbf{0} \text{ is the } n \times 1 \text{ zero matrix}$$

Since $\mathbf{x} \neq \mathbf{0}$, it follows from this equation that the $n \times n$ matrix $\mathbf{A} - \lambda\mathbf{I}$ is non-invertible and therefore

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This is called the **characteristic equation** of the matrix $\mathbf{A}$.

The eigenvalues of the matrix $\mathbf{A}$ can be found by solving the characteristic equation for λ. This is demonstrated in the next example.

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Finding the eigenvalues and eigenvectors of a 2×2 matrix

Example 3

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

Solution

Finding the eigenvalues

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This becomes

$$\det\left(\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

i.e. $$\det\begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix} = 0$$

Using the formula for the determinant, this becomes

$$\begin{aligned}(2-\lambda)(4-\lambda) - 3 &= 0 \\ 8 - 6\lambda + \lambda^2 - 3 &= 0 \\ \lambda^2 - 6\lambda + 5 &= 0 \\ (\lambda - 1)(\lambda - 5) &= 0\end{aligned}$$

The eigenvalues are the solutions of this equation: $\lambda = 1$ and $\lambda = 5$.

The eigenvectors with eigenvalue $\lambda = 1$

If $\lambda = 1$, then the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

So we obtain the simultaneous equations

$$\begin{aligned} x + y &= 0 \qquad (1) \\ 3x + 3y &= 0 \qquad (2) \end{aligned}$$

Both of these equations give $x = -y$.

Hence the eigenvectors with eigenvalue $\lambda = 1$ are given by $\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

The eigenvectors with eigenvalue $\lambda = 5$

If $\lambda = 5$, then the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}$$

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We obtain

$$-3x + y = 0 \qquad (1)$$
$$3x - y = 0 \qquad (2)$$

Both of these equations give $x = \frac{1}{3}y$.

Hence the eigenvectors with eigenvalue $\lambda = 5$ are given by $\mathbf{x} = \begin{bmatrix} k \\ 3k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

The characteristic polynomial

In Example 3, when we expanded the expression $\det(\mathbf{A} - \lambda\mathbf{I})$ from the characteristic equation of $\mathbf{A}$, we obtained the quadratic $\lambda^2 - 6\lambda + 5$.

In general, for an $n \times n$ matrix $\mathbf{A}$, the expression $\det(\mathbf{A} - \lambda\mathbf{I})$ expands into a polynomial in λ. This is called the **characteristic polynomial** of $\mathbf{A}$.

Here are three properties of the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$:

- The characteristic polynomial has degree n.
- The coefficient of the leading term is $(-1)^n$.
- The constant term is $\det(\mathbf{A})$.

The determinant of a 3 × 3 matrix

In this course, you are expected to use technology to find the determinant of an $n \times n$ matrix when $n \geq 3$. However, in order to find the eigenvalues of a 3×3 matrix by hand, we need to be able to find the determinant of a 3×3 matrix by hand.

Consider a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of $\mathbf{A}$ can be defined as follows:

$$\begin{aligned} \det(\mathbf{A}) &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

This formula comes from working through the first row of $\mathbf{A}$:

- The first 2×2 matrix is obtained by deleting the row and column containing a_{11}.
- The second 2×2 matrix is obtained by deleting the row and column containing a_{12}.
- The third 2×2 matrix is obtained by deleting the row and column containing a_{13}.

We can obtain equivalent formulas for the determinant by using any row or column of $\mathbf{A}$ in a similar way. For example, working through the first column of $\mathbf{A}$:

$$\det(\mathbf{A}) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

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The sign of the a_{ij} term in a formula for the determinant is determined by $(-1)^{i+j}$. These signs can also be determined using the following array:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, working through the second row of **A**:

$$\det(\mathbf{A}) = -a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22} \det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Example 4

Consider the 3×3 matrix $\mathbf{A} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}$.

Find the determinant of **A** by expanding from:

a the first row **b** the first column **c** the second row.

Solution

a $\det(\mathbf{A}) = 4(3 \times 5 - 1 \times 1) - 2(2 \times 5 - 1 \times 4) + 0 = 44$

b $\det(\mathbf{A}) = 4(3 \times 5 - 1 \times 1) - 2(2 \times 5 - 0 \times 1) + 4(2 \times 1 - 0 \times 3) = 44$

c $\det(\mathbf{A}) = -2(2 \times 5 - 0 \times 1) + 3(4 \times 5 - 0 \times 4) - 1(4 \times 1 - 2 \times 4) = 44$

Finding the eigenvalues and eigenvectors of a 3×3 matrix

Example 5

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

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Solution

Finding the eigenvalues

The characteristic equation is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This becomes

$$\det\left(\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

i.e.

$$\det\begin{bmatrix} 2-\lambda & 2 & -7 \\ 2 & 1-\lambda & 2 \\ 0 & 1 & -3-\lambda \end{bmatrix} = 0$$

By expanding out the determinant, we obtain

$$-\lambda^3 + 13\lambda - 12 = 0$$

The eigenvalues are the solutions of this equation: $\lambda = 1$, $\lambda = 3$ and $\lambda = -4$.

The eigenvectors with eigenvalue $\lambda = 1$

If $\lambda = 1$, then the equation $\mathbf{Ax} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives the simultaneous equations

$$x + 2y - 7z = 0 \qquad (1)$$

$$2x + 2z = 0 \qquad (2)$$

$$y - 4z = 0 \qquad (3)$$

From these equations, we have $y = 4z$ and $x = -z$.

Hence the eigenvectors with eigenvalue $\lambda = 1$ are given by $\mathbf{x} = \begin{bmatrix} -k \\ 4k \\ k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

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The eigenvectors with eigenvalue $\lambda = 3$

If $\lambda = 3$, then the equation $\mathbf{Ax} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives

$$\begin{aligned} -x + 2y - 7z &= 0 \qquad (1) \\ 2x - 2y + 2z &= 0 \qquad (2) \\ y - 6z &= 0 \qquad (3) \end{aligned}$$

From these equations, we have $y = 6z$ and $x = 5z$.

Hence the eigenvectors with eigenvalue $\lambda = 3$ are given by $\mathbf{x} = \begin{bmatrix} 5k \\ 6k \\ k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

The eigenvectors with eigenvalue $\lambda = -4$

If $\lambda = -4$, then the equation $\mathbf{Ax} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -4\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives

$$\begin{aligned} 6x + 2y - 7z &= 0 \qquad (1) \\ 2x + 5y + 2z &= 0 \qquad (2) \\ y + z &= 0 \qquad (3) \end{aligned}$$

From these equations, we have $y = -z$ and $x = \frac{3}{2}z$.

Hence the eigenvectors with eigenvalue $\lambda = -4$ are given by $\mathbf{x} = \begin{bmatrix} 3k \\ -2k \\ 2k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

The eigenvalues of a power of a matrix

We now prove a theorem that will be used in the next section.

Theorem

Let $\mathbf{A}$ be a square matrix and let n be a positive integer. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^n$ with eigenvalue λ^n.

Proof Assume that $\mathbf{Ax} = \lambda\mathbf{x}$. Then multiplying both sides on the left by $\mathbf{A}$ gives

$$\mathbf{A}^2\mathbf{x} = \lambda\mathbf{Ax}$$

$$\therefore \quad \mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$$

Multiplying by $\mathbf{A}$ again gives $\mathbf{A}^3\mathbf{x} = \lambda^3\mathbf{x}$, and so on. You can use mathematical induction to prove that $\mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$, for each positive integer n.

Eigenvalues of a 2 × 2 probability matrix

An example of a 2×2 probability matrix is

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

In general, a square matrix $\mathbf{A}$ is called a **probability matrix** (or a **stochastic matrix**) if all the entries are non-negative and each column sums to 1.

We will look at some applications of probability matrices in the next section.

Example 6

Find the eigenvalues and the corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

Solution

Finding the eigenvalues

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ implies

$$\det\left(\begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

i.e. $$\det\begin{bmatrix} \frac{2}{3} - \lambda & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} - \lambda \end{bmatrix} = 0$$

Hence the characteristic equation becomes

$$\left(\frac{2}{3} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \frac{1}{4} = 0$$

$$\frac{1}{6} - \frac{11}{12}\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$12\lambda^2 - 11\lambda - 1 = 0$$

$$(12\lambda + 1)(\lambda - 1) = 0$$

The eigenvalues are $\lambda = 1$ and $\lambda = -\frac{1}{12}$.

The eigenvectors with eigenvalue $\lambda = 1$

If $\lambda = 1$, then the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

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This gives the simultaneous equations

$$-\frac{1}{3}x + \frac{3}{4}y = 0 \qquad (1)$$

$$\frac{1}{3}x - \frac{3}{4}y = 0 \qquad (2)$$

Solving the simultaneous equations, we have $x = \frac{9}{4}y$.

Hence the eigenvectors with eigenvalue $\lambda = 1$ are given by $\mathbf{x} = \begin{bmatrix} 9k \\ 4k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

The eigenvectors with eigenvalue $\lambda = -\frac{1}{12}$

If $\lambda = -\frac{1}{12}$, then the equation $\mathbf{Ax} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{12}\begin{bmatrix} x \\ y \end{bmatrix}$$

This gives the equations

$$\frac{3}{4}x + \frac{3}{4}y = 0 \qquad (1)$$

$$\frac{1}{3}x + \frac{1}{3}y = 0 \qquad (2)$$

Solving these equations, we have $x = -y$.

Hence the eigenvectors with eigenvalue $\lambda = -\frac{1}{12}$ are given by $\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

The general case for 2×2 probability matrices

In general, a 2×2 probability matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \quad \text{where } 0 \le a \le 1 \text{ and } 0 \le b \le 1$$

To find expressions for the eigenvalues of $\mathbf{A}$, we consider

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This becomes

$$\det\left(\begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

i.e. $$\det\begin{bmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{bmatrix} = 0$$

So the characteristic equation is

$$(a-\lambda)(1-b-\lambda) - b(1-a) = 0$$

$$\lambda^2 - (1+a-b)\lambda + (a-b) = 0$$

$$(\lambda - 1)((\lambda - (a-b)) = 0$$

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Hence the eigenvalues of $\mathbf{A}$ are $\lambda = 1$ and $\lambda = a - b$.

In the case that $a - b \neq 1$, you can show that the corresponding eigenvectors are given by:

- $\lambda = 1$: $\mathbf{x} = \begin{bmatrix} kb \\ k(1-a) \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$
- $\lambda = a - b$: $\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

We will see an application of this general description in the next section.

Revisiting the definition of eigenvalues and eigenvectors

Complex eigenvalues

We have seen that the characteristic equation of an $n \times n$ matrix $\mathbf{A}$ is a polynomial equation of degree n. In general, such an equation may have solutions over the complex numbers that are not real numbers. It is useful to extend the definition of an eigenvector of $\mathbf{A}$ with eigenvalue λ by allowing λ to be a complex number.

Using technology

The eigenvalues and eigenvectors of a square matrix can be found using a package such as *Mathematica*. Some calculators also have the ability to find eigenvalues and eigenvectors. This is demonstrated in Section B4.

Further reading

Any undergraduate textbook on linear algebra will give more information about the importance of eigenvalues in the study of matrices; for example, see the book *Elementary Linear Algebra: Applications Version* by Howard Anton and Chris Rorres. This book is also a good source of information on applications of matrices.

Exercise B2

Example 3
Example 6

1 For each of the following 2×2 matrices, find:

- **i** the characteristic polynomial
- **ii** the eigenvalues
- **iii** the eigenvectors corresponding to each eigenvalue.

a $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{4} \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

Example 5

2 For each of the following 3×3 matrices, find:

- **i** the characteristic polynomial
- **ii** the eigenvalues
- **iii** the eigenvectors corresponding to each eigenvalue.

a $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 3 & 0 \\ 6 & 7 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}$

B3 Markov chains

Displaying conditional probabilities with matrices

Matrices can be used to display conditional probabilities. For example, suppose that in a certain town the probability that it rains on a Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability that it rains on Tuesday is 0.30.

Let A = the event that it rains on Monday

B = the event that it rains on Tuesday

Then

$$\begin{aligned} \Pr(B \mid A) &= \Pr(\text{rain on Tuesday} \mid \text{rain on Monday}) = 0.83 \\ \Pr(B' \mid A) &= \Pr(\text{fine on Tuesday} \mid \text{rain on Monday}) = 1 - \Pr(B \mid A) = 0.17 \\ \Pr(B \mid A') &= \Pr(\text{rain on Tuesday} \mid \text{fine on Monday}) = 0.30 \\ \Pr(B' \mid A') &= \Pr(\text{fine on Tuesday} \mid \text{fine on Monday}) = 1 - \Pr(B \mid A') = 0.70 \end{aligned}$$

These conditional probabilities can be arranged in a matrix as follows:

$$\begin{bmatrix} \Pr(B \mid A) & \Pr(B \mid A') \\ \Pr(B' \mid A) & \Pr(B' \mid A') \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} A & A' \end{array} \\ \begin{array}{c} B \\ B' \end{array} & \begin{bmatrix} 0.83 & 0.30 \\ 0.17 & 0.70 \end{bmatrix} \end{array}$$

Here, the *row* indicates the event for which the probability is given, and the *column* indicates the event that is being assumed. Note that this is a *probability matrix*: all the entries are non-negative and each column sums to 1.

Finding probabilities using matrix multiplication

We can find $\Pr(B)$ and $\Pr(B')$ by using the law of total probability. Writing the law in terms of this problem gives

$$\Pr(B) = \Pr(B \mid A)\Pr(A) + \Pr(B \mid A')\Pr(A')$$

and $$\Pr(B') = \Pr(B' \mid A)\Pr(A) + \Pr(B' \mid A')\Pr(A')$$

These two equations can be written as a single matrix equation:

$$\begin{bmatrix} \Pr(B) \\ \Pr(B') \end{bmatrix} = \begin{bmatrix} \Pr(B \mid A) & \Pr(B \mid A') \\ \Pr(B' \mid A) & \Pr(B' \mid A') \end{bmatrix} \begin{bmatrix} \Pr(A) \\ \Pr(A') \end{bmatrix}$$

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The probability of rain on Monday has been given as 0.21, so

$$\begin{bmatrix}\Pr(B)\\ \Pr(B')\end{bmatrix} = \begin{bmatrix}0.83 & 0.30\\ 0.17 & 0.70\end{bmatrix}\begin{bmatrix}0.21\\ 0.79\end{bmatrix}$$

$$= \begin{bmatrix}0.4113\\ 0.5887\end{bmatrix}$$

Example 7

Suppose that a netball team has a probability of 0.8 of winning its next game if the team won its last game, and a probability of only 0.5 of winning if it lost its last game. If the probability of the team winning the first game of the season is 0.5, then what is the probability that the team wins the second game?

Solution

Let A = the event that the team wins game 1

B = the event that the team wins game 2

From the given information, we have

$\Pr(B \mid A) = 0.8$ $\quad\quad$ $\Pr(B \mid A') = 0.5$

$\Pr(B' \mid A) = 1 - 0.8 = 0.2$ $\quad\quad$ $\Pr(B' \mid A') = 1 - 0.5 = 0.5$

We are also given that $\Pr(A) = 0.5$ and so $\Pr(A') = 0.5$.

Therefore

$$\begin{bmatrix}\Pr(B)\\ \Pr(B')\end{bmatrix} = \begin{bmatrix}\Pr(B \mid A) & \Pr(B \mid A')\\ \Pr(B' \mid A) & \Pr(B' \mid A')\end{bmatrix}\begin{bmatrix}\Pr(A)\\ \Pr(A')\end{bmatrix}$$

$$= \begin{bmatrix}0.8 & 0.5\\ 0.2 & 0.5\end{bmatrix}\begin{bmatrix}0.5\\ 0.5\end{bmatrix}$$

$$= \begin{bmatrix}0.65\\ 0.35\end{bmatrix}$$

Thus the probability that the team wins game 2 is 0.65.

Continuing with Example 7, we now consider the outcome of the netball team's third game.

Let C be the event that the team wins game 3. Then

$$\begin{bmatrix}\Pr(C)\\ \Pr(C')\end{bmatrix} = \begin{bmatrix}\Pr(C \mid B) & \Pr(C \mid B')\\ \Pr(C' \mid B) & \Pr(C' \mid B')\end{bmatrix}\begin{bmatrix}\Pr(B)\\ \Pr(B')\end{bmatrix}$$

Note that the entries in the two probability matrices

$$\begin{bmatrix}\Pr(B \mid A) & \Pr(B \mid A')\\ \Pr(B' \mid A) & \Pr(B' \mid A')\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}\Pr(C \mid B) & \Pr(C \mid B')\\ \Pr(C' \mid B) & \Pr(C' \mid B')\end{bmatrix}$$

are the same, as they give the probabilities of each outcome given the outcome of the previous game, which are constant.

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Thus

$$\begin{bmatrix} \Pr(C) \\ \Pr(C') \end{bmatrix} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} \Pr(B) \\ \Pr(B') \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} \Pr(A) \\ \Pr(A') \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}^2 \begin{bmatrix} \Pr(A) \\ \Pr(A') \end{bmatrix}$$

This process can be continued, so that the probability of winning games further ahead can be determined by continuing to multiply by the matrix of probabilities.

Example 8

If the probability of the netball team from Example 7 winning the first game of the season is 0.5, then what is the probability that they win the fourth game?

Solution

Let A = the event that the team wins game 1

D = the event that the team wins game 4

Then

$$\begin{bmatrix} \Pr(D) \\ \Pr(D') \end{bmatrix} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}^3 \begin{bmatrix} \Pr(A) \\ \Pr(A') \end{bmatrix} = \begin{bmatrix} 0.722 & 0.695 \\ 0.278 & 0.305 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.7085 \\ 0.2915 \end{bmatrix}$$

Thus the probability that the team wins game 4 is 0.7085.

Probability matrices and Markov chains

There are many situations in which the probability of an individual outcome of a trial depends only on the outcome of the preceding trial. For example, it seems logical that the probability that it rains today depends on whether or not it rained yesterday.

In this section, we consider sequences of repeated trials such that:

- each step of the sequence has the same possible outcomes
- the probability of each possible outcome is conditional only on the outcome in the previous step
- the conditional probabilities for each possible outcome are the same at each step (that is, the same probability matrix is used at each step).

Such a sequence is called a **Markov chain**, named after the Russian mathematician Andrey Markov (1856–1922).

Specifying a Markov chain

The possible outcomes at each step of a Markov chain are usually called **states**. A Markov chain with m states is specified by:

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- an **initial state vector** $\mathbf{S}_0$, which is an $m \times 1$ matrix that gives the probability of each state at step 0
- a **transition matrix** $\mathbf{T}$, which is an $m \times m$ probability matrix that gives the conditional probability of each state given the state at the previous step.

Information about the Markov chain at step n is given in the state vector $\mathbf{S}_n$, which is an $m \times 1$ matrix that gives the probability of each state at step n.

The sequence of state vectors $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots$ can be determined from the initial state vector $\mathbf{S}_0$ and the transition matrix $\mathbf{T}$ as follows:

$$\mathbf{S}_1 = \mathbf{T}\mathbf{S}_0$$
$$\mathbf{S}_2 = \mathbf{T}\mathbf{S}_1 = \mathbf{T}(\mathbf{T}\mathbf{S}_0) = \mathbf{T}^2\mathbf{S}_0$$
$$\mathbf{S}_3 = \mathbf{T}\mathbf{S}_2 = \mathbf{T}(\mathbf{T}^2\mathbf{S}_0) = \mathbf{T}^3\mathbf{S}_0$$

and so on. We can see that

$$\mathbf{S}_n = \mathbf{T}\mathbf{S}_{n-1} = \mathbf{T}^n\mathbf{S}_0$$

Two-state Markov chains

We begin by considering two-state Markov chains ($m = 2$). Suppose that we call the two possible states 'success' and 'failure'. Then

$$\mathbf{S}_n = \begin{bmatrix} \text{probability of observing a success at step } n \\ \text{probability of observing a failure at step } n \end{bmatrix}$$

Example 9

Suppose the probability that a train is late, given that it is late the previous day, is 0.1; while the probability that it is on time, given that it is on time the previous day, is 0.95.

a Find the probability matrix that can be used to represent this information.

b If the train is on time on day 0, find the probability (correct to four decimal places) that:

i the train is late on day 4 **ii** the train is late on day 5.

Solution

a In this case, the two states are 'late' (L) and 'on time' (O). The given information is represented by the probability matrix

$$\mathbf{T} = \begin{array}{c c} & \begin{array}{cc} L & O \end{array} \\ \begin{array}{c} L \\ O \end{array} & \begin{bmatrix} 0.1 & 0.05 \\ 0.9 & 0.95 \end{bmatrix} \end{array}$$

b If the train is on time on day 0, then $\mathbf{S}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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i To find the probabilities associated with day 4, calculate $\mathbf{S}_4 = \mathbf{T}^4\mathbf{S}_0$:

$$\mathbf{S}_4 = \begin{bmatrix} 0.1 & 0.05 \\ 0.9 & 0.95 \end{bmatrix}^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.0526 & 0.0526 \\ 0.9474 & 0.9474 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.0526 \\ 0.9474 \end{bmatrix}$$

Thus the probability of the train being late on day 4 is 0.0526.

ii To find the probabilities associated with day 5, calculate $\mathbf{S}_5 = \mathbf{T}^5\mathbf{S}_0$:

$$\mathbf{S}_5 = \begin{bmatrix} 0.1 & 0.05 \\ 0.9 & 0.95 \end{bmatrix}^5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.0526 & 0.0526 \\ 0.9474 & 0.9474 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.0526 \\ 0.9474 \end{bmatrix}$$

Thus the probability of the train being late on day 5 is also 0.0526.

The steady state of a probability matrix

Interestingly, the two probabilities determined in Example 9 are identical to four decimal places. This seems to indicate that, for this example, after a while the entries of the probability matrix will settle down to constant values. This is called the **steady state** of the probability matrix. The associated state vector is called the **steady state vector**.

As another example, consider the probability matrix from Examples 7 and 8:

$$\mathbf{T} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

Using a calculator, it is easy to compute successive powers of $\mathbf{T}$:

$$\mathbf{T}^2 = \begin{bmatrix} 0.74 & 0.65 \\ 0.26 & 0.35 \end{bmatrix}, \quad \mathbf{T}^3 = \begin{bmatrix} 0.722 & 0.695 \\ 0.278 & 0.305 \end{bmatrix}, \quad \mathbf{T}^4 = \begin{bmatrix} 0.7166 & 0.7085 \\ 0.2834 & 0.2915 \end{bmatrix},$$

$$\mathbf{T}^5 = \begin{bmatrix} 0.7150 & 0.7126 \\ 0.2850 & 0.2875 \end{bmatrix}, \quad \mathbf{T}^6 = \begin{bmatrix} 0.7145 & 0.7138 \\ 0.2855 & 0.2862 \end{bmatrix}, \quad \mathbf{T}^7 = \begin{bmatrix} 0.7143 & 0.7141 \\ 0.2857 & 0.2859 \end{bmatrix},$$

$$\mathbf{T}^8 = \begin{bmatrix} 0.7143 & 0.7142 \\ 0.2857 & 0.2858 \end{bmatrix}, \quad \mathbf{T}^9 = \begin{bmatrix} 0.7143 & 0.7143 \\ 0.2857 & 0.2857 \end{bmatrix}, \quad \mathbf{T}^{10} = \begin{bmatrix} 0.7143 & 0.7143 \\ 0.2857 & 0.2857 \end{bmatrix}$$

and so on.

It can be seen that, by step 9, the probability matrix has settled into a steady state (to four decimal places). So the sequence of state vectors $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \dots$ will approach a limiting state vector, which we denote by $\mathbf{S}^*$.

Example 10

Consider the probability matrix $\mathbf{T} = \begin{bmatrix} 0.73 & 0.35 \\ 0.27 & 0.65 \end{bmatrix}$.

a Find the steady state matrix of $\mathbf{T}$.

b Find the steady state vector if the initial state is:

i $\mathbf{S}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **ii** $\mathbf{S}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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Solution

a Try computing $\mathbf{T}^{10}$ and $\mathbf{T}^{20}$:

$$\mathbf{T}^{10} = \begin{bmatrix} 0.73 & 0.35 \\ 0.27 & 0.65 \end{bmatrix}^{10} = \begin{bmatrix} 0.5645 & 0.5645 \\ 0.4355 & 0.4355 \end{bmatrix}$$

$$\mathbf{T}^{20} = \begin{bmatrix} 0.73 & 0.35 \\ 0.27 & 0.65 \end{bmatrix}^{20} = \begin{bmatrix} 0.5645 & 0.5645 \\ 0.4355 & 0.4355 \end{bmatrix}$$

The probabilities are no longer changing. So, correct to four decimal places, the steady state of the probability matrix is

$$\mathbf{T}_s = \begin{bmatrix} 0.5645 & 0.5645 \\ 0.4355 & 0.4355 \end{bmatrix}$$

b **i** $\mathbf{S}^* = \mathbf{T}_s \mathbf{S}_0 = \begin{bmatrix} 0.5645 & 0.5645 \\ 0.4355 & 0.4355 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5645 \\ 0.4355 \end{bmatrix}$

ii $\mathbf{S}^* = \mathbf{T}_s \mathbf{S}_0 = \begin{bmatrix} 0.5645 & 0.5645 \\ 0.4355 & 0.4355 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5645 \\ 0.4355 \end{bmatrix}$

Using eigenvalues and eigenvectors to understand steady states

We have seen several examples where the sequence of state vectors of a Markov chain approaches a steady state vector, $\mathbf{S}^*$. This state vector must satisfy the equation

$$\mathbf{TS}^* = \mathbf{S}^*$$

So we see that the steady state vector is an eigenvector of $\mathbf{T}$ with eigenvalue $\lambda = 1$.

In the previous section, we found expressions for the eigenvalues and eigenvectors of a general 2×2 probability matrix. We can use this to understand steady states for two-state Markov chains.

Steady state of a 2×2 probability matrix

Consider the general 2×2 probability matrix

$$\mathbf{T} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \quad \text{where } 0 \le a \le 1 \text{ and } 0 \le b \le 1$$

We have seen that the eigenvalues of $\mathbf{T}$ are $\lambda = 1$ and $\lambda = a - b$. We have also seen that, if $a - b \neq 1$, then the corresponding eigenvectors are given by:

- $\lambda = 1$: $\mathbf{x} = \begin{bmatrix} kb \\ k(1-a) \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$
- $\lambda = a - b$: $\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

The steady state vector, $\mathbf{S}^*$, is an eigenvector of $\mathbf{T}$ with eigenvalue $\lambda = 1$. Therefore

$$\mathbf{S}^* = \begin{bmatrix} kb \\ k(1-a) \end{bmatrix}$$

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for some $k \in \mathbb{R} \setminus \{0\}$. The entries of a state vector must sum to 1, as they represent the probabilities of the states. So we have

$$\begin{aligned} kb + k(1-a) &= 1 \\ k(b+1-a) &= 1 \\ \therefore \quad k &= \frac{1}{1-a+b} \end{aligned}$$

Hence we have shown that, in the case $a - b \neq 1$, there is a unique steady state vector

$$\mathbf{S}^* = \begin{bmatrix} kb \\ k(1-a) \end{bmatrix}, \quad \text{where } k = \frac{1}{1-a+b}$$

In Example 10, we have $a = 0.73$ and $b = 0.35$. So this formula gives

$$\mathbf{S}^* = \frac{1}{1-0.73+0.35}\begin{bmatrix} 0.35 \\ 1-0.73 \end{bmatrix} = \begin{bmatrix} 0.5645 \\ 0.4355 \end{bmatrix}$$

as found in Example 10.

Approaching the steady state for a 2 × 2 probability matrix

We are now interested in whether the sequence of state vectors $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \dots$ will actually approach the steady state vector $\mathbf{S}^*$. Using your geometric knowledge of vectors, we can show that this happens in nearly all cases.

Theorem

Consider a two-state Markov chain with transition matrix

$$\mathbf{T} = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} \quad \text{where } 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1$$

If $\mathbf{T}$ satisfies $-1 < a - b < 1$, then the sequence of state vectors $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \dots$ approaches the steady state vector $\mathbf{S}^*$.

Proof In general, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors in two-dimensional space that are not parallel, then any other vector in two-dimensional space can be represented as a 'linear combination' of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Now consider $\mathbf{x}_1 = \begin{bmatrix} b \\ 1-a \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

From the previous section, we know that $\mathbf{x}_1$ is an eigenvector of $\mathbf{T}$ with eigenvalue $\lambda_1 = 1$, and that $\mathbf{x}_2$ is an eigenvector of $\mathbf{T}$ with eigenvalue $\lambda_2 = a - b$.

We are assuming that $a - b < 1$ and therefore $1 - a \neq -b$. It follows that $\mathbf{x}_1$ and $\mathbf{x}_2$ represent non-zero vectors in two-dimensional space that are not parallel. So the initial state vector $\mathbf{S}_0$ can be written as a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$. That is,

$$\mathbf{S}_0 = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \quad \text{for real constants } \alpha \text{ and } \beta$$

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We now have

$$\begin{aligned}\mathbf{S}_n &= \mathbf{T}^n\mathbf{S}_0 \\ &= \mathbf{T}^n(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) \\ &= \alpha\mathbf{T}^n\mathbf{x}_1 + \beta\mathbf{T}^n\mathbf{x}_2 \\ &= \alpha\lambda_1^n\mathbf{x}_1 + \beta\lambda_2^n\mathbf{x}_2 && \text{(as } \mathbf{x}_1 \text{ and } \mathbf{x}_2 \text{ are eigenvectors)} \\ &= \alpha\mathbf{x}_1 + \beta(a-b)^n\mathbf{x}_2 && \text{(as } \lambda_1 = 1 \text{ and } \lambda_2 = a - b\text{)}\end{aligned}$$

We are assuming that $-1 < a - b < 1$, and therefore

$$\lim_{n\to\infty} (a-b)^n = 0$$

It follows that

$$\lim_{n\to\infty} \mathbf{S}_n = \alpha\mathbf{x}_1$$

Note that $\alpha\mathbf{x}_1$ is a steady state vector, since $\mathbf{T}(\alpha\mathbf{x}_1) = \alpha\mathbf{T}\mathbf{x}_1 = \alpha\mathbf{x}_1$. We have seen that there is a unique steady state vector, $\mathbf{S}^*$. Hence

$$\lim_{n\to\infty} \mathbf{S}_n = \mathbf{S}^*$$

which completes the proof of the theorem.

Not all two-state Markov chains approach a steady state. For example, if we take

$$\mathbf{S}_0 = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

then the sequence of state vectors oscillates as follows:

$$\mathbf{S}_0 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{S}_1 = \begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{S}_3 = \begin{bmatrix}0\\1\end{bmatrix}, \quad \mathbf{S}_4 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \dots$$

Markov chains with more than two states

An m-state Markov chain works in the same way as a two-state Markov chain, but now the transition matrix $\mathbf{T}$ is an $m \times m$ probability matrix and each state vector $\mathbf{S}_n$ is an $m \times 1$ matrix.

The next example involves a three-state Markov chain ($m = 3$).

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Example 11

Suppose that a soccer team can win, draw or lose each match, and that the probabilities for the outcome of one match depend only on the outcome of the previous match. These conditional probabilities are given by the following matrix:

$$\begin{array}{c} \\ W \\ D \\ L \end{array}\begin{array}{c} \begin{array}{ccc} W & D & L \end{array} \\ \begin{bmatrix} 0.7 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{bmatrix} \end{array} \qquad (W = \text{win}, D = \text{draw}, L = \text{loss})$$

a What is the probability that the team wins game 3 of the season if they won the last game of the previous season?

b What is the probability that the team wins game 4 of the season if they drew the last game of the previous season?

Solution

a If the team won game 0, then $\mathbf{S}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and

$$\mathbf{S}_3 = \begin{bmatrix} 0.7 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.614 & 0.606 & 0.594 \\ 0.262 & 0.270 & 0.274 \\ 0.124 & 0.124 & 0.132 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.614 \\ 0.262 \\ 0.124 \end{bmatrix}$$

Thus the probability that the team wins game 3 if they won game 0 is 0.614.

b If the team drew game 0, then $\mathbf{S}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and

$$\mathbf{S}_4 = \begin{bmatrix} 0.7 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}^4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6104 & 0.6088 & 0.6056 \\ 0.2648 & 0.2664 & 0.2680 \\ 0.1248 & 0.1248 & 0.1264 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6088 \\ 0.2664 \\ 0.1248 \end{bmatrix}$$

Thus the probability that the team wins game 4 if they drew game 0 is 0.6088.

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Approaching the steady state for an $m \times m$ probability matrix

We can apply the techniques of eigenvalues and eigenvectors to find the steady state vector for the probability matrix from Example 11:

$$\mathbf{T} = \begin{bmatrix} \frac{7}{10} & \frac{1}{2} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix}$$

Note that we have rewritten the entries of the probability matrix as fractions in order to find the characteristic equation in a good form.

The characteristic equation of $\mathbf{T}$ is

$$-\lambda^3 + \frac{7\lambda^2}{5} - \frac{11\lambda}{25} + \frac{1}{25} = 0$$

i.e. $$(\lambda - 1)(5\lambda - 1)^2 = 0$$

The eigenvectors of $\mathbf{T}$ with eigenvalue $\lambda = 1$ are given by

$$\mathbf{x} = \begin{bmatrix} 39k \\ 17k \\ 8k \end{bmatrix}, \quad \text{for } k \in \mathbb{R} \setminus \{0\}$$

Since the entries of a state vector must sum to 1, we want $k = \frac{1}{64}$. Therefore the unique steady state vector is

$$\mathbf{S}^* = \begin{bmatrix} \frac{39}{64} \\ \frac{17}{64} \\ \frac{1}{8} \end{bmatrix} \approx \begin{bmatrix} 0.6094 \\ 0.2656 \\ 0.125 \end{bmatrix}$$

By experimentation, you can see that this steady state vector is rapidly approached starting from any initial state.

We include the following theorem without proof. It indicates that there are many situations where these techniques can be applied.

- **Definition**

  A probability matrix $\mathbf{A}$ is said to be **regular** if there exists a positive integer n such that the matrix $\mathbf{A}^n$ has all positive entries.

- **Theorem**

  If the transition matrix $\mathbf{T}$ of a Markov chain is regular, then the sequence of state vectors $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots$ approaches the unique steady state vector $\mathbf{S}^*$.

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Exercise B3

Example 7

1 In a certain country, the probability of a child being female is 0.6 if the preceding child in the family is female, and 0.45 if the preceding child is male. Let F_i be the event that the ith child is female, and let M_i be the event that the ith child is male. We can write

$$\begin{bmatrix} \Pr(F_{i+1}) \\ \Pr(M_{i+1}) \end{bmatrix} = \mathbf{T} \begin{bmatrix} \Pr(F_i) \\ \Pr(M_i) \end{bmatrix}$$

where $\mathbf{T}$ is a 2×2 matrix.

a Write down the matrix $\mathbf{T}$.

b If the probability of a first child being female is 0.52, find the probability that the second child will be female.

Example 8

2 Suppose that the outcomes in a series of cricket matches between Australia and England are such that

$$\Pr(A_1) = 0.6 \qquad \Pr(A_{i+1} \mid A_i) = 0.7 \qquad \Pr(A_{i+1} \mid E_i) = 0.5$$
$$\Pr(E_1) = 0.4 \qquad \Pr(E_{i+1} \mid A_i) = 0.3 \qquad \Pr(E_{i+1} \mid E_i) = 0.5$$

where A_i represents the event that Australia wins the ith match and E_i represents the event that England wins the ith match.

a Write down a matrix equation relating $\Pr(A_{i+1})$ and $\Pr(E_{i+1})$ to $\Pr(A_1)$ and $\Pr(E_1)$.

b Use this matrix equation to determine the probability that Australia wins:

i the second match **ii** the third match **iii** the fifth match.

Example 9

3 A factory has a large number of machines which can be in one of two states, operating or broken down. The probability that an operating machine breaks down by the end of the day is 0.04, and the probability that a machine that has broken down is repaired by the end of the day is 0.98.

a Find the probability matrix that can be used to represent this information.

b Find the percentage of machines which are operating at the end of day 3 if 8% of machines are broken down initially (day 0).

Example 10

4 Consider the probability matrix $\mathbf{T} = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$ and initial state vector $\mathbf{S}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Calculate $\mathbf{S}_n = \mathbf{T}^n \mathbf{S}_0$ for $n = 10, 15, 20, 25$ to find the steady state vector.

5 Consider the probability matrix $\mathbf{T} = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}$ and initial state vector $\mathbf{S}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Calculate $\mathbf{S}_n = \mathbf{T}^n \mathbf{S}_0$ for $n = 10, 15, 20, 25$ to find the steady state vector.

6 A large company has 1640 employees, 60% of whom currently work full-time and 40% of whom currently work part-time. Every year 20% of full-time workers move to part-time work, and 14% of part-time workers move to full-time work.

a Determine the probability matrix $\mathbf{T}$ that can be used to represent this information.

b Find the estimated number of people working full-time in the long term, assuming that the total number of employees remains constant.

7 A tennis player can serve to the forehand or the backhand of his opponent. If he serves to the forehand, there is a 75% chance that his next serve will be to the backhand. If he serves to the backhand, there is a 65% chance that his next serve will be to the forehand.

a Find the probability matrix that can be used to represent this information.

b What is the probability that the tennis player will serve the third serve to the backhand of his opponent:

i if the first serve was to the forehand

ii if the first serve was to the backhand?

c What is the steady state probability that the player will serve to the forehand of his opponent?

8 Suppose that there are two doctors in a country town, Dr Black and Dr White. Each year, 13% of Dr Black's patients switch to Dr White, while 19% of Dr White's patients switch to Dr Black. Suppose that initially 33% of all patients go to Dr Black and 67% go to Dr White. Find the percentage of patients who will eventually be attending each doctor if this pattern continues indefinitely.

9 A Markov chain has probability matrix $\mathbf{T} = \begin{bmatrix} 0.15 & 0.45 & 0.2 \\ 0.35 & 0.25 & 0.45 \\ 0.5 & 0.3 & 0.35 \end{bmatrix}$.

a For each of the following initial state vectors, find an approximate steady state vector $\mathbf{S}^*$ by calculating $\mathbf{S}_n = \mathbf{T}^n\mathbf{S}_0$ for $n = 10, 15, 20, 25$. Give your answers to four decimal places.

i $\mathbf{S}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **ii** $\mathbf{S}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **iii** $\mathbf{S}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

b Compare your answers to parts **i**, **ii** and **iii**.

10 Suppose that the trees in a forest are classified into three age groups: young trees (0–15 years), middle-aged trees (16–30 years) and old trees (more than 30 years). A time period is 15 years, and it is assumed that in each time period:

- 10% of young trees, 20% of middle-aged trees and 40% of old trees die
- surviving trees enter into the next age group; old trees remain old
- dead trees are replaced by young trees.

This information is summarised in the following probability matrix:

$$\begin{array}{c} \\ Y \\ M \\ O \end{array} \begin{array}{c} \begin{array}{ccc} Y & M & O \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.4 \\ 0.9 & 0 & 0 \\ 0 & 0.8 & 0.6 \end{bmatrix} \end{array}$$

Suppose that 20 000 young trees are planted in the forest. What is the eventual age distribution of the trees? Give your answers to the nearest whole number.

11 The temperature on any day can be described as 'cool', 'mild', 'warm' or 'hot'. The likelihood of a day being in any one of these categories is dependent on the temperature the day before. The conditional probabilities associated with each state are given by the following probability matrix:

$$\begin{array}{c} \\ C \\ M \\ W \\ H \end{array} \begin{array}{c} \begin{array}{cccc} C & M & W & H \end{array} \\ \begin{bmatrix} 0.4 & 0.2 & 0.2 & 0.3 \\ 0.3 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

What percentages of days are expected to be cool, mild, warm or hot in the long term, correct to one decimal place?

B4 Further approaches to Leslie matrices

In Chapter 9, we used Leslie matrices to model changes in the sizes of different age groups within a population. In this section, we will use our understanding of eigenvalues and eigenvectors to help in our investigations with Leslie matrices.

Review of Leslie matrices

Recall that an $n \times n$ **Leslie matrix** has the form

$$\mathbf{L} = \begin{bmatrix} B_1 & B_2 & B_3 & \cdots & B_{n-1} & B_n \\ S_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & S_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & S_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & S_{n-1} & 0 \end{bmatrix}$$

where:

- n is the number of age groups being considered
- S_m is the proportion of the population in age group m that progress to age group $m+1$
- B_m is the average number of female offspring from a mother in age group m during one time period.

The population matrix $\mathbf{P}_k$ is an $n \times 1$ matrix representing the size of each age group after the kth time period. This is calculated using

$$\mathbf{P}_k = \mathbf{L}\mathbf{P}_{k-1} = \mathbf{L}^k\mathbf{P}_0$$

Note: For this section, we add two basic assumptions to the definition of a Leslie matrix. We assume that each S_m is non-zero and that at least one B_m is non-zero. (This ensures that some offspring are born and that some survive to age group n.)

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Eigenvalues and eigenvectors of Leslie matrices

We start by considering a general 3×3 Leslie matrix

$$\mathbf{L} = \begin{bmatrix} B_1 & B_2 & B_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix}$$

The characteristic equation

The characteristic equation $\det(\mathbf{L} - \lambda\mathbf{I}) = 0$ becomes

$$\det\begin{bmatrix} B_1 - \lambda & B_2 & B_3 \\ S_1 & -\lambda & 0 \\ 0 & S_2 & -\lambda \end{bmatrix} = 0$$

Expanding from the first row gives

$$(B_1 - \lambda)\lambda^2 + B_2S_1\lambda + B_3S_1S_2 = 0$$

i.e. $$-\lambda^3 + B_1\lambda^2 + B_2S_1\lambda + B_3S_1S_2 = 0$$

The unique positive eigenvalue

The eigenvalues of $\mathbf{L}$ are the solutions of the characteristic equation:

$$-\lambda^3 + B_1\lambda^2 + B_2S_1\lambda + B_3S_1S_2 = 0$$

For positive values of λ, we can write this equation as

$$-1 + \frac{B_1}{\lambda} + \frac{B_2S_1}{\lambda^2} + \frac{B_3S_1S_2}{\lambda^3} = 0$$

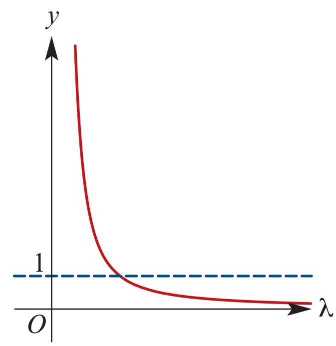
i.e. $$\frac{B_1}{\lambda} + \frac{B_2S_1}{\lambda^2} + \frac{B_3S_1S_2}{\lambda^3} = 1$$

Now consider the graph of

$$y = \frac{B_1}{\lambda} + \frac{B_2S_1}{\lambda^2} + \frac{B_3S_1S_2}{\lambda^3}$$

for positive values of λ.

The y-values are always positive, and decrease from very large to very small as $\lambda \to \infty$. Hence there is a unique positive solution to the equation $y = 1$. This means that the characteristic equation of $\mathbf{L}$ has only one positive solution.



Hence every 3×3 Leslie matrix $\mathbf{L}$ has a unique positive eigenvalue. This result can be generalised to $n \times n$ Leslie matrices:

> Every Leslie matrix has a unique positive eigenvalue.

We will see that, in many cases, the unique positive eigenvalue of a Leslie matrix gives the eventual rate of increase or decrease of the population.

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Example 12

Consider the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{P}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & \frac{6}{5} & \frac{12}{5} \\ \frac{3}{5} & 0 & 0 \\ 0 & \frac{3}{5} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{bmatrix} 20 \\ 25 \\ 22 \end{bmatrix}$$

a Find the unique positive eigenvalue of the matrix $\mathbf{L}$.

b Describe the family of eigenvectors corresponding to this eigenvalue.

c Calculate $\mathbf{P}_{20}$ and $\mathbf{P}_{21}$. Hence find the rate of increase in the population during the 21st time period and the approximate ratio of the sizes of the three age groups.

d Find the complex eigenvalues of the matrix $\mathbf{L}$.

Solution

a The characteristic equation is

$$\det(\mathbf{L} - \lambda\mathbf{I}) = 0$$

i.e. $$\det\begin{bmatrix} -\lambda & \frac{6}{5} & \frac{12}{5} \\ \frac{3}{5} & -\lambda & 0 \\ 0 & \frac{3}{5} & -\lambda \end{bmatrix} = 0$$

This gives

$$125\lambda^3 - 90\lambda - 108 = 0$$

$$(5\lambda - 6)(25\lambda^2 + 30\lambda + 18) = 0$$

$$\therefore \quad \lambda = \frac{6}{5} \quad \text{or} \quad 25\lambda^2 + 30\lambda + 18 = 0$$

The quadratic equation has no real solutions, since $\Delta = 30^2 - 4 \times 25 \times 18 < 0$.

Hence the unique positive eigenvalue is $\lambda = \dfrac{6}{5}$.

b If $\lambda = \dfrac{6}{5}$, then the equation $\mathbf{L}\mathbf{x} = \lambda\mathbf{x}$ becomes

$$\begin{bmatrix} 0 & \frac{6}{5} & \frac{12}{5} \\ \frac{3}{5} & 0 & 0 \\ 0 & \frac{3}{5} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{6}{5}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This gives

$$\tfrac{6}{5}y + \tfrac{12}{5}z = \tfrac{6}{5}x \qquad (1)$$

$$\tfrac{3}{5}x = \tfrac{6}{5}y \qquad (2)$$

$$\tfrac{3}{5}y = \tfrac{6}{5}z \qquad (3)$$

From these equations, we have $y = 2z$ and $x = 4z$.

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Hence the eigenvectors with eigenvalue $\lambda = \frac{6}{5}$ are given by $\mathbf{x} = \begin{bmatrix} 4k \\ 2k \\ k \end{bmatrix}$, for $k \in \mathbb{R} \setminus \{0\}$.

c ▪ $\mathbf{P}_{20} = \mathbf{L}^{20}\mathbf{P}_0 = \begin{bmatrix} 1749.15 \\ 874.01 \\ 436.65 \end{bmatrix}$; the total population is 3059.82

▪ $\mathbf{P}_{21} = \mathbf{L}^{21}\mathbf{P}_0 = \begin{bmatrix} 2096.78 \\ 1049.49 \\ 524.41 \end{bmatrix}$; the total population is 3670.68

Therefore

$$\text{rate of increase} = \frac{3670.68}{3059.82} = 1.1996 \approx \frac{6}{5}$$

The sizes of the three age groups are approximately in the ratio 4 : 2 : 1 by this stage.

d Solve the quadratic equation $25\lambda^2 + 30\lambda + 18 = 0$ from part **a**:

$$\begin{aligned} \lambda &= \frac{-30 \pm \sqrt{30^2 - 4 \times 25 \times 18}}{50} \\ &= \frac{-30 \pm \sqrt{-900}}{50} \\ &= \frac{-3 \pm \sqrt{-9}}{5} \\ &= \frac{3}{5}(-1 \pm i) \end{aligned}$$

In the previous example, the eigenvalues of the Leslie matrix **L** are

$$\lambda_1 = \frac{6}{5}, \quad \lambda_2 = \frac{3}{5}(-1 + i) \quad \text{and} \quad \lambda_3 = \frac{3}{5}(-1 - i)$$

Note that the unique positive eigenvalue λ_1 has the greatest modulus, since

$$|\lambda_2| = |\lambda_3| = \frac{3\sqrt{2}}{5} < \frac{6}{5}$$

Dominant eigenvalues

We have seen that each Leslie matrix **L** has a unique positive eigenvalue λ_1. It can be shown that $\lambda_1 \geq |\lambda_i|$, for any other eigenvalue λ_i of **L**. (Here we consider the modulus of λ_i, since eigenvalues can be complex numbers.)

We say that λ_1 is the **dominant eigenvalue** if we have

$$\lambda_1 > |\lambda_i|$$

for any other eigenvalue λ_i of **L**. This does happen in many cases. The following theorem gives a very useful sufficient condition; its proof is beyond the scope of this course.

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Theorem

If two successive entries B_i and B_{i+1} in the first row of a Leslie matrix $\mathbf{L}$ are non-zero, then the unique positive eigenvalue λ_1 of $\mathbf{L}$ is dominant.

The following theorem indicates the use of the dominant eigenvalue λ_1; its proof is also beyond the scope of this course.

Theorem

Let $\mathbf{L}$ be a Leslie matrix such that the unique positive eigenvalue λ_1 is dominant. Then

$$\mathbf{P}_{k+1} \approx \lambda_1 \mathbf{P}_k \quad \text{for large } k$$

So the proportion of the population in each age group stabilises over time.

Moreover, the eventual age-group proportions can be found from the proportions of the entries in any eigenvector $\mathbf{x}_1$ with eigenvalue λ_1.

It is clear from this theorem that a dominant eigenvalue λ_1 gives valuable information about the behaviour of the population over time:

- The population is eventually increasing if $\lambda_1 > 1$.
- The population is eventually decreasing if $\lambda_1 < 1$.
- The population eventually stabilises if $\lambda_1 = 1$.

Example 13

Consider the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{P}_0$:

$$\mathbf{L} = \begin{bmatrix} 1.6 & 1.5 & 0.25 & 0.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The dominant eigenvalue of $\mathbf{L}$ is $\lambda_1 = 2.1742$, correct to four decimal places.

a Calculate $\mathbf{P}_5$ and $\mathbf{P}_6$.

b Show that $\mathbf{P}_6 \approx \lambda_1 \mathbf{P}_5$. Experiment with larger values of k to show that $\mathbf{P}_{k+1} \approx \lambda_1 \mathbf{P}_k$.

c Show that $\mathbf{P}_5 \approx \lambda_1^5 \mathbf{x}_1$, for some eigenvector $\mathbf{x}_1$ of $\mathbf{L}$ with eigenvalue λ_1. Experiment with larger values of k.

Solution

a $\mathbf{P}_5 = \mathbf{L}^5 \mathbf{P}_0 = \begin{bmatrix} 71.8589 \\ 26.412 \\ 6.102 \\ 0.825 \end{bmatrix}$ and $\mathbf{P}_6 = \mathbf{L}^6 \mathbf{P}_0 = \begin{bmatrix} 156.2002 \\ 57.4871 \\ 13.206 \\ 1.8306 \end{bmatrix}$

b We calculate

$$\lambda_1 \mathbf{P}_5 = 2.1742 \begin{bmatrix} 71.8589 \\ 26.412 \\ 6.102 \\ 0.825 \end{bmatrix} = \begin{bmatrix} 156.2356 \\ 57.4250 \\ 13.2670 \\ 1.7937 \end{bmatrix} \approx \mathbf{P}_6$$

For larger values of k, the approximation $\mathbf{P}_{k+1} \approx \lambda_1 \mathbf{P}_k$ gets better.

c Dividing each entry of $\mathbf{P}_5$ by λ_1^5, we can write

$$\mathbf{P}_5 = \lambda_1^5 \mathbf{x}_1, \quad \text{where } \mathbf{x}_1 = \begin{bmatrix} 1.4791 \\ 0.5436 \\ 0.1256 \\ 0.0170 \end{bmatrix}$$

We calculate

$$\mathbf{L}\mathbf{x}_1 = \begin{bmatrix} 3.215 \\ 1.183 \\ 0.272 \\ 0.038 \end{bmatrix} \quad \text{and} \quad \lambda_1 \mathbf{x}_1 = \begin{bmatrix} 3.216 \\ 1.182 \\ 0.273 \\ 0.037 \end{bmatrix}$$

We have $\mathbf{L}\mathbf{x}_1 \approx \lambda_1 \mathbf{x}_1$. So $\mathbf{x}_1$ approximates an eigenvector of $\mathbf{L}$ with eigenvalue λ_1. By using larger values of k, we obtain better approximations to this eigenvector.

Using technology

Finding eigenvalues and eigenvectors is computationally difficult, so it is helpful to be able to use a mathematical package or an appropriate calculator.

Here we demonstrate the use of *Mathematica* and the TI-Nspire.

Example 14

Consider again the Leslie matrix from Example 13:

$$\mathbf{L} = \begin{bmatrix} 1.6 & 1.5 & 0.25 & 0.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}$$

Use technology to find the dominant eigenvalue λ_1 and a corresponding eigenvector.

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Solution using Mathematica

Eigenvalues

Input the command:

$$\textbf{Eigenvalues}\left[\begin{pmatrix} 1.6 & 1.5 & 0.25 & 0.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{pmatrix}\right]$$

Rounded to four decimal places, the output is

$$\{2.1742 + 0.i,\ -0.4992 + 0.i,\ -0.0375 + 0.0982i,\ -0.0375 - 0.0982i\}$$

Hence $\lambda_1 = 2.1742$ is the dominant eigenvalue.

Eigenvectors

Input the command:

$$\textbf{Eigenvectors}\left[\begin{pmatrix} 1.6 & 1.5 & 0.25 & 0.1 \\ 0.8 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{pmatrix}\right]$$

Rounded to four decimal places, the output is

$$\begin{pmatrix} 0.9355 + 0.i & 0.3442 + 0.i & 0.0792 + 0.i & 0.0109 + 0.i \\ -0.3759 + 0.i & 0.6025 + 0.i & -0.6035 + 0.i & 0.3627 + 0.i \\ -0.0081 + 0.0042i & 0.0517 + 0.0463i & 0.1178 - 0.3082i & -0.9414 + 0.i \\ -0.0081 - 0.0042i & 0.0517 - 0.0463i & 0.1178 + 0.3082i & -0.9414 + 0.i \end{pmatrix}$$

The first row gives an eigenvector corresponding to λ_1:

$$\mathbf{x}_1 = \begin{bmatrix} 0.9355 \\ 0.3442 \\ 0.0792 \\ 0.0109 \end{bmatrix}$$

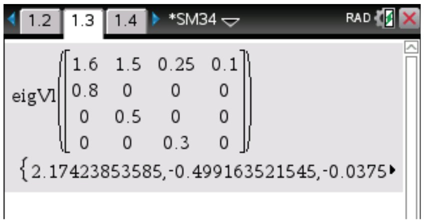
(The other eigenvectors corresponding to λ_1 are the non-zero scalar multiples of $\mathbf{x}_1$.)

Using the TI-Nspire CX non-CAS

Eigenvalues

Use menu > **Matrix & Vector** > **Advanced** > **Eigenvalues** as shown.

Hence $\lambda_1 = 2.1742$ is the dominant eigenvalue.



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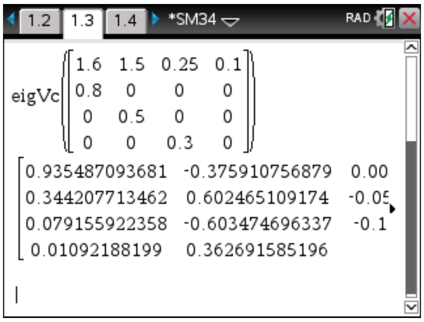
■ Eigenvectors

Use (menu) > **Matrix & Vector** > **Advanced** > **Eigenvectors** as shown.

The first column gives an eigenvector corresponding to λ_1:

$$\mathbf{x}_1 = \begin{bmatrix} 0.9355 \\ 0.3442 \\ 0.0792 \\ 0.0109 \end{bmatrix}$$

(The other eigenvectors corresponding to λ_1 are the non-zero scalar multiples of $\mathbf{x}_1$.)



Exercise B4

Example 13
Example 14

1 Consider the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{P}_0$:

$$\mathbf{L} = \begin{bmatrix} 0.5 & 1 & 0.25 & 0.1 \\ 0.7 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

a Using technology, find the dominant eigenvalue λ_1 of $\mathbf{L}$ and a corresponding eigenvector $\mathbf{x}_1$.

b Calculate $\mathbf{P}_{20}$ and $\mathbf{P}_{21}$.

c Show that $\mathbf{P}_{21} \approx \lambda_1 \mathbf{P}_{20}$.

d Show that there is a constant c such that $\mathbf{P}_{20} \approx \lambda_1^{20}(c\mathbf{x}_1)$. Experiment with larger values of k to show that $\mathbf{P}_k \approx \lambda_1^k(c\mathbf{x}_1)$.

2 Consider the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{P}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & 3 & 8 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}$$

a The rate of increase in the population over each time period eventually stabilises. Find this long-term growth rate by comparing $\mathbf{P}_k$ and $\mathbf{P}_{k+1}$ for large k.

b The percentage of the population in each age group eventually stabilises. Find these long-term percentages by computing $\mathbf{P}_k$ for large k.

c Using your answers to parts **a** and **b**, write down the dominant eigenvalue λ_1 of $\mathbf{L}$ and a corresponding eigenvector $\mathbf{x}_1$.

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3 Repeat Question 2 for the following Leslie matrix $\mathbf{L}$ and initial population matrix $\mathbf{P}_0$:

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 3 & 8 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_0 = \begin{bmatrix} 100 \\ 0 \\ 100 \\ 0 \end{bmatrix}$$

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B5 Leontief matrices

In this section, we apply matrices to modelling economies. In particular, we consider the problem of balancing the input and output of interdependent industries. The models we use were developed by Wassily Leontief, who won the Nobel Prize for Economics in 1973.

A normal economy involves a large number of industries, such as agriculture, mining, energy, manufacturing, construction, transport, and so on. To introduce Leontief models, we will start with a very simple economy consisting of two industries.

Leontief open model

Consider two industries that use each other's output and also sell some of their output elsewhere. We can use the coal and steel industries, as the steel industry needs coal for its furnaces and the coal industry needs steel for structures and transport in its mines. The two industries also have external demand for their output (e.g. for export).

Consumption

The coal and steel industries use each other's output. Suppose that:

- to produce one unit of coal, the coal industry uses 0.1 units of coal and 0.4 units of steel
- to produce one unit of steel, the steel industry uses 0.3 units of coal and 0.2 units of steel.

We represent this information in the **consumption matrix**:

$$\mathbf{A} = \begin{array}{c} \\ C \\ S \end{array}\begin{array}{c} \begin{array}{cc} C & S \end{array} \\ \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{bmatrix} \end{array} \qquad (C = \text{coal industry},\ S = \text{steel industry})$$

Here entry a_{ij} is the amount of product i that is needed to produce one unit of product j.

Production and demand

We want to determine the total amount of coal and steel that should be produced to meet a given external demand. (Since the coal and steel industries use each other's output, the total production needs to be more than the external demand.)

We define the **production vector**, **P**, and **demand vector**, **Y**, by

$$\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \text{units of coal produced} \\ \text{units of steel produced} \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \text{units of coal required externally} \\ \text{units of steel required externally} \end{bmatrix}$$

For production to exactly meet demand, we must have

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We can write this equation as

$$\mathbf{P} - \mathbf{AP} = \mathbf{Y}$$

$$\therefore \quad (\mathbf{I} - \mathbf{A})\mathbf{P} = \mathbf{Y}$$

This is called the **Leontief equation**, and the matrix **I − A** is called the **Leontief matrix**.

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If the inverse matrix $(\mathbf{I} - \mathbf{A})^{-1}$ exists and has non-negative entries, then we can write

$$\mathbf{P} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{Y}$$

For this example, we have

$$(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{3}{2} \end{bmatrix}$$

and therefore we can determine the total production using

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Note that, in this example, it is possible for production to exactly meet any external demand.

Example 15

Consider an open economy with four sectors – agriculture (A), construction (C), energy (E) and manufacturing (M) – such that the consumption matrix, $\mathbf{A}$, and demand vector, $\mathbf{Y}$, are given by

$$\mathbf{A} = \begin{array}{c} \\ A \\ C \\ E \\ M \end{array} \begin{array}{c} \begin{array}{cccc} A & C & E & M \end{array} \\ \begin{bmatrix} 0.2 & 0.4 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0.3 & 0.1 & 0.1 & 0.3 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{Y} = \begin{array}{c} A \\ C \\ E \\ M \end{array} \begin{bmatrix} 21\,900 \\ 27\,000 \\ 25\,000 \\ 33\,000 \end{bmatrix}$$

Find the production vector, given that production exactly meets demand.

Solution

Using a calculator, we find

$$(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 3.44639 & 2.65864 & 2.57112 & 2.34136 \\ 2.13348 & 3.07440 & 2.06783 & 1.92560 \\ 2.30853 & 2.35230 & 3.46827 & 2.64770 \\ 2.11160 & 1.91466 & 1.89278 & 3.08534 \end{bmatrix}$$

and we can compute

$$\mathbf{P} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{Y} = \begin{bmatrix} 288\,802 \\ 244\,973 \\ 288\,150 \\ 247\,075 \end{bmatrix}$$

We state the following result without proof.

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Theorem

If **A** is a consumption matrix such that each column sums to less than 1, then the inverse matrix $(\mathbf{I} - \mathbf{A})^{-1}$ exists and has non-negative entries.

Leontief closed model

The model is called **closed** when everything that is produced by the industries being considered is used by these industries.

We represent a closed economy with two industries by an **input–output matrix** of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where entry a_{ij} is the fraction of the total output of industry j that is purchased by industry i. This is a *probability matrix*: all the entries are non-negative and each column sums to 1.

Define the **price vector**, **P**, by

$$\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \text{price charged by industry 1 for its total output} \\ \text{price charged by industry 2 for its total output} \end{bmatrix}$$

In order for the expenditure of each industry to be equal to its income, the following matrix equation must be satisfied:

$$\mathbf{AP} = \mathbf{P}$$

That is, the price vector **P** must be an eigenvector of **A** with eigenvalue $\lambda = 1$.

For example, consider the input–output matrix $\mathbf{A} = \begin{bmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{bmatrix}$.

The equation $\mathbf{AP} = \mathbf{P}$ becomes

$$\begin{bmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

We obtain the simultaneous equations

$$-0.8p_1 + 0.3p_2 = 0 \qquad (1)$$
$$0.8p_1 - 0.3p_2 = 0 \qquad (2)$$

Hence $p_1 = \frac{3}{8}p_2$. So the family of possible price vectors is given by $\mathbf{P} = \begin{bmatrix} 3k \\ 8k \end{bmatrix}$, for $k > 0$.

For example, we could take $\mathbf{P} = \begin{bmatrix} \$30\ 000 \\ \$80\ 000 \end{bmatrix}$.

Example 16

Consider a closed economy with three sectors – energy (E), manufacturing (M) and services (S). The input–output matrix, **A**, and price vector, **P**, are given by

$$\mathbf{A} = \begin{array}{c} \\ E \\ M \\ S \end{array}\!\!\begin{array}{c} \begin{array}{ccc} E & M & S \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.5 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.5 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{P} = \begin{array}{c} E \\ M \\ S \end{array}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Find the family of possible price vectors such that the expenditure of each sector is equal to its income.

Solution

The equation $\mathbf{AP} = \mathbf{P}$ becomes

$$\begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.5 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

This gives the equations

$$-0.8p_1 + 0.3p_2 + 0.2p_3 = 0 \qquad (1)$$
$$0.5p_1 - 0.8p_2 + 0.3p_3 = 0 \qquad (2)$$
$$0.3p_1 + 0.5p_2 - 0.5p_3 = 0 \qquad (3)$$

From these equations, we have $p_1 = \dfrac{25}{49}p_3$ and $p_2 = \dfrac{34}{49}p_3$.

Hence the possible price vectors are given by $\mathbf{P} = \begin{bmatrix} 25k \\ 34k \\ 49k \end{bmatrix}$, for $k > 0$.

Note: Since the input–output matrix **A** is a probability matrix, we can see that the equation $\mathbf{AP} = \mathbf{P}$ for a price vector corresponds to the equation $\mathbf{TS}^* = \mathbf{S}^*$ for the steady state of a Markov chain. So the theorem about regular probability matrices from the end of Section B3 can be used to give a condition under which an input–output matrix has a valid price vector that can be found using iteration.

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Exercise B5

Example 15

1 Consider an open economy with four sectors – agriculture (A), construction (C), energy (E) and manufacturing (M) – such that the consumption matrix, $\mathbf{A}$, and demand vector, $\mathbf{Y}$, are given by

$$\mathbf{A} = \begin{array}{c} \\ A \\ C \\ E \\ M \end{array} \begin{array}{c} \begin{array}{cccc} A & C & E & M \end{array} \\ \begin{bmatrix} 0.1 & 0.5 & 0.2 & 0.2 \\ 0.4 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{Y} = \begin{array}{c} A \\ C \\ E \\ M \end{array} \begin{bmatrix} 21\,900 \\ 27\,000 \\ 25\,000 \\ 33\,000 \end{bmatrix}$$

Find the production vector, given that production exactly meets demand.

Example 16

2 Consider a closed economy with three sectors – energy (E), manufacturing (M) and services (S). The input–output matrix, $\mathbf{A}$, and price vector, $\mathbf{P}$, are

$$\mathbf{A} = \begin{array}{c} \\ E \\ M \\ S \end{array} \begin{array}{c} \begin{array}{ccc} E & M & S \end{array} \\ \begin{bmatrix} 0.3 & 0.2 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.1 & 0.6 & 0.6 \end{bmatrix} \end{array} \quad \text{and} \quad \mathbf{P} = \begin{array}{c} E \\ M \\ S \end{array} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Find the family of possible price vectors such that the expenditure of each sector is equal to its income.

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B6 Cryptology with matrices

Cryptology is concerned with keeping communications private by encrypting a message before it is transmitted and then decrypting the message at its destination. For example, when you use online banking, the information that you send to the bank is encrypted by your computer and then decrypted by the bank's computer.

One very simple encryption method (certainly not secure enough for online banking) uses a square matrix that has an inverse.

An example of a matrix cipher

For example, suppose that we want to send the message:

YOUR SECRET IS SAFE

We shall use the following invertible 3×3 matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{bmatrix} -3 & 0 & -2 \\ -5 & -1 & -3 \\ 3 & 1 & 2 \end{bmatrix}$$

The sender of the message uses the matrix $\mathbf{A}$ and the receiver uses the matrix $\mathbf{A}^{-1}$. Both matrices must be kept secret.

We assign a unique number to each required symbol (including 'space'). For example:

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z	space
14	15	16	17	18	19	20	21	22	23	24	25	26	30

The unencrypted message (**plaintext**) becomes:

25, 15, 21, 18, 30, 19, 5, 3, 18, 5, 20, 30, 9, 19, 30, 19, 1, 6, 5

Encrypting the message

We write the plaintext message down the columns of a $3 \times n$ matrix (filling out the entries with spaces) and then pre-multiply this matrix by $\mathbf{A}$:

$$\begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 25 & 18 & 5 & 5 & 9 & 19 & 5 \\ 15 & 30 & 3 & 20 & 19 & 1 & 30 \\ 21 & 19 & 18 & 30 & 30 & 6 & 30 \end{bmatrix}$$

$$= \begin{bmatrix} -47 & -80 & -37 & -95 & -89 & 5 & -115 \\ 46 & 37 & 23 & 35 & 39 & 25 & 35 \\ 58 & 111 & 53 & 140 & 129 & -17 & 170 \end{bmatrix}$$

The following encrypted message (**ciphertext**) is sent:

−47, 46, 58, −80, 37, 111, −37, 23, 53, −95, 35, 140, −89, 39, 129, 5, 25, −17, −115, 35, 170

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Decrypting the message

The receiver puts the ciphertext into a $3 \times n$ matrix and pre-multiplies by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} -3 & 0 & -2 \\ -5 & -1 & -3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} -47 & -80 & -37 & -95 & -89 & 5 & -115 \\ 46 & 37 & 23 & 35 & 39 & 25 & 35 \\ 58 & 111 & 53 & 140 & 129 & -17 & 170 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 18 & 5 & 5 & 9 & 19 & 5 \\ 15 & 30 & 3 & 20 & 19 & 1 & 30 \\ 21 & 19 & 18 & 30 & 30 & 6 & 30 \end{bmatrix}$$

So the receiver can now reconstruct the original plaintext message.

Breaking the cipher

Suppose that an unauthorised person who has intercepted the ciphertext message can also obtain or guess at part of the plaintext message. For example, the plaintext and ciphertext corresponding to *YOUR SECR* are

$$25, 15, 21, 18, 30, 19, 5, 3, 18 \qquad \text{and} \qquad -47, 46, 58, -80, 37, 111, -37, 23, 53$$

If an unauthorised person knows that

$$\mathbf{A}\begin{bmatrix} 25 & 18 & 5 \\ 15 & 30 & 3 \\ 21 & 19 & 18 \end{bmatrix} = \begin{bmatrix} -47 & -80 & -37 \\ 46 & 37 & 23 \\ 58 & 111 & 53 \end{bmatrix}$$

then they can calculate

$$\mathbf{A} = \begin{bmatrix} -47 & -80 & -37 \\ 46 & 37 & 23 \\ 58 & 111 & 53 \end{bmatrix}\begin{bmatrix} 25 & 18 & 5 \\ 15 & 30 & 3 \\ 21 & 19 & 18 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 3 \end{bmatrix}$$

This allows them to decrypt the rest of this message and any future messages that are encrypted using the same matrix $\mathbf{A}$.

Exercise B6

1 Again consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 3 \end{bmatrix}$$

Use this matrix to encrypt and then decrypt the message *THE COAST IS CLEAR*.

2 A 3×3 matrix $\mathbf{A}$ was used to produce the following ciphertext:

$$-12, 22, -15, -79, 122, -73, -58, 89, -61, -100, 150, -80,$$
$$-92, 127, -40, -24, 45, -41, -43, 67, -33, -136, 196, -90$$

Find $\mathbf{A}$ and hence decipher the message, given that it starts with *MEET ME AT*.

3 Encrypt and then decrypt a message by using a suitable 2×2 matrix.

B7 Spreadsheets and matrices

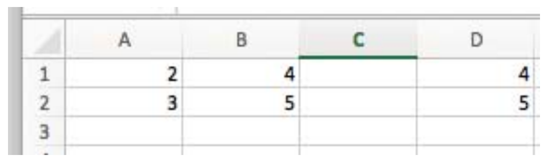
When carrying out investigations with matrices, it is convenient to use computer software. You may have access to a specialised mathematical package such as *Mathematica*. Otherwise, it is also possible to perform some matrix calculations using a spreadsheet.

In this section, we give some examples of performing basic matrix calculations in Excel. Note that the output obtained in Excel can be copied into a Word document.

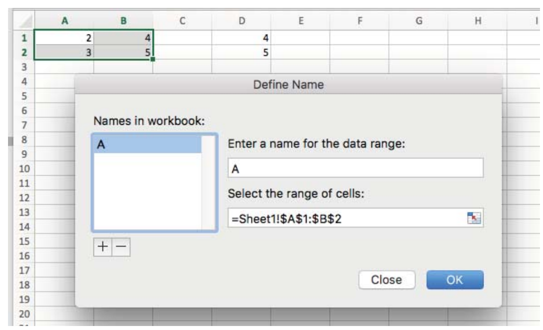
Naming matrices

To assign names to the matrices $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$:

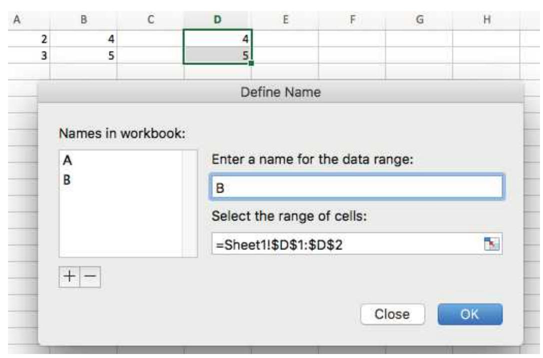
- Put the entries of **A** into cells A1:B2.
- Put the entries of **B** into cells D1:D2.



- To define matrix **A**, select cells A1:B2.
- Go to the **Formulas** menu and choose **Define Name**.
- Enter the name A in the dialogue box and click OK.



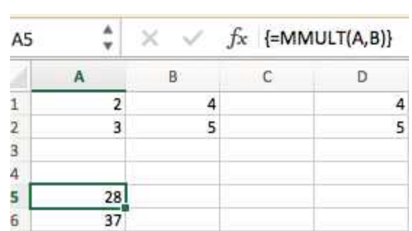
- To define matrix **B**, select cells D1:D2.
- Go to the **Formulas** menu and choose **Define Name**.
- Enter the name B in the dialogue box and click OK.



Matrix multiplication

To find the product **AB**:

- Select cells A5:A6 as the location for the product.
- In the formula bar, enter: =MMULT(A,B)
- Press Control-Shift-Enter.
- Hence $\mathbf{AB} = \begin{bmatrix} 28 \\ 37 \end{bmatrix}$.

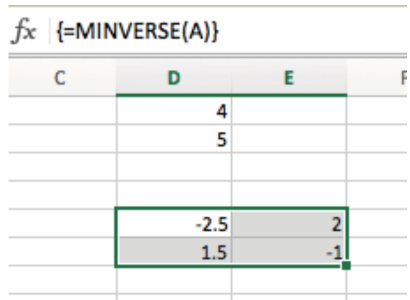


Determinant of a matrix

To find the determinant of **A**:

- Select one cell to be the location of the determinant.
- In the formula bar, enter =MDETERM(A) and then press Enter.

Inverse of a matrix



To find the inverse of **A**:

- Select cells D5:E6 as the location for the inverse.
- In the formula bar, enter: =MINVERSE(A)
- Press Control-Shift-Enter.
- Hence $\mathbf{A}^{-1} = \begin{bmatrix} -2.5 & 2 \\ 1.5 & -1 \end{bmatrix}$.

Multiplication by a scalar

To find 3**A**:

- Select cells D10:E11 as the location for the matrix 3**A**.
- In the formula bar, enter =3*A and press Control-Shift-Enter.

Addition and subtraction of matrices

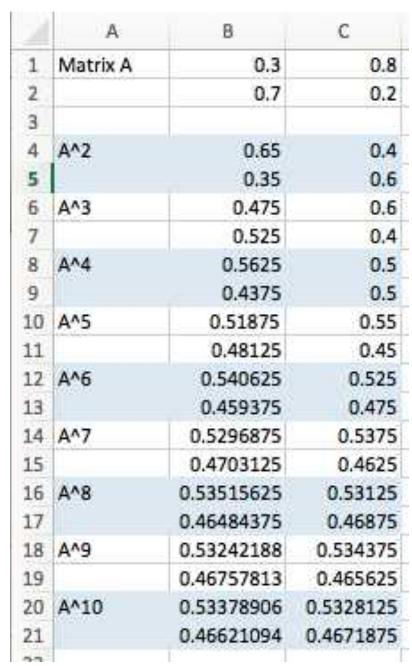
To find **A** + **D** or **A** − **D**, where **D** is a given 2×2 matrix:

- Enter and name a 2×2 matrix **D**.
- Select a 2×2 block of cells as the location for the result.
- In the formula bar, enter =A+D or =A-D and press Control-Shift-Enter.

Powers of matrices

You can use iteration to find powers of square matrices. That is, use the fact that $\mathbf{A}^{n+1} = \mathbf{A}^n\mathbf{A}$.

- In cells B1:C2 of a new spreadsheet, enter the 2×2 matrix
$$\mathbf{A} = \begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix}$$
- Name this matrix A.
- Select cells B4:C5 and obtain the matrix $\mathbf{A}^2$ using the formula =MMULT(A,A)
- Select cells B6:C7 and obtain the matrix $\mathbf{A}^3$ using the formula =MMULT(B4:C5,A)
- Select cells B6:C7 and fill down to obtain successive powers as shown.



	A	B	C
1	Matrix A	0.3	0.8
2		0.7	0.2
3			
4	A^2	0.65	0.4
5		0.35	0.6
6	A^3	0.475	0.6
7		0.525	0.4
8	A^4	0.5625	0.5
9		0.4375	0.5
10	A^5	0.51875	0.55
11		0.48125	0.45
12	A^6	0.540625	0.525
13		0.459375	0.475
14	A^7	0.5296875	0.5375
15		0.4703125	0.4625
16	A^8	0.53515625	0.53125
17		0.46484375	0.46875
18	A^9	0.53242188	0.534375
19		0.46757813	0.465625
20	A^10	0.53378906	0.5328125
21		0.46621094	0.4671875

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Answers

Exercise B1

1 D, A, C, B

2 Answers will vary

Exercise B2

1a **i** $\lambda^2 - 7\lambda + 10$

ii $\lambda = 5$ and $\lambda = 2$

iii $\lambda = 5$: $\mathbf{x} = \begin{bmatrix} k \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 2$: $\mathbf{x} = \begin{bmatrix} -k \\ 2k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

b **i** $\lambda^2 - \dfrac{13\lambda}{12} + \dfrac{1}{12}$

ii $\lambda = 1$ and $\lambda = \dfrac{1}{12}$

iii $\lambda = 1$: $\mathbf{x} = \begin{bmatrix} 3k \\ 8k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = \dfrac{1}{12}$: $\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

c **i** $\lambda^2 - 7\lambda + 10$

ii $\lambda = 5$ and $\lambda = 2$

iii $\lambda = 5$: $\mathbf{x} = \begin{bmatrix} k \\ 2k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 2$: $\mathbf{x} = \begin{bmatrix} -k \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

2a **i** $-\lambda^3 + 12\lambda^2 - 47\lambda + 60$

ii $\lambda = 5$, $\lambda = 4$ and $\lambda = 3$

iii $\lambda = 5$: $\mathbf{x} = \begin{bmatrix} 8k \\ k \\ 2k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 4$: $\mathbf{x} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 3$: $\mathbf{x} = \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

b **i** $-\lambda^3 + 12\lambda^2 - 47\lambda + 60$

ii $\lambda = 5$, $\lambda = 4$ and $\lambda = 3$

iii $\lambda = 5$: $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 4$: $\mathbf{x} = \begin{bmatrix} -k \\ -5k \\ 41k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 3$: $\mathbf{x} = \begin{bmatrix} 0 \\ -2k \\ 7k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

c **i** $-\lambda^3 + 12\lambda^2 - 42\lambda + 44$

ii $\lambda = 5 + \sqrt{3}$, $\lambda = 5 - \sqrt{3}$ and $\lambda = 2$

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iii $\lambda = 5+\sqrt{3}$: $\mathbf{x} = \begin{bmatrix} 2\sqrt{3}k \\ (3+\sqrt{3})k \\ (9+\sqrt{3})k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 5-\sqrt{3}$: $\mathbf{x} = \begin{bmatrix} (\sqrt{3}-1)k \\ (\sqrt{3}-2)k \\ (2\sqrt{3}-5)k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$;

$\lambda = 2$: $\mathbf{x} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix}, k \in \mathbb{R} \setminus \{0\}$

Exercise B3

1a $\begin{bmatrix} 0.6 & 0.45 \\ 0.4 & 0.55 \end{bmatrix}$ **b** 0.528

2a $\begin{bmatrix} \Pr(A_{i+1}) \\ \Pr(E_{i+1}) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}^i \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

b **i** 0.620 **ii** 0.624 **iii** 0.625

3a $\begin{bmatrix} 0.96 & 0.98 \\ 0.04 & 0.02 \end{bmatrix}$ **b** 96.1%

4 $\begin{bmatrix} 0.7143 \\ 0.2857 \end{bmatrix}$

5 $\begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}$

6a $\begin{bmatrix} 0.80 & 0.14 \\ 0.20 & 0.86 \end{bmatrix}$ **b** 675 people

7a $\begin{bmatrix} 0.25 & 0.65 \\ 0.75 & 0.35 \end{bmatrix}$

b **i** 0.45 **ii** 0.61

c 0.464

8 59.4% Dr Black, 40.6% Dr White

9a

i $\begin{bmatrix} 0.2743 \\ 0.3521 \\ 0.3735 \end{bmatrix}$ **ii** $\begin{bmatrix} 0.2743 \\ 0.3521 \\ 0.3735 \end{bmatrix}$ **iii** $\begin{bmatrix} 0.2743 \\ 0.3521 \\ 0.3735 \end{bmatrix}$

b All the same

10 5405 young, 4865 middle-aged, 9730 old

11 28.4% cool, 22.3% mild, 21.6% warm, 27.6% hot

Exercise B4

1a $\lambda_1 = 1.1524$, $\mathbf{x}_1 = \begin{bmatrix} 0.8461 \\ 0.5139 \\ 0.1338 \\ 0.0464 \end{bmatrix}$

b $\mathbf{P}_{20} = \begin{bmatrix} 132.240 \\ 80.324 \\ 20.910 \\ 7.258 \end{bmatrix}$, $\mathbf{P}_{21} = \begin{bmatrix} 152.398 \\ 92.568 \\ 24.097 \\ 8.364 \end{bmatrix}$

d $c \approx 9.16$

2a 1.246

b 78.1%, 15.7%, 6.3%

c $\lambda_1 = 1.246$, $\mathbf{x}_1 = \begin{bmatrix} 0.781 \\ 0.157 \\ 0.063 \end{bmatrix}$

3a 1

b 60.6%, 24.2%, 12.1%, 3.0%

c $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{bmatrix} 0.606 \\ 0.242 \\ 0.121 \\ 0.030 \end{bmatrix}$

Exercise B5

1 $\mathbf{P} = \begin{bmatrix} 193\,545 \\ 178\,852 \\ 125\,585 \\ 188\,737 \end{bmatrix}$

2 $\mathbf{P} = \begin{bmatrix} 10k \\ 13k \\ 22k \end{bmatrix}$ for $k > 0$

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Exercise B6

1 Ciphertext: $-6, 25, -1, -6, 45, -6, -77, 21, 115, -26, 49, 24, 0, 42, -15, -33, 23, 47$

2 $\mathbf{A} = \begin{bmatrix} 1 & -2 & -3 \\ -1 & 3 & 4 \\ 0 & -2 & -1 \end{bmatrix}$,
MEET ME AT THE STATION

3 Answers will vary

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