PEARSON

MATHEMATICAL METHODS

QUEENSLAND

STUDENT BOOK

11

UNITS 1 & 2

QCE 2019 SYLLABUS

PEARSON

MATHEMATICAL METHODS

QUEENSLAND

STUDENT BOOK

11

UNITS 1 & 2

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Writing and development team

We are grateful to the following people for their time and expertise in contributing to **Pearson Mathematical Methods 11**.

Nicola Silva
Retired teacher, QLD
Author

Daniel Hernandez Navas
Lead publisher
Content and learning specialist
Pearson Australia

Lindy Bayles
Mathematics teacher, VIC
Development editor

Rodney Anderson
Mathematics teacher, QLD
Reviewer and answer checker

David Coffey
Mathematics author, VIC
Answer checker
Teacher support material author

Elizabeth Constable
Mathematics teacher, QLD
Answer checker
Teacher support material author

Shelley Cross
Mathematics teacher, QLD
Teacher support material author
Reviewer

Jennifer Dawes
Mathematics teacher, QLD
Reviewer and answer checker

Andrew Duncan
Mathematics teacher and HoD, QLD
Reviewer and answer checker

Emily Frazer
Mathematics teacher, QLD
Teacher support material author
Reviewer

Paul Grace
Mathematics teacher, VIC
Answer checker

Amy Hawke
Mathematics teacher, QLD
Teacher support material author

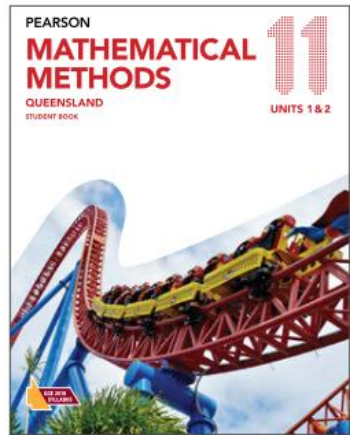
Kerri Hill
Mathematics teacher and HoD, QLD
Reviewer

Emily Hobbs
Mathematics teacher and HoD, VIC
Reviewer
Teacher support material author

Karleigh Nicholls
Senior Mathematics teacher, QLD
Reviewer

Robert Paterson
Mathematics teacher and HoD, QLD
Reviewer

Julie Watson-Hayward
Associate lecturer, Griffith University, QLD
Answer checker



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Supporting the integrating of technology

Students are supported with the integration of technology in a number of ways. The eBook includes 'How to' user guides covering all basic functionality for the following three graphing calculators:

- TI-84 Plus CE
- TI-Nspire CX (non CAS)
- CASIO fx-CG50AU

Throughout the student book are Technology worked examples strategically placed within the theory. These are suitable for both the TI-Nspire CX (non CAS) and CASIO fx-CG50AU. The examples clearly demonstrate how the technology can be used effectively and efficiently for the content covered in that chapter.

Graphing calculators are not the only technology integrated throughout the Pearson Queensland senior mathematics series. Spreadsheets, Desmos and interactive widgets have been included to provide students with the opportunity to visualise concepts, consolidate their understanding and make mathematical connections.

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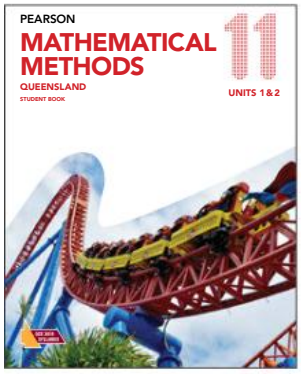


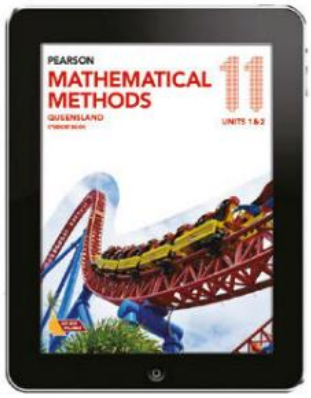
Mathematical Methods 11
Student book

Student book

The student book has been written by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next-generation eBook. This is an electronic textbook that students can access on any device, online or offline, and is linked to features, interactives and visual media that will help consolidate their understanding of concepts and ideas, as well as other useful content developed specifically for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks and Making connections interactive widgets, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



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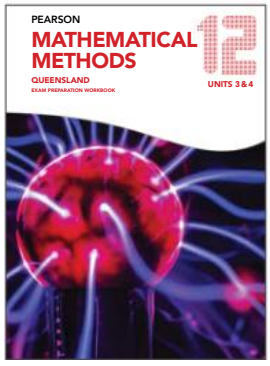


Mathematical Methods 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson Mathematical Methods 11 Queensland Units 1 & 2

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teacher support.

Key information

Key information and rules are highlighted throughout the chapter.

Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.

Making connections

This eBook feature provides teachers and students with a visual interactive of specific mathematics concepts or ideas to aid students in their understanding.

Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars)

The visibility of this grading helps ensure all levels of difficulty are well covered.

(c) (-3.5, -1.2)

1 Calculate the value of $x^2 + y^2$.	For (-3.5, -1.2): $x^2 + y^2 = (-3.5)^2 + (-1.2)^2 = 12.25 + 1.44 = 13.69$
2 Compare $x^2 + y^2$ to r^2.	$r^2 = 25$; $x^2 + y^2 < r^2$; $13.69 < 25$
3 Interpret the result.	$x^2 + y^2 < r^2$, so the point is inside the circle.

Transformations and the graph of the circle with equation $(x-a)^2 + (y-b)^2 = r^2$

If a translation of a units parallel to the x-axis and b units parallel to the y-axis is applied to the relation $x^2 + y^2 = r^2$, the general form of a circle with centre (a, b) and radius r is obtained from Pythagoras' theorem, where $(x-a)^2 + (y-b)^2 = r^2$.

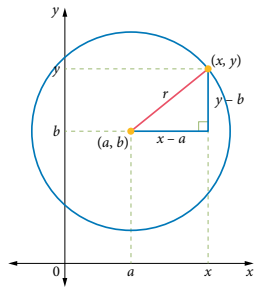
The general form of a circle of radius r is $(x-a)^2 + (y-b)^2 = r^2$, where the relation $x^2 + y^2 = r^2$ has undergone:

- a horizontal translation of a units parallel to the x-axis. If $a > 0$, the translation is right. If $a < 0$, the translation is left
- a vertical translation of b units parallel to the y-axis. If $b > 0$, the translation is up. If $b < 0$, the translation is down.

The centre of the circle is at (a, b).

WARNING
The centre of a circle is often displayed on the graph, but it is not a point belonging to the graph of the circle.

Technology worked example
Graphing circles



Making connections
Transforming a circle
Move the sliders to view the effect each parameter has on the relation $(x-a)^2 + (y-b)^2 = r^2$.

13 Sketch circles with centre (a,b)

Consider the relation with equation $(x-2)^2 + (y+1)^2 = 4$.

(a) Determine the centre and radius of the circle.

THINKING	WORKING
1 Write the equation in the form $(x-a)^2 + (y-b)^2 = r^2$.	$(x-2)^2 + (y+1)^2 = 4$; $(x-2)^2 + (y+1)^2 = 2^2$
2 Determine the coordinates of the centre and the length of the radius.	The coordinates of the centre are (2, -1) and the length of the radius is 2 units.

Technology worked examples

These worked examples offer support in using technology such as spreadsheets, graphing calculators and graphing software, and include technology-focused worked examples and activities.

Meeting the needs of the QCE Syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

(b) Determine the axes intercepts.

1 Determine the coordinates of the x-intercepts.

x-intercept, let $y = 0$:

$$(x-2)^2 + (1)^2 = 4$$
$$(x-2)^2 = 3$$
$$x - 2 = \pm\sqrt{3}$$
$$x = 2 \pm \sqrt{3}$$

The coordinates of the x-intercepts are $(2-\sqrt{3}, 0)$ and $(2+\sqrt{3}, 0)$.

2 Determine the coordinates of the y-intercept(s).

y-intercept, let $x = 0$:

$$(-2)^2 + (y+1)^2 = 4$$
$$4 + (y+1)^2 = 4$$
$$(y+1)^2 = 0$$
$$y + 1 = 0$$
$$y = -1$$

The coordinates of the y-intercept are (0, -1).

Tech-free questions to practice and maintain by-hand skills

Every exercise includes 'tech-free' questions to help students continuously practice their by-hand skills in preparation for Exam Paper 1.

6 Consider the cubic expressions in expanded form:

(a) For $-2x(x+11)(x-11)$ the number of terms will be:

A 1 B 2 C 3 D 4

(b) For $4(x-4)^3$ the number of terms will be:

A 1 B 2 C 3 D 4

(c) Explain the common error made by a student who chose the second incorrect option in part (b).

Worked solutions

Fully worked solutions are provided for every question in the student textbook and can be accessed from the accompanying eBook.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

Additional information

These interactives appear in the eBook in two forms, as videos explaining specific concepts or as interactive questions to check students' understanding.

Additional information

The vertex of a parabola

This activity explores how the coordinates of the turning points can be derived from the general quadratic expression.

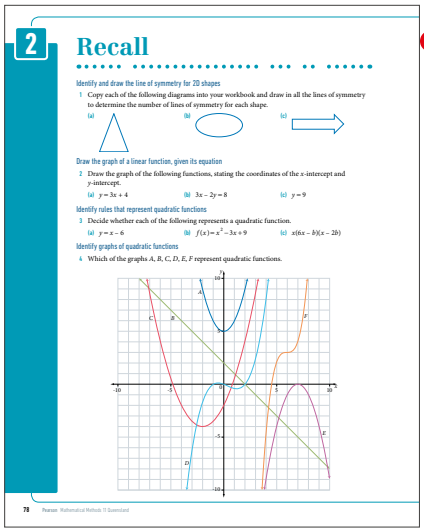
Explore further

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

Explore further

Exploring cubic functions in the form $y = ax^3 + bx^2 + cx + d$

Explore the effect of varying the coefficients on the graph of a cubic polynomial expressed in the general form.

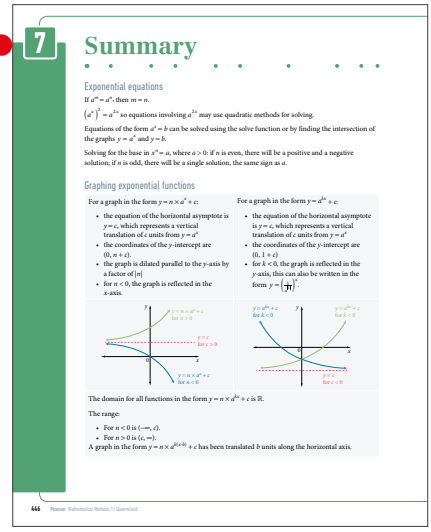


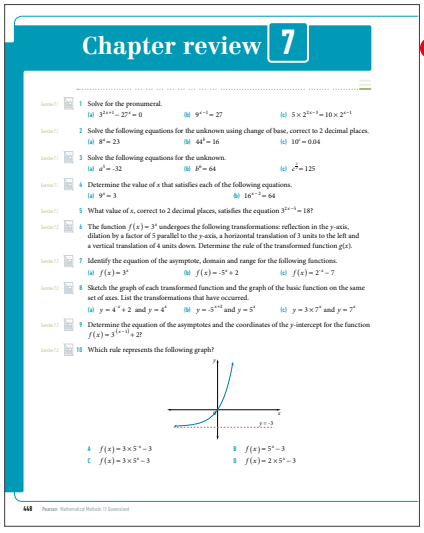
Recall

Each chapter begins with a review of assumed knowledge for the chapter.

Summary

At the end of each chapter, there is a summary of the key facts and rules used in the chapter.





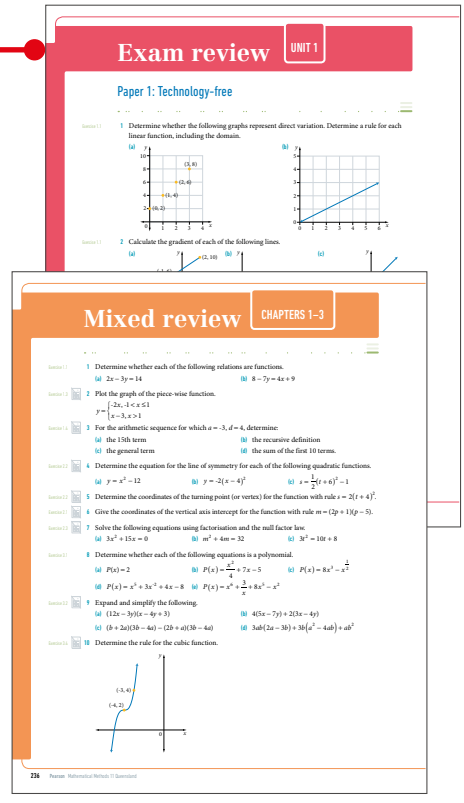
Chapter review

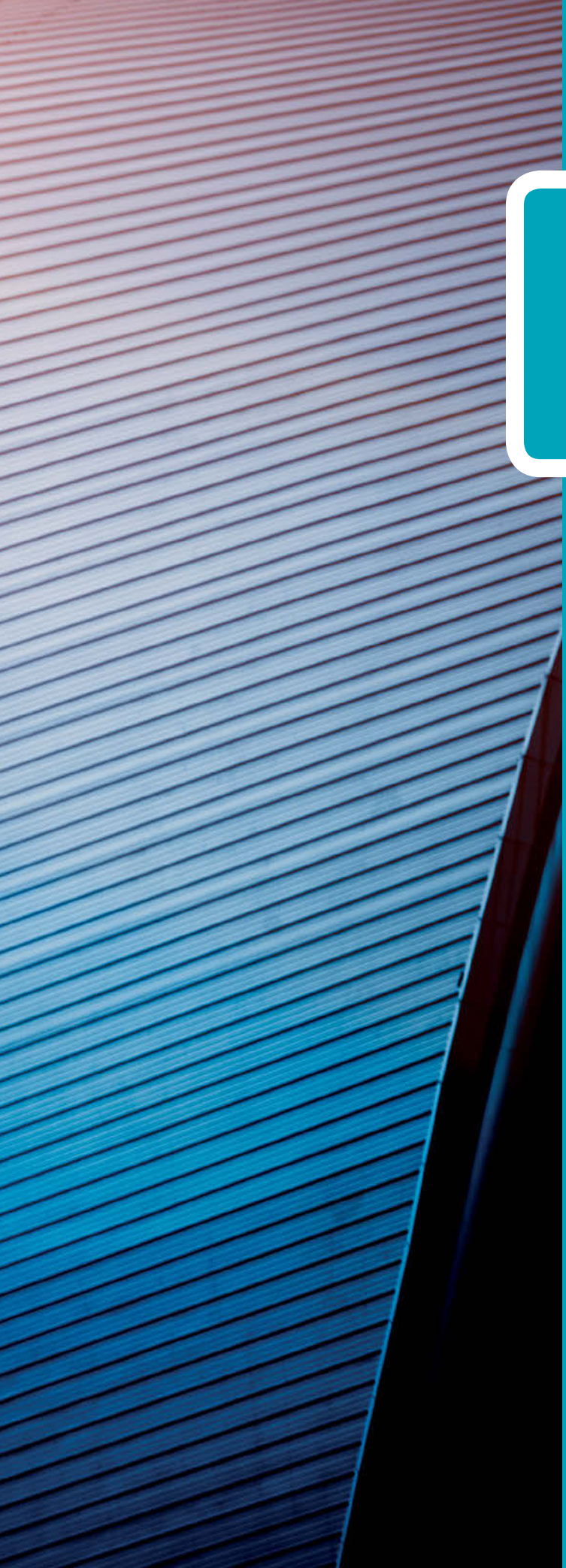
Every chapter review follows the QCAA examination proportions for levels of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.

Mixed and Exam review

Exam reviews give cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each unit.

As well, Mixed reviews provide cumulative revision that mixes content from previous chapters placed midway through each unit.





1

Linear functions

Recall 4

1.1 Linear relations and functions 6

1.2 Solving simultaneous linear equations 25

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1.4 Arithmetic sequences and series 49

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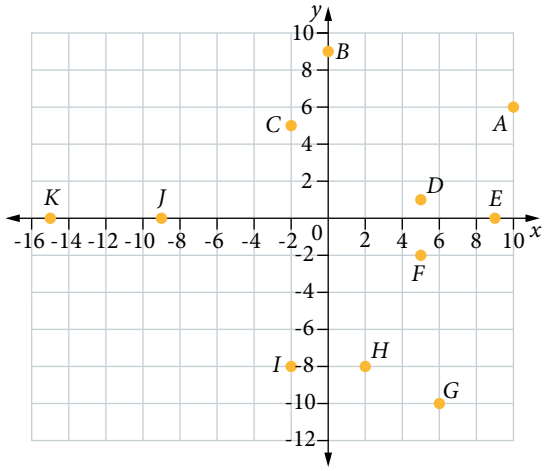
1 Recall

Plotting points on the Cartesian plane

1 Identify the coordinates of the points on the graph.



Determining the coordinates of the *x*- and *y*-intercepts

2 Determine the coordinates of the *x*- and *y*-intercepts for the following linear equations.

(a) $y = 2x + 3$ (b) $2x + 3y = 6$ (c) $y = -2x$

Complete a table of values for a linear function

3 Complete each table of values, given that the relationship between x and y is linear.

(a)

x	-1	0	1		3	4
y	1	5	9	13		

(b)

x	-3	-1			5	7
y	-2	-1	0	1		

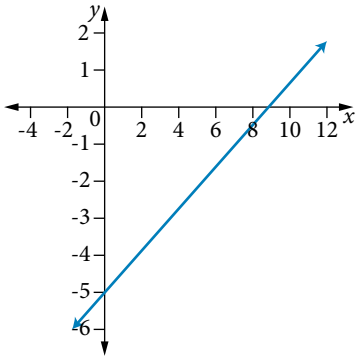
(c)

x	-0.5		0.5	1		2
y		-2	-1	0	1	

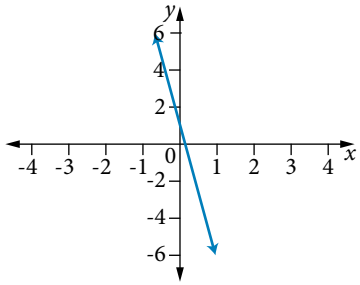
Distinguish between a negative, zero, undefined and positive gradient

4 Determine whether the gradient of the linear function is negative, zero or positive.

(a)



(b)



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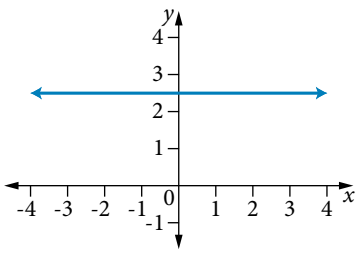
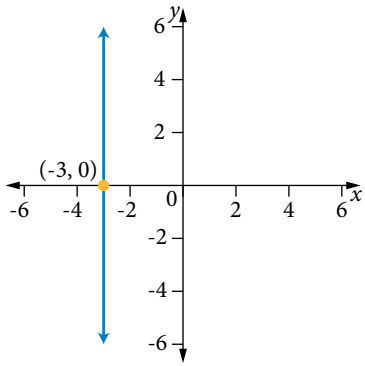


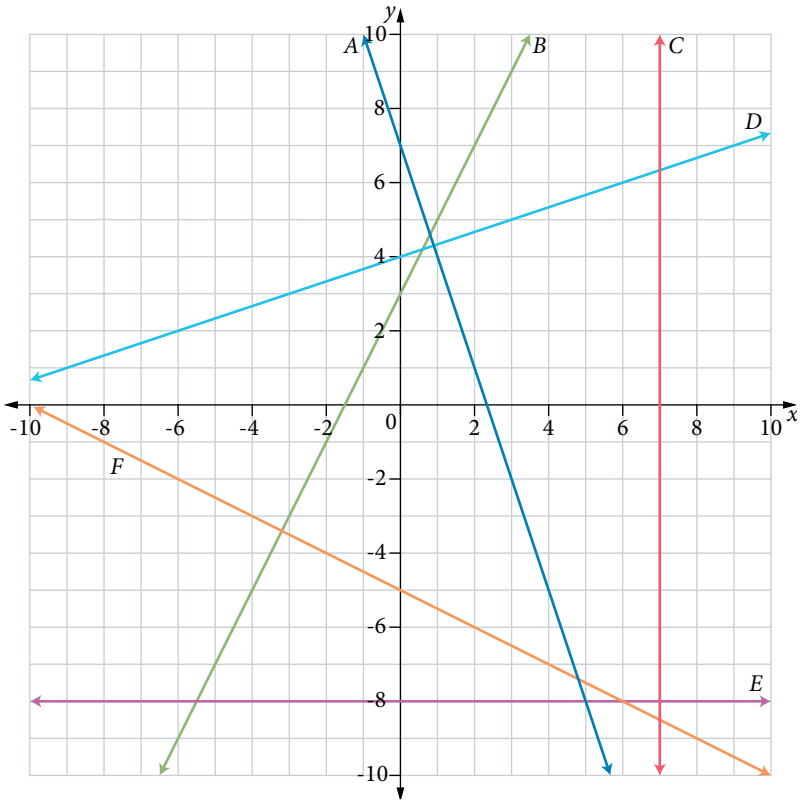
(d)



Determine the gradient of a straight line

5 Determine the gradient of each line.

(a)



(b) line through (4, 1) and (10, -2)

(c) line through (-9, -6) and (18, 0)

Determine the gradient and *y*-intercept of a straight line with an equation in $y = mx + c$ form

6 Identify the gradient m and the y-intercept c for the graphs of the following linear equations.

(a) $y = \frac{5}{3}x + 2$ **(b)** $y = -3x + 4$ **(c)** $3x + 4y + 8 = 0$

Solve quadratic equations from a graph

7 By graphing $y = 3x^2 - 10x + 5$ on your technology, solve each of the following, correct to 2 decimal places.

(a) $3x^2 - 10x + 5 = 0$ **(b)** $3x^2 - 10x + 5 < 0$ **(c)** $3x^2 - 10x + 5 > 0$

Expand brackets

8 Expand and simplify each algebraic expression.

(a) $(x+3)^2$ **(b)** $(x-6)^2 + 3$ **(c)** $(x+8)^2 - 20$

Factorise monic quadratics

9 Factorise each of the following quadratic expressions.

(a) $x^2 + 8x + 15$ **(b)** $x^2 - 14x + 49$ **(c)** $x^2 - 7x - 18$

1.1 Linear relations and functions

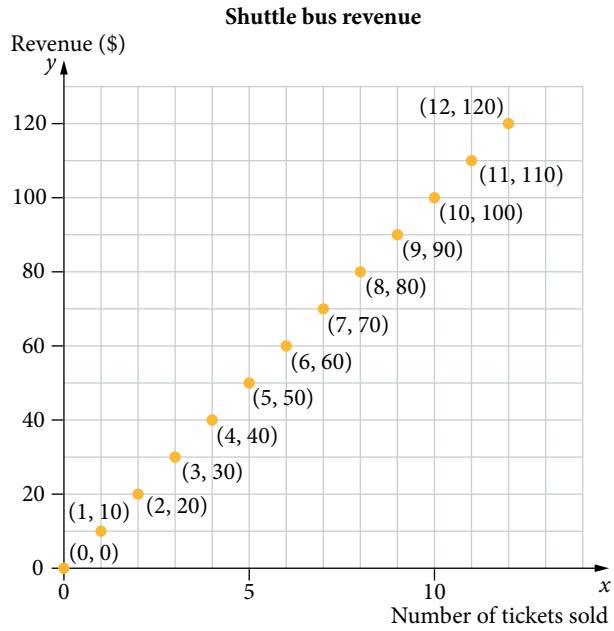
Domain and range

In the equation for a linear relation $y = mx + c$, the dependent variable y is defined in terms of the independent variable x.

The x-values for which a relation is defined are referred to as the domain. The matching set of values for y is the range.

Discrete domains

Consider a shuttle bus with a maximum capacity of 12 passengers and tickets that cost \$10 per trip. The independent variable is the number of tickets purchased (passengers) and the dependent variable is the revenue (income from ticket sales). The equation for a linear relation is $y = 10x$ and the domain is discrete.



- The domain is the set of independent values on the horizontal axis, which represents the number of tickets sold:
  $x : \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ or
  $x \in \{0 \le \mathbb{Z} \le 12\}$, where $\mathbb{Z}$ represents the set of integers (or whole numbers).
  The minimum number of tickets sold is 0 and the maximum is 12. Only whole tickets can be purchased (you cannot buy 1.8 tickets).
- The corresponding range is the set of dependent values on the vertical axis, which represent the revenue:
  $y : \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120\}$
  The minimum revenue is \$0 if no passengers are on board and the maximum revenue is $12 \times 10 = \$120$ if all seats are taken.

Continuous domains

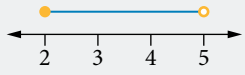
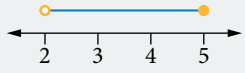
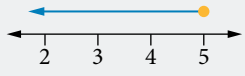
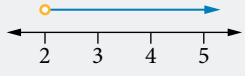
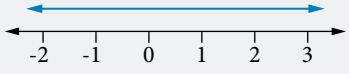
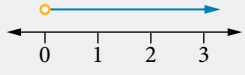
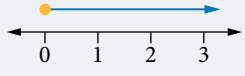
In situations where no context is given, then assume the maximal domain, which is any real number. So you say 'x is an element of $\mathbb{R}$' or $x \in \mathbb{R}$. In set notation you could write $\{x: -\infty < x < \infty\}$ or simply $-\infty < x < \infty$.

Interval notation uses square and round brackets to represent continuous subsets of real numbers. It is often used to describe the domain and range of functions or the domain and range restrictions in modelling contexts.

When describing domain and range of a function the following symbols are often used:

Symbol	Meaning
[ ]	Square brackets are used to show included endpoints within a set of values and are represented graphically with a closed dot. ●
( )	Round brackets are used to show excluded endpoints within a set of values and are represented graphically with an open dot. ○
$\mathbb{R}$	All real numbers from $-\infty$ to ∞. $\mathbb{R}^+$ is the set of all positive real numbers and $\mathbb{R}^-$ is the set of all negative real numbers.
$\mathbb{Z}$	The set of integers (whole numbers). $\mathbb{Z}^+$ is the set of positive whole numbers {1, 2, 3, 4...} $\mathbb{Z}^-$ is the set of negative whole numbers {-1, 2, -3, -4...}
$\cup$	The union symbol can be read as 'or'. The union symbol is often used to link two sets of values. For example: $\mathbb{Z}^+ \cup 0$ or {0, 1, 2, 3...} is the set of positive integers or 0.

The following table shows examples of set notation, interval notation and the representation on a number line.

Subset of the real numbers	Interval notation	Representation on the number line
$2 \le x < 5$	$[2,5)$	
$2 < x \le 5$	$(2,5]$	
$x \le 5$	$(-\infty,5]$	
$x > 2$	$(2, \infty)$	
$x \in \mathbb{R}$	$(-\infty, \infty)$	
$x \in \mathbb{R}^+$	$(0, \infty)$	
$x \in \mathbb{R}^+ \cup \{0\}$	$[0,\infty)$	

WARNING

Do not use square brackets with either ∞ or -∞ because these are indefinite continuations of sets and do *not* indicate a specific endpoint.

Linear relations

A linear relation is a set of ordered pairs that forms a straight line when graphed. A linear relation can be expressed in a number of ways, and each form has advantages and disadvantages when determining the features of a graph. The different forms to be considered are:

- the general form $ax + by + c = 0$
- the gradient–intercept form $y = mx + c$

The general form of a linear relation

The general form of a linear relation in the Cartesian plane is $ax + by + c = 0$ where the coefficients a, b and c are real numbers and (x, y) are the coordinates of the points that satisfy the relation.

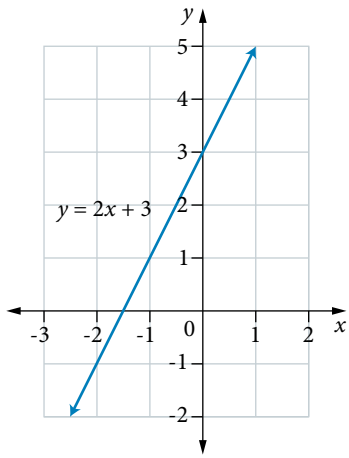
WARNING

Do not assume that the general form of a linear relation $ax + by + c = 0$ is unique for a relation. Multiplying each term by any constant (other than zero) would give an equivalent equation. For example, $y = 2x + 3$ and $2y = 4x + 6$ and $3y = 6x + 9$ produce the same line when graphed.

1 General form of a linear relation

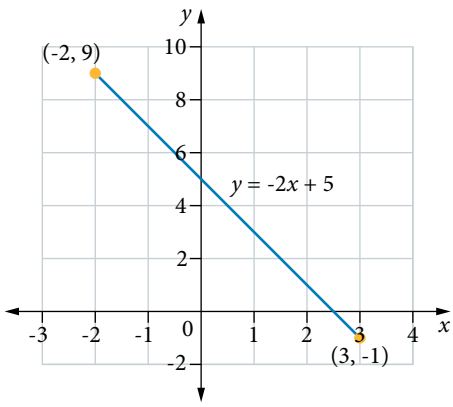
Determine the equation of the linear relation for each graph in the form of $ax + by + c = 0$. Include the domain of the relation.

(a)



THINKING	WORKING
1 Define the equation in the general form, starting with a positive coefficient for the first term: $ax + by + c = 0$	$y = 2x + 3$ $2x - y + 3 = 0$
2 Identify the domain of the relation.	$x \in \mathbb{R}$ Domain: $(-\infty, \infty)$

(b)



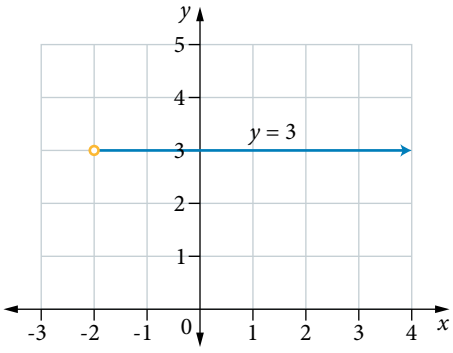
1 Define the equation in the general form, starting with a positive coefficient for the first term: $ax + by + c = 0$

$y = -2x + 5$
$2x + y - 5 = 0$

2 Identify the domain of the relation.

$-2 \le x \le 3$
Domain: $[-2, 3]$

(c)



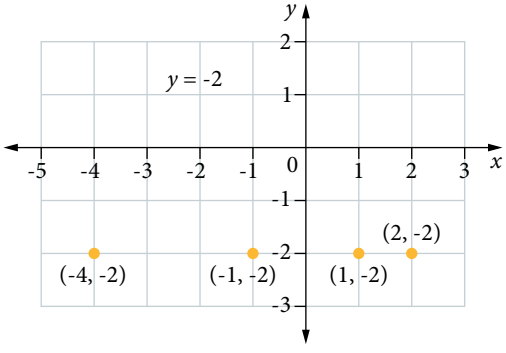
1 Define the equation in the general form, starting with a positive coefficient for the first term: $ax + by + c = 0$

$y = 3$
$y - 3 = 0$

2 Identify the domain of the relation.

$x > -2$
Domain: $(-2, \infty)$

(d)



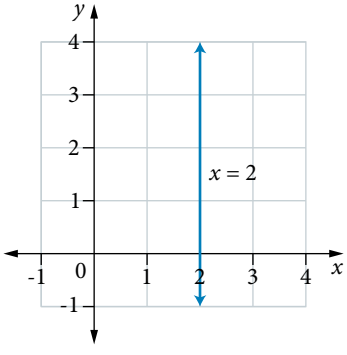
1 Define the equation in the general form, starting with a positive coefficient for the first term: $ax + by + c = 0$

$y = -2$
$y + 2 = 0$

2 Identify the domain of the relation.

$x \in \{-4, -1, 1, 2\}$

(e)



1 Define the equation in the general form, starting with a positive coefficient for the first term: $ax + by + c = 0$

$x = 2$
$x - 2 = 0$

2 Identify the domain of the relation.

$x \in \{2\}$

The range is often not stated for a relation, so you may need to draw a graph or substitute the values of the endpoints to determine the range of the relation.

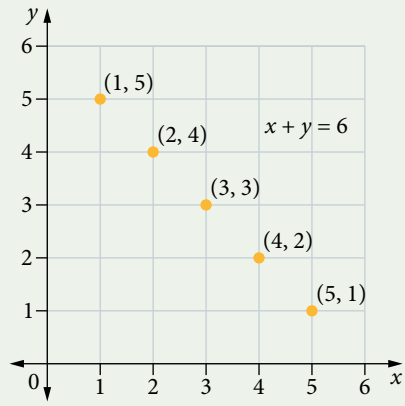
2 Range of a linear relation

Determine the range of the linear relation.

(a) $x + y = 6, x \in \{1, 2, 3, 4, 5\}$

THINKING	WORKING
1 Rearrange the equation to make y the subject.	$x + y = 6$ $y = 6 - x$
2 Substitute each x-value from the domain and calculate the corresponding y-value.	
3 Determine the range of the function by listing the resulting y-values in order.	Range: $y \in \{1, 2, 3, 4, 5\}$
4 Verify your result by plotting points on a graph.	

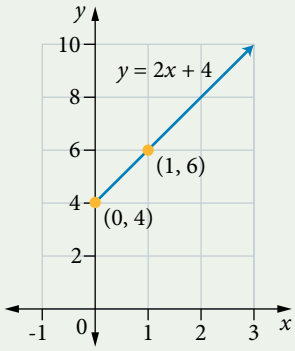
x	1	2	3	4	5
y	$y = 6 - 1$ $= 5$	$y = 6 - 2$ $= 4$	$y = 6 - 3$ $= 3$	$y = 6 - 4$ $= 2$	$y = 6 - 5$ $= 1$



(b) $y = 2x + 4, x \geq 0$

1 Substitute the x-value from the endpoint of the domain and calculate the corresponding y-value.	For $x = 0$: $y = 2 \times 0 + 4$ $= 4$ The coordinates of the y-intercept are (0, 4).
2 Substitute another value of x from the domain and calculate the corresponding y-value.	For $x = 1$: $y = 2 \times 1 + 4 = 6$ This gives the point (1, 6).
3 Determine the range in the same style as the given domain.	Range: $y \geq 4$

4 Verify your result by drawing a graph.



(c) $y = 2x$, $-3 < x < 5$

1 Substitute the x-value from the endpoint of the domain and calculate the corresponding y-value.

For $x = -3$:

$y = 2 \times (-3) = -6$

The excluded endpoint is (-3, -6).

For $x = 5$:

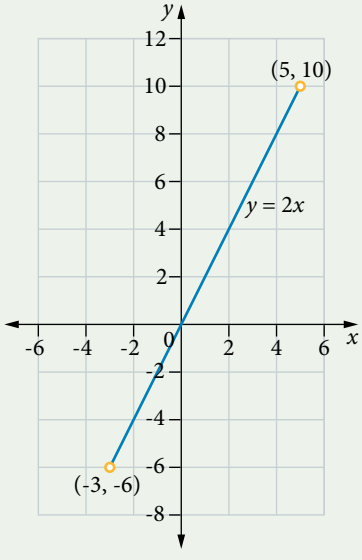
$y = 2 \times 5 = 10$

The excluded endpoint is (5, 10).

2 Determine the range in the same style as the given domain.

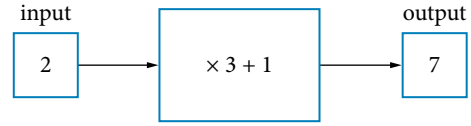
Range: $-6 < y < 10$

3 Verify your result by drawing a graph.



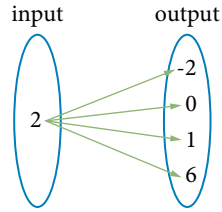
Linear functions

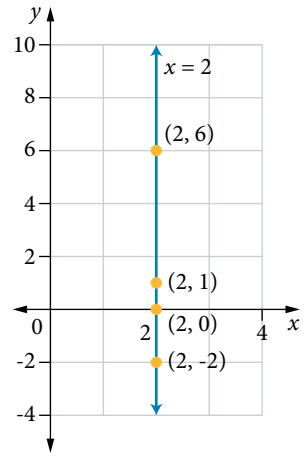
For a relation to be a function, each value of the independent variable x in the given domain maps onto a unique value of the dependent variable y. Using an input–output machine analogy, each input results in a single output. For example, if Output = 3 × Input + 1, and 2 is the input, then 7 is the output. Each output is unique, so a 'function' can be produced:
$f = \{(2,7), (3, 10), (10, 31), (250,751)\}$



Vertical lines are not functions because the input number can produce endless output results.

For example, $g = \{(2,-2), (2,1), (2, 0) (2, 6)\}$ is not a function because 2 is mapped onto -2, 0, 1 and 6.



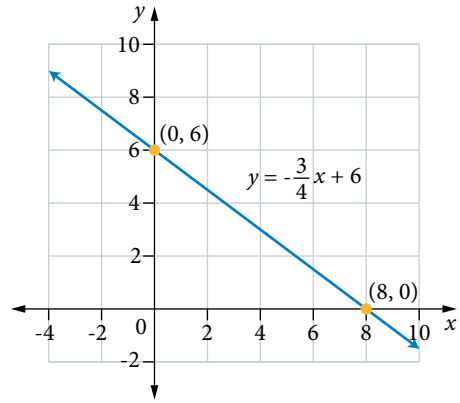


Lines with positive, negative or zero gradients are functions.

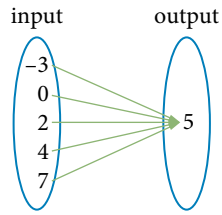
For the general equation of a line $ax + by + c = 0$:

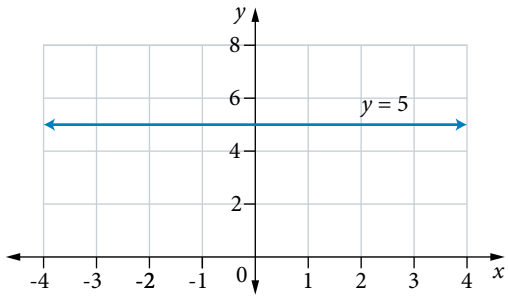
- If $b = 0$, then the line is vertical and of the form $x = -\frac{c}{a}$.
  The gradient of a vertical line is undefined.
  The horizontal run is zero so $m = \frac{\Delta y}{0}$ is undefined.
- If $a \neq 0$ and $b \neq 0$, then the equation of the line can be transposed and written in gradient–intercept form.
  The line is oblique (not vertical nor horizontal). For example, the equation $3x + 4y - 24 = 0$ can be written in gradient–intercept form by making y the subject. The domain and range of this function are both $\mathbb{R}$.

  $$3x + 4y = 24$$
  $$4y = -3x + 24$$
  $$y = -\frac{3}{4}x + 6$$



- If $a = 0$ then the line is horizontal. For example, the equation $y = 5$ is a horizontal line with gradient 0 and y-intercept at (0, 5). The domain of this function is $\mathbb{R}$ and the range is {5}. The vertical rise is zero so $m = \frac{0}{\Delta x}$ is zero.





Function notation

For the equation $y = mx + c$, the dependent y-variable is a function of the independent variable x, or $y = f(x)$.

The equation, in function notation is $f(x) = mx + c$. This notation is useful for conveying more information in a simpler form. For example, $x = 5$, $y = -3$ can be written as $f(5) = -3$.

3 Function notation

For the equation $f(x) = 3x - 5$, determine each of the following:

(a) the value of $f(2)$

THINKING	WORKING
1 Recall the equation and identify the substitution to take place.	$f(x) = 3x - 5$ for $x = 2$
2 Substitute for x and calculate the answer.	$f(2) = 3 \times 2 - 5$ $= 1$
3 Interpret the result.	The value of the function for $x = 2$ is 1.

(b) the value of x where $f(x) = 2$

1 Recall the equation and identify the substitution to take place.	$f(x) = 3x - 5$ for $f(x) = 2$
2 Substitute for $f(x)$ and solve for x.	$3x - 5 = 2$ $3x = 7$ $x = \frac{7}{3}$
3 Interpret the result.	The value of x that makes the function $f(x)$ equal to 2 is $2\frac{1}{3}$.

The gradient–intercept form

Linear relations can be expressed in the gradient–intercept form of $y = mx + c$. This form is useful for identifying the gradient and the y-intercept of the line.

Features of the graph of $y = mx + c$

Consider the graph of the function $y = 2x - 3$.

x	-1	0	1	2	3	4
y	$y = 2 \times -1 - 3$ $= -2 - 3$ $= -5$	$y = 2 \times 0 - 3$ $= 0 - 3$ $= -3$	$y = 2 \times 1 - 3$ $= 2 - 3$ $= -1$	$y = 2 \times 2 - 3$ $= 4 - 3$ $= 1$	$y = 2 \times 3 - 3$ $= 6 - 3$ $= 3$	$y = 2 \times 4 - 3$ $= 8 - 3$ $= 5$
(x, y)	(-1,-5)	(0,-3)	(1,-1)	(2,1)	(3,3)	(4,5)

Key features for the graph of this function:

- The points all lie on a straight line.
- The line has an x-intercept and a y-intercept. The value of the y-intercept is represented by c in $f(x) = mx + c$.

For $x = 0$:

$y = 2x - 3$

$= 2 \times 0 - 3$

$= -3$

The y-intercept is (0, -3).

For $y = 0$:

$2x - 3 = 0$

$x = \frac{3}{2}$

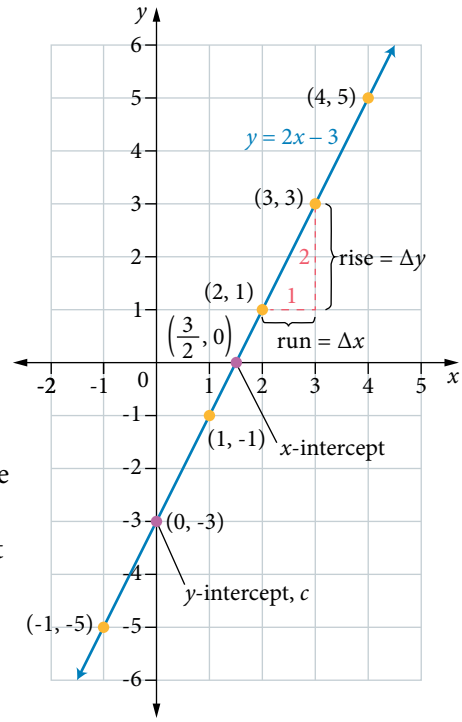
The x-intercept is $\left(\frac{3}{2}, 0\right)$.

- The constant gradient of the line is a measure of its steepness and is defined by the ratio $m = \frac{\text{rise}}{\text{run}}$, where the *rise* is the change in vertical height Δy and the *run* is the change in horizontal distance Δx between any two points. The value of the gradient is represented by m in $f(x) = mx + c$. To calculate the gradient of the line between the two points (2, 1) and (3, 3):

$m = \frac{\text{rise}}{\text{run}}$

$= \frac{2}{1}$

$= 2$



Additional information

Slope

This video explains the slope in the context of a car travelling over a hill on roads with a gradient.

For a linear function in the form $y = mx + c$:

m (the coefficient of the x-term) represents the value of the gradient.

c (the constant term) represents the value of the y-intercept.

To sketch a linear graph you only need two points. You can start with the y-intercept and then use the gradient to determine a second point. It is conventional to show the x-intercept as well, if appropriate to the context.

4 Sketch linear functions from the gradient and y-intercept

Identify the gradient and the y-intercept for each function and hence sketch its graph. Mark the x-intercept on each graph.

(a) $y = 9 - 5x$

THINKING	WORKING
1 Write in gradient–intercept form and identify the values of m and c.	$y = -5x + 9$ $m = -5, c = 9$
2 Determine the coordinates of the y-intercept.	The coordinates of the y-intercept are (0,9).
3 Write the gradient m as a simple fraction, placing the negative sign, if any, with the numerator.	$m = -5$ $= \frac{-5}{1}$

4 Interpret the gradient.

For every horizontal step of 1 unit, the slope has a vertical drop of 5 units.

5 Locate a second point by moving to the right by the number of units in the denominator, and then up (+) or down (-) by the number of units in the numerator.

From the y-intercept (0, 9) move 1 unit to the right and 5 units down.

$(0 + 1, 9 - 5) \rightarrow (1, 4)$

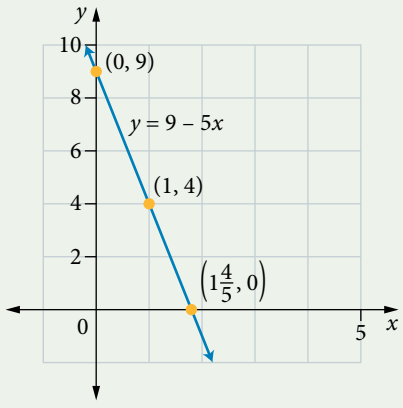
6 Determine the coordinates of the x-intercept.

For $y = 0$:

$$y = 9 - 5x$$
$$0 = 9 - 5x$$
$$5x = 9$$
$$x = \frac{9}{5}$$

The coordinates of the x-intercept are $\left(\frac{9}{5}, 0\right)$.

7 Plot the three points and draw a straight line passing through the points.



(b) $2x - 3y = 6$

1 Write the rule in gradient–intercept form.

$$2x - 3y = 6$$
$$3y = 2x - 6$$
$$y = \frac{2}{3}x - 2$$

2 Identify the values of m and c.

$m = \frac{2}{3}$, $c = -2$

3 Interpret the gradient.

The gradient $\frac{2}{3}$ represents a line where for every horizontal step of 3 units there is a vertical rise of 2 units.

4 Determine the coordinates of the y-intercept.

The coordinates of the y-intercept are (0,-2).

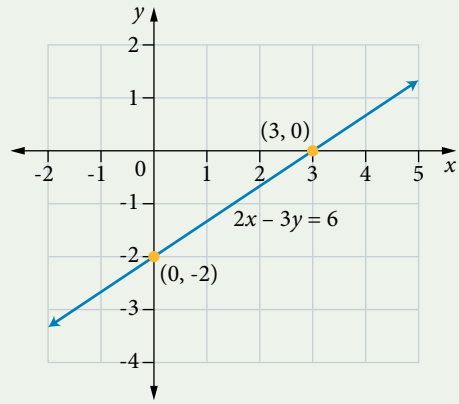
5 Locate a second point using the gradient by moving to the right by the number of units in the denominator, and then up (+) or down (-) by the number of units in the numerator.

6 Plot the two points and draw a straight line passing through the points.

From the y-intercept (0,-2) move 3 units to the right and 2 units up.

$(0+3, -2+2) \rightarrow (3,0)$

This is the x-intercept.



You can sketch graphs on your technology as a first step, when technology is allowed, or to check your working.

Determining the equation of a straight line

You will need two pieces of information to determine the equation of a line. The rule for the gradient (rise over run) can be transposed to a useful form for determining the equation of a line.

Technology worked example
Sketching graphs

$y - y_1 = m(x - x_1)$, where (x_1, y_1) represents the coordinates of a known point and m is the gradient.

5 Determine the equation of a line, given the gradient and a point

Determine the equation of the line that has a gradient of 3 and passes through the point (-2,1).

METHOD 1

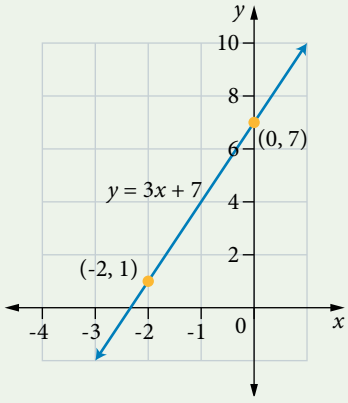
THINKING	WORKING
1 Substitute the gradient and the given point into the equation $y = mx + c$ and solve for c.	$m = 3$ and (-2, 1): $y = 3x + c$ $1 = 3(-2) + c$ $c = 7$
2 Write the equation by substituting for m and c in $y = mx + c$.	$y = 3x + 7$

METHOD 2

1 Substitute the gradient and the given point into the equation $y - y_1 = m(x - x_1)$, writing your answer in the form $y = mx + c$.

$$m = 3$$
$$(x_1, y_1) \rightarrow (-2, 1)$$
$$y - 1 = 3(x - (-2))$$
$$y - 1 = 3(x + 2)$$
$$y - 1 = 3x + 6$$
$$y = 3x + 7$$

2 Sketch the graph to justify your answer.



When two points are given, use the gradient formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ first, then solve for c using either of the given points.

6 Determine the equation of a line, given two points that lie on the line

Determine the equation of the line that passes through the points (-1, 6) and (5, -6).

1 Calculate the gradient m between the two points using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

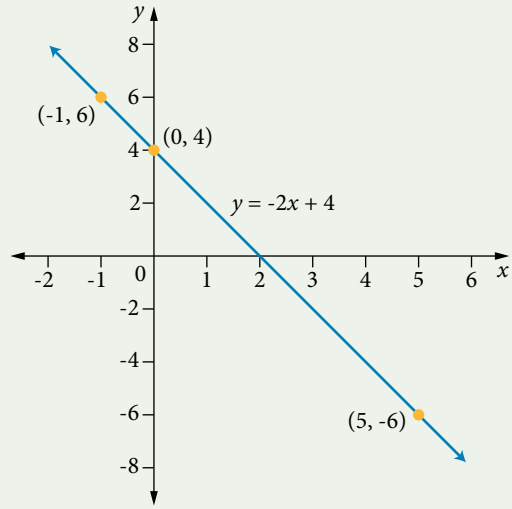
For (-1, 6) and (5, -6):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-6 - 6}{5 - (-1)}$$
$$= \frac{-12}{6}$$
$$= -2$$

2 Substitute the gradient and the given point into the equation $y - y_1 = m(x - x_1)$, writing your answer in the form $y = mx + c$.

$$m = -2$$
$$(x_1, y_1) \rightarrow (-1, 6)$$
$$y - 6 = -2(x - (-1))$$
$$y - 6 = -2(x + 1)$$
$$y - 6 = -2x - 2$$
$$y = -2x + 4$$

3 Sketch the graph to justify your answer.

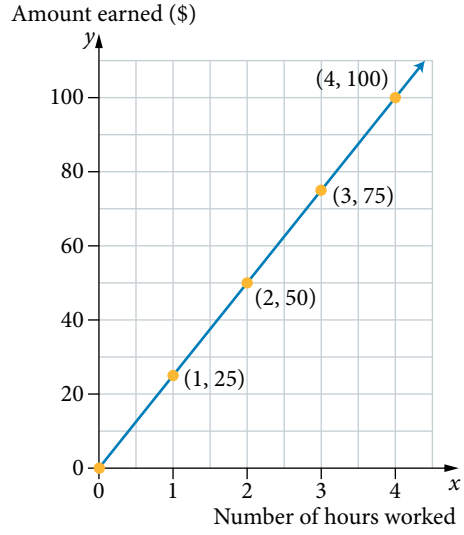


You can determine the equation of a line, given two points using technology.

Technology worked example
Determine the equation of a line, given two points that lie on the line

Direct variation

Linear functions with graphs that pass through the origin $O(0, 0)$ have particular properties. If the ratio of two variables is constant, then they are said to be in *direct variation* or vary directly. For example, the relationship between the number of hours worked and the amount earned represents direct variation if the rate of pay is constant.



If the variables are x and y, and the ratio of y to x is a positive constant, then $\frac{y}{x} = k$, where $k > 0$. Hence $y = kx$.

Explore further
Direct variation
Explore the characteristics of a graph with direct variation.

The graph of $y = kx$:

- is a straight line
- passes through the origin (0,0)
- has a gradient of k
- passes through the point $(1, k)$.

Direct variation is a special case of $y = mx + c$. If a linear function does not involve direct proportion, then $y = mx + c$ should be sufficient to determine a rule for in all other instances.

Additional information
Direct variation
This video explains direct variation where $k < 0$.

7 Determine direct variation

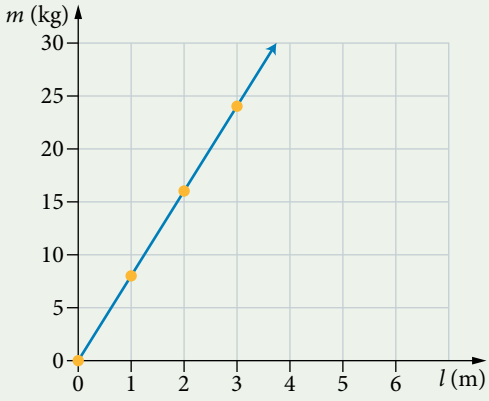
Determine the rule and the domain for each of the following and say whether the relationship involves direct variation. Justify your answer with a table and graph.

(a) The mass and length of timber, if each 1 m length has a mass of 8 kg.

THINKING	WORKING
1 Construct a table of values for four values of the variable.	

Length l (m)	0	1	2	3
Mass m (kg)	0	8	16	24

2 Plot the points. Join the points with a line if the variable is continuous. Draw an arrow on each end if the domain is not restricted.



3 Determine whether the points form a straight line and pass through the origin.

The points are on a straight line that passes through the origin so this is direct variation.

4 Direct variation is of the form $y = kx$, where k represents the ratio between the variables.

For $l = 1$, $m = 8$:

$m = kl$

$8 = k(1)$

$k = 8$

5 Write the rule of the function.

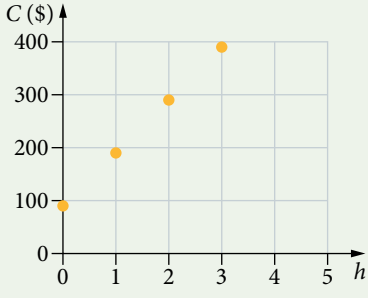
$m = 8l, l \geq 0$

(b) An electrician charges a call-out fee of \$90 and \$100 per hour of work.

1 Construct a table of values for four values of the variable.

No. of hours h	0	1	2	3
Total cost \$$C$	90	190	290	390

2 Plot the points.



3 Determine whether the points form a straight line and pass through the origin.

The points lie on a straight line that does *not* pass through the origin so this is *not* direct variation.

4 Determine the rule of the function, identifying an appropriate domain.

Use $y = mx + c$

For $m = 100$, $c = 90$:

$C = 100h + 90, h \in \{0, 1, 2\ldots\}$

EXERCISE 1.1 Linear relations and functions

Worked Example 1

1 The following table of values represents a function.

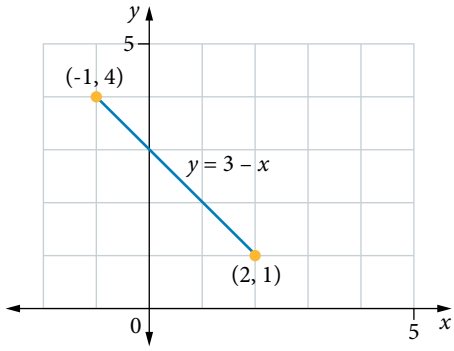
x	-7	-5	3	4
y	5	3	-4	1

(a) Determine the domain of this function.

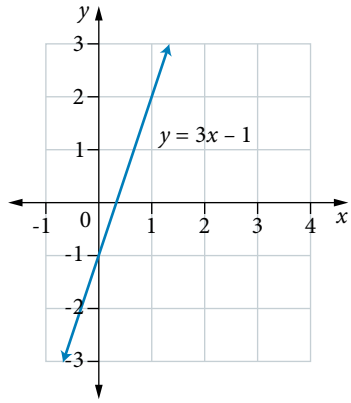
(b) Determine the range of this function.

2 Determine the equation of each linear relation in the general form $ax + by + c = 0$. Include the domain of the relation.

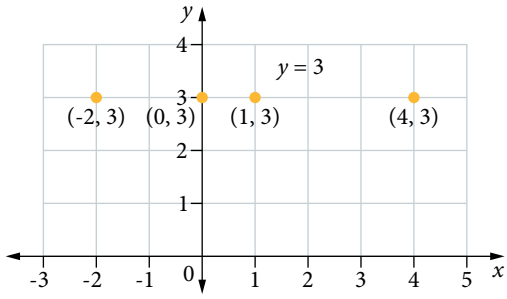
(a)



(b)



(c)



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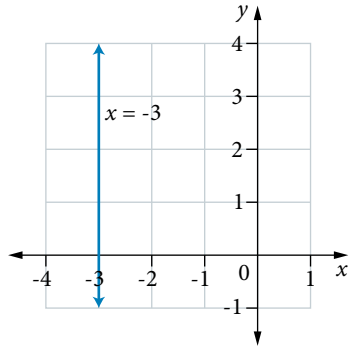

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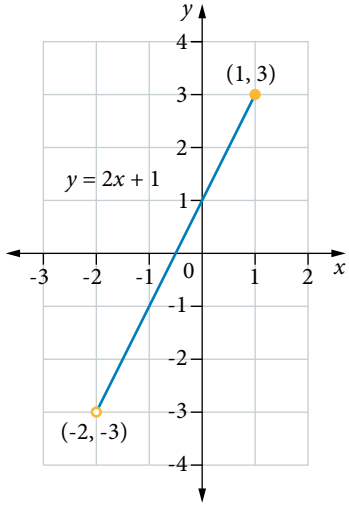

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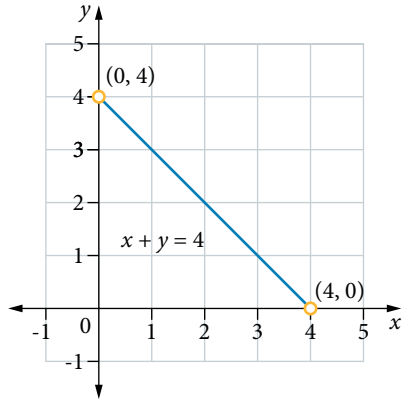
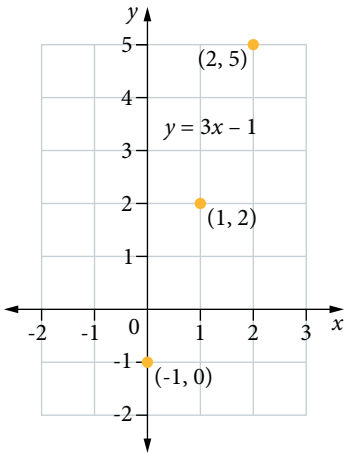
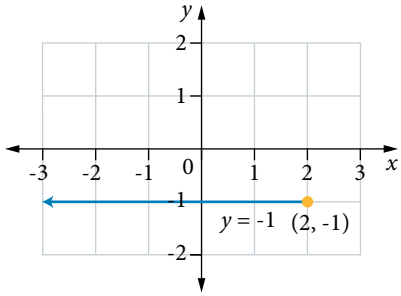


(g)



(h)



Worked Example

3 Determine the range of each linear relation. 2

(a) $x + y = 5, x \in \{0, 1, 2, 3\}$

(b) $y = x + 1, -1 < x < 4$

(c) $y = 3x - 2, x \geq -2$

(d) $y = x + 3, -4 < x < 2$

(e) $2x + y = 5, x \in \{0, 1, 2, 3\}$

(f) $y = -2x, x \leq -3$

(g) $x + 2y = 8, x \in \{-2, 0, 2, 4, 6\}$

(h) $y = 3 - 3x, -1 < x < 2$

4 For the equation $f(x) = 2x + 6$, determine the value of: 3

(a) $f(4)$

(b) $f(0)$

(c) $f(-1)$

(d) x when $f(x) = 3$

(e) x when $f(x) = 0$

(f) x when $f(x) = -5$

5 For the equation $f(x) = \frac{2}{3}x - 1$, determine the value of:

(a) $f(3)$

(b) $f(0)$

(c) $f(-6)$

(d) x when $f(x) = 1$

(e) x when $f(x) = 0$

(f) x when $f(x) = -9$

6 Consider the line with equation $2y = 4x - 8$.

(a) The gradient is:

A -4 B -8 C 2 D -2

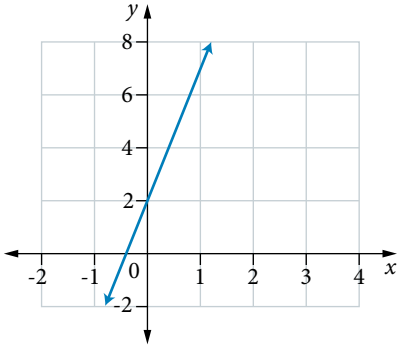
(b) Explain the common error made by a student who thought the answer was 4.

7 The y-intercept of the line with equation $3x = 2y + 6$ is:

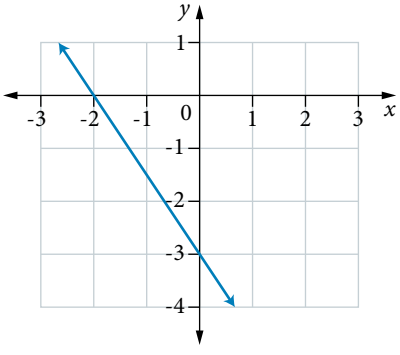
A 3 B -3 C $\frac{2}{3}$ D $\frac{3}{2}$.

8 Identify the gradient of the line for each of the following.

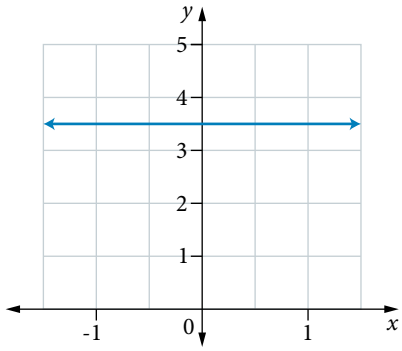
(a)



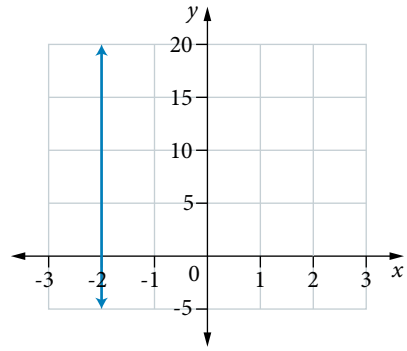
(b)



(c)



(d)



9 Determine the gradient of the line $2y = 9x - 12$.

10 Determine the y-intercept of the line $x - 5y = 20$.

11 Determine the gradient of the line $5x - 10y = -6$.

12 Determine the coordinates of the y-intercept of the line $3x - 6y - 9 = 0$.

13 Determine gradient of the line $x + 4y = 12$.

Worked Example

14 The equation of the line that passes through the point (4, 0) with a gradient of 2 is:

A $y = 4x + 2$ **B** $y = 2x - 4$ **C** $y = 2x$ **D** $y = 2x - 8$

5

15 Determine the equation for each of the following lines.

(a) $m = 2$ and passes through the point (2, 1)

(b) $m = 3$ and passes through the point (2, -6)

(c) $m = \frac{1}{5}$ and passes through the point (-8, 10)

(d) $m = -\frac{2}{3}$ and passes through the point (0, 9)

(e) $m = 8$ and passes through the point (-1, 4)

(f) $m = -\frac{1}{2}$ and passes through the point (-10, 4)

(g) $m = \frac{3}{4}$ and passes through the point (8, 8)

(h) $m = \frac{1}{7}$ and passes through the point (5, 1)

16 Determine the equation of the line that passes through each of the following pairs of points.

(a) (-3, 0) and (0, 6)

(b) (1, 3) and (0, -1)

(c) (4, 0) and (0, 4)

(d) (0, 0) and (10, 0)

(e) (3, -8) and (3, 8)

17 Determine the coordinates of the x- and y-intercepts for the line with equation $9x + 21y + 35 = 0$.

18 The line $2x + 3y + 2 = 0$ passes through the point $(h, 2)$. Determine the value of h.

19 A line with gradient 0.25 passes through the points $(t, 1)$ and (9, 3). Determine the value of t.

20 (a) The gradient of the line passing through the points (2, 2) and (0, -4) is:

A -4 **B** $-\frac{1}{2}$ **C** 3 **D** -2

(b) The gradient of the line passing through the points (-2, 0) and (0, -6) is:

A -6 **B** 2 **C** 3 **D** -3

21 Identify whether each relationship involves direct variation and, where possible, determine the rule and the domain. 7

(a) The number of laptop computers purchased n and the total cost of the purchase C ($), given that each laptop costs $658.

(b) The number of litres l of petrol purchased and the total cost C ($) when cost of petrol is $1.42 per litre.

(c) The total cost C ($) of repairing a washing machine and the number of hours h taken for the repair, given that the cost is charged on a per hour basis only at $80 per hour.

(d) The number of hours h training and the number of wins w for a season.

(e) The amount of money collected C ($) and the number of cars sold n in a car yard.

(f) The maximum temperature for the day T and the number of millimetres of rain for the day n.

(g) The number of kicks at the goal posts k and the number of goals scored g.

22 Rena decides to catch a taxi from her home to the airport. The taxi fare includes a $3.90 flagfall and a rate of $1.59 per kilometre.

(a) Calculate the fare for Rena if the airport is 30 km from her home.

(b) Write an equation to represent the total cost C ($) of a fare for any given distance d (km).

(c) Construct a table of values and then plot the points on a set of axes.

(d) Identify the gradient of the line. What does the gradient represent in this situation?

(e) Identify the coordinate on the vertical axis. What does this intercept represent in this situation?

23 Identify the gradient m and the y-intercept c for each function and hence sketch its graph. Mark the x-intercept on each graph, where appropriate. 4

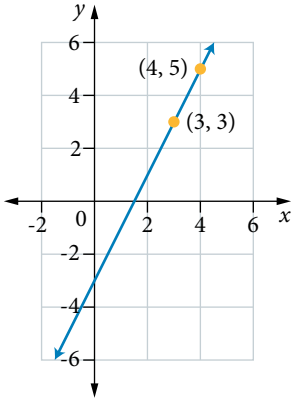
(a) $y = 5x + 9$ **(b)** $y = 3 - 9x$ **(c)** $y = -8x$ **(d)** $y = -1.3x$

(e) $y = 0.2 - 4.5x$ **(f)** $y = 9$ **(g)** $y = -15$ **(h)** $3y = 6x - 4$

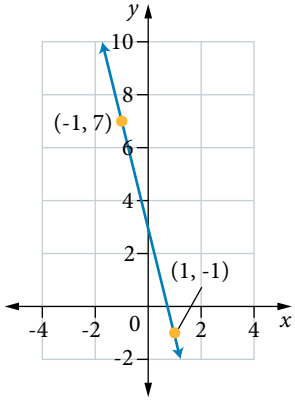
(i) $2y = 9 + 8x$ **(j)** $4y - 3x = 20$ **(k)** $x = 5y - 24$

24 Determine the equation of the line for each of the following graphs.

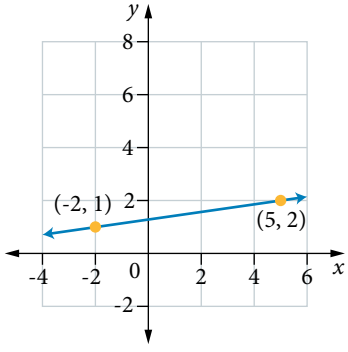
(a)



(b)



(c)



25 Determine the equation of the line that passes through each of the following pairs of points. 6

(a) (8, 1) and (4, -3) **(b)** (1, 1) and (2, 7) **(c)** (-3, 5) and (2, 0)

(d) (2, 11) and (7, 1) **(e)** (-3, 1) and (2, 11)

26 Determine the equation of the line passing through the points (1.3, 2) and (2.8, 2.75).

27 Determine the equation of the line passing through the points (4.2, -1.5) and (5.4, 4.5).

28 Determine the equation of the line passing through the points $\left(-4\frac{2}{3}, 2\frac{1}{3}\right)$ and $\left(\frac{1}{3}, -7\frac{2}{3}\right)$.

29 Given that the line passing through the points (2, p) and (12, -2) has a gradient of $-\frac{6}{5}$, determine the value of p.

30 Ahmed is a cyclist who loves to ride his bicycle in the country. When out riding one day, he sees a sign that indicates the hill ahead has a gradient of 3:25 (i.e. $\frac{3}{25}$). If he knows the height of the hill is 50 m, calculate the horizontal distance along the base of the hill (i.e. the 'run') correct to 1 decimal place.

31 Cyclists often refer to the gradient of a climb in terms of a percentage: $\frac{\text{vertical climb}}{\text{horizontal distance}} \times 100\%$. One of the climbs in the Tour de France has a gradient of 5.1%. If the climb starts at an altitude of 546 m and ends at an altitude of 2068 m, calculate the length of the climb, to the nearest kilometre. State any assumptions you had to make to answer the question.

32 A woman is currently 5 years older than triple her daughter's age. Her daughter is a primary school student. Create a function of the mother's age in terms of her daughter's age. Determine a suitable domain for the function and draw the graph. Comment on the complete set of values, restricting the domain and range if necessary. Identify any assumptions necessary to answer the question.

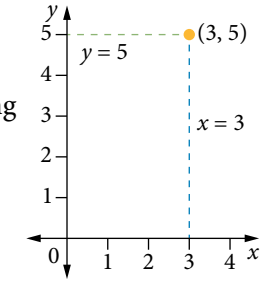
33 The line passing through the points (-2, p) and (1, 11p) has a gradient of $-3\frac{1}{3}$. Determine the equation of the line.

34 The overheads at a factory are \$500 per week. Each item costs \$1.15 to produce and sells for \$4.20. If profit is a function of the number of items sold each week, determine the domain of the function, given that the profit ranges between \$300 and \$400 from week to week. Explain any assumptions you had to make to answer the question.

Solving simultaneous linear equations 1.2

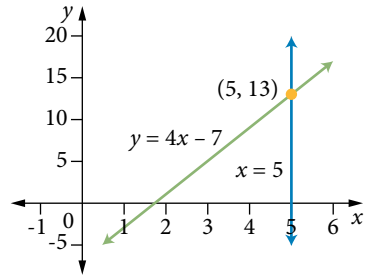
Coordinates

Plotting a point using its coordinates is a simple application of determining the intersection of two lines. For example, to plot the point (3, 5) move along the vertical line $x = 3$ until it intersects the horizontal line $y = 5$.



Substitution

If you substitute a particular x-value into the rule of a linear function, then the resulting calculation determines the corresponding y-value. For example, if $x = 5$ and $y = 4x - 7$, then $y = 4(5) - 7 = 13$. Graphically, this corresponds to the intersection of the vertical line $x = 5$ and the line $y = 4x - 7$ at (5, 13).



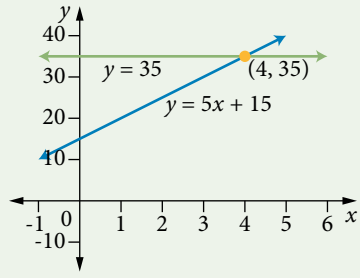
Solving linear equations

To solve the linear equation $5x + 15 = 35$, you need to perform a combination of arithmetic operations (addition, subtraction, multiplication and division) to isolate the variable x. This equation is in the form of $mx + c = k$, so the solution is the x-value of the coordinates of intersection of the graphs of $y = k$ and $y = mx + c$, as shown in the following worked example.

> The solution of a linear equation in the form of $mx + c = k$ can be represented graphically as the x-coordinate of the intersection of the horizontal line $y = k$ and the line $y = mx + c$.

8 Solving linear equations algebraically and graphically

Solve the equation $5x + 15 = 35$ algebraically. Verify the solution graphically.

THINKING	WORKING
1 Transpose the equation to make x the subject. Identify the solution, which is the value of x that makes the equation true.	$5x + 15 = 35$ $5x + 15 - 15 = 35 - 15$ $\frac{5x}{5} = \frac{20}{5}$ $x = 4$
2 Write two equations: $y =$ LHS and $y =$ RHS	$y = 5x + 15$ and $y = 35$
3 Verify your answer by plotting graphs of the equations and interpreting the answer.	 The point of intersection is (4, 35). The coordinates show that $5x + 15$ equals 35 for $x = 4$.

You can use technology to check the answers by drawing two graphs and determining the x-value of the point of intersection.

9 Solve linear equations

Solve each of the following equations algebraically.

(a) $9(4-2m)=6m-12$

THINKING	WORKING
1 Expand the brackets.	$9(4-2m)=6m-12$ $36-18m=6m-12$
2 Collect like terms.	$36+12=6m+18m$ $48=24m$
3 Divide both sides by the coefficient of the variable and state the solution.	$m=\frac{48}{24}$ $m=2$
4 Evaluate the reasonableness of the solution.	$\text{LHS}=9(4-2\times(2))$ $=9\times(4-4)$ $=0$ $\text{RHS}=6\times(2)-12$ $=12-12$ $=0$ LHS = RHS so the solution is $m=2$.

(b) $\frac{3x+4}{2}=5x-3$

1 Multiply both sides of the equation by the denominator.	$\frac{3x+4}{2}=5x-3$ $2\times\frac{3x+4}{2}=2\times(5x-3)$ $3x+4=10x-6$
2 Collect like terms.	$4+6=10x-3x$ $10=7x$
3 Divide both sides by the coefficient of the variable and state the solution.	$\frac{10}{7}=\frac{7x}{7}$ $x=\frac{10}{7}$
4 Evaluate the reasonableness of the solution.	$\text{LHS}=\frac{3\times\left(\frac{10}{7}\right)+4}{2}$ $=\frac{\frac{30}{7}+4}{2}$ $=\frac{29}{7}$ $\text{RHS}=5\times\left(\frac{10}{7}\right)-3$ $=\frac{50}{7}-\frac{21}{7}$ $=\frac{29}{7}$ LHS = RHS so the solution is $x=\frac{10}{7}$.

(c) $\dfrac{4x+3}{2} = \dfrac{-38-3x}{5}$

1 Multiply both sides of the equation by the denominators.	$\dfrac{4x+3}{2} = \dfrac{-38-3x}{5}$ $\dfrac{4x+3}{2} \times 2 = \dfrac{2(-38-3x)}{5}$ $5(4x+3) = \dfrac{2(-38-3x)}{5} \times 5$ $5(4x+3) = 2(-38-3x)$ $20x+15 = -76-6x$
2 Collect like terms.	$20x+6x = -76-15$ $26x = -91$
3 Solve the equation and find the solution by transposing to make the unknown the subject.	$\dfrac{26x}{26} = \dfrac{-91}{26}$ $x = -3.5$
4 Evaluate the reasonableness of the solution.	$\text{LHS} = \dfrac{4(-3.5)+3}{2} = \dfrac{-14+3}{2} = \dfrac{-11}{2} = -5.5$ $\text{RHS} = \dfrac{-38-3(-3.5)}{5} = \dfrac{-38+10.5}{5} = \dfrac{-28.5}{5} = -5.5$ LHS = RHS so the solution is $x = -3.5$.

Simultaneous linear equations

The coordinates of the point of intersection of lines represent the graphical solution to simultaneous linear equations. The intersection is a unique point that satisfies both equations and is determined by the linear equations.

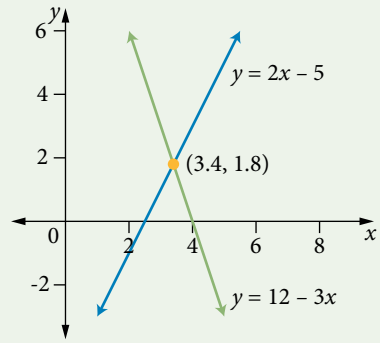
You can determine the solution to simultaneous linear equations algebraically or graphically, using technology.

Technology worked example
Solving linear equations

10 Solution of simultaneous equations where *y* is the subject

Solve the simultaneous equations $y = 2x - 5$ and $y = 12 - 3x$ algebraically. Verify the solution graphically.

THINKING	WORKING
1 Identify the simultaneous linear equations and equate.	$y = 2x - 5 \quad [1]$ $y = 12 - 3x \quad [2]$
2 Equate the equations and solve for x by collecting like terms to make x the subject.	$2x - 5 = 12 - 3x$ $2x = 17 - 3x$ $5x = 17$ $x = \dfrac{17}{5}$ $x = 3.4$

3 Substitute into one of the original equations to calculate the corresponding y-value.	Substitute $x = 3.4$ into [1]: $y = 2x - 5$ $= 2 \times 3.4 - 5$ $= 6.8 - 5$ $= 1.8$
4 Identify the simultaneous solution.	$x = 3.4$ and $y = 1.8$
5 Check the reasonableness of the solution by substituting the x-value into the other equation.	Substitute $x = 3.4$ into [2]: $y = 12 - 3x$ $= 12 - 3 \times 3.4$ $= 12 - 10.2$ $= 1.8$
6 Sketch both graphs on the same set of axes using graph paper or a graphics calculator.	
7 The solution is where the two graphs intersect. Read off the coordinates.	The coordinates of the intersection are (3.4, 1.8). The simultaneous solution is $x = 3.4$ and $y = 1.8$.

Additional information

Solving linear equations graphically

View the video for a fully worked example of solving linear equations graphically.

Solution of simultaneous equations by substitution

The substitution method is most efficient when one of the variables has a coefficient of 1. To solve two linear equations using substitution:

- Make the variable with a coefficient of 1 the subject of the equation.
- Substitute the expression into the other equation to determine the value of the corresponding variable.
- Substitute the value of that variable into either equation to determine the value of the other variable.

11 Solution of simultaneous equations using substitution

Solve $y = 3x + 7$ and $x + 2y = 7$ simultaneously by substitution.

THINKING	WORKING
1 Label the equations [1] and [2].	$y = 3x + 7$ [1] $x + 2y = 7$ [2]

2 Substitute equation [1] into equation [2] and solve for x (or y).	Substitute [1] into [2]: $x + 2(3x + 7) = 7$ $x + 6x + 14 = 7$ $7x = -7$ $x = -1$
3 Substitute the value for x (or y) into one of the equations to determine the corresponding y (or x) value.	Substitute $x = -1$ into [1]: $y = 3(-1) + 7$ $y = 4$
4 Identify the simultaneous solution.	The simultaneous solution is $x = -1$ and $y = 4$.
5 Check the reasonableness by substituting the solution into one of the equations.	For [2], $x + 2y = 7$: LHS $= x + 2y$ $= -1 + 2 \times 4$ $= 7$ LHS = RHS so the solution is correct.

Solution of simultaneous equations by elimination

The elimination method is most effiecient when the equations are in the form of $ax + by = c$ and neither variable has a coefficient of 1. To solve two linear equations using elimination:

- Rearrange the equations so that the coefficients of one variable are identical or have the same magnitude but different sign. You may have to multiply all terms in the equation by a constant first.
- If the coefficients are *identical*, then *subtract* the two equations to eliminate the variable.
- If the coefficients have *opposite signs*, then *add* the two equations to eliminate the variable.

12 Solution of simultaneous equations using elimination

Solve the simultaneous equations algebraically using the elimination method.

(a) $3y = 2x + 3$ and $2x + 5y = -11$

THINKING	WORKING
1 Define the equations in sequence with variables in corresponding positions.	$3y = 2x + 3 \quad [1]$ $5y = -2x - 11 \quad [2]$
2 Determine which variable to eliminate.	The coefficient of x is equal in magnitude but opposite in sign in each equation and so can be eliminated by adding.
3 Add the equations and solve for the unknown variable.	[1] + [2]: $8y = -8$ $y = -1$

4 Substitute the value into either of the original equations and solve for the second variable.	Substitute $y = -1$ into [1]: $3(-1) = 2x + 3$ $-3 = 2x + 3$ $2x = -6$ $x = -3$
5 Write the simultaneous solution.	The solution to the simultaneous equations is $x = -3$ and $y = -1$.
6 Check the reasonableness by substituting the solution into the other equation.	For [2], $2x + 5y = 11$: $\text{LHS} = 2x + 5y$ $= 2(-3) + 5(-1)$ $= -11$ LHS = RHS so the solution is correct.

(b) $5x - 4y = -11$ and $-3x - 8y = 4$

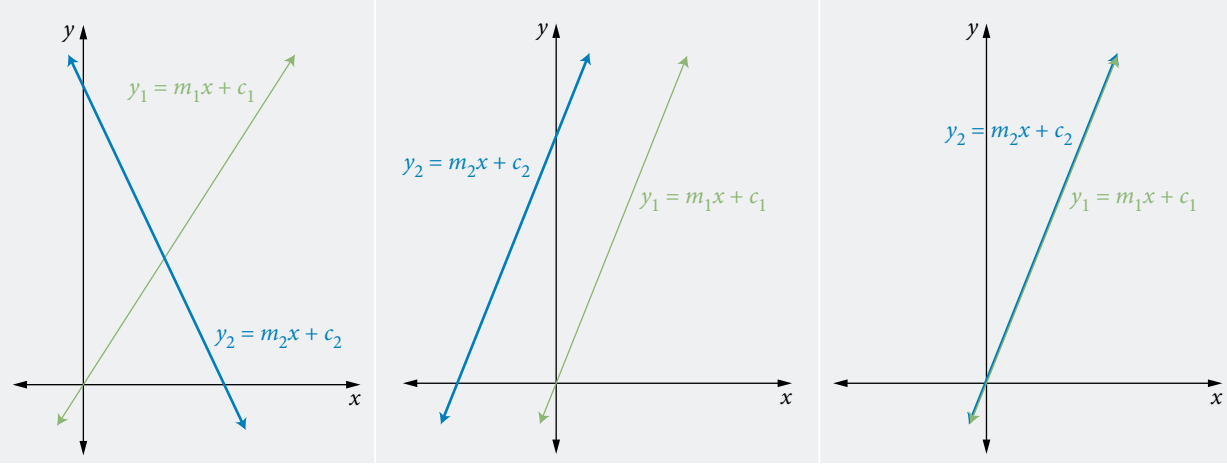
1 Define the equations in sequence with variables in corresponding positions.	$5x - 4y = -11 \quad [1]$ $-3x - 8y = 4 \quad [2]$
2 Determine which variable to eliminate. Multiply each equation by a constant (if necessary) so that the coefficients of the selected variable (y in this case) are equal.	The coefficient of y in [2] is twice the coefficient of y in [1]: $[3] = 2 \times [1]$: $10x - 8y = -22 \quad [3]$ $-3x - 8y = 4 \quad [2]$
3 Subtract one equation from the other (with an equal coefficient) and then solve for the unknown variable.	[2] − [3]: $-13x = 26$ $x = -2$
4 Substitute the value into either of the original equations and solve for the second variable.	Substitute $x = -2$ into [1]: $5(-2) - 4y = -11$ $-10 - 4y = -11$ $-4y = -1$ $y = \frac{1}{4}$
5 Write the simultaneous solution.	The solution to the simultaneous equations is $x = -2$ and $y = \frac{1}{4}$.
6 Check the reasonableness by substituting the solution into the other equation.	For [2], $-3x - 8y = 4$: $\text{LHS} = -3x - 8y$ $= -3 \times (-2) - 8 \times \left(\frac{1}{4}\right)$ $= 6 - 2$ $= 4$ LHS = RHS so the solution is correct.

(c) $3x + 2y = 5$ and $4x + 3y = 8$

1 Define the equations in sequence with variables in corresponding positions.	$3x + 2y = 5$ [1] $4x + 3y = 8$ [2]
2 Determine which variable to eliminate. Multiply each equation by a constant so that the coefficients of the selected variable (y in this case) are the same.	$3 \times [1]$: $9x + 6y = 15$ [3] $2 \times [2]$: $8x + 6y = 16$ [4]
3 Subtract one equation from the other (with an equal coefficient) and then solve for the unknown variable.	$[3] - [4]$: $x = -1$
4 Substitute the value into either of the original equations and solve for the second variable.	$3(-1) + 2y = 5$ $2y = 8$ $y = 4$
5 Write the simultaneous solution.	Solution: $x = -1$, $y = 4$
6 Check the reasonableness by substituting the solution into the other equation.	For [2], $4x + 3y = 8$: LHS $= 4x + 3y$ $= 4 \times (-1) + 3 \times 4$ $= 8$ LHS = RHS so the solution is correct.

Number of solutions for the system of linear equations

A system of simultaneous solutions has a unique solution (a single point of intersection) if $m_1 \neq m_2$.	A system of simultaneous equations will have no solution if $m_1 = m_2$ but $c_1 \neq c_2$. Graphically the equations will produce parallel lines.	A system of simultaneous equations will have infinite solutions if $m_1 = m_2$ and $c_1 = c_2$ because $y_1 = y_2$ for every value of x.



Technology worked example
Solving simultaneous equations graphically

Solving problems

Simultaneous equations can be used to determine unknown values. For two unknowns, you will require two pieces of information to define two variables and form two equations. Use your technology or algebraic methods to solve the equations, and then interpret the solution in terms of the original problem.

Additional information

The number of solutions for the system of linear equations

Sometimes no unique solution exists; view the reasoning in this activity.

13 Problem solving using simultaneous equations

Three sandwiches and a drink cost \$19.45. Two sandwiches and two drinks cost \$17.50. Determine the individual cost of a sandwich and a drink.

THINKING	WORKING
1 State any assumptions you need to make to solve the problem.	Assume all drinks cost the same amount, and all sandwiches cost the same amount.
2 Define the variables.	Let the cost of a sandwich be x. Let the cost of a drink be y.
3 Construct equations corresponding to each piece of information.	$3x + y = 19.45$ [1] $2x + 2y = 17.50$ [2]
4 Transpose one of the equations, making x or y the subject.	$y = 19.45 - 3x$ [1]
5 Substitute one equation into the other and solve for the unknown variable.	Substitute [1] into [2]: $2x + 2(19.45 - 3x) = 17.50$ $2x + 38.9 - 6x = 17.5$ $-4x = -21.4$ $x = \frac{-21.4}{-4}$ $= 5.35$
6 Substitute the value into either of the original equations to determine the value of the second variable.	Substitute $x = 5.35$ into [1]: $y = 19.45 - 3x$ $= 19.45 - 3 \times 5.35$ $= 3.4$
7 Interpret the solution, stating the solution to the original problem.	A sandwich costs \$5.35 and a drink costs \$3.40.
8 Check the reasonableness of the solution by substituting the values into the original equations.	$3 \times \$5.35 + \$3.40 = \$19.45$ $2 \times \$5.35 + 2 \times \$3.40 = \$17.50$

EXERCISE 1.2 Solving simultaneous linear equations

Worked Example

1 Solve each of the following equations for x. Verify the solution graphically. 8

(a) $3x + 8 = 20$ (b) $15 - 4x = 31$ (c) $4(x + 2) = 16$

(d) $-3(x - 8) = -16$ (e) $4(6x - 9) = -48$ (f) $3(x - 8) + 20 = -25$

2 Solve for the unknown.

(a) $\frac{2x}{5} = 8$ (b) $\frac{a}{7} = 9$ (c) $\frac{3m}{10} = 5$

(d) $\frac{h + 6}{2} = 12$ (e) $\frac{y - 8}{3} = 14$

3 Solve for the unknown value. 9

(a) $8y - 10 = 4y$ (b) $23x + 2 = 17x - 34$ (c) $27x + 17 = 42 - 13x$

(d) $5 - 22a + 16 = 3a - 34$ (e) $10b + 9 = 3b - 12$ (f) $2(4h - 6) = 3(2h + 6)$

(g) $2(15 - 5d) = 37 - 9d$ (h) $12x + 3 = 13 - 3x$ (i) $4(3y + 6) = 10y + (9y - 4)$

(j) $4(16a + 5) = 28$ (k) $15 - 4m = 3(3 - 2m)$

4 Solve for the unknown.

(a) $34.8 = 9h + 16.1$ (b) $5 - 0.2(8 - 6x) = 8 + 4(0.5x + 1)$

(c) $15(2 - 3m) = 47$ (d) $\frac{5x - 7.5}{3.5} = 15$

5 Solve for the unknown.

(a) $\frac{3g}{2} = 10g + 1$ (b) $\frac{3h + 5}{2} = 4h - 15$ (c) $\frac{24 - 3m}{3} = -1 - 2m$

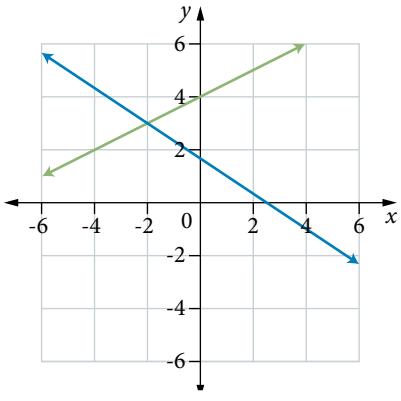
(d) $\frac{16 + 3x}{4} = x + 2$ (e) $\frac{4 + 2k}{3} = \frac{3 - 8k}{7}$ (f) $\frac{13 + 12y}{10} = \frac{8y + 6}{5}$

6 Select the correct solution to the equation $\frac{2x + 6}{3} = 6$.

A $x = 2$ B $x = 0$ C $x = 6$ D $x = 1\frac{1}{2}$

7 Determine the solution for the simultaneous equations displayed on each graph.

(a)



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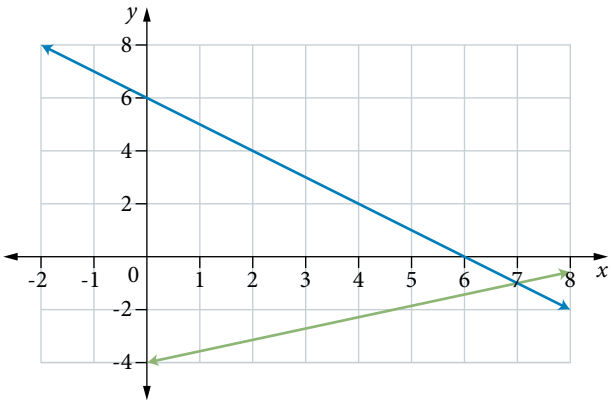

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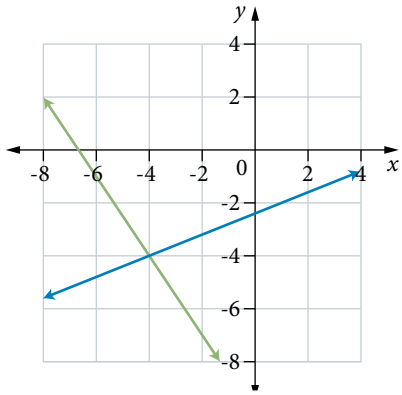
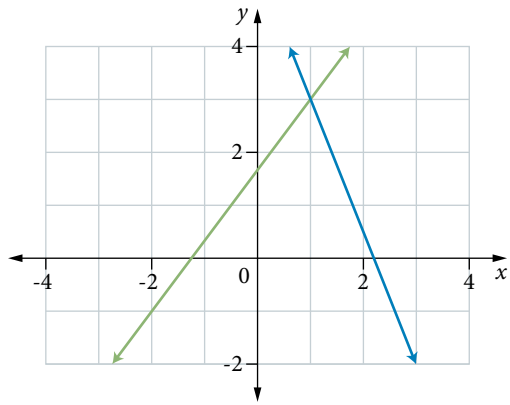


(d)



Worked Example

10

8 Solve the following simultaneous linear equations. Verify your solution graphically.

(a) $y = 3x + 2$ and $y = 4x + 10$

(b) $y = -2x + 4$ and $y = 3x - 16$

(c) $y = 0.5x + 2$ and $y = -6x + 2$

(d) $y = 11x$ and $y = -19x + \frac{2}{3}$

11

9 Solve the following simultaneous linear equations using the substitution method.

(a) $5x + y = -10$ and $4x - 3y = -27$

(b) $x - 8y - 8 = 0$ and $6x + 7y - 48 = 0$

(c) $x + 10y = 66$ and $2x + 3y - 30 = 0$

(d) $3x - y = -11$ and $5x - 2y = -21$

10 Determine the solution to the following simultaneous linear equations.

(a) $3x - y = 1$ and $-3x + 4y = 14$

(b) $5x - y = 1$ and $4x + y + 10 = 0$

(c) $4x + 5y = 30$ and $2x - 5y = 12$

(d) $2x - 4y = 24$ and $2x + 3y + 18 = 0$

11 Sharna made an error when solving each of the following equations. Identify her mistake(s) and write the correct solution.

(a) $2(2x + 4) = 32$ divide by 2

$x + 2 = 16$ subtract 2

$x = 14$

(b) $\frac{4(m-6)}{3} = 8$

$4m - 6 = 24$

$4m = 30$

$m = 7.5$

12

12 Solve each of the following simultaneous linear equations using the elimination method.

(a) $3x + 2y = 9$ and $4x - 2y = 26$

(b) $5x + 4y = 62$ and $6y - 5x = 18$

(c) $3x + 4y + 26 = 0$ and $2x - 3y = 11$

(d) $6x - 5y = 34$ and $4x + 3y = 10$

(e) $x + y = 12$ and $3x - 2y = 16$

13 Determine the solution to the following simultaneous linear equations.

(a) $2x + 3y = 5$ and $2x - 4y = -16$

(b) $3x + 3y = 6$ and $-2x + 4y = -16$

(c) $2x - 5y = 12$ and $3x + 2y = -20$

(d) $5x + 2y = 11$ and $4x - 3y = -5$

14 Solve each pair of simultaneous linear equations, correct to 2 decimal places.

(a) $5x + 2.7y = -35.4$ and $3.6x - 4.3y = -37.3$

(b) $0.72x - 8y = 28.54$ and $4.11x + 5.03y = 56.21$

(c) $\frac{5}{7}x + 3y = 14$ and $\frac{2}{5}y - \frac{3}{4}x = -\frac{19}{4}$

(d) $\frac{4}{5}x - \frac{2}{3}y = -10$ and $\frac{3}{4}x + \frac{1}{2}y = 20$

(e) $-1.36x - 5.20y = 0.40$ and $0.72y + 2.81x = -7.70$

(f) $-0.05x - 5.20y = 0.40$ and $0.72y + 0.01x = -7.70$

15 (5, -3) is the point of intersection of the graphs of the line $3x + y = 12$ and which other line?

A $5x - 3y = 12$ B $2x - 4y = 22$ C $3x + y = 15$ D $3x - 5y = 22$

16 Write an equation and determine the value of the original number for each of the following.

(a) When I double a number and add fourteen, the result is fifty-two.

(b) One-quarter of a number is added to three times the same number and the result is sixty-five.

(c) When a number is multiplied by three and added to nine, the result is the same as adding three to the original number and then multiplying the result by five.

(d) When seven is subtracted from the product of four and a number, and the result is divided by three, the answer equals the original number.

(e) If a number is multiplied by three and then ten is subtracted, the answer is the same as doubling the original number and adding two.

Worked Example 13

17 Form a pair of simultaneous equations and solve them to answer each of the following problems.

(a) Gerhard is five times the age of his son, Benny, and the sum of their ages is 42 years. How old are Gerhard and Benny?

(b) Two drinks and three ice-creams cost \$14.80, and three drinks and four ice-creams cost \$20.60. Determine the cost of each drink and each ice-cream.

(c) Two apples and five bananas cost \$7.90 and six apples and two bananas cost \$8.10. Determine the cost of each apple and each banana.

(d) The total weight of six chairs and one table is 112 kg and the total weight of two chairs and two tables is 179 kg. Determine the weight of each chair and each table.

18 A straight line with the equation $y = mx + c$ passes through the points (10, 2) and (-5, -3). Determine the value of m and c and hence determine the equation of the line.

19 Company A charges \$10 per hour for the hire of one of their canoes, plus a fixed charge for insurance of \$20. Company B charges \$8 per hour for the hire of their canoes and an insurance charge of \$30.

(a) Sketch each of the graphs of total cost C (\$) against time t (hours) for the two companies.

(b) Determine the time t when the charge for both companies will be the same.

(c) Determine which company is the cheaper for 6 hours hire, and by how much.

20 Guildford School plans to take a number of students on a day excursion. If the students travel by public transport, the cost will be \$3.50 per student. If the students travel by a private charter bus, the cost is \$1.50 per student plus \$40 for the hire of the bus.

(a) Use graphs to determine how many students are needed for the bus to be the cheaper option.

(b) If two buses are needed, then another \$40 would need to be paid. Use graphs to determine how many students are needed for the bus to be the cheaper option if two buses are required.

21 Ashlee and Tahnee live 48 km apart. At 9 am they both set out to ride to each other's home and travel the same route but in the opposite directions. Ashlee rides at a constant speed of 20 km/h and Tahnee rides at a constant speed of 18 km/h.

(a) Use equations and graphs to determine the time at which they will meet.

(b) How far has each of them travelled, to the nearest kilometre, by the time they meet?

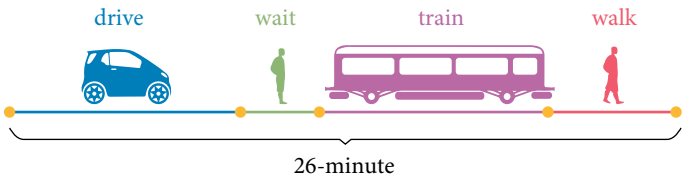
1.3 Piece-wise functions

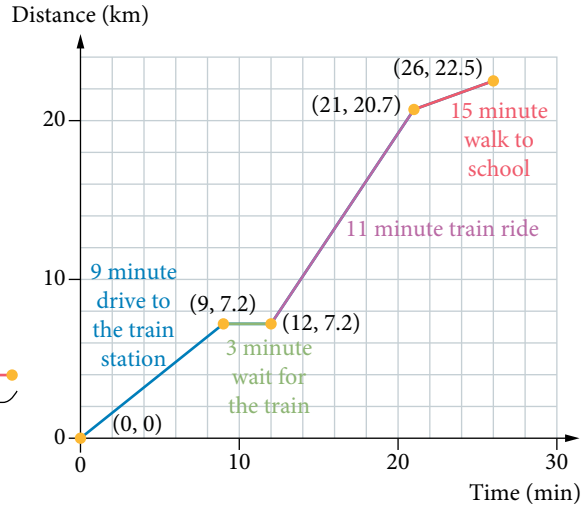
Continuous piece-wise functions

A *piece-wise function* is made up of different functions over subsets of its domain. Each segment has its own rule and subdomain. The endpoints of the domains are sometimes called critical points. A graph is considered to be continuous if the subdomains of adjacent segments meet at the same coordinate point. The graph will be continuous if it has no gaps and you can draw the line without lifting your pen.

14 Determine the rule of a continuous piece-wise graph

Determine the rule of the piece-wise function for a 22.5 km journey over a 26-minute period.





THINKING	WORKING
1 Define the variables and examine each segment of the graph to determine an appropriate domain for each, ensuring that none of the subdomains overlap.	Let x be the time in minutes. Let $f(x)$ be the distance travelled in kilometres. $f(x) = \begin{cases} \text{drive}, & 0 \le x \le 9 \\ \text{wait}, & 9 < x \le 12 \\ \text{train}, & 12 < x \le 21 \\ \text{walk}, & 21 < x \le 26 \end{cases}$
2 Use the coordinates of the endpoints to determine the equations of the drive and waiting segments.	Drive segment (0, 0) to (9, 7.2): $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7.2 - 0}{9 - 0} = 0.8$ $y_1 = 0.8x$ Waiting segment (9, 7.2) to (12, 7.2): The waiting segment is the horizontal line: $y_2 = 7.2$

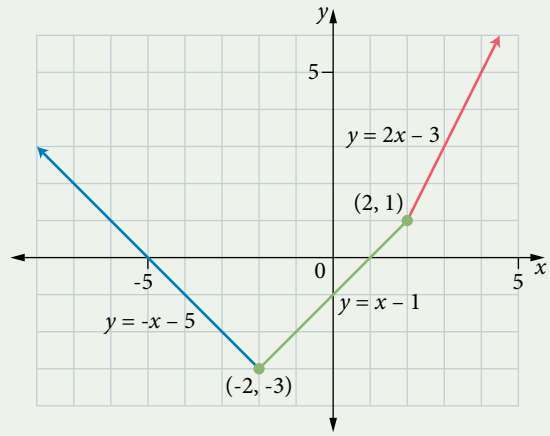
3 Use the coordinates of the endpoints to determine the gradient and y-intercept of the train and walk segments, and write in the form $y = mx + c$.	Train segment (12, 7.2) to (21, 20.7): $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{20.7 - 7.2}{21 - 12}$ $= \frac{13.5}{9}$ $= 1.5$ $m = 1.5$ and (12, 7.2): $y - y_1 = m(x - x_1)$ $y - 7.2 = 1.5(x - 12)$ $y - 7.2 = 1.5x - 18$ $y = 1.5x - 10.8$ $y_3 = 1.5x - 10.8$	Walk segment (21, 20.7) to (26, 22.5): $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{22.5 - 20.7}{26 - 21}$ $= \frac{1.8}{5}$ $= 0.36$ $m = 0.36$ and (21, 20.7): $y - y_1 = m(x - x_1)$ $y - 20.7 = 0.36(x - 21)$ $y - 20.7 = 0.36x - 7.56$ $y = 0.36x + 13.14$ $y_4 = 0.36x + 13.14$
4 Check the reasonableness of the function by checking that the endpoints meet.	For $x = 9$: $y_1 = 0.8 \times 9 = 7.2$ $y_2 = 7.2$ $y_1 = y_2$ so the graph is continuous at $x = 9$. For $x = 12$: $y_2 = 7.2$ $y_3 = 1.5 \times 12 - 10.8 = 7.2$ $y_2 = y_3$ so the graph is continuous at $x = 12$.	For $x = 21$: $y_3 = 1.5 \times 21 - 10.8$ $= 20.7$ $y_4 = 0.36 \times 21 + 13.14$ $= 20.7$ $y_3 = y_4$ so the graph is continuous at $x = 21$.
5 Construct a piece-wise function by combining the rules of each segment.	$f(x) = \begin{cases} 0.8x, & 0 \le x < 9 \\ 7.2 & 9 \le x \le 12 \\ 1.5x - 10.8 & 12 < x \le 21 \\ 0.36x + 13.14 & 21 < x \le 26 \end{cases}$	

To sketch a piece-wise graph, calculate the critical points where the rule changes at the end of each segment, and then plot the segments.

15 A continuous piece-wise graph

(a) Plot the graph of $f(x)$ and identify the coordinates of the axes intercepts.

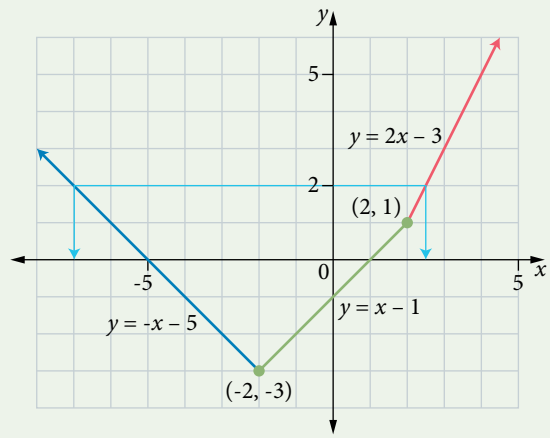
$$f(x) = \begin{cases} -x - 5, & x < -2 \\ x - 1, & -2 \le x \le 2 \\ 2x - 3, & x > 2 \end{cases}$$

THINKING	WORKING
1 Determine the coordinates of the endpoint of each segment by substituting the extreme values of the domain into the appropriate rule. Where the extreme is not specified, choose a representative value.	For $x < -2$: $x = -5, y = 0$: $(-5, 0)$ $y = -x - 5$ $x = -2, y = -3$: $(-2, -3)$ For $-2 \le x \le 2$: $x = -2, y = -3$: $(-2, -3)$ $y = x - 1$ $x = 2, y = 1$: $(2, 1)$ For $x > 2$: $x = 2, y = 1$: $(2, 1)$ $y = 2x - 3$ $x = 3, y = 3$: $(3, 3)$
2 Construct a set of axes, scaled to cover the required domain. Plot the calculated points and join using straight lines. Include arrowheads where required.	 The axes intercepts are (-5, 0), (0, -1) and (1, 0).

(b) Calculate $f(-4)$.

1 Determine the subdomain with the given value of x.	$x = -4$ is in the subdomain $x < -2$ for the equation $f(x) = -x - 5$.
2 Substitute x into the expression for the subdomain and calculate.	$f(-4) = -x - 5$ $= -(-4) - 5$ $= -1$

(c) Calculate the value(s) of x for $f(x) = 2$.

1 Estimate the value from the graph.	 $f(x) = 2$ at approximately $x = -7$ and $x = 2.5$.

2 Solve for x algebraically and interpret the result.	For $f(x) = 2$ and $f(x) = -x - 5$: $-x - 5 = 2$ $-x = 7$ $x = -7$ For $f(x) = 2$ and $f(x) = 2x - 3$: $2x - 3 = 2$ $2x = 5$ $x = 2.5$ Therefore $f(x) = 2$ at $x = -7$ and $x = 2.5$.

Remember that technology can be used to determine intersections of lines or to read off values from individual graphs.

Technology worked example
A continuous piece-wise function

Discontinuous piece-wise functions

If adjacent segments of a piece-wise graph do not meet at the same coordinate point, then the graph will be discontinuous at that critical point. Always check to see whether each endpoint of a segment is included in the segment.

16 A discontinuous piece-wise graph

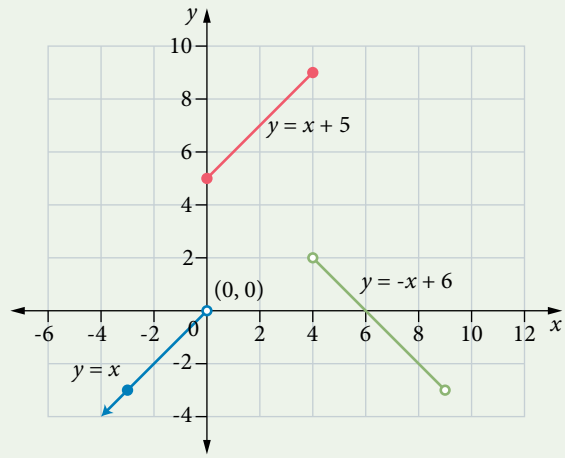
Plot the following piece-wise graph.

$$y = \begin{cases} x, & x < 0 \\ x + 5, & 0 \le x \le 4 \\ -x + 6, & 4 < x < 9 \end{cases}$$

THINKING	WORKING
1 Determine the endpoints for each segment by substituting in the extreme values of the domain. Use technology to calculate the endpoints, when allowed.	For $x < 0$: $y = x$ $x = -3, y = -3$: $(-3, -3)$ $x = 0, y = 0$: $(0, 0)$ For $0 \le x \le 4$: $y = x + 5$ $x = 0, y = 5$: $(0, 5)$ $x = 4, y = 9$: $(4, 9)$ For $4 < x < 9$: $y = -x + 6$ $x = 4, y = 2$: $(4, 2)$ $x = 9, y = -3$: $(9, -3)$

2 Construct a set of axes, scaled to cover the required domain. Plot the calculated points and join using straight lines.

Use a closed circle if a point is included and an open circle if not included. Draw arrowheads on lines with no endpoints.



It may be possible to choose the value of a parameter, usually denoted by a, such that a piece-wise function is made to be continuous.

Explore further

Piece-wise functions

Explore the construction of continuous and discontinuous piece-wise functions.

17 Ensure continuity

Determine the value of a parameter a that makes the function continuous over the domain. Justify your answer graphically.

$$y = \begin{cases} 2x+1, & x \le 1 \\ a - x, & x > 1 \end{cases}$$

THINKING

1 For the function to be continuous at the critical x-value the function must be the same for each subdomain.

WORKING

For the function to be continuous at $x = 1$:

$$\begin{aligned} 2x+1 &= a - x \\ 2\times 1 + 1 &= a - 1 \\ 3 &= a - 1 \\ a &= 4 \end{aligned}$$

$$y = \begin{cases} 2x+1, & x \le 1 \\ 4 - x, & x > 1 \end{cases}$$

2 Determine the axes intercepts for the $y = 2x + 1$ segment.

For $x = 0$:

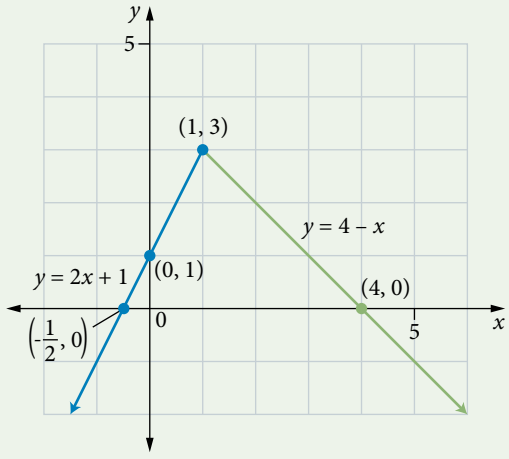
$$\begin{aligned} y &= 2x+1 \\ &= 2\times 0 + 1 \\ &= 1 \end{aligned}$$

The y-intercept is at $(0, 1)$.

For $y = 0$:

$$\begin{aligned} y &= 2x+1 \\ 0 &= 2x+1 \\ x &= -\frac{1}{2} \end{aligned}$$

The x-intercept is at $\left(-\frac{1}{2}, 0\right)$.

3 Determine the x-intercept and the critical point for the $y = 4 - x$ segment.	For $y = 0$: $y = 4 - x$ $0 = 4 - x$ $x = 4$ The x-intercept is at (4, 0).	For $x = 1$: $y = 4 - x$ $= 4 - 1$ $= 3$ The critical point is at (1, 3).
4 Draw the graph.		
5 Interpret the answer.	The piece-wise function is continuous for $a = 4$.	

Step graphs

A step graph is a line graph with horizontal segments or 'steps'. The endpoints of adjacent segments line up vertically to create a step. The graph is discontinuous at the endpoint of a segment, between the steps.

An example of a step graph is the cost of a taxi ride that depends on the distance travelled. There is usually an initial charge (flagfall) and a cost per kilometre. If the total cost were a continuous function of distance, it would be a linear function with flagfall as the constant term (or vertical intercept) and the charge per kilometre as the gradient of the line. In reality, a taxi meter 'clicks over' incrementally at the beginning of each kilometre (or other set distance), so that the charge is per whole kilometre, or part thereof of any unfinished kilometre.

18 Step graphs

A taxi fare includes a flagfall of \$2.90 and then a charge of \$2.17 per kilometre (or part). Draw the step graph of the total cost for a taxi journey of up to 4 m.

THINKING	WORKING
1 Define the variables.	Independent variable: Let distance be d (km). Dependent variable: Let total cost be C (\$).

2 Determine endpoint values for each kilometre of the journey. Remember the charge for distance is incurred at the start of the increment of 1 km. This means that you pay for the first kilometre immediately.

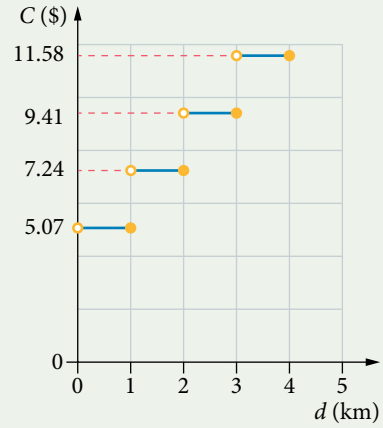
$0 < d \leq 1$: \$2.90 + \$2.17 = \$5.07
Plot (0, 5.07), (1, 5.07)

$1 < d \leq 2$: \$5.07 + \$2.17 = \$7.24
Plot (1, 7.24), (2, 7.24)

$2 < d \leq 3$: \$7.24 + \$2.17 = \$9.41
Plot (2, 9.41), (3, 9.41)

$3 < d \leq 4$: \$9.41 + \$2.17 = \$11.58
Plot (3, 11.58), (4, 11.58)

3 Plot the calculated points and form horizontal line segments. The closed circles at the end of each segment indicate that the incremental fare is paid up to and including the end of each kilometre.



Step graphs such as these are often approximated to continuous linear graphs.

Varying rates

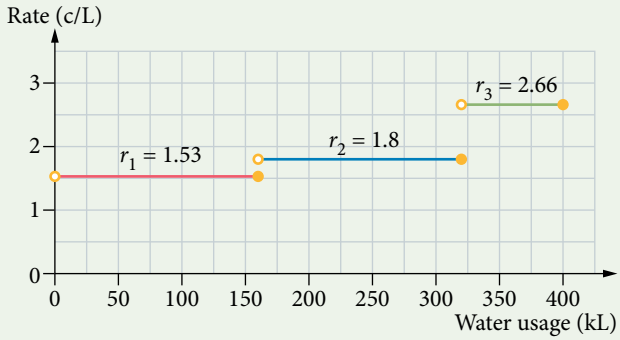
A water supply company may charge different rates, depending on how much water a household uses. To encourage water conservation, the cost per litre may increase above a certain level.

19 Display costs from variable rates

The charges for half-yearly water usage are given in the table.

Water usage (L)	Cost (c/L)
0–160 000	1.53
160 000–320 000	1.8
More than 320 000	2.66

(a) Construct a step graph for the cost rate and water usage for up to 400 000 L.

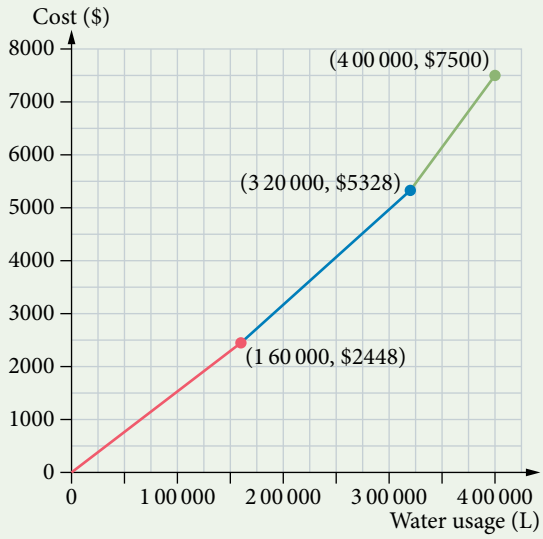
THINKING	WORKING
1 Define the variables used and construct an appropriate function.	Let r represent the cost rate and w represent the volume of water used in L. $$r = \begin{cases} 1.53, & 0 < w \leq 160\,000 \\ 1.8, & 160\,000 < w \leq 320\,000 \\ 2.66, & 320\,000 < w \leq 400\,000 \end{cases}$$
2 Plot the cost per litre over the appropriate domain.	

(b) Draw a piece-wise linear graph to represent the charges for consumption from 0 to 400 000 L.

1 Determine the cost of usage at the endpoints of the graph and the critical points where the costs change.

Water usage (L)	Cost ($)
0	0
160 000	$160\,000 \times 0.0153 = \$2448$
320 000	$2448 + 160\,000 \times 0.018 = \$5328$
400 000	$5328 + 40\,000 \times 0.0266 = \$6932$

2 Construct a suitably scaled set of axes by plotting the critical points. These segments are linear so connect the points with straight lines.



(c) Determine the rule for the piece-wise function.

1 Define the variables.

Let water usage be w (L) and total cost be C ($).

2 Construct individual rules for each part of the domain, where the gradient of each segment is the rate.

For $0 < w \le 160\,000$: $C_1 = 0.0153w$

For $160\,000 < w \le 320\,000$: $C_2 = 0.018w + k$

$(160\,000, 2448)$:

$2448 = 0.018 \times 160\,000 + k$

$2448 = 2880 + k$

$k = -432$

$C_2 = 0.018w - 432$

For $w > 320\,000$: $C_3 = 0.0266w + c$

$(320\,000, 5328)$:

$5328 = 0.0266 \times 320\,000 + c$

$5328 = 8512 + c$

$c = -3184$

$C_3 = 0.0266w - 3184$

3 Determine the rule of the piece-wise function.

$$C = \begin{cases} 0.0153w, & 0 < w \le 160\,000 \\ 0.018w - 432, & 160\,000 < w \le 320\,000 \\ 0.0266w - 3184, & w > 320\,000 \end{cases}$$

WARNING

A common error is to take the gradient form $y = mx + c$ and substitute the rate as m and the cost as c.

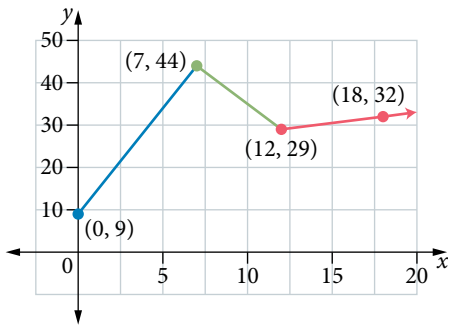
Be careful that subdomains do not overlap at the endpoints.

EXERCISE 1.3 Piece-wise functions

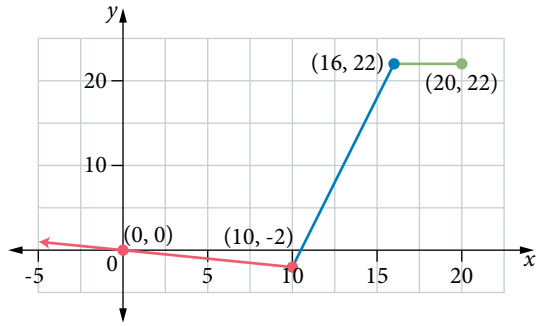
Worked Example 14

1 Determine the rule of each piece-wise function.

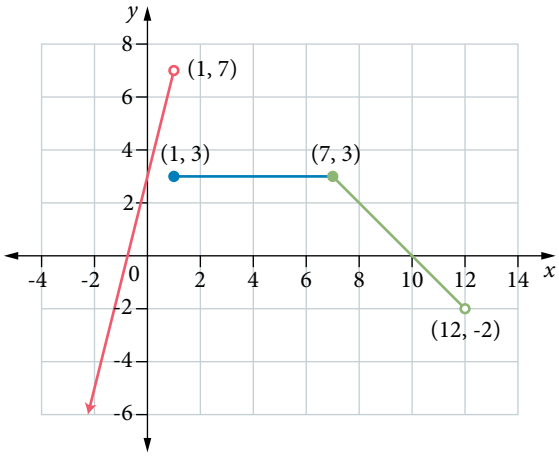
(a)



(b)



(c)



Worked Example 15

2 Sketch the graph of each piece-wise function, identifying the coordinates of axes intercepts.

$$f(x) = \begin{cases} -x - 1, & x < 0 \\ 2x - 1, & x \geq 0 \end{cases}$$

3 Plot each of the following piece-wise graphs.

(a) $y = \begin{cases} x + 1, & x \leq 0 \\ 5x + 1, & 0 < x \leq 10 \\ -x + 61, & x > 10 \end{cases}$

(b) $y = \begin{cases} 3x, & x \leq 0 \\ -x, & 0 < x \leq 5 \\ 3x - 20, & 5 < x \leq 8 \end{cases}$

4 Let $f(x) = \begin{cases} -2x - 6, & x < -2 \\ -2, & -2 \leq x \leq 3 \\ -2x + 4, & x > 3 \end{cases}$

The graph of $f(x)$ is:

A discontinuous at $x = -2$

B discontinuous at $x = 3$

C discontinuous at $x = -2$ and at $x = 3$

D continuous for all values of x.

5 Determine the values of x for which $h(x)$ is discontinuous.

$$h(x) = \begin{cases} x, & x \leq -1 \\ 2x + 1, & -1 < x \leq 1 \\ x, & x > 1 \end{cases}$$

6 For the piece-wise function:

$$h(x) = \begin{cases} 3x - 2, & x < 0 \\ 3x + 2, & 0 \leq x < 2 \\ x - 5, & 2 \leq x < 4 \\ 5 - x, & x \geq 4 \end{cases}$$

(a) the correct calculation for $h(2)$ is:

A $3 \times 2 - 2$

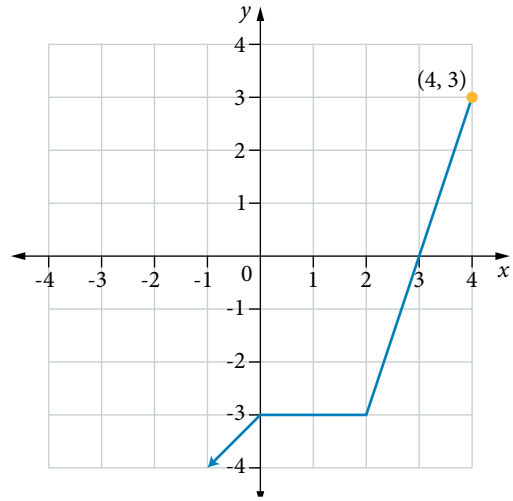
B $3 \times 2 + 2$

C $5 - 2$

D $2 - 5$

(b) Explain the common error made by a student who thought the answer was $3 \times 2 + 2$.

7 Identify the equation and domain for each part of the piece-wise function shown.



Worked Example

16

8 Plot the following piece-wise graphs.

(a) $y = \begin{cases} 2, & x < 0 \\ 2x - 3, & 0 \le x \le 5 \\ \frac{9}{5}x - 16, & 5 < x < 10 \end{cases}$

(b) $y = \begin{cases} -x, & x \le 0 \\ 4, & 0 < x \le 5 \\ 2x + 3, & 5 < x \le 10 \end{cases}$

9 Electricity prices increase according to usage every 3 months as outlined in the table. Draw a piece-wise graph for the cost of electricity for usage between 0 and 1600 kWh.

First 1000 kWh	21.8680 c\kWh
Remaining kWh	23.2760 c\kWh

10 The local water supply company charges different rates depending on the volume of water a household uses. Consider the table of charges on the right.

Draw a piece-wise graph to represent the total cost ($) for consumption from 0 to 500 000 L.

Consumption (L)	Charge (c/L)
0–150 000	0.12
150 000–350 000	0.19
More than 350 000	0.32

11 Natural gas bills are charged at the following rates according to usage over a 2 month period.

Usage (MJ)	Cost
0–4000	0.8457c for each MJ
4000–8000	3382.8c plus 0.8322c for each MJ over 4000
More than 8000	6711.6 plus 0.8182c for each MJ over 8000

(a) Draw a piece-wise graph for cost C($) for up to 10 000 MJ of gas.

(b) Given that x is the volume of gas used in MJ and C is the cost of gas in dollars, determine the cost equation.

(c) Using the graph, what is the approximate cost of using 5000 MJ of gas? Write your answer to the nearest dollar.

(d) Using the formula, what is the actual cost of 5000 MJ of gas?

18

12 The cost of hiring a taxi in the evening involves a booking fee of $1.50 and a flagfall of $4.30 with the additional charge of $2.17 per kilometre (or part). Draw the step graph of the total cost for a taxi journey over the first four kilometres.

13 The cost of hiring a taxi at the airport after midnight incurs an airport rank fee of $3.15 and a flagfall of $6.30 with the additional charge of $2.17 per kilometre (or part). Draw the step graph of the total cost for a taxi journey for between 15 km and 20 km.

Worked Example

14 Determine the value of a such that $f(x)$ is continuous for all values of x. Justify your answer with a graph. 17

$$f(x)=\begin{cases} a-2x, & x\le 1 \\ 2-\frac{1}{2}x, & x>1 \end{cases}$$

15 Determine the values of a and b such that $f(x)$ is continuous for all values of x. Justify your answer with a graph.

$$f(x)=\begin{cases} 2x+a, & x<-2 \\ 3-x, & -2\le x\le 4 \\ 2x+b, & x>4 \end{cases}$$

16 The table on the right gives the information of variable speed over time. 19

Time (seconds)	Speed (m/s)
$0\le t<5$	6
$5\le t<15$	4
$15\le t\le 30$	2

(a) Draw the piece-wise linear graph of total distance travelled against time.

(b) Write a rule for the piece-wise function distance D (m) in terms of time t (s).

17 The table on the right gives the information of variable rate of pay over a 16 hour day.

Time (hours)	Rate of pay ($/h)
$0\le t\le 8$	16
$8<t\le 12$	24
$12<t\le 16$	32

(a) Draw the piece-wise linear graph for total pay against time.

(b) Write a rule for the piece-wise function for pay P ($) in terms of time t (hours).

18 The income tax schedule for Italy in a certain year is shown in the following table, where taxable income is always a whole number of euros.

Income bracket (euro)	Tax rate
0–15 000	0.23€ for every euro
15 001–28 000	3450€ + 0.27€ for every euro over 15 000
28 001–55 000	6960 € + 0.38€ for every euro over 28 000
55 001–75 000	17 220 € + 0.4€ for every euro over 55 000
75 001 and over	25 420€ + 0.43€ for every euro over 75 000

(a) Write an equation and relevant domain for the tax payable on income from 0 to 55 000 euro, where y is the tax payable in euro and x is taxable income in euro.

(b) Explain whether the function is continuous or not.

19 Waterworks supplies water to households. They charge a fixed quarterly maintenance fee of $40, and then charge $2.50 a kilolitre for up to 50 kL of water supplied per quarter. After 50 kL the price rises to $2.95 for every additional kilolitre.

(a) Write the rule for the cost c ($) of water per quarter and the number of kilolitres supplied n.

(b) Draw a graph for up to 100 kL supplied per quarter.

(c) Ashan receives a bill for close to $200. Use your graph to determine his water usage, to the nearest kilolitre, for the quarter. Verify your answer algebraically.

20 The income tax rates in Australia are said to be progressive because the rate increases as your income goes above set thresholds. For a certain taxation year, the following rates of tax applied to the lowest three thresholds.

Income bracket	Tax
0–\$6000	\$0
\$6001–\$37 000	15c for each \$1 over \$6000
\$37 001–\$80 000	\$4650 plus 30c for each \$1 over \$37 000

(a) Draw a piece-wise graph for the tax paid on incomes from 0 to \$50 000.

(b) Write the rule for the tax paid, P (\$) in terms of taxable income I (\$) for incomes up to \$80 000.

21 In an ironman contest, Derek swims 4 km at a steady rate of 3 km/h, and then rides his bike for 180 km at an average speed of 30 km/h. For the final leg, Derek runs the 40 km marathon at a constant speed of 12 km/h.

(a) Write a rule for the piece-wise function that represents Derek's journey.

(b) Draw the graph of total distance travelled d (km) against time elapsed t (hours).

(c) How far has Derek travelled after 5 hours?

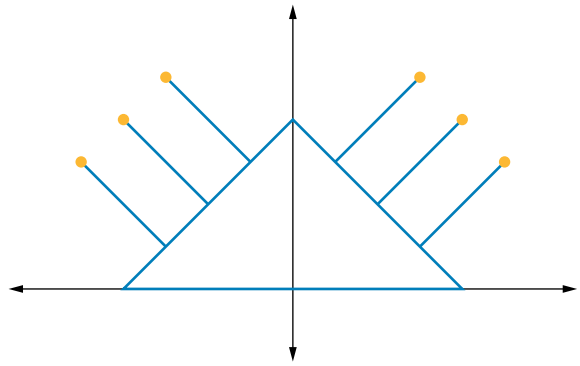
(d) Determine the time it takes for Derek to travel 200 km.

22 The function $y = |x|$ (the absolute value of x) can be written as the piece-wise function $y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

(a) Write a rule for the piece-wise function $y = |2x - 1|$.

(b) Draw the graph of the function.

23 The following diagram can be produced by combining the graphs of several functions. The equation of one of the segments is $x + y = 4$. Use your knowledge of piece-wise functions to write rules for the diagram, using as few functions as possible. Make any assumptions required to answer the question.



24 In 2009 a new flat taxation rate of 20 cents per dollar was proposed to replace the progressive tax rate for 2009–10, as shown in the table below.

Income bracket	Tax
\$0–6000	\$0
\$6001–35 000	15c for each \$1 over \$6000
\$35 001–80 000	\$4350 plus 30c for each \$1 over \$35 000
\$80 001–180 000	\$17 850 plus 38c for each \$1 over \$80 000
\$180 001 and over	\$55 850 plus 45c for each \$1 over \$180 000

Discuss the advantages and disadvantages of the proposal and the income levels that will be affected. Justify your solution with a graph.

Arithmetic sequences and series

1.4

Arithmetic sequences

An *arithmetic sequence* is a set of numbers that have a constant difference between consecutive terms. An arithmetic sequence is sometimes also called an arithmetic progression.

Consider the sequence 5, 8, 11, 14, 17, 20, …
The first term t_1 is 5 and the common difference between consecutive terms is 3.

In the card stack shown, the number of cards increases by two at each level (not counting cards that may be used to form the horizontal bases).

Additional information

Arithmetic sequences

Develop your understanding of arithmetic sequences.

20 Determine whether a sequence is arithmetic

Determine whether 345, 320, 295, … is an arithmetic sequence.

THINKING	WORKING
1 Define the values in the sequence.	$t_1 = 345, t_2 = 320, t_3 = 295$
2 Determine the common difference between consecutive terms: $t_{n+1} - t_n = d$	$d = t_2 - t_1 = 320 - 345 = -25$ $d = t_3 - t_2 = 295 - 320 = -25$
3 Conclude whether or not it is an arithmetic sequence.	There is a common difference of -25 so this is an arithmetic sequence.

Terms of an arithmetic sequence

If you know the initial term and the amount by which the sequence progresses, you can calculate any term in the sequence.

The formula $t_{n+1} - t_n = d$ gives the *recursive definition* of an arithmetic sequence. Rearranging to $t_{n+1} = t_n + d$ allows you to generate subsequent terms after t_1.

A sequence is arithmetic if $t_{n+1} - t_n = d$ for $n \geq 1$, where:
n is the term number
t_n is the nth term
d is the *common difference*

To generate subsequent terms use:
$t_{n+1} = t_n + d$, for $n \geq 1$
$t_n = t_{n-1} + d$, for $n \geq 2$

Alternatively, the terms of the sequence can be expressed as:

$t_1, t_2 = t_1 + d, t_3 = t_1 + 2d \ldots, t_n = t_1 + (n-1)d$

The general term of an arithmetic sequence is: $t_n = t_1 + (n-1)d$ where: t_1 is the first term, or $t_n = a + (n-1)d$ where: a is the first term.

21 Terms of an arithmetic sequences

Consider the sequence 350, 341, 332, …

(a) Determine a formula for the nth term.

THINKING	WORKING
1 Define the values in the sequence.	$t_1 = 350, t_2 = 341, t_3 = 332$
2 Determine the common difference between consecutive terms: $t_{n+1} - t_n = d$	$d = t_2 - t_1 = 341 - 350 = -9$ $d = t_3 - t_2 = 332 - 341 = -9$
3 Identify the value of the first term.	$a = t_1 = 350$
4 Create a formula to generate the terms in the sequence using $t_n = a + (n-1)d$ and simplify.	$t_n = a + (n-1)d$ $= 350 + (n-1)(-9)$ $= 350 - 9n + 9$ $= 359 - 9n$

(b) Determine the value of t_{98}.

1 Substitute the given value of n into the formula.	For $n = 98$: $t_n = 359 - 9n$ $t_{98} = 359 - 9 \times 98$ $= -523$
2 Interpret the result.	The 98th term, t_{98} is -523.

(c) Determine the first negative term of the sequence.

1 Write an inequality to match the given condition. Solve for n, which will indicate how many terms it will take to reach the condition.	$359 - 9n < 0$ $-9n < -359$ $n > 39\frac{8}{9}$ The 40th term is the first negative term.
2 Substitute the calculated value of n into the sequence formula to evaluate the reasonableness of your prediction.	$t_{40} = 359 + 40 \times (-9)$ $= 359 - 360$ $= -1$ The consecutive terms decrease by 9, so $t_{39} = 8$ and the 40th term is the first negative term.

22 Generate terms in a sequence

Using the given information, generate the first four terms of the arithmetic sequence.

(a) $a = -4$, $d = 2$

THINKING	WORKING
1 Start with the first term and then add multiples of d for the next three terms.	$t_1, t_1 + d, t_1 + 2d, t_1 + 3d\ldots$ $-4, -4 + 2, -4 + 2 \times 2, -4 + 3 \times 2\ldots$ $-4, -2, -4 + 4, -4 + 6\ldots$ $-4, -2, 0, 2\ldots$
2 Interpret the result.	The first four terms of the sequence: $-4, -2, 0, 2$

(b) $t_4 = -19$, $d = -2$

1 Start with the first term and then add multiples of d for the next three terms.	$t_4 - 3d, t_4 - 2d, t_4 - d, t_4\ldots$ $-19 - 3(-2), -19 - 2 \times (-2), -19 - (-2), -19\ldots$ $-19 + 6, -19 + 4, -19 + 2, -19\ldots$ $-13, -15, -17, -19\ldots$
2 Interpret the result.	The first four terms of the sequence: $-13, -15, -17, -19$

(c) $t_4 = 9$, $t_7 = 18$

1 Calculate the common difference between terms, recognising that each term differs from the first by n multiples of the common difference d.	There are three steps from t_4 to t_7: $t_4 = 9$ and $t_7 = 18$: $3d = t_7 - t_4$ $3d = 18 - 9$ $3d = 9$ $\therefore d = 3$
2 Check the reasonableness of the common difference calculated by recalling the information given.	$9, 12, 15, 18\ldots$ Adding multiples of 3 to $t_4 = 9$ gives the correct term $t_7 = 18$, so the common difference $d = 3$ is correct.
3 Given t_4, subtract multiples of d to determine the third, second and first terms of the sequence.	$t_4 - 3d, t_4 - 2d, t_4 - d, t_4\ldots$ $9 - 3 \times 3, 9 - 2 \times 3, 9 - 3, 9\ldots$ $9 - 9, 9 - 6, 6, 9\ldots$ $0, 3, 6, 9\ldots$
4 Interpret the result.	The first four terms of the sequence: $0, 3, 6, 9$

23 Solve for d and then generate a sequence

Given $a = 7$, $t_8 = 6t_3$, determine the first four terms of the arithmetic sequence.

THINKING	WORKING
1 Construct two equations of the form $t_n = a + (n-1)d$.	Given $a = 7$: $t_8 = 7 + (8-1)d$ $= 7 + 7d$ $t_3 = 7 + (3-1)d$ $= 7 + 2d$
2 Substitute the simplified expressions into the given equation.	Given $t_8 = 6t_3$: $t_8 = 6t_3$ $7 + 7d = 6(7 + 2d)$ $7 + 7d = 42 + 12d$ $-35 = 5d$ $d = -7$
3 Identify the first term in the sequence t_1 and the common difference d to generate terms in an arithmetic sequence using $t_{n+1} = t_n + d$.	$t_1, t_1 + d, t_1 + 2d, t_1 + 3d\ldots$ $7, 7 + (-7), 7 + 2 \times -7, 7 + 3 \times -7\ldots$ $7, 0, 7 - 14, 7 - 21\ldots$ $7, 0, -7, -14\ldots$
4 Recall the original information given to check the reasonableness of the values in the sequence using $t_n = a + (n-1)d$.	Check that $t_8 = 6t_3$, given that $t_3 = -7$. $t_8 = 7 + (8-1) \times -7$ $\quad$ $t_8 = 6 \times t_3$ $= 7 + 7 \times -7$ $\quad$ $= 6 \times -7$ $= 7 - 49$ $\quad$ $= -42$ $= -42$ Therefore the sequence generated meets the conditions given in the question.
5 Interpret the result.	The first four terms of the sequence: $7, 0, -7, 14$

24 Determine the number of terms in a sequence

Determine the number of terms in the arithmetic sequence 2, 9, 16, … 128.

THINKING	WORKING
1 Identify the first term and calculate the common difference between consecutive terms, $d = t_n - t_{n-1}$.	$a = 2$ $d = t_2 - t_1$ $\quad$ $d = t_3 - t_2$ $= 9 - 2$ $\quad$ $= 16 - 9$ $= 7$ $\quad$ $= 7$ The first term, $a = 2$ and the common difference, $d = 7$.

2 Determine the rule for the nth term by substituting a and d into $t_n = a + (n-1)d$.	For $a = 2$ and $d = 7$: $t_n = 2 + 7(n-1)$ $= 2 + 7n - 7$ $= 7n - 5$
3 Substitute the value of the last term and solve for n to determine the number of terms in the sequence.	Let $t_n = 128$ $128 = 7n - 5$ $7n = 133$ $n = 19$
4 Interpret the result.	There are 19 terms in the sequence.

Arithmetic series

As a child, Carl Friedrich Gauss amazed his classmates and teacher by calculating the total of the numbers 1 to 100 in a matter of seconds.

There were no calculators in 1785, so the 8 year old used a mental maths technique of pairing the numbers: 1 + 100 = 101, 2 + 99 = 101, and so on. The total for the 50 pairs of numbers is $101 \times 50 = 5050$. Simple!

This technique can be used to develop a formula for the sum of an arithmetic sequence. A series is the sum of the terms in a sequence.

If $\{t_1, t_2, t_3, \ldots\}$ is a sequence, then the sums of the series $t_1 + t_2 + t_3 + \ldots$ are:

$$S_1 = t_1, \quad S_2 = t_1 + t_2, \quad \ldots \quad S_n = t_1 + t_2 + \ldots + t_n$$

From the equation for the general term, an arithmetic sequence with n terms can be expressed algebraically as:

$$t_n = t_1 + (n-1)d$$

To determine the sum of this series, you can denote the sum of the first n terms as S_n.

The nth term (last term) is t_n, so:

$$S_n = t_1 + (t_1 + d) + (t_1 + 2d) + \ldots + (t_n - 2d) + (t_n - d) + t_n$$

In reverse order:

$$S_n = t_n + (t_n - d) + (t_n - 2d) + \ldots + (t_1 + 2d) + (t_1 + d) + t_1$$

Adding the two equations for S_n together:

$$\begin{aligned} S_n + S_n &= (t_1 + t_n) + (t_1 + d + t_n - d) + (t_1 + d + t_n - d) + \ldots + (t_n + t_1) \\ 2 \times S_n &= (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \ldots + (t_1 + t_n) \\ 2S_n &= n(t_1 + t_n) \qquad \text{(as there are } n \text{ terms in the sequence)} \\ S_n &= \frac{n}{2}(t_1 + t_n) \end{aligned}$$

Given that $t_n = t_1 + (n-1)d$, the sum of the first n terms can also be written as:

$$S_n = \frac{n}{2}(t_1 + t_1 + (n-1)d)$$

The formula for the sum of the first n terms in an arithmetic sequence can be written as:

$S_n = \frac{n}{2}(t_1 + t_n)$, where $t_n = t_1 + (n-1)d$:

$S_n = \frac{n}{2}(2t_1 + (n-1)d)$, where t_1 is the first term

$S_n = \frac{n}{2}(2a + (n-1)d)$, where a is the first term

25 Determine the sum of a sequence

Determine the sum of the first 11 terms of an arithmetic sequence starting with 1.7 and consecutive terms increasing in value by 2.3.

THINKING	WORKING
1 Identify the required information.	For $n = 11$, $a = 1.7$ and $d = 2.3$
2 Substitute the given values into the formula $S_n = \frac{n}{2}(2a + (n-1)d)$.	$S_{11} = \frac{11}{2}(2 \times 1.7 + (11-1) \times 2.3)$ $= 5.5(3.4 + 10 \times 2.3)$ $= 5.5 \times 26.4$ $= 145.2$
3 Interpret the answer and check that the answer makes sense.	The sum of the first 11 terms is 145.2. $1.7 + 4 + 6.3 + \ldots 24.7 = 145.2$

Explore further

Arithmetic sequences

Use technology to determine and verify sums of arithmetic sequences.

Technology worked example

Solving quadratic equations from a graph

26 Determine the number of terms from the sum

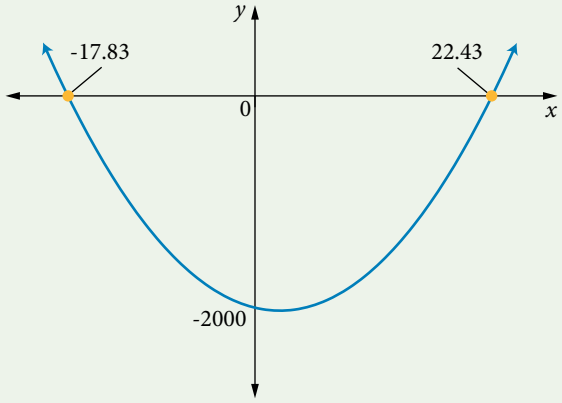
Determine the number of terms for the sum of the series $(-9) + (-4) + 1 + 6 + \ldots$ to exceed 1000.

THINKING	WORKING
1 Identify the required information.	$a = -9$, $d = 5$
2 Write an inequality using $S_n = \frac{n}{2}(2a + (n-1)d)$ to represent the given condition. Substitute the values for a and d.	$S_n = \frac{n}{2}(2a + (n-1)d)$, so $\frac{n}{2}(2a + (n-1)d) > 1000$ $\frac{n}{2}(2(-9) + (n-1)5) > 1000$ $-18n + n(n-1)5 > 2000$ $-18n + 5n^2 - 5n > 2000$ $5n^2 - 23n - 2000 > 0$

3 Solve the inequality for n.

You will need to use technology to determine the required value of n.

Graph $y = 5x^2 - 23x - 2000$ and determine the roots.



$5n^2 - 23n - 2000 = 0$ for $n = 22.43$ or $n = -17.83$

4 Rewrite the inequality in terms of the valid domain.

There must be a positive number of terms in the sequence, therefore:
$S_n > 1000$ for $n > 22.43$

5 Conclude by stating the number of terms. Check the reasonableness of the solution.

As 23 is the smallest positive integer greater than 22.43, 23 terms will be required to reach a sum of at least 1000.

$$S_{23} = \frac{23}{2}\left(2 \times (-9) + (23-1) \times 5\right)$$
$$= 1058$$

EXERCISE 1.4 Arithmetic sequences and series

Worked Example 20

1 Determine which of the following sequences are arithmetic. For those sequences that are arithmetic, determine the common difference.

(a) 5, 7, 9, 11, …
(b) 3, 6, 12, 24, …
(c) 1, 4, 9, 16, …
(d) -4, -2, 0, 2, …
(e) $t_{n+1} = t_n + 6$, $t_1 = 3$
(f) $\frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \ldots$
(g) $t_n = 4n^2$
(h) $c, c+p, c+2p, c+3p\ldots$ (c and p are constants)

2 Determine which of the following does *not* represent an arithmetic sequence.

A 0, -16, -32, -48 …
B $t_n = \frac{3}{4}n - 5$
C 4, -1, -4, -9 …
D $2h, 3h+k, 4h+2k, 5h+3k \cdots$

3 Generate the first four terms of an arithmetic sequence in which $a = 3$ and $d = 7$.

4 Determine the value of n for an arithmetic sequence for $a = 52$, $d = -5$ and $t_n = -3$.

5 Determine the value of the indicated term in each of the given arithmetic sequences.

(a) 1, 3, 5, … ; 14th term

(b) 27, 21, 15, … ; 15th term

(c) $2m, 3m - n, 4m - 2n, \ldots$; 7th term (m and n are constants)

Worked Example

(d) 3, 0, -3, … ; nth term

21 6 The following questions refer to the sequence 223, 201, 179, …

(a) Show that the sequence is an arithmetic sequence.

(b) Determine a formula for the nth term.

(c) Determine the value of the 42nd term.

(d) Determine the first negative term of the sequence.

22 7 Generate the first four terms of each arithmetic sequence, using the given information.

(a) $a = 5, d = -2$

(b) $t_5 = -28, d = 3$

(c) $t_3 = 31,\ t_9 = -17$

8 The 8th term of an arithmetic sequence is 55 and the 15th term is 97. Determine the values of a and d.

9 In an arithmetic sequence, the fourth term is 42 and the eighth term is 70. Determine the 25th term.

10 In an arithmetic sequence, the fourth term is -4.5 and $t_7 = 0$. Determine the 20th term.

11 Determine the value of p if $p + 5, 4p + 3, 8p - 2$ are successive terms of an arithmetic sequence.

24 12 Determine the number of terms in each of the following arithmetic sequences.

(a) 1, 7, 13, … 115

(b) 3, -2, -7, … -97

(c) $m, m + 3n, m + 6n, \ldots m + 27n$

(d) $2x + y, 3x + 4y, 4x + 7y, \ldots 10x + 25y$

13 For the arithmetic sequence 20, 22, 24, …

(a) the sum of the first seven terms is given by:

A $20 + 22 + 24$

B $20 + 7 \times 2$

C $\frac{7}{2}(40 + 6 \times 2)$

D $\frac{7}{2}(40 + 7 \times 2)$

(b) Explain the common error made by a student who thought the answer was $20 + 6 \times 2$.

14 In an arithmetic sequence, the sum of the first 20 terms is 650. Determine the first term if the common difference is 3.

25 15 Determine the sum of the number of terms indicated for each of the following arithmetic sequences.

(a) 1, 2, 3, ... ; first 12 terms

(b) 3, 7, 11, ... ; first 10 terms

(c) 11, 5, (-1), … ; first 10 terms

(d) 30, 18, 6, … ; first 20 terms

(e) 1, 3, 5, … ; first n terms

(f) (-7), (-2), 3, … ; first n terms

16 Determine the first three terms of the arithmetic sequence if $t_4 = 17$ and $S_{16} = 632$.

17 Determine the first three terms of the sequence if $t_4 = 13$ and $t_8 = 25$.

23 18 Determine the first four terms of an arithmetic sequence if $a = 6$ and $t_7 = -2t_3$.

19 Determine x and the common difference d if the following number sequences form arithmetic sequences.

(a) $7, x, 11, \ldots$

(b) $x+2, 2(x-1), 2x+1, \ldots$

(c) $x-2, x+2, x^2, \ldots$

(d) $\frac{1}{x+3}, \frac{1}{x}, \frac{1}{x-1}, \ldots$

Worked Example 26

20 Determine the number of terms of the arithmetic series $2.5 + 4 + 5.5 + \ldots$ that must be added for the sum to exceed 150.

21 If the numbers $1, 3a-2, a^2$ are an arithmetic sequence, determine all possible values of a.

22 Consider the arithmetic sequence 5, 23, 41, ...

(a) Determine the value of d.

(b) Calculate t_6.

(c) Calculate t_{19}.

(d) How many of the terms in the sequence are between 100 and 150?

23 Ayla is training for a long-distance running event. She runs every day, increasing the distance by 500 m each time she runs. Her initial run was 4 km in length.

(a) How far will Ayla run on the fifth day of training?

(b) On which day of training will Ayla first run more than 12 km?

(c) How many days of training will be needed for Ayla to have run a total of at least 20 km?

(d) What total distance will Ayla have run after 20 days of training?

24 You will be asked to draw conclusions about the values of the expressions $t_1 + t_4$ and $t_2 + t_3$ for an arithmetic sequences.

(a) Determine the value of each expression for each of the following arithmetic sequences

(i) 1, 2, 3, 4, ...

(ii) 5, 10, 15, 20, ...

(iii) $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

(iv) -7, -4, -1, 2, ...

(b) Based on the results in part **(a)** what equality could you write if you were told $j, k, l, m, \ldots$ was the start of an arithmetic sequence?

(c) Using a and d notation, verify your answer to part **(b)**.

25 Consider the arithmetic sequence 3, 5, 7, ...

(a) Model the terms of the sequence with a linear equation of the form $y = mx + c$ where $x \in \{0, 1, 2, \ldots\}$.

(b) Model the sum of x terms of the series with an equation. State whether this function is linear.

26 Determine the value of n for which the sum of the terms of the arithmetic sequences -8, -3, 2, … and 20, 22, 24, ... are either equal or as close as they can be.

27 The following questions apply to consecutive terms of an arithmetic sequences.

(a) Prove that the mean of any three terms is the same as the middle term.

(b) Determine a generalisation for the mean of any five terms.

(c) Determine a generalisation for the mean of any six terms.

1.5 Linear growth and decay

Simple linear growth models

Any situation in which a starting amount changes by the same amount over a regular period can be modelled by an arithmetic sequence. If the values increase at a constant rate, then this is an example of linear growth. If the values decrease at a constant rate, then it is an example of linear decay.

27 Calculate linear growth

A new high school starts with 200 students and anticipates increasing in capacity by 150 students each year until the original Year 7 cohort reach Year 12.

(a) Write a formula to model the number of students at the school in terms of the number of years since its start.

THINKING	WORKING
1 Write the function values as an arithmetic sequence.	Number of students each year: 200, 350, 500, 650, 800, 950
2 Define the variables.	Let t be the number of years since the start. Let n be the number of students at the school.
3 Use $y = mx + c$ to create the formula.	Let $n = mt + c$ When $t = 0$, $n = 200$, so $c = 200$. The annual increase is 150, so $m = 150$. Hence $n = 150t + 200$.

(b) Determine the domain for which this linear model is valid.

1 Decide whether the domain is discrete or continuous. Determine the lowest and highest values for which the formula applies.	Let $n = 0$ represent the year the class starts in Year 7. When $n = 1$, the cohort will be in Year 8. $n = 5$ represents the year the cohort reach Year 12. $n = 0, 1, 2, \ldots 5$
2 Check the reasonableness of your answer with a description in words.	The school will reach its capacity after 5 years.

Simple interest

A common example of linear growth is *simple interest*, where the same amount of interest is added each time to an amount of money. The interest is calculated on the *principal*, the initial investment, and it is called 'simple' because the interest is constant. Simple interest is sometimes referred to as a 'flat' rate.

28 Calculate simple interest

Sachin deposits \$3500 into a Super Saver account that pays simple interest at a rate of 5% p.a.

(a) Calculate the interest Sachin earns in a year.

THINKING	WORKING
1 Determine the percentage of the original deposit.	$\$3500 \times \frac{5}{100} = \$175$
2 Interpret the answer.	The simple interest paid each year is \$175.

(b) Calculate the value of the investment after one year.

1 Add the interest to the original deposit.	$3500 + 175 = \$3675$
2 Interpret the answer.	The value of the investment is \$3675 after a year.

(c) Generate the first six terms of a sequence (as numbers only) that represent the balance of the account from the start.

1 Calculate the value of each term of the sequence. In practical applications, it may be useful to label the first term or invested amount t_0.	$t_0 = 3500$ $t_1 = 3500 + 175$ $= 3675$ $t_2 = 3675 + 175$ $= 3850$	$t_3 = 3850 + 175$ $= 4025$ $t_4 = 4025 + 175$ $= 4200$ $t_5 = 4200 + 175$ $= 4375$
2 Write the sequence.	3500, 3675, 3850, 4025, 4200, 4375	

(d) Interpret the value of t_5.

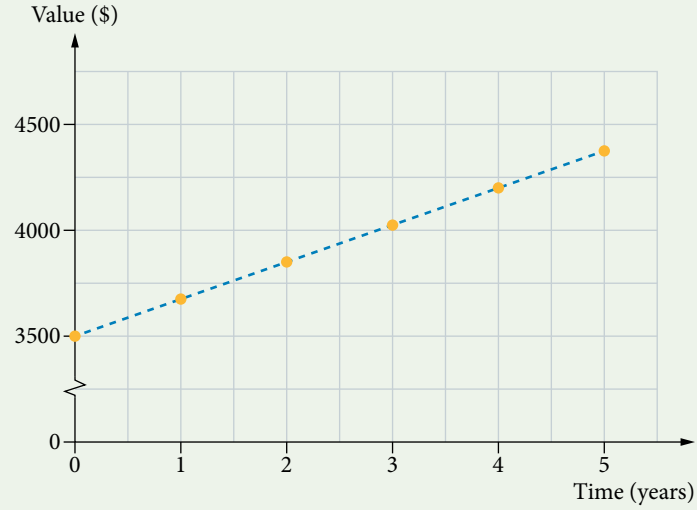
1 Write the term value.	$t_5 = 4375$
2 Interpret the meaning from your knowledge of arithmetic sequences.	t_5 is the sixth account balance from the when the investment began at t_0. It gives the value of the investment after five lots of interest have been added. \$4375 is the value of the investment at the end of 5 years.

(e) Plot the value of the investment against time to demonstrate the linear nature of the function.

1 Make a table of values, beginning when time is zero.

Time (years)	0	1	2	3	4	5
Value (\$)	3500	3675	3850	4025	4200	4375

2 Plot the points, joining them with a dotted line.



The calculation of simple interest can be represented as follows.

$I = Pin$, where:

I is the amount of interest paid ($) (or charged if you are borrowing money)

P is the principal ($) (the amount of money invested or borrowed)

i is the interest rate for the same time period as n, expressed as a decimal.

n is the number of periods for which the interest is paid

$A = P + I$, where A is the account balance ($).

Term deposits usually involve simple interest. The rates and terms (period of the investment) are agreed beforehand and the same amount of interest is added each year throughout the term of the investment. Simple interest is not common in the finance sector. In most other investments, compound interest is calculated because the amount of interest compounds as the account balance grows.

29 Simple interest growth in an investment

A principal amount of $5000 is invested in a term deposit at 7% p.a. simple interest. The balance of the account at the end of n years is A_n.

(a) Calculate the amounts of A_1, A_2, A_3, and A_4.

THINKING	WORKING
1 Determine the fixed interest for each time period by substituting the given values for P and i, and $n = 1$ (the end of the first year), into the formula $I = Pin$.	$I = Pin$ $= 5000 \times 0.07 \times 1$ $= 350$

2 Calculate A_1 by adding the principal to the interest value. Generate the terms A_2, A_3, A_4 in the sequence.	Let $A_0 = 5000$ $A_1 = 5000 + 350$ $= \$5350$ $A_2 = A_1 + 350$ $= 5350 + 350$ $= \$5700$ $A_3 = A_2 + 350$ $= 5700 + 350$ $= \$6050$ $A_4 = A_3 + 350$ $= 6050 + 350$ $= \$6400$

(b) Determine the formula for A_n and hence calculate A_{15}.

1 Use the arithmetic sequence formula for the nth term: $t_n = a + (n-1)d$.	For $a = 5350$ and $d = 350$ $A_n = 5350 + (n-1) \times 350$ $= 5350 + 350n - 350$ $= 5000 + 350n$
2 Substitute $n = 15$ to calculate the value of the 15th term in the sequence.	$A_{15} = 5000 + 350 \times 15$ $= 10\,250$
3 Interpret the result.	After 15 years, and the accumulation of 15 interest payments, the investment will be worth \$10 250.

(c) Calculate the number of years it will take for the balance to exceed \$12 500.

1 Construct an inequality using the formula found for A_n and the required value. Solve for n.	$5000 + 350n > A_n$ $5000 + 350n > 12\,500$ $350n > 7500$ $n > 21.429$ (3 d.p)
2 Conclude by interpreting the result within the valid domain.	It will take 22 years before the balance exceeds \$12 500.
3 Check the reasonableness of the solution by substituting the values into the simple interest formula.	$P = \$5000$, $i = 0.07$ p.a., $n = 22$ $I = Pin$ $= 5000 \times 0.07 \times 22$ $= \$7700$ In 22 years, the account will have grown by \$7700, giving an investment balance of $7700 + 5000 = \$12\,700$.

Financing the purchase of a car

At some stage in your life you will probably make a large purchase for a car, property or business.

You may need to take out a loan in order to make that purchase.

Personal loans for the purchase of a car or holiday are always paid off in the short term, generally no more than 5 years, and typically set up using simple interest calculations. Regular repayments are set up with amounts rounded, so that a one-off smaller amount is paid at the end. The amount owing, and the amount paid, both form arithmetic sequences.

Explore further

Simple interest

Explore the effects of changing principal amounts, interest rates and number of years in a simple interest setting.

30 Repay a personal loan

Oliver buys his first car for \$18 500. He has saved a deposit of \$5000 and needs finance for the rest. He is offered terms of 7.54% p.a. simple interest over 4 years with monthly payments of \$370.

(a) Determine the number of payments of \$370 that will be paid and calculate the amount of the final one-off payment at the end of the term, assuming it is less than the regular payment.

THINKING	WORKING
1 Determine the amount borrowed P by subtracting the deposit from the price of the car.	Amount borrowed = price − deposit $P = 18\,500 - 5000$ $= \$13\,500$
2 Calculate the interest charged, using $I = Pin$.	$I = Pin$ $= 13500 \times 0.0754 \times 4$ $= \$4071.60$
3 Calculate the total amount to be repaid, in instalments, using $A = P + I$.	$A = P + I$ $= 13\,500 + 4071.60$ $= \$17\,571.60$
4 Calculate the number of repayments needed by dividing the total amount by the amount of each payment. *Note:* Here you would expect a number slightly less than $4 \times 12 = 48$.	Repayments needed: $\frac{\$17\,571.60}{\$370} = 47.4\ldots$ 47 payments of \$370 will be made.
5 Determine the final balance to be paid as a one-off smaller instalment.	Final instalment: $17\,571.60 - 47 \times 370$ $= 17\,571.60 - 17\,390$ $= \$181.60$

(b) Model the amount owing as an arithmetic sequence, making an exception of the final term.

1 Write the terms, in order, from the amount to be repaid by instalments, decreasing by the instalment amount, up to the final payment.	$a = 17\,571.60$, $d = -370$ 17 571.60, 17 201.60, 16 831.60, …, 181.60
2 Determine the limitations of this model.	The final term is \$181.60. Any further terms will be negative and do not apply in this context.
3 Hence, determine the number of terms in the arithmetic sequence, and comment on the final term. Note that the first term represents zero payments.	There are 48 terms in the sequence, where $t_1 = 13\,500$ and the final value of the sequence is $t_{48} = 181.60$.

(c) Model the amount paid as an arithmetic sequence, making an exception of the final term. Include the deposit in your model.

1 Calculate the total amount paid for the car, including the deposit.	Total amount paid = price + interest = 18 500 + 4071.60 = \$22 571.60
2 Calculate the total amount paid prior to the one-off instalment.	Total amount paid after 47 instalments: 22 571.60 − 181.60 = \$22 390
3 Write the terms, in order, from the deposit, increasing by the instalment amount, up to the total amount paid, apart from the final payment.	$a = 5000$, $d = 370$ 5000, 5370, 5740, … , 22 390
4 State the number of terms in the sequence, and comment on the final term. Note that the first term represents zero payments.	There are 48 terms in the sequence, with a final term of 22 571.60 that is not part of the sequence.

Straight line depreciation

Depreciation of assets, like growth of investments, is often calculated as a percentage of the previous year's value, so you would use compound interest formulas. The 'paper' value of assets is an exception because these are depreciated by a constant amount each year throughout their lifetime, for taxation purposes.

If an item such as a computer is bought for the purpose of earning an income, the amount of its depreciation each year can be claimed as a deduction up until the full purchase price has been claimed.

31 Straight line depreciation

A computer bought for \$3892 is to be claimed as a tax deduction using straight line depreciation of 15% of its original value each year until the total value has been claimed.

(a) Calculate the first year's deduction amount, to the nearest dollar.

THINKING	WORKING
1 Calculate the percentage of the original value.	15% of \$3892 = 0.15 × \$3892 = \$583.80
2 Interpret the result to the nearest dollar.	A deduction of \$584 is claimed in the first year.

(b) Construct a formula to calculate t_n and hence determine the number of years that the equal deduction amounts will be claimed.

1 Construct the formula $t_n = a + (n-1)d$ from the information given and then solve the inequality $t_n < 0$.	Let $t_0 = 3892$ $t_1 = 3892 - 584$ $= 3308$ For $a = 3308$ and $d = -584$: $t_n = 3308 - 584(n-1)$ The deductions will take place until $t_n < 0$. $3308 - 584(n-1) < 0$ $3308 - 584n + 584 < 0$ $3892 - 584n < 0$ $-584n < -3892$ $-n < -6.664\ldots$ $n > 6.664\ldots$ So 6 complete deductions are required (and then one part payment).
2 Check the reasonableness of the solution.	$n = 0$ represents the initial value of \$3892. $n = 1$ represents the value at the end of year 1 after 1 deduction. So $n = 6$ represents the value at the end of year 6 and is the sixth deduction.
3 Interpret the result, rounding down to the nearest whole year.	The deduction of \$584 will be claimed each year for 6 years.

(c) Calculate the final year's deduction amount.

1 Subtract the deductions so far from the purchase price.	6 payments of \$584: $6 \times 584 = \$3504$ $3892 - 3504 = \$388$	Using the formula: $t_n = 3308 - 584(n-1)$ $t_6 = 3308 - 584(5)$ $= \$388$
2 Conclude by interpreting the result.	In the 7th year the final deduction will be \$388.	

Displaying straight line depreciation

The value of items that depreciate by a constant amount for a fixed period can be represented using a linear graph in the form $y = mx + c$, where $m < 0$. A spreadsheet or graphics calculator can be used.

32 Display straight line depreciation

Model the paper value of the computer from the previous example as a straight line with a limited domain. Determine the equation to the linear function.

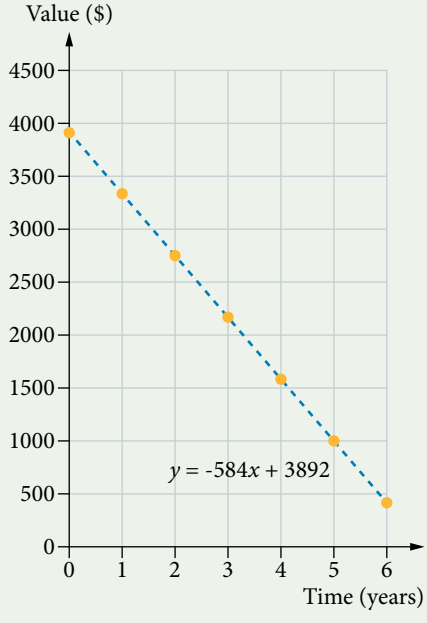
THINKING

1 Make a table of value ($) against time (years) for the linear part of the function.

WORKING

Time (years)	Value ($)
0	3892
1	3308
2	2724
3	2140
4	1556
5	972
6	388

2 Enter the pairs of values into a spreadsheet, or as lists in the statistics mode of your calculator. Graph the function and determine the line of best fit.



$y = 3892 - 584x$, $x \in \{0,1,2,3,4,5,6\}$ where y represents value ($) and x represents time (years) after the computer is purchased.

WARNING

Be careful when determining initial values in linear graphs that are a function of time. The first term is usually when time is zero, but this depends on how the variables in the rules are defined.

EXERCISE 1.5 Linear growth and decay

Worked Example

27

1 Adam has 30 football cards and plans to collect 2 each week until he has 1000.

(a) Write a formula to model the number of cards in Adam's possession in terms of the number of weeks from now.

(b) Determine the domain for the linear function.

2 When Heather opens an antique store she has a collection of 300 souvenir teaspoons. The spoons are sold at a rate of two teaspoons each week until she has only 40 remaining.

(a) Write a formula to model the number of spoons in terms of the number of weeks after the opening.

(b) Determine the domain for the linear function.

28

3 A bank pays simple interest at a rate of 3% p.a. provided no withdrawals or deposits are made. Mahalia deposits \$10 000 into this type of account for three years.

(a) Calculate the interest Mahalia earns in a year.

(b) Generate the terms of a sequence (as numbers only) that represent the balance of the account from the start to the end of the 3 years.

(c) Interpret the value of t_4.

(d) Plot the value of the investment against time to demonstrate the linear nature of the function.

4 \$7800 is invested at a fixed rate of 2.32% p.a. for 4 years.

(a) Determine the final balance.

A \$7076.16 B \$7802.32 C \$7809.28 D \$8523.84

(b) Explain the common error made by a student who thought the answer was \$723.84.

5 Calculate the simple interest and the account balance for each of the following:

(a) \$8000 invested for 10 years at 7.5% p.a.

(b) \$600 invested for 3.5 years at 4% p.a.

(c) \$2100 deposited for nine months at 12.25% p.a.

(d) \$565 deposited for eight months at 8.30% p.a.

29

6 A principal of \$9400 is invested in a term deposit at 6.5% p.a. simple interest. Let A_n be the account balance at the end of n years.

(a) Determine the values of A_1, A_2, and A_3.

(b) Determine the formula for A_n and A_{10}.

(c) Determine the number of years will it take before the balance exceeds \$12 000.

7 A bank offers a simple interest rate of 10.30% p.a. for 30 months. How much money needs to be invested to give a final balance of at least \$2000? Give your answer to the nearest ten cents.

8 Complete the following table.

Simple interest	Principal	Simple interest rate (p.a.)	Time	Account balance
\$4675	\$8500	11%		\$13175
	\$2800		15 months	\$2957.50
\$180		9%	1 month	

9 Which annual simple interest rate will double an investment in 20 years?

A 4% B 5% C 8% D 10%

Worked Example

10 A mower bought for $8468 is claimed as a tax deduction, using straight line depreciation of 10% of its original value each year, until the total value has been claimed. 31

(a) Calculate the first year's deduction amount, to the nearest dollar.

(b) Determine the number of years that the equal deduction amounts will be claimed.

(c) Calculate the final year's deduction amount.

11 Model the paper value of the mower from the previous question as a straight line with a limited domain. Determine the equation to the linear function. 32

12 A factory replaces its heavy machinery for $125 934. The accountant determines that the paper value should decrease by $20 000 each year until the value is less than $50 000. After that it is anticipated that the machinery will be sold as scrap and replaced.

(a) Determine the anticipated number of years that the machinery will be used.

(b) Calculate the paper value of the equipment at the time it will be scrapped.

(c) Write a linear function for the paper value of the machinery with a restricted domain.

13 Maxine finances an $8725 cruise with a personal loan, having saved just $2000 for the deposit. She is offered terms of 9.62% p.a., at a flat rate, over 2 years with monthly payments of $340. 30

(a) Determine the number of payments of $340 that will be paid and calculate the amount of the final one-off payment at the end of the term, assuming it is less than the regular payment.

(b) Model the amount owing as an arithmetic sequence, making an exception of the final term.

(c) Model the amount paid as an arithmetic sequence, making an exception of the final term. Include the deposit in your model.

14 Michael buys a fridge on finance for $786, having only $100 as a deposit. He is offered terms of 12.88% p.a., at a flat rate, over 20 weeks with weekly instalments, rounded to the nearest dollar.

(a) Determine the value of the regular instalments to the nearest $10.

(b) When the amount owing is less than the instalment amount, the final payment is boosted to include the small amount. Calculate the number of regular payments and the amount of the final payment.

(c) Model the amount owing as an arithmetic sequence, excluding the final term.

(d) Model the amount paid as an arithmetic sequence, excluding the final term. Include the deposit in your model.

15 Hanh was given $3000 on his 16th birthday. He would like to deposit it into an account that would give him the best return by his 21st birthday. Which is the better investment and by how much?

- 2.80% p.a. simple interest for 5 years
- 2.55% p.a. simple interest for 4 years, then roll over the principal and interest to 2.25% p.a. simple interest for 1 year.

16 Siobhan inherited $20 000 from a relative. She invests the money at a fixed rate of 4.25% p.a. for 10 years. At the end of this time she rolls over (reinvests) the final balance as well as another $5000 she has managed to save to a 5-year term deposit at a fixed rate of 2.85% p.a. Siobhan plans to use her savings for a deposit on a unit. Assuming her rate of saving is constant, determine the amount she will have for the deposit at the end of the investment period.

17 An initial sum of $15 000 is invested in a term deposit for 4 years at a fixed rate of 2.64% p.a. with the same amount of interest added at the end of every year. At the end of the term, the entire balance is rolled over for another 4 years with an improved rate of 2.81% p.a.
Graph the nine values the investment takes over the 8 years. Write a discrete linear piece-wise function for the growth of the investment.

18 George has $350 000 invested in his superannuation account. The interest earned is 7.5% p.a., paid at the end of the year, and George withdraws 6% of the balance at the beginning of each year.

Year	Amount at beginning of year ($)	Amount at end of year ($)
1	350 000	
2		
3		
4		
5		

(a) Use this information to complete the following table in your workbook or on a spreadsheet. Round the values in your table to the nearest dollar.

(b) Describe what is happening to the balance in the fund over the 5 years.

(c) Complete a table in your workbook for the reversed situation, where the interest rate is 6% and there is a withdrawal of 7.5% of the balance each year.

(d) What is happening to the balance in the account this time?

19 Before about 1780, forests covered a total area of about 5900 million hectares worldwide. Figures for the last 30 or so years are given in the table below.

World forest cover, 1990–2010			
Region	Total forest cover (million hectares)		
	1990	2000	2010
Africa	749	709	674
Asia	576	570	593
Europe	989	998	1,005
North and Central America	708	705	705
Oceania	199	198	191
South America	946	904	864
World	**4168**	**4085**	**4033**

Source: Compiled by Earth Policy Institute from U.N. Food and Agriculture Organisation, **Forest Resources Assessment 2010: Global Tables** (Rome, 2010), www.fao.org/forestry/fra/fra2010/en/.

(a) Create a piece-wise linear function to model the number of millions of hectares of forests worldwide up until 2000. Choose a suitable subdomain for the initial function and explain your choice. Use suitable rounding.

(b) What amount of coverage would your model predict if the final equation was used for 2010? Compare this with the actual figure for that year, and suggest some factors that limit the use of your model to predict coverage by forests into the future.

(c) Choose the region best suited to a linear growth model. Explain your choice. Create a rule for your model with a suitable domain.

(d) Choose the region best suited to a linear decay model. Explain your choice. Create a rule for your model with a suitable domain.

Summary

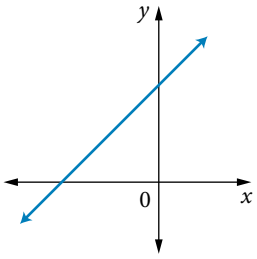
Functions

A function is a relation for which there is only one value of the dependent variable in the range corresponding to each value of the independent variable in the domain.

Variables may be discrete or continuous.

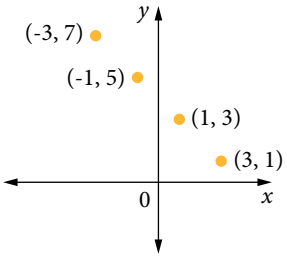
The graph of a function can be increasing, constant or decreasing over subsets of the domain of the function.

$y = x + 1, x \in \mathbb{R}$



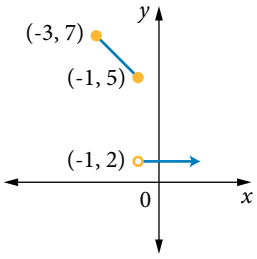
Increasing continuous function

$y = 4 - x, x \in \{-3, -1, 1, 3\}$



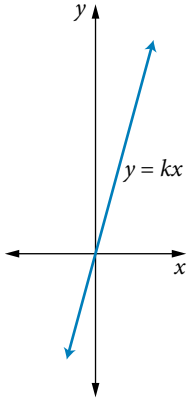
Decreasing discrete function

$y = \begin{cases} 4 - x, & -3 \le x \le -1 \\ 2, & x > -1 \end{cases}$

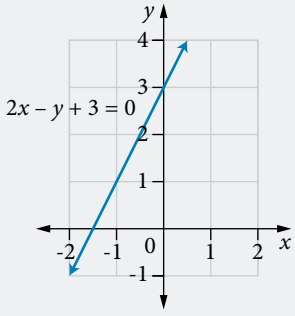
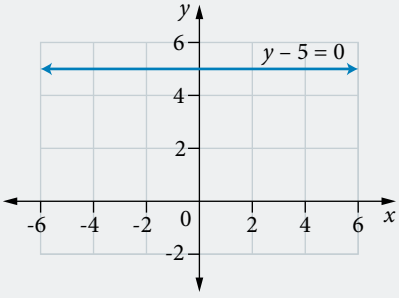
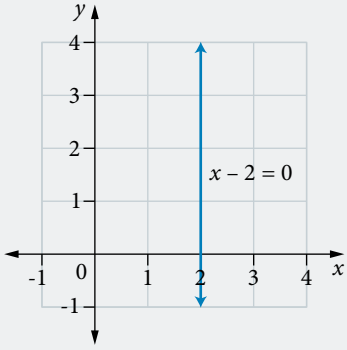


Piece-wise function

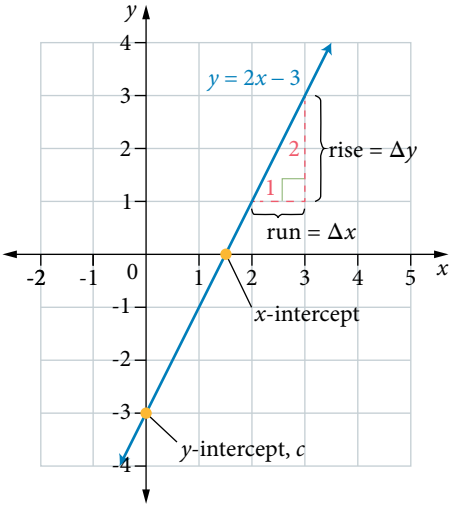
Direct variation is when two variables are related such that the ratio of the variables is constant. The graph of the variables is a straight line with an equation in the form $y = kx$. The line passes through the origin with a gradient equal to the constant ratio k.



A linear relation has a rule of the form $ax + by + c = 0$, and the graph of a linear relation is a straight line.

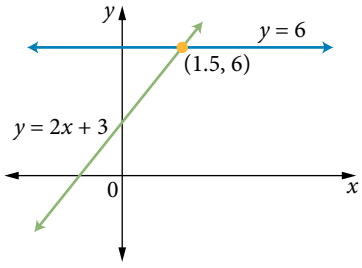
If a and b are both non-zero, then this will be an oblique straight line.	If $a = 0$, then the graph will be a horizontal straight line.	If $b = 0$, then the graph will be a vertical straight line. This relation is not a function.
		

A linear function can always be expressed with rule of the form $y = mx + c$ or $f(x) = mx + c$. This is called the gradient–intercept form of the rule as m represents the gradient $= \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$ of the corresponding graph and c represents the y-intercept.



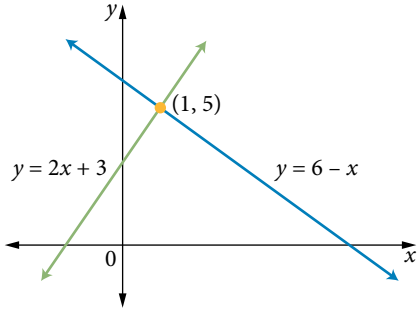
The rule of a linear function can be determined, given the gradient m and one point; or the coordinates of two points with different x-values.

Any linear equation of the form $mx + c = k$, where k is a real number, can be solved graphically by drawing the graphs of $y = mx + c$ and $y = k$ on the same set of axes and identifying the coordinates of the point of intersection or algebraically by rearranging to make x the subject of the equation. The solution to the equation $2x + 3 = 6$ is $x = 1.5$; graphically this is given by the point (1.5, 6).



Two simultaneous linear equations can be solved graphically by drawing both graphs on the same set of axes and identifying the coordinates of the point of intersection (unless the lines have the same gradient) or algebraically by equating, or using substitution or elimination approaches.

The simultaneous solution to the equations $y = 6 - x$ and $y = 2x - 3$ is $x = 1$ and $y = 5$. Graphically this is represented by the point of intersection at (1, 5).

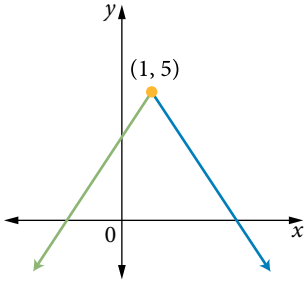


Piece-wise functions

A piece-wise function is made up of different functions over subsets of its domain.

Each subdomain has its own rule.

A continuous function is one you can draw without lifting your pen off the page.



$$y = \begin{cases} 2x + 3, & x \le 1 \\ 6 - x, & x > 1 \end{cases}$$

Arithmetic sequences and series

An *arithmetic sequence* is a set of numbers in which the difference between consecutive terms is constant.

Therefore, a sequence is arithmetic if $t_{n+1} - t_n = d$ for $n \geq 1$, where d is a constant called the *common difference*.

The *recursive definition* of an arithmetic sequence can be expressed as $t_{n+1} = t_n + d, t_1 = a$ for $n \geq 1$, where a is the first term.

The formula for the general term of an arithmetic sequence is given by $t_n = a + (n-1)d$, and it can be used to determine any term in the sequence t_n.

A series is the sum of the terms in a sequence. If $t_1, t_2, t_3, \ldots$ is a sequence, then $S_1 = t_1$, $S_2 = t_1 + t_2, \ldots$, $S_n = t_1 + t_2 + \ldots + t_n$ are partial sums of the sequence $t_1 + t_2 + t_3 + \ldots$

To determine the sum of n terms in an arithmetic sequence, use the formula: $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

Financial mathematics

Simple interest: $I = Pin$, where:

I is the amount of interest paid ($) (or charged if you are borrowing money).
n is the number of periods for which the interest is paid.
P is the principal ($) (the amount of money invested or borrowed).
i is the interest rate expressed as a decimal for the same time period as n.

$A = P + I$, where A is the account balance ($).

Straight line depreciation of assets refers to the paper value of assets, depreciated by a constant amount each year throughout their intended lifetime.

Chapter review 1

Exercise 1.1 **1** Determine whether the following situations describe direct variation between the variables.

(a) The number of hours worked at the local newsagent and the (before tax) pay received.

(b) The number of pets owned and the number of children a family has.

Exercise 1.1 **2** Determine the value of the gradient and the coordinates of the y-intercept in each of the following linear equations.

(a) $y = 14 - 3x$

(b) $2x + y - 5 = 0$

(c) $y = -\frac{5}{7}x - 20$

(d) $y = x$

Exercise 1.1 **3** Determine the rule of the linear equation defined by the given information.

(a) gradient of 3 and y-intercept at (0, -2)

(b) gradient of $\frac{3}{7}$ and y-intercept at $\left(0, \frac{1}{2}\right)$

(c) passing through (2, 5) and (0, 7)

Exercise 1.1 **4** Consider the linear function with rule $3x + 4y = 48$.

(a) Determine the range if the domain is restricted to $5 \le x \le 15$, where x is a real number.

(b) Determine the domain if the range is restricted to $20 \le y \le 30$, where y is a real number.

Exercise 1.1 **5** Determine the equation of the line with gradient $-\frac{2}{5}$ and passing through the point (4, 4).

Exercise 1.1 **6** Determine the equation of the line passing through the point (-2, 0) with gradient $\frac{4}{7}$.

Exercise 1.2 **7** Solve the following equations.

(a) $3x + 35 = 9 - 10x$

(b) $\frac{5a - 2}{4} = \frac{1 + 2a}{3}$

Exercise 1.2 **8** Solve the following simultaneous linear equations.

(a) $3x - y = 7$ and $2x + y = 2$

(b) $5x + 2y = -8$ and $x - 3y = -5$

(c) $3y = 2x - 3$ and $x + 2y = 6$

9 Select the statement that best describes the difference between a continuous and discontinuous piece-wise graph. Exercise 1.3

A A continuous graph is a straight line.

B A continuous graph has a y-value for every x-value.

C A discontinuous graph has limits at one or both ends.

D A discontinuous graph cannot be drawn without lifting your pencil from the page.

10 Sketch the graph of the piece-wise function. Exercise 1.3

$$y = \begin{cases} x, & x < 5 \\ 5, & 5 \le x \le 10 \\ -2x + 15, & x > 10 \end{cases}$$

11 Determine the value of p if $p + 7$, $6p + 4$, $10p + 2$ are three consecutive terms of an arithmetic sequence. Exercise 1.4

12 A sequence is defined by $t_n = 135 - 6n$. Which term of the sequence is the first negative value? Exercise 1.4

13 Which one of the following sequences is *not* arithmetic? Exercise 1.4

A $t_n = 3n + 4$

B 35, 26, 17, 8 ...

C $\frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2} \ldots$

D $t_{n+1} = 7 - t_n$, $t_1 = 4$

14 Determine the sum of the first 10 terms of the sequence 3, 8, 13, 18, … Exercise 1.4

15 The lengths of the 10 rungs on a clown's circus ladder increase by the same quantity on each ascending rung. If the fourth rung is 34 cm long and the ninth rung is 59 cm long, determine: Exercise 1.4

(a) the difference in length between each consecutive rung on the ladder

(b) the length of the first and tenth rungs on the ladder.

16 A taxi fare consists of a flagfall fee of \$4 to which is added \$2.50 for every kilometre travelled. Calculate how much a taxi ride from Cladstone to Auberry will cost if the distance between these suburbs is 25 km. Exercise 1.4

17 Council puts out 4500 new green bins initially, with 825 new bins to come every week until every household has a new bin. Exercise 1.5

(a) Write a formula to model the number of bins that have been rolled out in terms of the number of weeks since the start of the process.

(b) Determine the domain for which this linear model is valid, given that there are 102 925 households in the council area.

Exercise 1.5 **18** A laptop purchased for work costs \$2840. For tax purposes depreciation of 20% of the initial value is to be claimed as a deduction each year.

(a) Determine the amount of the deduction and the number of years it will be claimed.

(b) Model the paper value of the laptop as a straight line with a limited domain. Determine the equation to the linear function.

Exercise 1.5 **19** \$5000 is invested at 6% p.a. simple interest for 4 years. Determine the balance of the account at the end of the 4 years.

Exercise 1.1 **20** Determine the rule of the linear equation that passes through:

(a) (-2, 11) and (-1, 22)

(b) (2, 3) and (4, -1)

(c) (-5, -9) and (-1, 1)

Exercise 1.1 **21** The equation of the line passing through the points (5, -3) and (2, 4) is:

A $7x + 3y - 26 = 0$

B $7x + 3y + 26 = 0$

C $y = \frac{7}{3}x + \frac{26}{3}$

D $y = -\frac{3}{7}x + \frac{26}{7}$

Exercise 1.2 **22** Solve the simultaneous linear equations $2x + 3y + 15 = 0$ and $3y = 5x - 36$.

Exercise 1.2 **23** The pair of simultaneous equations $y = x + a$ and $y = bx - 1$ intersect at (-1, 4). The values of a and b, respectively, are:

A -1, 4 **B** 1, -4 **C** -5, 5 **D** 5, -5

Exercise 1.2 **24** Graeme sets a challenge for Chloe: 'I think of a number, double it and then add nine. The result is the same as subtracting one from the original number and multiplying the result by three.' Determine the original number.

Exercise 1.2 **25** Kiara attempts a puzzle that she reads in the local newspaper:
'I think of a number, add eight to it, multiply this result by five and then get one half of the result. This answer will produce the same result as multiplying the original number by three and then adding four.'
Determine the original number.

Exercise 1.2 **26** Determine a pair of numbers whose sum is 53 and whose difference is 17.

Exercise 1.2 **27** Bethany quotes \$65 per hour to complete an electrical job for a customer. Angela quotes a call-out fee of \$42 plus \$58 per hour for the same job.

(a) Create an equation to represent Bethany's cost \$$C$ for the job taking h hours.

(b) Create an equation to represent Angela's cost \$$C$ for the job taking h hours.

(c) Calculate the total cost Angela will charge for a job that takes $3\frac{1}{2}$ hours.

(d) At what time will the cost for a job be the same for both Angela and Bethany?

28 In an arithmetic series, the sum of the first n terms is 120. If the series is $30 + 26 + 22 + \ldots$, determine the value(s) of n. Exercise 1.4

29 A principal of \$9600 is deposited into an 8.75% annual simple interest rate account. Let A_n denote the total amount at the end of n years. Exercise 1.5

(a) Determine the formula for A_n, hence calculate A_6.

(b) How many years will it take before the total amount is double the original principal?

30 Tamara borrows money to purchase a \$15 607 car, having saved just \$6000 for the deposit. She is offered terms of 7.1% p.a., at a flat rate, over 18 months with equal monthly instalments.

(a) Determine the amount of the regular instalments, rounding to the nearest \$20. Calculate the amount of the final one-off payment at the end of the term, assuming it is less than the regular payment.

(b) Model the amount owing as an arithmetic sequence, making an exception of the final term.

(c) Model the amount paid as an arithmetic sequence, making an exception of the final term. Include the deposit in your model.

31 Use equations $y = 2x - 4$, $y = x + 3$ and $x + y = 10$ to create a piece-wise continuous graph over $0 \le x \le 8$. Your graph must be above the x-axis throughout the domain. Write the individual domain for each piece of the graph. Exercise 1.3

32 The last three terms of an arithmetic sequence are 61, 63, 65. Determine the first term, given that the sum of the entire series containing these terms is 1085. Exercise 1.4

33 Determine the value of n for which the sum of the terms of the arithmetic sequences -20, -11, -2, ... and 12.2, 12.4, 12.6, ... are either equal or as close as they can be. Exercise 1.4

2

Quadratic functions

Recall 78

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2.4 Determine the rule of a quadratic function 126

Summary 138

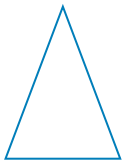
Chapter review 140

2 Recall

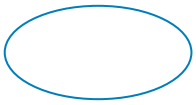
Identify and draw the line of symmetry for 2D shapes

1 Copy each of the following diagrams into your workbook and draw in all the lines of symmetry to determine the number of lines of symmetry for each shape.

(a)



(b)



(c)

Draw the graph of a linear function, given its equation

2 Draw the graph of the following functions, stating the coordinates of the x-intercept and y-intercept.

(a) $y = 3x + 4$ (b) $3x - 2y = 8$ (c) $y = 9$

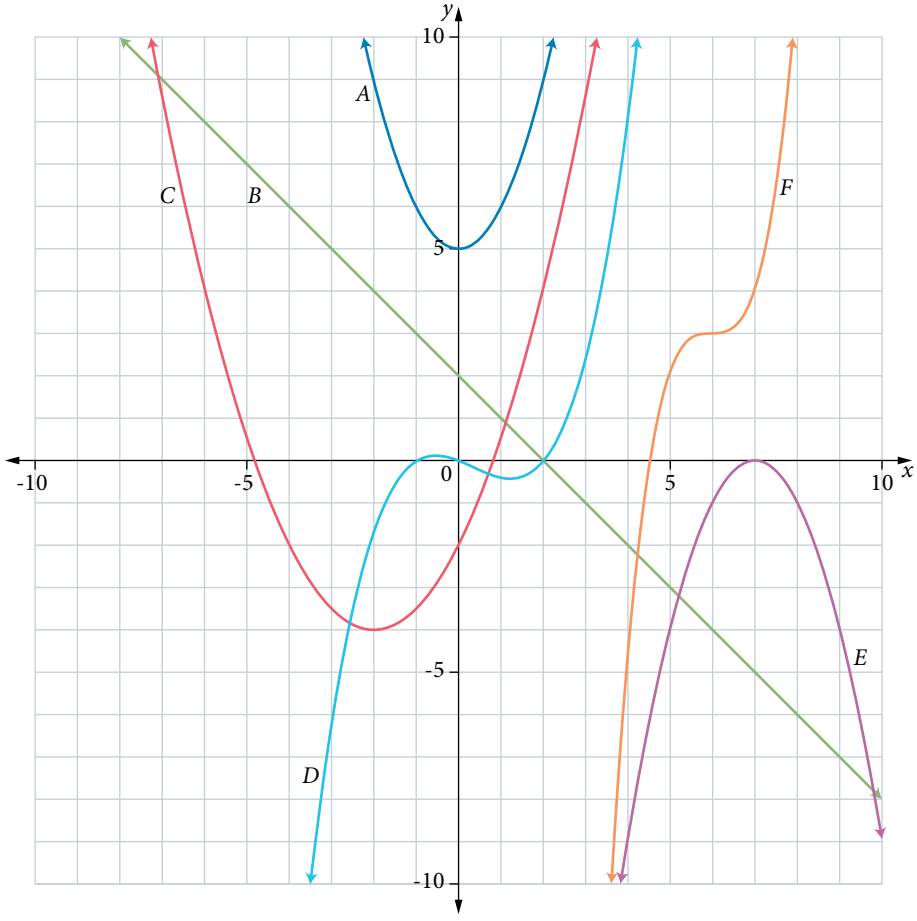
Identify rules that represent quadratic functions

3 Decide whether each of the following represents a quadratic function.

(a) $y = x - 6$ (b) $f(x) = x^2 - 3x + 9$ (c) $x(6x - b)(x - 2b)$

Identify graphs of quadratic functions

4 Which of the graphs A, B, C, D, E, F represent quadratic functions.



Simplify algebraic expressions

5 Simplify the following expressions by collecting like terms.

(a) $3x + y - 5y + 7x$ **(b)** $8a^2 - b^2 - 5b^2 + 7ab$ **(c)** $4xy - 9xy + x^2 - yx - 3x^2$

Expand algebraic expressions

6 Expand each of the following expressions.

(a) $5(x - 7)$ **(b)** $-x(11 - x)$ **(c)** $(t + 2)(t - 5)$

Factorise algebraic expressions

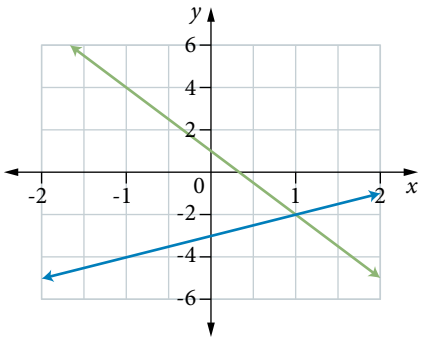
7 Factorise each of the following expressions.

(a) $6y^2 - 3y$ **(b)** $2a + 6b - 12$ **(c)** $-8x + 20$

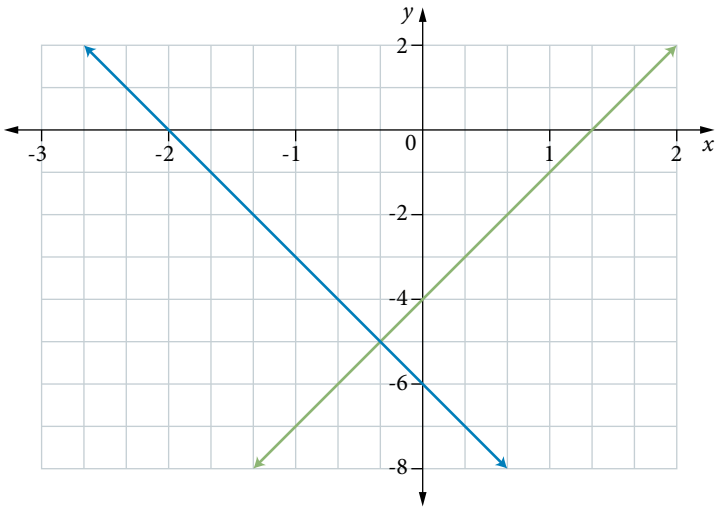
Solve simultaneous equations

8 Determine the simultaneous solution for each of the following.

(a)



(b)



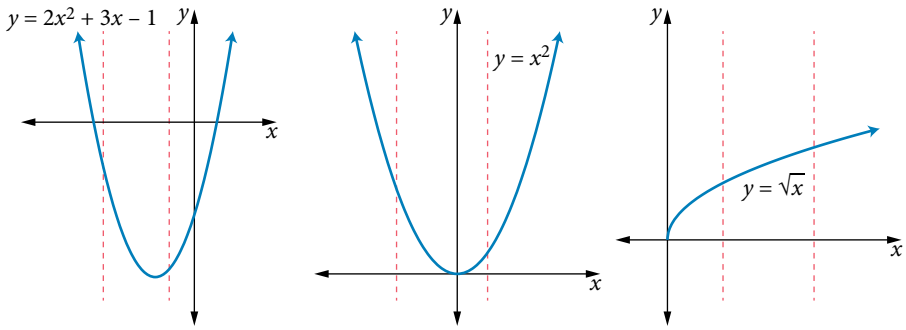
(c) $y = 2x + 11$ and $y = 3x + 7$

2.1 Introduction to quadratic functions

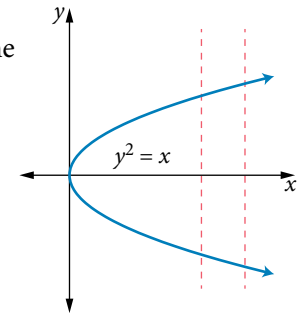
Functions and relations

Consider the graph of a relation and a set of vertical lines of the form $x = a$, where the value of a varies across the domain of the relation given.

- A relation is a function if each x-value has only one corresponding y-value. You can determine whether any relation is a function from its graph using a vertical line test. For example, $y = 2x^2 + 3x - 1$, $y = x^2$ and $y = \sqrt{x}$ are all functions. The vertical lines intersect the graph only once for every point in the domain, so the relation is a function. Each x-value has only one corresponding y-value.



- A relation is not a function if an x-value has more than one corresponding y-value. For example, $y^2 = x$ is not a function. The vertical lines intersect the graph more than once in the domain, so the relation is not a function. Each x-value has more than one corresponding y-value.



Explore further

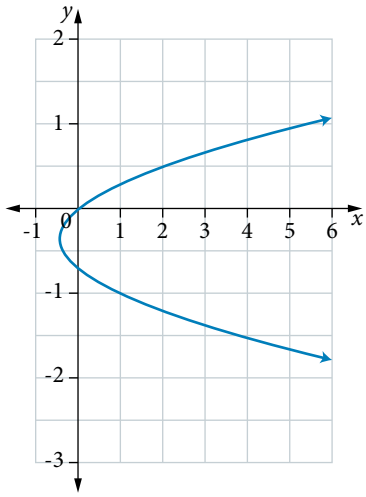
Quadratics and functions

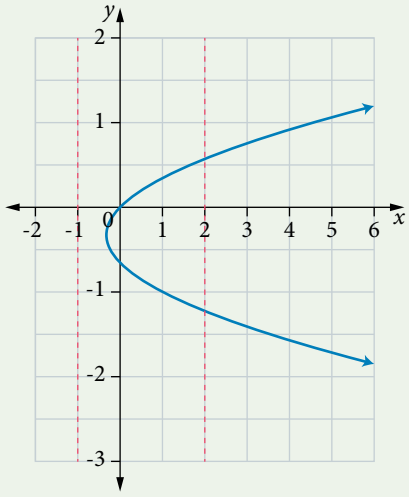
Explore the vertical line test with different functions and relations.

1 Vertical line test for a function

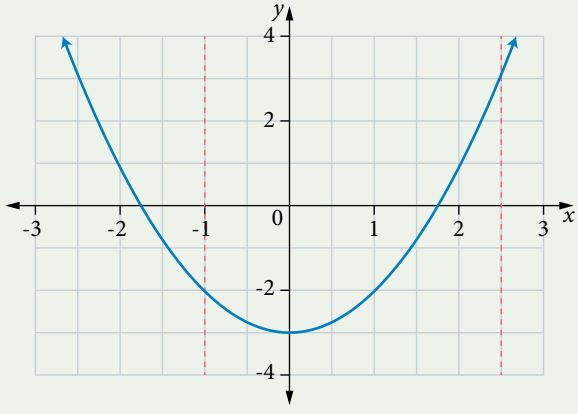
Determine whether each of the following quadratic relations is a function.

(a) $x = 3y^2 + 2y$



THINKING	WORKING
1 Draw vertical lines on the graph. If a vertical line can cut the graph more than once, then the graph does not represent a function. Otherwise the relation passes the vertical line test to be a function.	
2 Interpret your results and write your conclusion.	The vertical line to the right of the vertex cuts the graph twice, so each x-value is assigned more than one y-value. Hence the relation is not a function.

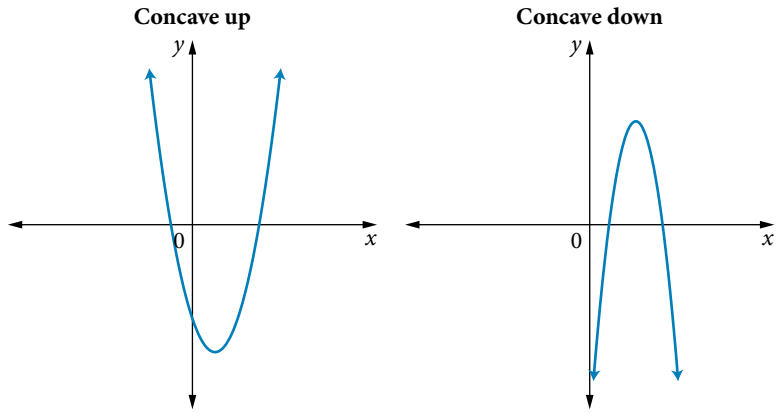
(b) $y = x^2 - 3$

1 Recall that any relation of the form $y = f(x)$ will be a function.	$y = x^2 - 3$ is a function.
2 Justify your answer with a graph. Use technology to sketch the graph. Draw vertical lines on the graph to determine if it is possible for any vertical line to cut the graph more than once.	 It is not possible for a vertical line to cut the graph more than once; this means that each x-value is assigned only one y-value.
3 Confirm your original statement.	The relation is a function.

Quadratic functions

Quadratic functions can be expressed in the general form of $y = ax^2 + bx + c, a \neq 0$.

The graph of a quadratic function is a parabola, as illustrated in the following graphs. The graph on the left is said to be *concave up* and the graph on the right is said to be *concave down*.



Quadratic functions are used to model parabolic shapes such as arches, satellite dishes, the parabolic trajectories of balls thrown through the air and the path of a water fountain.

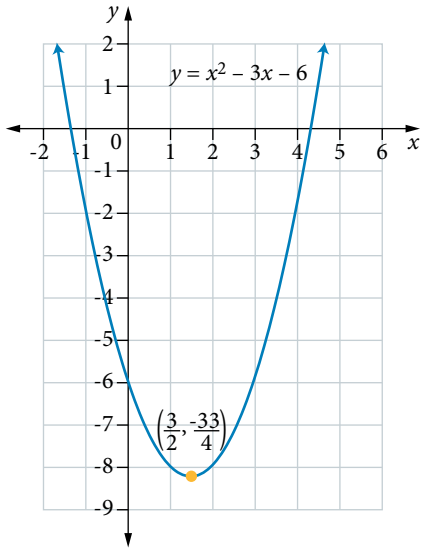
Domain and range

In general, a quadratic function is defined for all real values of the independent variable. That is, its largest possible (or maximal) domain is the set of all real numbers $\mathbb{R}$.

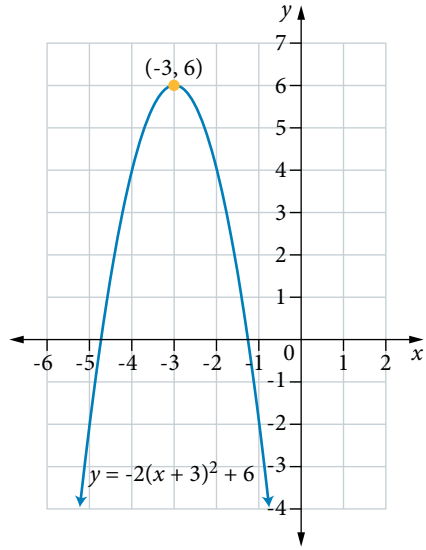
To specify the range of a quadratic function, it is convenient to use interval notation.

The range of a quadratic function depends on the location of its vertex and whether its graph is concave up ($a > 0$) or concave down ($a < 0$).

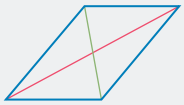
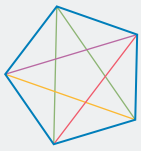
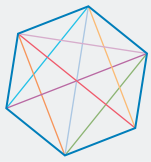
For domain $\mathbb{R}$, the range of $y = x^2 - 3x - 6$ is the set of all real numbers greater than or equal to $-\frac{33}{4}$ or $\left[-\frac{33}{4}, \infty\right)$.



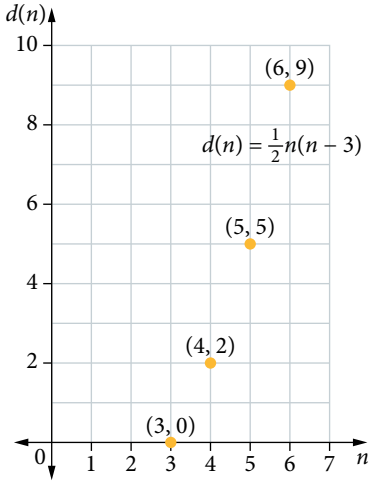
The range of $f(x) = -2(x+3)^2 + 6$ is the set of all real numbers less than or equal to 6 or $(-\infty, 6]$.



If a quadratic function represents a real-life situation, then it will likely have restrictions on the domain. In some modelling contexts a quadratic function may involve discrete variables. For example, if a polygon has n sides, then n must be an integer greater than or equal to 3. The number of diagonals d for a polygon with n sides is given by $d(n)=\frac{1}{2}n(n-3)$, as shown in the following table.

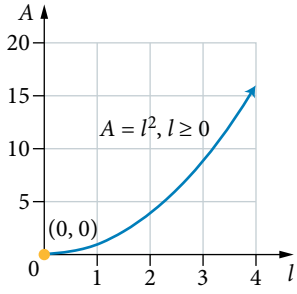
	Triangle	Quadrilateral	Pentagon	Hexagon
Shape				
n	3	4	5	6
$d(n)$	0	2	5	9

The domain of the function $d(n)=\frac{1}{2}n(n-3)$ is $\{3, 4, 5, 6...\}$ and the range of the function is $\{0, 2, 5, 9...\}$. The graph of the function is the set of discrete points that are on a parabolic arc, as shown in part on the right.



Quadratic functions can be used in practical situations such as representing the area of a square. The area of a square A with side length l is $A = l^2$, $l \geq 0$.

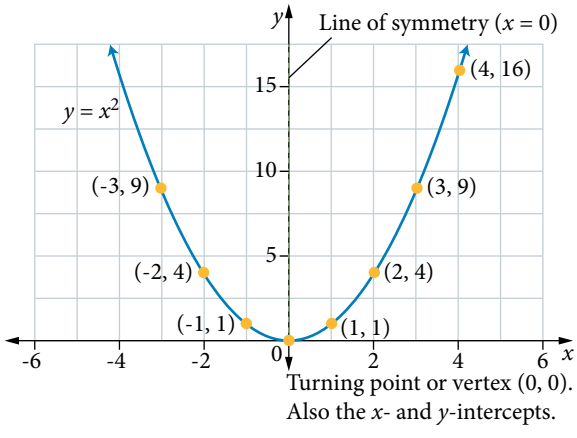
The graph representing part of the corresponding function is not defined for negative values of l, so the domain and the range of the function is $[0, \infty)$.



Quadratic graphs

The basic quadratic function is $y = x^2$ or $f(x) = x^2$, where $x \in \mathbb{R}$. To draw its graph, choose values for the independent variable x and then calculate the corresponding value of the dependent variable y to obtain the coordinates of the corresponding point (x, y). Plot these points and then join them with a continuous (no breaks) and smooth (no sudden changes of direction) curve.

x	-3	-2	-1	0	1	2	3	4
$y = x^2$	$(-3)^2 = 9$	$(-2)^2 = 4$	$(-1)^2 = 1$	$0^2 = 0$	$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$

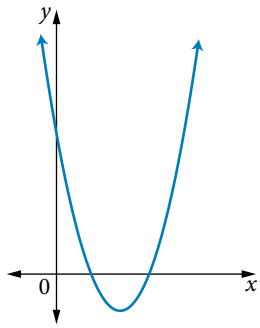


Features of the quadratic graph

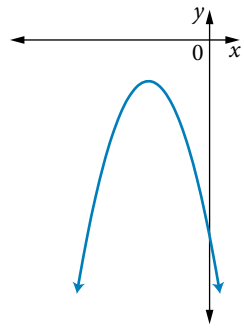
The graphs of quadratic functions are called *parabolas*. All parabolas are transformations of the basic quadratic graph $y = x^2$, which has the following key features:

- A vertical line of symmetry divides the parabola into two mirror images. The graph of $y = x^2$ has a line of symmetry with equation $x = 0$ on the y-axis.
- Over an unrestricted domain, the turning point (vertex) of a parabola is either a maximum point (when the graph is concave down), or a minimum point (when the graph is concave up). The turning point (or vertex) is always on the axis of symmetry.
- The graph of $y = x^2$ decreases smoothly and without breaks from the left to the right, until it reaches a minimum value at the turning point at (0, 0) and then increases smoothly and without breaks.
- The graph of $y = x^2$ is a special case because the turning point, x-intercept and y-intercept all coincide at the origin.
- The graphs of quadratic functions in the form of $y = ax^2 + bx + c$ have a turning point with an x-value of $x = \frac{-b}{2a}$.
- The y-intercept of a parabola is where the graph crosses the y-axis. The y-intercept can be found by substituting $x = 0$ into the rule to determine the value of y or by finding $f(0)$ for the rule $y = f(x)$. All parabolas have exactly one y-intercept.
- The x-intercepts of a parabola are where the graph crosses the x-axis. The x-intercepts can be found by solving $y = 0$ or $f(x) = 0$ for x. A parabola may have zero, one or two x-intercepts (or roots).

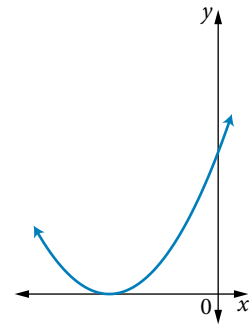
2 x-intercepts



no x-intercepts



1 x-intercept



Additional information

The vertex of a parabola

This activity explores how the coordinates of the turning points can be derived from the general quadratic expression.

2 Determine features of a quadratic graph

Consider the function $y = x(x + 2)$.

(a) Construct a table of values using x-values from -4 to 2.

THINKING

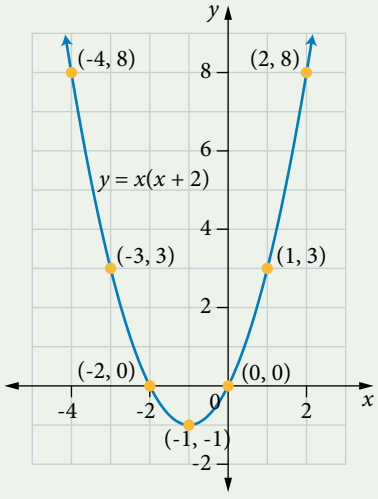
Use the rule to complete a table of values.

WORKING

x	-4	-3	-2	-1	0	1	2
y	8	3	0	-1	0	3	8

(b) Plot the points and join with a smooth curve.

Draw a set of axes, plot the coordinate points and join using a smooth curve.



(c) Identify whether the graph is concave up or concave down.

If the graph has a minimum turning point then it is concave up, if the graph has a maximum turning point it is concave down.	The graph is concave up.

(d) Determine the coordinates of the turning point.

Determine the equation of the axis of symmetry, hence write the coordinates of the turning point. If this cannot be read from the graph, the turning point of a function in the form $y = ax^2 + bx + c$ occurs at $x = \frac{-b}{2a}$.	The equation of the axis of symmetry is $x = -1$. The coordinates of the turning point are (-1, -1).

(e) Determine the coordinates of the y-intercept.

If possible, read the coordinates off the graph. If not, substitute $x = 0$ and calculate the y-coordinate.	The coordinates of the y-intercept are (0, 0).

(f) Determine the range of the function.

The y-coordinate of the turning point provides one end point of the range. Assuming the domain is not restricted the graph will approach $-\infty$ or ∞ depending on the shape of the graph.	The range is $[-1, \infty)$.

Continuity and smoothness

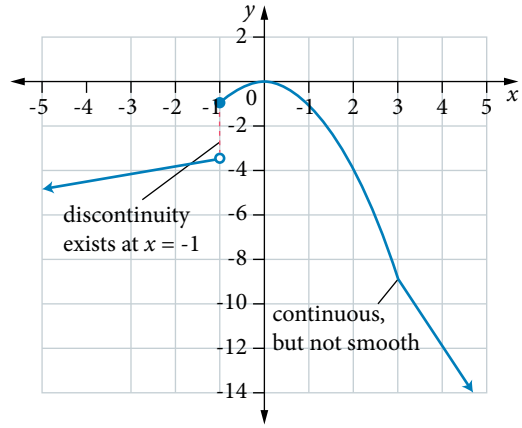
Technology worked example
Determine the features of a quadratic function

Quadratic functions with unrestricted domains will always be continuous. Piece-wise functions that include parabolas are said to be continuous if there are no breaks or gaps and smooth if there are no kinks, sharp bends or pointy bits (change in concavity). Check piece-wise functions at critical points, where subdomains meet, to ensure they are continuous and smooth.

This graph is continuous for all x-values except $x = -1$.

At $x = 3$ there is a kink or bend, but there is no gap. So the graph is continuous but not smooth at $x = 3$.

A graph is smooth if the gradient is the same to the left and right of a critical point. For now you will mainly need to rely on 'looks' to see whether a graph is smooth; however, this can be tested using derivatives in Chapters 9 and 10.



3 Determine continuity and smoothness of a piece-wise graph

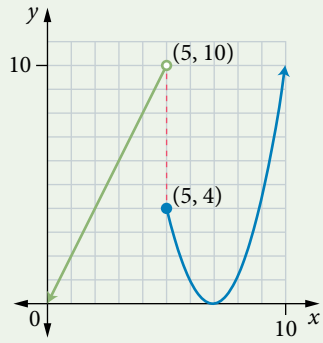
For each piece-wise function determine any values of x in the domain for which the function is not continuous or not smooth.

(a) $y = \begin{cases} 2x, & x < 5 \\ (x-7)^2, & x \geq 5 \end{cases}$

THINKING

1 Draw graphs for each expression using technology.
Determine critical values at the ends of domains.

WORKING

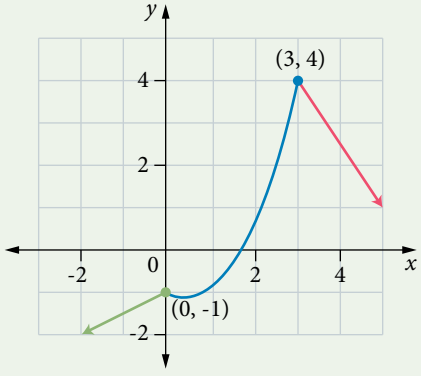


2 To be continuous at a point, the y-values of the expressions either side must be the same for the x-value at that point.

For $x = 5$:
$y_1 = 2 \times 5$
$= 10$
An excluded endpoint exists at (5, 10).
For $x = 5$:
$y_2 = (5-7)^2$
$= 2^2$
$= 4$
An included endpoint exists at (5, 4).
There is a discontinuity at $x = 5$.

(b) $y = \begin{cases} 0.5x - 1, & x \le 0 \\ \frac{5}{9}x^2 - 1, & 0 < x \le 3 \\ -1.5x + 8.5, & x > 3 \end{cases}$

1 Draw graphs for each expression using technology.
Determine critical values at the ends of domains.



2 To be continuous at a point, the y-values of the expressions either side must be the same for the x-value at that point.

The function is continuous at $x = 0$ and $x = 3$ and therefore continuous over $\mathbb{R}$.

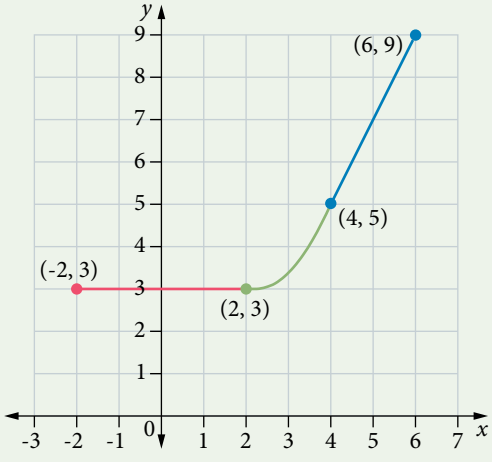
3 To be smooth, the graph must be continuous and have no kinks, sharp bends or pointy bits.

The graph is not smooth at $x = 0$ and $x = 3$.

(c) $f(x) = \begin{cases} 3, & -2 \le x < 2 \\ \frac{1}{2}(x-2)^2 + 3, & 2 \le x < 4 \\ 2x - 3, & 4 \le x \le 6 \end{cases}$

Technology worked example
A discontinuous piece-wise graph

1 Draw graphs for each expression using technology.
Determine critical values at the ends of domains.



2 To be continuous at a point, the y-values of the expressions either side must be the same for the x-value at that point.

The graph is continuous at both $x = 2$ and $x = 4$ and therefore continuous over the domain $[-2, 6]$.

3 To be smooth, the graph must be continuous and have no kinks, sharp bends or pointy bits.

The graph appears to be smooth at both $x = 2$ and $x = 4$.

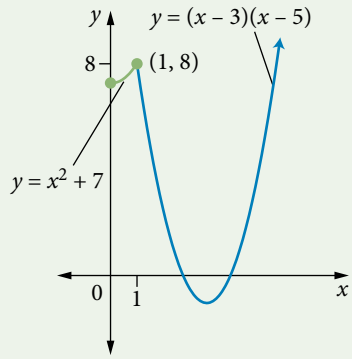
You can determine the value of a variable to ensure continuity in a function.

4 Determine the value of a variable to ensure continuity of a piece-wise function

Determine the value of a for which the function is continuous throughout its domain.

$$y = \begin{cases} x^2 + 7, & 0 \le x \le 1 \\ (x-a)(x-5), & x > 1 \end{cases}$$

Verify your answer with a graph.

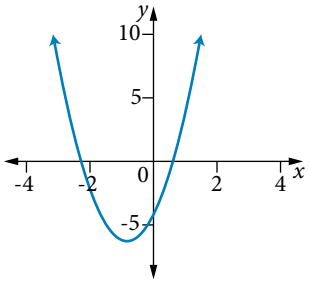
THINKING	WORKING
1 Substitute the critical value of x into both expressions.	For $x = 1$: $x^2 + 7 = 1^2 + 7$ $= 8$ The coordinates of the included endpoint are (1, 8). $(x-a)(x-5) = (1-a)(1-5)$ $= -4(1-a)$ $= 4a - 4$ The coordinates of the excluded endpoint are $(1, 4a - 4)$.
2 Equate the expressions and solve for the variable.	$4a - 4 = 8$ $4a = 1$ $a = 3$
3 Draw both graphs using technology. Determine critical values at the ends of domains.	
4 Make a statement of continuity involving the rule for the function.	The function is continuous at $x = 1$ for $a = 3$.

EXERCISE 2.1 Introduction to quadratic functions

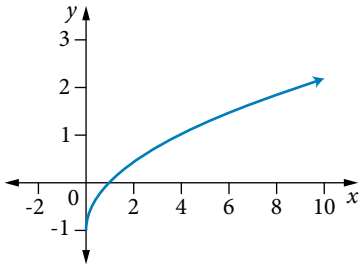
Worked Example

1 Determine whether the following graphs represent functions. 1

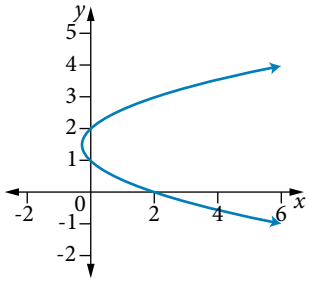
(a)



(b)



(c)



2 The relation $y - 4 = 3x - x^2$ represents a function because:

A for each y-value there is a unique x-value

B for each x-value there is one y-value

C each y-value corresponds to the same x-value

D for some x-values there is more than one y-value.

3 Consider the relations $y = 1, x = 2y + 3, x = y^2, y = x^3, x = y^3, x = 1$.

(a) Which option lists all of the relations that are not functions?

A $x = 2y + 3, x = y^2, x = y^3, x = 1$

B $y = 1, x = 1$

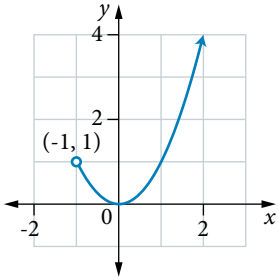
C $x = y^3, x = 1$

D $x = y^2, x = 1$

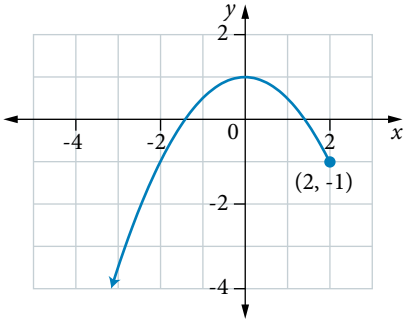
(b) Explain the common error made by a student who chose the first incorrect option.

4 Determine the domain and range of the following graphs using interval notation. 2

(a)

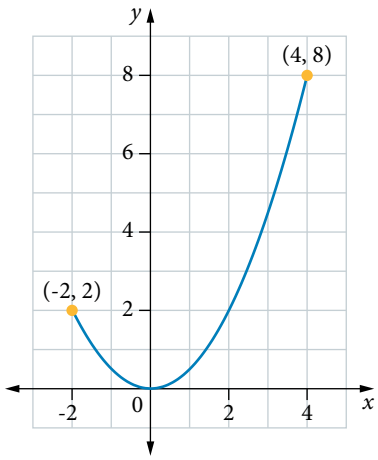


(b)

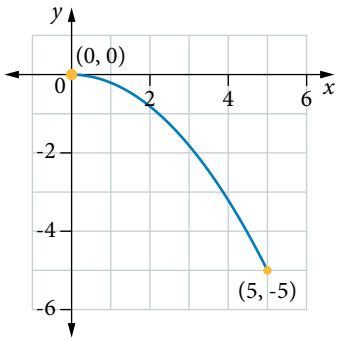


5 Determine the domain and range of the following graphs using inequality symbols.

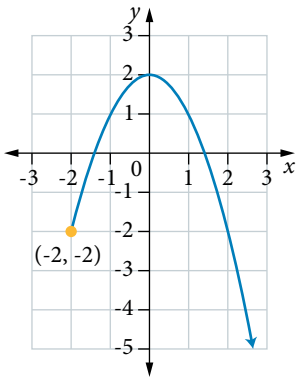
(a)



(b)

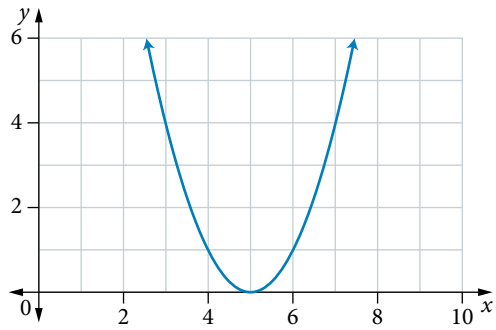


(c)

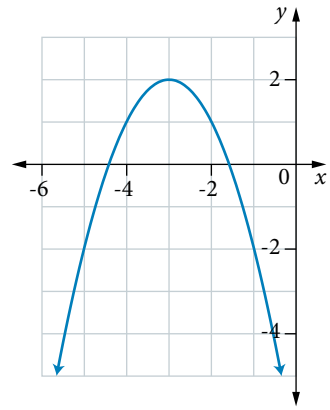


6 Determine the equation of the line of symmetry for the following graphs.

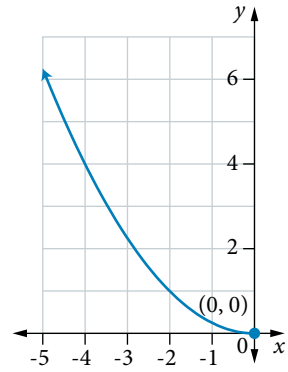
(a)



(b)



7 Determine the domain and range of the graph shown in terms of the set of real numbers $\mathbb{R}$ or the subsets $\mathbb{R}^+$, $\mathbb{R}^-$ and $\{0\}$. Use the union of sets symbols $\cup$ to link two subsets.



8 Consider the function $y = x(x - 4)$.

(a) Complete the table of values using x-values from -1 to 5. Then plot the points and join with a smooth curve.

x	-1	0	1	2	3	4	5
y							

(b) Determine whether the graph is concave up or concave down.

(c) Determine the coordinates of the turning point.

(d) Determine the coordinates of the y-intercept.

(e) Write the range of the function using interval notation.

9 Consider the function $f(x) = -(x - 3)(x + 3)$.

(a) Complete the table of values for the function, using x-values from -4 to 4. Then plot the points and join with a smooth curve.

x	-4	-3	-2	-1	0	1	2	3	4
y									

(b) Determine whether the graph is concave up or concave down.

(c) Determine the coordinates of the turning point.

(d) Determine the coordinates of the y-intercept.

(e) Write the range of the function using inequality symbols.

10 Consider the function $y = -(x+5)^2$.

(a) Construct a table of values for the function using x-values from -8 to 0. Plot the points and join with a smooth curve.

x	-8	-7	-6	-5	-4	-3	-2	-1	0
y									

(b) Identify whether the graph is concave up or concave down.

(c) Determine the coordinates of the turning point.

(d) Determine the coordinates of the y-intercept.

(e) Write the range of the function using interval notation.

11 Consider the discrete piece-wise function.

$$f(x) = \begin{cases} 5-x, & x \in \{-5,-4,-3\} \\ x^2, & x \in \{-2,-1,0,1,2\} \\ 5-x, & x \in \{3,4,5\} \end{cases}$$

(a) Draw the graph of the function.

(b) Determine the range of the function.

12 Consider the discrete piece-wise function.

$$f(x) = \begin{cases} 3x^2+6x+2, & x \in \{-2,-1,0\} \\ 4-x^2, & x \in \{1,2,3\} \\ x^2-2x-12, & x \in \{4,5\} \end{cases}$$

(a) Draw the graph of the function.

(b) Determine the range of the function.

Worked Example 3

13 The graph of the following piece-wise function is made up of a line segment and a parabola.

$$h(x) = \begin{cases} 3x-2, & x < 0 \\ (x-2)(x+2), & x \geq 0 \end{cases}$$

(a) Draw the graph of the function.

(b) Determine the following:

(i) $h(2)$ **(ii)** $h\left(-\frac{1}{3}\right)$ **(iii)** $h(0)$

(c) Determine which of the following would be the correct working out for calculating $h(1)$: $3 \times 1 - 2$ or $(1-2)(1+2)$.

14 Consider the piece-wise function.

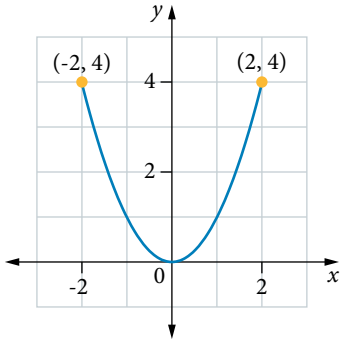
$$f(x) = \begin{cases} (x+7)(x+5), & x < -5 \\ x+5, & x \geq -5 \end{cases}$$

(a) Draw the graph of the function.

(b) Determine whether the graph of $f(x)$ is continuous for all values of x.

(c) Determine whether the graph of $f(x)$ is smooth for all values of x.

15 The following graph shows $f(x) = x^2$ over a given domain.



(a) Determine the rule for the function, including the domain.

(b) If the domain is changed to $[-3, 5)$, determine the range of the function.

(c) If the range was [0, 16], explain why you cannot state with certainty what the domain would be. Verify your answer with examples.

4

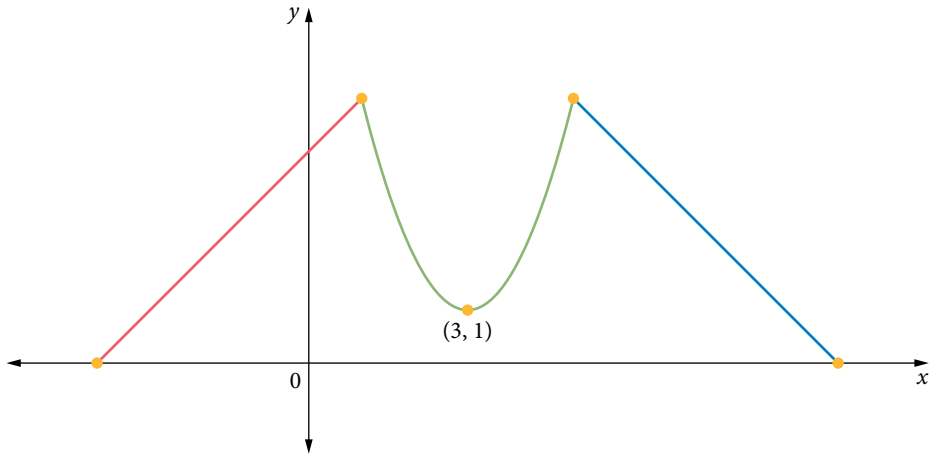
16 Determine the value for a such that $f(x)$ is continuous for all values of x. Write the rule for the continuous function and verify your answer with a graph.

$$f(x)=\begin{cases} x^2-1, & x\le 1 \\ -\frac{1}{2}x+a, & x>1 \end{cases}$$

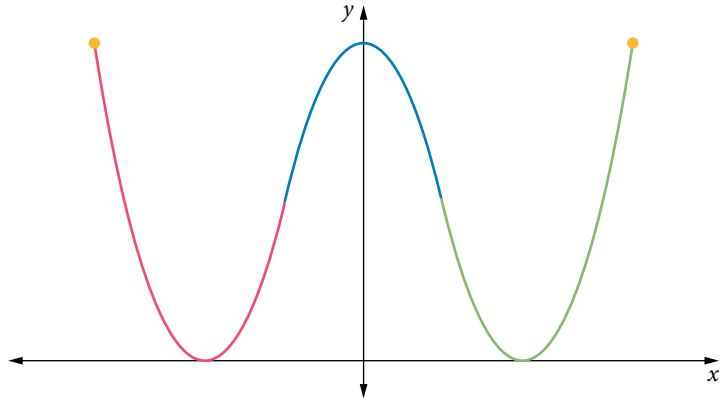
17 Determine the value for a such that $f(x)$ is continuous for all values of x. Write the rule for the continuous function and verify your answer with a graph.

$$f(x)=\begin{cases} (x-2)(x+1), & x\le 1 \\ (x+1)(a-2x), & x>1 \end{cases}$$

18 A symmetrical diagram is drawn on the Cartesian plane using a piece-wise function that involves the rules $y=(x-3)^2+1$, $y=x+a$ and one other. Determine the piecewise function given that the intersection points and a are whole numbers.



19 A symmetrical graph is drawn on the Cartesian plane, as shown, with three points at the maximum value and smooth joins at the intersections of the pieces. Determine the piece-wise function that involves the rules $y=(x-6)^2$, $y=a-x^2$ and one other, given that a is a whole number. Give the domain correct to 2 decimal places.



20 Consider the discrete function on the right.

Determine values for a and b such that the function is a piece-wise linear function. Determine the rule for the function and verify your answer with a graph.

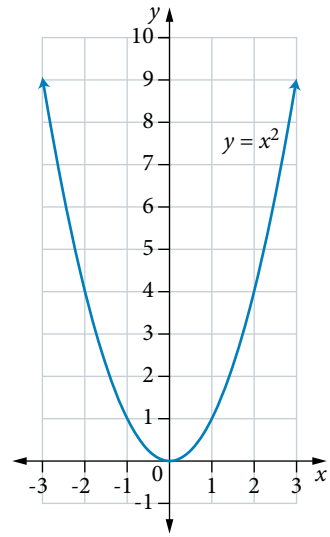
$$f(x)=\begin{cases} x^2+2x-a, & x\in\{-2,-1,0\} \\ 4-x^2, & x\in\{1,2\} \\ x(x-b), & x\in\{3,4,5\} \end{cases}$$

Transformations of quadratic functions

2.2

The basic quadratic function

The basic quadratic function is $y = x^2$ over domain $\mathbb{R}$, and the graphs of other quadratic functions can be described in terms of transformations (translations, dilations, reflections) of this function. The graph of this function, called a parabola, is shown on the right.



Explore further

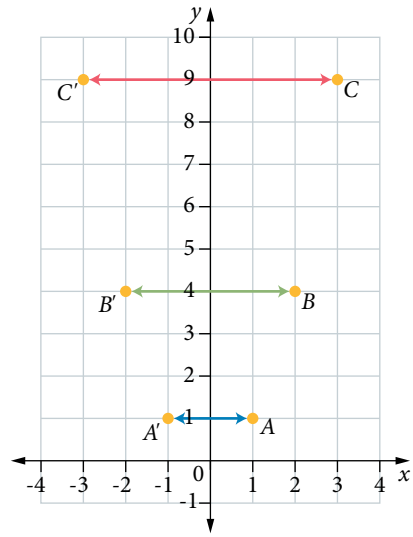
The function $y = a(x - b)^2 + c$

Explore the effect of the parameters a, b and c on the graph $y = a(x - b)^2 + c$.

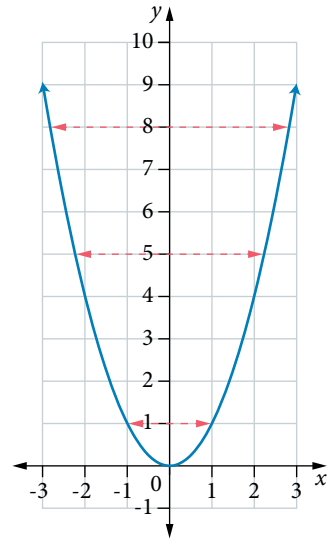
Reflection

Reflection in the y-axis

A reflection of the graph of a function in the vertical or y-axis is similar to the reflection observed in a mirror or window, where left-hand and right-hand orientation is reversed. Under this transformation, a point with coordinates (x, y) is mapped onto the point with coordinates $(-x, y)$. For example, the points $A(1, 1)$, $B(2, 4)$, $C(3, 9)$ on the graph of $y = x^2$ are respectively mapped onto the points $A'(-1, 1)$, $B'(-2, 4)$, $C'(-3, 9)$ and vice versa.



The graph of $f(x) = x^2$ is said to have vertical axis of symmetry about $x = 0$, and $f(-x) = f(x)$ for all x in its domain $\mathbb{R}$.



For a reflection in the y-axis, the x-coordinate changes sign and the y-coordinate remains unchanged.

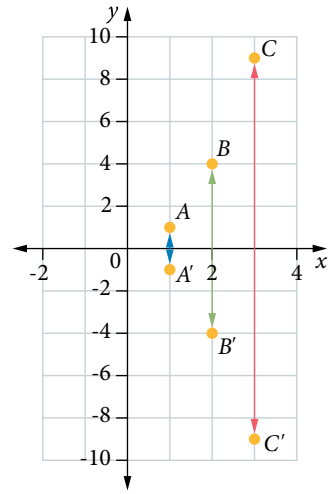
The point (x, y) is mapped onto the point with coordinates $(-x, y)$.

The image of any function reflected in the y-axis can be written as:

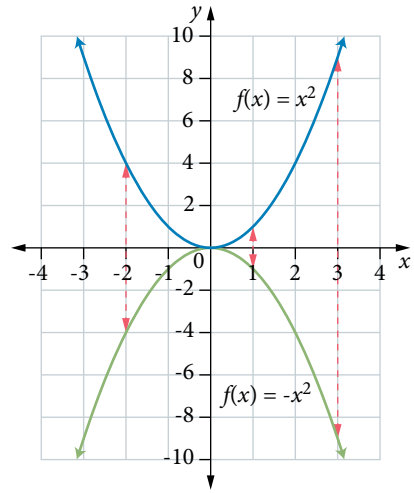
$y = f(x)$ becomes $y = f(-x)$

Reflection in the *x*-axis

A reflection of the graph of a function in the horizontal or x-axis is similar to the reflection observed in a flat surface, where top and bottom orientation are reversed. Under this transformation, a point with coordinates (x, y) is mapped onto the point with coordinates $(x, -y)$. For example, the points $A(1, 1)$, $B(2, 4)$, $C(3, 9)$ on the graph of $y = x^2$ are respectively mapped onto the points $A'(1, -1)$, $B'(2, -4)$, $C'(3, -9)$ and vice versa.



The graph of $f(x) = x^2$ is mapped onto the graph of $f(x) = -x^2$, for all x in its domain $\mathbb{R}$. The function $f(x) = -x^2$ is sometimes referred to as *concave down* parabola, the *upside down* parabola, *negative* parabola or *maximum* parabola.



For a reflection in the x-axis, the x-coordinate remains unchanged and the y-coordinate changes sign.

The point (x, y) is mapped onto the point $(x, -y)$.

The image function for any function reflected in the x-axis can be written as:

$y = f(x)$ becomes $-y = f(x)$ or $y = -f(x)$.

Dilation

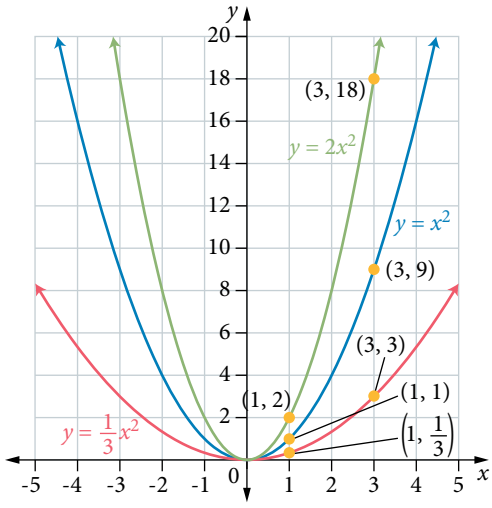
Vertical dilation

A vertical dilation involves stretching or shrinking a graph by a non-zero scale factor parallel to the y-axis and away from the x-axis.

Compare the graph of $y = x^2$ with the graph of $y = 2x^2$.

The basic quadratic graph has been stretched vertically parallel to the y-axis by a scale factor of 2. Each new point has been 'stretched' to twice the distance from the x-axis as its original position.

- The point (1, 1) has been mapped onto (1, 2).
- The point (3, 9) has been mapped onto (3, 18).
- The point at the origin (0, 0) is unchanged.



Compare the graph of $y = x^2$ and the graph of $y = \frac{1}{3}x^2$.

The basic quadratic graph has been dilated parallel to the y-axis by a scale factor of $\frac{1}{3}$. Each new point has been 'shrunk' to $\frac{1}{3}$ the distance from the x-axis as its original position.

- The point (1, 1) has been mapped onto $\left(1, \frac{1}{3}\right)$.
- The point (3, 9) has been mapped onto (3, 3).
- The point at the origin (0, 0) is unchanged.

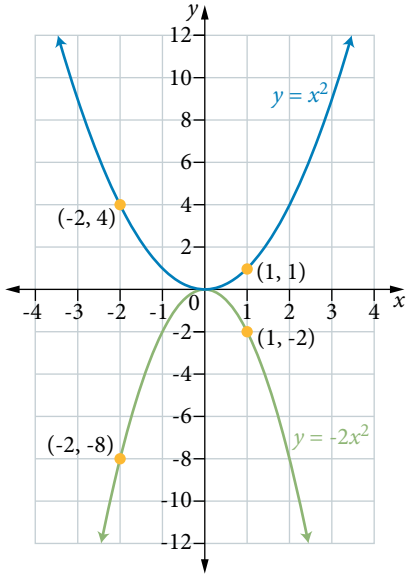
Vertical dilation and reflection

The order in which you reflect and dilate is not important. (You only need to consider order when a vertical translation is involved).

Compare the graph of $y = x^2$ with the graph of $y = -2x^2$.

The basic quadratic graph has been reflected in the x-axis and then stretched parallel to the y-axis by a scale factor of 2.

- The point (1, 1) has been mapped onto (1, -2).
- The point (-2, 4) has been mapped onto (-2, -8).
- The point at the origin (0, 0) is unchanged.



The transformations applied to the graph of the basic quadratic function $y = x^2$ with domain $\mathbb{R}$ to obtain the graph of $y = ax^2$ are:

- dilation parallel to the y-axis by a scale factor of a for $a > 0$
- for $a < 0$, reflection in the x-axis and dilation parallel to the y-axis by a scale factor of $|a|$ (the magnitude or size of a)
- if $|a| > 1$ the graph is narrower than $y = x^2$
- if $|a| < 1$ the graph is wider than $y = x^2$.

The range of the function is:

- $[0, \infty)$, if $a > 0$ (concave up)
- $(-\infty, 0]$, if $a < 0$ (concave down).

The transformations of reflection and dilation to obtain the graph of $y = ax^2$ for $a < 0$ from the graph of $y = x^2$ can occur in either order.

5 Dilation and reflection of quadratic graphs

Consider the quadratic function $s = -2.75t^2$ with domain $\mathbb{R}$.

(a) Describe the transformations required to obtain the graph of $s = -2.75t^2$ from $s = t^2$.

THINKING	WORKING
1 Identify the value of a by comparing the function in the form $y = ax^2$.	$a = -2.75$
2 Describe the transformations.	To obtain the graph of $s = -2.75t^2$, the graph of $s = t^2$ has been reflected in the x-axis and dilated by a factor of 2.75 parallel to the y-axis.

(b) Determine the coordinates of the vertical intercept.

Dilation does not shift the graph away from the origin, so the vertical intercept remains unchanged from the basic graph of $y = x^2$.	The coordinates of the s-intercept are (0, 0).

(c) Determine the range of $s = -2.75t^2$.

Identify the coordinates of the turning point and the shape of the graph.	The graph is concave down and the turning point is at (0, 0), so the range is $(-\infty, 0]$.

(d) Sketch the graph of the function.

1 Determine the coordinates of another point.

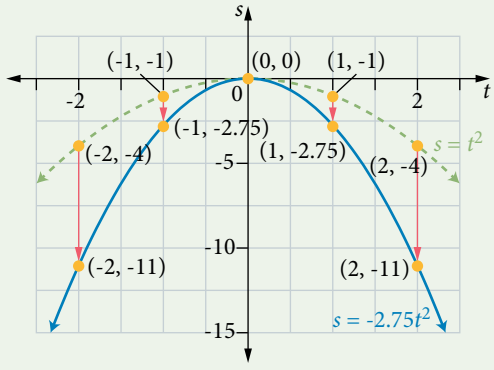
$s = -2.75t^2$ at $t = 1$

$s = -2.75 \times 1^2$

$= -2.75$

The coordinates for the point are (1, -2.75) and by symmetry, another known point is (-1, -2.75).

2 Include all of the detail found in the previous parts, and one other point to show the scale.



Horizontal dilation

A horizontal dilation involves stretching or shrinking a graph by a non-zero scale factor parallel to the x-axis and away from the y-axis.

What happens when the graph of $y = x^2$ is dilated horizontally by a factor of 3?

The x-values will increase by a factor of 3:

$$x_{\text{image}} = 3 \times x_{\text{original}}$$

$$x_{\text{original}} = \frac{x_{\text{image}}}{3}$$

$$y = x^2$$

$$y_{\text{image}} = \left(\frac{x_{\text{image}}}{3}\right)^2$$

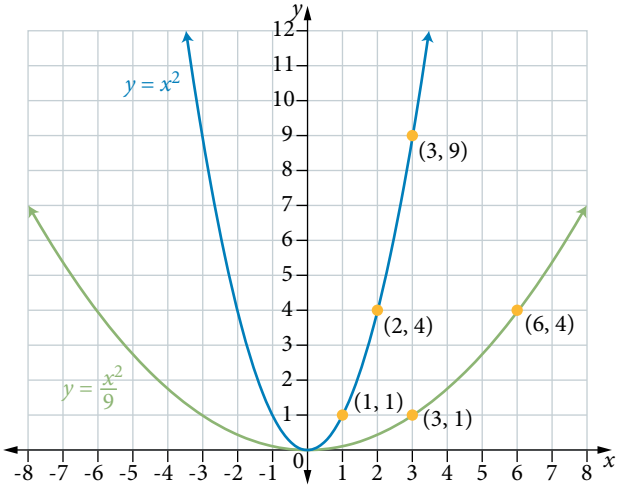
$$y = \frac{x^2}{9}$$

For a basic quadratic function $y = x^2$, a horizontal dilation by a factor of 3 parallel to the x-axis is equivalent to a vertical dilation by a factor of $\frac{1}{9}$ parallel to the y-axis.

Compare the graph of $y = x^2$ with the graph of $y = \frac{x^2}{9}$.

The basic quadratic graph has undergone a horizontal dilation and has been stretched parallel to the x-axis by a factor of 3 (or a vertical dilation by a factor of $\frac{1}{9}$).

- The point (1, 1) has been mapped onto (3, 1).
- The point (2, 4) has been mapped onto (6, 4).
- The point at the origin (0, 0) is unchanged.



Combined horizontal and vertical dilations

What happens when the graph of $y = x^2$ is dilated parallel to the x-axis by a factor of 3, and dilated parallel to the y-axis by a factor of 2?

The x-values will increase by a factor of 3:

$$x_{\text{image}} = 3 \times x_{\text{original}}$$

$$x_{\text{original}} = \frac{x_{\text{image}}}{3}$$

The y-values will increase by a factor of 2:

$$y_{\text{image}} = 2 \times y_{\text{original}}$$

$$y_{\text{original}} = \frac{y_{\text{image}}}{2}$$

$$y = x^2$$

$$\frac{y_{\text{image}}}{2} = \left(\frac{x_{\text{image}}}{3}\right)^2$$

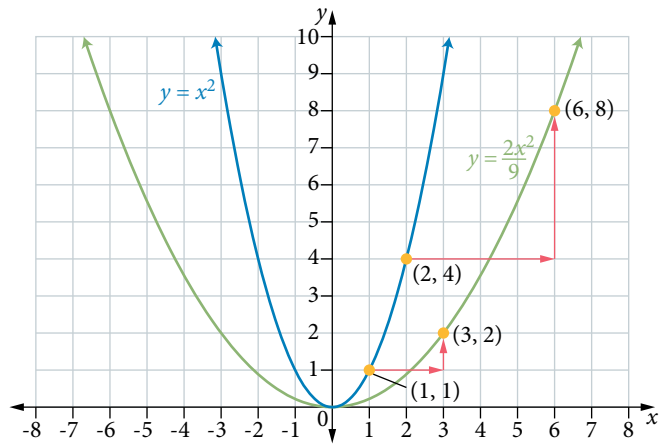
$$y = \frac{2x^2}{9}$$

For a basic quadratic function $y = x^2$, a horizontal dilation of 3 is equivalent to a vertical dilation of $\frac{1}{9}$. So a horizontal dilation of 3 and a vertical dilation of 2 is equivalent to a vertical dilation of $\frac{2}{9}$.

Compare the graph of $y = x^2$ with the graph of $y = \frac{2x^2}{9}$.

The basic quadratic graph has undergone a horizontal dilation by a factor of 3 and a vertical dilation of factor of 2.

- The point (1, 1) has been mapped onto (3, 2).
- The point (2, 4) has been mapped onto (6, 8).
- The point at the origin (0, 0) is unchanged.



For $y = x^2$, a combined vertical dilation of factor a and horizontal dilation of factor k can be replaced by a single vertical dilation of factor $\frac{a}{k^2}$.

Translation

A *translation* of the graph involves shifting the graph to the left, right, up or down or a combination of a vertical shift and a horizontal shift.

Vertical translation

The general form for vertical translation is $f(x) = x^2 + c$:

- the vertical translation is c units up
- the coordinates of the turning point are $(0, c)$
- the range is $[c, \infty)$.

For graphs in the form of $f(x) = ax^2 + c$ and $a < 0$:

- reflect the graph in the x-axis before you translate
- translate vertically c units up
- the coordinates of the turning point are $(0, c)$
- the range is $(-\infty, c]$.

6 Vertical translations of quadratic graphs

Consider the quadratic function $y = x^2 - 7$ with domain $\mathbb{R}$.

(a) Determine the translation for $y = x^2 - 7$ when compared to $y = x^2$.

THINKING	WORKING
1 Identify the value of c when the function is in the form $y = x^2 + c$.	$c = -7$
2 Describe the transformation.	The graph of $y = x^2 - 7$ has been translated down 7 units from the graph of $y = x^2$.

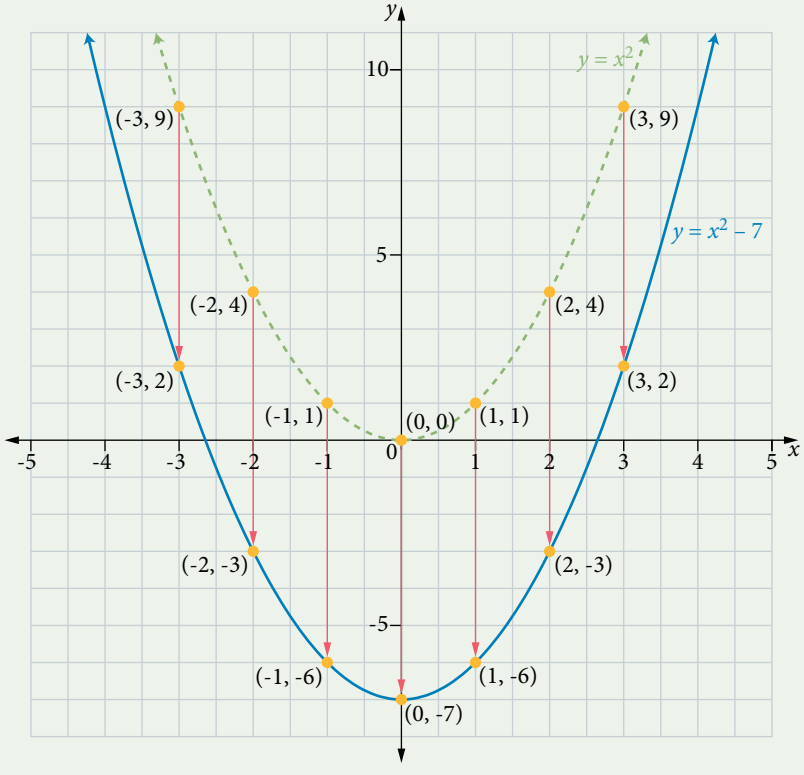
(b) Determine the equation of the line of symmetry.

A vertical shift up or down does not change the line of symmetry.	The equation of the line of symmetry is $x = 0$.

(c) Sketch the graph, stating the coordinates of the turning point and identifying the range of the function.

1 Sketch the graph using the information provided.

The basic parabola has been translated 7 units down.



2 Determine the coordinates of the turning point hence identify the range using interval notation.

The coordinates of the turning point are (0, -7).

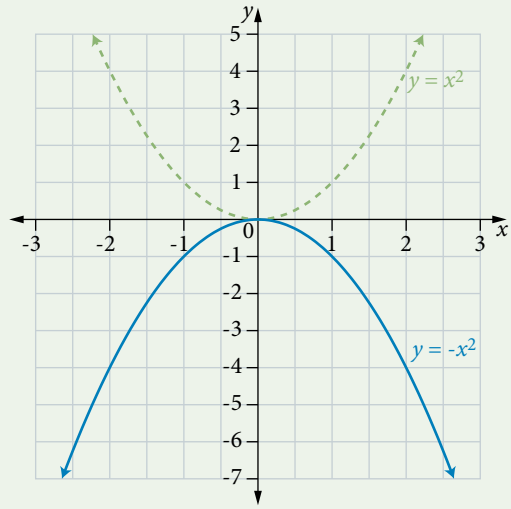
Range: $[-7, \infty)$

(d) Compare the functions $y = x^2 - 7$ and $y = -x^2 - 7$.

1 In the form $y = ax^2$ where $a < 0$, identify the reflection and/or dilation from the graph of $y = x^2$.

$a = -1$

The graph of $y = x^2$ has been reflected in the x-axis creating the concave down parabola $y = -x^2$.



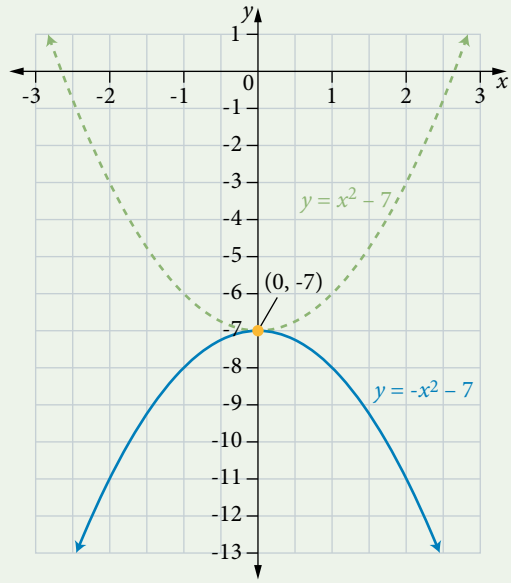
2 Determine the value of c by comparing the rule of the function to $y = ax^2 + c$, hence determine the location of the turning point.

$c = -7$

The graph is translated 7 units down from the basic function $y = ax^2$.

The coordinates of the turning point of the graph $y = -x^2 - 7$ are $(0, -7)$.

3 Graph each function on the same axes and compare the features.



The graphs of $y = x^2 - 7$ and $y = -x^2 - 7$ share a turning point at $(0, -7)$. Both graphs have been translated 7 units in the negative direction of the y-axis.

The graph of $y = x^2 - 7$ is concave up.

The graph of $y = -x^2 - 7$ is concave down.

Horizontal translation

The general form for horizontal translation is $f(x) = (x - b)^2$:

- the horizontal translation is b units to the right
- the coordinates of the turning point are $(b, 0)$
- the equation of the line of symmetry is $x = b$.

7 Horizontal translation of graphs of quadratic functions

Compare the quadratic function $f(x) = (x + 12)^2$ with that of $f(x) = x^2$ over the domain $\mathbb{R}$.

(a) Determine the translation for $f(x) = (x + 12)^2$.

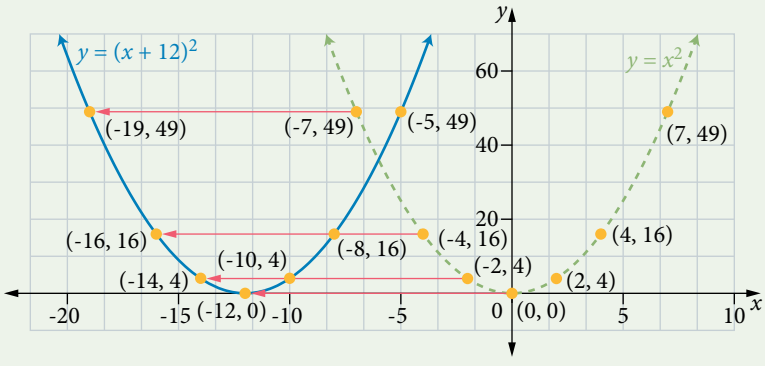
THINKING	WORKING
1 Identify the value of b by comparing the rule of the function to $y = (x - b)^2$.	$f(x) = (x + 12)^2$ $= (x - (-12))^2$ Therefore $b = -12$.
2 Describe the transformation.	The graph of $y = (x + 12)^2$ has been translated left 12 units from the graph of $y = x^2$.

(b) Determine the equation of the line of symmetry.

A horizontal shift affects the line of symmetry. The line of symmetry is translated b units to the right.	The equation of the line of symmetry has been translated 12 units left to $x = -12$.

(c) Sketch the graph and write the coordinates of the turning point and the range of the function.

1 Sketch the graph using the information provided.



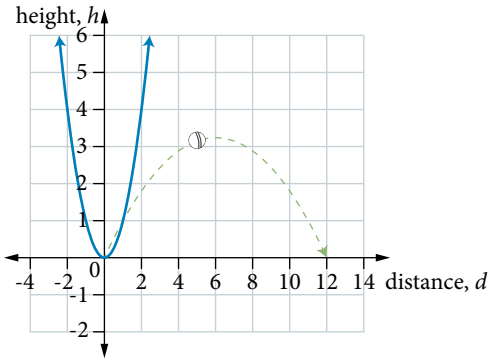
2 Determine the coordinates of the turning point and hence identify the range using interval notation.

The coordinates of the turning point are (-12, 0).

Range: $[0, \infty)$

Combination of transformations of graphs of quadratic functions

Combinations of transformations of the basic quadratic function can be used to model real-life situations. To model the path of a ball would require the basic quadratic function to be reflected in the x-axis, dilated, and shifted both vertically and horizontally.



For quadratic functions in the completed square or turning point form $f(x) = a(x-b)^2 + c$:

- the coordinates of the turning point (or vertex) are (b, c)
- the equation of the line of symmetry is $x = b$
- the range is $[c, \infty)$ if $a > 0$ and $(-\infty, c]$ if $a < 0$
- the graph is dilated parallel to the y-axis by a scale factor of a for $a > 0$
- the graph is reflected in the x-axis and dilated parallel to the y-axis by a scale factor of $|a|$ for $a < 0$.

So far you have only explored graphing quadratic functions in the completed square (or turning point) form. This is because the turning point (or vertex) of the parabola can be readily determined from this form. Graphing quadratic functions in the general form and factorised form will be covered later in this chapter.

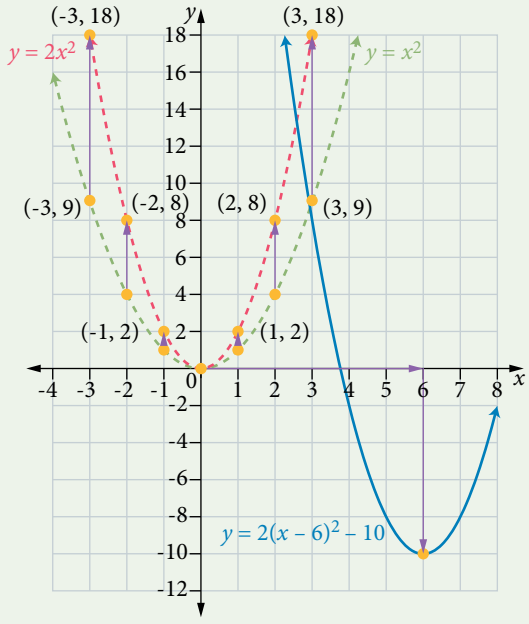
8 Combined transformations of quadratic graphs

Consider the quadratic function $f(x) = 2(x-6)^2 - 10$ with domain $\mathbb{R}$.

(a) Determine the transformations to produce the graph of $f(x) = 2(x-6)^2 - 10$ from $g(x) = x^2$ and sketch the graph.

THINKING	WORKING
1 Compare the given rule to $f(x) = a(x-b)^2 + c$ and identify the values of a, b and c.	$a = 2$ $b = 6$ $c = -10$
2 Interpret these values in terms of transformations.	From $g(x) = x^2$ the graph of $f(x) = 2(x-6)^2 - 10$ is dilated by a factor of 2 parallel to the y-axis; translated 6 units right and translated 10 units down.

3 Use the values for a, b and c to sketch the graph.



(b) Determine the equation of the line of symmetry.

When written in turning point form $y = a(x-b)^2 + c$, the equation of the line of symmetry is $x = b$.

The equation of the axis of symmetry is $x = 6$.

(c) Determine the coordinates of the turning point and the range of the function using interval notation.

1 When written in turning point form $y = a(x-b)^2 + c$, the turning point is at (b, c).

The coordinates of the turning point are (6,-10).

2 The range of the function is:

$[c, \infty)$ if $a > 0$ and $(-\infty, c]$ if $a < 0$.

Range: $[-10, \infty)$

(d) Determine the coordinates of the y-intercept.

1 The y-intercept occurs at $x = 0$. Subsitute $x = 0$ into the given rule to calculate the value of the y-intercept.

$$\begin{aligned} y &= 2(0-6)^2 - 10 \\ &= 2(-6)^2 - 10 \\ &= 72 - 10 \\ &= 62 \end{aligned}$$

2 Interpret the answer.

The coordinates of the y-intercept are (0, 62).

9 Applications of quadratic functions

Brett throws a ball to his cousin Lewis, who is 14 m away. The ball follows a parabolic path, reaching a maximum height 5 m above its point of release from Brett's hand.

Refer to the function as $p(x)$, where p is the height of the ball and x is the horizontal distance, in metres, from Brett's hand at the origin (0, 0). Assume the throw and catch are at the same height.

(a) Determine the rule for the graph.

THINKING	WORKING
1 Recall the general rule for a parabola, in turning point form.	$p(x) = a(x-b)^2 + c$
2 Identify the information known.	The ball travels a total horizontal distance of 14 m and has a maximum vertical displacement of 5 m. Therefore the ball passes through the point (7, 5), which represents the turning point of its parabolic path.
3 Substitute the values into the general rule.	$p(x) = a(x-7)^2 + 5$
4 Use the starting point as another piece of information to determine the value of a.	Substitute the values of the coordinates (0, 0): $0 = a(0-7)^2 + 5$ $0 = 49a + 5$ $-5 = 49a$ $a = -\frac{5}{49}$
5 Determine the rule of the function.	$p(x) = -\frac{5}{49}(x-7)^2 + 5$

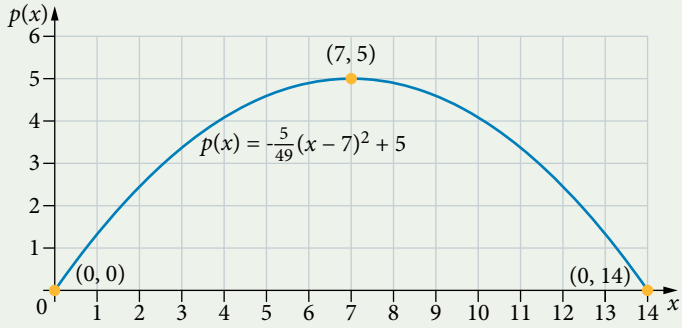
(b) Determine the domain and range.

Determine the minimum and maximum values of each variable.	Domain: $0 \le x \le 14$ Range: $0 \le p \le 5$

(c) Sketch the graph.

Use the known information to sketch the graph.

Plot the endpoints (0, 0) and (14, 0) as well as the turning point (7, 5).



EXERCISE 2.2 Transformations of quadratic functions

Worked Example

1 List the transformations required for each function from the graph of $f(x) = x^2$, then sketch each graph, stating the coordinates of the y-intercept and the turning point, and hence the range. 5

(a) $f(x) = -9x^2$

(b) $y = 0.75x^2$

2 Consider the quadratic function $f(x) = -x^2 + 14$ with domain $\mathbb{R}$. 6

(a) Determine the translation of the graph when compared to $f(x) = x^2$.

(b) Determine the equation of the line of symmetry.

(c) Sketch the graph to determine the coordinates of the turning point and the range of the function.

3 Consider the quadratic function $s = t^2 - 3.9$ with domain $\mathbb{R}$.

(a) Determine the translation of the graph from $s = t^2$.

(b) Write the equation of the line of symmetry.

(c) Sketch of the graph in your working to determine the coordinates of the turning point and the range of the function.

4 Consider the function $y = -x^2 + 3$.

(a) Sketch the graph of the function.

(b) A student sketches the graph based on their understanding of transformations, however has mistakenly translated the graph of $y = x^2$ and then reflected the resulting graph through the x-axis. Sketch the incorrect graph and determine the formula for the graph produced.

(c) Explain how to determine the reasonableness of a sketch from the formula.

5 Consider the quadratic function $f(x) = -2(x + 3)^2$ with domain $\mathbb{R}$. 7

(a) Determine whether the graph is concave up or concave down.

(b) Determine any translations that have occurred to $f(x) = x^2$ to give this function.

(c) Determine the turning point, the coordinates of y-intercept and the range of the function. Provide a sketch to justify your answer.

6 Consider the quadratic function $h = \frac{2}{5}(t-5)^2$ with domain $\mathbb{R}$.

(a) Determine whether the graph is concave up or concave down.

(b) Determine any translations that have occurred from the graph of $h(t) = t^2$ to give this function.

(c) Determine the turning point, the coordinates of y-intercept and the range of the function. Provide a sketch to justify your answer.

7 Determine the transformation of the function from $m = p^2$ to $m = p^2 + 21$.

8 Determine the equation for a function that has undergone a translation of 6 units to the left from $f(x) = x^2$.

9 Determine the transformation of the function from $s = t^2$ to $s = (t+18)^2$.

10 Determine the transformation of the function from $m = p^2$ to $m = (p-20)^2$.

11 Determine the rule for the quadratic function that represents a dilation parallel to the y-axis by a factor of 3 from the function $f(x) = x^2$.

12 Determine the quadratic function that represents a dilation parallel to the vertical axis with scale factor $\frac{1}{2}$ from the function $m(p) = p^2$.

13 Choose the option that gives the rule for the quadratic function that represents a reflection in the horizontal axis and a dilation parallel to the vertical axis with scale factor $\frac{2}{3}$.

A $s = -\frac{2}{3}t^2$ B $s = -\frac{3}{2}t^2$ C $s = \frac{3}{2}t^2$ D $s = \frac{2}{3}t^2$

14 (a) Determine the rule of the image of the function $y = x^2$ after the transformation:

(i) A vertical dilation by factor 5.

(ii) A horizontal dilation by factor 2.

(iii) A vertical dilation by factor 5 and a horizontal dilation by factor 2.

(b) Sketch $y = x^2$ and the three image graphs on the same set of axes. Label each graph.

15 Determine the equation of a quadratic function that demonstrates each of the following transformations.

(a) A translation of 5 units to the left and 2 units up.

(b) A translation of 10 units to the right and 3 units up.

(c) A translation of 12 units to the right and 9 units down.

(d) A dilation parallel to the y-axis by a scale factor of 9.

(e) A dilation parallel to the vertical axis by a scale factor of $\frac{1}{10}$.

(f) A reflection in the horizontal axis and a dilation parallel to the vertical axis, scale factor 2.

(g) A reflection in the horizontal axis and a dilation parallel to the vertical axis, scale factor $\frac{2}{7}$.

16 Describe the transformation(s) indicated by each of the following functions.

(a) $y = (x-0.5)^2$ (b) $m = p^2 - 0.12$ (c) $f(x) = (x+10)^2 - 12$

(d) $s = -t^2 + 16$ (e) $y = -18 + (x+6)^2$ (f) $d = 24 - e^2$

17 For each of the following quadratic functions determine:

(i) the transformation of the graph and use the information to draw a sketch of the function

(ii) the equation of the line of symmetry

(iii) the coordinates of the turning point

(iv) the coordinates of the y-intercept

(v) the range of the function.

(a) $y = -2x^2$ (b) $y = x^2 - 6$ (c) $y = (x-3)^2$

18 Construct the equation of a quadratic function that demonstrates each of the following transformations.

(a) A dilation parallel to the y-axis by a scale factor of 4 and translated 9 units down and 5 units left.

(b) A dilation parallel to the y-axis by a scale factor of $\frac{2}{5}$ and translated 16 units up and 3 units right.

19 Describe the transformation(s) indicated by each of the following functions.

(a) $f(x) = 3x^2 - 6$

(b) $f(x) = 4(x-5)^2 - 14$

(c) $m = -(p+8)^2 - 9$

(d) $s = -\frac{3}{5}(t-5)^2 + 2$

20 A function is transformed from $f(x) = x^2$ to $f(x) = (x+12)^2 - 6$.

(a) Choose the correct description of the transformations.

A translation 6 units left and 12 units up

B translation 6 units right and 12 units up

C translation 12 units left and 6 units down

D translation 6 units right and 12 units down

(b) Explain the common error made by a student who thought it was a translation 12 units right and 6 units down.

21 Consider the quadratic function $y = (x+3)^2 + 2$ with domain $\mathbb{R}$.

(a) List the transformations the graph of $f(x) = x^2$ has undergone to produce the graph of this function.

(b) Determine the equation of the line of symmetry, the turning point and the range, then draw the graph.

22 Consider the quadratic function $f(x) = -\frac{1}{4}(x-1)^2 - 3$ with domain $\mathbb{R}$. Worked Example 8

(a) List the transformations the graph of $f(x) = x^2$ has undergone to produce the graph of this function.

(b) Determine the equation of the line of symmetry, the turning point and the range, and then sketch the graph.

23 A string of decorative lights hangs symmetrically in the shape of a parabola between two poles that are 2 m apart. The lights fall at most 20 cm below the points where each end is attached to the poles. Let the point on the left-hand pole be the origin (0, 0). Refer to the function as $l(x)$, where l(m) is the distance above the connection point and x(m) is the distance to the right of the left-hand pole. 9

(a) Determine the rule of the graph that describes the path of the lights, including the domain. Explain any assumptions you had to make to answer the question.

(b) Determine the range of the function and sketch the graph.

24 Rahul built his front fence so that the pickets form parabolic curves between the posts. The posts are 1.75 m apart and the middle picket is 12 cm below the level of the imaginary line connecting the tops of the posts. Let the connection point on the left-hand post be the origin (0, 0). Refer to the function as $p(x)$, where p(m) is the distance above the top of the first picket and x(m) is the distance to the right of the first picket on the left.

Determine the rule of the graph, in turning point form, that describes the path of the fence. Give the value of a accurate to 4 decimal places.

25 For each of the following quadratic functions determine:

(i) the transformations from the graph of $y = x^2$, justifying your answer with a sketch of the function.

(ii) the equation of the line of symmetry

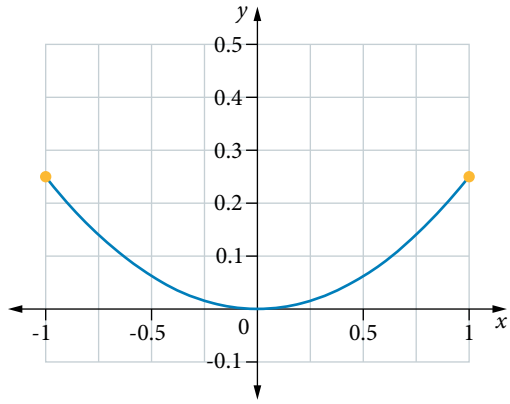
(iii) the coordinates of the turning point

(iv) the coordinates of the y-intercept

(v) the range of the function.

(a) $y = -2(x-3)^2 + 4$ (b) $f(x) = \frac{3}{4}(x+5)^2 - 2$ (c) $s = -\left(t - \frac{1}{2}\right)^2 + \frac{1}{2}$

26 Frank is turning a bowl from a cylindrical shaped block of wood. The quadratic equation $y = 0.25x^2$ can be used to describe the cross-section of the bowl. The graph drawn shows the shape of the bowl. Using a scale of 1 unit representing 100 mm, if the width of the bowl is from the x-value of -1 to 1, what is the depth of the bowl?



27 The path of a stream of water from a fountain can be described by the quadratic equation $h = -d(d-3)$, where d(m) is the horizontal distance from the base of the fountain to the point at which the water hits the ground and h(m) is the height of the stream of water.
Use technology to draw the graph of this quadratic function and use the graph to answer the following questions.

(a) Determine and interpret the coordinates of the turning point.

(b) Determine the horizontal distance from the base of the spout to where the water hits the ground.

28 The graph of $y = (x-3)^2 + 3, 1 \le x \le 4$ undergoes the following transformations in the order given: horizontal dilation of factor 2 from the axis of symmetry, reflection in the x-axis, translation up 4, left 3. Determine the rule for the final image.

29 A small rock is thrown upwards from a platform that juts out from a cliff. The platform is 40 m above the ground and the rock reaches its maximum height 10 m out and 5 m up from the platform. Determine where the rock hits the ground relative to the platform. State any assumptions you had to make to answer the question.

30 Adele sets up a volleyball net in her yard. She punches the ball so that it clears the net by 10 cm and hits the ground 2 m from the net on the other side. Adele estimates that she punched the ball 190 cm above the ground and half a metre from the net.
Determine the height of the net above the ground. State any assumptions you had to make to answer the question. Give your answer in metres, correct to 2 decimal places.

Solving quadratic equations

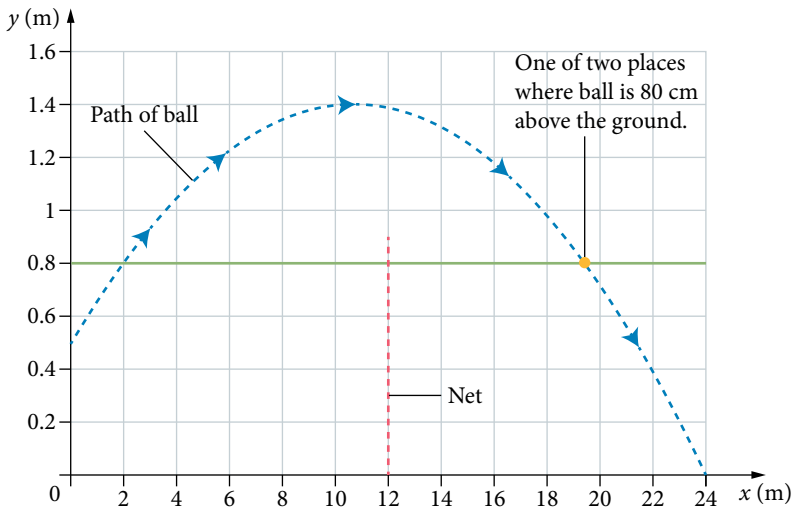
2.3

Introduction

Quadratic equations can be used to represent real-life situations. The properties of the graph of a quadratic function can be used to solve aspects of these practical situations. For example, how does the distance from a goal affect the angle of the shot?

A ball-launching machine can deliver a tennis ball at varying angles depending on the settings for practice.

A ball-launching machine on the baseline of a tennis court, 12 m from the net, delivers balls on a path $h(x) = -0.008x^2 + 0.17x + 0.5$, where h(m) is the ball height and x is the horizontal distance, up to about 24 m, from the ball-launching machine. If the ball is released 50 cm above the ground, where is the best place to stand to practise backhands at hip height, 80 cm from the ground?



To determine the best place to stand, you need to solve the equation $h(x) = 0.8$ either by hand or with technology.

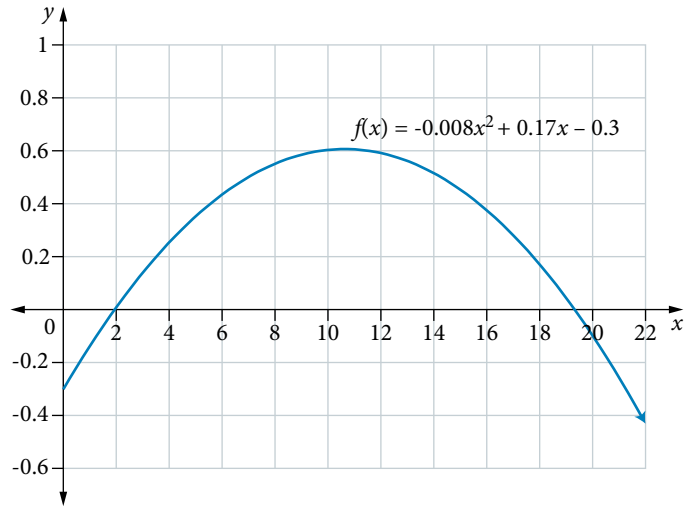
In practical situations, where decimal coefficients are involved, technology is more efficient. Either a graph function or a solve function can be used.

To solve the quadratic equation $h(x) = 0.8$ by hand you can equate to zero and then use the quadratic formula.

$$-0.008x^2 + 0.17x + 0.5 = 0.8$$
$$-0.008x^2 + 0.17x + 0.5 - 0.8 = 0$$
$$-0.008x^2 + 0.17x - 0.3 = 0$$
$$x = 1.94, 19.31$$

The solution corresponds to where the graph of $f(x) = -0.008x^2 + 0.17x - 0.3$ cuts the x-axis. The second solution (19.31, 0.8) corresponds to where the player would stand to practise their backhands.

So a player should stand about 19.3 m from the machine, or 7.3 m from the net.



You may have already learnt several ways of solving quadratic equations by hand such as factorising and using the null factor law, completing the square and the quadratic formula. The following examples will revise these skills.

Additional information

Solving a quadratic equation

This activity explores the quadratic equation.

Factorising quadratic expressions

The technique used for factorising depends on the type of quadratic expression. Always look for common factors first, then explore quadratic factorising techniques if necessary.

10 Factorise the quadratic expressions

Factorise each of the following quadratic expressions.

(a) $12x - 4x^2$

THINKING	WORKING
1 A quadratic expression in the form $ax^2 + bx$ will factorise in the form $x(ax + b)$. Determine the highest common factor (HCF).	The highest common factor of x and x^2 is x. The factors of: $12 = \{1, 2, 3, \underline{4}, 6, 12\}$ $4 = \{1, 2, \underline{4}\}$ Therefore the highest common factor of $12x$ and $4x^2$ is $4x$.

2 Factorise the expression using the distributive law to express each term as the product of the highest common factor (HCF) and another term.

$$12x - 4x^2 = 4x \times 3 - 4x \times x$$
$$= 4x(3 - x)$$

(b) $x^2 - 9$

Apply the rule for factorising a difference of squares:
$a^2 - b^2 = (a + b)(a - b)$

$$x^2 - 9 = x^2 - 3^2$$
$$= (x + 3)(x - 3)$$

(c) $x^2 - 2x + 1$

METHOD 1

For a quadratic expression in the form $x^2 + bx + c$ where $c = \left(\frac{b}{2}\right)^2$, then $x^2 + bx + c$ is a perfect square and will factorise to $\left(x + \frac{b}{2}\right)^2$.

For $x^2 - 2x + 1$, $b = -2$

Therefore $\frac{b}{2} = -1$ and

$$\left(\frac{b}{2}\right)^2 = (-1)^2$$
$$= 1$$
$$= c$$
$$x^2 - 2x + 1 = (x - 1)^2$$

METHOD 2

1 For a quadratic trinomial in the form $x^2 + bx + c$, determine the two values that multiply to give c and add to give b.

$$-1 + (-1) = -2$$
$$-1 \times -1 = 1$$

2 Factorise by splitting the middle term.

$$x^2 - 2x + 1$$
$$= x^2 - x - x + 1$$
$$= (x^2 - x) - (x - 1)$$
$$= x(x - 1) - 1(x - 1)$$
$$= (x - 1)(x - 1)$$
$$= (x - 1)^2$$

(d) $2x^2 - 18x + 28$

1 Check whether there is a common factor for all terms.

$$2x^2 - 18x + 28 = 2(x^2 - 9x + 14)$$

2 For a quadratic trinomial in the form $x^2 + bx + c$ where $\left(\frac{b}{2}\right)^2 \neq c$, determine the two values that multiply to give c and add to give b.

$$-7 + (-2) = -9$$
$$-7 \times -2 = 14$$

3 Factorise by splitting the middle term.	$2x^2 - 18x + 28$ $= 2(x^2 - 9x + 14)$ $= 2(x^2 - 7x - 2x + 14)$ $= 2\left((x^2 - 7x) - (2x - 14)\right)$ $= 2(x(x - 7) - 2(x - 7))$ $= 2(x - 2)(x - 7)$

(e) $3x^2 + 19x - 14$

1 For a quadratic in the form $ax^2 + bx + c$ where $a \neq 1$, determine which two numbers multiply to give $a \times c$ and add to give b.	$b = 19$ $a \times c = 3 \times -14 = -42$ The two numbers are -2 and 21: $-2 \times 21 = -42$ $-2 + 21 = 19$
2 Split the middle term and factorise.	$3x^2 + 19x - 14$ $= 3x^2 + 21x - 2x - 14$ $= \left(3x^2 + 21x\right) - (2x + 14)$ $= 3x(x + 7) - 2(x + 7)$ $= (3x - 2)(x + 7)$

You can use factorising techniques in conjunction with the null factor law to solve quadratic equations.

Additional information

Factorising quadratic expressions

Practise factorising quadratic expressions with these activities.

Using the null factor law to solve a quadratic equation

This method only applies when a quadratic expression can be factorised.

- If the product of two factors is equal to 0, then the value of at least one of the factors (and possibly both) must be 0.
- If $a \times b = 0$ and a is any real number, then $b = 0$. Conversely, if it is assumed that b can have any real value, then $a = 0$. It could also be the case that $a = b = 0$. For example:

$4x(2x - 5) = 0$

$4x = 0$ or $2x - 5 = 0$

So $x = 0$ or $x = \frac{5}{2}$ or 2.5.

$(x + 3)^2 = 0$

$(x + 3)(x + 3) = 0$

$(x + 3) = 0$ so $x = -3$

(this is called a *repeated* root)

The null factor law states that if the product of two real numbers a and b is zero, $a \times b = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.

WARNING

The null factor law only applies when a product is equal to zero.

If $a \times b = 0$, then $a = 0$ and/or $b = 0$.

11 Solve a quadratic equation using factorisation and the null factor law

Solve each of the following quadratic equations using factorisation and the null factor law.

(a) $p^2 - 10p - 24 = 0$

THINKING	WORKING
1 Determine which two numbers add together to give the middle term and multiply to give the constant.	$p^2 - 10p - 24 = 0$ $-12 + 2 = -10$ $-12 \times 2 = -24$
Given the constant term is negative, one number will be positive and one will be negative.	The linear factors are $(p - 12)$ and $(p + 2)$.
2 Write the equation in factorised form.	$(p - 12)(p + 2) = 0$
3 Solve the equation using the null factor law.	$p - 12 = 0$ $\quad$ $p + 2 = 0$ $p = 12$ $\quad$ $p = -2$
4 Interpret the answer.	The solutions to the equation $p^2 - 10p - 24 = 0$ are $p = 12$ or $p = -2$.

(b) $9x^2 + 4 = 12x$

1 Rearrange the equation so all terms are on one side.	$9x^2 - 12x + 4 = 0$
2 For a quadratic in the form $ax^2 + bx + c$ where $a \neq 1$, determine which two numbers multiply to give $a \times c$ and add to give b.	$b = -12$ $a \times c = 9 \times 4 = 36$ $36 = 1 \times 36, 2 \times 18, 3 \times 12, 4 \times 9, 6 \times 6$ The middle term is negative, so the two factors will be negative: $-6 \times -6 = 36$ $-6 + -6 = -12$
3 Split the middle term and factorise the quadratic expression.	$9x^2 - 12x + 4 = 0$ $9x^2 - 6x - 6x + 4 = 0$ $(9x^2 - 6x) - (6x - 4) = 0$ $3x(3x - 2) - 2(3x - 2) = 0$ $(3x - 2)(3x - 2) = 0$ $(3x - 2)^2 = 0$

4 Solve the equation using the null factor law.	$3x - 2 = 0$ $3x = 2$ $x = \frac{2}{3}$
5 Interpret the answer.	The solution to the equation $9x^2 - 12x + 4 = 0$ is $x = \frac{2}{3}$.

(c) $6k^2 = 15k$

1 Rearrange the equation so all terms are on one side.	$6k^2 - 15k = 0$ $3k \times 2k - 3k \times 5 = 0$
2 Take out the highest common factor and write the equation in factorised form.	$3k(2k - 5) = 0$
3 Solve the equation using the null factor law.	$3k = 0$ $\quad$ $2k - 5 = 0$ $k = 0$ $\quad$ $2k = 5$ $k = \frac{5}{2}$
4 Interpret the answer.	The solutions to the equation $6k^2 = 15k$ are $k = 0$ or $k = \frac{5}{2}$.

The null factor law can be used to graph quadratics of the form $y = a(x - b)(x - c)$.

The graph of $y = a(x - b)(x - c)$, $a \neq 0$ has the following features:

- x-intercepts at $(b, 0)$ and $(c, 0)$
- axis of symmetry: $x = \frac{b + c}{2}$
- concave up for $a > 0$
- concave down for $a < 0$.

12 Graph a quadratic function from factorised form

Draw the graph of $y = 2(x + 3)(x - 1)$, showing key features of a parabola and all axes intercepts.

THINKING	WORKING
1 Determine the concavity of the parabola from the coefficient of x^2 in the expanded form of the expression.	$y = 2(x + 3)(x - 1)$ In expanded form the coefficient of x^2 is 2 ($a > 0$) so the parabola is concave up.
2 Determine the y-intercept by substituting $x = 0$ and calculating the value of y.	For $x = 0$: $y = 2(0 + 3)(0 - 1)$ $= -6$ The coordinates of the y-intercept are (0, -6).

3 Determine the x-intercepts by letting $y = 0$ and using the null factor law to solve for x.

For $y = 0$:

$2(x+3)(x-1) = 0$

$x + 3 = 0$ or $x - 1 = 0$

$x = -3$ or $x = 1$

The coordinates of the x-intercepts are (-3, 0) and (1, 0).

4 The axis of symmetry occurs exactly halfway between the x-intercepts.

Axis of symmetry:

$$x = \frac{-3+1}{2}$$
$$= -1$$

The equation of the axis of symmetry is $x = -1$.

5 Determine the coordinates of the turning point by substituting the x-value from the axis of symmetry to calculate the value of y.

For $x = -1$:

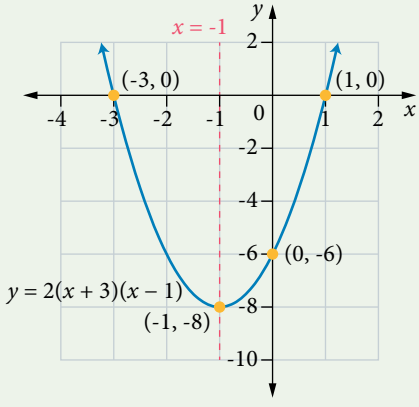
$y = 2(-1+3)(-1-1)$

$= 2 \times 2 \times -2$

$= -8$

The coordinates of the turning point are (-1, -8).

6 Sketch the parabola, using a smooth curve passing through the points symmetrical about the parabolic axis of symmetry.



Completing the square

Completing the square allows you to write a quadratic expression in turning point form. This form is easy to read off the turning point, as well as readily see the transformations that have been applied to the basic quadratic function $f(x) = x^2$. By solving a given equation equal to 0, you can calculate the x-intercepts. Completing the square uses your knowledge of the perfect square $(x+a)^2 = x^2 + 2ax + a^2$.

Completing the square involves writing the x^2 and x terms of the given equation as a perfect square, and balancing out the constant term. A common *rule of thumb* when the coefficient of x^2 is 1 is 'halve the middle term and square it'. Any quadratic trinomial can be changed into an equivalent completed square form.

	x	$+a$
x	$x \times x = x^2$	$x \times a = ax$
$+a$	$a \times x = ax$	$a \times a = a^2$

13 Use completing the square to solve quadratic equations

Complete the square and then solve each of the following quadratic equations.

(a) $x^2 - 6x + 5 = 0$

THINKING	WORKING
1 To complete the square, write the x^2 and x terms of the given expression as a perfect square.	$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9$
2 Balance out the constant term so that the equation remains true and is equal to the equation originally given.	$x^2 - 6x + 5 = 0$ $\left(x^2 - 6x + 9\right) - 9 + 5 = 0$
3 Solve the equation in completed square form equal to zero.	$\left(x^2 - 6x + 9\right) - 4 = 0$ $(x-3)^2 - 4 = 0$ $(x-3)^2 = 4$ $x - 3 = \pm\sqrt{4}$ $x - 3 = \pm 2$ $x = 3 \pm 2$
4 Write the solutions.	$x = 5$ or $x = 1$

(b) $x^2 + 7x + 8 = 0$

1 To complete the square, write the x^2 and x terms of the given expression as a perfect square.	$x^2 + 7x + \left(\frac{7}{2}\right)^2 = x^2 + 7x + \frac{49}{4}$
2 Balance out the constant term so that the equation remains true and is equal to the equation originally given.	$\left(x^2 + 7x + \frac{49}{4}\right) - \frac{49}{4} + 8 = 0$
3 Solve the equation in completed square form equal to zero.	$\left(x + \frac{7}{2}\right)^2 - \frac{17}{4} = 0$ $\left(x + \frac{7}{2}\right)^2 = \frac{17}{4}$ $x + \frac{7}{2} = \pm\frac{\sqrt{17}}{2}$ $x = -\frac{7}{2} \pm \frac{\sqrt{17}}{2}$ $x = \frac{1}{2}\left(-7 \pm \sqrt{17}\right)$
4 Write the solutions.	$x = \frac{1}{2}\left(-7 + \sqrt{17}\right)$ or $x = \frac{1}{2}\left(-7 - \sqrt{17}\right)$

(c) $3x^2 + 12x - 1 = 0$

1 To complete the square, write the x^2 and x terms of the given expression as a perfect square. If the coefficient of the x^2 term is not 1, then take its value out as a factor of the first two terms.

$$3\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) = 3\left(x^2 + 4x + 4\right)$$

2 Balance out the constant term so that the equation remains true and is equal to the equation originally given.

$$3\left(x^2 + 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right) - 1 = 0$$

3 Solve the equation in completed square form equal to zero.

$$3\left((x+2)^2 - 2^2\right) - 1 = 0$$
$$3(x+2)^2 - 3(2^2) - 1 = 0$$
$$3(x+2)^2 - 12 - 1 = 0$$
$$3(x+2)^2 - 13 = 0$$
$$(x+2)^2 = \frac{13}{3}$$
$$(x+2) = \pm\sqrt{\frac{13}{3}}$$
$$x = -2 \pm \sqrt{\frac{13}{3}}$$

4 Write the solutions.

$$x = -2 - \sqrt{\frac{13}{3}} \text{ or } x = -2 + \sqrt{\frac{13}{3}}$$

The quadratic formula

Many quadratics do not have rational factors so it is not always possible to solve a quadratic equation using factorisation and the null factor law. It is always possible to complete the square, although this method can be complicated when done by hand.

Additional information

Completing the square

Try the activity to consolidate your understanding of completing the square.

The quadratic formula can be used to solve quadratic equations in the general form $ax^2 + bx + c = 0$, where $a \neq 0$, and is derived by completing the square for the general case, as follows.

$ax^2 + bx + c = 0$

Divide the equation by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

To complete the square, write the x^2 and x terms of the given equation as a perfect square:

$$x^2 + \frac{b}{a}x + ?$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2$$

Balance out the constant term so that the equation is equivalent to the given equation.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4a}{4a} \times \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

If a quadratic equation has been written in the form $ax^2 + bx + c = 0$ where $a \neq 0$, then the quadratic formula can be used to solve the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a \neq 0$$

Solve the equation for x.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

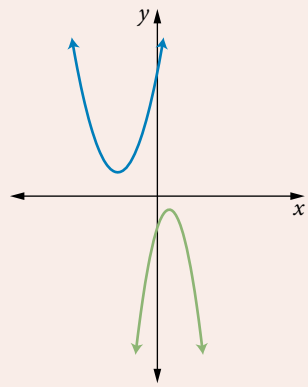
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

WARNING

Not every quadratic function can be solved using factorisation or by completing the square or using the quadratic formula. Remember that the solutions represent the x-intercepts and the graphs of some quadratic functions that have undergone vertical translations will not cross the x-axis.



The discriminant

The part of the quadratic formula under the square root symbol (radical), $b^2 - 4ac$, is called the *discriminant* and is represented by the symbol Δ(delta). The discriminant informs you whether a quadratic equation has 0, 1 or 2 real solutions. The quadratic formula is also sometimes written in the form:

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}, \text{ where } \Delta = b^2 - 4ac \text{ and } a \neq 0.$$

Explore further

The discriminant

Explore the relationship between the discriminant and number of solutions.

Any quadratic equation of a single real variable can be written in the form $ax^2 + bx + c = 0$. The discriminant $\Delta = b^2 - 4ac$ indicates the number and nature of roots and corresponding x-axis intercepts.

Discriminant	Interpretation
$\Delta < 0$	no real roots (no intersection of the graph with the x-axis)
$\Delta = 0$	exactly 1 real root (exactly 1 point of intersection of the graph with the x-axis and it will be the turning point)
$\Delta > 0$	two distinct real roots (two points of intersection of the graph with the x-axis)
Δ is a perfect square	the roots are rational

14 Use the quadratic formula to solve quadratic equations

Use the quadratic formula to solve the following equations. Give answers in exact form and then interpret the answer graphically.

(a) $3x^2 - 5x = 10$

THINKING	WORKING
1 Rearrange the equation into the form $ax^2 + bx + c = 0$ and identify the values of a, b and c.	$3x^2 - 5x - 10 = 0$ $a = 3,\ b = -5,\ c = -10$
2 Calculate the value of the discriminant, $b^2 - 4ac$, and hence determine the number of solutions of the equation.	$\Delta = b^2 - 4ac$ $= (-5)^2 - 4(3)(-10)$ $= 25 - (-120)$ $= 145$ $\Delta > 0$ therefore two real solutions exist.
3 Use the quadratic formula to solve the equation $ax^2 + bx + c = 0$ for x: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-(-5) \pm \sqrt{145}}{2 \times 3}$ $= \dfrac{5 \pm \sqrt{145}}{6}$ $x = \dfrac{5 + \sqrt{145}}{6}$ or $x = \dfrac{5 - \sqrt{145}}{6}$
4 Interpret the answer.	The graph of $y = 3x^2 - 5x - 10$ has two x-intercepts at approximately (2.84, 0) and (-1.17, 0).

(b) $-0.25x^2 + 2x - 4 = 0$

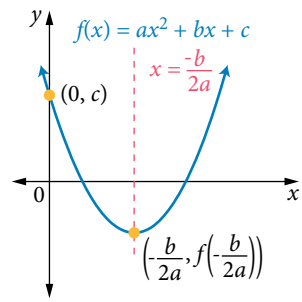
1 Rearrange the equation into the form: $ax^2 + bx + c = 0$ Identify the values of a, b and c.	$-0.25x^2 + 2x - 4 = 0$ $a = -0.25,\ b = 2,\ c = -4$
2 Calculate the value of the discriminant, $b^2 - 4ac$, and hence determine the number of solutions of the equation.	$\Delta = b^2 - 4ac$ $= (2)^2 - 4(-0.25)(-4)$ $= 4 - 4$ $= 0$ $\Delta = 0$ therefore 1 solution exists.
3 Use the quadratic formula to solve the equation $ax^2 + bx + c = 0$ for x: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-2 \pm \sqrt{0}}{-0.5}$ $= 4$

4 Interpret the answer.	The graph of $y = -0.25x^2 + 2x - 4$ has one x-intercept at (4, 0), which means the maximum turning point is sitting on the x-axis.

(c) $-2x^2 - 20x - 54 = 0$

1 Rearrange the equation into the form: $ax^2 + bx + c = 0$ Identify the values of a, b and c.	$-2x^2 - 20x - 54 = 0$ $a = -2$, $b = -20$, $c = -54$
2 Calculate the value of the discriminant, $b^2 - 4ac$, and hence determine the number of solutions of the equation.	$\Delta = b^2 - 4ac$ $= (-20)^2 - 4(-2)(-54)$ $= 400 - 432$ $= -32$ $\Delta < 0$ therefore no real solution exists.
3 Interpret the answer.	The graph of $y = -2x^2 - 20x - 54$ is concave down because $a < 0$. The turning point is below the x-axis and hence no x-intercepts (real solutions) exist.

You can draw graphs of the general form $f(x) = ax^2 + bx + c$ using the quadratic formula to determine the x-intercepts.
The coordinates of the y-intercept are $(0, c)$.
$x = \frac{-b}{2a}$ gives both the equation of the axis of symmetry and the x-coordinate of the turning point.



15 Graph a quadratic function in the general form

Draw the graph of each quadratic function, showing key features of a parabola and all axes intercepts. Round values correct to 2 decimal places, where necessary.

(a) $y = 8 + 4x - x^2$

THINKING	WORKING
1 Rewrite the equation of the function in the form $y = ax^2 + bx + c$. Determine the concavity of the parabola by identifying the coefficient of x^2.	$y = -x^2 + 4x + 8$ $a = -1$ The coefficient of x^2 is -1 so the parabola will be concave down.
2 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the value of y.	For $x = 0$: $y = 8$ The coordinates of the y-intercept are (0, 8).

3 Determine the coordinates of the x-intercepts by letting $y = 0$ and using the quadratic formula to solve for x.

For $y = 0$:

$$8 + 4x - x^2 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(-1) \times 8}}{2(-1)}$$

$$= \frac{-4 \pm \sqrt{48}}{-2}$$

$$= -1.46 \text{ or } 5.46 \ (2 \text{ d.p.})$$

The coordinates of the x-intercepts are (-1.46, 0) and (5.46, 0).

4 Identify the values of a and b, and recall the formula $x = \frac{-b}{2a}$ to determine the equation of the axis of symmetry.

Axis of symmetry:

$$x = \frac{-b}{2a}$$

$$= \frac{-4}{2 \times -1}$$

$$= \frac{-4}{-2}$$

$$= 2$$

The equation of the axis of symmetry is $x = 2$.

5 Substitute the x-value from the axis of symmetry into the original equation to determine the coordinates of the turning point.

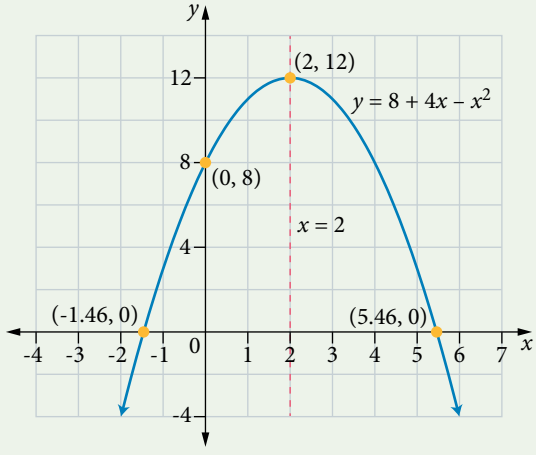
For $x = 2$:

$$y = 8 + 4 \times 2 - 2^2$$

$$= 12$$

The coordinates of the turning point are (2, 12).

6 Sketch a parabola with a smooth shape through the points symmetrical about the axis of symmetry.



(b) $y = x^2 + 2x + 5$

1 Determine the concavity of the parabola by identifying the coefficient of x^2.

$y = x^2 + 2x + 5$

The coefficient of x^2 is 1 so the parabola will be concave up.

2 Determine the coordinates of the y-intercept by letting $x = 0$ and calculating the value of y.

For $x = 0$: $y = 5$

The coordinates of the y-intercept are (0, 5).

3 Determine the coordinates of the x-intercepts by letting $y = 0$ and using the quadratic formula to solve for x.

For $y = 0$:

$x^2 + 2x + 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$$
$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$\Delta < 0$, therefore no real solutions exist.

There are no x-intercepts.

4 Identify the values of a and b, recall the formula $x = \frac{-b}{2a}$ to determine the equation of the axis of symmetry.

Axis of symmetry:

$$x = \frac{-b}{2a}$$
$$= \frac{-2}{2}$$
$$= -1$$

The equation of the axis of symmetry is $x = -1$.

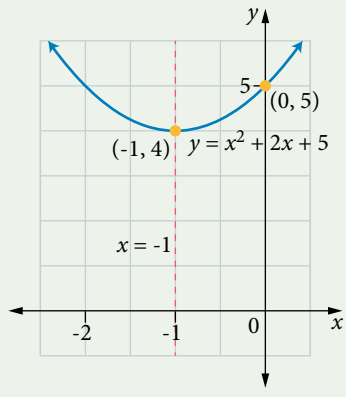
5 Substitute the x-value from the axis of symmetry into the original equation to determine the turning point by substitution.

For $x = -1$:

$$y = (-1)^2 + 2(-1) + 5$$
$$= 1 - 2 + 5$$
$$= 4$$

The coordinates of the turning point are (-1, 4).

6 Sketch a parabola with a smooth shape through the points symmetrical about the axis of symmetry.



EXERCISE 2.3 Solve quadratic equations

Worked Example

1 Factorise each of the following quadratic expressions. 10

(a) $6y^2 - 3y$ (b) $a^2 + a - 12$ (c) $b^2 - 15b + 54$

(d) $2x^2 - 7x - 15$ (e) $4x^2 + 16x + 15$ (f) $m^2 - 16$

2 Solve each of the following quadratic equations. Use factorisation and the null factor law to determine the solutions. 11

(a) $a^2 + 12a + 36 = 0$ (b) $a^2 + 2a - 35 = 0$ (c) $a^2 - 10a + 16 = 0$ (d) $-a^2 + 48 = 2a$

(e) $2a^2 + 5a = 12$ (f) $-3a^2 + 14a - 8 = 0$ (g) $2a^2 + 6a = 20$ (h) $3a^2 = 2a + 1$

3 (a) Which of the following equations could be represented by the graph shown?

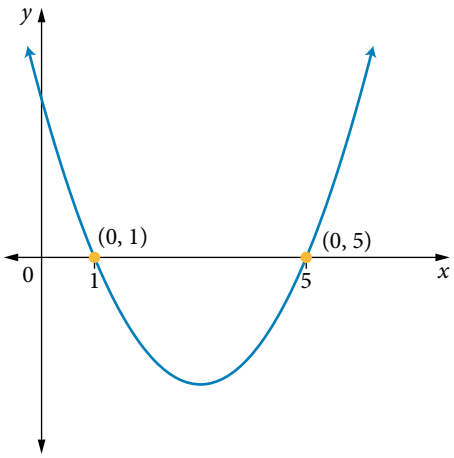
A $y = 2(x-3)^2 - 2$

B $y = 3(x-1)(x-5)$

C $y = 2(x+1)(x+5)$

D either A or B

(b) For the correct equation(s), determine the y-intercept and the turning point.



4 Determine the discriminant and hence identify the number of solutions for each of the following quadratic equations.

(a) $x^2 - 4x - 20 = 0$ (b) $3m^2 - m - 20 = 0$ (c) $c^2 - 10c + 25 = 0$

(d) $-4x^2 - 12x - 9 = 0$ (e) $h^2 - 4h + 9 = 0$

5 Solve each of the following quadratic equations using the quadratic formula. 14

(a) $x^2 + x - 20 = 0$ (b) $x^2 + 14x + 48 = 0$ (c) $2x^2 + x - 6 = 0$

(d) $3a^2 + 2a - 1 = 0$ (e) $x^2 + 8x = 20$

6 The number of solutions of the equation $5x^2 + 7x = k$, where $k > 0$ is:

A 0 B 1 C 2 D not possible to tell

7 Use the quadratic formula to solve $x^2 - x = 3$, correct to 3 decimal places.

8 Determine the solutions to the equations, where possible, correct to 3 decimal places as necessary.

(a) $-x^2 + 2x = 13$ (b) $\frac{2}{3}x^2 - 5x - 18 = 0$ (c) $1.5x^2 - 9.76 = 0$ (d) $-2x^2 - 11x - 1 = 0$

(e) $3x^2 = 20$ (f) $\frac{x^2}{3} - \frac{x}{3} - 2 = 0$ (g) $x^2 = 17x$

9 Determine the solutions to the quadratic equation $x^2 + x = 12$.

10 Determine the solutions to the quadratic equation $x^2 + 2x = 5$, correct to 2 decimal places.

11 Solve the following quadratic equations in exact form.

(a) $3a^2 = 18a + 5$ **(b)** $-2b^2 + 15b = -12$ **(c)** $\frac{2}{7}c^2 - 8c = 9$ **(d)** $3.4x^2 - 9x + 2 = 0$

12 Determine the exact solutions to the equation $7t + 5 = 2t^2$.

13 Rajesh uses the completing the square method to solve a quadratic equation, as shown below. Determine whether Rajesh made an error and if so, write the correct solution.

$$\begin{aligned} x^2 + 6x + 1 &= 0 \\ \left(x^2 + 6x + 9\right) - 9 + 1 &= 0 \\ (x+3)^2 - 9 + 1 &= 0 \\ (x+3)^2 &= 8 \\ x + 3 &= 8 \text{ or } x + 3 = -8 \\ x &= 5 \text{ or } x = -11 \end{aligned}$$

14 Emil used the method of completing the square incorrectly to solve the equation shown below. Determine where Emil made the mistake in his working out and then write the correct solution.

$$\begin{aligned} x^2 - 8x + 3 &= 0 \\ (x^2 - 8x + 4) - 4 + 3 &= 0 \\ (x+2)^2 &= 1 \\ x + 2 &= \pm 1 \\ x &= 3 \text{ or } x = -1 \end{aligned}$$

15 Use the quadratic formula to solve the simultaneous equations: $y = x^2 - 4x - 6$ and $y = 0$. Give your answer in simplified form.

Worked Example

13 **16** Complete the square to determine the solutions of the following quadratic equations. Give your answer as a single expression in exact form using the ± symbol.

(a) $a^2 + 6a - 17 = 0$ **(b)** $a^2 + 10a - 14 = 0$ **(c)** $a^2 + 5a - 9 = 0$

(d) $a^2 + 9a = -6$ **(e)** $a^2 - 7a + 11 = 0$ **(f)** $2a^2 - 8a = 9$

(g) $6a^2 + 18a + 9 = 0$

17 Determine the exact solutions to the equation $x^2 = 4x + 1$ by completing the square.

18 Determine the solutions to each equation by completing the square.

(a) $m^2 - 3m - 9 = 0$ **(b)** $-x^2 + 10x - 14 = 0$ **(c)** $t^2 + 9t - 23 = 0$

12 **19** Sketch the graph of each quadratic function, showing the key features.

(a) $y = (x+2)(x+4)$ **(b)** $y = (3+x)(x-5)$ **(c)** $y = 3(x-1)(x-3)$ **(d)** $y = 2x^2 + 16x + 14$

15 **20** Sketch the graph of each quadratic function, showing the key features.

(a) $y = x^2 - 3x - 6$ **(b)** $y = x^2 + 4x + 5$ **(c)** $y = x - 7 - 2x^2$ **(d)** $y = 3x^2 + 2x - 4$

21 Daniel likes to ride his skateboard at the local park. Work has just been completed on one of the ramps for the park, the ramp approximating the shape of a symmetrical section of a parabola. The shape of the ramp can be approximated by the function $h(x) = 0.072x^2 - 0.72x + 1.8$, where x is the horizontal distance of the ramp from one of the platforms and h is the height of the ramp above ground level.

(a) Sketch a graph of the ramp.

(b) The ramp has been constructed with the lowest point being at ground level and both sides of the ramp rising up to the same height above ground level. Determine the maximum height of the ramp above ground level.

(c) Determine the maximum horizontal distance spanned by the ramp.

22 A ball-launching machine, placed on the baseline of a tennis court, delivers balls on a path approximated by the function $h(x) = -0.008x^2 + 0.17x + 0.5$, where h(m) represents the ball height and x represents the horizontal distance, up to 24 m, from the ball-launching machine. You want to practice your volleys closer to the net, hitting the ball at a height of 1.3 m before it hits the ground. Given the ball is released 50 cm above the ground, find where you should stand, in metres, from the ball-launching machine. Justify your solution by sketching a graph of the function.

23 One of the fireworks for Brisbane's New Year's Eve celebrations is launched from near the top of the Story Bridge at a height of 70 m, exploding after 4 seconds at the highest point of its trajectory.

The height of the firework over time can be modelled by the quadratic equation $h(t) = -4.9t^2 + bt + c$, where t is the time elapsed (in seconds) from the time the firework is launched, $h(t)$ is the height above the river (in metres) of the firework and b and c are constants.

(a) Determine the equation of the quadratic model. Determine an appropriate domain and range and then graph the function.

(b) At what time is the firework at a height of 100 m above the river?

(c) If the firework failed to explode, when will it hit the surface of the river?

24 For the quadratic and linear functions $y = x^2 - 2$ and $y = 2x - 4$:

(a) Determine whether the graphs intersect and if so, the coordinates of the point of intersection. Justify your solution with a graph.

(b) Construct a piece-wise function using the two expressions such that the domain is all real numbers. Make the critical values as close to each other as possible.

25 Determine values of k for which $f(x) = 3x^2 - kx + 5$ will have 0, 1 and 2 real roots.

2.4 Determine the rule of a quadratic function

Different forms of the quadratic function

The rule of a quadratic function can be represented in three main forms.

Form	General expression	Example
General or trinomial form	$y = ax^2 + bx + c$	$y = -2x^2 + 3x - 4$
Turning point or completed square form	$y = a(x-b)^2 + c$	$y = 5(x-3)^2 + 6$
Factorised form	$y = a(x-b)(x-c)$	$y = 4(x-2)(x+1)$

When determining the rule of a quadratic function based on some given coordinates, or from a sketch of its graph, use the form best suited to the given information. In each of the forms there are three unknowns. This means three pieces of information are required to determine the required rule.

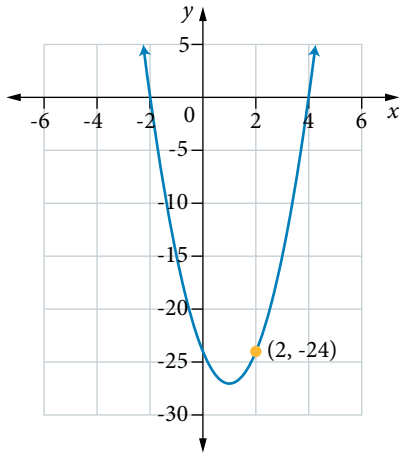
Explore further

Different forms of the quadratic function

Move the sliders to explore the properties that can be obtained from a graph in each form.

16 Determine a quadratic function in factorised form

Determine the rule of the quadratic function shown in the following graph.

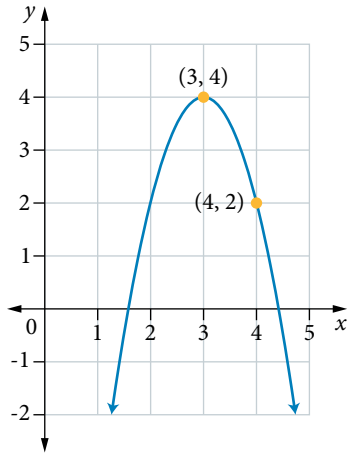


THINKING	WORKING
1 Identify the information shown on the graph.	The x-intercepts are at (-2, 0) and (4, 0). Another point: (2, -24)
2 Recall the appropriate form of the quadratic function rule.	Two x-intercepts are given so use the factorised form: $y = a(x-b)(x-c)$
3 Write the rule using the known values.	$y = a(x-(-2))(x-4)$ $= a(x+2)(x-4)$

4 Substitute the coordinates of the other point and solve to determine the value of the unknown.	For (2, -24): $-24 = a(2+2)(2-4)$ $-24 = a(4)(-2)$ $-24 = -8a$ $a = 3$
5 Check the reasonableness of a.	The graph is concave up so $a > 0$ is expected.
6 Write the rule in factorised form.	$y = 3(x+2)(x-4)$

17 Determine a quadratic function in turning point form

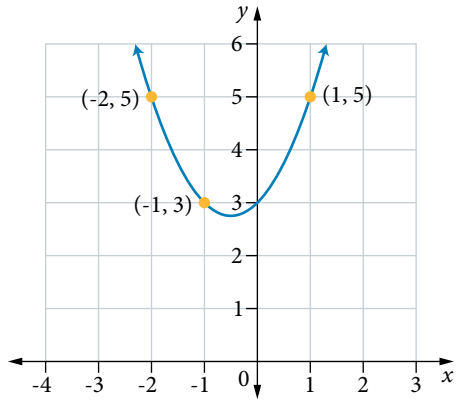
Determine the equation of the quadratic function shown in the following graph.



THINKING	WORKING
1 Identify the information shown on the graph.	The coordinates of the turning point: (3, 4) Another point: (4, 2)
2 Recall the appropriate form of the quadratic function rule.	The turning point is given so use turning point form: $y = a(x-b)^2 + c$
3 Write the rule using the known values.	$y = a(x-3)^2 + 4$
4 Substitute the coordinates of the other point and solve to determine the value of the unknown.	For (4, 2): $2 = a(4-3)^2 + 4$ $2 = a + 4$ $a = -2$
5 Check the reasonableness of a.	The graph is concave down (has a maximum turning point) so $a < 0$ is expected.
6 Write the rule in in turning point form.	$y = -2(x-3)^2 + 4$

18 Determine a quadratic function in the general form

Determine the rule of the quadratic function shown in the following graph.



THINKING	WORKING
1 Identify the information shown on the graph.	The coordinates of three points are (-2, 5), (-1, 3) and (1, 5).
2 Recall the appropriate form of the quadratic function rule.	Three points are given, none of which is the vertex or an x-intercept, so use trinomial form: $y = ax^2 + bx + c$
3 Write three simultaneous equations using the three points.	For $(-2,5)$: $5 = a(-2)^2 - 2b + c$ $5 = 4a - 2b + c \quad [1]$ For $(-1,3)$: $3 = a(-1)^2 - b + c$ $3 = a - b + c \quad [2]$ For $(1,5)$: $5 = a(1)^2 + b + c$ $5 = a + b + c \quad [3]$
4 Solve the system of equations.	$2 = 2b \quad [3]-[2]$ $b = 1$ $2 = 3a - b \quad [1]-[2]=[4]$ Substitute $b = 1$ into [4]: $2 = 3a - 1$ $3 = 3a$ $a = 1$ Substitute $a = 1$ and $b = 1$ into [2]: $3 = 1 - 1 + c$ $c = 3$
5 Write the function rule in general form.	$y = x^2 + x + 3$

A graphics calculator, spreadsheet and other technologies can be used to determine the equation most efficiently. After entering the three points into a table, find the quadratic of best fit.

Technology worked example
Using quadratic regression

Converting between forms of the quadratic function rule

Sometimes information may be given that leads to a quadratic written in one form but the question requires it to be written in a different form. Converting either factorised form $y = a(x-b)(x-c)$ or turning point form $y = a(x-b)^2 + c$ to trinomial form $y = ax^2 + bx + c$ requires expanding and simplifying.

19 Convert to trinomial form

Convert the following quadratic expressions to trinomial form.

(a) $y = -2(x+5)(x-3)$

THINKING	WORKING
Expand and simplify the given expression.	$y = -2(x+5)(x-3)$ $= -2[x(x-3)+5(x-3)]$ $= -2(x^2-3x+5x-15)$ $= -2(x^2+2x-15)$ $= -2x^2-4x+30$

(b) $y = 3(x-5)^2 - 2$

Expand using the perfect square $(a+b)^2 = a^2 + 2ab + b^2$ and simplify the given expression.	$y = 3(x-5)^2 - 2$ $= 3(x^2-10x+25) - 2$ $= 3x^2 - 30x + 75 - 2$ $= 3x^2 - 30x + 73$

Not every quadratic function can be written in factorised form. If a quadratic function is written in trinomial form, then the discriminant can be used to determine whether it can be factorised. Recall that if factors exist, then the graph of the corresponding quadratic function will have x-axis intercepts.

- If $\Delta \geq 0$, then real factors exist.
- If Δ is a perfect square, then the factors will be rational.
- If the quadratic function is written in turning point form, then the existence of factors can be determined from the turning point and the sign of a.

20 Convert to factorised form

Convert $y = 2x^2 + 2x - 4$ to factorised form.

THINKING	WORKING
1 Identify the values of a, b and c when comparing to the graph of $y = ax^2 + bx + c$.	$a = 2,\ b = 2,\ c = -4$
2 Determine the value of the discriminant to determine if factors exist.	$\Delta = b^2 - 4ac$ $= 4 - 4 \times 2 \times (-4)$ $= 4 + 32$ $= 36$ $\Delta > 0$ so two real solutions exist. Δ is a perfect square, so the solutions will be rational.
3 Factorise the given expression using one of the known techniques.	$y = 2x^2 + 2x - 4$ $= 2\left(x^2 + x - 2\right)$ $= 2(x+2)(x-1)$

When quadratic functions are written in turning point form, you can see the sequence of transformations that map the graph of $y = x^2$ onto the graph of $y = a(x-b)^2 + c$. You can also determine the exact values of irrational roots of the equations without the use of technology.

21 Converting to turning point form

Complete the square to convert the following quadratic expressions into turning point form.

(a) $y = 3x^2 + 2x - 4$

THINKING	WORKING
Use the completing the square technique.	$y = 3x^2 + 2x - 4$ $= 3\left(x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right) - 4$ $= 3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) - 4$ $= 3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - 3 \times \frac{1}{9} - 4$ $= 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3} - 4$ $= 3\left(x + \frac{1}{3}\right)^2 - \frac{13}{3}$

(b) $y = 2(x-3)(x-2)$

1 Expand to get the expression into the general form $y = ax^2 + bx + c$.	$y = 2(x-3)(x-2)$ $= 2(x^2 - 5x + 6)$
2 Use the completing the square technique.	$= 2\left(\left(x^2 - 5x + \left(\frac{5}{2}\right)^2\right) - \left(\frac{5}{2}\right)^2 + 6\right)$ $= 2\left(\left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{24}{4}\right)$ $= 2\left(\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}\right)$ $= 2\left(x - \frac{5}{2}\right)^2 - \frac{1}{2}$

Quadratic functions can be used to solve real-life problems such as modelling a projectile that follows a parabolic path. When technology is allowed, values may be obtained more efficiently than using by-hand methods. Check your solutions by sketching a graph, wherever possible.

22 Determine a quadratic function from an application

A projectile is launched from a point on the ground. The projectile is 18 m high when it has travelled a horizontal distance of 1 m and it is 32 m high at 8 m.

(a) Determine a quadratic function that models the path of the projectile with horizontal distance x (m) and the vertical distance y (m) from the launch position.

THINKING	WORKING
1 Identify the given information as coordinates.	The origin: (0, 0) 18 m high at 1 m: (1, 18) 32 m high at 8 m: (8, 32)
2 Recall an appropriate form of the quadratic equation.	The coordinates of three points are known so use the form $y = ax^2 + bx + c$.

3 Write three equations using the three points and solve simultaneously.	For (0, 0): $c = 0$
	For (1, 18):
	$a(1)^2 + b(1) = 18$
	$a + b = 18$
	$b = 18 - a \quad [1]$
	For (8, 32):
	$a(8)^2 + b(8) = 32$
	$64a + 8b = 32$
	$8a + b = 4 \quad [2]$

	Substitute [1] into [2] and solve for a: $8a + (18 - a) = 4$ $7a + 18 = 4$ $7a = -14$ $a = -2$ Substitute $a = -2$ into [1] and solve for b: $b = 18 - (-2)$ $= 20$
4 Write the equation by substituting the values of a, b and c.	$y = -2x^2 + 20x$
5 Determine the domain. Here, the positions where the projectile is at ground level determine the ends of the domain.	End points: $y = 0$: $-2x^2 + 20x = 0$ $2x(10 - x) = 0$ $x = 0$ or $x = 10$ Domain: $0 \le x \le 10$
6 Write a statement including the answer.	The path of the projectile is modelled by the graph of the function: $y = -2x^2 + 20x, 0 \le x \le 10$

(b) Determine the highest point reached by the projectile and the horizontal value at which it occurs.

1 The turning point determines the highest point. Here use the symmetry of the endpoints.	Axis of symmetry: $x = \frac{0 + 10}{2}$ $= 5$ For $x = 5$: $y = -2 \times 5^2 + 20 \times 5$ $= 50$ The coordinates of the turning point are (5, 50).
2 Interpret the result.	The highest point reached by the projectile is 50 m, which occurs at a horizontal distance 5 m from the launch position.

(c) Determine the horizontal distance the projectile travels before it again reaches ground level.

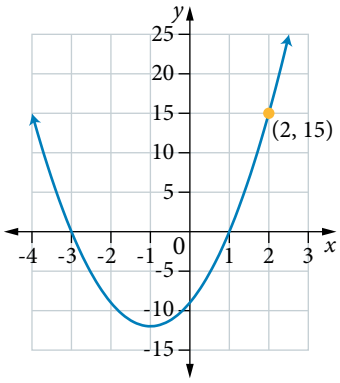
This will be the distance from the origin to the point at which the projectile again hits the ground.	$10 - 0 = 10$ m The projectile travels 10 m horizontally before it again reaches the ground.

EXERCISE 2.4 Determine the rule of a quadratic function

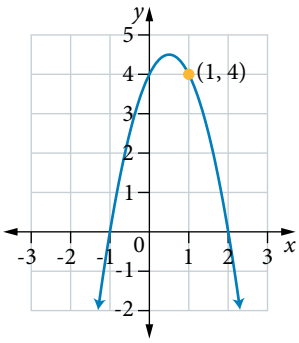
Worked Example

1 Determine the quadratic equation in factorised form for each of the following graphs. 16

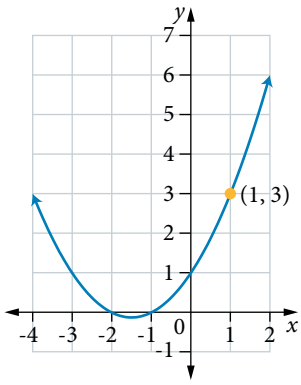
(a)



(b)

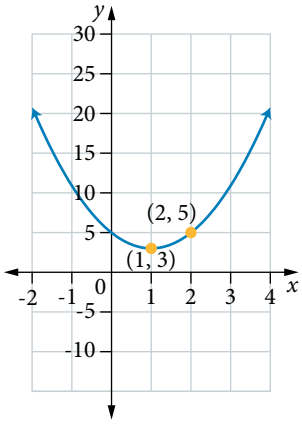


(c)

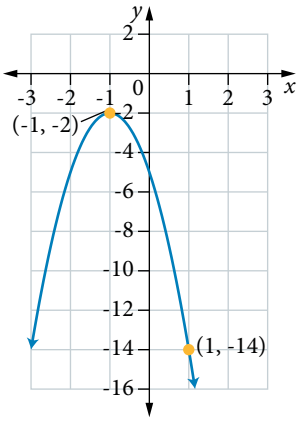


2 Determine the quadratic equation for each of the following graphs. 17

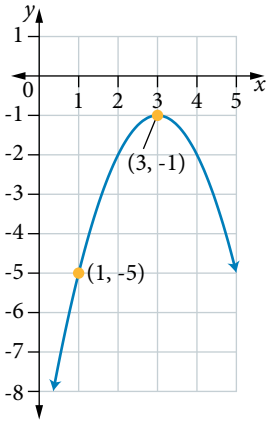
(a)



(b)

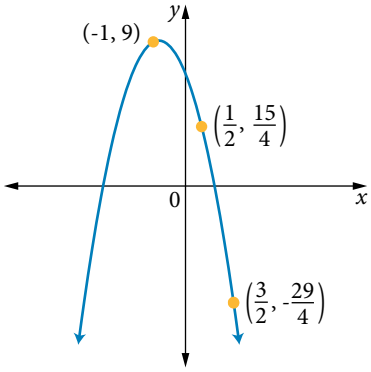


(c)

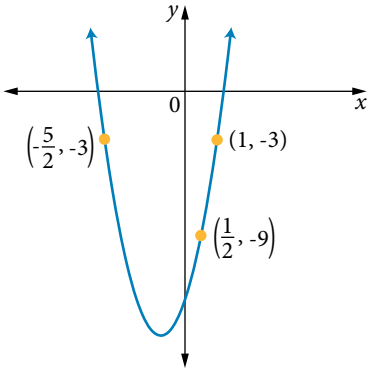


3 Use technology to determine the quadratic equation for each of the following graphs.

(a)



(b)



19

4 Convert the following quadratic equations to trinomial form.

(a) $y = 3(x-2)(x+4)$ (b) $y = -2(x+4)^2 - 3$ (c) $y = -2(x+5)(x-3)$ (d) $y = 5(x-3)^2 - 2$

5 Convert the following quadratic equations to trinomial form.

(a) $y = \dfrac{(2x-3)(x+2)}{5}$ (b) $y = -\dfrac{2}{3}(x-3)^2 - 6$

(c) $y = \dfrac{(5x+6)(2x-9)}{7}$ (d) $y = \dfrac{5}{7}(2x-1)^2 + 2$

20

6 Convert the following quadratic equations to factorised form.

(a) $y = 3x^2 - 6x - 24$ (b) $y = -2(x-1)^2 - 4$

(c) $y = -4x^2 + 32x - 60$ (d) $y = -3(x+2)^2 + 48$

7 For the quadratic equations given:

(a) which one could *not* be written with rational factors?

A $y = 3x^2 - 5x - 2$ B $y = 3x^2 - 5x + 2$ C $y = -3x^2 + 5x + 2$ D $y = -3x^2 - 5x - 3$

(b) which one could be written with rational factors?

A $y = -2(x-3)^2 + 2$ B $y = 3(x-2)^2 + 1$ C $y = -3(x-2)^2 + 1$ D $y = -3(x+2)^2 + 1$

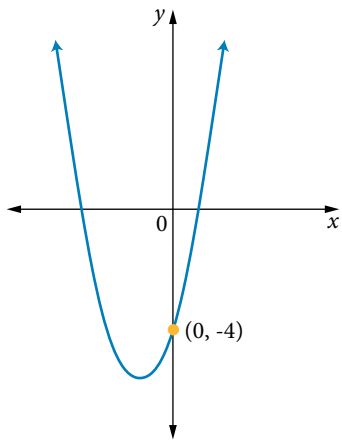
8 Which of the following equations is equivalent to $y = 2\left(\dfrac{3x-5}{2}\right)^2 - 4$?

A $y = \dfrac{9x^2}{2} - 15x + \dfrac{33}{2}$ B $y = \dfrac{9x^2}{2} - 15x + \dfrac{17}{2}$

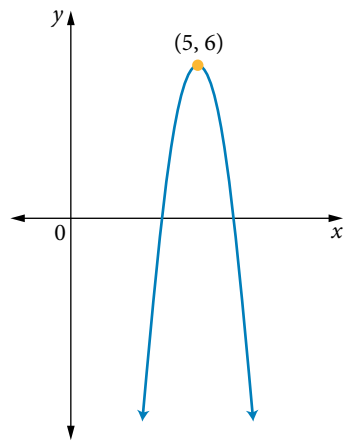
C $y = -\dfrac{9x^2}{2} + 15x - \dfrac{33}{2}$ D $y = \dfrac{9x^2}{2} + 15x + \dfrac{17}{2}$

9 For each of the following quadratic graphs, identify the key information shown in terms of turning point, y-intercept, x-intercepts.

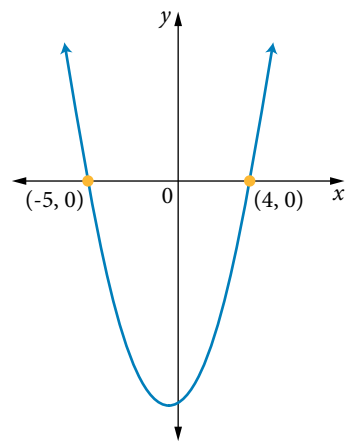
(a)



(b)

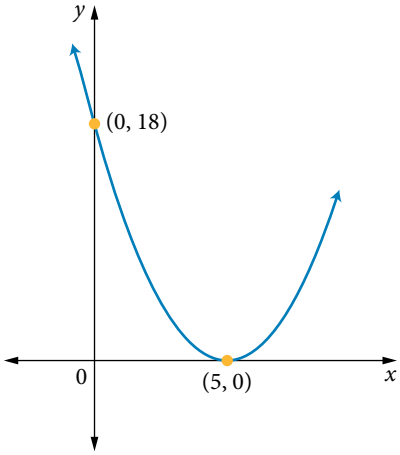


(c)

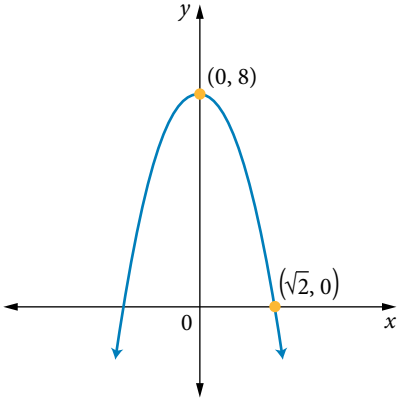


10 Determine the equation for each of the following quadratic graphs.

(a)



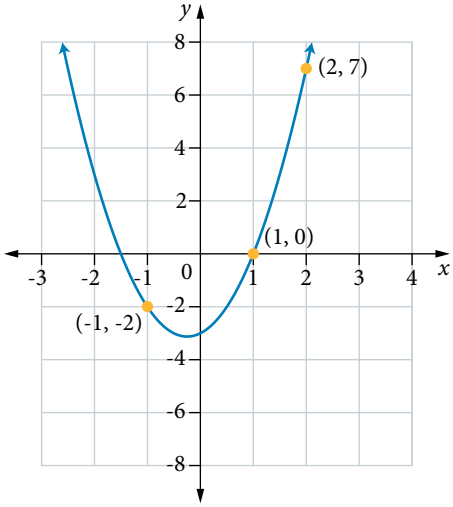
(b)



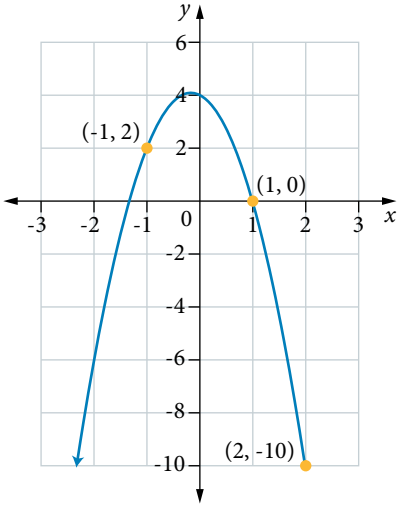
Worked Example

11 Determine the quadratic equation for each of the following graphs. 18

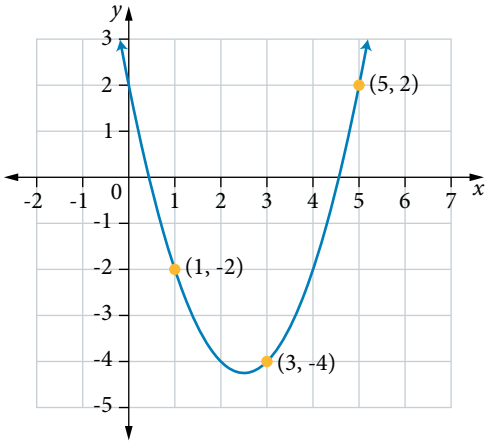
(a)



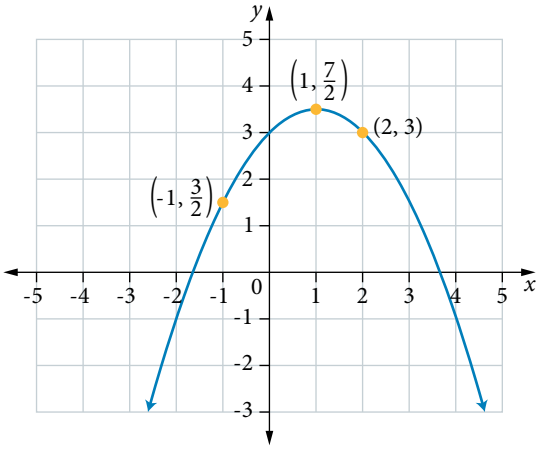
(b)



(c)



(d)



12 Convert the following quadratic equations to factorised form.

(a) $y = \frac{2x^2}{3} - \frac{2x}{3} - 8$ **(b)** $y = \frac{4}{5}(x-1)^2 - \frac{64}{5}$ **(c)** $y = -\frac{2x^2}{5} + \frac{8x}{5} + \frac{24}{5}$

13 Convert the following quadratic equations to turning point form. 21

(a) $y = -2x^2 + 4x - 3$ **(b)** $y = 3(x-5)(x+3)$ **(c)** $y = 5x^2 - 4x + 6$ **(d)** $y = -2(x+6)(x+2)$

14 Convert the following quadratic equations to turning point form.

(a) $y = -2x^2 + \frac{4x}{3} - 3$ **(b)** $y = \frac{3}{4}(2x - 5)(x + 2)$

15 For each of the following quadratic equations determine:

(i) the exact coordinates of the x-intercepts, if they exist

(ii) the coordinates of the y-intercept

(iii) the coordinates of the turning point

(iv) the equation of the line of symmetry.

(a) $y = x^2 - 4x + 6$ **(b)** $y = -3(x - 4)^2 + 2$ **(c)** $y = -2(x + 4)(x - 3)$

Worked Example

(d) $y = -3x^2 + 2x + 4$ **(e)** $y = \frac{(x - 5)^2}{3} - 1$ **(f)** $y = \frac{(2x - 5)(x + 3)}{2}$

22

16 A projectile is launched from a point that can be considered the origin. It passes through the points $\left(1, \frac{59}{60}\right)$ and $\left(2, \frac{29}{15}\right)$. The horizontal and vertical distances are measured in metres.

(a) Construct an equation in trinomial form for the path of the projectile.

(b) Determine the maximum height reached by the projectile and the horizontal distance at which it occurs.

(c) Determine the horizontal distance the projectile travels before it again reaches the ground.

17 The height h (m) reached by a ball after t (s) travelling vertically upwards is given by the equation $h = 20t - 4.9t^2$. Calculate the following correct to 2 decimal places.

(a) The time taken to first reach a height of 15 m.

(b) The maximum height reached by the ball and the time at which this happens.

(c) The time that the ball is in the air.

18 Time-lapse photography is used to track the path of an object. Based on an arbitrary location of a set of axes, it is determined that the object has passed through the points (1, 1.63), (2, 4.91) and (3, 2.64).

(a) Determine the quadratic equation of the path of the object.

(b) Determine, correct to 2 decimal places, the maximum height reached by the object, assuming the unit is metres.

(c) Compared to the arbitrary origin, calculate correct to 2 decimal places the horizontal values (in metres) where the object crossed the horizontal axis.

19 The graph of a quadratic equation has a y-intercept at (0, 3) and a turning point at (-2, 5).

(a) Sketch a graph showing the information given.

(b) Determine the quadratic equation in turning point form.

(c) If the graph exists on or above the horizontal axis, write the domain in exact form.

20 A shot put competitor's throw is modelled by the equation $y = -0.027x^2 + 0.33x + 1.83$, where x and y are horizontal and vertical distances, in metres, from the edge of the plate.

(a) Determine the effect that 5 cm difference in height (shorter or taller) would have on the person's result. Explain any assumptions you had to make to answer the question.

(b) Determine the maximum height of the put in each case.

21 A ball is rolled across the top of a table, which is 120 cm wide, and then falls in a parabolic arc to the floor, which is 80 cm below the table. The ball lands 50 cm away from the table and then bounces up to $\frac{3}{4}$ of the height of the previous bounce. It follows translations of the parabolic arc until it is caught at the top of the second rebound. Create a piece-wise model for the motion of the ball, where x and y are horizontal and vertical distances in centimetres. Give the position where the ball is caught.

22 Gina's great-nana is 98 years old. Gina looks up the number of 98-year-old women there have been in Australia over several years and comes up with the following data:

2000	2005	2015	2016	2017
1538	1753	2811	2993	3034

Gina thinks the number of 98-year-old women could be modelled as a quadratic function over time.

(a) Use the statistics mode of your technology to determine the quadratic equation of best fit, using 2000 as the base year. Comment on whether the model appears to be suitable.

(b) Predict the number of 98-year-old women in Australia in 2025 using your model. The ABS prediction is 4828. Is it likely that their statisticians are using a similar model?

(c) Gina will be 98 years old in 2098, if she lives that long. Determine the number of women that will reach that age in Australia, according to the model. Comment on the reliability of this estimate.

2 Summary

Functions

If it is possible for a vertical line to intersect the graph of a relation more than once for any point in the domain, then the relation is *not* a function, as for each x-value there is more than one corresponding y-value.

Quadratic functions with unrestricted domains are continuous and smooth. Piece-wise functions need to be checked for continuity and smoothness at critical points, where subdomains meet.

General or trinomial form of a quadratic function

$y = ax^2 + bx + c$, where $a \neq 0$ and $x \in \mathbb{R}$.

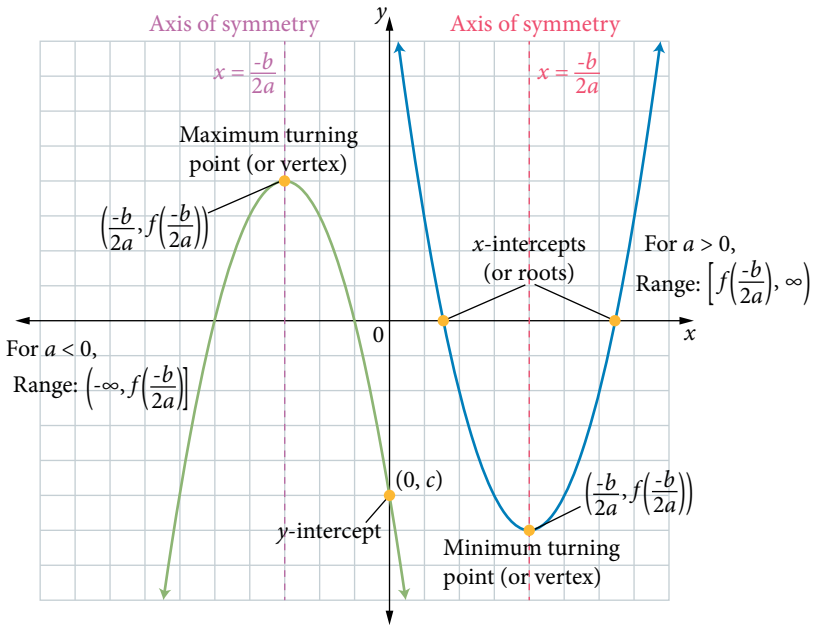
- If $a > 0$, then the graph of the function is concave up and the y-coordinate of the vertex corresponds to the minimum value of the function.
- If $a < 0$, then the graph of the function is concave down and the y-coordinate of the vertex corresponds to the maximum value of the function.

The equation of the axis of symmetry and x-value of the turning point are given by $x = -\dfrac{b}{2a}$.

The coordinates of the y-intercept are $(0, c)$.

There may be 0, 1 or 2 x-intercepts according to the number of solutions of $ax^2 + bx + c = 0$.

The maximal domain of a quadratic function is $\mathbb{R}$.



x-intercept form of a quadratic function

The rules of some quadratic functions can be written as a product of two linear factors with rational constants: $y = a(x - b)(x - c)$, where $(b, 0)$ and $(c, 0)$ are the x-intercepts.

The axis of symmetry and x-value of the turning point are located half-way between the intercepts and are given by $x = \dfrac{b + c}{2}$.

Turning point or completed square form of a quadratic function

$y = a(x-b)^2 + c$, where (b, c) is the vertex or turning point.

The equation of the line of symmetry is $x = b$.

The range is $[c, \infty)$ if $a > 0$ or $(-\infty, c]$ if $a < 0$.

Transformations of the basic quadratic function $y = x^2$

When compared to the graph of $y = x^2$, quadratic functions in the completed square or turning point form $f(x) = a(x-b)^2 + c$ are:

- dilated parallel to the y-axis by a scale factor of a when $a > 0$.
- reflected in the x-axis and dilated parallel to the y-axis by a scale factor of $|a|$ when $a < 0$.
- translated b units horizontally and c units vertically.

For any function, $y = f(kx)$, the graph of $y = x^2$ is dilated horizontally by $\frac{1}{k^2}$, which is equivalent to a vertical dilation of k^2.

Solving quadratic equations

Any quadratic equation of a single variable can be written in the form $ax^2 + bx + c = 0$.

The roots or solutions of this equation, and correspond to the location of the x-intercepts. The discriminant $\Delta = b^2 - 4ac$ indicates the number and nature of these roots and corresponding x-intercepts.

Discriminant	Interpretation
$\Delta < 0$	no roots (no intersection of the graph with the x-axis)
$\Delta = 0$	exactly 1 root (exactly 1 point of intersection of the graph with the x-axis and it will be the turning point)
$\Delta > 0$	two distinct roots (two points of intersection of the graph with the x-axis)
Δ is a perfect square	roots are rational

A quadratic equation can be solved exactly or approximately using several approaches.

Without technology

Use the discriminant $\Delta = b^2 - 4ac$ to check if there are roots or not; if there are no roots, there is no need to proceed further.

If Δ is a perfect square, then the function has rational roots and can be factorised. The null factor law can then be applied: if $(x-a)(x-b) = 0$, then $x = a$ or $x = b$.

Solutions are given by the quadratic formula: $x = \frac{-b \pm \sqrt{\Delta}}{2a}$.

Complete the square to obtain the form $f(x) = a(x-b)^2 + c$ and solve $f(x) = 0$ for x.

Chapter review 2

Exercise 2.1

1 Explain how you know that the equation $y - x^2 = 3x - 10$ represents a function.

Exercise 2.2

2 For each of the following quadratic functions determine the equation of the line of symmetry and the coordinates of the turning point.

(a) $y = 3x^2 - 8$ **(b)** $y = (x - 12)^2$ **(c)** $y = -(x + 7)^2 - 11$

(d) $y = 4(x + 20)^2$ **(e)** $y = -\dfrac{x^2}{2} + 16$ **(f)** $y = 2.5x^2 - 10x - 3.9$

Exercise 2.2

3 Consider the function $y = x^2 + 9$ and determine:

(a) the vertical translation from the graph of $y = x^2$

(b) the equation of the line of symmetry

(c) the coordinates of the turning point

(d) the coordinates of the y-intercept.

Exercise 2.2

4 Determine whether each quadratic function has a maximum or minimum turning point.

(a) $y = 3x^2 - 4x$ **(b)** $y = 15 - 3x - x^2$ **(c)** $y = 2x^2 + 8x$

(d) $y = -6x^2$ **(e)** $2y = 14 - 9x + 3x^2$ **(f)** $\dfrac{1}{2}y - 3x = 2 + \dfrac{2}{3}x^2$

Exercise 2.2

5 Determine the equation of each quadratic function following transformations of the base function $y = x^2$. Identify the range of the function using interval notation.

(a) a translation of 3 units upwards

(b) a translation of 4 units to the left

(c) a dilation, parallel to the y-axis by a scale factor of $\dfrac{2}{3}$

(d) a reflection in the x-axis, a dilation parallel to the y-axis of scale factor $\dfrac{3}{5}$ and a translation 3 units to the right

Exercise 2.2

6 Determine the equation for the line of symmetry for the function $y = (x + 9)^2 - 3$.

Exercise 2.2

7 Determine the coordinates for the turning point of the function $y = 3(x - 7)^2$.

Exercise 2.2

8 Determine the transformation for the function $y = (x - 9)^2 + 3$ compared to $y = x^2$.

Exercise 2.2

9 Choose the most appropriate description for the transformation for the function $y = 2(x + 4)^2 - 8$ compared to $y = x^2$.

A dilation parallel to the y-axis, scale factor 2; translated 4 units right; translated 8 units down

B dilation parallel to the y-axis, scale factor 4; translated 2 units left; translated 8 units up

C dilation parallel to the y-axis, scale factor 2; translated 4 units left; translated 8 units down

D dilation parallel to the y-axis, scale factor 4; translated 8 units left; translated 2 units up

10 Determine the rule of the function represented by the transformation of $y = x^2$: a translation 12 units to the left; dilation parallel to the y-axis of scale factor 4 and translation 6 units down. Exercise 2.2

11 Determine the coordinates of the turning point for the quadratic function $y = (x + 4)^2 - 2$. Exercise 2.2

12 Solve each of the following quadratic equations using factorisation and the null factor law. Exercise 2.3

(a) $x^2 - 3x - 54 = 0$ **(b)** $2x^2 - x - 15 = 0$ **(c)** $15x^2 - 7x - 36 = 0$

13 Use technology to solve each of the following quadratic equations. Give your answer correct to 2 decimal places if necessary. Exercise 2.3

(a) $-3x^2 + 8x - 1 = 0$ **(b)** $y = \dfrac{3x^2 - 5x - 19}{3}$ **(c)** $x^2 + 30x = -180$

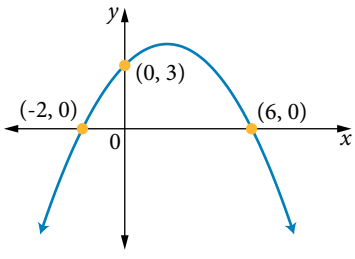
14 Determine the exact roots of quadratic equation $y = 2(x + 4)^2 - 3$. Exercise 2.3

15 Determine the roots of the quadratic function $y = (x - 4)(x + 8)$. Exercise 2.3

16 Determine the roots of the quadratic function $y = x^2 - 9$. Exercise 2.3

17 Determine the equation of the quadratic functions in the following graphs. Exercise 2.4

(a)



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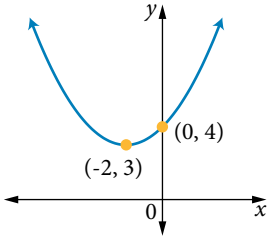

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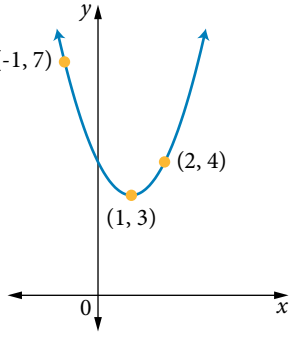


18 Determine which of the following quadratic equations has no roots. Exercise 2.4

A $y = -3(x - 4)^2 + 2$ **B** $y = \dfrac{(x - 4)}{3^2} - 4$ **C** $y = 2(x + 2)^2 - 3$ **D** $y = -4(x + 2)^2 - 1$

19 Write $y = \dfrac{(2x - 3)(x + 4)}{2}$ in the general form. Exercise 2.4

20 Consider the function $y = x^2 + 2x - 24$. Exercise 2.4

(a) Determine the equation of the line of symmetry.

(b) Determine the coordinates of the turning point.

(c) Determine the roots of the quadratic.

21 Consider the function $y = \dfrac{1}{2}x^2 - 8x + 3$. Exercise 2.4

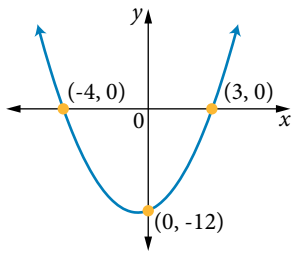
(a) Determine the equation of the line of symmetry.

(b) Determine the coordinates of the turning point.

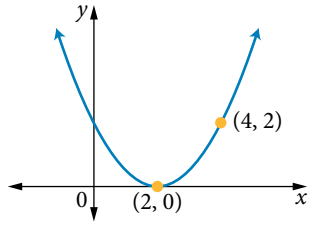
(c) Determine the roots of the quadratic.

Exercise 2.4 **22** Determine the rule for each quadratic function in the general form.

(a)



(b)



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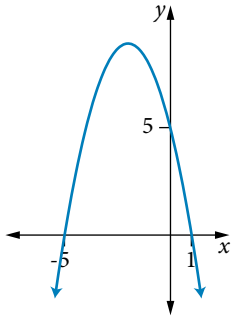


Exercise 2.1 **23** Which of the following is false for the graph of the following piece-wise function?

$$f(x)=\begin{cases} x+5, & x<-5 \\ x^2-25, & -5\le x\le 5 \\ x-5, & x>5 \end{cases}$$

A The graph of $f(x)$ is continuous for all values of x.

B The graph is discontinuous at $x=-5$ and $x=5$.

C The graph is not smooth at $x=-5$ and $x=5$.

D The graph has negative gradient for $-5\le x\le 0$.

Exercise 2.1 **24** The graph of the piece-wise function given below is continuous.

$$f(x)=\begin{cases} x+a, & x<-3 \\ (x-4)(x+5), & x\ge -3 \end{cases}$$

Choose the correct value of a. Sketch the graph to confirm your answer.

A -11 **B** -14 **C** 11 **D** 14

Exercise 2.3 **25** Solve each of the following quadratic equations by completing the square. Give your answers in exact form.

(a) $x^2-8x+5=0$ **(b)** $-x^2-10x-2=0$ **(c)** $x^2-5x+7=0$

Exercise 2.3 **26** For the quadratic function $f(x)=-2.5x^2+14x-22$, determine the axis of symmetry, the coordinates of the turning point and all axes intercepts and hence sketch its graph.

Exercise 2.3 **27** Determine the integer values of k for which $f(x)=2x^2+kx+7$ has 2 roots.

Exercise 2.4 **28** Convert $y=x^2+3x+2$ to turning point form.

Exercise 2.4 **29** Greg is planning to carve a bowl from a block of wood. He determines that the middle cross-section of the bowl will approximate the shape of the graph of $y=0.01x^2$.

(a) The internal measurement of the bowl is to have a maximum diameter of 30 cm. Calculate the depth of the bowl.

(b) If Greg wishes to make the bowl so that it is 5 mm thick, determine the minimum depth of the block of wood that he should use for carving the bowl.

30 A missile is launched from a launch pad 3 m from the ground and follows a path approximating the function $h(x) = -0.05x^2 + 15.8x + 3$, where x is the horizontal distance from the launch site and $h(x)$ is the height of the missile in metres. Give your answers correct to 1 decimal place. Exercise 2.4

(a) Determine the maximum height reached by the missile.

(b) The target for the missile is an object located on a hill and is at a height of 850 m above ground level. Determine the horizontal distance of this object from the launch site, assuming the missile is descending when it strikes.

31 Time-lapse photography is used to track the height of an object in metres. The following heights were recorded at times 0.2 s, 1.2 s and 2.1 s respectively: 9.203 m, 14.344 m and 10.591 m. Exercise 2.4

(a) Determine the quadratic equation of the height $y(t)$ of the object over time, in seconds.

(b) Determine, correct to 2 decimal places, the maximum height reached by the object.

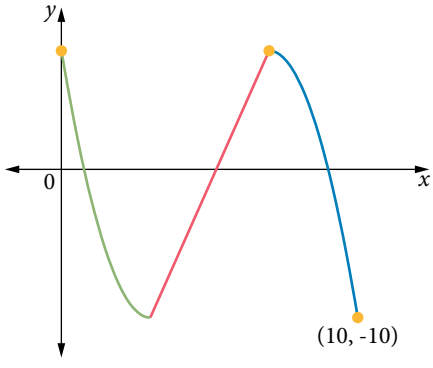
(c) If the object was launched from a height of 5 m and continued until it hit the ground, determine the valid domain for the function and interpret this in terms of the timing of the photography and the time the object was in the air.

(d) Determine the new rule for the function, including the domain, so that $t = 0$ corresponds to the time the object was launched. Round your answer appropriately.

32 A piece-wise function shown below undergoes the following transformations in the order given: translation of 5 units to the right, reflection in the x-axis, horizontal dilation from the x-axis of factor 2. Determine a rule for the new piece-wise function and state its range. Exercise 2.1

$$y = \begin{cases} 2x, & -3 \le x \le -1 \\ (x-1)^2, & -1 < x \le 3 \end{cases}$$

33 The symmetric diagram is drawn from a piece-wise function consisting of two quadratics and a line joining the turning points. The shape begins at the y-axis and ends at the point (10, -10) as shown. The equation for one of the quadratics is given by $y = 2x^2 - 12x + c$. Determine the rule for the piece-wise function giving both quadratics in turning point form. Exercise 2.1



3

Polynomial functions

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3 Recall

Substitute a given value into an expression

1 Substitute $x = -2$ into each rule and calculate the value of y.

(a) $y = 2x + 6$ (b) $y = 2x^2 - 4x + 3$ (c) $y = -\frac{1}{2}x^2 + 7x - 10$

Rearrange an equation

2 Rearrange the following equations to make y the subject.

(a) $3x = 2y + 9$ (b) $4x^2 = 3y - 4$ (c) $2y^2 = x - 4$

Determine where the graph of a relation will intersect the vertical axis

3 For the following relations determine the coordinates of the y-axis intercept.

(a) $y = 3x^2 + 5x - 4$ (b) $5x + 3y = 10$ (c) $y = 2(x - 4)^2 + 1$

Factorise a quadratic expression

4 Factorise the following quadratic expressions.

(a) $x^2 - 2x$ (b) $x^2 + 4x + 4$ (c) $x^2 - 7x + 12$

Expand brackets using the distributive law

5 Expand the brackets and simplify the following expressions.

(a) $2(x + 4)$ (b) $-\frac{1}{3}(x - 4)$ (c) $3x(x - 6)$

Expand binomial products

6 Expand each of the binomial products and then simplify.

(a) $(x + 4)(x - 2)$ (b) $2(x - 5)(x + 1)$ (c) $-4(x - 3)^2$

Describe the geometric transformations applied to a function

7 Describe the geometric transformations that are applied to the first function to produce the second. In some cases more than one transformation may apply.

(a) $y = x^2$ is transformed to $y = x^2 + 3$.

(b) $y = x^2$ is transformed to $y = (x - 1)^2$.

(c) $y = x^2$ is transformed to $y = -x^2 + 4$.

Determine the domain and range of a function

8 Determine the domain and range of each function.

(a) $4x - 7y = 15$ (b) $y = 4x - x^2 - 5$ (c) $y = 2(x + 3)^2 - 9$

Polynomials

3.1

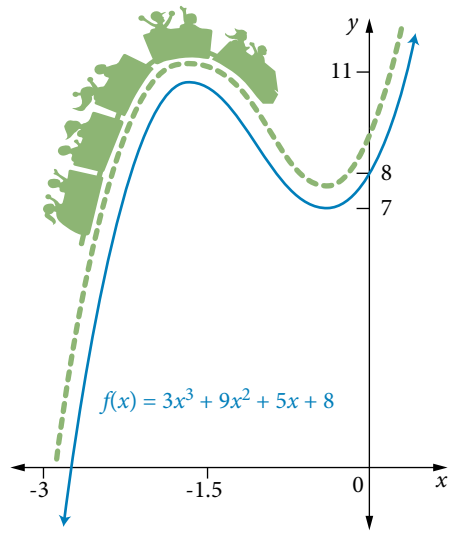
Polynomial functions

In mathematics it is important to recognise and make generalisations about patterns such as in the decimal number system. See the example below:

The number **3958** can be written in terms of thousands, hundreds, tens and units (that is, in its *expanded form*).	$3958 = 3000 + 900 + 50 + 8$
Each term can be separated into power-of-ten components (a process similar to counting out a sum of money in different denominations of notes and coins).	$3 \times 1000 + 9 \times 100 + 5 \times 10 + 8$
Each power of 10 can be written in index form.	$3 \times 10^3 + 9 \times 10^2 + 5 \times 10^1 + 8 \times 10^0$
Replacing the base number 10 with a variable such as x, an expression is formed.	$3x^3 + 9x^2 + 5x^1 + 8x^0$
The numbers 3, 9, 5 and 8 are the coefficients of x^3, x^2, x^1 and x^0 respectively. The powers $x^1 = x$ and $x^0 = 1$ are not written explicitly, so $5x^1$ is written as $5x$, and $8x^0$ is written as 8. The final expression is:	$3x^3 + 9x^2 + 5x + 8$

A function such as $f(x) = 3x^3 + 9x^2 + 5x + 8$ is said to be a *polynomial* function of a single real variable x.

Polynomials have many real-world applications: in the modelling of measurement problems involving length, area and volume; describing the motion of a particle; calibration and error modelling in instrumentation; generating curves in graphics design packages (called Bézier curves); and approximating other more complicated functions.



A polynomial is an algebraic expression that is a sum of terms that consist of a constant multiplied by a variable raised to a non-negative integer power.

Polynomials are constructed using only the operations of addition, subtraction and multiplication on the variables. Expressions that involve division, reciprocals, square roots or other operations on the variables are not polynomials.

Aspects of polynomial functions

A polynomial function P of a single real variable x is defined as:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

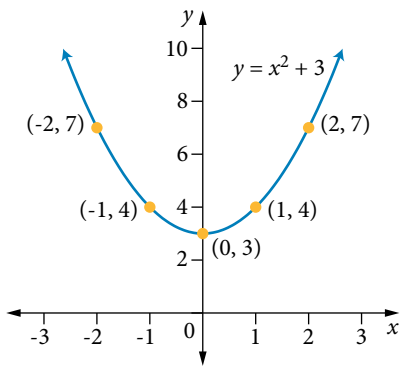
where n is a non-negative integer, the coefficients a_0, a_1, a_{n-1}, a_n are real constants, a_n is not zero and x is a real variable. All polynomial functions have maximal domain $\mathbb{R}$, so that $P(x)$ is defined for all real values of x.

Some common types of polynomials are constant (degree 0), linear (degree 1), quadratic (degree 2), cubic (degree 3), and quartic polynomials (degree 4).

Type	Degree	Polynomial	Example
quartic	4	$P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0$	$P(x) = 2x^4 + 7x^3 - x^2 + 7x - 8$
cubic	3	$Q(x) = a_3x^3 + a_2x^2 + a_1x^1 + a_0$	$Q(x) = -3x^3 + 6x^2 + 2x - 1$
quadratic	2	$R(x) = a_2x^2 + a_1x^1 + a_0$	$R(x) = 4x^2 + 3$
linear	1	$S(x) = a_1x^1 + a_0$	$S(x) = -5x + 4$
constant	0	$T(x) = a_0$	$T(x) = 9$

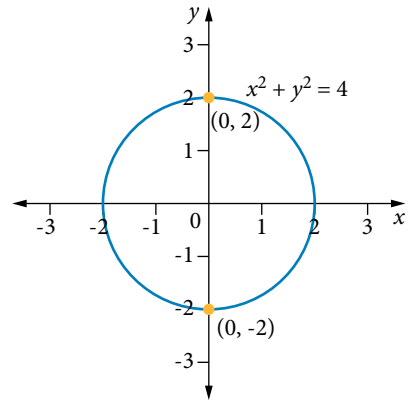
The polynomials here are all written with descending powers of x. However, you should note that polynomials may also be written with ascending powers of x, or less commonly with 'jumbled' powers of x. Similarly, some polynomials do not contain all the 'expected' terms that could comprise the function. For example, the quartic $P(x) = 2x^4 + 7x^3 - x^2 - 8$ is written with four terms. The x-term raised to the power of 1 has a coefficient of zero and therefore is not written.

A graph of the polynomial function $y = x^2 + 3$ is shown. Each x-coordinate has exactly one corresponding y-coordinate. The graph of the function shows its parabolic shape, its turning point at (0, 3) and the axis of symmetry with equation $x = 0$.



x	-2	-1	0	1	2
y	7	4	3	4	7

The graph of the relation with the rule $x^2 + y^2 = 4$ is shown. You can see from the graph and from the table of values that many x-coordinates have two corresponding y-coordinates (e.g. for $x = 0$, y is both +2 and -2). This relation is *not* a function.



x	-2	-1	0	1	2
y	0	$\pm\sqrt{3}$	± 2	$\pm\sqrt{3}$	0

1 Recognise polynomial expressions

Is the expression a polynomial? If not, then provide a reason.

(a) $x^3 + 4x^2 - x + 7$

THINKING	WORKING
1 Write the expression.	$x^3 + 4x^2 - x + 7$
2 All terms in a polynomial expression have variables raised to non-negative integer powers.	$x^3 + 4x^2 - x + 7$ is a polynomial.

(b) $x^{\frac{3}{2}} + 2x^2 - 3x - 1$

1 Write the expression.	$x^{\frac{3}{2}} + 2x^2 - 3x - 1$
2 All terms in a polynomial expression have variables raised to non-negative integer powers.	The first term power is a non-integer value so $x^{\frac{3}{2}} + 2x^2 - 3x - 1$ is not a polynomial.

(c) $\left(2 + \frac{4}{x}\right)^2$

1 Expand the brackets and collect like terms. You could use the perfect square rule $(a+b)^2 = a^2 + 2ab + b^2$.	$\left(2 + \frac{4}{x}\right) \times \left(2 + \frac{4}{x}\right)$ $= 4 + \frac{8}{x} + \frac{8}{x} + \frac{16}{x^2}$ $= 4 + \frac{16}{x} + \frac{16}{x^2}$ $= 4 + 16x^{-1} + 16x^{-2}$
2 All terms in a polynomial expression have variables raised to non-negative integer powers.	Two terms have variables with negative powers so $\left(2 + \frac{4}{x}\right)^2$ is not a polynomial.

In any polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$:

- The *leading term* is the term containing the highest power of the variable (with a non-zero coefficient).
- The *degree* of the polynomial is the index of the leading term. P is said to be of degree (has the highest power) n, provided $a_n \neq 0$.
- The *leading coefficient* is the coefficient of the leading term.
- A *monic polynomial* is a polynomial whose leading coefficient is 1, so that $a_n = 1$.
- The *constant term* is the term of index zero: $a_0 x^0 = a_0 \times 1 = a_0$. This term is independent of x.

2 Identify features of polynomials

For the following polynomials, arrange the terms in descending order by degree, then identify:

(i) the leading term
(ii) the leading coefficient
(iii) whether the polynomial is monic
(iv) the degree
(v) the constant term.

(a) $Q(x) = 7x^2 + 10x^4 - 9 + \frac{2x}{3}$

THINKING	WORKING
1 Write the polynomial expression in descending order by degree, starting with the term with the highest power of x.	$Q(x) = 10x^4 + 7x^2 + \frac{2x}{3} - 9$
2 Determine the leading term and the leading coefficient. Therefore, identify whether it is a monic polynomial.	(i) Leading term: $10x^4$ (ii) Leading coefficient: 10 (iii) This is not a monic polynomial.
3 Identify the term with highest power of x and write this power as the degree.	(iv) Degree: 4
4 Identify the term with index 0 (the term independent of x).	(v) Constant term: -9

(b) $R(x) = x^5 - 4$

1 Determine the leading term and the leading coefficient. Therefore, identify whether it is a monic polynomial.	(i) Leading term: x^5 (ii) Leading coefficient: 1 (iii) This is a monic polynomial.
2 Identify the term with highest power of x and write this power as the degree.	(iv) Degree: 5
3 Identify the term with index 0 (the term independent of x).	(v) Constant term: -4

Substitution into polynomial expressions

A value for x can be substituted into a polynomial to obtain a corresponding value of the function. For example, $x = 2$ is substituted into $P(x) = 2x^4 + 7x^3 - x^2 + 7x - 8$:

$$\begin{aligned} P(2) &= 2(2)^4 + 7(2)^3 - (2)^2 - 8 \\ &= 32 + 56 - 4 - 8 \\ &= 76 \end{aligned}$$

The following table shows the polynomials $P(x)$, $Q(x)$, $R(x)$, $S(x)$, and $T(x)$ after $x = 2$ is substituted into each respective function. Note that for the constant function the substitution has no effect as this function is independent of x.

Example	For $x = 2$
$P(x) = 2x^4 + 7x^3 - x^2 + 7x - 8$	$P(2) = 76$
$Q(x) = -3x^3 + 6x^2 + 2x - 1$	$Q(2) = 3$
$R(x) = 4x^2 + 3$	$R(2) = 19$
$S(x) = -5x + 4$	$S(2) = -6$
$T(x) = 9$	$T(2) = 9$

Arithmetic operations with polynomial expressions

Arithmetic operations such as multiplication by a constant, addition and subtraction can be applied to polynomials.

3 Operations with polynomial functions

Given that $Q(x) = 3x + 2x^2 - 4 - x^3$ and $R(x) = 9 - x^2$ determine:

(a) $R(1)$

THINKING	WORKING
Substitute $x = 1$ into $R(x)$.	$R(1) = 9 - (1)^2$ $= 9 - 1$ $= 8$

(b) $2Q(-3)$

1 Substitute $x = -3$ into $Q(x)$ to give $Q(-3)$.	$Q(-3) = 3 \times -3 + 2 \times (-3)^2 - 4 - (-3)^3$
2 Multiply $Q(-3)$ by 2 to determine $2Q(-3)$.	$2Q(-3) = 2 \times (-9 + 18 - 4 + 27)$ $= 2 \times 32$ $= 64$

(c) $Q(x) + R(x)$

METHOD 1

1 To determine $Q(x) + R(x)$, add both functions together.	$Q(x) + R(x) = 3x + 2x^2 - 4 - x^3 + (9 - x^2)$
2 Simplify the expression by collecting like terms, starting with the term that has the highest power of x.	$Q(x) + R(x) = 3x + 2x^2 - x^2 - 4 + 9 - x^3$ $= 3x + x^2 + 5 - x^3$ $= -x^3 + x^2 + 3x + 5$

METHOD 2

1. To determine $Q(x) + R(x)$, arrange both functions according to descending powers of x. Then use the traditional arithmetic addition layout to add these functions. (Note that there is no carrying involved in this calculation.)

$$\begin{array}{lr} Q(x) & -x^3 + 2x^2 + 3x - 4 \\ R(x) + & (-1x^2 \qquad + 9) \\ \hline & -x^3 + x^2 + 3x + 5 \end{array}$$

2. Write in simplified terms, starting with the term that has the highest power of x.

$$Q(x) + R(x) = -x^3 + x^2 + 3x + 5$$

(d) $Q(x) - R(x)$

1. To determine $Q(x) - R(x)$, subtract the functions.

$$Q(x) - R(x) = 3x + 2x^2 - 4 - x^3 - \left(9 - x^2\right)$$

2. Simplify the expression by collecting like terms, starting with the term that has the highest power of x.

$$\begin{aligned} Q(x) - R(x) &= 3x + 2x^2 - 4 - x^3 - 9 + x^2 \\ &= -x^3 + 3x^2 + 3x - 13 \end{aligned}$$

(e) $2R(x) - 3Q(x)$

1. To calculate $2R(x) - 3Q(x)$, double $R(x)$ and subtract 3 times $Q(x)$.

$$\begin{aligned} &2R(x) - 3Q(x) \\ &= 2\left(9 - x^2\right) - 3\left(3x + 2x^2 - 4 - x^3\right) \end{aligned}$$

2. Expand the brackets and simplify.

$$\begin{aligned} &= 18 - 2x^2 - 9x - 6x^2 + 12 + 3x^2 \\ &= 3x^3 - 8x^2 - 9x + 30 \end{aligned}$$

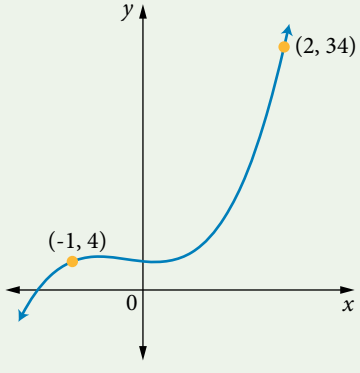
Determining the equation of a polynomal function

You can use simultaneous equations to determine unknown parameters in a polynomial equation, if you are given enough information.

- For a linear function $P(x) = ax + b$, the coordinates of two distinct points are sufficient to determine the coefficients a and b. Construct and solve two simultaneous linear equations in two unknowns.
- For a quadratic function $P(x) = ax^2 + bx + c$, the coordinates of three distinct points are sufficient to determine the coefficients a, b and c. Construct and solve three simultaneous linear equations in two unknowns.
- For a cubic function $P(x) = ax^3 + bx^2 + cx + d$, the coordinates of four distinct points are sufficient to determine the coefficients a, b, c and d. Construct and solve four simultaneous linear equations in two unknowns.

4 Determine coefficients of polynomials

Consider the polynomial function $P(x) = ax^3 + bx^2 - x + 4$. Given that $P(-1) = 4$ and $P(2) = 34$, determine the values of a and b. Confirm your answers with a graph.

THINKING	WORKING
1 Substitute $x = -1$ and $P(x) = 4$ into the equation and simplify. Label [1].	$P(-1) = a(-1)^3 + b(-1)^2 - (-1) + 4$ $4 = -a + b + 5$ $-1 = -a + b$ $a = b + 1 \quad [1]$
2 Substitute $x = 2$ and $P(x) = 34$ into the equation and simplify. Label [2].	$P(2) = a(2)^3 + b(2)^2 - 2 + 4$ $34 = 8a + 4b - 2 + 4$ $32 = 8a + 4b$ $8 = 2a + b \quad [2]$
3 Substitute [1] into [2] and solve for b.	$8 = 2(b + 1) + b$ $8 = 2b + 2 + b$ $6 = 3b$ $2 = b$
4 Substitute $b = 2$ into [1] to determine the value of a.	$a = 2 + 1$ $= 3$
5 Write the equation for the function by inserting the values of a and b.	$P(x) = 3x^3 + 2x^2 - x + 4$
6 Graph the function using technology and check that the given points lie on the graph.	

Additional information

Points on a polynomial

This video investigates the number of values defining a polynomial.

5 Use simultaneous equations to establish the rule of a cubic function, given 4 points

Determine the rule of the cubic function whose graph passes through the points (1, 10), (-1, -2), (2, 25) and (-3, -70).

THINKING	WORKING
1 Substitute the points into the general equation of a cubic function $y = ax^3 + bx^2 + cx + d$, labelling the resulting equations [1], [2], [3] and [4] respectively.	$10 = a(1)^3 + b(1)^2 + c(1) + d$ $= 1a + 1b + 1c + d$ $= a + b + c + d$ [1] $-2 = a(-1)^3 + b(-1)^2 + c(-1) + d$ $= -1a + 1b - 1c + d$ $= -a + b - c + d$ [2] $25 = a(2)^3 + b(2)^2 + c(2) + d$ $= 8a + 4b + 2c + d$ [3] $-70 = a(-3)^3 + b(-3)^2 + c(-3) + d$ $= -27a + 9b - 3c + d$ [4]
2 To obtain values for a, b, c and d, a series of calculations is performed. In the four equations from the previous step, the pronumeral d is the easiest to eliminate because the coefficient of d is 1. The simplest calculation involves subtractions to eliminate d and label new equations [5], [6] and [7] respectively.	[4] − [1]: $-70 = -27a + 9b - 3c + d$ $- \quad 10 = a + b + c + d$ $-80 = -28a + 8b - 4c$ [5] [3] − [1]: $25 = 8a + 4b + 2c + d$ $- \quad 10 = a + b + c + d$ $15 = 7a + 3b + c$ [6] [2] − [1]: $-2 = -a + b - c + d$ $- \quad 10 = a + b + c + d$ $-12 = -2a - 2c$ Simplify: $6 = a + c$ [7]
3 Equation [7] is written using two variables, so use this to eliminate another pronumeral, c, by using the substitution method. Rearrange equation [7] to make c the subject, then substitute this revised equation into equations [5] and [6].	[7] can be rearranged as: $6 = a + c$ $c = 6 - a$ Substitute c into equation [5]: $-80 = -28a + 8b - 4(6 - a)$ $= -28a + 8b - 24 + 4a$ $-56 = -24a + 8b$ [8]

Substitute c into equation [6]:

$$\begin{aligned} 15 &= 7a + 3b + (6 - a) \\ &= 6a + 3b + 6 \\ 9 &= 6a + 3b \\ 3 &= 2a + b \qquad [9] \end{aligned}$$

4 Equations [8] and [9] are written using two variables, so these can be used to eliminate another pronumeral, b.
In this example the elimination method is used. However, it is also possible to rearrange equation [9] and then use the substitution method. Multiply each equation by a constant to make the b terms the same: $1 \times [8]$ and $8 \times [9]$.

$$\begin{aligned} -56 &= -24a + 8b \qquad [8] \\ 3 &= 2a + b \qquad [9] \end{aligned}$$

$[8] - 8 \times [9]$:

$$\begin{aligned} -56 &= -24a + 8b \\ - \quad 24 &= 16a + 8b \\ \hline -80 &= -40a \\ a &= 2 \end{aligned}$$

5 Substitute $a = 2$ into equations [7] and [9] to solve for c and b respectively. Then substitute the values for a, b, and c into equation [1] to solve for d.

Substitute $a = 2$ into [7]:

$$\begin{aligned} c &= 6 - a \\ c &= 6 - 2 = 4 \end{aligned}$$

Substitute $a = 2$ into [9]:

$$\begin{aligned} 3 &= 2a + b \\ 3 &= 2(2) + b \\ b &= 3 - 4 \\ b &= -1 \end{aligned}$$

Substitute $b = -1$ and $c = 4$ into [1]:

$$\begin{aligned} 10 &= 2 + (-1) + (4) + d \\ 5 &= d \end{aligned}$$

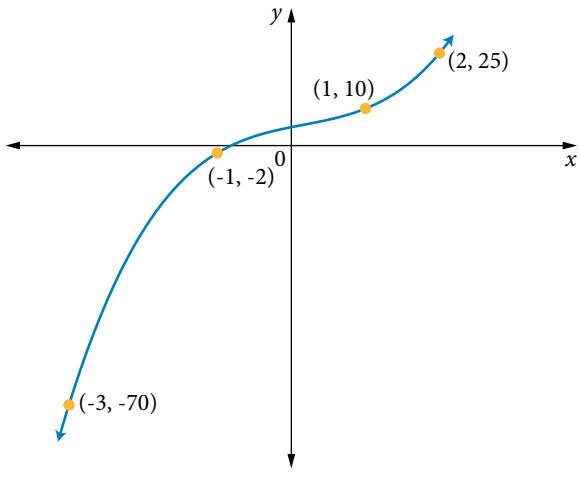
6 Define the cubic equation with all coefficients.

$$y = 2x^3 - x^2 + 4x + 5$$

Technology worked example

Using cubic regression

If technology is allowed, enter the four coordinate pairs from the previous worked example into a table in statistics mode. Determine the cubic function of best fit. The values of a, b, c and d will be given for the general cubic polynomial $y = ax^3 + bx^2 + cx + d$.



Making connections

Fitting a cubic to four given points

Vary the decimal values of the coefficients of a cubic polynomial to progressively fit the graph to four given points.

EXERCISE 3.1 Polynomials

Worked Example

1 **1** Determine whether the following expressions are polynomials. If not, then provide a reason.

(a) $3x+2-4x^2$ (b) $-5x^2+3x-7$ (c) $4x^{\frac{5}{2}}+\frac{5}{7}x^2-14$ (d) $x^7-\frac{7}{x}+3x^4-4x$

(e) $9x^5-4x^3+5$ (f) $x^3+2x^2-4x+\sqrt{x}$ (g) 5

2 **2** For each polynomial, arrange the terms in descending order by degree, then identify the leading term and the leading coefficient, the polynomial's degree and the constant term.

(a) $P(x)=4+6x-2x^2$ (b) $Q(x)=-5x^2+8x^4-6+\frac{5x}{6}$ (c) $R(x)=10$

(d) $S(x)=8+2x^3$ (e) $T(x)=4x^2-3x+x^4+\frac{2}{3}$ (f) $U(x)=-7x+1-\frac{3x^2}{4}$

3 **3** Given that $P(x)=3x^4+2x^3-x+1$ and $Q(x)=10-x^2$, determine the following.

(a) The degree of $P(x)$. (b) $P(2)$ (c) $Q(-1)$

(d) $Q(x)+P(x)$ (e) $Q(x)-P(x)$ (f) $2Q(-1)+3P(2)$

4 Given $T(x)=7x^6+4x^3-9x^4+7$, $T(-2)$ is equal to:

A 627 B 279 C 620 D 326

5 For each following polynomial, determine its degree, $P(1)$ and $P(-2)$.

(a) $P(x)=2x^3-x^2+4x-1$ (b) $P(x)=10x^4+x^3-3x^2+6$

(c) $P(x)=8-4x+13x^2-x^3$ (d) $P(x)=8x^9+4x^2-3$

(e) $P(x)=-5+7x-x^2+x^3$ (f) $P(x)=4x^2+5x^3-6x^7+4$

(g) $P(x)=6x^8-4x^7+3x^3-9$ (h) $P(x)=8$

6 Given that $Q(x)=7-x+2x^2$, $R(x)=-3x^3+2x^2-x+8$ and $T(x)=5x^4-7x+9$, determine the following.

(a) $Q(t)+T(t)$ (b) $2R(z)-3Q(z)$

(c) $R(2p)+Q(p)-T(p)$ (d) $3Q(z)-20$

7 Given $P(x)=cx^4-2x^3+3x^2-9$ and $P(-1)=-3$, determine the value of c.

8 Given $G(x)=3x^3-2x+x^n+1$, and $G(3)=103$, determine the value of n.

Worked Example

9 A polynomial is defined by $P(x) = ax^3 + bx^2 + 3x - 5$. Given $P(-1) = 13$ and $P(-3) = 121$, determine the values of a and b. Confirm your answers with a graph of the function. 4

10 For the polynomial $T(x) = Ax^4 + Bx^3 - 2x + 5$, $T(-2) = 17$ and $T(3) = 242$.

(a) The values of A and B are:

A $A = 2, B = -3$ B $A = -2, B = 3$ C $A = 2, B = 3$ D $A = 1, B = 3$

(b) Suggest the incorrect thinking for a student who chose either of the first two incorrect options.

11 Determine the values of a and b if $f(x) = ax^5 + bx^4 - x^3 + 2$, $3f(2) = -18$ and $2f(1) = 6$.

12 A polynomial $W(x) = x^3 + 3x^2 + 3x + 8$ can also be written as $W(x) = a(x+1)^3 + b$. Determine the values of a and b, given that $(x+1)^3 = x^3 + 3x^2 + 3x + 1$.

13 Construct the following polynomial examples.

(a) A non-monic polynomial $P(x)$ with degree 2, constant term 4, and with $P(1) = 6$.

(b) A monic cubic polynomial with constant term -5, and with $Q(-3) = -2$.

(c) A non-monic, quartic polynomial consisting of three terms, no constant term, and with $R(2) = 10$.

14 Determine the equation of the cubic function that passes through each set of points. 5

(a) (0, 0), (2, 18), (-3, 3) and (-1, 9).

(b) (-3, 24), (2, -6), (0, 12), and (1, 0)

(c) (1, 0), (-1, 8), (2, -4), and (0, 6)

15 Determine the equation of a cubic function whose graph passes through each set of points. Write your answers correct to 3 decimal places where necessary. Graph each function to confirm your answer.

(a) (4, 0), (1, 18), (0.5, 18.375) and (-2, -36)

(b) (-2.5, 0), (1.5, 2), (3.5, 2), and (0, 3)

(c) (7, 45), (-6, -52.5), (-2, 4.5), and (4, 0)

16 For polynomials $P(x)$, $Q(x)$ and $R(x)$: $P(x) = 2Q(x) + 3$ and $R(x) = 2P(x) + 3$.

(a) If $Q(x) = x^3 - 2x + 1$, determine $R(x)$ in simplest form.

(b) Determine points on the graph of $y = Q(x)$ and their image points on the graph of $y = R(x)$ for $x = 0$, 1 and 2.

(c) Write a direct relationship connecting $R(x)$ and $Q(x)$. Describe the relationship in terms of transformations of the graph of $y = Q(x)$ to the graph of $y = R(x)$. Demonstrate the sense of your solution in terms of the three image points in part (b).

17 The graphs of two different cubic functions intersect at $x = -2$, $x = -1$ and $x = 3$. The first passes through (-3, 6) and the second through (2, 1). Determine two possible pairs of equations that satisfy this description. Graph each pair to confirm your solutions.

3.2 Expanding cubic polynomials

Binomial cubics

Cubic expressions may have 1, 2, 3 or 4 terms. Expressions with only one term are of the form $P(x) = ax^3$. Binomial cubics have two terms, including ax^3 and one other term.

6 Expand to form a binomial cubic

Expand each of the following expressions.

(a) $3\left(x^3 + 7\right)$

THINKING	WORKING
Multiply each of the terms in the brackets by the factor at the front.	$3\left(x^3 + 7\right) = 3 \times x^3 + 3 \times 7$ $= 3x^3 + 21$

(b) $2x\left(x^2 - 5\right)$

Multiply each of the terms in the brackets by the factor at the front and then simplify.	$2x\left(x^2 - 5\right) = 2x \times x^2 + 2x \times -5$ $= 2x^3 - 10x$

(c) $6x(x + 5)(x - 5)$

Use the difference of two squares rule $(a + b)(a - b) = a^2 - b^2$. Multiply each of the terms in the brackets by the factor at the front and then simplify.	$6x(x + 5)(x - 5) = 6x\left(x^2 - 5^2\right)$ $= 6x\left(x^2 - 25\right)$ $= 6x^3 - 150x$

Multiplying polynomials

Any two polynomial expressions can be multiplied using the distributive property, and this process is sometimes called 'expanding'. Consider the example of multiplying $P(x) = 3x^2 + 2x - 7$ by $G(x) = 2x - 4$, which could be written as $\left(3x^2 + 2x - 7\right) \times (2x - 4)$ or simply as $\left(3x^2 + 2x - 7\right)(2x - 4)$. As multiplication is commutative, it doesn't matter which of the two terms is written first: $\left(3x^2 + 2x - 7\right)(2x - 4) = (2x - 4)\left(3x^2 + 2x - 7\right)$

Three different methods may be useful for multiplying polynomials:

- Multiplying out brackets is convenient for polynomial expressions of low degree.
- Arithmetic multiplication is similar to the usual method for division of polynomials, so it shows how traditional multiplication and division is applied to polynomial expressions in algebra.
- Lattice (grid) expansion makes it easier to keep track of like terms with equal powers when polynomial expressions of higher degrees are involved.

Multiplying out brackets

When multiplying out brackets, a common approach is to multiply each term in the first factor by every term in the second factor and then collect like terms, as shown:

$$
\begin{aligned}
(3x^2 + 2x - 7)(2x - 4) &= 3x^2(2x - 4) + 2x(2x - 4) - 7(2x - 4) \\
&= 3x^2 \times 2x + 3x^2 \times -4 + 2x \times 2x + 2x \times -4 + (-7) \times 2x + (-7) \times -4 \\
&= 6x^3 - 12x^2 + 4x^2 - 8x - 14x + 28 \\
&= 6x^3 - 8x^2 - 22x + 28
\end{aligned}
$$

This could also be expanded as:

$$
\begin{aligned}
(2x - 4)\left(3x^2 + 2x - 7\right) &= 2x\left(3x^2 + 2x - 7\right) - 4\left(3x^2 + 2x - 7\right) \\
&= 2x \times 3x^2 + 2x \times 2x + 2x \times -7 - 4 \times 3x^2 - 4 \times 2x - 4 \times -7 \\
&= 6x^3 + 4x^2 - 14x - 12x^2 - 8x + 28 \\
&= 6x^3 - 8x^2 - 22x + 28
\end{aligned}
$$

Additional information

Expanding three factors

Multiply out three sets of brackets in this activity.

Arithmetic multiplication

This method uses the same process and setting out as for multiplying two multiple-digit numbers.

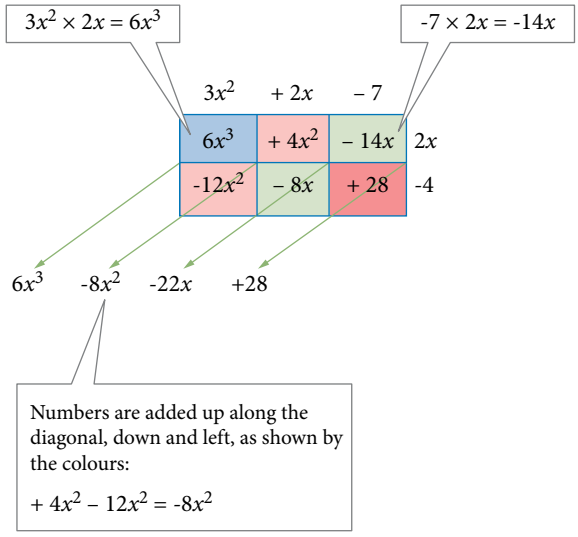
$$
\begin{array}{rrrrr}
 & & 3x^2 & +\ 2x & -\ 7 \\
\times & & & 2x & -\ 4 \\
\hline
 & & -12x^2 & -\ 8x & +28 \\
+ & 6x^3 & +\ 4x^2 & -14x & +\ 0 \\
\hline
 & 6x^3 & -8x^2 & -22x & +28
\end{array}
$$

Note that there are no 'carries' as each term is multiplied; the product of the terms multiplied is placed in the vertical lines of working out.

Lattice (vertical grid) expansion

In this method, the two expressions to be multiplied sit across the top and down the right of a grid, as shown. The size of the grid corresponds to the number of terms in each expression.

After the terms are multiplied, the final result is found by adding like terms along the diagonals.



Additional information

Multiplying polynomials

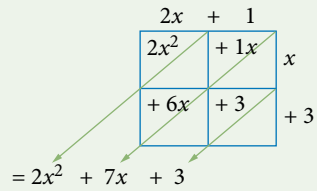
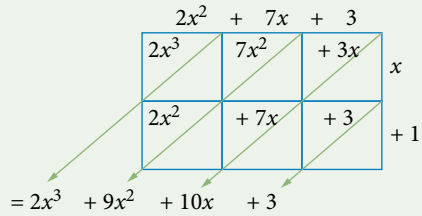
Watch the video to explore arithmetic multiplication and multiplying out brackets in more detail.

7 Expand brackets and simplify by collecting like terms

For the cubic expression $(2x+1)(x+3)(x+1)$:

(a) Expand and simplify by collecting like terms.

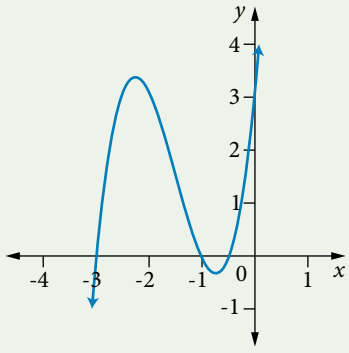
METHOD 1

THINKING	WORKING
1 Identify the expression to be factorised.	$(2x+1)(x+3)(x+1)$
2 Use the lattice method to multiply the first two sets of brackets. Multiply each element according to column and row, and then add the resulting terms along the diagonals.	
3 Multiply the answer from the previous step by the remaining factor $(x+1)$. A 3×2 lattice grid will be needed to multiply the polynomials.	

METHOD 2

1 Multiply each term in the second set of brackets by each term in the third set of brackets. Simplify the expression by collecting like terms.	$(2x+1)(x+3)(x+1)$ $=(2x+1)\left[x(x+1)+3(x+1)\right]$ $=(2x+1)\left(x^2+x+3x+3\right)$ $=(2x+1)\left(x^2+4x+3\right)$
2 Multiply each term in the first set of brackets by each term in the second set of brackets. Simplify the expression by collecting like terms.	$=2x\left(x^2+4x+3\right)+1\left(x^2+4x+3\right)$ $=2x^3+8x^2+6x+x^2+4x+3$ $=2x^3+9x^2+10x+3$

(b) Use technology to verify the result for **(a)** graphically.

THINKING	WORKING
1 Write each expression as a function to be graphed.	Graph on the same coordinate axes: $y_1 = (2x+1)(x+3)(x+1)$ $y_2 = 2x^3 + 9x^2 + 10x + 3$
2 Confirm that the functions give the same graph.	 The graphs are the same, so the factorised and expanded expressions are equivalent.

Pascal's triangle and the binomial theorem

Pascal's triangle can be used to expand polynomials that are a product of repeated factors. It provides a visual representation of the coefficients in the expansion of $(a+b)^n$.

Pascal's triangle can be illustrated by the following:

$(a+b)^0 = 1$				1				row 0
$(a+b)^1 = 1a + 1b$			1		1			row 1
$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$		1		2		1		row 2
$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$	1		3		3		1	row 3

Each element in Pascal's triangle is formed by adding the two values directly above it, where each row begins and ends with a 1. The top value, comprising only the element 1, is called row 0.

In a binomial expansion of $(a+b)^n$:

$$(a+b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots \binom{n}{n}a^0b^n$$

- the powers of the first term a decrease from n to 0 from left to right
- the powers of the second term b increase from 0 to n from left to right
- the numbers in Pascal's triangle are the coefficients of each of the terms
- the powers of a and b for each term sum to n
- there are $(n+1)$ terms in the expansion.

C Making connections

Binomial expansion

Watch the binomial expansions.

8 Expand brackets using binomial expansion

Use binomial expansion to expand each of the factorised expressions.

(a) $(x+y)^2$

THINKING	WORKING
1 Identify the coefficients from row 2 in Pascal's triangle.	1, 2, 1
2 Expand by decreasing the powers of the first term and increasing the powers of the second term from left to right.	$(x+y)^2$ $= 1(x)^2(y)^0 + 2(x)^1(y)^1 + 1(x)^0(y)^2$
3 Simplify the expression.	$= x^2 + 2xy + y^2$

(b) $(2m-3n)^2$

1 Identify the coefficients from row 2 in Pascal's triangle.	1, 2, 1
2 Expand by decreasing the powers of the first term and increasing the powers of the second term from left to right.	$(2m-3n)^2$ $= 1(2m)^2(-3n)^0 + 2(2m)^1(-3n)^1 + 1(2m)^0(-3n)^2$
3 Simplify the expression.	$= (2m)^2 + 2\times(2m)(-3n) + (-3n)^2$ $= 4m^2 - 18mn + 9n^2$

(c) $(2x-1)^3$

1 Identify the coefficients from row 3 in Pascal's triangle.	1, 3, 3, 1
2 Expand by decreasing the powers of the first term and increasing the powers of the second term from left to right.	$(2x-1)^3$ $= 1(2x)^3(-1)^0 + 3(2x)^2(-1)^1 + 3(2x)^1(-1)^2 + 1(2x)^0(-1)^3$
3 Simplify the expression.	$= (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3$ $= 8x^3 - 12x^2 + 6x - 1$

(d) $(2p+3q)^3$

1 Identify the coefficients from row 3 in Pascals triangle.	1, 3, 3, 1
2 Expand by decreasing the powers of the first term and increasing the powers of the second term from left to right.	$(2p+3q)^3$ $= 1(2p)^3 + 3(2p)^2(3q)^1 + 3(2p)^1(3q)^2 + 1(3q)^3$
3 Simplify the expression.	$= (2p)^3 + 3(2p)^2(3q) + 3(2p)(3q)^2 + (3q)^3$ $= 8p^3 + 36p^2q + 54pq^2 + 27q^3$

Four identities are useful in the expansion of quadratic and cubic functions. These are the *square of a sum* and the *square of a difference* for quadratics, and similarly the *cube of a sum* and the *cube of a difference* for cubics.

Square of a sum: $(a+b)^2 = a^2 + 2ab + b^2$

Square of a difference: $(a-b)^2 = a^2 - 2ab + b^2$

Cube of a sum: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Cube of a difference: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

WARNING

Be careful when expanding the squares and cubes of sums. Avoid these common mistakes:

$$(a \pm b)^2 \neq a^2 \pm b^2$$
$$(a \pm b)^3 \neq a^3 \pm b^3$$

When there is a product of a binomial power and another factor, it is usual to expand the power first.

9 Product of a binomial power and other factors

Expand the following expressions and simplify by collecting like terms.

(a) $2(x+3)^3$

THINKING	WORKING
1 Expand the power first using the cube of a sum $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.	$2(x+3)^3$ $= 2\left(x^3 + 3x^2 \times 3 + 3x \times 3^2 + 3^3\right)$ $= 2\left(x^3 + 9x^2 + 27x + 27\right)$
2 Multiply each of the terms in the brackets by the factor at the front.	$= 2x^3 + 18x^2 + 54x + 54$

(b) $(3x-2)(x-7)^2$

1 Expand the square of a difference $(a-b)^2 = a^2 - 2ab + b^2$.	$(3x-2)(x-7)^2$ $= (3x-2)\left(x^2 - 14x + 49\right)$
2 Multiply each of the terms in the brackets by the factor at the front, then collect like terms.	$= 3x\left(x^2 - 14x + 49\right) - 2\left(x^2 - 14x + 49\right)$ $= 3x^3 - 42x^2 + 147x - 2x^2 + 28x - 98$ $= 3x^3 - 44x^2 + 175x - 98$

(c) $-x(3x-2)^2$

1 Expand the square of a difference $(a-b)^2 = a^2 - 2ab + b^2$.	$-x(3x-2)^2$ $= -x\left((3x)^2 - 2(3x)(2) + 2^2\right)$ $= -x\left(9x^2 - 12x + 4\right)$
2 Multiply each of the terms in the brackets by the factor at the front.	$= -9x^3 + 12x^2 - 4x$

EXERCISE 3.2 Expanding cubic polynomials

Worked Example

6 **1** Expand each of the following cubic polynomials.

(a) $5(x^3-3)$ (b) $-2(x^3+4)$ (c) $3x(2x^2+9)$

(d) $-x(3x^2+4)$ (e) $-x(x+3)(x-3)$ (f) $7x(x-2)(x+2)$

2 Use your preferred method to expand the following expressions and simplify by collecting like terms.

(a) $(x+1)(x-3)$ (b) $(y-1)(y+4)$ (c) $(7-m)(m+6)$

(d) $(10+p)(10-p)$ (e) $2(n+4)(n-3)$ (f) $-3(4-t)(t+5)$

8 **3** Expand each of the following, using binomial expansion.

(a) $(c+d)^2$ (b) $(m-n)^2$ (c) $(k+l)^3$

(d) $(s-t)^3$ (e) $(z+2)^3$ (f) $(a-5)^3$

(g) $(2x+1)^2$ (h) $(3y-2)^2$ (i) $(4z+1)^3$

(j) $(5a-3)^3$ (k) $(x+4)^2$ (l) $(6-b)^2$

(m) $(y-3)^3$

4 Complete the following.

(a) Expand $(a+b)^3$ to demonstrate that $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

(b) Demonstrate that $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

5 What is $(7-3x)^3$ when fully expanded and simplified?

6 Consider the cubic expressions in expanded form:

(a) For $-2x(x+11)(x-11)$ the number of terms will be:

A 1 B 2 C 3 D 4

(b) For $4(x-4)^3$ the number of terms will be:

A 1 B 2 C 3 D 4

(c) Explain the common error made by a student who chose the second incorrect option in part (b).

9 **7** Expand the following expressions and simplify by collecting like terms.

(a) $3(x-1)^3$ (b) $-3(x+5)^3$

(c) $(x+4)(2x-1)^2$ (d) $(x-1)(x+3)^2$

(e) $2x(x-5)^2$ (f) $-5x(2x+3)^2$

8 Which of the following is *not* a possible expansion of the polynomial $F(x)=(x-2)(x+3)(2x-1)$?

A $(x-2)(2x^2+5x-3)$ B $(2x^2-5x+2)(x+3)$

C $2x^3+x^2+7x+6$ D $(x^2+x-6)(2x-1)$

Worked Example 7

9 Use your preferred method to expand the following expressions and simplify by collecting like terms. In your working, check your result by graphing functions for the original and final expressions.

(a) $(x+1)(x-4)(x-7)$

(b) $(y-3)(y+2)(y+4)$

(c) $(5-p)(p+2)(p+1)$

(d) $(3-m)(m+2)(3+m)$

10 Use your preferred method to expand the following expressions and simplify by collecting like terms.

(a) $3(x+2)(x-1)(x+5)$

(b) $-2(x-1)(x-2)(x-3)$

(c) $-(x+3)(x+1)(x+2)$

(d) $6(x-2)(x+1)(x+3)$

11 Simplify $(x+y)^3+(xy)^3+2(x-y)^2(x+y)-3(x+y)^2(x-y)$.

12 The coefficient of x^2 is 4 in the expansion of $(x+m)^2(x-2)$. Calculate the value of m.

13 Given $(5x^2+3x+2)(x-3)=Ax^3+(-2A+B)x^2+(3B-1)x-C$, determine the values of A, B and C.

14 Determine the value of the integer x if $\left(\sqrt{2}+1\right)^3=\sqrt{x}+\sqrt{x-1}$.

15 Two simultaneous equations are given: $a+b=1$ and $a^3+b^3=19$.

(a) Determine the solutions for a and b that make these equations true.

(b) Calculate the value of a^2+b^2.

16 A feeding trough is made from a rectangular sheet of metal 80 cm by 52 cm by cutting squares of side length x cm from each of the four corners, then folding the sides up to make an open rectangular prism.

(a) Create a formula in terms of x for the volume $V(x)$ of the trough (in cm^3).

(b) Determine the x-intercepts for $V(x)$.

(c) Using technology, or otherwise, sketch the graph of V against x. Clearly identify all intercepts and turning points.

(d) Plot the graph over the valid domain.

(e) For the domain specified in the previous part, determine the value of x for which V is maximum. Hence determine the maximum volume of the trough.

3.3 Factorising cubic polynomials

Expanding and factorising

Expanding and factorising are *inverse* processes. Expanding is the process of multiplying numbers or algebraic expressions together. Factorising is the process of writing a number or algebraic expression as the product of factors. For example, $3458 = 2 \times 7 \times 13 \times 19$ and $294 = 2 \times 3 \times 5^2$ are factorised forms.

Factorising cubic polynomials when *x* is a factor

To determine whether x is a factor of a polynomial expression, look for a constant term. For example, x is a factor of $5x^3 - 7x = x \times 5x^2 - x \times 7$ but it is *not* a factor of the expression $5x^3 - 7x - 8$, because it is not a factor of the constant term.

10 Factorise cubics when *x* is a factor

Factorise the following cubic expressions.

(a) $2x^3 + x$

THINKING	WORKING
1 Factorise the expression by firstly taking out the highest common factor.	$2x^3 + x = x \times 2x^2 + x \times 1$ $= x\left(2x^2 + 1\right)$
2 Check the quadratic expression for factors, by inspection or using the determinant $\Delta = b^2 - 4ac$.	For the factor $2x^2 + 1$: $b^2 - 4ac = 0 - 4 \times 2 \times 1$ $= 0 - 8$ $= -8$ Given that $\Delta < 0$, there are no real factors.

(b) $x^3 + 4x^2 + 3x$

1 Factorise the expression by firstly taking out the highest common factor.	$x^3 + 4x^2 + 3x = x\left(x^2 + 4x + 3\right)$
2 Check the quadratic expression for factors, by inspection.	$= x\left(x^2 + x + 3x + 3\right)$ $= x\left(x(x+1) + 3(x+1)\right)$ $= x(x+1)(x+3)$

(c) $-3x^3 - 12x^2 - 6x$

1 Factorise the expression by firstly taking out the highest common factor.	$-3x^3 - 12x^2 - 6x = -3x\left(x^2 + 4x + 2\right)$

2 Check the quadratic expression for factors, by inspection or using the determinant $\Delta = b^2 - 4ac$.
If Δ is a perfect square, then there will be rational factors.

For the factor $x^2 + 4x + 2$:

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4 \times 1 \times 2 \\ &= 16 - 8 \\ &= 8 \end{aligned}$$

Given that $\Delta = 8$, the expression is not a perfect square, so further factorisation will involve surds.

Polynomial expressions can be factorised using linear factors, constants and quadratic factors.

Factors and remainders

To decide whether one number is a factor of another, simply divide and see if there is a remainder, as shown in the following examples.

Is 11 a factor of 2497?

$2497 \div 11 = 227$

The answer is *yes* because 11 divides into 2497 exactly.

There is no remainder (zero remainder):

$2497 = 11 \times 227 + 0 = 11 \times 227$

Therefore 11 is a factor of 2497.

Is 7 a factor of 2497?

$2497 \div 7 = 356.7\ldots$

The answer is *no* because there is a non-zero decimal remainder part.

$2497 \div 7 = 356\frac{5}{7}$ or

$2497 \div 7 = 356$ with remainder 5.

This can be rewritten as $2497 = 7 \times 356 + 5$.

Re-expression of a polynomial in terms of $(x - a)$

A similar approach can be developed for polynomials and their factors. There is a straightforward way to determine whether a linear expression $(x - a)$ is a factor of a polynomial $P(x)$ or not. The first step is to write $P(x)$ in the form:

$P(x) = (x - a)Q(x) + R$

where $Q(x)$ is a polynomial of degree one less than the degree of $P(x)$, and R the remainder is a real number.

Any polynomial $P(x)$ of degree $n > 0$ can be written in the form:

$P(x) = (x - a)Q(x) + R$,

where $Q(x)$ is a polynomial of degree $n - 1$ and R the remainder is a real number.

If $P(x)$ is divided by $(x - a)$, then:

$$\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$$

The term $(x - a)$ is called the divisor and $Q(x)$ is called the quotient.

To express $P(x)$ in the form of $P(x) = (x - a)Q(x) + R$, you can use two methods: equating coefficients of like terms or long division. The following examples demonstrate how to use both methods to express $P(x) = 2x^3 - 4x^2 + 6x - 7$ in the form $(x + 1)Q(x) + R$.

Method 1: Re-expression by equating coefficients

Let $2x^3 - 4x^2 + 6x - 7 = ax^2(x+1) + bx(x+1) + c(x+1) + d$

$a = 2$ to obtain $2x^3$ on the right-hand side:	$2x^3 - 4x^2 + 6x - 7$ $= 2x^2(x+1) + bx(x+1) + c(x+1) + d$
Expanding the first term would also give $2x^2$, but $-4x^2$ is required, so $6x^2$ must be subtracted. This means $b = -6$:	$= 2x^2(x+1) + -6x(x+1) + c(x+1) + d$
Expanding the second term would also give $-6x$, but $+6x$ is required, so $12x$ must be added. This means $c = 12$:	$= 2x^2(x+1) - 6x(x+1) + 12(x+1) + d$
Expanding the third term would also give $+12$, but -7 is required, so 19 must be subtracted. This means $d = -19$:	$= 2x^2(x+1) - 6x(x+1) + 12(x+1) - 19$
Collecting the terms with factor $(x+1)$ together gives:	$= (x+1)(2x^2 - 6x + 12) - 19$

With practice these steps can be done as two lines of working: one line for determining the values of a, b, c and d, and the second line. Written in the form $Q(x)(x-a) + R$: The quotient $Q(x)$ is $2x^2 - 6x + 12$ and the remainder is -19.

Method 2: Long division

Set up a long division. $x+1$ is the divisor and $2x^3 - 4x^2 + 6x - 7$ is the dividend	$x+1\overline{)2x^3 - 4x^2 + 6x - 7}$
Divide the first term of the dividend by the first term of the divisor: $\frac{2x^3}{x} = 2x^2$ This is the first term of the quotient, so write on top of the division line.	$\begin{array}{r} 2x^2 \\ x+1\overline{)2x^3 - 4x^2 + 6x - 7} \end{array}$
Calculate the product of this term of the quotient by the divisor. $2x^2(x+1) = 2x^3 + 2x^2$ Write these terms under the like terms in the dividend.	$\begin{array}{r} 2x^2 \\ x+1\overline{)2x^3 - 4x^2 + 6x - 7} \\ -\underline{(2x^3 + 2x^2)} \end{array}$
Complete the subtraction and drop down the next term in the dividend.	$\begin{array}{r} 2x^2 \\ x+1\overline{)2x^3 - 4x^2 + 6x - 7} \\ -\underline{(2x^3 + 2x^2)} \downarrow \\ 0 - 6x^2 + 6x \end{array}$
Divide the first term of the subtraction by the first term of the divisor: $\frac{-6x^2}{x} = -6x$ This is the second term of the quotient, so write on top of the division line.	$\begin{array}{r} 2x^2 - 6x \\ x+1\overline{)2x^3 - 4x^2 + 6x - 7} \\ -\underline{(2x^3 + 2x^2)} \\ 0 \quad -6x^2 + 6x \end{array}$

Calculate the product of this term of the quotient by the divisor. $-6x(x+1) = -6x^2 - 6x$ Write this expression under the like terms in the dividend.	$\begin{array}{r} 2x^2 - 6x \\ x+1\overline{\big)2x^3 - 4x^2 + 6x - 7} \\ -\underline{\left(2x^3 + 2x^2\right)} \\ 0 - 6x^2 + 6x \\ -\underline{\left(-6x^2 - 6x\right)} \end{array}$
Complete the subtraction and drop down the next term in the dividend.	$\begin{array}{r} 2x^2 - 6x \\ x+1\overline{\big)2x^3 - 4x^2 + 6x - 7} \\ -\underline{\left(2x^3 + 2x^2\right)} \\ 0 - 6x^2 + 6x \\ -\underline{\left(-6x^2 - 6x\right)} \downarrow \\ 0 \quad + 12x - 7 \end{array}$
Divide the first term of the subtraction by the first term of the divisor: $\frac{12x}{x} = 12$ This is the third term of the quotient, so write on top of the division line.	$\begin{array}{r} 2x^2 - 6x + 12 \\ x+1\overline{\big)2x^3 - 4x^2 + 6x - 7} \\ -\underline{\left(2x^3 + 2x^2\right)} \\ 0 - 6x^2 + 6x \\ -\underline{\left(-6x^2 - 6x\right)} \\ 0 \quad + 12x - 7 \end{array}$
Calculate the product of this term of the quotient by the divisor. $12(x+1) = 12x + 12$ Write these terms under the like terms in the dividend.	$\begin{array}{r} 2x^2 - 6x + 12 \\ x+1\overline{\big)2x^3 - 4x^2 + 6x - 7} \\ -\underline{\left(2x^3 + 2x^2\right)} \\ 0 - 6x^2 + 6x \\ -\underline{\left(-6x^2 - 6x\right)} \\ 0 \quad + 12x - 7 \\ -\underline{(12x + 12)} \end{array}$
Complete the subtraction to calculate the remainder.	$\begin{array}{r} 2x^2 - 6x + 12 \\ x+1\overline{\big)2x^3 - 4x^2 + 6x - 7} \\ -\underline{\left(2x^3 + 2x^2\right)} \\ 0 - 6x^2 + 6x \\ -\underline{\left(-6x^2 - 6x\right)} \\ 0 + 12x - 7 \\ -\underline{(12x + 12)} \\ 0 \quad - 19 \end{array}$

Written in the form $Q(x)(x-a)+R$:

$$P(x)=(x-1)\left(2x^2-6x+12\right)-19$$

The quotient is $2x^2-6x+12$ and the remainder is -19.

With practice these steps can be done as a single set of working, followed by identifying the divisor, quotient and remainder and writing in the form $P(x)=(x-a)Q(x)+R$.

WARNING

Do not ignore 'zero terms' in polynomials when determing factors or remainders. For example, the expression x^3+2 must be written as x^3+0x^2+0x+2.

Additional information

Long division of polynomials

Practise completing long division of polynomials.

11 Divide a polynomial by a linear factor

Determine the quotient $Q(x)$ and the remainder R when $x^3-8x^2+20x-15$ is divided by $(x-3)$, using re-expression by equating coefficients.

THINKING	WORKING
1 Write the polynomial in the form $P(x)=Q(x)(x-3)+R$. Expand the polynomial and highlight corresponding terms.	$x^3-8x^2+20x-15$ $=ax^2(x-3)+bx(x-3)+c(x-3)+d$ $=ax^3-3ax^2+bx^2-3bx+cx-3c+d$ $=ax^3+(b-3a)x^2+(c-3b)x+(d-3c)$
2 Equate the x^3-terms to determine the value of a.	$x^3=ax^3$ $a=1$
3 Equate the x^2-terms to determine the value of b.	$-8x^2=(b-3a)x^2$ $-8=(b-3(1))$ $b=-5$
4 Equate the x-terms to determine the value of c.	$20x=(c-3b)x$ $20=c-3(-5)$ $20=c+15$ $c=5$
5 Equate the constant terms to determine the value of d.	$-15=d-3c$ $-15=d-3(5)$ $-15=d-15$ $d=0$
6 Collect the $(x-3)$ terms. Write the answer in the form $P(x)=(x-a)Q(x)+R$ and identify the value of the quotient and remainder.	$x^3-8x^2+20x-15=(x-3)\left(x^2-5x+5\right)$ The quotient is $Q(x)=x^2-5x+5$. The remainder is 0.

12 Divide a polynomial by a linear factor

Determine the quotient $Q(x)$ and the remainder R when $x^3 - 8x^2 + 20x - 15$ is divided by $(x - 3)$ using long division.

THINKING	WORKING
1 Set up the long division template and divide the divisor into the dividend.	$$\begin{array}{r l} & x^2 - 5x + 5 \quad [Q(x)] \\ x-3 \overline{\big)\, x^3 - 8x^2 + 20x - 15} & [P(x)] \\ -\underline{(x^3 - 3x^2)} & \\ -5x^2 + 20x & \\ -\underline{(-5x^2 + 15x)} & \\ 5x - 15 & \\ -\underline{(5x - 15)} & \\ 0 & [R] \end{array}$$
2 Write the answer in the form $P(x) = (x - a)Q(x) + R$ and identify the value of the quotient and remainder.	$x^3 - 8x^2 + 20x - 15 = (x - 3)(x^2 - 5x + 5)$ The quotient is $Q(x) = x^2 - 5x + 5$ The remainder is 0.

Remainder and factor theorems

Remainder theorem

If a polynomial is written in terms of a linear expression $(x - a)$ and $P(x) = (x - a)Q(x) + R$, then the remainder R must either be zero or some non-zero number.

- If $R = 0$, then $(x - a)$ is a factor of $P(x)$ and $P(x) = (x - a)Q(x)$.
- If $R \neq 0$, then $(x - a)$ is not a factor of $P(x)$ and there is a non-zero remainder.

When $P(x)$ is divided by $(x - a)$, the remainder is $P(a) = R$.

Factor theorem

The factor theorem is a special case of the remainder theorem for $R = 0$.
If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$ and the remainder is zero. Similarly if $(x + a)$ is a factor of $P(x)$ then $P(-a) = 0$ and the remainder is 0.

You can test whether $(x - a)$ is a factor of $P(x)$ without having to do the long division or the re-expression, simply by evaluating $P(a)$.

- If $P\left(\frac{b}{a}\right) = 0$, then $(ax - b)$ or $a\left(x - \frac{b}{a}\right)$ is a factor of $P(x)$.
- If $P\left(\frac{-b}{a}\right) = 0$, then $(ax + b)$ or $a\left(x + \frac{b}{a}\right)$ is a factor of $P(x)$.

Rational root theorem

The rational root theorem can be used to determine the integer values of a and b for which $(ax+b)$ is a factor of a cubic polynomial $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

Consider $a_3x^3 + a_2x^2 + a_1x + a_0 = (ax+b) \times Q(x)$, where $Q(x)$ is a quadratic polynomial. If you expand the right-hand side of this expression, you would see that a is a factor of a_3 and b is a factor of a_0. You can test all possible sets of integer factors for $\pm\frac{b}{a}$:

- If $P\left(\frac{-b}{a}\right) = 0$ and a is an integer factor of a_3 and b is an integer factor of a_0, then there is a rational root.
- If $P\left(\frac{-b}{a}\right) \neq 0$ for any combination, then there is no rational root.

For $P(x) = 2x^3 - x^2 - 14x + 7$, $a_3 = 2$ and $a_0 = 7$. Therefore a is a factor of 2 and b is a factor of 7. Possible combinations of a are ±1 or ±2 and b are ±1 or ±7, so test $\pm\frac{b}{a}$: $\pm1, \pm\frac{1}{2}, \pm7$ and $\pm\frac{7}{2}$.

Trying all of these combinations shows that only $P\left(\frac{1}{2}\right) = 0$, therefore $(2x-1)$ is a factor of $P(x)$.

$$\begin{aligned} P(x) &= 2x^3 - x^2 - 14x + 7 \\ &= (2x-1)(x^2-7) \end{aligned}$$

Using re-expression or polynomial division, the quadratic factor of $x^2 - 7$ is found, which has no rational factors.

Additional information

Rational root theorem

Try the activity to practise using the rational root theorem.

A polynomial $P(x)$ can always be written in the form: $P(x) = (x-a)Q(x) + R$

- If $P(x) = (x-a)Q(x) + R$, then $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$.

The remainder theorem states that $P(a) = R$.

The factor theorem states that if $P(a) = 0$, then $(x-a)$ is a factor of $P(x)$.

- If $P(-a) = 0$, then $x = -a$ is a solution and $(x+a)$ is a factor of $P(x)$.
- If $P\left(\frac{b}{a}\right) = 0$, then $x = \frac{b}{a}$ is a solution and $(ax-b)$ is a factor of $P(x)$.
- If $P\left(\frac{-b}{a}\right) = 0$, then $x = -\frac{b}{a}$ is a solution and $(ax+b)$ is a factor of $P(x)$.

13 Determine the remainder when a polynomial is divided by a linear term

Determine the remainder when $P(x) = 2x^3 - 3x^2 + x + 1$ is divided by:

(a) $x - 3$

THINKING	WORKING
1 Using the remainder theorem, determine the value of x to be substituted.	$x - 3 = 0$ $x = 3$

2 Substitute $x = a$ to calculate $P(a)$. $P(x)$ is divided by a linear expression of the form $(x - a)$, so calculate $P(a)$.	$P(x) = 2x^3 - 3x^2 + x + 1$ $P(3) = 2(3)^3 - 3(3)^2 + (3) + 1$ $= 2 \times 27 - 3 \times 9 + 3 + 1$ $= 54 - 27 + 3 + 1$ $= 31$
3 Interpret the result.	When $P(x) = 2x^3 - 3x^2 + x + 1$ is divided by $(x - 3)$ the remainder is 31.

(b) $4x + 1$

1 Using the remainder theorem, determine the value of x to be substituted. $P(x)$ is divided by a linear expression of form $(ax + b)$, calculate $P\left(-\frac{b}{a}\right)$.	$4x + 1 = 0$ $4x = -1$ $x = -\frac{1}{4}$
2 Substitute $x = -\frac{b}{a}$ to determine $P\left(-\frac{b}{a}\right)$.	$P(x) = 2x^3 - 3x^2 + x + 1$ $P\left(-\frac{1}{4}\right) = 2\left(-\frac{1}{4}\right)^3 - 3\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right) + 1$ $= 2 \times -\frac{1}{64} - 3 \times \frac{1}{16} - \frac{1}{4} + 1$ $= -\frac{1}{32} - \frac{3}{16} - \frac{1}{4} + 1$ $= -\frac{1}{32} - \frac{6}{32} - \frac{8}{32} + \frac{32}{32}$ $= \frac{17}{32}$
3 Interpret the result.	When $P(x) = 2x^3 - 3x^2 + x + 1$ is divided by $(4x + 1)$ the remainder is $\frac{17}{32}$.

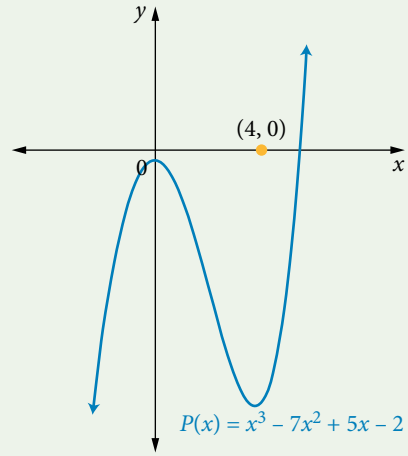
14 Determine whether a polynomial has a given linear term as a factor

Determine whether the linear term $(x - 4)$ is a factor of each polynomial. Verify your answer graphically using technology.

(a) $P(x) = x^3 - 7x^2 + 5x - 2$

THINKING	WORKING
1 Calculate $P(a)$. If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.	For $x - 4$ to be a factor of $P(x)$, $P(4)$ must equal 0. $P(4) = 4^3 - 7\left(4^2\right) + 5(4) - 2$ $= -30$
2 Interpret the result.	$P(4) \neq 0$, so $(x - 4)$ is not a factor of $P(x)$.

3 Verify your answer graphically using technology. If $(x - a)$ is a factor, then $(a, 0)$ is an x-intercept of the graph.



4 Interpret the result.

The point (4, 0) is not an x-intercept therefore $x = 4$ is not a solution to $P(x) = 0$ and $(x - 4)$ is not a factor of $P(x)$.

(b) $P(x) = x^3 - 3x^2 - 2x - 8$

1 Calculate $P(a)$. If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.

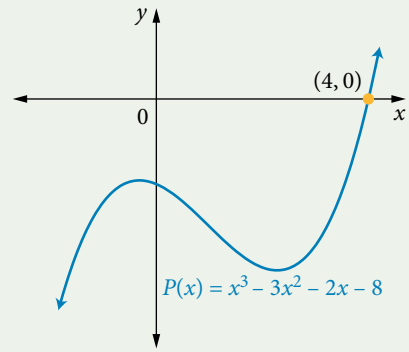
For $x - 4$ to be a factor of $P(x)$, $P(4)$ must equal 0.

$$\begin{aligned} P(4) &= 4^3 - 3(4)^2 - 2(4) - 8 \\ &= 64 - 48 - 8 - 8 \\ &= 0 \end{aligned}$$

2 Interpret the result.

$(x - 4)$ is a factor of $P(x)$.

3 Verify your answer graphically using technology. If $(x - a)$ is a factor, then $(a, 0)$ is an x-intercept of the graph.



4 Interpret the result.

The point (4, 0) is an x-intercept therefore $x = 4$ is a solution to $P(x) = 0$ and $(x - 4)$ is a factor of $P(x)$.

Factorising cubic polynomials using the factor theorem

The factor theorem can be used to determine an initial factor of a cubic expression. You can continue using the factor theorem to determine the remaining factors, or you can use re-expression or polynomial division or the factorisation methods if the quotient is a quadratic.

15 Factorise cubics using the factor theorem

Factorise each of the following cubic equations.

(a) $P(x) = x^3 - x^2 - 9x + 9$

THINKING	WORKING
1 Consider the constant term to determine values of a to try as linear factors.	The constant term is 9. The factors of 9 are $\{1, 3, 9\}$. Try $a = \pm 1, \pm 3, \pm 9$
2 If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.	$P(x) = x^3 - x^2 - 9x + 9$ $P(1) = 1^3 - 1^2 - 9(1) + 9$ $= 1 - 1 - 9 + 9$ $= 0$ $\therefore x = 1$ is a solution to $P(x) = 0$ and $(x - 1)$ is a linear factor of $P(x)$.
3 Determine the quadratic factor.	$\begin{array}{r\|l} & x^2 \qquad -9 \\ \hline x-1 & x^3 - x^2 - 9x + 9 \\ & -(x^3 - x^2) \\ \hline & 0 \quad -9x \\ & -(-9x + 9) \\ \hline & 0 \end{array}$
4 Factorise the quadratic factor, if possible.	$x^2 - 9 = x^2 - 3^2$ $= (x + 3)(x - 3)$
5 Write the fully factorised expression.	$P(x) = (x - 1)(x + 3)(x - 3)$

(b) $P(x) = 6x^3 + 9x^2 + 9x + 3$

1 Take out the common factor.	$6x^3 + 9x^2 + 9x + 3 = 3(2x^3 + 3x^2 + 3x + 1)$
2 If $(x - a)$ is a factor of $P(x)$: $P(a) = 0$. Determine values of a to try.	Let $G(x) = 2x^3 + 3x^2 + 3x + 1$ Constant term is 1 and coefficient of x^3 is 2, so try $a = \pm\frac{1}{2}, \pm 1$. $G\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right) + 1$ $= -\frac{1}{4} + \frac{3}{4} - \frac{3}{2} + 1$ $= 0$ $\therefore x = -\frac{1}{2}$ is a solution to $G(x) = 0$ and $(2x + 1)$ is a linear factor of $G(x)$.

3 Use long division or re-expression to determine the quadratic factor.

$$
\begin{array}{r}
x^2 + x + 1 \\
2x+1 \overline{\big) 2x^3 + 3x^2 + 3x + 1} \\
-\underline{\left(2x^3 + x^2\right)} \\
2x^2 + 3x \\
-\underline{\left(2x^2 + x\right)} \\
2x + 1 \\
-\underline{\left(2x + 1\right)} \\
0
\end{array}
$$

4 Factorise the quadratic factor, if possible.

$x^2 + x + 1$ has no apparent factors.

$b^2 - 4ac = 1^2 - 4(1)(1)$
$= -3$

No further factorisation is possible.

5 Write the fully factorised expression.

$P(x) = 3(2x+1)\left(x^2 + x + 1\right)$

Factorise from a graph

Using technology, the roots of a cubic expression will appear as the x-intercepts on a graph of the function. Graphs will determine all real roots, rational as well as decimal approximations to surds.

16 Factorise cubics from a graph

Factorise the following cubic equations by first graphing the function. Round to 2 decimal places where necessary.

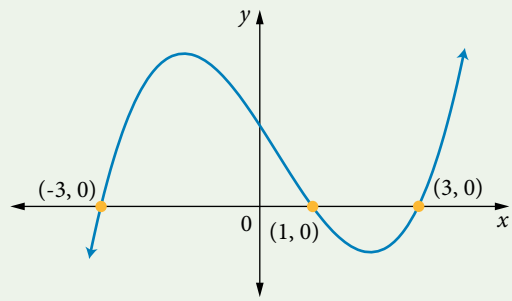
(a) $P(x) = x^3 - x^2 - 9x + 9$

THINKING

1 Graph the function and read the roots from the x-intercepts.

WORKING

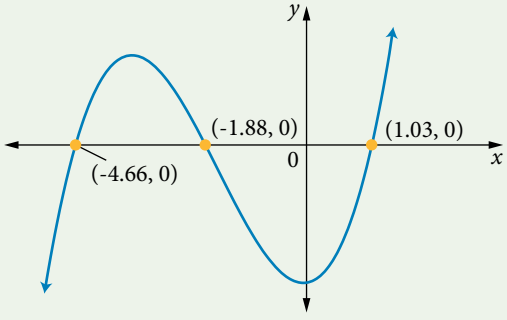
$y = x^3 - x^2 - 9x + 9$



The coordinates of the x-intercepts are (-3, 0), (1, 0) and (3, 0).

2 Create factors from the roots.	(-3, 0): $P(-3) = 0 \quad \therefore (x+3)$ is a factor. (1, 0): $P(1) = 0 \quad \therefore (x-1)$ is a factor. (3, 0): $P(3) = 0 \quad \therefore (x-3)$ is a factor.
3 Write the factors as part of the expression for $P(x)$, allowing for the coefficient of x^3.	$P(x) = a(x-1)(x+3)(x-3)$
4 Determine the value of a.	The coefficient of x^3 is 1 so $a = 1$.
5 Write the fully factorised expression.	$P(x) = (x-1)(x+3)(x-3)$

(b) $P(x) = 2x^3 + 11x^2 + 4x - 18$

1 Graph the function and read the roots from the x-intercepts.	$y = 2x^3 + 11x^2 + 4x - 18$  Correct to 2 decimal places, the x-intercepts are at (-4.66, 0), (-1.88, 0), (1.03, 0).
2 Create factors from the roots.	(-4.66, 0): $P(-4.66) = 0$, so $(x + 4.66)$ is a factor. (-1.88, 0): $P(-1.88) = 0$, so $(x + 1.88)$ is a factor. (1.03, 0): $P(1.03) = 0$, so $(x - 1.03)$ is a factor.
3 Write the factors as part of the expression for $P(x)$, allowing for the coefficient of x^3.	$P(x) = a(x + 4.66)(x + 1.88)(x - 1.03)$
4 Determine the value of a.	The coefficient of x^3 is 2 so $a = 2$.
5 Write the fully factorised expression.	$P(x) = 2(x + 4.66)(x + 1.88)(x - 1.03)$

An application of remainder theorem

Under certain circumstances, you can use your knowledge of the remainder theorem to determine the rule of a polynomial function.

17 Use remainders from linear terms to determine coefficients of a polynomial

Determine the values of unknowns a, b and c if $P(x) = x^3 + ax^2 + bx + c$ has remainders of 9, 21 and 6 when divided by $(x + 2)$, $(x - 1)$ and $(x + 3)$ respectively.

THINKING	WORKING
1 Identify the given information.	$(x + 2)$ gives remainder 9: $P(-2) = 9$ $(x - 1)$ gives remainder 21: $P(1) = 21$ $(x + 3)$ gives remainder 6: $P(-3) = 6$
2 Create three equations by substituting the given information.	For $P(-2) = 9$: $-8 + 4a - 2b + 1c = 9$ $4a - 2b + c = 17$ [1] For $P(1) = 21$: $1 + 1a + 1b + 1c = 21$ $a + b + c = 20$ [2] For $P(-3) = 6$: $-27 + 9a - 3b + 1c = 6$ $9a + 3b + c = 33$ [3]
3 Solve simultaneously for a, b and c.	[1] − [2]: $3a - 3b = -3$ $a - b = -1$ [4] [3] − [2]: $8a - 4b = 13$ [5] [4] × 4: $4a - 4b = -4$ [6] [5] − [6]: $4a = 17$ $a = 4.25$ Substitute $a = 4.25$ into [4]: $a - b = -1$ $4.25 - b = -1$ $b = 5.25$ Substitute $a = 4.25$ and $b = 5.25$ into [2]: $a + b + c = 20$ $4.25 + 5.25 + c = 20$ $c = 10.5$
4 Conclude by stating the results.	$a = 4.25$, $b = 5.25$, $c = 10.5$

Special patterns and factorising

You have already seen how to expand the cube of a sum, and the cube of a difference (See Worked example 9). These expansion identities can be applied in reverse to factorise a polynomial.

As well as these there are two new rules: the difference of cubes, and the sum of cubes.

Factorising the cube of a sum: $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$

Factorising the cube of a difference: $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$

Difference of cubes: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

Sum of cubes: $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

You can check that these rules are true by expanding the right-hand side of the equation in each case and showing that it results in the left-hand side of the equation.

Additional information

Factorisation examples

Practise applying the rules for factorisation.

18 Factorise cubic expressions

Factorise each of the following expressions.

(a) $x^3 + 1$

THINKING	WORKING
1 Write the expression as a sum of cubes and use the sum of cubes rule: $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$	$x^3 + 1 = x^3 + 1^3$
2 Apply the rule to factorise the expression.	$= (x+1)\left(x^2 - x \times 1 + 1^2\right)$ $= (x+1)\left(x^2 - x + 1\right)$

(b) $8p^3 - 27q^3$

1 Write the expression as a difference of cubes and use the difference of cubes rule: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$	$8p^3 - 27q^3 = (2p)^3 - (3q)^3$
2 Apply the rule to factorise the expression.	$= (2p - 3q)\left((2p)^2 + 2p \times 3q + (3q)^2\right)$ $= (2p - 3q)\left(4p^2 + 6pq + 9q^2\right)$

(c) $x^3 + 6x^2 + 12x + 8$

1 Recall the cube of a sum rule to be used to factorise the expression.	$a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$
2 Rewrite all terms so that they 'fit' the rule more closely.	$x^3 + 6x^2 + 12x + 8$ $= x^3 + 3x^2(2) + 3x(2)^2 + 2^3$
3 Apply the rule to factorise the expression completely.	$= (x+2)^3$

(d) $4c^3 - 24c^2d + 48cd^2 - 32d^3$

1 Recall the cube of a difference rule to be used to factorise the expression.	$a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$
2 Rewrite all terms so that they 'fit' the rule more closely. Write the expression and take out the highest common factor.	$4c^3 - 24c^2d + 48cd^2 - 32d^3$ $= 4\left(c^3 - 6c^2d + 12cd^2 - 8d^3\right)$ $= 4\left(c^3 - 3c^2(2d) + 3c(2d)^2 - (2d)^3\right)$
3 Apply the rule to factorise the expression completely.	$= 4(c-2d)^3$

(e) $8x^3 - 36x^2 + 54x - 27$

1 Recall the rule to be used to factorise the expression.	$a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$
2 Rewrite all terms so that they 'fit' the rule more closely.	$8x^3 - 36x^2 + 54x - 27$ $= (2x)^3 - 3\left(12x^2\right) + 3(18x) - (3)^3$ $= (2x)^3 - 3(2x)^2(3) + 3(2x)(3)^2 - (3)^3$
3 Apply the rule to factorise the expression completely.	$= (2x-3)^3$

EXERCISE 3.3 Factorising cubic polynomials

Worked Example

1 Factorise the following cubic expressions. Your answers should not involve surds. 10

(a) $3x^3+5x$ (b) $-4x^3-12x^2+24x$ (c) x^3+11x^2+30x

(d) $2x^3-50x$ (e) $-4x^3+5x^2-x$ (f) $4x^3+4x^2+12x$

2 Use re-expression to determine the quotient $Q(x)$ and the remainder R in the form $P(x)=(x-a)Q(x)+R$ when each polynomial is divided by the given linear expression. 11

(a) $2x^3+3x^2-3x+5$ divided by $2x+5$ (b) x^3-2x^2+5x-2 divided by $x-4$

(c) $x^3+9x^2+26x+24$ divided by $x+4$ (d) x^4-13x^3+36 divided by $x-2$

3 Use long division to determine the quotient $Q(x)$ and the remainder R in the form $P(x)=(x-a)Q(x)+R$ when each polynomial is divided by the given linear expression. 12

(a) x^3+2x^2-x+6 divided by $x-3$ (b) $x^3-x^2-10x+8$ divided by $x+3$

(c) $3x^3-x^2-x-1$ divided by $x-1$ (d) x^4-12x^2+35 divided by $x-1$

4 For each of the following polynomials, determine $P(1)$, $P(-1)$, $P(2)$ and $P\left(-\frac{1}{2}\right)$.

(a) $P(x)=x^3+2x^2-x+6$ (b) $P(x)=x^3-2x^2+5x-2$

(c) $P(x)=x^3+9x^2+26x+24$ (d) $P(x)=x^3-x^2-10x+8$

(e) $P(x)=3x^3-x^2-x-1$ (f) $P(x)=2x^3+3x^2-3x+5$

(g) $P(x)=x^4-12x^2+35$ (h) $P(x)=x^4-13x^3+36$

5 Determine the remainder when each of the following polynomials is divided by the given linear term. 13

(a) $P(x)=x^3+9x^2+26x-30$ divided by $x-3$ (b) $A(x)=x^3-4x^2+x+6$ divided by $x+1$

(c) $F(x)=x^3+2x^2-x-2$ divided by $x+1$ (d) $H(x)=x^3+7x^2-2x+4$ divided by $2x+1$

(e) $y=5x^3+12x^2-36x-16$ divided by $x+2$ (f) $y=6x^3-5x^2-2x+3$ divided by $x+5$

(g) $y=x^4-x^3-5x^2-2x+8$ divided by $x+2$ (h) $y=3x^4+2x^3-3x^2-4x+4$ divided by $3x-1$

6 Consider the polynomial function $f(x)=2x^4+x^3-4x^2-5x+6$. Which of the following is a factor of $f(x)$?

A x B $x-1$ C $x+3$ D $x+1$

7 Determine whether $D(x)=x+2$ is a factor of the following polynomials. 14

(a) $f(x)=x^2+4x+4$ (b) $g(x)=x^3+2x^2-2x+1$

(c) $h(x)=2x^3-7x^2-x+4$ (d) $f(x)=x^4+x^3-5x^2-2x+8$

8 Use a graphing method to determine whether $D(x)=2x+5$ is a factor of the following polynomials.

(a) $f(x)=2x^3+3x^2-9x-10$ (b) $f(x)=x^4+x^3-5x^2-2x+8$

(c) $h(x)=2x^3-7x^2-x+14$ (d) $g(x)=2x^3-9x^2-11x+60$

9 (a) Which of the following expressions cannot be factorised?

A x^2+1 B x^2-1 C x^3+1 D x^3-1

(b) Use the factor theorem to explain why x^3+a^3 can be factorised but x^2+a^2 cannot.

10 Factorise the following expressions.

(a) x^3+8 (b) $1-n^3$ (c) $2c^3+16$

(d) $8e^3+27f^3$ (e) $64-y^3$ (f) $5p^3-40q^3$

Worked Example

(g) $2m^2+2n^2$ (h) $4p^3-4q^3$

18 11 Factorise the following expressions completely, using the cube of a sum and cube of a difference rules as a guide.

(a) $x^3+3x^2y+3xy^2+y^3$ (b) $p^3+3p^2q+3pq^2+q^3$

(c) $2c^3+6c^2d+6cd^2+2d^2$ (d) $8e^3+24e^2f+24ef^2+8f^3$

(e) $x^3-3x^2y+3xy^2-y^3$ (f) $c^3-3c^2d+3cd^2-d^3$

(g) $2m^3-6m^2n+6mn^2-2n^3$ (h) $4p^3-12p^2q+12pq^2-4q^3$

(i) $8x^3+36x^2+54x+27$ (j) $27x^3-27x^2+9x-1$

16 12 Factorise the following cubic equations by first graphing the function. Round to 2 decimal places where necessary.

(a) $P(x)=x^3+4x^2-4x-16$ (b) $P(x)=3x^3+39x^2+141x+105$

(c) $P(x)=2x^3+5x^2-2x-4$ (d) $P(x)=-5x^3-6x^2+7x+5$

15 13 Fully factorise the following polynomials.

(a) $f(x)=x^3-4x^2+x+6$ (b) $f(x)=x^3+2x^2-x-2$

(c) $f(x)=x^3+4x^2-3x-18$ (d) $f(x)=x^3+3x^2-4x-12$

(e) $f(x)=2x^3-5x^2+x+2$ (f) $f(x)=3x^3-13x^2-32x+12$

(g) $f(x)=2x^3-3x^2-11x+6$ (h) $f(x)=2x^3+x^2-18x-9$

17 14 Determine the values of the constants p, q and r if px^2+qx+r has remainders of 10, 24.5 and 72 when divided by $(x-1)$, $(x-2)$ and $(x+3)$ respectively.

15 The polynomial $g(x)=x^4+px^3+qx^2+3x+2$ has one linear factor of $(x+2)$. If the polynomial has a remainder of 20 when divided by $(x-3)$, determine the values of p and q.

16 Given that there is a remainder of $1-k$ when $x^3+(2k-1)x^2+(k-1)x-3$ is divided by $x+k$, determine the value of k. Assume k is rational.

17 For a polynomial function of degree 3 that has factors of $(2x+1)$ and $(x-3)$:

(a) Determine four possible rules for the function, and then graph on the same set of axes. Try to choose functions that produce a variety of shapes for the cubic graphs.

(b) Describe graphs of such functions in general terms.

Graphs of power functions

3.4

Power functions

Polynomials include linear, quadratic, cubic and quartic functions. All of these polynomial functions have domain $\mathbb{R}$ and can be represented graphically on a Cartesian plane.

The simplest polynomial functions have rules of the form $P(x) = ax^n$, where n is an element of $\mathbb{Z}^+$, as well as the simple transformations of these. The graphs of these functions can be readily sketched on the Cartesian plane. A common feature of functions with rules of the form $P(x) = ax^n$ is that over an unrestricted domain $\mathbb{R}$, their graphs all pass through the origin (0, 0). Polynomial functions of this type are special types of *power functions*. These polynomial functions can be distinguished by their behaviour as x becomes a very large positive or negative value.

Explore further

Power graphs in the form $y = ax^n$

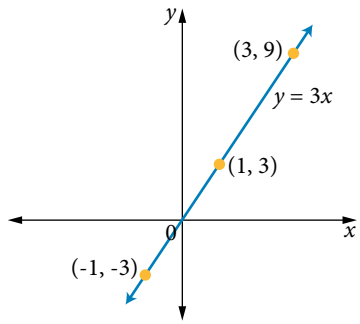
Explore the shapes of the functions in the form $y = ax^n$, where n is an element of $\mathbb{Q}$, the rational numbers.

Linear power functions

Linear functions are polynomials of degree 1 with rules of the form $P(x) = ax$. Their graphs are lines that pass through the origin (0, 0) with gradient a. For any linear power function with a positive gradient $a > 0$:

- as the x-values become large and positive and approach positive infinity ($x \to +\infty$), the y-values also become large and positive ($y \to +\infty$)
- as the x-values become large and negative and approach negative infinity ($x \to -\infty$), the y-values also become large and negative ($y \to -\infty$).

The following example shows a linear power function with a positive gradient, $y = 3x$.

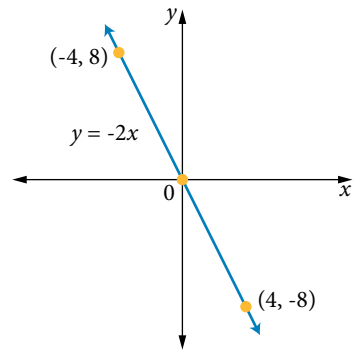


Linear power functions in the form $y = ax$ with a positive gradient $a > 0$ have a graph that behaves as follows:

as $x \to +\infty$, $y \to +\infty$, and as $x \to -\infty$, $y \to -\infty$.

The graph of $y = ax$ where $a > 0$ can be obtained from the graph of $y = x$ by a dilation parallel to the y-axis. This can also be described as a dilation from the x-axis.

The following example shows a linear power function with a negative gradient, $y = -2x$.



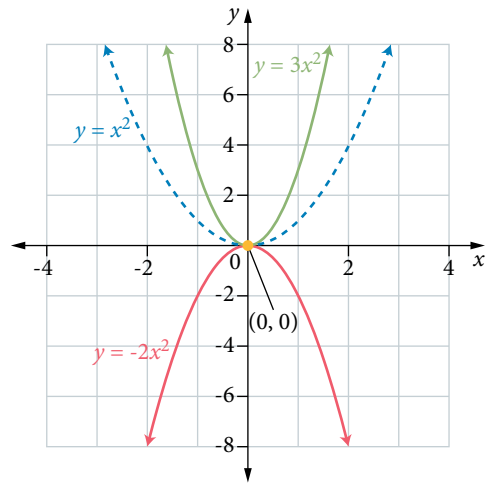
Linear power functions in the form $y = ax$ with a negative gradient $a < 0$ have a graph that behaves as follows:

as $x \to +\infty$, $y \to -\infty$, and as $x \to -\infty$, $y \to +\infty$.

The graph of $y = ax$ where $a < 0$ can be obtained from the graph of $y = x$ by a dilation parallel to the y-axis, followed by reflection in the x-axis.

Quadratic power functions

Quadratic power functions are polynomial functions of degree 2 with rules of the form $P(x) = ax^2$, where $a \neq 0$. Their graphs are parabolas that have a stationary point, or vertex, at the origin (0, 0) and have a line of symmetry at $x = 0$.



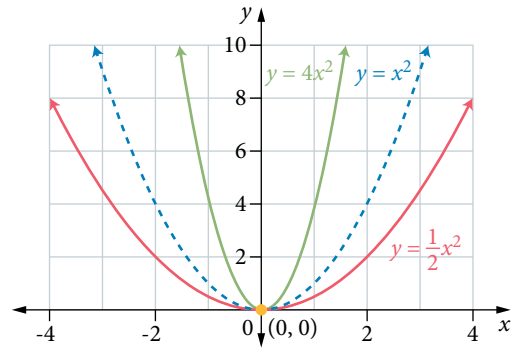
Concavity: sign of *a*

The sign of a in the quadratic function $y = ax^2$ where $a \neq 0$ indicates the *concavity* of the graph. Concavity refers to the overall shape and curvature of the parabola. The two types of concavity are *concave up* and *concave down*.

When a is positive ($a > 0$), the graph is concave up; when a is negative ($a < 0$), the graph is concave down. For example, the graph of $y = 3x^2$ is concave up and the graph of $y = -2x^2$ is concave down.

Dilation: magnitude of *a*

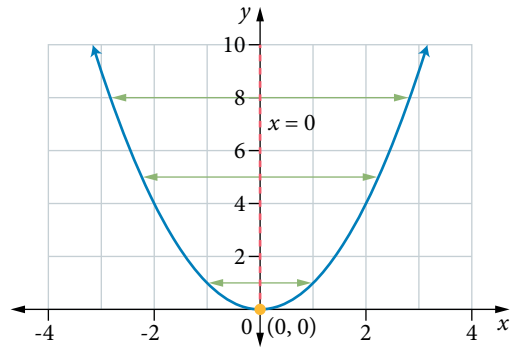
When the value of a changes in the rule $y = ax^2$ where $a \neq 0$, the graph is dilated parallel to the y-axis, which means that the graph becomes steeper or flatter than the original graph of $y = x^2$.



- If $a > 1$ or if $a < -1$, then the graph is stretched parallel to the y-axis. For example, the graph of $y = 4x^2$ has been stretched by a factor of 4 from the x-axis, so the graph is steeper than $y = x^2$.
- If $0 < a < 1$ or if $-1 < a < 0$ so $0 < |a| < 1$, then the graph is compressed parallel to the y-axis parallel. For example, the graph of $y = \frac{1}{2}x^2$ has been compressed by a factor of $\frac{1}{2}$ closer to the x-axis, so the graph is flatter than $y = x^2$.

The line of symmetry

The line of symmetry is the vertical line that bisects a parabola into two symmetrical mirror images. The quadratic power function graphs can be 'bisected' on the y-axis, or on the vertical line with equation $x = 0$.



Quadratic graphs as $x \to \pm\infty$

For any quadratic power function $y = ax^2$:

- if $a > 0$, the y-values become large and positive ($y \to +\infty$) as the x-values increase ($x \to +\infty$) and as the x-values decrease ($x \to -\infty$)
- if $a < 0$, the y-values become large and negative ($y \to -\infty$) as the x-values increase ($x \to +\infty$) and as the x-values decrease ($x \to -\infty$).

For quadratic power functions $P(x) = ax^2$ where $a \neq 0$, the behaviour can be summarised as follows:

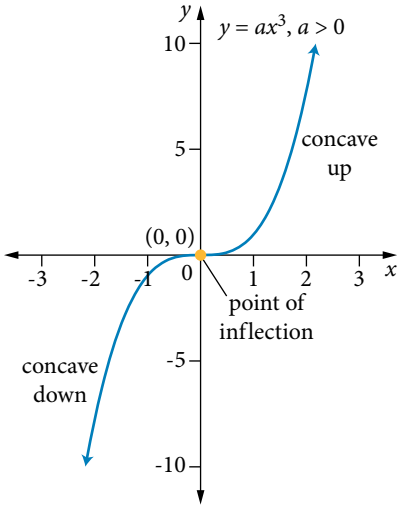
There is a line of symmetry at $x = 0$.

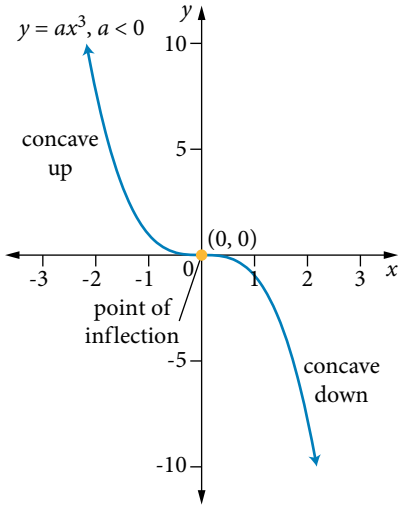
- For $a > 0$: As $x \to \infty$, $y \to \infty$; and as $x \to -\infty$, $y \to \infty$.
  The shape of the graph is 'concave up', like a ∪.
  The minimum turning point, or vertex, is at (0, 0).
- For $a < 0$: As $x \to \infty$, $y \to -\infty$; and as $x \to -\infty$, $y \to -\infty$.
  The shape of the graph is 'concave down', like a ∩.
  The maximum turning point, or vertex, is at (0, 0).

Cubic power functions

Cubic power functions are polynomials of degree 3 with rules of the form $P(x) = ax^3$, where $a \neq 0$. Their graphs have a point of inflection and half-turn rotational symmetry about the origin.

A *point of inflection* is a point on a graph of a function where the concavity of the graph changes. A point of inflection may be 'flat' where the gradient of the graph at that point is zero, or it may be 'oblique', where the gradient of the graph at that point is non-zero. Cubic power functions have a *stationary* point of inflection at the origin (0, 0). 'Stationary' means that the gradient of the tangent at that point is zero.

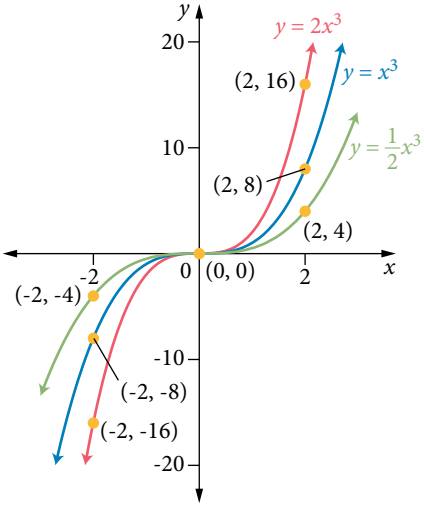




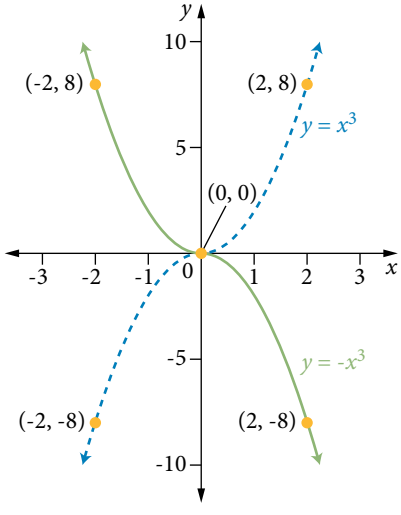
The value of *a*

As the value of a changes in the rule $y = ax^3$ where $a \neq 0$, the graph is dilated parallel to the y-axis (from the x-axis), which means that the graph becomes steeper or flatter than the original graph of $y = x^3$.

- If $a > 1$ or $a < -1$, so $|a| > 1$, then the graph is stretched parallel to the y-axis and becomes more steep. For example, the graph of $y = 2x^3$ has been stretched by a factor of 2 from the x-axis.
- If $0 < a < 1$ or if $-1 < a < 0$, so $0 < |a| < 1$, then the graph is compressed parallel to the y-axis. For example, the graph of $y = \frac{1}{2}x^3$ has been compressed by a factor of $\frac{1}{2}$ closer to the x-axis.



- If $a < 0$, then the graph of $y = x^3$ is dilated by a factor of $|a|$ parallel to the y-axis and is reflected in the x-axis.



C Making connections

The graph of $y = ax^3$

View the dilations and reflections of the cubic function in the form $y = ax^3$.

Cubic graphs as $x \to \pm\infty$

For any cubic power function $y = ax^3$ where $a > 0$:

- as x-values become large and positive ($x \to +\infty$), the y-values also become large and positive ($y \to +\infty$)
- as x-values become large and negative ($x \to -\infty$), the y-values also become large and negative ($y \to -\infty$)

If $a < 0$, the behaviour is different:

- as x-values become large and positive ($x \to +\infty$), the y-values become large and negative ($y \to -\infty$)
- as x-values become large and negative ($x \to -\infty$), the y-values become large and positive ($y \to +\infty$).

This behaviour is true for cubic functions in general and is not restricted to cubic power functions, so it holds true for all transformations of $y = ax^3$.

For cubic power functions $y = ax^3$ where $a \neq 0$:

- if $a > 0$: as $x \to \infty$, $y \to \infty$; and as $x \to -\infty$, $y \to -\infty$
- if $a < 0$: as $x \to \infty$, $y \to -\infty$; and as $x \to -\infty$, $y \to \infty$
- a stationary point of inflection is at (0, 0), where the gradient is zero and the concavity of the graph changes from 'concave up' to 'concave down' or vice versa.

19 Sketch graphs of polynomial power functions

Sketch the following functions on a Cartesian plane by finding several key points.

(a) $y = 3x$

THINKING

1 Substitute $x = -1, 0, 1$ into the equation to determine points on the curve, or consider that a gradient of +3 represents a vertical increase of 3 units for every horizontal increase of 1 unit.

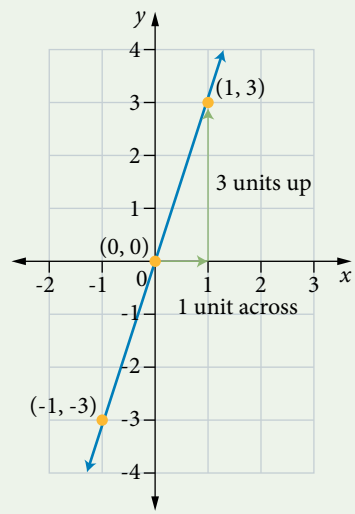
2 Plot points on the plane and connect them with a smooth continuous line. As this is a linear graph, extend the line in both directions.

WORKING

For $x = -1$:
$y = 3(-1)$
$= -3$
(-1, -3)

For $x = 0$:
$y = 3(0)$
$= 0$
(0, 0)

For $x = 1$:
$y = 3(1)$
$= 3$
(1, 3)



(b) $y = \frac{1}{2}x^2$

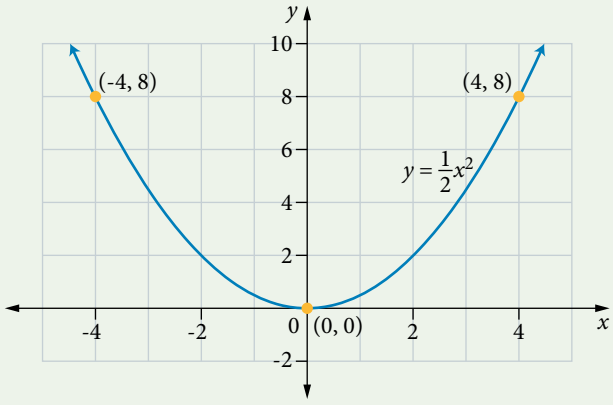
1 Substitute $x = -4, 0, 4$ into the equation to determine points on the curve.

For $x = -4$:
$y = \frac{1}{2}(-4)^2$
$= 8$
$(-4, 8)$

For $x = 0$:
$y = \frac{1}{2}(0)^2$
$= 0$
$(0, 0)$

For $x = 4$:
$y = \frac{1}{2}(4)^2$
$= 8$
$(4, 8)$

2 Plot these points on the plane and connect them with a smooth curve and an axis of symmetry at $x = 0$.



(c) $y = -2x^3$

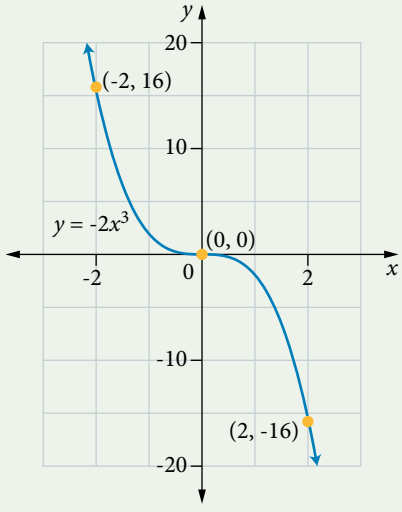
1 Substitute $x = -2, 0, 2$ into the equation to determine points on the curve.

For $x = -2$:
$y = -2(-2)^3$
$= 16$
$(-2, 16)$

For $x = 0$:
$y = -2(0)^3$
$= 0$
$(0, 0)$

For $x = 2$:
$y = -2(2)^3$
$= -16$
$(2, -16)$

2 Sketch these points on the plane and connect them with a smooth symmetrical curve.

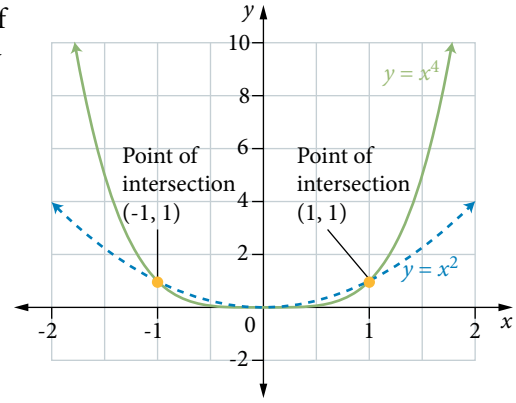


Quartic power functions

Quartic power functions are polynomials of degree 4 with rules of the form $P(x) = ax^4$, where $a \neq 0$. Their graphs have a stationary point at the origin and a line of symmetry at $x = 0$.

Note that as $\left(x^2\right)^2 = x^4$, the graph of $y = ax^4$ has some similarities to the graph of $y = ax^2$.

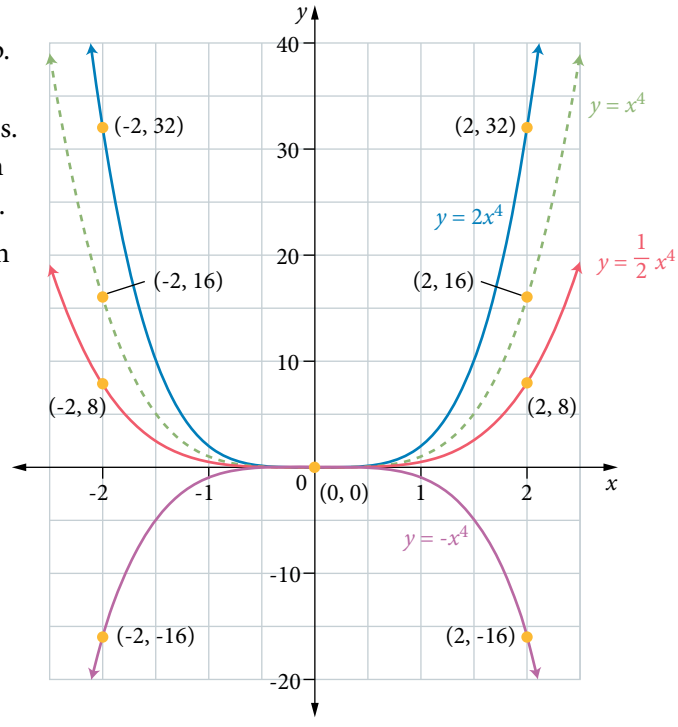
Consider the graphs of the two power functions $y = x^4$ and $y = x^2$ shown. The graph of $y = x^4$ is flatter than the graph of $y = x^2$ over the domain $-1 < x < 1$. Conversely, the graph of $y = x^4$ is steeper than $y = x^2$ over the domains $x < -1$ and $x > 1$. The graphs intersect at the points (-1, 1) and (1, 1).



The value of *a*

When the value of a changes in the rule $y = ax^4$ where $a \neq 0$, the graph is dilated parallel to the y-axis, which means that the graph becomes more steep or less steep than the original graph of $y = x^4$.

- If $a > 1$ or $a < -1$, then the graph is stretched parallel to the y-axis and becomes more steep. For example, the graph of $y = 2x^4$ has been stretched by a factor of 2 away from the x-axis.
- If $0 < a < 1$ or if $-1 < a < 0$, so $0 < |a| < 1$, then the graph is compressed parallel to the y-axis. For example, the graph of $y = \frac{1}{2}x^4$ has been compressed by a factor of $\frac{1}{2}$ closer to the x-axis.
- If $a < 0$, then the graph of $y = x^4$ has been dilated by a factor of $|a|$ parallel to the y-axis and is reflected in the x-axis.



Making connections

The graph of $y = ax^4$

View the dilations and reflections of the quartic function in the form $y = ax^4$.

For quartic power functions $y = ax^4$ where $a \neq 0$, their behaviour can be summarised as follows:

There is a line of symmetry at $x = 0$.

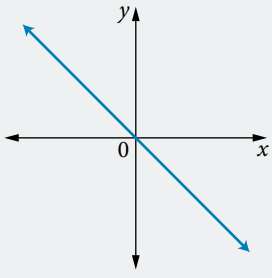
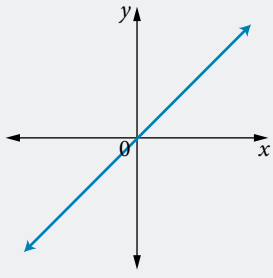
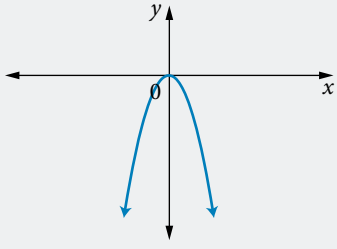
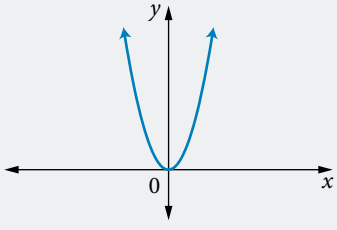
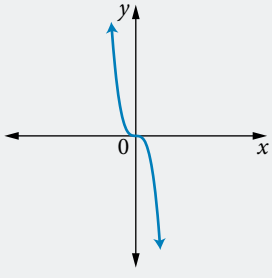
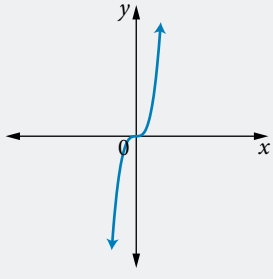
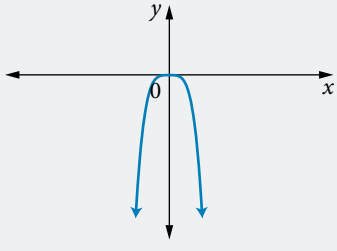
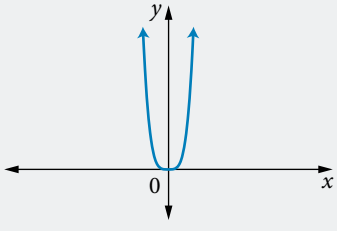
- For $a > 0$: as $x \to \infty$, $y \to \infty$; and as $x \to -\infty$, $y \to \infty$
  The shape of the graph is 'concave up', like a $\cup$ with a flatter base than a quadratic graph, and the minimum turning point, or vertex, is at (0, 0).
- For $a < 0$: as $x \to \infty$, $y \to -\infty$; and as $x \to -\infty$, $y \to -\infty$
  The shape of the graph is 'concave down', like a $\cap$ with a flatter top than a quadratic graph, and the maximum turning point or vertex is at (0, 0).

Odd and even functions

Graphs of functions with odd powers such as $y = ax^1$ and $y = ax^3$ have half-turn symmetry about the origin. The left-hand side of the graph is the negative of the right-hand side of the graph, so that $f(-x) = -f(x)$. Functions whose graphs have this property are said to be *odd* functions.

Graphs of functions with even powers such as $y = ax^2$ and $y = ax^4$ have reflection symmetry in the y-axis. The left-hand side of the graph is the same as the right-hand side of the graph, so that $f(-x) = f(x)$. Functions whose graphs have this property are said to be *even* functions.

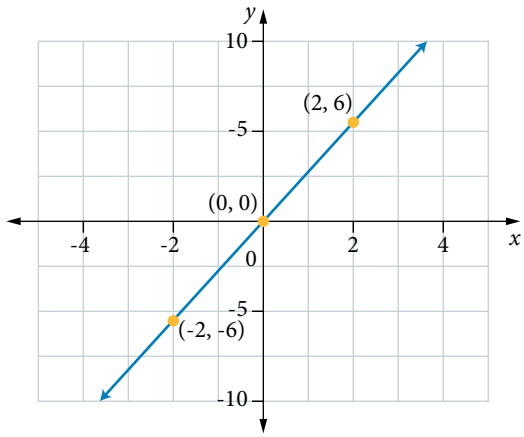
Here is a summary of the power functions $y = x^n$, for $n \in \{1, 2, 3, 4\}$.

Equation (rule) of graph	$a < 0$	$a > 0$
$y = ax$ Odd function		
$y = ax^2$ Even function		
$y = ax^3$ Odd function		
$y = ax^4$ Even function		

20 Determine the equation of a power function from a graph

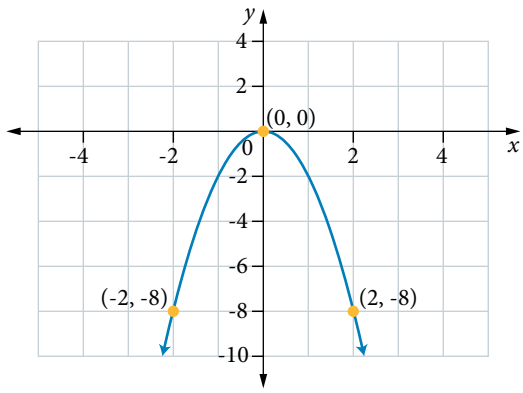
For each graph, determine the rule of the corresponding polynomial.

(a)



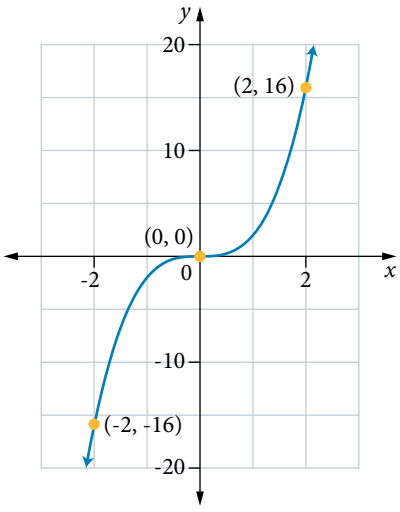
THINKING	WORKING
1 Identify the type of graph given and recall the equation in the form $y = ax^n$.	Linear function: $y = ax$.
2 Substitute a given point into the basic equation and solve for a.	Substitute (2, 6) into $y = ax$: $6 = a(2)$ $6 = 2a$ $a = 3$
3 Write the equation of the function.	$y = 3x$

(b)



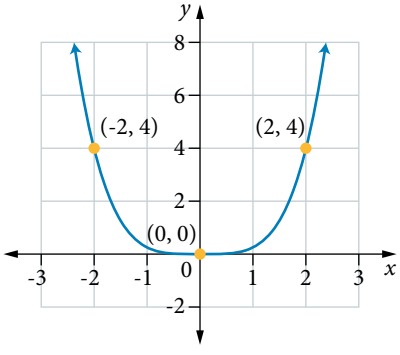
1 Identify the type of graph given and recall the equation in the form $y = ax^n$.	Quadratic function: $y = ax^2$.
2 Substitute a given point into the basic equation and solve for a.	Substitute (2, -8) into $y = ax^2$: $-8 = a(2)^2$ $-8 = 4a$ $a = -2$
3 Write the equation of the function.	$y = -2x^2$

(c)



1 Identify the type of graph given and recall the equation in the form $y = ax^n$.	Cubic function: $y = ax^3$.
2 Substitute a given point into the basic equation and solve for a.	Substitute (2, 16) into $y = ax^3$ $16 = a(2)^3$ $16 = 8a$ $a = 2$
3 Write the equation of the function.	$y = 2x^3$

(d)



1 Identify the type of graph given and recall the equation in the form $y = ax^n$.	Quartic function: $y = ax^4$.
2 Substitute a given point into the basic equation and solve for a.	Substitute (-2, 4) into $y = ax^4$ $4 = a(-2)^4$ $4 = 16a$ $a = \frac{1}{4}$
3 Write the equation of the function.	$y = \frac{1}{4}x^4$

Transformation form for a cubic function

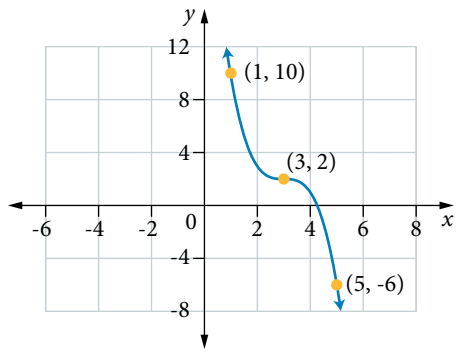
In general, the graph of any cubic power function $y = a(x-b)^3 + c$ where $a \neq 0$ can be obtained from the graph of $y = x^3$ by a combination of the following transformations in this order:

- for $a > 0$, a dilation by a factor of a parallel to the y-axis (from the x-axis)
- for $a < 0$, a dilation by a factor of $|a|$ parallel to the y-axis (from the y-axis) and a reflection in the x-axis
- a translation parallel to one or both axes, horizontally by b units or vertically by c units.

The coordinates of the point of inflection are (b, c).

21 Determine the rule of a cubic function in transformation form

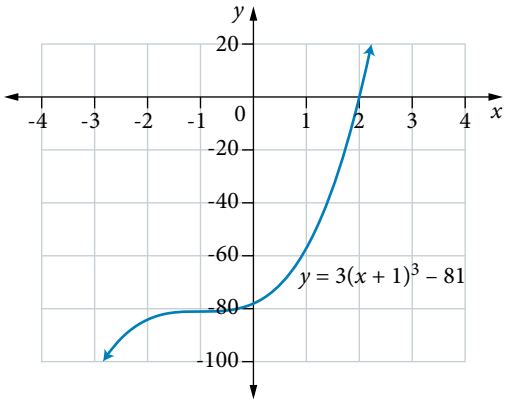
Determine the rule of the following cubic function.



THINKING	WORKING
1 Identify the type of graph and determine the equation in the form $y = a(x-b)^3 + c$ with a point of inflection at (b, c).	For (3, 2): $y = a(x-3)^3 + 2$ The shape of the graph indicates that $a < 0$.
2 Substitute a given point into the rule and solve for a.	Substitute (1, 10): $y = a(x-3)^3 + 2$ $10 = a(1-3)^3 + 2$ $-8 = 8a$ $a = -1$
3 Write the rule of the function.	$y = -(x-3)^3 + 2$

22 Determine key aspects of graphed cubic functions

Determine the coordinates of the y-intercept and the x-intercept, and the behaviour of $f(x)$ as $x \to \pm\infty$.



THINKING	WORKING
1 The y-intercept is found by substituting $x = 0$ into the rule and calculating the value of $f(0)$.	$f(0) = 3(0+1)^3 - 81$ $= -78$ The coordinates of the y-intercept are (0, -78).
2 The x-intercept(s) are found by substituting $y = 0$ into the rule and solving for x.	$0 = 3(x+1)^3 - 81$ $81 = 3(x+1)^3$ $27 = (x+1)^3$ $\sqrt[3]{27} = x+1$ $3 = x+1$ $x = 2$ The coordinates of the x-intercept are (2, 0).
3 Examining the shape of the graph, comment on the behaviour as $x \to \pm\infty$.	As $x \to \infty$, $y \to \infty$ And as $x \to -\infty$, $y \to -\infty$

23 Determine the parameters of cubic functions in transformation form

Determine the values of a, b, and c if the graph of $y = a(x-b)^3 + c$ has a stationary point of inflection at (1, 4) and passes through the point (2, 6).

THINKING	WORKING
1 Use the stationary point of inflection to substitute values for two of the parameters.	Use $y = a(x-b)^3 + c$ with (b, c) as (1, 4). $y = a(x-1)^3 + 4$
2 Substitute the given point to determine the value of a.	For (2, 6): $6 = a(2-1)^3 + 4$ $2 = 1a$ $a = 2$
3 Conclude by writing the values of the parameters.	$a = 2$, $b = 1$, $c = 4$
4 Write the rule of the function.	$y = 2(x-1)^3 + 4$

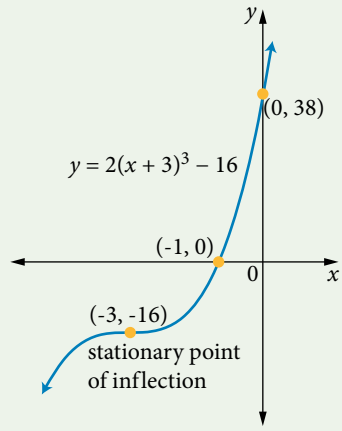
To sketch a graph in the form $y = a(x-b)^n + c$ determine the coordinates of:

- the stationary point (b, c)
- the y-intercept by letting $x = 0$ and calculating the value of y
- the x-intercept by letting $y = 0$ and solving for x.

Then plot the points on a set of labelled axes joining with a smooth curve.

24 Sketch cubic functions in transformation form, showing all relevant features

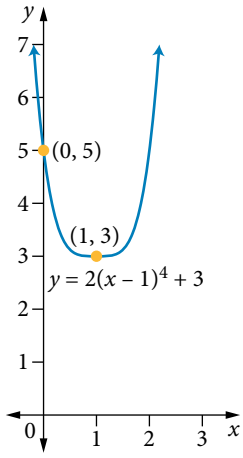
Sketch the graph of $y = 2(x+3)^3 - 16$, showing all relevant features.

THINKING	WORKING
1 Locate the stationary point of inflection (b, c) by comparing to the transformation form of a cubic $y = a(x-b)^3 + c$.	$y = 2(x+3)^3 - 16$ The coordiantes of the stationary point of inflection are (-3, -16).
2 Determine the coordinates of the y-intercept.	Let $x = 0$ to determine the y-intercept: $y = 2(0+3)^3 - 16$ $= 2 \times 27 - 16$ $= 54 - 16$ $= 38$ The coordinates of the y-intercept are (0, 38).
3 Determine the coordinates of the x-intercept.	Let $y = 0$ to determine the x-intercept: $0 = 2(x+3)^3 - 16$ $2(x+3)^3 = 16$ $(x+3)^3 = 8$ $x + 3 = \sqrt[3]{8}$ $x + 3 = 2$ $x = 2 - 3$ $x = -1$ The coordinates of the x-intercept are (-1, 0).
4 Sketch the graph using the stationary point of inflection, y-intercept and x-intercept.	

Transformation form for a quartic function

The transformation form of the basic quartic function $y = x^4$ is:
$P(x) = a(x-b)^4 + c,\ a \neq 0$.

Not all quartic polynomial functions can be written in this form. For quartics of this form, the stationary point is a turning point. For example, the equation $f(x) = 2(x-1)^4 + 3$ has a stationary point at (1, 3) and a y-intercept at (0, 5) as shown.



In general, the graph of any quartic function with rule $y = a(x-b)^4 + c$ where $a \neq 0$ can be obtained from the graph of $y = x^4$ by a combination of the following transformations:

- dilation parallel to the y-axis (from the x-axis), for $a > 0$
- dilation parallel to the y-axis and reflection in the x-axis, for $a < 0$
- translation parallel to one or both axes, horizontally by b units and vertically by c units.

The coordinates of the turning point are (b, c).

The transformation form of a polynomial $y = x^n$ is written as $y = a(x-b)^n + c$. When compared to the graph of $y = x^n$:

a dilates the graph parallel to the y-axis.

- If $a > 0$, the graph is concave up for even values of n.
- If $a < 0$, the graph is concave down for even values of n.
- If $a > 1$ or $a < -1$, the graph is stretched away from the x-axis.
- If $-1 < a < 1$, the graph is compressed towards the x-axis.

b translates the graph horizontally.

- If $b > 0$, the graph moves to the right.
- If $b < 0$, the graph moves left.
- If n is even, then $x = b$ is the equation of the line of symmetry.

c translates the graph vertically.

- If $c > 0$, the graph moves up.
- If $c < 0$, then the graph moves down.

Making connections

Transforming a quartic

Explore the transformations of a quartic function in the form $y = a(x-b)^4 + c$.

25 Identifying transformations applied to quartic functions

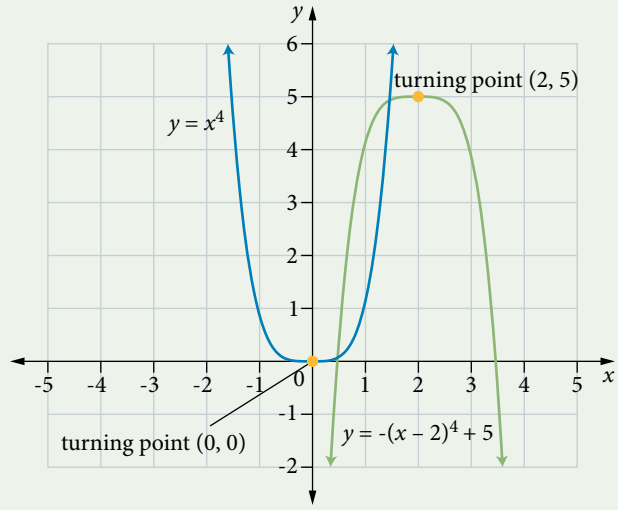
Describe the transformations that would map $y = x^4$ to $y = -(x-2)^4 + 5$ and sketch both functions on the same set of axes.

THINKING	WORKING
1 Determine the value of a in the form $y = a(x-b)^4 + c$. Hence identify any dilation or reflection in the x-axis.	For $y = -(x-2)^4 + 5$: $a = -1$, the graph of $y = x^4$ has been reflected in the x-axis.

2 Determine the coordinates of the turning point. Hence identify the horizontal and vertical translations.

3 Sketch both graphs on the same axes.

The turning point is (2, 5), so a horizontal translation of 2 units to the right and a vertical translation of 5 units up.



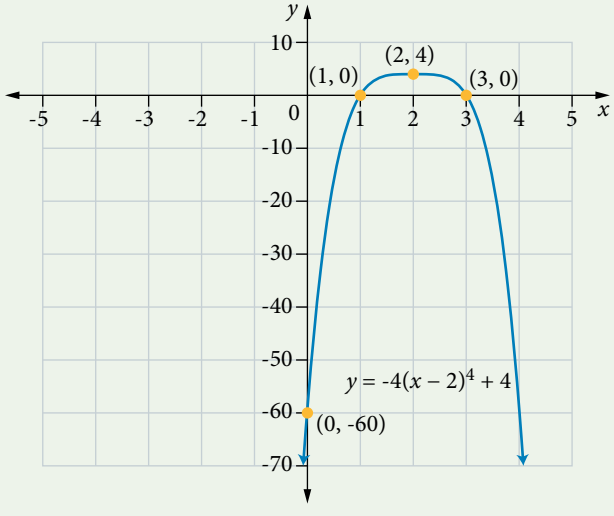
26 Determine the equation of a quartic function

The graph of $y = x^4$ has been transformed so that it has a turning point at (0.5, -5) and passes through the point (0, -6). Determine the rule for the function.

THINKING	WORKING
1 Determine the rule in turning point form $y = a(x-b)^4 + c$.	Turning point is (0.5, -5): $y = a(x-0.5)^4 - 5$
2 Determine the value of a.	Substitute (0, -6): $-6 = a(0-0.5)^4 - 5$ $-1 = \frac{a}{16}$ $a = -16$
3 Write the rule of the function in turning point form.	$y = -16\left(x - \frac{1}{2}\right)^4 - 5$
4 Simplify, if possible.	$y = -2^4\left(x - \frac{1}{2}\right)^4 - 5$ $= -\left(2\left(x - \frac{1}{2}\right)\right)^4 - 5$
5 Write the rule of the function.	$y = -(2x-1)^4 - 5$

27 Sketch the graph of a quartic function

Sketch the graph of $y = -4(x-2)^4 + 4$, showing all relevant features.

THINKING	WORKING
1 Locate the stationary point by comparing to the transformation form of a quartic $y = a(x-b)^4 + c$.	$y = -4(x-2)^4 + 4$ The coordinates of the turning point are (2, 4).
2 Determine the coordinates of the y-intercept by substituting $x = 0$ and calculating the value of y.	Let $x = 0$: $y = -4(0-2)^4 + 4$ $= -60$ The coordinates of the y-intercept are (0, -60).
3 Determine the coordinates of the x-intercept(s), if any. To do this, substitute $y = 0$ into the equation and solve for x.	Let $y = 0$: $0 = -4(x-2)^4 + 4$ $(x-2)^4 = 1$ $x - 2 = \pm 1$ $x = 3$ or $x = 1$ The coordinates of the x-intercepts are (1, 0) and (3, 0).
4 Sketch the graph using the turning point and intercepts.	

WARNING

Do not assume that all cubic or quartic graphs have the same shape as the cubic or quartic power graphs. They don't! Unlike linear and quadratic functions, many cubic or quartic polynomial functions cannot be expressed in transformation form, and these graphs have a different shape to the power graphs, as you will see in the following modules.

EXERCISE 3.4 Graphs of power functions

Worked Example

19

1 Sketch each function on a Cartesian plane, include the point corresponding to the specified values of x or y and describe the behaviour of the graph as $x \to +\infty$ and as $x \to -\infty$.

(a) $y = 2x$: $x = 2$ **(b)** $y = -3x^2$: $y = -3$ **(c)** $y = \frac{1}{4}x^3$: $y = 2$, $y = -2$

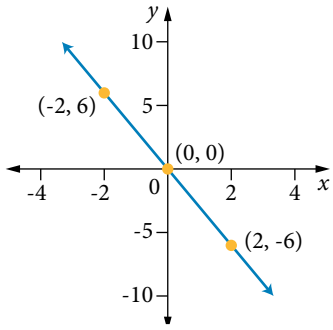
(d) $y = -2x^4$: $y = -32$ **(e)** $y = -3x$: $y = -6$ **(f)** $y = \frac{1}{5}x^2$: $y = 5$

(g) $y = 2x^3$: $y = 16$, $y = -16$ **(h)** $y = \frac{1}{3}x^4$: $y = 27$

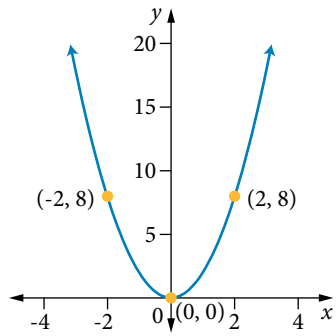
20

2 For each graph, determine the rule of the polynomial function.

(a)



(b)



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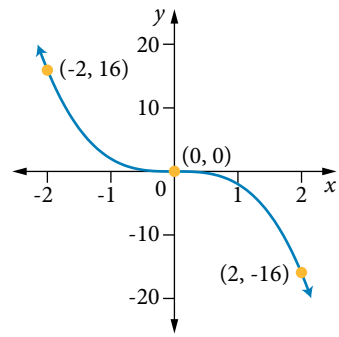
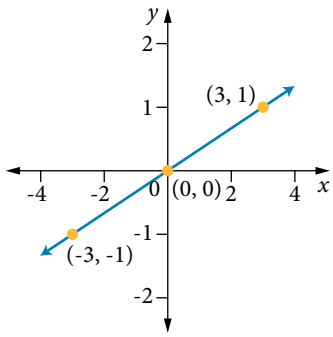


(d)



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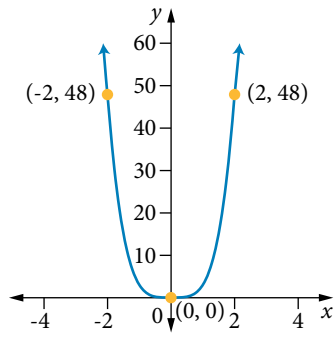

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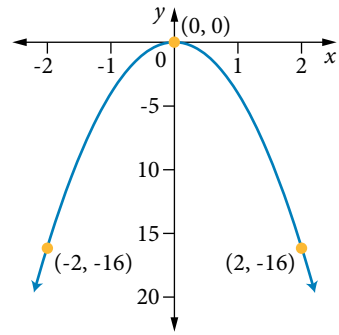


3 Which of the following polynomial functions include the points (-1, 1), (0, 0) and (1, 1)?

A $y = 2x$ **B** $y = x^4$ **C** $y = x^3$ **D** $y = \frac{1}{2}x^2$

4 Consider the polynomial functions listed below.

(a) Which one satisfies the conditions: as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to -\infty$?

A $f(x) = x^4$ **B** $f(x) = -x^3$ **C** $f(x) = x^3$ **D** $f(x) = 2x^2$

(b) Explain the error made by a student who thought the answer was $f(x) = -2x^3$.

5 A polynomial function is defined as $g(x) = -\frac{1}{4}x^4$.

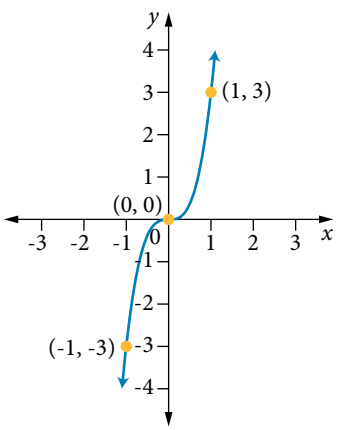
(a) Determine the values of $g(0)$, $g(-1)$ and $g(1)$.

(b) Determine the equation of the line of symmetry.

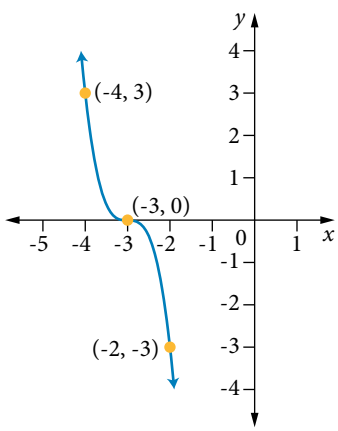
(c) Determine the trend in the function values for extreme values of x.

6 Determine the rules for each cubic function represented in the following graphs.

(a)



(b)



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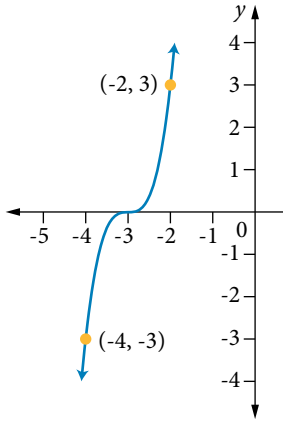
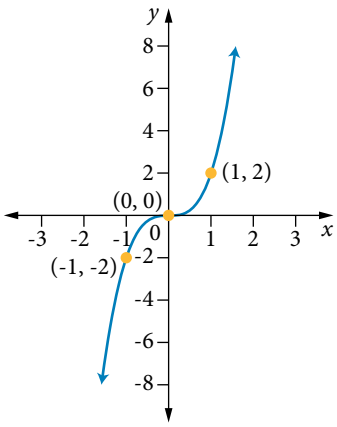


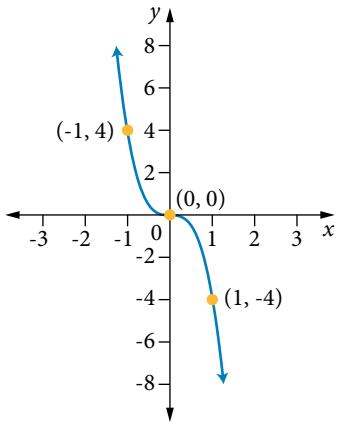
Worked Example 21

7 For each graph, determine the rule of the cubic function.

(a)



(b)



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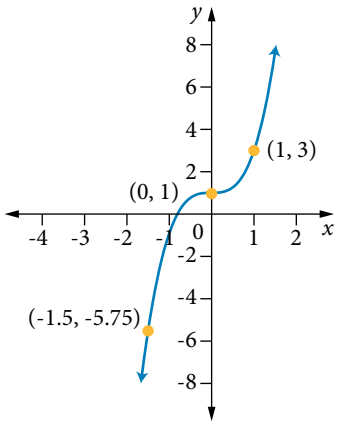
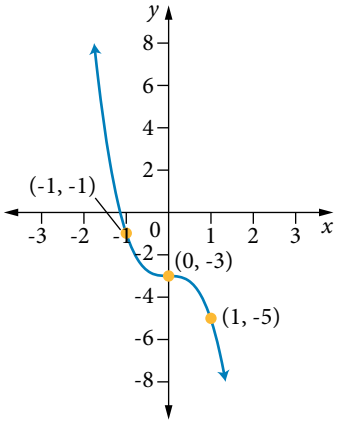
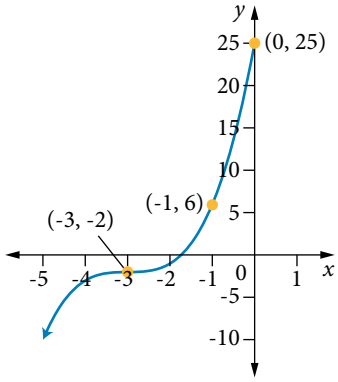


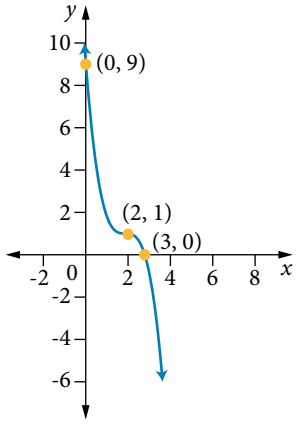
(d)



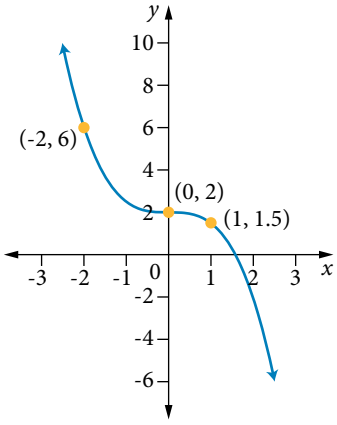
(e)



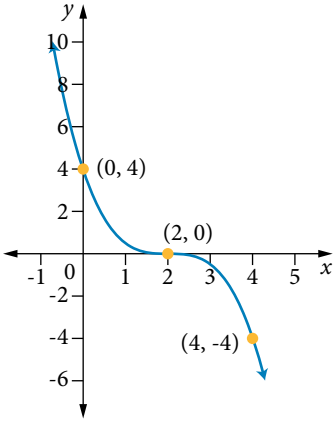
(f)



(g)



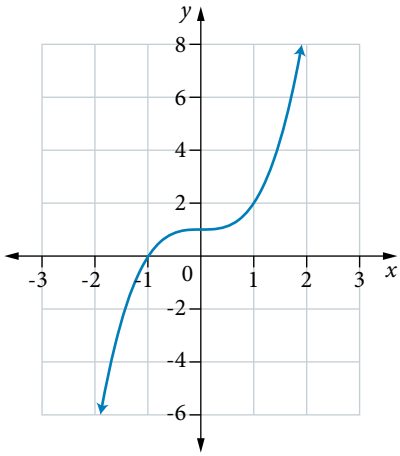
(h)



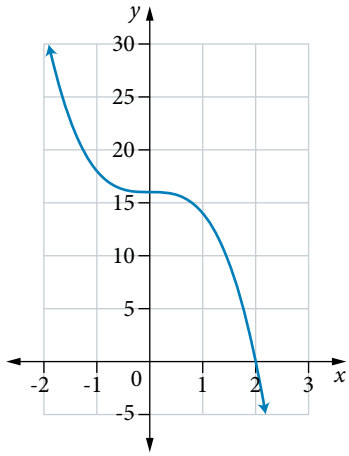
22

8 For each graph, determine the coordinates of the y-intercept and x-intercept, and describe the behaviour of the function as $x \to -\infty$ and $x \to \infty$.

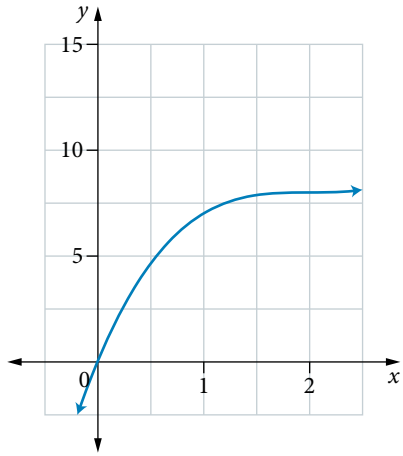
(a) $f(x) = x^3 + 1$



(b) $f(x) = -2x^3 + 16$



(c) $h(x) = (x-2)^3 + 8$



9 Describe the transformations that would map the cubic function $y = 2x^3$ to $y = 2(x-3)^3 + 1$.

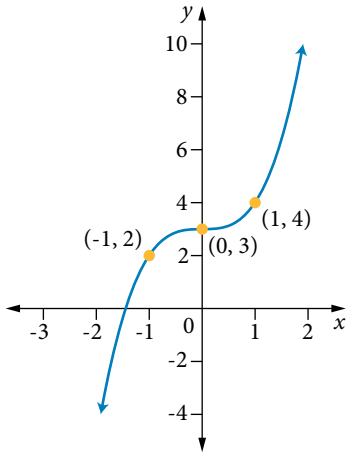
10 Describe the transformations that would map the cubic function $y = 2x^3$ to $y = 4(x-3)^3 + 1$.

23

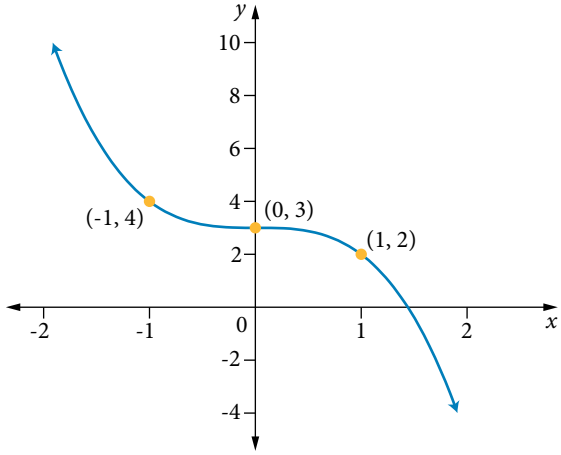
11 The graph of $y = x^3$ is transformed to give $y = a(x-b)^3 + c$. Determine the values of a, b and c, given that the graph now has a stationary point of inflection at (-1, 3) and passes through (-2, 11).

12 Determine the rule for each cubic function.

(a)



(b)



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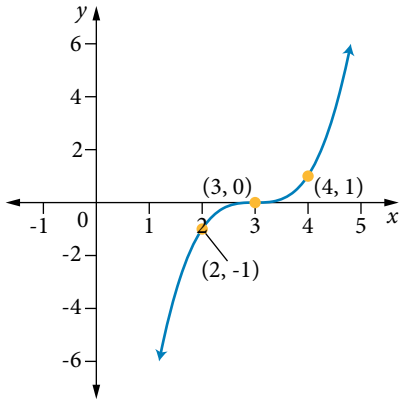

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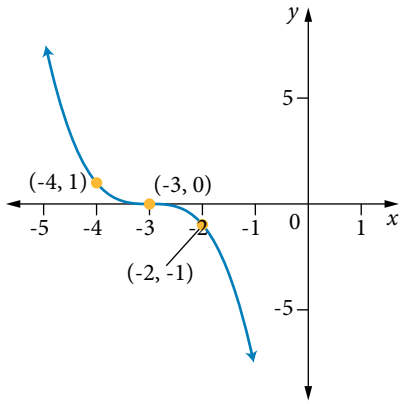


13 Describe the transformations that would change $y = x^4$ to each of the following graphs. 25

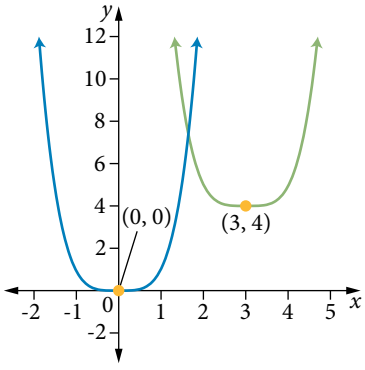
(a) $y = -2x^4$ (b) $y = 3x^4 + 5$ (c) $y = 0.5x^4 - 2$

(d) $y = (x-2)^4 - 5$ (e) $y = -(x+4)^4 - 1$ (f) $y = -3(x+3)^4 - 2$

14 The graph of $y = x^4$ is transformed to give $y = a(x-b)^4 + c$. Determine the values of a, b and c if the graph has a turning point at (-1, 3) and passes through (-2, 11). 26

15 Examine the graphs below and think about the transformation(s) that have taken place for the original polynomial function ($y = x^4$) to move to the new location. Hence, provide the equation of the transformed polynomial function. Assume that the function has retained its original shape (i.e. there are no dilations parallel to the x-axis or y-axis).

(a)



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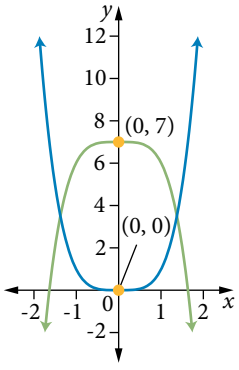

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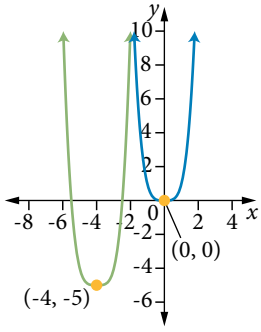


16 The graph of $y = -2(x+3)^4 - 2$ is transformed by a horizontal translation of 5 units to the left and a vertical translation of 2 units down. Determine the coordinates of the new turning point.

17 A function is said to be even with reflection symmetry in the y-axis if $f(-x) = f(x)$. A function is said to be odd with half-turn rotational symmetry about the origin if $f(-x) = -f(x)$. Test the following functions to identify whether they are even, odd or neither.

(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$ (d) $y = x^4$

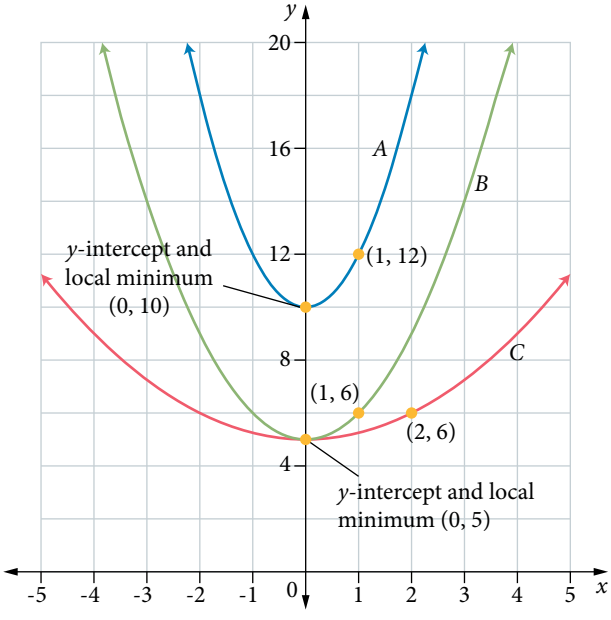
18 Three polynomial functions (A, B, C) are graphed on the Cartesian plane shown.

(a) Match each of the equations with a polynomial graph shown.

$y = x^2 + 5$, $y = a(x^2 + 5)$, $y = (bx)^2 + 5$

(b) Determine the values of a and b.

(c) Describe the effects of a and b in the previous graphs, as transformations of $y = x^2 + 5$.



19 Sketch the graph of each cubic function, showing all intercepts and critical points. 24

(a) $y = 2(x-1)^3 - 2$ (b) $y = (4-x)^3$ (c) $p(x) = -2(x-1)^3 - 16$

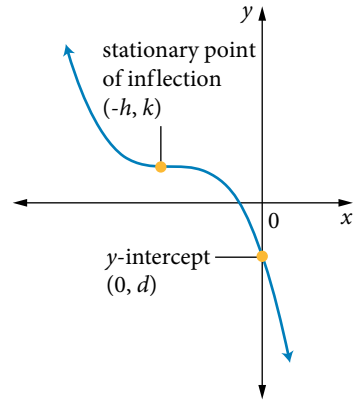
27

20 Sketch the graph of each of the quartic polynomials, showing all key features. Write values correct to 2 decimal places where necessary.

(a) $y = -x^4 + 16$ **(b)** $y = 4(x-3)^4 - 12$

21 Graph the power functions $y = x$, $y = x^2$, $y = x^3$ and $y = x^4$ over the domain $-3 \le x \le 3$. Identify end points and points of intersection on your graph.

22 Determine the equation of the cubic function represented by the graph. Write your answer in terms of the pronumerals given. Discuss the sign of each pronumeral and hence determine the dilation factor of the graph when compared with $y = x^3$.



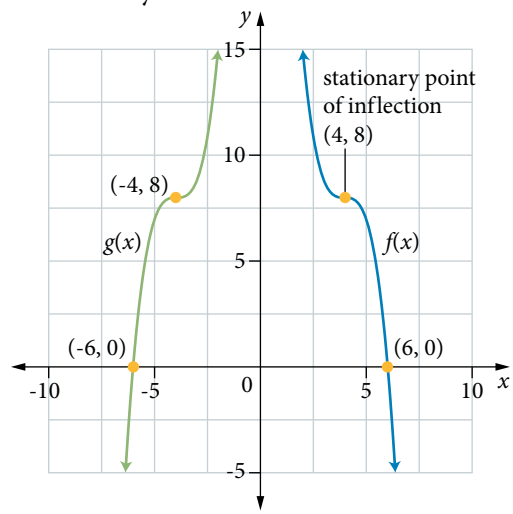
23 Consider the cubic function $y = (x-2)^3 + 3$.

(a) Create a piece-wise quadratic function that, when graphed, could be mistaken for the cubic function. Sketch both graphs on the same set of axes showing all points of intersection.

(b) Improve your sketch by calculating the y-intercept for each function.

(c) Describe the graph of the piece-wise function in terms of continuity and smoothness.

24 A conical flask is used to heat liquids to a very high temperature. To ensure optimal gas flow, the flask has a particular shape that is modelled on a cubic function. The cross-section of a flask of height 15 cm is shown below. This cross-section is represented by the cubic function $f(x)$ and its reflection across the y-axis, $g(x)$.



(a) Given $f(x) = a(x-b)^3 + c$, determine the values of a, b and c. Hence, write the rule for $f(x)$.

(b) Determine the equation for $g(x)$.

(c) What is the width of the flask when it is filled to the 10 cm mark? Write this in exact form and as a decimal correct to 2 decimal places.

25 A half-pipe is a piece of equipment used by skateboarders. The U-shape of this half-pipe can be represented by the rule $H(x) = a(x-h)^4 + k$, where $H(x)$ is the vertical height (in metres) of the half-pipe from ground level at any horizontal distance x.

(a) Determine the equation $H(x)$ that represents this particular half-pipe if the entry positions are 9 m above ground level and 20 m apart, and the flat section of the pipe is 1 m above ground level. The horizontal distance is measured from the edge of the left-hand side of the half-pipe.

(b) Sketch a graph of $H(x)$ on the Cartesian plane, showing the turning point and y-intercept.

(c) Determine the domain of the function.

(d) What is the height of the half-pipe when the horizontal distance is 5 m?

(e) An average skateboard is approximately 15 cm high (above the ground), and the height of Gianni, a tall skateboarder, is approximately 185 cm. Determine a function $G(x)$ that describes the top of Gianni's head as he skates along the half-pipe, assuming that he remains perfectly upright.

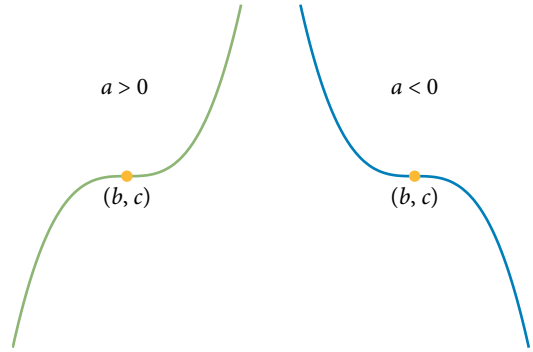
(f) What is the range of $G(x)$?

Graphs of cubic functions

3.5

The shapes of cubic functions

You should now be familiar with cubic power functions of the form $P(x) = ax^3$, where $a \neq 0$, with the centre of rotational symmetry and point of horizontal inflection at the origin, (0, 0). You have also seen simple transformations of these graphs. Only cubic graphs that can be expressed in the form $P(x) = a(x-b)^3 + c$ have this shape.



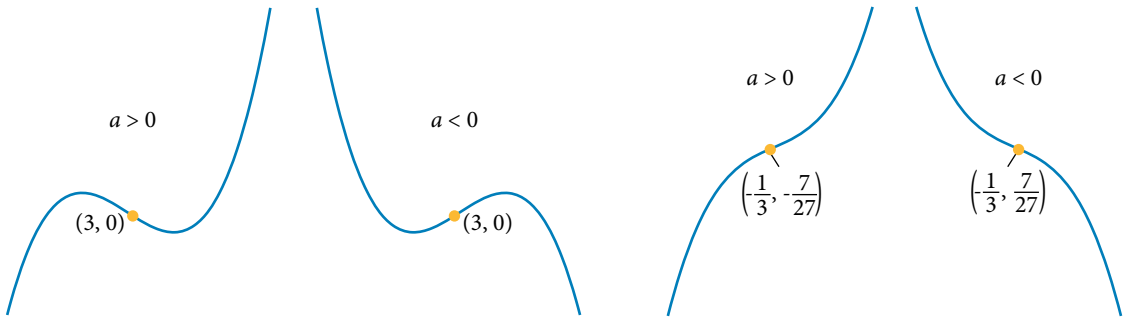
There is a single change of concavity at the centre of rotational symmetry. This is a point of horizontal inflection at (b, c).

The point at the centre of symmetry has gradient of 0. The rest of the curve has a gradient either greater than 0 throughout or less than 0 throughout.

All cubic functions can be expressed in the general form $y = ax^3 + bx^2 + cx + d$.

The shape of cubic graphs that cannot be expressed in $P(x) = a(x-b)^3 + c$ have the following features:

- The point at the centre of rotational symmetry has an oblique point of inflection, where the concavity changes. The gradient at this point is not 0.
- There may be a local maximum on one side and a local minimum on the other, or the curve may have a gradient that is either greater than 0 throughout or less than 0 throughout.



Determining the turning points and point of oblique inflection will be covered in a calculus unit later in the year. For now you can observe these features using technology.

WARNING

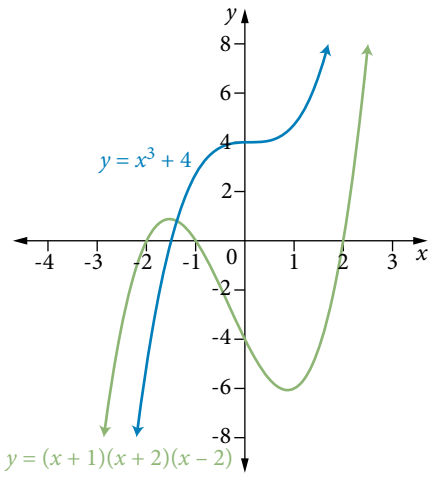
Do not assume that turning points are midway between any two points with the same y-value in graphs of cubic functions. This is true for parabolas, but is not the case for all cubics.

General form

The general form of a cubic is $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$ and the domain is $x \in \mathbb{R}$.

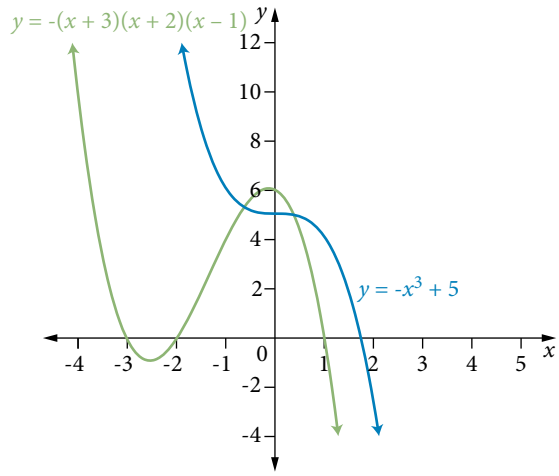
If a is positive ($a > 0$), then the cubic function is considered 'positive'. The behaviour of a positive cubic function is as follows:

as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$



If a is negative ($a < 0$), then the cubic function is considered 'negative'. The behaviour of a negative cubic function is as follows:

as $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to \infty$



An advantage of cubic functions written in general form is that the y-intercept can be easily found. By substituting $x = 0$, it can be seen that the y-intercept for the function $y = ax^3 + bx^2 + cx + d$ is $(0, d)$. The x-axis intercept(s) can be determined by solving $y = 0$ for x, which can be done in several ways, including by technology.

Explore further

Exploring cubic functions in the form $y = ax^3 + bx^2 + cx + d$

Explore the effect of varying the coefficients on the graph of a cubic polynomial expressed in the general form.

Additional information

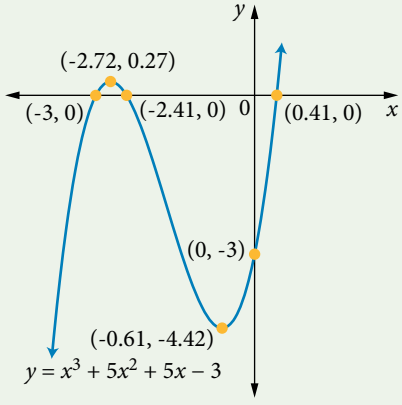
Behaviour of polynomials

Consolidate your understanding of the end behaviour of polynomials.

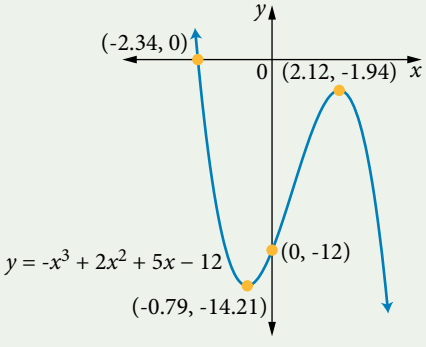
28 Sketch cubics using technology

Sketch each of the following cubic functions. Use technology to determine all axis intercepts and turning points, correct to 2 decimal places, where necessary.

(a) $y = x^3 + 5x^2 + 5x - 3$

THINKING	WORKING
1 Enter the equation into your technology and draw the graph. Determine the roots, rounding if necessary.	x-intercepts: (-3, 0), (-2.41, 0), (-0.41, 0)
2 Determine the y-intercept.	y-intercept: (0, -3)
3 Determine the coordinates of the maximum and minimum turning points.	Maximum turning point: (-2.72, 0.27) Minimum turning point: (-0.61, -4.42)
4 Sketch and label the graph, noting all determined points and indicating behaviour at the extremes.	

(b) $y = -x^3 + 2x^2 + 5x - 12$

1 Enter the equation into your technology and draw the graph. Determine the roots, rounding if necessary.	x-intercept: (-2.34, 0)
2 Determine the y-intercept.	y-intercept: (0, -12)
3 Determine the coordinates of the maximum and minimum turning points.	Minimum turning point: (-0.79, -14.21) Maximum turning point: (2.12, -1.94)
4 Sketch and label the graph, noting all determined points and indicating behaviour at the extremes.	

Factorised form

Some cubic functions can be written in factorised form of $y = k(x - a)(x - b)(x - c)$, where $k \neq 0$.

The sign of k determines whether the function is positive or negative.

a, b and c are the roots of the function and correspond to x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$ respectively.

A linear factor in the form of $ax + b$ can be written as $a\left(x + \frac{b}{a}\right)$.

For example, $f(x) = (x-1)(2x+3)(3x-5)$ can be re-written as:

$f(x) = (x-1) \times 2\left(x + \frac{3}{2}\right) \times 3\left(x - \frac{5}{3}\right)$ or $f(x) = 6(x-1)\left(x + \frac{3}{2}\right)\left(x - \frac{5}{3}\right)$.

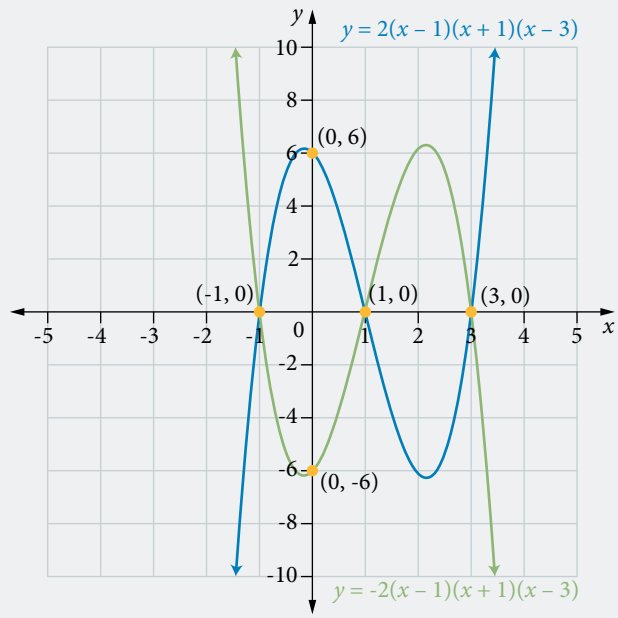
Three distinct factors: 3 x-intercepts

When a cubic function has three different factors, then the graph will cut the x-axis at three different points.

For example, the graphs of $y = 2(x-1)(x+1)(x-3)$ and $y = -2(x-1)(x+1)(x-3)$ each have three x-intercepts at (1, 0), (-1, 0) and (3, 0).

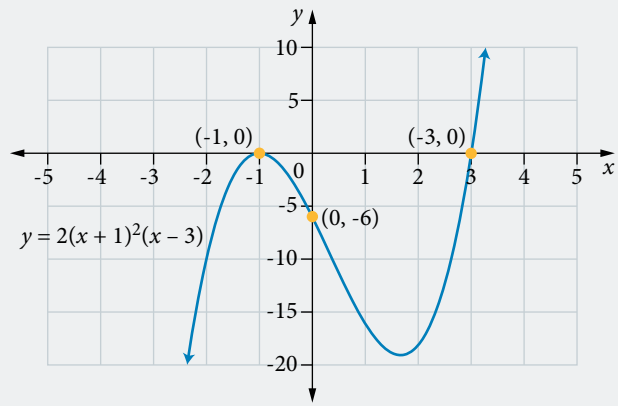
The functions are the same except for the negative sign at the front, so the second graph is the reflection in the x-axis of the first graph. The coordinates of the y-intercepts are (0, 6) and (0, -6), as shown.

For $y = 2(x-1)(x+1)(x-3)$ the local maximum is at (-0.15, 6.16) and the local minimum is at (2.15, -6.16).



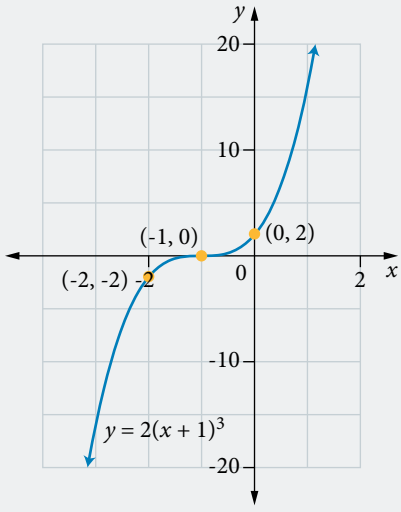
Two distinct factors: 2 x-intercepts

When one of the linear factors is repeated, this indicates that the cubic function has a turning point on the x-axis. For example, the graph of $y = 2(x+1)^2(x-3)$ has a repeated factor of $(x+1)$, so its turning point is on the x-axis at (-1, 0).



One distinct factor: 1 x-intercept

When the cubic function has the same linear factor three times, this is the same as the transformation form seen previously, because all three x-intercepts coincide. For example, $y = 2(x+1)^3$ has a point of inflection on the x-axis at (-1, 0).



Some cubic functions can be written in factorised form of $y = k(x-a)(x-b)(x-c)$, where $k \neq 0$.

The sign of k determines whether the function is positive or negative.

$y = (x-a)(x-b)(x-c)$ has three distinct roots and the graph cuts the x-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

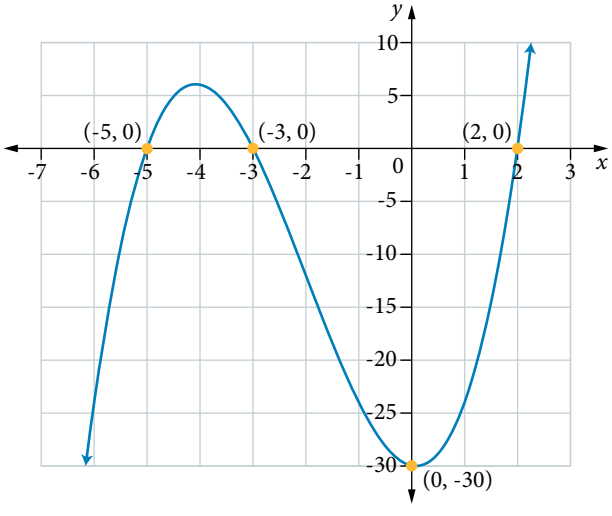
$y = (x-a)^2(x-b)$ has two distinct roots and the graph has a turning point at $(a, 0)$ and cuts the x-axis at $(b, 0)$.

$y = (x-a)^3$ has one distinct root and the graph has a point of inflection at $(a, 0)$.

29 Determining the rule of a cubic function

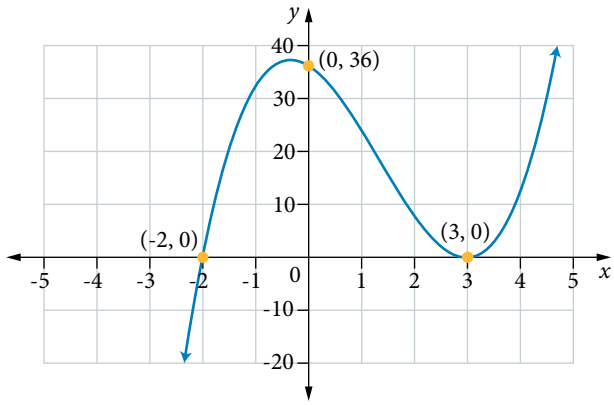
For each of the following graphs, determine the rule of the cubic function.

(a)



THINKING	WORKING
1 Identify the type of graph and recall the rule in intercept or factorised form.	$y = k(x - a)(x - b)(x - c)$
2 Substitute the x-values of the x-intercepts into the basic rule.	The x-intercepts are (-3, 0), (2, 0), (-5, 0) so the equation becomes: $y = a(x + 3)(x - 2)(x + 5)$
3 Substitute another given point and solve for a.	Substitute (0, -30): $-30 = a(0 + 3)(0 - 2)(0 + 5)$ $-30 = -30a$ $a = 1$
4 Write the rule of the function.	$y = (x + 3)(x - 2)(x + 5)$

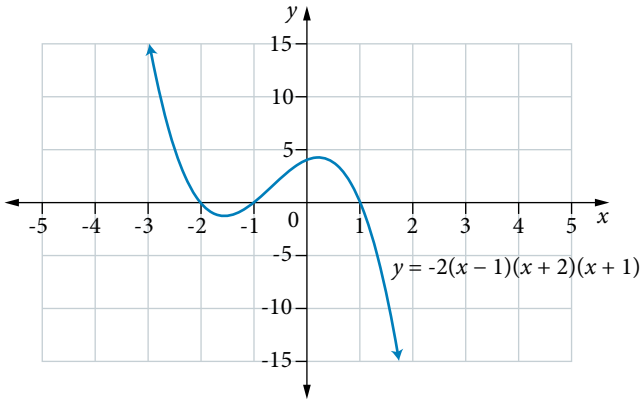
(b)



1 Identify the type of graph and recall the rule in intercept or factorised form.	$y = k(x - a)(x - b)(x - c)$
2 Substitute the x-values of the x-intercepts into the basic rule.	The x-intercept is (-2, 0). The turning is at (3, 0). $y = a(x + 2)(x - 3)^2$
3 Substitute another given point and solve for a.	Substitute (0, 36): $36 = a(0 + 2)(0 - 3)^2$ $36 = 18a$ $a = 2$
4 Write the rule of the function.	$y = 2(x + 2)(x - 3)^2$

30 Determine key aspects of graphed cubic functions

For the following graph, determine the coordinates of the y-intercept and the x-intercept(s), and the behaviour of $f(x)$ as $x \to \pm\infty$.



THINKING	WORKING
1 The y-intercept is found by substituting $x = 0$ into the rule.	$f(0) = -2(0-1)(0+2)(0+1)$ $= 4$ The coordinates of the y-intercept are (0, 4).
2 The x-intercept(s) are found by substituting $y = 0$ into the rule and using the null factor law to solve for x.	$0 = -2(x-1)(x+2)(x+1)$ $x = 1$, $x = -2$ or $x = -1$ The coordinates of the x-intercepts are (1, 0), (-2, 0) and (-1, 0).
3 Examining the shape of the graph, comment on the behaviour as $x \to \pm\infty$. Note that the approximate locations of the local maximum and local minimum could be found using technology.	As $x \to \infty$, $y \to -\infty$. As $x \to -\infty$, $y \to \infty$.

Technology worked example
Sketching cubic functions

31 Determine the rule of a cubic function from given information

Determine the rule of a cubic function with the following roots.

(a) 1, -2, 2

THINKING	WORKING
1 Determine the linear factors from these roots.	The linear factors are: $(x-1)$, $(x+2)$, $(x-2)$
2 Multiply the factors together to write the cubic polynomial. Include the coefficient k as there is no extra information given to completely determine the polynomial.	$\therefore f(x) = k(x-1)(x+2)(x-2)$

(b) 0, -3, 4

1 Determine the linear factors from these roots.	The linear factors are: x, $(x+3)$, $(x-4)$
2 Multiply the factors together to write the cubic polynomial. Include the coefficient k as there is no extra information given to completely determine the polynomial.	$\therefore f(x) = kx(x+3)(x-4)$

(c) -3, with a turning point on the x-axis at (1, 0) and a y-intercept at (0, 9)

1 Determine the linear factors from these roots.	The linear factors are: $(x-1)$, $(x-1)$, $(x+3)$
2 Multiply the factors together to write the cubic polynomial. Include the coefficient k at the start of the polynomial.	$\therefore f(x) = k(x+3)(x-1)^2$
3 Determine the value of k by substituting the values from another known point.	For the y-intercept (0, 9): $9 = k(0+3)(0-1)^2$ $9 = 3k$ $k = 3$
4 Conclude by writing the equation of the polynomial.	$f(x) = 3(x+3)(x-1)^2$

(d) $-\frac{1}{2}, \frac{1}{2}, 1$

1 Determine the linear factors from these roots.	$x = -\frac{1}{2}$ $2x = -1$ $2x + 1 = 0$ $x = \frac{1}{2}$ $2x = 1$ $2x - 1 = 0$ The linear factors are: $(2x-1)$, $(2x+1)$, $(x-1)$
2 Multiply the factors together to write the cubic polynomial. Include the coefficient k as there is no extra information given to completely determine the polynomial.	$f(x) = k(2x+1)(2x-1)(x-1)$

(e) Point of horizontal inflection at the single root $x = -7$ and a point on the curve at (1, 1024)

1 Determine the linear factors from these roots.	The only linear factor is $(x+7)$.
2 Multiply the factors together to write the cubic polynomial. Include the coefficient k at the start of the polynomial.	$f(x) = k(x+7)^3$
3 Determine the value of k by substituting the values from another known point.	For $(1,1024)$: $1024 = k(1+7)^3$ $1024 = 512k$ $k = 2$
4 Conclude by writing the equation of the polynomial.	$f(x) = 2(x+7)^3$

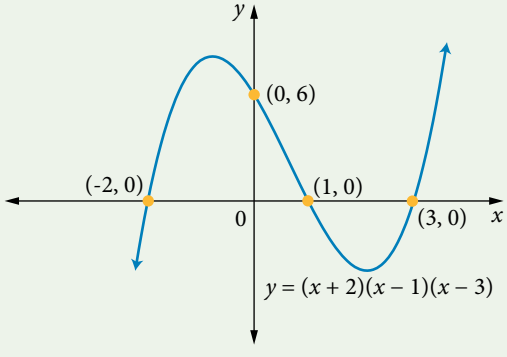
Sketch graphs from factorised form

Factorised form allows you to determine the coordinates of the x-intercepts of a cubic function with ease, without using technology. With the y-intercept as a guide, you will be able to draw the general shape. Accuracy with the placement of turning points is not expected.

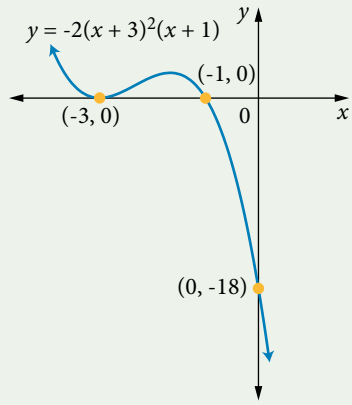
32 Sketch cubic functions from factorised form

Sketch each of the following cubic functions.

(a) $y = (x+2)(x-1)(x-3)$

THINKING	WORKING
1 Determine whether the function is positive or negative.	In expanded form, the coefficient of x^3 will be 1, so the cubic is positive.
2 Determine the roots, by substituting $y = 0$ and using the null factor law.	For $y = 0$: $y = (x+2)(x-1)(x-3)$ $0 = (x+2)(x-1)(x-3)$ $x = -2$, $x = 1$ or $x = 3$ The x-intercepts are at (-2, 0), (1, 0) and (3, 0).
3 Determine the y-intercept by substituting $x = 0$.	For $x = 0$: $y = (0+2)(0-1)(0-3) = 6$ The coordinates of the y-intercept are (0, 6).
4 Sketch the graph with a smooth curve through all determined points and indicating behaviour at the extremes.	

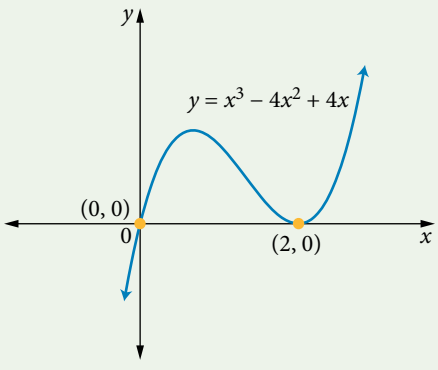
(b) $y = -2(x+3)^2(x+1)$

1 Determine whether the function is positive or negative.	In expanded form, the coefficient of x^3 will be -2, so the cubic is negative.
2 Determine the roots by substituting $y = 0$ and using the null factor law to solve for x.	For $y = 0$: $y = -2(x+3)^2(x+1)$ $0 = -2(x+3)^2(x+1)$ $x = -3$ or $x = 1$ The coordinates of the x-intercepts are (-3, 0) and (-1, 0).
3 Determine the coordinates of the y-intercept by substituting $x = 0$.	For $x = 0$: $y = -2(0+3)^2(0+1)$ $= -2 \times 9$ $= -18$ The coordinates of the y-intercept are (0, -18).
4 Sketch the graph with a smooth curve through all determined points and indicating behaviour at the extremes. The repeated factor results in a turning point at the corresponding x-intercept.	

33 Sketch cubic functions from general form

First factorise and then sketch each of the following cubic functions.

(a) $y = x^3 - 4x^2 + 4x$

THINKING	WORKING
1 Factorise the expression, taking out the common factor first.	$y = x^3 - 4x^2 + 4x$ $= x\left(x^2 - 4x + 4\right)$
2 Factorise the quadratic, if possible.	$= x(x-2)^2$
3 Determine whether the function is positive or negative. Keep this in mind as a check of a reasonable solution.	The cubic is positive.
4 Determine the roots by substituting $y = 0$ and using the null factor law.	For $y = 0$: $0 = x(x-2)^2$ $x = 0$ or $x = 2$ The coordinates of the x-intercepts are (0, 0) and (2, 0).
5 Determine the coordinates of the y-intercept by substituting $x = 0$.	The coordinates of the y-intercept are (0, 0).
6 Sketch the graph with a smooth curve through all determined points and indicating behaviour at the extremes. The repeated factor results in a turning point at the corresponding x-intercept. This is a positive function so as $x \to \infty$, $y \to \infty$.	The coordinates of the turning point on the x-axis are (2, 0). 

(b) $y = x^3 - x^2 - 7x + 7$

1 Factorise the expression, taking out the common factor first.

Try $x - 1$:

$P(1) = 0 \therefore x - 1$ is a factor.

2 Factorise the quadratic, if possible.

$$\begin{array}{r|l} & x^2 - 7 \\ \hline x - 1 & x^3 - x^2 - 7x + 7 \\ & -\left(x^3 - x^2\right) \downarrow \\ \hline & 0 - 7x \downarrow \\ & -\left(-7x + 7\right) \\ \hline & 0 \end{array}$$

$y = (x - 1)\left(x^2 - 7\right)$

$x^2 - 7$ cannot be factorised with rational values.

3 Determine whether the function is positive or negative. Keep this in mind as a check of a reasonable solution.

The cubic is positive.

4 Determine the roots by substituting $y = 0$ and using the null factor law to solve for x.

For $y = 0$:

$0 = (x - 1)\left(x^2 - 7\right)$

$x - 1 = 0$ or $x^2 - 7 = 0$

$x = 1$

$x^2 = 7$

$x = \pm\sqrt{7}$

The coordinates of the x-intercepts are $(1, 0)$ $\left(-\sqrt{7}, 0\right)$ and $\left(\sqrt{7}, 0\right)$.

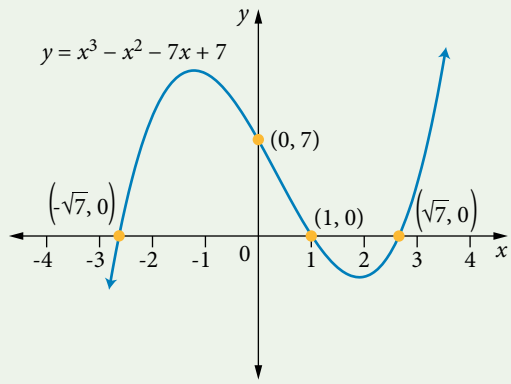
5 Determine the y-intercept by substituting $x = 0$.

For $x = 0$:

The coordinates of the y-intercept are $(0, 7)$.

6 Sketch the graph with a smooth curve through all determined points and indicating behaviour at the extremes.

$\sqrt{7}$ lies between 2 and 3.



EXERCISE 3.5 Graphs of cubic functions

Worked Example

1 Sketch each of the following cubic functions. Use technology to determine the coordinates of all axis intercepts and turning points, rounded correct to 2 decimal places, where necessary. 28

(a) $y = x^3 - 4x^2 + 2x + 1$

(b) $y = -x^3 - 5x^2 - 5x + 2$

(c) $y = 2x^3 + 4x^2 + 5$

(d) $y = -x^3 + x^2 + 6x + 5$

2 For the cubic function $y = -(x - 1)(x + 2)(x - 3)$:

(a) the coordinates of the axes intercepts are:

A (0, 1), (0, -2), (0, 3) **B** (-1, 0), (-2, 0), (-3, 0) **C** (1, 0), (2, 0), (3, 0) **D** (1, 0), (-2, 0), (3, 0)

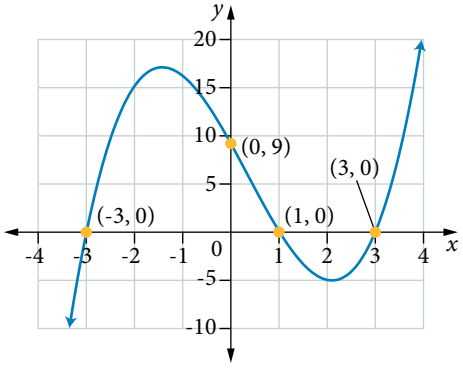
(b) Explain the common error made by a student who stated the x-intercepts in part **(a)** as (-1, 0), (2, 0), (-3, 0).

3 The cubic function $y = (x + 4)(2 - x)(x + 1)$ has a maximum turning point:

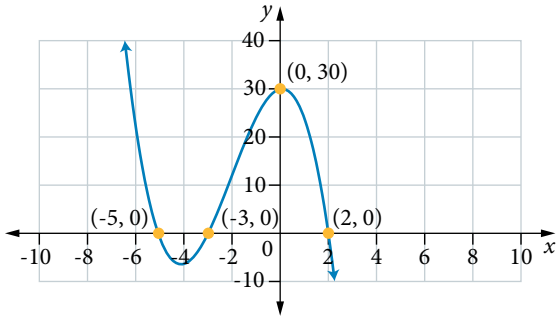
A where $x = 0.5$ **B** near $x = 0.5$ **C** where $x = -2.5$ **D** near $x = -2.5$

4 Determine the rule of each cubic function. 29

(a)

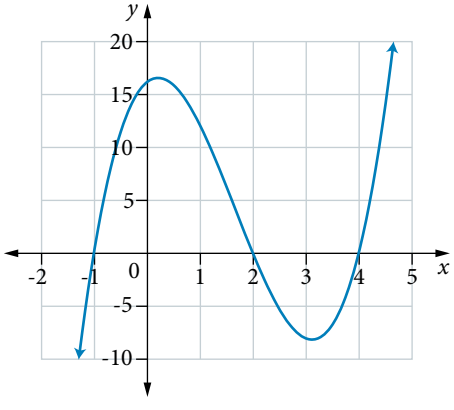


(b)

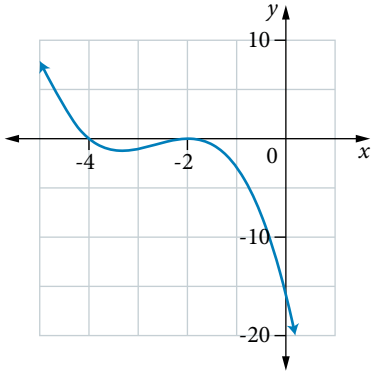


5 For each polynomial graph, determine the coordinates of the y-intercept and the x-intercept(s) and describe the behaviour at the extremes. 30

(a) $g(x) = 2(x + 1)(x - 2)(x - 4)$



(b) $g(x) = -(x + 2)^2(x + 4)$



31

6 Determine a cubic equation for each function, given the following information.

(a) Roots: -1, 2, 3 (b) Roots: -3, 0, 4 (c) Roots: $-\frac{1}{2}, -\frac{1}{3}, 2$ (d) Roots: $\frac{2}{3}, \frac{3}{2}, -1$

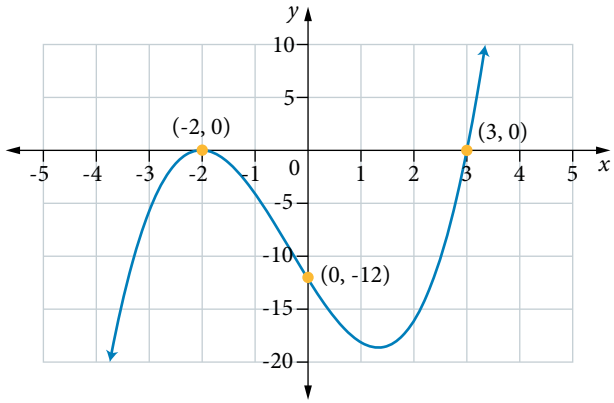
(e) Roots: $-3, 0, \frac{2}{3}$ (f) Roots: $\frac{1}{3}, \frac{2}{5}, 2$, y-intercept (0, 8)

(g) Touches the x-axis at 1 and passes through the origin

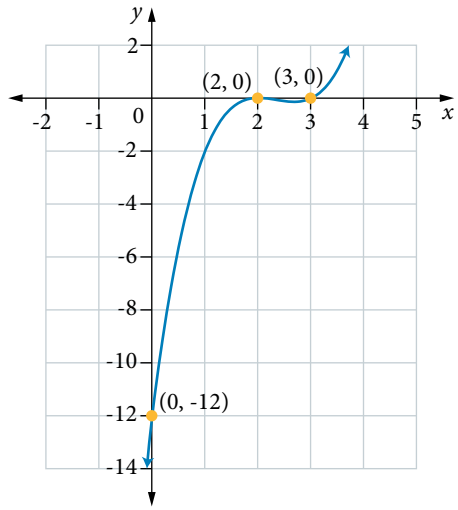
(h) Point of horizontal inflection at the single root $x = 2$, point on curve: (-2, -128)

7 Determine the rule for each graph.

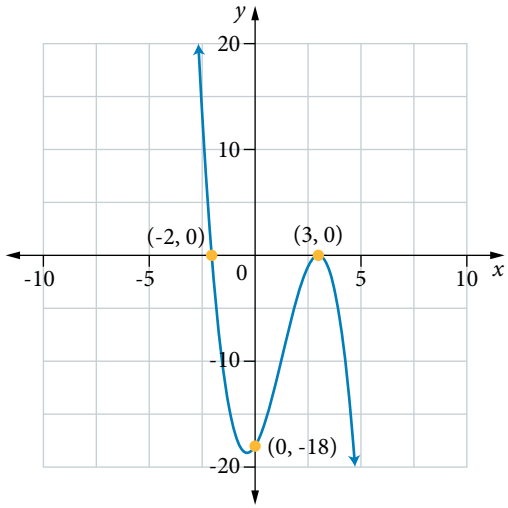
(a)



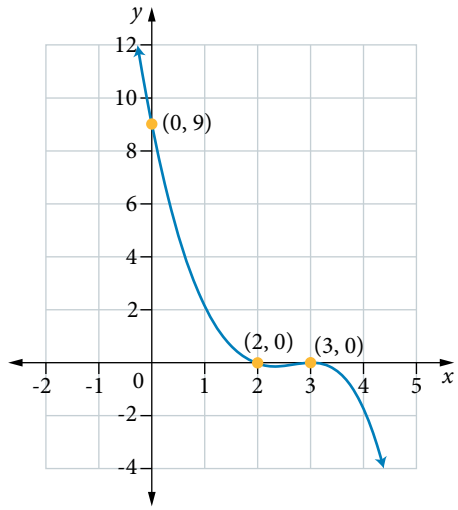
(b)



(c)



(d)



32

8 Sketch the graph of each cubic function, clearly showing all intercepts.

(a) $y = (x+5)(x-1)(x-2)$ (b) $y = 2(x-1)(x-3)^2$

(c) $y = (2-x)(x-1)(x+4)$ (d) $h(x) = -(x+4)(x-1)^2$

9 The function $f(x) = x^3 + (a-b)x^2 + (-a-b)x - 3$ has x-intercepts at (1, 0) and (3, 0). Determine the values of a and b and hence the rule for the function.

10 The functions $f(x) = x^3 - (a-b)x^2 + x + a$ and $g(x) = -x^3 + 5x^2 - (6-b)x - 2(a+b)$ share an x-intercept of (-1, 0). Determine the values of a and b and hence the rules of each function.

Worked Example 33

11 Sketch the graph of each cubic function, clearly showing all intercepts.

(a) $y = x^3 + x^2 - 4x - 4$

(b) $f(x) = x^3 + 7x^2 + 12x$

(c) $g(x) = -x^3 + 8x^2 - 21x + 18$

(d) $y = x^3 + 4x^2 - 3x - 12$

12 A sheet of A3 cardboard is 297 mm × 420 mm. An open box is to be made from a sheet of the cardboard by cutting equal squares from the corners and folding up the sides.

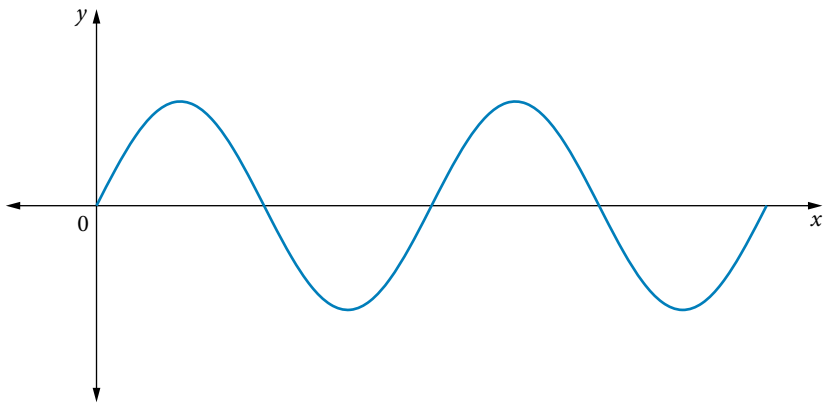
(a) Determine the formula for the volume of the box in mm^3 if the edge length of the square is x mm.

(b) Sketch the function in a suitable domain.

(c) Determine the maximum volume of the box and the length of x required for this volume, to the nearest mm.

13 Translate the graph of $f(x) = 2(x+3)(x-1)(x-2)$ up 1 unit, right 2 units. Write a rule for the new function $g(x)$. Graph the functions on grid paper using the same set of axes clearly showing key points.

14 A symmetrical sinusoidal graph is given below. The first $\frac{1}{8}$ portion of the graph is congruent to each section that follows. Create a piece-wise cubic function with as close to the same general appearance of this graph as you can. Discuss your strategy in your working. Discuss the limitations of your attempts and make suggestions for improvements.



15 A block of wood is to be cut so that the sum of the perimeter of its square end and its length is 250 cm.

Determine the maximum volume of the block of wood. Round your answer appropriately.

3.6 Solving polynomial equations

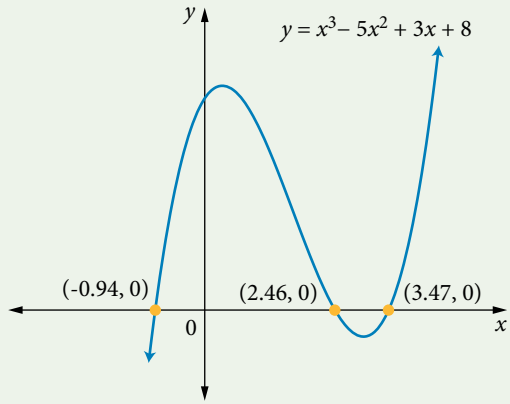
Solve equations from graphs

When technology is allowed, there are quick and efficient methods to determine solutions of equations. Graphs are particularly useful for confirming that the values are reasonable.

34 A cubic expression equal to a constant

Solve the equations, giving values of x correct to 2 decimal places.

(a) $x^3 - 5x^2 + 3x + 8 = 0$

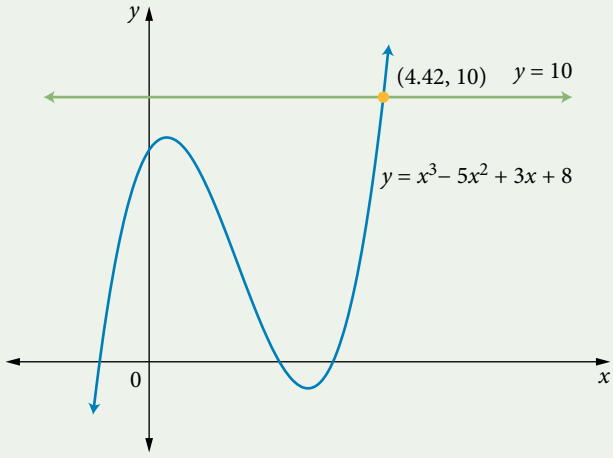
THINKING	WORKING
1 Construct linear functions from the expressions on either side of the equals sign in the form $y = \text{LHS}$ and $y = \text{RHS}$.	$y = x^3 - 5x^2 + 3x + 8$ $y = 0$
2 Sketch the graph, noting the x-intercepts.	 The x-intercepts, are at (-0.94, 0), (2.46, 0) and (3.47, 0), correct to 2 decimal places
3 Interpret the answer.	The solutions to the equation $x^3 - 5x^2 + 3x + 8 = 0$ are $x = -0.94$, $x = 2.46$ or $x = 3.47$.

(b) $x^3 - 5x^2 + 3x + 8 = 10$

METHOD 1

1 Construct linear functions from the expressions on either side of the equals sign in the form $y = \text{LHS}$ and $y = \text{RHS}$.	$y = x^3 - 5x^2 + 3x + 8$ $y = 10$

2 Sketch the graph, noting the point(s) of intersection.



3 Interpret the answer.

The solution to the equation $x^3 - 5x^2 + 3x + 8 = 10$ is $x = 4.42$.

METHOD 2

1 Use subtraction so you have an equation of the form $P(x) = 0$.

$x^3 - 5x^2 + 3x + 8 = 10$

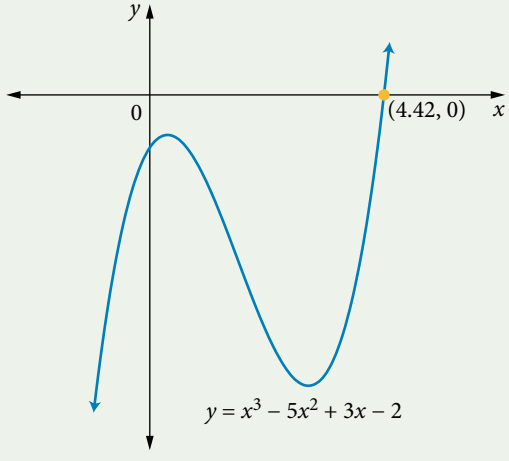
$x^3 - 5x^2 + 3x - 2 = 0$

2 Identify the strategy for using technology. Here, draw the graph and read the roots.

Graph $y = x^3 - 5x^2 + 3x - 2$.

Read the x-value(s) for $y = 0$.

3 Sketch the graph noting the x-intercept.



4 Interpret the answer.

The solution to the equation $x^3 - 5x^2 + 3x + 8 = 10$ is $x = 4.42$.

A cubic function will always have at least one point in common with any linear function, and may have as many as three. If you let the cubic function be $C(x)$ and the linear function be $L(x)$, then the x-values of the points of intersection can be found by solving $C(x) = L(x)$ or $C(x) - L(x) = 0$.

The expression $C(x) - L(x)$ must be a cubic expression, so there will be at least one and up to three solutions.

Explore further

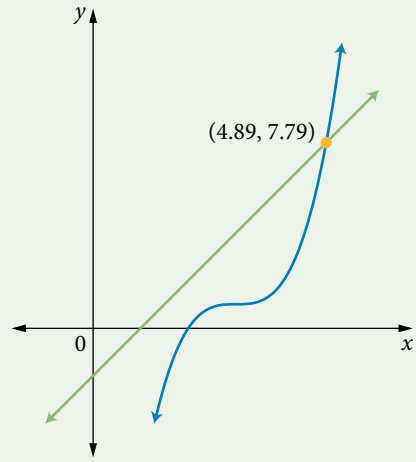
Solutions to cubic equations

Explore the shapes and roots of cubic functions with a given number of linear factors.

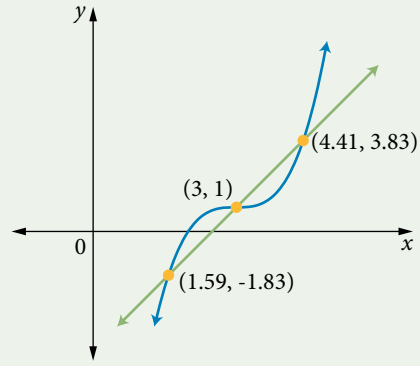
35 A cubic expression equal to a linear expression

Solve the equations for x, correct to 2 decimal places.

(a) $(x-3)^3+1=2x-2$

THINKING	WORKING
1 Construct and graph linear functions from the expressions on either side of the equals sign in the form $y=$ LHS and $y=$ RHS.	Graph $y=(x-3)^3+1$ and $y=2x-2$. 
2 Identify the coordinates of the point of intersection.	The coordinates of the point of intersection are (4.89, 7.79), correct to 2 decimal places.
3 Interpret the answer.	The solution to $(x-3)^3+1=2x-2$ is $x=4.89$.

(b) $(x-3)^3+1=2x-5$

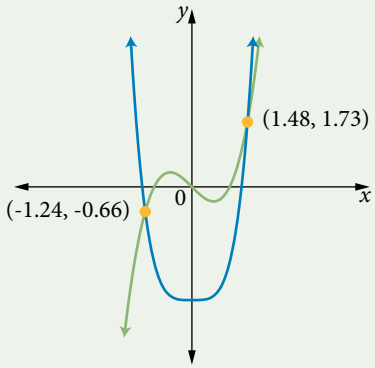
1 Construct and graph linear functions from the expressions on either side of the equals sign in the form $y=$ LHS and $y=$ RHS.	Graph $y=(x-3)^3+1$ and $y=2x-5$. 
2 Identify the coordinates of the point of intersection.	The coordinates of the points of intersections are (1.59, -1.83), (3, 1), (4.41, 3.83), correct to 2 decimal places
3 Interpret the answer.	The solutions to $(x-3)^3+1=2x-5$ are $x=1.59$, $x=3$ or $x=4.41$.

When graphing $y = P(x)$ and $y = Q(x)$, the x-value of each point of intersection gives a solution to $P(x) = Q(x)$, and the corresponding y-value gives the equal values of LHS and RHS in the equation.

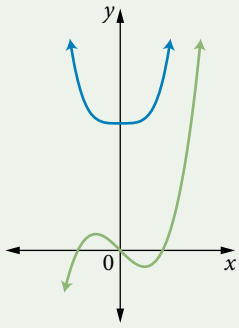
36 A quartic expression equal to a cubic expression

Solve the equations, giving values of x correct to 2 decimal places.

(a) $x^4 - 3 = x\left(x^2 - 1\right)$

THINKING	WORKING
1 Construct and graph linear functions from the expressions on either side of the equals sign in the form $y =$ LHS and $y =$ RHS.	Graph $y = x^4 - 3$ and $y = x\left(x^2 - 1\right)$. 
2 Identify the coordinates of the point of intersection.	The coordinates of the points of intersection are (-1.24, -0.66) and (1.48, 1.73), correct to 2 decimal places,
3 Interpret the answer.	The solutions to $x^4 - 3 = x\left(x^2 - 1\right)$ are $x = -1.24$ or $x = 1.48$.

(b) $x^4 + 3 = x\left(x^2 - 1\right)$

1 Construct and graph linear functions from the expressions on either side of the equals sign in the form $y =$ LHS and $y =$ RHS.	Graph $y = x^4 + 3$ and $y = x\left(x^2 - 1\right)$. 
2 Identify the coordinates of the point of intersection.	There are no points of intersection.
3 Interpret the answer.	$x^4 + 3 = x\left(x^2 - 1\right)$ has no real solutions.

A cubic function may have as many as four points of intersection with a quartic function. If you let the cubic function be $C(x)$ and the quartic function be $Q(x)$, the x-values of the points of intersection can be solved using $Q(x) = C(x)$ or $Q(x) - C(x) = 0$. The expression $Q(x) - C(x)$ must be a quartic expression so there could be no solutions or up to four solutions.

Notice that quartic graphs become steeper than cubic graphs at the extreme values, so there are no possibilities of solutions outside the domain where the graphs diverge.

Algebraic techniques

When solving $P(x) = Q(x)$ algebraically you should first subtract and equate to zero, so $P(x) - Q(x) = 0$. The exception to this is when both expressions are linear, allowing solving by isolating the variable.

If the resulting expression is in factorised form, then the null factor law can be used to solve the equation.

37 Solve using common factors

Solve each of the following equations.

(a) $x^3 + 3x^2 = x^2 + 8x$

THINKING	WORKING
1 Write in the form $P(x) - Q(x) = 0$.	$x^3 + 3x^2 = x^2 + 8x$ $x^3 + 3x^2 - x^2 - 8x = 0$ $x^3 + 2x^2 - 8x = 0$
2 Factorise by taking out the highest common factor of x and then factorising the quadratic expression.	$x^3 + 2x^2 - 8x = 0$ $x(x^2 + 2x - 8) = 0$ $x(x+4)(x-2) = 0$
3 Use the null factor law to solve for x.	$x = 0$ $\quad x + 4 = 0$, $x = -4$ $\quad x - 2 = 0$, $x = 2$
4 Write the solutions.	$x = 0$, $x = -4$ or $x = 2$

(b) $x^2(x+3) = 4(x+3)$

1 Write in the form $P(x) - Q(x) = 0$.	$x^2(x+3) = 4(x+3)$ $x^2(x+3) - 4(x+3) = 0$
2 Factorise by taking out the highest common factor of $(x - 3)$ and then factorise the quadratic the expression using the difference of perfect squares: $(a-b)^2 = a^2 - 2ab + b^2$	$(x+3)(x^2 - 4) = 0$ $(x+3)(x+2)(x-2) = 0$
3 Use the null factor law to solve.	$x + 3 = 0$, $x = -3$ $\quad x + 2 = 0$, $x = -2$ $\quad x - 2 = 0$, $x = 2$
4 Write the solutions.	$x = -3$, $x = -2$ or $x = 2$

If a cubic expression can be factorised, use the factor theorem followed by long division or re-expression to begin the process. The roots of quadratic expressions (if they exist) can be determined from factors or from the quadratic formula.

38 Solve after first factorising

Solve the equations, stating the exact value(s) of x.

(a) $x^3 + x = 5x^2 - 2x - 9$

THINKING	WORKING
1 Write in the form $P(x) - Q(x) = 0$.	$x^3 + x = 5x^2 - 2x - 9$ $x^3 + x - (5x^2 - 2x - 9) = 0$ $x^3 + x - 5x^2 + 2x + 9 = 0$ $x^3 - 5x^2 + 3x + 9 = 0$
2 Use the factor theorem to determine an initial root.	Try $x = -1$: $P(1) = (-1)^3 - 5(-1)^2 + 3(-1) + 9$ $= -1 - 5 - 3 + 9$ $= 0$ $\therefore x + 1$ is a factor.
3 Use re-expression to start the process of factorisation. Equate the coefficients of the x^2 term.	$(x+1)(x^2 + kx + 9)$ $= x^3 + kx^2 + 9x + x^2 + kx + 9$ $= x^3 + (k+1)x^2 + (k+9)x + 9$ $= x^3 - 5x^2 + 3x + 9$ Equate the coefficients of either term for k: $k + 1 = -5$ or $k + 9 = 3$ $k = -6$ $\quad$ $k = -6$
4 Factorise the quadratic, if possible.	$x^3 - 5x^2 + 3x + 9 = 0$ $(x+1)(x^2 - 6x + 9) = 0$ $(x+1)(x-3)^2 = 0$
5 Solve for x using the null factor law.	$x + 1 = 0$ $\quad$ $x - 3 = 0$ $x = -1$ $\quad$ $x = 3$
6 Write the solutions.	$x = -1$ or $x = 3$

(b) $x^2(3x+4) = 2x^3 + 5$

1 Write in the form $P(x) - Q(x) = 0$. Expand and simplify.	$x^2(3x+4) = 2x^3+5$ $x^2(3x+4) - 2x^3 - 5 = 0$ $3x^3 + 4x^2 - 2x^3 - 5 = 0$ $x^3 + 4x^2 - 5 = 0$
2 Use the factor theorem to determine an initial root.	Try $x = 1$: $P(1) = (1)^3 + 4(1)^2 - 5$ $= 1 + 4 - 5$ $= 0$ $\therefore x - 1$ is a factor.
3 Use long division or re-expression to start the process of factorisation.	$(x-1)(x^2 + kx + 5)$ $= x^3 + kx^2 + 5x - x^2 - kx - 5$ $= x^3 + (k-1)x^2 + (-k+5)x - 5$ $= x^3 + 4x^2 - 5$ Equating the coefficients gives: $k - 1 = 4 \qquad -k + 5 = 0$ $k = 5 \qquad k = 5$ $\therefore x^3 + 4x^2 - 5 = (x-1)(x^2 + 5x + 5)$
4 Factorise the quadratic, if possible. Use the discriminant to check for rational roots.	$b^2 - 4ac = 5^2 - 4 \times 1 \times 5$ $= 25 - 20$ $= 5$ Given that $\Delta > 0$, two real roots exist. 5 is not a perfect square, so the roots are not rational and will contain surds.
5 Write the equation with the LHS in fully factorised form.	$(x-1)(x^2 + 5x + 5) = 0$
6 Use the null factor law to determine the root of the linear factor.	$x - 1 = 0$ $x = 1$
7 Use the quadratic formula to solve for x in the quadratic factor.	$x^2 + 5x + 5 = 0$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 5}}{2 \times 1}$ $= \dfrac{-5 \pm \sqrt{5}}{2}$
8 Write the solutions.	$x = 1,\ x = \dfrac{-5 + \sqrt{5}}{2}$ or $x = \dfrac{-5 - \sqrt{5}}{2}$

EXERCISE 3.6 Solving polynomial equations

Worked Example

1 Solve the equations, giving values of x correct to 2 decimal places. 34

(a) $x^3 + 4x^2 - 6x + 7 = 0$ (b) $x^3 + 4x^2 - 6x + 7 = 20$

2 Solve the equations, giving values of x correct to 2 decimal places.

(a) $-2x^3 + 10x^2 - 3x - 9 = 0$ (b) $-2x^3 + 10x^2 - 3x - 9 = 2$

3 Solve the equations, giving values of x correct to 2 decimal places.

(a) $3(x+2)^4 + 15 = 0$ (b) $3(x+2)^4 + 15 = 30$

4 Solve the equations, giving values of x correct to 2 decimal places. 35

(a) $4x^2 + 8 = x^3 + 6x$ (b) $0.3x^3 + 5.62x^2 - 14.83x - 7.191 = 5.07x - 7$

(c) $x^3 = x^2 + 9x - 12$ (d) $x^3 + x^2 - 6 = 2x - 5$

5 Solve the equations, giving values of x correct to 2 decimal places. 36

(a) $x^4 - 5 = (x+2)(x-1)(x-3)$ (b) $x^4 + 5 = (x+2)(x-1)(x-3)$

6 Determine whether the given x-value is a solution to the accompanying equation.

(a) $x^3 + 9x^2 + 26x + 24 = 0;\ x = -2$ (b) $x^3 - 6x^2 + 9x + 1 = 0;\ x = 4$

(c) $x^3 + 3x = 6x^2 + 5;\ x = -1$ (d) $x^3 + 24 = 20x - 2x^2;\ x = 2$

7 How many real solutions does the equation $(x+2)(x^2 + 6x + 10) = 0$ have?

A 1 B 2 C 3 D 4

8 (a) How many solutions does the equation $x^4 - 10 = x^2 + 2$ have?

A 4 B 3 C 2 D 1

(b) Explain the error made by a student who stated that $x^4 - 10 = x^2 + 2$ has no solutions.

9 For the cubic polynomial $f(x) = 6x^3 - 37x^2 + 5x + 6$, it is known that $f(6) = 0$.

(a) Write $f(x)$ as the product of a linear factor and a quadratic factor.

(b) If possible, write $f(x)$ as the product of three linear factors.

(c) Determine all of the solutions to the equation $f(x) = 0$.

10 Solve the equations. 37

(a) $2x^3 + 8x^2 = 10x$ (b) $2x^3 = 11x^2 + 21x$ (c) $x^3 = 4x$ (d) $2x^3 = 50x$

(e) $3x^3 = 27x^2$ (f) $4x^3 + 60x^2 = 0$ (g) $x^3 + 13x^2 = 30x$ (h) $x^2(x+3) = 49(x+3)$

(i) $x(x^2+1) = 5(x^2+1)$ (j) $x^2(x+2) = 2x(x+2) - (x+2)$

11 A large container holding water is being emptied. The volume V_1 (in litres) at any time t minutes is given by the equation $V_1 = 6(8-t)^3$, where $0 \le t \le 8$.

(a) Determine the volume at the following times: $t = 0$, $t = 3$ and $t = 7$.

(b) Sketch the graph of V_1 against t for $0 \le t \le 8$.

(c) Determine the times when the volume of the container is: 1296 L, 384 L, 162 L.

A similar container is also being emptied and its volume of water V_2 (in litres) at any time t minutes is given by the equation $V_2 = 2(4+t)(t-8)^2$, where $0 \le t \le 8$.

(d) Determine the volume at the following times: $t = 0$, $t = 3$, $t = 7$.

(e) Sketch the graph of V_2 against t for $0 \le t \le 8$ on the same set of axes as the graph for V_1.

(f) Determine the time when the volume of the second container is: 490 L, 162 L, 80 L.

(g) The containers V_1 and V_2 start being emptied at the same time. Determine the time(s) when the volume of water in both containers is the same.

Worked Example

38

12 Solve the equations, giving values of x correct to 2 decimal places.

(a) $x^3 - 5x^2 + 8x - 4 = 0$ (b) $3x^3 = 4x^2 + 13x + 6$ (c) $x^3 + x = 3 - x^2$

(d) $x^3 + 2x = 3x^2 - x + 1$ (e) $3x^3 + 4x - 6 = 2x^3 + 6x^2 - 7x$

(f) $x^3 - 2x = 2x^2 - x - 2$ (g) $x^2(x+2) = 3x + 6$

13 The equation $x^3 + ax^2 + bx - 64 = 0$ has solutions $x = 4$ and $x = -2$. Determine the values of a and b.

14 A system of equations is given by $y = x^3 + 9x^2 + 26x + 16$ and $y - 3x = 1$. Determine the simultaneous solutions. Write your answers as coordinate pairs.

15 The graphs of the functions $h(x) = 3x + 1$ and $g(x) = 2x^3 - 5x^2 + x + 6$ intersect. Determine the coordinates of these points.

16 (a) For the functions $p(x) = -2x + 3$ and $q(x) = x^3 - 6x^2 + 6x + 3$, determine all solutions to the equation $p(x) = q(x)$.

(b) If graphs were drawn of $y = p(x)$ and $y = q(x)$, what would be the points of intersection?

17 Determine all solutions to the equation $x^4 - 1 = 3x(x^2 - 1)$. Justify your solution with algebraic calculations and a graph.

18 The equation $x(x^2 - 7) = 2x + c$ has exactly two solutions.

(a) Estimate the value(s) of c, writing your answer(s) in the form $c_1 < c < c_2$, where c_1 and c_2 are integers. Justify your working with graphs.

(b) If the equation $x(x^2 - 7) = mx + 15$ has exactly two solutions, estimate the value of m, writing your answer(s) in the form $m_1 < m < m_2$, where m_1 and m_2 are integers. Justify your working with graphs.

19 The equation $x^4 + 5 = P(x)$, where $P(x)$ is a cubic expression, has four solutions. Create a possible expression for $P(x)$. Determine the solutions for the equation, correct to 2 decimal places.

Summary

3

Polynomials

A polynomial function P is defined as: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a non-negative integer, the numerical coefficients $a_0, a_1, \ldots a_{n-1}, a_n$ are real numbers and the domain is $\mathbb{R}$.

The leading term of any polynomial is the term with the highest index (with a non-zero coefficient).

The degree of the polynomial is the index of the leading term.

A monic polynomial is a polynomial whose leading coefficient is 1.

The constant term is the term where the variable has an index of zero. This is the term not containing a variable and is called the term independent of the variable.

Linear functions

Linear functions are odd polynomials of degree 1 and are represented by a straight-line graph.

Linear functions with a positive gradient, and all odd polynomials with a positive coefficient of the leading term, behave as follows:

- as $x \to +\infty$, $y \to +\infty$, and as $x \to -\infty$, $y \to -\infty$.

Linear functions with a negative gradient, and all odd polynomials with a negative coefficient of the leading term, behave as follows:

- as $x \to +\infty$, $y \to -\infty$ and as $x \to -\infty$, $y \to +\infty$.

Quadratic functions

Quadratic functions are even polynomials of degree 2 that are represented by a parabolic graph.

For any function of the form $y = ax^2$:

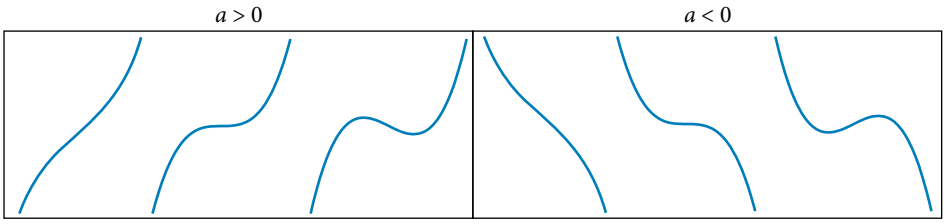
- the equation of the line of symmetry is $x = 0$
- the coordinates of the turning point are (0, 0).

Cubic functions

Cubic functions are odd polynomials of degree 3 that are represented by a cubic-curve graph.

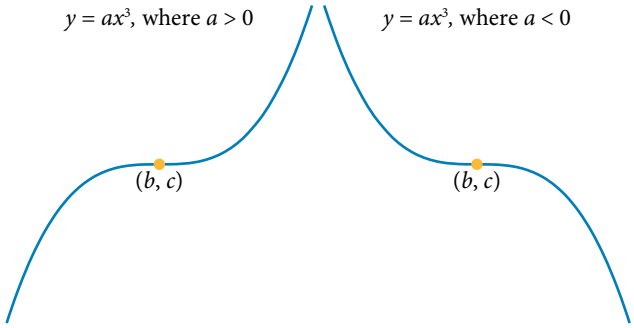
Cubic polynomial equations can have 1, 2 or 3 solutions.

The general form of a cubic is: $y = ax^3 + bx^2 + cx + d$. All cubic graphs change from concave down to concave up, for $a > 0$, or concave up to concave down, for $a < 0$, at a point of oblique inflection.



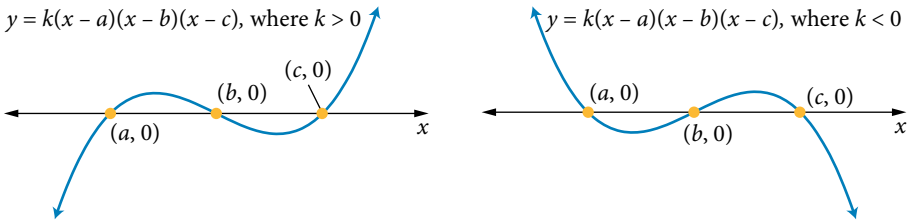
The basic cubic, $y = x^3$, can be transformed into $y = a(x-b)^3 + c$, where:

- b represents the horizontal translation of the function b units in the x-direction
- c represents the vertical translation of the function c units in the y-direction
- (b, c) are the coordinates of the stationary point of inflection
- the size of a, $|a|$ represents the dilation factor parallel to the y-axis
- if a is negative, $a < 0$, the graph is reflected in the x-axis.



Cubic functions can also be written in the form $y = k(x-a)(x-b)(x-c)$, where:

a, b, and c represent x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$.

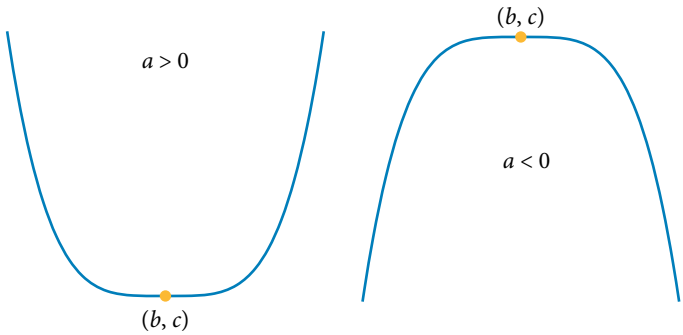


Quartic functions

Quartic functions are even polynomials of degree 4.

The basic quartic, $y = x^4$ can be transformed into $y = a(x-b)^4 + c$, where:

- b represents the horizontal translation of the function b units in the x-direction
- c represents the vertical translation of the function c units in the y-direction
- (b, c) are the coordinates of the turning point
- $x = b$ is the equation of the line of symmetry
- the size of a $|a|$ represents the dilation factor parallel to the y-axis
- if a is negative ($a < 0$), the graph is reflected in the x-axis.



Even polynomial functions

For the graphs of all polynomials in the form of $y = ax^n$, where n is even:

- if $a > 0$, then as $x \to \infty$, $y \to \infty$; as $x \to -\infty$, $y \to \infty$
- if $a < 0$, then as $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to -\infty$.

Expanding and factorising

Square of a sum: $(a+b)^2 = a^2 + 2ab + b^2$

Square of a difference: $(a-b)^2 = a^2 - 2ab + b^2$

Cube of a sum: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Cube of a difference: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Difference of squares: $a^2 - b^2 = (a-b)(a+b)$

Difference of cubes: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

Sum of cubes: $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

Division of polynomials

When one polynomial $P(x)$ is divided by another polynomial $D(x)$, the result can be written as:

$P(x) = \mathrm{Q}(x) + \dfrac{R(x)}{D(x)}$, where $\mathrm{Q}(x)$ is the quotient, $R(x)$ is the remainder, and $D(x)$ is the divisor.

When $P(x)$ is divided by $(ax+b)$, the remainder is $P\left(\dfrac{-b}{a}\right)$.

The factor theorem states:

- if $(x-a)$ is a factor of $P(x)$, then $P(a) = 0$
- if $(ax+b)$ is a factor of $P(x)$, then $P\left(\dfrac{-b}{a}\right) = 0$.

Solving equations

Equations of the form $P(x) = Q(x)$ can be solved by:

- graphing $y = P(x)$ with $y = Q(x)$. The x-value of each point of intersection gives a solution
- subtracting and equating to zero, so $T(x) = 0$ for $T(x) = P(x) - \mathrm{Q}(x)$. Now factorise and use the null factor theorem to solve.

Chapter review 3

Exercise 3.1

1 For each of the following polynomials determine its degree, $P(-1)$ and $P(2)$.

(a) $P(x) = -2x^3 + 4x^2 - x + 5$

(b) $P(x) = 3x^2 - 2x^4 + \frac{1}{2}x - 3$

(c) $P(x) = -\frac{1}{2}x^5 + 3x^2 - 2$

(d) $P(x) = -2(x+1)(x-3)(3x-2)$

(e) $P(x) = -6$

(f) $P(x) = 4x^6 - 5x^4 + 3x^2 - 1$

Exercise 3.1

2 Given $A(x) = 3x^2 - 5x^3 + \frac{1}{3}x - 5$, determine the value of $A(-3)$.

Exercise 3.1

3 A polynomial is defined by $f(x) = x^3 - ax^2 - bx + b$. Determine the values of a and b, given $f(1) = 0$ and $f(-2) = 63$.

Exercise 3.1

4 Determine the equation of a cubic function that includes the following points.

(a) (2, -22), (2.5, -43.5), (-1, 2) and (-2, 6)

(b) (2, -28), (2.5, -52.375), (-1, -13) and (-2, -4)

Exercise 3.1

5 Determine the equation of a cubic function whose graph passes through the points (-3, -1), (1, 1), (5, 3) and (6, -2). Write your answers correct to 2 decimal places where necessary. Graph each function, showing the four points, axis intercepts and turning points.

Exercise 3.2

6 Expand and simplify by collecting like terms.

(a) $3x(x^2 + 4)$

(b) $(y+7)(2y^2 - 1)$

(c) $-2x(x+5)^2$

(d) $z(z-1)^2$

(e) $(x+1)(x-2)(x+3)$

(f) $(y-3)^2(y+2)$

(g) $(g+3)(4-g)(g+5)$

(h) $(a-1)(2-a)(a+1)$

Exercise 3.2

7 Which of the following is not a possible expansion of $P(x) = (x-1)(x+2)(3x-4)$?

A $(x^2 + x - 2)(3x - 4)$

B $(3x^2 - 7x + 4)(x + 2)$

C $(3x^2 + 7x - 4)(x + 2)$

D $(3x^2 + 2x - 8)(x - 1)$

Exercise 3.3

8 Factorise $8x^3 + 125y^3$.

Exercise 3.3

9 For the following polynomials determine whether or not $A(x)$ is a factor of $P(x)$.

(a) $P(x) = x^3 + x^2 - 9x - 9$; $A(x) = (x-1)$

(b) $P(x) = x^3 + x^2 - 9x - 9$; $A(x) = (x+1)$

(c) $P(x) = x^3 - 2x^2 - 5x + 6$; $A(x) = (x-1)$

(d) $P(x) = x^3 - 2x^2 - 5x + 6$; $A(x) = (x-2)$

(e) $P(x) = 2x^3 - 3x^2 + 4x - 3$; $A(x) = (x-1)$

(f) $P(x) = 3x^3 - 3x^2 - 15x + 6$; $A(x) = (x+2)$

Exercise 3.3

10 Consider the polynomial function $f(x) = 2x^3 - 6x^2 - 2x + 6$.

(a) Determine the remainder when this polynomial is divided by $(x-5)$.

(b) Determine the quotient $Q(x)$ when the polynomial is divided by $(x-5)$.

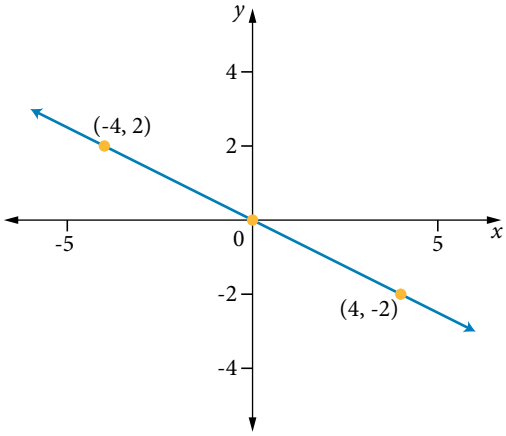
Exercise 3.3

11 Factorise $P(x) = -2x^3 + 7x + 1$ by first graphing the function. Round to 2 decimal places where necessary.

12 For each of the graphs below, state the rule of the function.

Exercise 3.4

(a)



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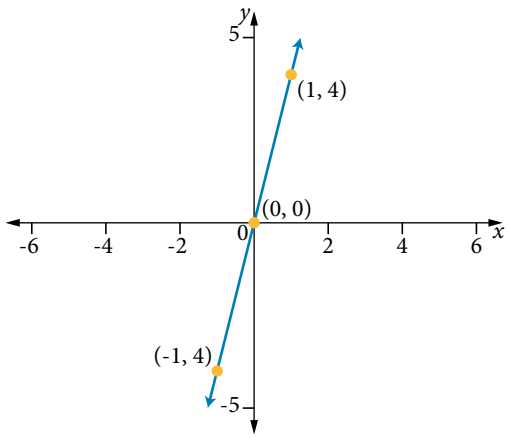

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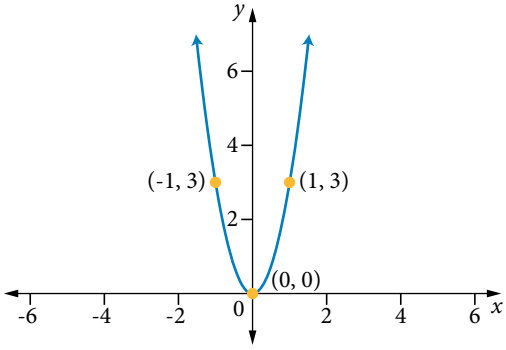

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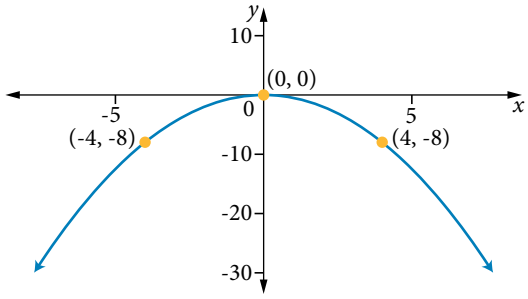

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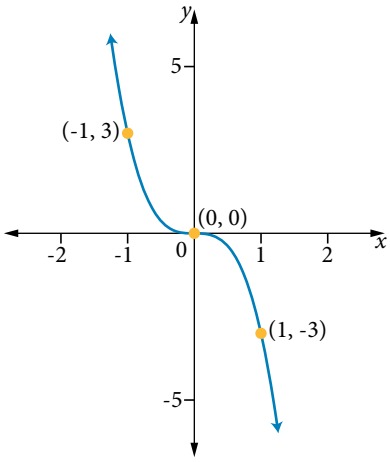

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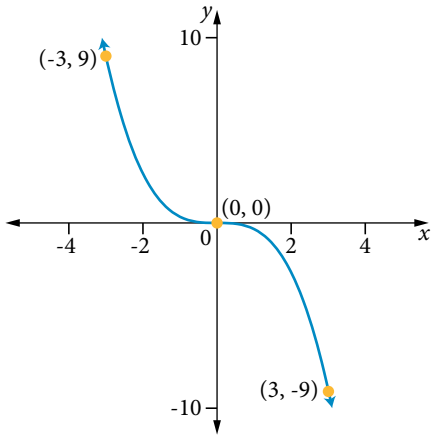
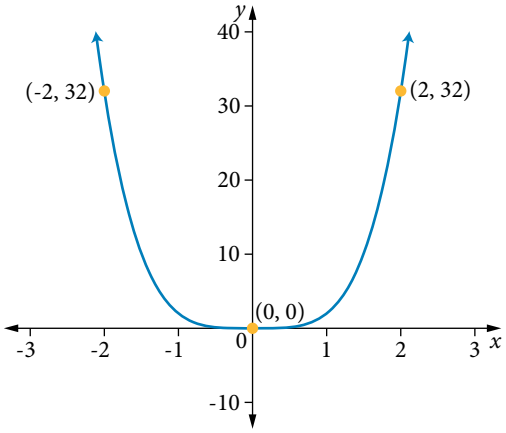
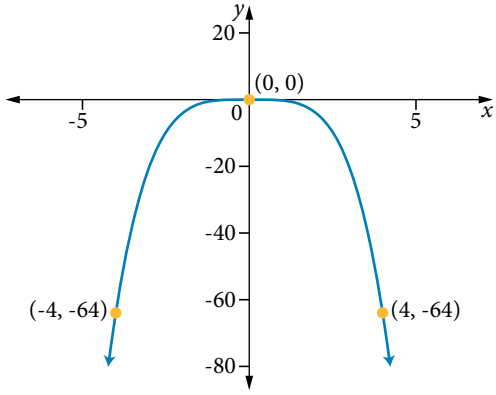


(g)



(h)



Exercise 3.4 **13** The graph of $y = x^3$ has been transformed to give $y = a(x-b)^3 + c$. Determine the values of a, b and c if the graph has a stationary point of inflection at (2, -4) and passes through (5, 23).

Exercise 3.4 **14** Determine the equation of the polynomial after each set of transformations are applied to $y = x^3$.

(a) Reflection in the x-axis followed by a translation 4 units down.

(b) Reflection in the x-axis followed by a translation 4 units up.

(c) Translation 3 units right, followed by a translation 4 units up.

(d) Dilation parallel to y-axis of scale factor 2, followed by a translation of 4 units down.

Exercise 3.4 **15** A polynomial function is defined as $f(x) = -2x^4$. Which of the following statements is not true?

A $f(-2) = -32$

B $f\left(-\frac{1}{2}\right) = -\frac{1}{8}$

C As $x \to \infty$, $y \to -\infty$

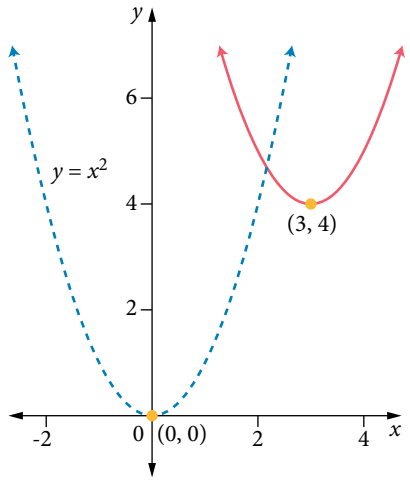
D As $x \to -\infty$, $y \to \infty$

Exercise 3.4 **16** Describe the sequence of transformations to change $y = x^3$ to $y = -4(x-1)^3 + 2$.

Exercise 3.4 **17** A graph with equation $y = x^4$ is transformed into $y = a(x-b)^4 + c$ by the following steps: dilation parallel to y-axis scale factor 3, a reflection in the x-axis, horizontal translation 2 units left, vertical translation 6 units up.
Determine the equation of the transformed function.

Exercise 3.4 **18** By examining the graphs, list the transformation(s) for the original polynomial function (dotted curve) to transform to the new function (solid line). Provide the rule of the transformed polynomial function. Assume that the function has retained its original shape (so there are no dilations parallel to either axis).

(a)



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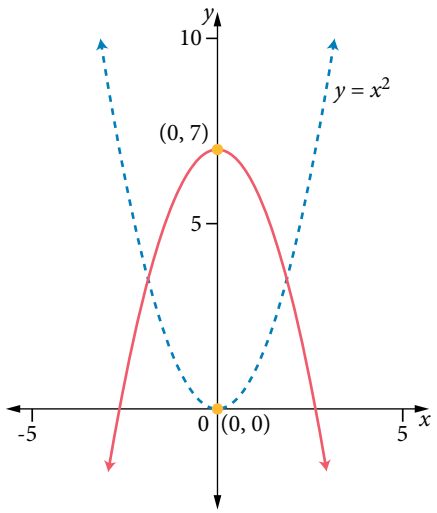

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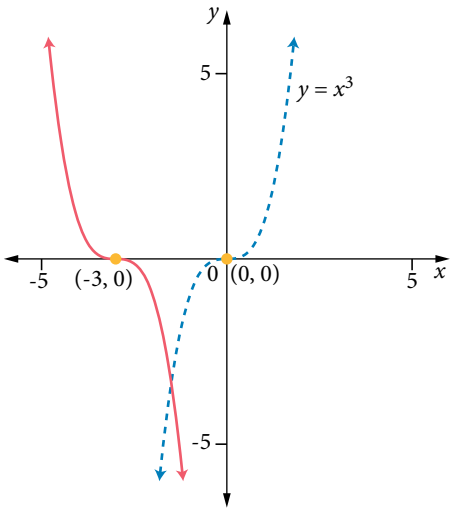

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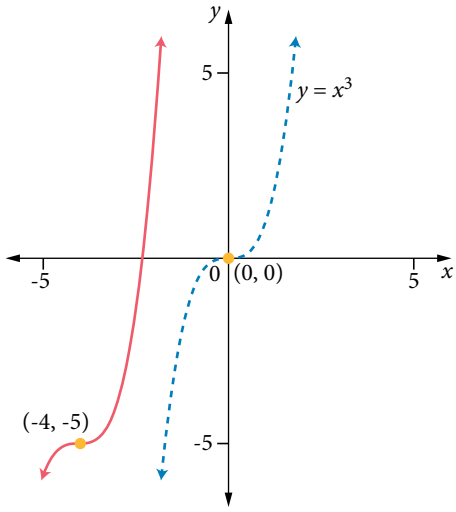

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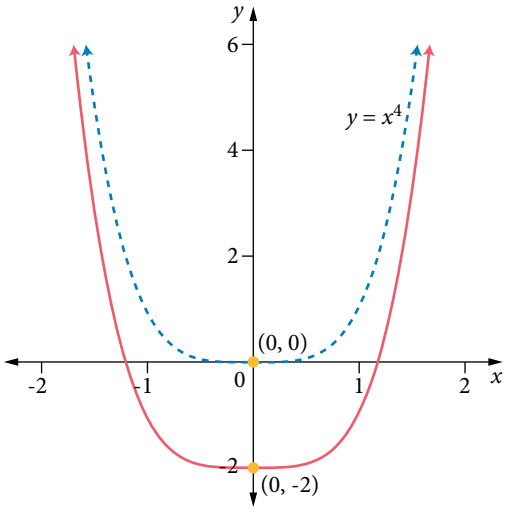

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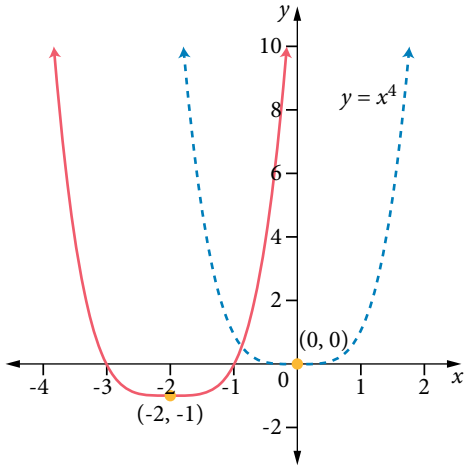


19 Sketch the graph of each cubic function, clearly showing all intercepts and critical points. Exercise 3.5

(a) $y = -(4 - x)(x - 2)(x + 4)$ (b) $y = 3(x - 2)^3 - 4$ (c) $y = (3 - x)^3$

(d) $g(x) = x^3 - 4x^2 + x + 6$ (e) $h(x) = -(x - 3)(x + 2)^2$ (f) $p(x) = -(x - 2)^3 + 3$

20 A cubic polynomial intersects the x-axis at the x-values of m, n and P. Determine the cubic equation that displays each of the following features. Exercise 3.5

(a) $m = -4$, $n = 1$, $p = 2.5$ and the coordinates of the y-intercept are (0, -15).

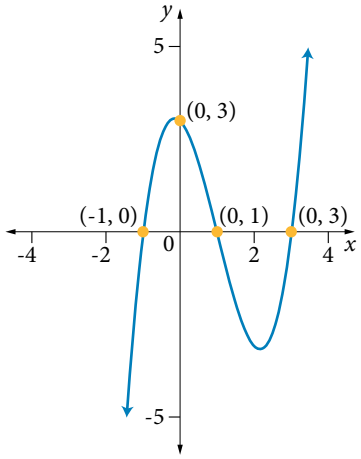
(b) $p = 4$ and coordinates of the turning points are (1, 0) and (3, -8).

(c) $m = n = p = -3$, and the coordinates of the y-intercept are (0, -3).

21 For each cubic function, determine the coordinates of the x- and y-intercepts. Exercise 3.5

(a) $y = (x + 4)(4 - x)(x + 2)$ (b) $y = (x + 4)(x^2 - 4)$

Exercise 3.5 **22** Determine the rule of the function shown in the graph.



Exercise 3.5 **23** A cubic polynomial has x-intercepts at (-2, 0), (-1, 0) and (3, 0), and a y-intercept at (0, 3).

(a) Determine the equation of the polynomial function.

(b) Determine the nature and coordinates of the turning points. State the coordinates correct to 2 decimal places.

Exercise 3.5 **24** A cubic polynomial has x-intercepts at (-3, 0), (1, 0) and (5, 0) and y-intercept at (0, 5). Determine the equation for this function.

Exercise 3.6 **25** Solve the equations, giving values of x correct to 2 decimal places.

(a) $2x^3 + 5x^2 - 6x - 7 = 3x - 2$

(b) $25 - (x-2)^4 = 2(1-x)(x-3)^2$

Exercise 3.6 **26** The volume of a wooden crate is $756\ \text{m}^3$. The dimensions of the crate in metres are length $= x$, width $= (3x + 9)$, height $= (4x - 7)$.
Determine the value of x for this volume. Hence, provide the dimensions of the crate.

Exercise 3.1 **27** A polynomial is defined by $h(x) = mx^4 + nx^3 - 3x^2 - 4x - 7$. Determine the value of m and n, given that $h(1) = -2$ and $h(2) = 235$.

Exercise 3.1 **28** A polynomial is defined by $P(x) = gx^2 - 5x - 3$ and it passes through the points (1, -6) and (f, 4). The values of g and f are:

A $g = 2, f = -1$ or 3.5

B $g = -2, f = -1$ or 3.5

C $g = -2, f = 2$ or -1

D $g = 1, f = -2$

Exercise 3.1 **29** The function $f(x) = x^3 - (a+2)x^2 + 11x + (b-5)$ has an x-intercept at (1, 0) and a y-intercept at (0, -6). Determine the values of a and b.

Exercise 3.3 **30** $(x - 1)$ and $(x - 4)$ are factors of the polynomial $3x^3 + bx^2 + cx + 8$. The values of b and c are:

A $b = 13, c = 2$

B $b = 15, c = 3$

C $b = -13, c = 2$

D $b = 15, c = -3$

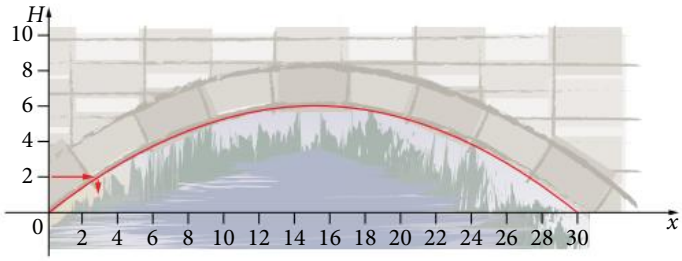
Exercise 3.3 **31** Fully factorise the following polynomial expressions.

(a) $6x^3 - x^2 - 15x$

(b) $x^3 + 4x^2 + 6x + 4$

(c) $x^3 - 5x^2 - 4x + 20$

32 A bridge is to be built over a river using a quadratic model for its shape and dimensions. The cross-section of the bridge can be modelled by the graph of a quadratic function: $H(x) = a(x-b)^2 + c$, where $H(x)$ is the height of the bridge at a horizontal distance x from the origin (0, 0). The base of the bridge is 30 m wide, and it has a maximum height of 6 m. The cross-section of the bridge is shown on the graph above in red. Exercise 3.4



(a) Determine the values of a, b and c. Hence write the rule for $H(x)$ including its domain and range.

(b) A floating footpath will run along the left side of the river and pass under the bridge. How far does the left edge of the footpath need to be away from the origin to ensure that a person 2 m tall does not bump their head on the bridge?

33 Determine the coordinates of the points of intersection of $y = x^3 + 9x^2 + 26x + 16$ and $y - 3x = 1$. Exercise 3.6

34 To determine the volume of a pyramid, use the formula: Exercise 3.6

$V = \frac{1}{3} \times$ area of base $\times$ perpendicular height.

A rectangular-based pyramid has a volume of 112 m^3. If the dimensions of the base are $(2x-5)$ m and $(x+2)$ m, and the perpendicular height is x m, determine the numerical dimensions of the pyramid by solving for x.

35 Solve the following equations for x, by first fully factorising where possible. Exercise 3.6

(a) $x^3 + 3x^2 - 6x - 8 = 0$

(b) $x^3 + 2x^2 - 21x + 18 = 0$

(c) $x^3 + 4x^2 + 7x + 6 = 0$

(d) $2x^3 + 9x^2 + 3x - 4 = 0$

36 Solve the equations. Exercise 3.6

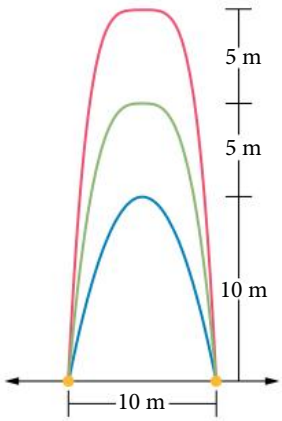
(a) $2x^3 = 11x^2 - 5x$

(b) $(x+3)(x^2+2) = x(x+3)$

(c) $x^3 + 8x^2 - 8 = x^2 - 16x - 20$

37 Determine values of the integers x, y and z, given $\left(x - \sqrt{y}\right)\left(z + 3\sqrt{y}\right) = 27 - 5\sqrt{5}$. Exercise 3.2

38 A piece of artwork is made making use of polynomial functions. Write equations with suitable domains to model the artwork. Demonstrate your solution with a graph. Exercise 3.4



39 Create a piece-wise quadratic function to mimic the function $y = (x-2)(x-4)(x-7) + 20$, $1 \le x \le 8$. Explain your strategy in your working, and comment on the limitations of your model. Exercise 3.4, 3.5

Mixed review

CHAPTERS 1–3

Exercise 1.1

1 Determine whether each of the following relations are functions.

(a) $2x - 3y = 14$ **(b)** $8 - 7y = 4x + 9$

Exercise 1.3

2 Plot the graph of the piece-wise function.

$$y = \begin{cases} -2x, -1 < x \le 1 \\ x - 3, x > 1 \end{cases}$$

Exercise 1.4

3 For the arithmetic sequence for which $a = -3$, $d = 4$, determine:

(a) the 15th term **(b)** the recursive definition

(c) the general term **(d)** the sum of the first 10 terms.

Exercise 2.2

4 Determine the equation for the line of symmetry for each of the following quadratic functions.

(a) $y = x^2 - 12$ **(b)** $y = -2(x - 4)^2$ **(c)** $s = \frac{1}{2}(t + 6)^2 - 1$

Exercise 2.2

5 Determine the coordinates of the turning point (or vertex) for the function with rule $s = 2(t + 4)^2$.

Exercise 2.1

6 Give the coordinates of the vertical axis intercept for the function with rule $m = (2p + 1)(p - 5)$.

Exercise 2.3

7 Solve the following equations using factorisation and the null factor law.

(a) $3x^2 + 15x = 0$ **(b)** $m^2 + 4m = 32$ **(c)** $3t^2 = 10t + 8$

Exercise 3.1

8 Determine whether each of the following equations is a polynomial.

(a) $P(x) = 2$ **(b)** $P(x) = \frac{x^2}{4} + 7x - 5$ **(c)** $P(x) = 8x^3 - x^{\frac{1}{2}}$

(d) $P(x) = x^5 + 3x^{-2} + 4x - 8$ **(e)** $P(x) = x^6 + \frac{3}{x} + 8x^5 - x^2$

Exercise 3.2

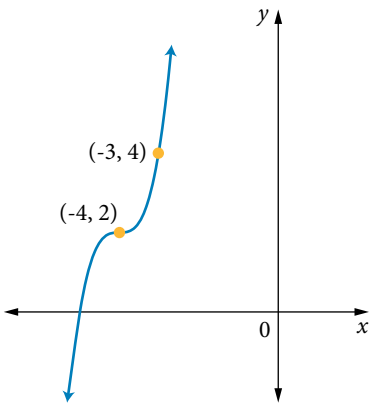
9 Expand and simplify the following.

(a) $(12x - 3y)(x - 4y + 3)$ **(b)** $4(5x - 7y) + 2(3x - 4y)$

(c) $(b + 2a)(3b - 4a) - (2b + a)(3b - 4a)$ **(d)** $3ab(2a - 3b) + 3b(a^2 - 4ab) + ab^2$

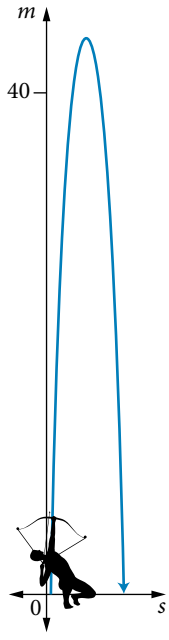
Exercise 3.4

10 Determine the rule for the cubic function.



11 Determine the solution to the simultaneous equations $3x - 2y = 12$ and $5x + 3y = 20$. Exercise 1.2

12 The height of an arrow shot into the air is modelled by the equation $h = 1.5 + 29.4t - 4.9t^2$ where h (m) is the height above the ground and t (s) is the time after the arrow is released. Determine the domain and range of the function and calculate the amount of time the arrow is above a height of 40 m. Write your answer correct to 2 decimal places. Exercise 2.1



13 Determine the equation of a cubic function whose graph passes through the points (-10, -20), (2, -3), (10, 8) and (12, 20). Write your answers correct to 2 decimal places where necessary. Graph the function showing the plotted points, axis points and turning points. Exercise 3.1

14 Sketch the graph of $y = -2x^3 - 2x^2 + 10x - 6$, clearly showing all intercepts. Exercise 3.5

15 A bank account with an initial balance of \$1200 has \$300 deposited at the end of every month. Interest at 2% p.a. is paid into the account after a year, based on the minimum monthly balance throughout the year. Exercise 1.5

Model the monthly account balance with a piece-wise linear function for a period of 18 months.

16 A rectangular area is to be created in a park to allow dogs to play off leash. 80 m of fencing has been allocated and an existing fence is to be used as one side of the enclosure. Create a model for the area of the enclosure, $A\left(\text{m}^2\right)$ in terms of the length of existing fence to be used, x (m). Use your model to determine the dimensions that will maximise the area of the enclosure. Exercise 2.4

4

Advanced functions and relations

Recall 240

4.1 Inverse proportion and graphs of rectangular hyperbolas 242

4.2 The graph of $x = y^2$ 255

4.3 Circles 266

Summary 278

Chapter review 280

4 Recall

Calculate values using algebra

1 For the rules below, determine the corresponding y-value for the given x-values.

(a) $y = \dfrac{1}{x+2}$ for $x = 0$

(b) $y = 2 + \sqrt{x+4}$ for $x = 5$

(c) $x^2 + y^2 = 25$ for $x = 3$

Simplify surds

2 Simplify each of the following surds.

(a) $\sqrt{18}$

(b) $\dfrac{\sqrt{125}}{5}$

(c) $2\sqrt{32}$

(d) $2\sqrt{5} - 5\sqrt{5}$

Rationalise the denominator

3 Simplify the following surds and rationalise the denominator.

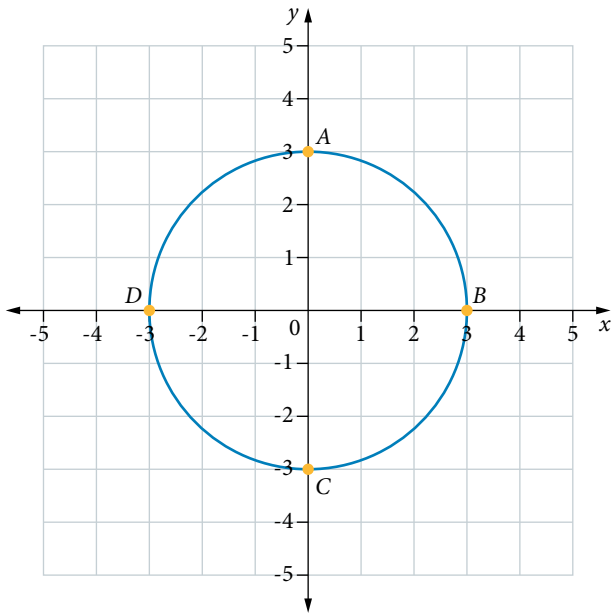
(a) $\dfrac{2}{\sqrt{5}}$

(b) $\dfrac{\sqrt{3}}{\sqrt{2}}$

(c) $\dfrac{3\sqrt{6}}{2\sqrt{7}}$

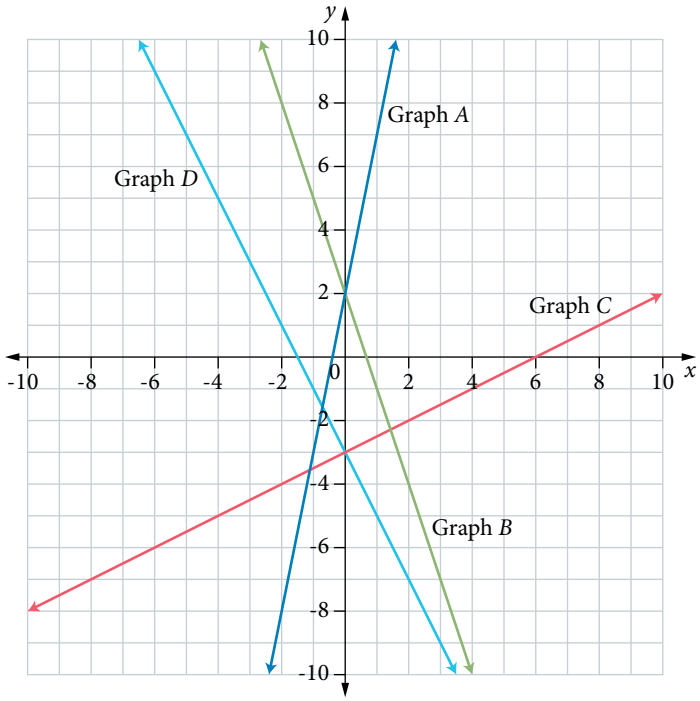
Label points on a graph

4 Determine the coordinates of the points A, B, C and D.



Determine the equations of lines from their graphs

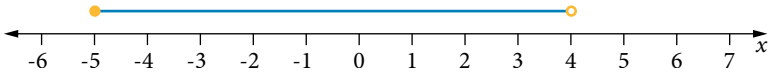
5 Determine the equation for each of the linear graphs, A, B, C and D, shown on the graph below.



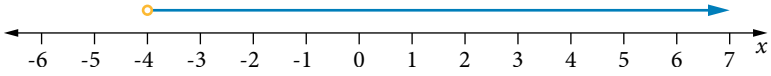
Label intervals of the real number line using set and interval notation

6 Describe each interval using inequality symbols and interval notation.

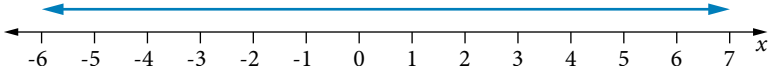
(a)



(b)



(c)



Complete a table of values

7 Complete the table of values for $y = \frac{1}{x}$.

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y						

Solve equations

8 Solve the following equations for x.

(a) $\frac{3x-4}{4} = 2$ **(b)** $\sqrt{x+4} = 1$ **(c)** $\frac{2}{x+4} = 1$

Determine the y-intercept of functions

9 Determine the coordinates of the y-intercept for each function.

(a) $y = 2x - 6$ **(b)** $y = \frac{1}{x-2} + 3.5$ **(c)** $y = \sqrt{x+1}$

4.1 Inverse proportion and graphs of rectangular hyperbolas

Inverse proportion

Inverse proportion is a situation in which an increase in one variable produces a decrease in another related variable. Examples of inverse proportion are:

- the further the distance a missile travels above the Earth, the less the force of gravity it experiences
- the more air that is blown into a balloon, the thinner the rubber becomes
- the further away from a rock band speaker, the softer the sound of the music
- the longer the time after being given an antibiotic injection, the less the chemical concentration in your bloodstream.

Two variables are inversely proportional if an increase in one variable produces a decrease in the other variable.

If y is inversely proportional to x, then $y = \frac{k}{x}$, where x and y are variables and k is a constant.

Additional information

Introduction to inverse proportion

Watch the video to further explore inverse proportion.

1 Average speed

A helicopter travels to a destination 60 km from its starting point. The rule $s = \frac{60}{t}$ can be used to determine the average speed of the helicopter s (km/h) for a trip of t (hours).

(a) Determine the speeds of the helicopter for trips that took:

(i) 30 minutes **(ii)** 1 hour **(iii)** 2 hours

THINKING

Substitute the time values into the equation $s = \frac{60}{t}$ to determine the average speed.

WORKING

(i) 30 minutes

$= \frac{1}{2}$ hour

$s = \frac{60}{t}$

$= 60 \div \frac{1}{2}$

$= 60 \times \frac{2}{1}$

$= 120$ km/h

(ii) 1 hour

$s = \frac{60}{t}$

$= \frac{60}{1}$

$= 60$ km/h

(iii) 2 hours

$s = \frac{60}{t}$

$= \frac{60}{2}$

$= 30$ km/h

(b) Describe how the average speed of the helipoter varies as the time for the trips changes.

Two variables are inversely proportional if an increase in one variable produces a decrease in the other variable.

As the time taken to complete the 60 km trip increases, the average speed decreases.

(c) If the average speed of the helicopter was 45 km/h, determine the time it took to complete the journey. Write your answer in hours and minutes.

1 Recall the formula $s = \frac{60}{t}$ and substitute the given speed. Solve for t, writing the time in hours and minutes.

$$45 = \frac{60}{t}$$
$$t = \frac{60}{45}$$
$$t = 1\frac{1}{3}\text{ h}$$
$$t = 1\text{ h } 20\text{ min}$$

2 Interpret the result.

The helicopter took 1 h 20 min to complete the journey.

2 Inverse proportion and graphs

Four contestants in different events complete a time trial of a 120 km course. They travel with average speeds of 240 km h^{-1}, 120 km h^{-1}, 60 km h^{-1} and 30 km h^{-1} respectively for the journey. The equation for the time of travel t (hours) is $t = \frac{120}{s}$ and contestants travel at an average speed of s (km h^{-1})

(a) Complete the table of values for the time taken for the cars to complete the journey.

Speed (km h^{-1})	240	120	60	30
Time (hours)				

THINKING

Recall and transpose the formula $s = \frac{d}{t}$ to calculate the time taken for each car to complete the trial.

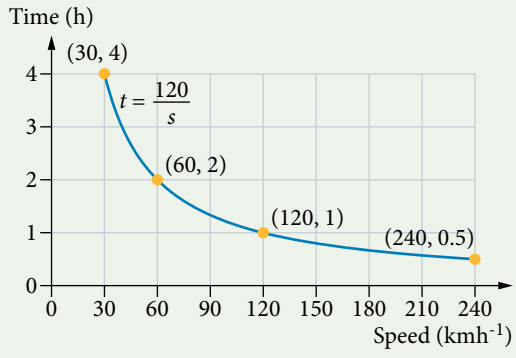
WORKING

$t = \frac{d}{s}$

Speed (km h^{-1})	240	120	60	30
Time (hours)	$\frac{120}{240} = 0.5$	$\frac{120}{120} = 1$	$\frac{120}{60} = 2$	$\frac{120}{30} = 4$

(b) Draw a graph of time versus speed.

Plot the points and join them with a smooth curve.

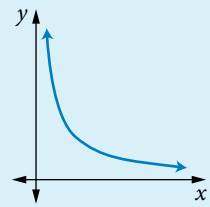


(c) Describe the shape of the graph.

Describe the shape of the graph.	The graph is a smooth decreasing curve: t increases as s decreases, and t decreases as s increases. The slower the average speed, the longer time it takes to complete the course, with the graph approaching the t-axis. The faster the average speed, the shorter the time it takes to complete the course, with the graph approaching the s-axis. The graph does not touch either of the axes.

The graph of $y = \frac{k}{x}$, where $k > 0$ is in the first quadrant only.

The graph approaches, but does not touch either axis.



You can determine a specific rule from the general form, as long as sufficient information is given. In the case of inverse proportion, with general rule $y = \frac{k}{x}$, where $x \neq 0$, one distinct point is sufficient to determine k.

Additional information

Graphs of inverse proportion

Explore additional explanations of graphs of inverse proportion.

3 Establish the inverse proportion rule, given a point that belongs to the function

The table on the right shows the time t (seconds) it takes for a standard tablet to dissolve in a fixed quantity of water at temperature T°C.

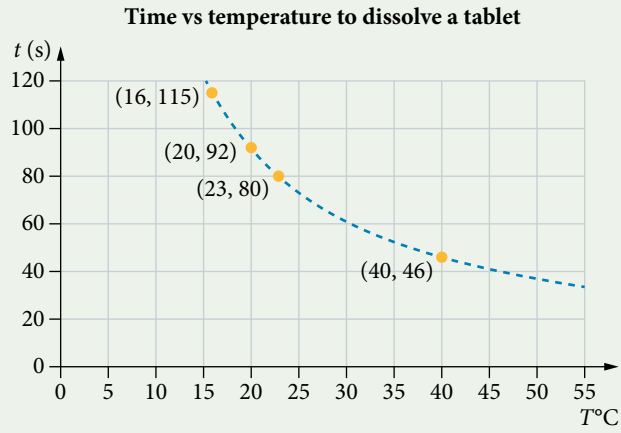
Plot the points and determine the rule of the function connecting T°C and t (s).

T°C	t (s)
16	115
20	92
23	80
40	46

THINKING

1 Consider an appropriate scale for each axis and plot the points.

WORKING



2	The general equation for inverse proportion is $y = \frac{k}{x}$, where $x \neq 0$. Identify the given information to determine the value of k.	Using (40, 46): $46 = \frac{k}{40}$ $k = 46 \times 40$ $= 1840$
3	Write the rule.	The rule is $t = \frac{1840}{T}$ where $t > 0$.

The rectangular hyperbola

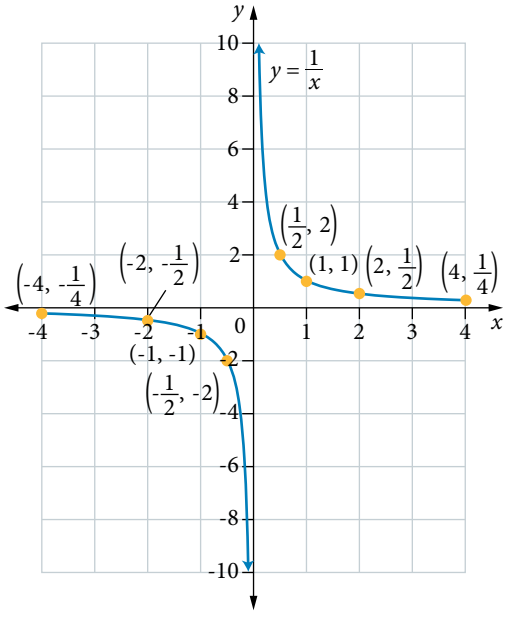
A graph showing inverse proportion is called a rectangular hyperbola. The simplest hyperbola is in the form $f(x) = \frac{1}{x}$, $x \neq 0$.

A table of values can display the coordinates on the graph, which can be used to explain its shape.

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y	$\frac{1}{-4} = -\frac{1}{4}$	$\frac{1}{-2} = -\frac{1}{2}$	$\frac{1}{-1} = -1$	$\frac{1}{-\frac{1}{2}} = -2$	$\frac{1}{0}$ is undefined	$\frac{1}{\frac{1}{2}} = 2$	$\frac{1}{1} = 1$	$\frac{1}{2}$	$\frac{1}{4}$

Examining the extreme values for x for $y = \frac{1}{x}$:

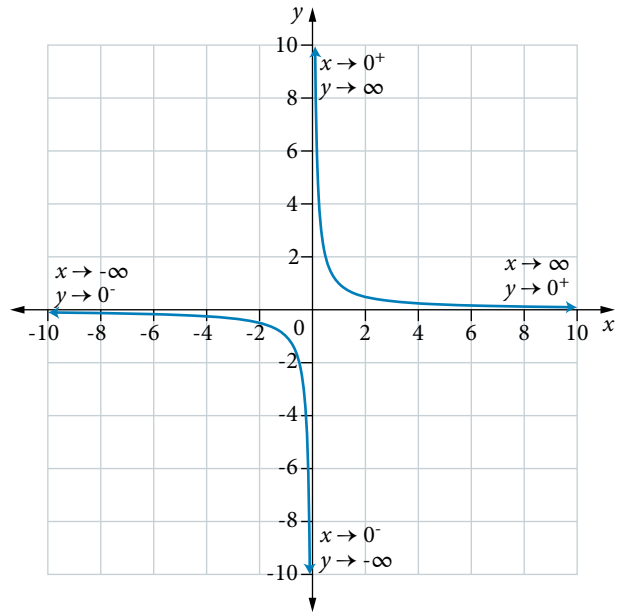
- As the x-values approach positive infinity, the y-values approach 0^+ (zero from above).
  As $x \to \infty$, then $y = \frac{1}{\infty}$ so $y \to 0^+$
- As the x-values approach negative infinity, the y-values approach 0^- (zero from below).
  As $x \to -\infty$, then $y = \frac{1}{-\infty}$ so $y \to 0^-$



Graphically, this means that the curve $y = \frac{1}{x}$ approaches the x-axis from above as $x \to \infty$ and from below as $x \to -\infty$, explaining the shape of the curve at these extremes. So the curve $y = \frac{1}{x}$ approaches but does not touch x-axis.

- As the x-values approach zero from the positive side, the y-values approach ∞.
  As $x \to 0^+$, then $y = \frac{1}{0^+}$ so $y \to \infty$
- As the x-values approach zero from the negative side, the y-values approach $-\infty$.
  As $x \to 0^-$, then $y = \frac{1}{0^-}$ so $y \to -\infty$.

Graphically, this means that as the curve $y = \frac{1}{x}$ approaches the value of $x = 0$ from above $y \to \infty$ and from below $y \to -\infty$, explaining the shape of the curve at these extremes. So the curve $y = \frac{1}{x}$ approaches but does not touch the y-axis.



An asymptote is a line that a curve approaches but doesn't touch. The graph of $y = \frac{1}{x}$ has two asymptotes: the x-axis, or the line with equation $y = 0$, and the y-axis, or the line with equation $x = 0$. In rectangular hyperbolas, the two asymptotes are at right angles.

The domain and the range of the function is $\mathbb{R} \setminus \{0\}$; all real numbers except for zero.

The shape of the curve for $f(x) = \frac{1}{x}, x \neq 0$, is the basic shape of a rectangular hyperbola that can be used to sketch other related curves using the transformations of dilation, reflection and translation.

Algebra is used to determine the axis intercepts to complete an accurate sketch of the function.

Making connections

Transforming a rectangular hyperbola

Move the sliders to view the effect of each parameter in the rule $f(x) = \frac{a}{(x-b)} + c$.

The general rule of a rectangular hyperbola is $f(x) = \frac{a}{(x-b)} + c$:

- $|a|$ dilates the curve parallel to the y-axis (or from the x-axis).
- $a < 0$ reflects the curve in the x-axis.
- b translates the graph in the x-direction (horizontally). The vertical asymptote is $x = b$.
- c translates the graph in the y-direction (vertically). The horizontal asymptote is $y = c$.
- The domain is $\mathbb{R} \setminus \{b\}$.
- The range is $\mathbb{R} \setminus \{c\}$.

4 Sketch rectangular hyperbolas

Sketch the graphs of $f(x) = \frac{1}{x}$ and each of the following on the same set of axes.

(a) $g(x) = \frac{5}{x}$

THINKING	WORKING
1 Write $g(x)$ in the form $f(x) = \frac{a}{x}$ and identify the values of a.	$g(x) = \frac{5}{x}$ $a = 5$
2 Identify any dilations or translations.	For $a = 5$, the graph of $f(x) = \frac{1}{x}$ has been dilated by a factor of 5 parallel to the y-axis, so the shape of $g(x)$ is steeper than $f(x)$.

3 Determine the equations of the asymptotes.

There are no translations, so the equations of the asymptotes are $x = 0$ and $y = 0$.

4 Check the reasonableness of the asymptotes.

An asymptote occurs where the denominator is 0: $x = 0$

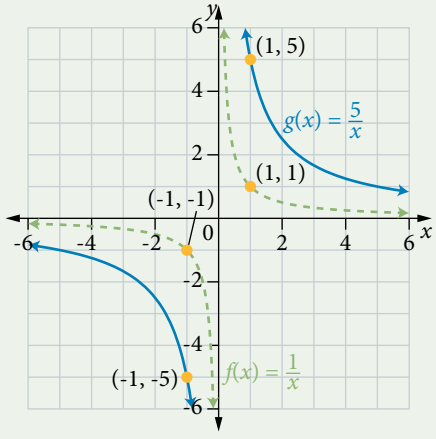
An asymptote occurs where the fraction is equal to 0: $y = 0$.

5 A hyperbola in the form $f(x) = \frac{a}{x}$ will have no axes intercepts. Therefore, determine the coordinates of a point on each arm.

For $x = 1$:
$g(1) = \frac{5}{1}$
$= 5$
Quadrant 1: (1, 5)

For $x = -1$:
$g(-1) = \frac{5}{-1}$
$= -5$
Quadrant 3: (-1, -5)

6 Sketch the graphs of $f(x)$ and $g(x)$, showing the major features.



(b) $g(x) = \dfrac{-2}{x-3}$

1 Write $g(x)$ in the form $f(x) = \frac{a}{(x-b)}$ and identify the values for a and b.

$g(x) = \dfrac{-2}{x-3}$
$a = -2$ and $b = 3$

2 Identify any dilations or translations.

$a < 0$ represents a reflection in the x-axis.

For $a = 2$, the dilation factor is 2 parallel to the y-axis, so the shape of $g(x)$ is steeper than $f(x)$.

For $b = 3$, the graph of $g(x)$ is translated 3units to the right from $f(x)$.

3 Determine the equations of the asymptotes using any translations from the basic curve $f(x) = \frac{1}{x}$.

As $g(x)$ is translated 3 units right, the vertical asymptote is $x = 3$.

4 Check the reasonableness of the solution.

A vertical asymptote occurs where the denominator is equal to 0: $x = 3$.

A horizontal asymptote occurs where the fraction is equal to 0: $y = 0$.

5 Determine the coordinates of the y-intercept of $g(x)$.

For $x = 0$:

$g(0) = \frac{-2}{(0-3)} = \frac{2}{3}$

The coordinates of the y-intercept of $g(x)$ are $\left(0, \frac{2}{3}\right)$.

6 For hyperbolas $f(x) = \frac{a}{(x-b)}$, the x-axis is an asymptote, so no x-intercepts.

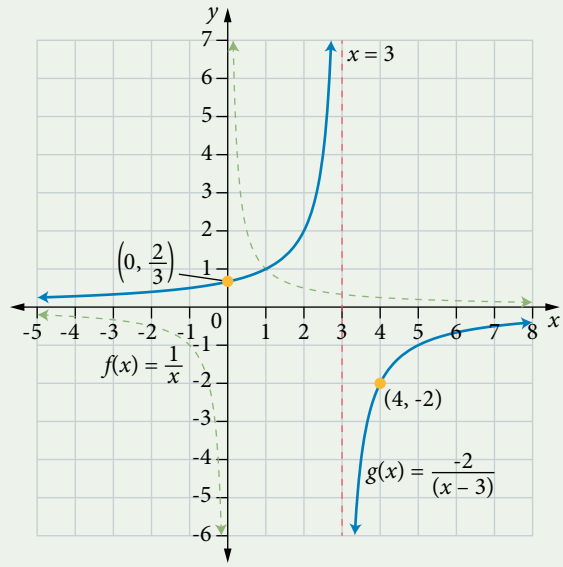
Determine the coordinates of a point on the second arm of the graph.

For $x = 4$:

$g(4) = \frac{-2}{(4-3)} = -2$

The point is at (4, -2).

7 Sketch the graphs of $f(x)$ and $g(x)$ showing the major features.



(c) $g(x) = \frac{-4}{x+2} + 1$

1 Write $g(x)$ in the form $f(x) = \frac{a}{(x-b)}$ and identify the values for a and b.

$g(x) = \frac{-4}{x+2} + 1$ $\quad a = -4$, $b = -2$ and $c = 1$

2 Identify and determine the significance of the a, b and c values relative to the graph of $f(x)$.

Compared to $y = \frac{1}{x}$:

For $|a| = 4$, the dilation factor is 4 parallel to the y-axis

$a < 0$ represents a reflection in the x-axis

$b = 2$ represents a horizontal translation 2 units left

$c = 1$ represents a vertical translation 1 unit up.

3 Determine the equations of the asymptotes using any translations from the basic curve $f(x) = \frac{1}{x}$.

As $g(x)$ is translated 2 units left, the equation of the vertical asymptote is $x = -2$.

As $g(x)$ is translated vertically 1 unit up, the equation of the horizontal asymptote is $y = 1$.

4 Check the reasonableness of the solution.

An asymptote occurs where the denominator is equal to 0: $x = -2$

An asymptote occurs where the fraction is equal to 0: $y = 1$

5 Determine the coordinates of the y-intercept of $g(x)$.

For $x = 0$:

$$g(0) = \frac{-4}{0+2} + 1 = -1$$

The coordinates of the y-intercept are (0, -1).

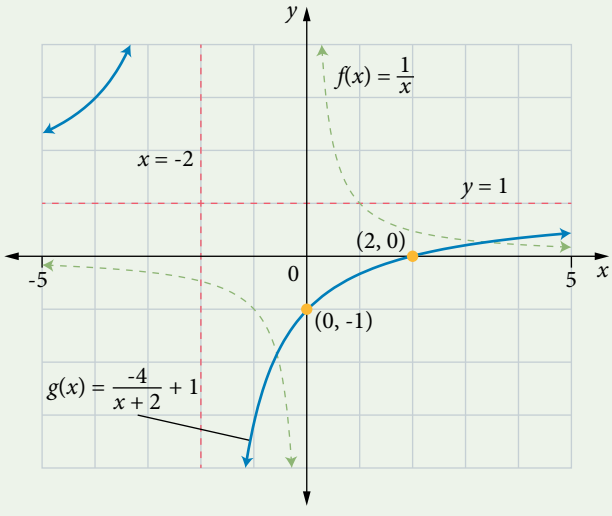
6 Determine the coordinates of the x-intercept of $g(x)$.

For $y = 0$:

$$0 = \frac{-4}{x+2} + 1$$
$$4 = x + 2$$
$$x = 2$$

The coordinates of the x-intercept are (2, 0).

7 Sketch the graphs of $f(x)$ and $g(x)$, showing the major features.



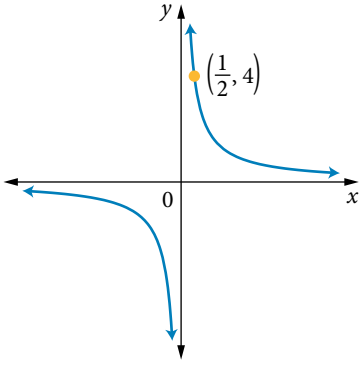
The rule for a rectangular hyperbola, when the asymptotes are known, can be determined from just one point.

Technology worked example
Graphing rectangular hyperbolas

5 Determine the rule for a rectangular hyperbola

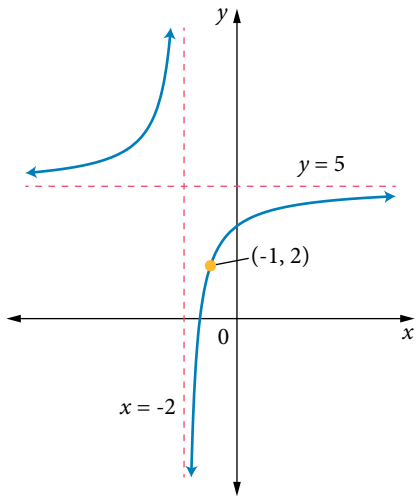
Determine the rule for each rectangular hyperbola.

(a)



THINKING	WORKING
1 The axes are asymptotes so use the general form for inverse proportion.	Let the equation be $y = \frac{k}{x}$.
2 Substitute the given point and solve for k.	For $\left(\frac{1}{2}, 4\right)$: $4 = \frac{k}{\frac{1}{2}}$ $k = 4 \times \frac{1}{2}$ $k = 2$
3 Write the equation.	$y = \frac{2}{x}$

(b)



1 When the asymptotes are not the axes, use the general form for rectangular hyperbola.	Let the equation be in the form $y = \frac{a}{(x-b)} + c$.
2 Substitute the parameters given by the asymptotes.	Vertical asymptote: $x = -2$, so $b = -2$ Horizontal asymptote: $y = 5$, so $c = 5$ The equation becomes $y = \frac{a}{x+2} + 5$
3 Substitute the given point and solve for a.	For (-1, 2): $2 = \frac{a}{-1+2} + 5$ $2 = a + 5$ $a = -3$
4 Write the equation.	$y = \frac{-3}{x+2} + 5$
5 Check the reasonableness of your answer.	The graph shows a reflection in the x-axis compared to the graph of $y = \frac{1}{x}$, so a negative value of a was expected.

Algebraic manipulation to match the general form

The general equation of a rectangular hyperbola is $f(x) = \frac{a}{(x-b)} + c$, but it can also be represented by a linear function divided by another linear function. In this case, you can use equivalent fractions to obtain the general equation.

6 Algebraic manipulation to match the general form

Write $h(x) = \dfrac{5x+21}{x-2}$ in the form of the general equation for a rectangular hyperbola.

THINKING	WORKING
1 Equate the given expression to $\frac{a}{(x-b)} + c$, making the denominators equivalent. Use a common denominator with the RHS. Simplify.	$\dfrac{5x+21}{x-2} = \dfrac{a}{x-2} + c$ $= \dfrac{a}{x-2} + \dfrac{x(x-2)}{x-2}$ $= \dfrac{a + c(x-2)}{x-2}$ $= \dfrac{cx + a - 2c}{x-2}$
2 Equate the coefficients of x in the numerator to solve for c.	$5x = cx$ $\therefore c = 5$
3 Equate the constant terms in the numerator and substitute for c to solve for a.	Given $c = 5$: $5x + 21 = cx + a - 2c$ $5x + 21 = 5x + a - 2 \times 5$ $a - 10 = 21$ $a = 31$
4 Write the function in the general form.	$h(x) = \dfrac{31}{x-2} + 5$

EXERCISE 4.1 Inverse proportion and graphs of rectangular hyperbolas

Worked Example

1 The time taken t (hours) to paint a mural on a large wall by n painters is given by the equation $t = \dfrac{36}{n}$.

(a) Determine the time it would take to paint the mural for each of the following numbers of painters.

(i) 3 (ii) 6 (iii) 9 (iv) 12

(b) Determine the number of painters required if the mural needs to be painted in 10 hours.

2 The relationship between the volume of an inflated balloon V (cm^3) with a surrounding air pressure P (kilopascals) is $V = \frac{200}{P}$.

(a) Calculate the volume of the balloon at sea level when the air pressure is 100 kilopascals.

(b) The balloon is inflated with helium and let go. As it flies higher, the surrounding air pressure decreases. Determine the volume of the balloon when the surrounding air pressure is:

(i) 50 kilopascals (ii) 25 kilopascals (iii) 1 kilopascal

Worked Example

3 The frequency of sound f (hertz) is related to the wavelength λ (m) by the relationship $f = \frac{300}{\lambda}$. Determine the frequency of sound with wavelength 0.2 m.

2

4 A time trial over the distance of 180 km involves five cars travelling at average speeds of 240 km/h, 180 km/h, 120 km/h, 90 km/h and 60 km/h.
The equation for the time of travel t (hours) for an average speed of s (km/h) is $t = \frac{180}{s}$.

(a) Complete the table of values for the time taken for the cars to complete the journey.

Average speed (km/h)	60	90	120	180	240
Time (h)					

(b) Sketch the graph and explain the shape of the curve.

3

5 Soluble vitamin tablets often dissolve quicker when the water is warmer. Sian dissolved some vitamin tablets in water at different temperatures, T°C, and measured the time t (s) it took for one tablet to dissolve in a fixed quantity of water.

T°C	t (s)
8	120
15	64
20	48

Determine the rule of the function connecting T°C and t (s).

6 y and x are inversely proportional, and the point (3.5, 41) is on this graph.

(a) Calculate y where $x = 17.5$.

(b) Calculate x where $y = 51.25$.

7 y and x are inversely proportional, and the point (0.8, 0.075) is on this graph.

(a) Calculate y (to 1 decimal place) where $x = 0.2$.

(b) Calculate x (to 4 decimal places) where $y = 37.5$.

8 For the function $y = -\frac{2}{x+1} + 1$, determine the y-value for $x = 1$.

9 As a piston is pushed down into a cylinder, the volume decreases and the pressure increases. The pressure P (Pa) inside the cylinder is inversely proportional to the volume V(cm^3). If the volume in the piston is 9 cm^3 when the pressure is 0.3 Pa, determine the pressure when the volume is 20 cm^3.

10 The resistance of an electrical appliance is inversely proportional to the current applied under constant voltage. The equation for the resistance R (ohm) when a current I (amp) is applied through an appliance is $R = \frac{360}{I}$.

(a) Complete the table of values for the resistance when the current is applied through the appliance.

(b) Draw a graph of resistance versus current.

(c) Explain the shape of the curve.

Current, I (amps)	10	30	60	120	360
Resistance, R (ohms)					

11 Determine the equations of the asymptotes for $f(x) = \dfrac{2}{x+3} - 4$.

A $x = 3, y = 2$ **B** $x = -3, y = 2$ **C** $x = 3, y = 4$ **D** $x = -3, y = -4$

12 Consider the function $g(x) = \dfrac{-4}{x-1} + 2$.

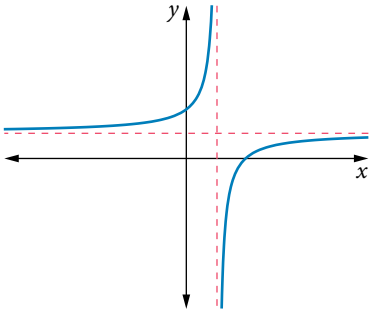
(a) Determine the coordinates of the x-intercept and y-intercept respectively.

A (3, 0), (0, 6) **B** (-3, 0), (0, 6) **C** (0, 3), (6, 0) **D** (4, 0), (0, 2)

(b) Explain the common error made by a student who thought the answer was $(1,0), (0,2)$.

13 Choose the graph that represents $f(x) = -\dfrac{1}{x-1} + 1$.

A



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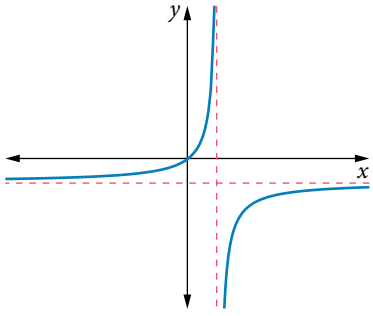
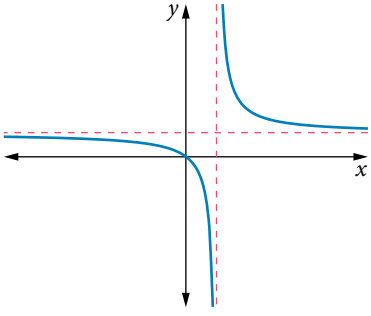
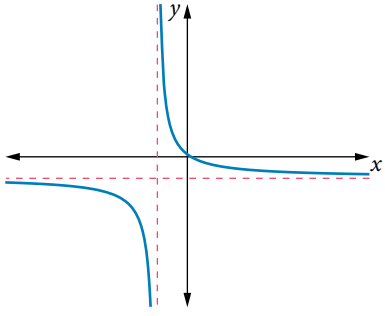


C



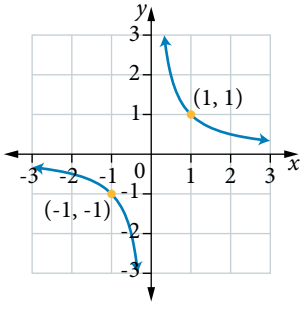
D



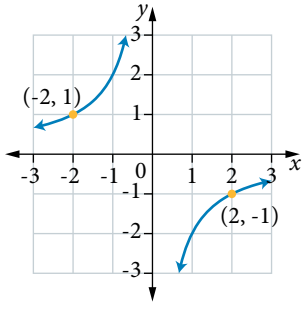
Worked Example 5

14 Determine the equation for each of the following hyperbolic graphs.

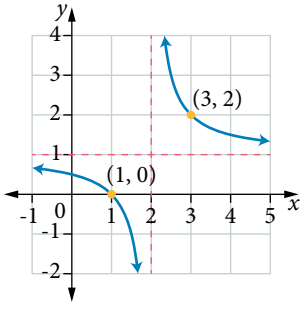
(a)



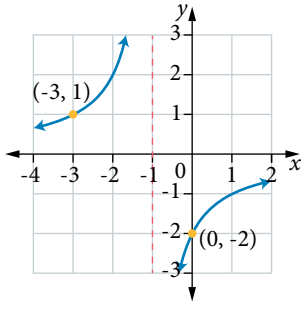
(b)



(c)



(d)



6

15 Express each of the following in the form $\frac{a}{x-b}+c$.

(a) $\frac{x+4}{x-5}$ (b) $\frac{x-1}{x+2}$ (c) $\frac{x}{x+3}$ (d) $\frac{2x+3}{x+1}$ (e) $\frac{5-x}{x+4}$

4

16 For each of the following functions, complete the table of values, write the equations of the asymptotes, sketch the graph of the function and label one point on the graph that belongs to it.

(a) $y=\frac{1}{x-4}$

x	1	2	3	4	5	6	7
y							

(b) $y=\frac{1}{x+6}$

x	-9	-8	-7	-6	-5	-4	-3
y							

17 For each of the following functions, complete the table of values and determine the equations of the asymptotes, sketch the graph of the function and label one point on the graph.

(a) $y=\frac{2}{x}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y						

(b) $y=\frac{1}{2x}$

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y						

(c) $y=\frac{1}{x}+4$

x	-3	-2	-1	0	1	2	3
y							

(d) $y=\frac{1}{x}-6$

x	-3	-2	-1	0	1	2	3
y							

18 Sketch the graph of each function, showing the coordinates of intercepts and equations of asymptotes.

(a) $y=\frac{1}{x-2}+4$ (b) $y=\frac{4}{x-1}-6$ (c) $y=2+\frac{2}{x+1}$ (d) $y=-3+\frac{3}{x+6}$

(e) $y=\frac{-2}{x+1}-1$ (f) $y=6-\frac{1}{x+4}$ (g) $y=-\frac{1}{x+4}-2$ (h) $y=2+\frac{1}{4-x}$

(i) $f(x)=\frac{3x+7}{x+2}$ (j) $f(x)=\frac{4x+11}{x-3}$ (k) $f(x)=\frac{3x-7}{x+2}$ (l) $f(x)=\frac{4x-11}{x+3}$

19 Sketch the graph of $y=\frac{2x+5}{3-4x}$, by first determining the equations to the asymptotes, and the coordinates of the x- and y-intercepts.

20 $f(x)=g(x)-h(x)$ where $g(x)=\frac{2}{x-1}+3$ and $h(x)=\frac{4}{3-x}$.

Determine the subdomains where $f(x)>0$.

The graph of $x = y^2$

4.2

The inverse of $y = x^2$

An inverse relation can be produced by reversing the independent and dependent variables. The relation $x = y^2$ is not a function because the equation can be written as $y = \pm\sqrt{x}$, $x \geq 0$, and hence there are two y-values for each x-value in the domain (except for zero).

Interchanging the x- and y-coordinates of every point in the Cartesian plane is a transformation of the plane that maps the graph of a relation onto the graph of its inverse relation, and vice versa. Geometrically, this corresponds to a reflection in the line $y = x$ as long as the axes have one-to-one scales.

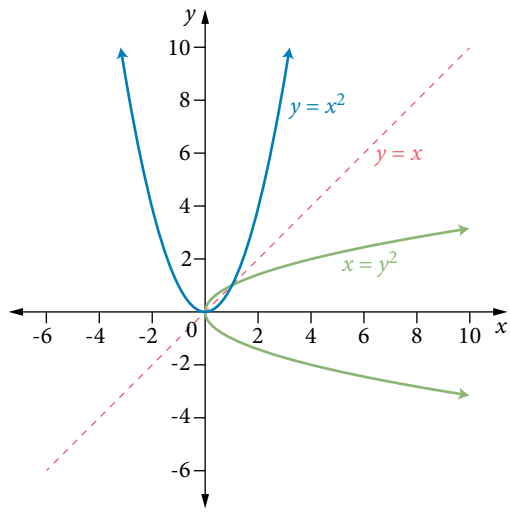
The following table of values represents part of the relation $y = x^2$.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

The corresponding values for the inverse relation $x = y^2$ or $y = \pm\sqrt{x}$ are shown in the table on the right.

x	0	1	4	9
y	0	±1	±2	±3

The following shows the graphs of both relations on the same set of axes, using a one-to-one ratio for the axes scales.



The graph of $y = x^2$ intersects with the graph of $x = y^2$ on the line $y = x$ at (0, 0) and (1, 1).

The function $y = \sqrt{x}$

The domain of the function $f(x) = \sqrt{x}$ is the set of non-negative numbers $[0, \infty)$ or $x \geq 0$.

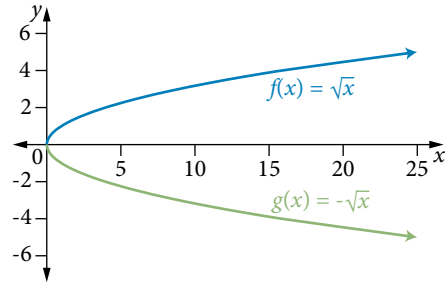
The following table shows some values for $f(x) = \sqrt{x}$.

x	0	1	4	9	16	25
y	0	1	2	3	4	5

The function $f(x) = \sqrt{x}$ corresponds to the positive (and zero) part of the relation $x = y^2$. The range is also the set of non-negative numbers $[0, \infty)$ or $y \geq 0$.

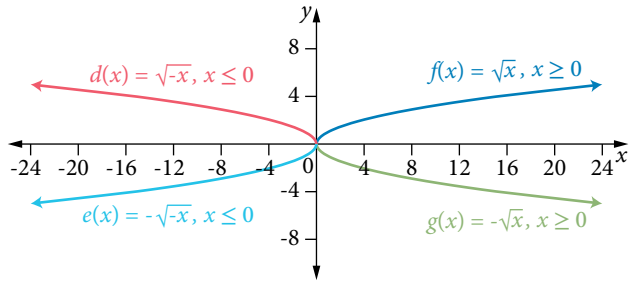
The function $g(x) = -\sqrt{x}$ corresponds to the negative (and zero) part of the relation $x = y^2$.

The graph of $g(x) = -\sqrt{x}$ is the reflection of the graph of $f(x) = \sqrt{x}$ in the x-axis.



The graph of the function $d(x) = \sqrt{-x}$ is the reflection of the graph of $f(x)$ in the y-axis.

The graph of the function $e(x) = -\sqrt{-x}$ is the reflection of the graph of $g(x)$ in the y-axis.



You can graph a relation such as $x = y^2$ by drawing both square root functions: the positive arm and the negative arm. The graphs of these relations are parabolas so have a horizontal axis of symmetry and a vertex.

The vertex of $x = y^2$ is (0, 0).

The line of symmetry is the x-axis: $y = 0$.

WARNING

The vertex of $x = y^2$ is not a turning point as the gradient of the tangent at the vertex is undefined.

7 Graph relations by forming two functions

Sketch the graph of the following relations.

(a) $x = y^2 + 2$

THINKING	WORKING
1 Use your understanding of the inverse relation and the graph of $y = x^2$ to predict the shape of the graph.	For $y = x^2 + 2$, the vertex is at (0, 2). The line of symmetry is the y-axis: $x = 0$. For $x = y^2 + 2$: the vertex is at (2, 0). The line of symmetry is the x-axis: $y = 0$.
2 Solve the equation to make y the subject and identify the domain.	$x = y^2 + 2$ $y^2 = x - 2$ $y = \pm\sqrt{x - 2}, x \geq 2$

3 Determine the coordinates of an additional point on each arm. Use your knowledge of perfect squares.

For $x = 3$:

$y = \pm\sqrt{3-2}$

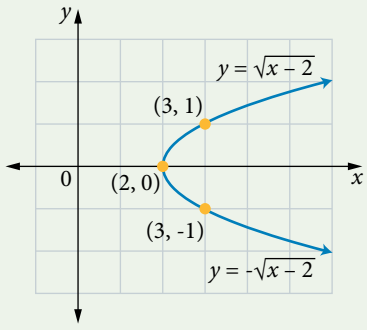
$= \pm\sqrt{1}$

$= \pm 1$

$y = \sqrt{x-2}$ passes through (3, 1).

$y = -\sqrt{x-2}$ passes through (3, -1).

4 Sketch the graph. Confirm the shape of the graph using technology.



(b) $x = (y+2)^2$

1 Use your understanding of the inverse relation and the graph of $y = x^2$ to predict the shape of the graph.

For $y = (x+2)^2$, the vertex is at (-2, 0). The line of symmetry is the line of $x = -2$.

For $x = (y+2)^2$, the vertex is at (0, -2). The line of symmetry is the line $y = -2$.

2 Solve the equation to make y the subject and identify the domain.

$x = (y+2)^2$

$y + 2 = \pm\sqrt{x}$

$y = -2 \pm \sqrt{x}$, $x \geq 0$

3 Determine the coordinates of an additional point on each arm. Use your knowledge of perfect squares.

For $x = 4$:

$y = -2 \pm \sqrt{4}$

$= -2 \pm 2$

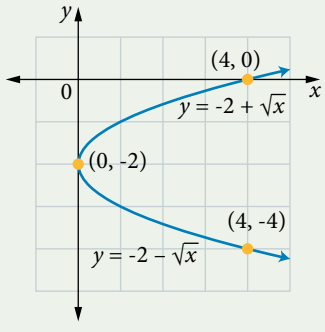
$= -4$ or 0

$y = -2 + \sqrt{x}$ passes through (4, 0).

$y = -2 - \sqrt{x}$ passes through (4, -4).

4 Sketch the graph. Confirm the shape of the graph using technology.

$y = -2 + \sqrt{x}$ and $y = -2 - \sqrt{x}$



(c) $x = -y^2$

1 Use your understanding of the inverse relation and the graph of $y = x^2$ to predict the shape of the graph.

For $y = -x^2$, the vertex is (0, 0).
The line of symmetry is the y-axis, $x = 0$.

For $x = -y^2$, the vertex is at (0, 0).
The line of symmetry is the x-axis, $y = 0$.

2 Solve the equation to make y the subject.

$$x = -y^2$$
$$y^2 = -x$$
$$y = \pm\sqrt{-x},\ x \le 0$$

3 Determine the coordinates of an additional point on each arm. Use your knowledge of perfect squares.

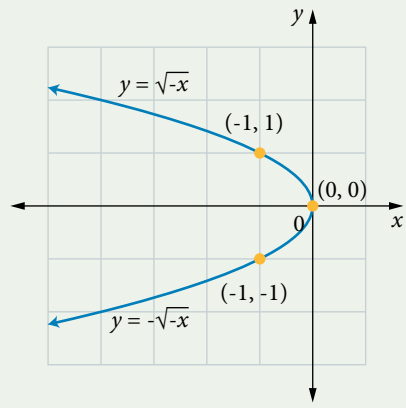
For $x = -1$

$$y = \pm\sqrt{-(-1)}$$
$$= \pm\sqrt{1}$$
$$= \pm 1$$

$y = \sqrt{-x}$ passes through (-1, 1).
$y = -\sqrt{-x}$ passes through (-1, -1).

4 Sketch the graph. Confirm the shape of the graph using technology.

$y = \sqrt{-x}$ and $y = -\sqrt{-x}$



Square root functions

Transformations of the basic square root function have many practical applications such as calculating the period of swing for a pendulum from its length, calculating the speed of a car from the length of the skid marks after an accident and determining the speed of sound from the air temperature.

You can create square root functions by changing the subject of a formula if one of the variables is squared. In practical situations where neither variable can be negative, this generally leads to a single function because you reject the negative expression.

8 Calculate a length, given the area

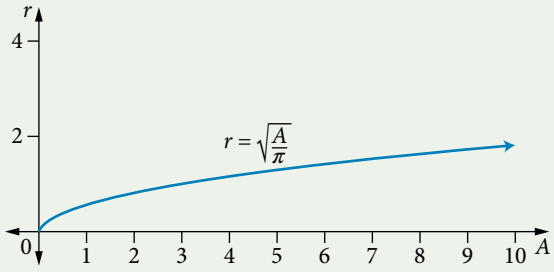
Consider the formula for area of a circle $A = \pi r^2$.

(a) Construct a formula for radius in terms of area.

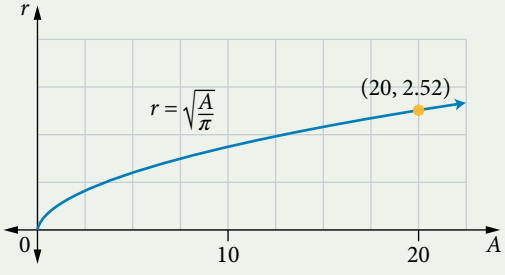
THINKING	WORKING
1 Transpose the equation to make the radius the subject.	$A = \pi r^2$ $r^2 = \dfrac{A}{\pi}$ $r = \pm\sqrt{\dfrac{A}{\pi}}$
2 Reject the negative expression, because lengths cannot be negative.	The radius r represents a length $\therefore r \geq 0$ $r = \sqrt{\dfrac{A}{\pi}}$

(b) Graph the function using technology.

Use technology to draw $y = \sqrt{\dfrac{x}{\pi}}$. Replace the variables as appropriate.



(c) Determine the radius, given that the area is $20\,\text{cm}^2$.

1 Solve algebraically by substitution.	For $A = 20$: $r = \sqrt{\dfrac{A}{\pi}}$ $= \sqrt{\dfrac{20}{\pi}}$ $= 2.52\ (2\text{ d.p.})$
2 Interpret the result, including units.	A circle of area $20\,\text{cm}^2$ has a radius of $2.52\,\text{cm}$.
3 Check the reasonableness of the coordinates by locating the point on your graph.	

Transformations of the square root function

The basic square root graph of $y = \sqrt{x}$ can be translated, dilated or reflected to give some recognisable general forms.

The general form for transformations of the function $y = \sqrt{x}$ is $f(x) = a\sqrt{(x-b)} + c$, where:

- $|a|$ is the vertical dilation factor parallel to the y-axis. When a is positive, the graph is the positive arm of the parabola and when a is negative, the graph is the negative arm of the parabola
- b translates the function horizontally, parallel to the x-axis
- c translates the function vertically, parallel to the y-axis.

The point (b, c) is the vertex of the graph.

The function $y = \sqrt{-x}$ is the reflection in the y-axis of $y = \sqrt{x}$.

The general form for transformations of the function $y = \sqrt{-x}$ is $f(x) = a\sqrt{-(x-b)} + c$, where:

- $|a|$ is the vertical dilation factor parallel to the y-axis. When a is positive, the graph is the positive arm of the parabola and when a is negative, the graph is the negative arm of the parabola
- b translates the function horizontally, parallel to the x-axis
- c translates the function vertically, parallel to the y-axis.

The point (b, c) is the vertex of the graph.

Making connections

Transforming the square root graph

Move the sliders to view the effect each parameter has on the graph of $y = a\sqrt{(x-b)} + c$.

Technology worked example

Graphing the square root function

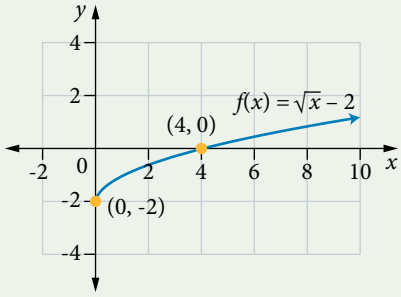
9 Sketch simple square root functions

Sketch the graph of $f(x)$, clearly showing the coordinates of the vertex and axes intercepts. Identify the domain and range.

(a) $f(x) = \sqrt{x} - 2$

THINKING	WORKING
1 Identify the transformations that have occurred from the standard square root function $f(x) = \sqrt{x}$.	From the graph of $y = \sqrt{x}$, $f(x) = \sqrt{x} - 2$ has been translated 2 units down.
2 Determine the coordinates of the vertex.	The coordinates of the vertex are (0, -2).
3 Determine the coordinates of the axes intercept of $f(x)$.	The y-intercept is the vertex. $\sqrt{x} - 2 = 0$ $\sqrt{x} = 2.$ $x = 4$ The coordinates of the x-intercept are (4, 0).

4 Sketch the graph of $f(x)$ showing the vertex and x-intercept.



5 Identify the domain and range of the function using interval and set notation.

The domain of $f(x)$ is $x \geq 0$ or $[0, \infty)$.

The range of $f(x)$ is $y \geq -2$ or $[-2, \infty)$.

(b) $f(x) = \sqrt{-x+2}$

1 Rewrite the rule so transformations can be identified and state the domain.

$$f(x) = \sqrt{-x+2}$$
$$= \sqrt{-(x-2)},\ x \leq 2$$

2 Identify the transformations that have occurred from the function $f(x) = \sqrt{-x}$.

From the graph of $f(x) = \sqrt{-x}$, $f(x) = \sqrt{-x+2}$ has been translated 2 units right.

3 Determine the coordinates of the vertex.

The coordinates of the vertex are (2, 0).

4 Determine the coordinates of the axes intercept of $f(x)$.

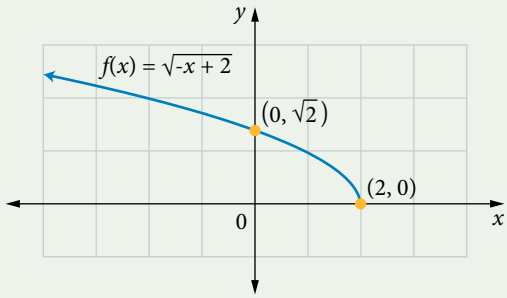
The x-intercept is the vertex.

For $x = 0$:

$$f(x) = \sqrt{-x+2}$$
$$f(0) = \sqrt{-0+2}$$
$$= \sqrt{2}$$

The coordinates of the y-intercept are $\left(0, \sqrt{2}\right)$.

5 Sketch the graph of $f(x)$ showing the vertex and x-intercept.



6 Identify the domain and range of the function using interval and set notation.

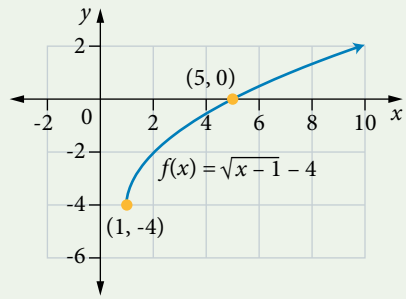
The domain of $f(x)$ is $x \leq 2$ or $(-\infty, 2]$.

The range of $f(x)$ is $y \geq 0$ or $[0, \infty)$.

10 Sketch square root functions with rule in the form $f(x) = a\sqrt{(x-b)} + c$

Consider the function $f(x) = 2\sqrt{x-1} - 4$.

(a) Sketch the graph of $f(x)$, clearly showing the coordinates of the vertex and axis intercept.

THINKING	WORKING
1 Write the rule in the form $f(x) = a\sqrt{(x-b)} + c$ and identify the domain.	$f(x) = 2\sqrt{(x-1)} - 4,\ x \geq 1$
2 Identify the values for a, b and c.	$a = 2$, $b = 1$ and $c = -4$
3 State the significance of the a, b and c values relative to the graph of $y = \sqrt{x}$.	For $a = 2$, the dilation factor is 2 parallel to the y-axis showing the upper arm of the parabola. For $b = 1$, the graph of $f(x)$ is translated 1 unit to the right from the origin. For $c = -4$, the graph of $f(x)$ is translated 4 units down.
4 Determine the coordinates of the vertex.	Considering the transformations stated above the coordinates of the vertex are (1, -4).
5 Determine the coordinates of the axes intercepts.	From the position of the vertex at (1, -4), the graph does not cut the y-axis. For $f(x) = 0$: $2\sqrt{x-1} - 4 = 0$ $2\sqrt{x-1} = 4$ $\sqrt{x-1} = 2$ $x - 1 = 4$ $x = 5$ The coordinates of the x-intercept are (5, 0).
6 Sketch the graph of $f(x)$ showing the vertex and x-intercept.	

(b) Identify the domain and range of the function.

Identify the domain and range of the function using interval and set notation.	The domain of $f(x)$ is $x \geq 1$ or $[1, \infty)$. The range of $f(x)$ is $y \geq -4$ or $[-4, \infty)$.

EXERCISE 4.2 The graph of $x = y^2$

Worked Example

1 Using technology or otherwise, sketch the graph of each relation. Identify the coordinates of the vertex and intercepts. 7

(a) $x = 3y^2$ **(b)** $y^2 = 3x$ **(c)** $x = y^2 + 3$ **(d)** $y^2 = x + 3$

2 Consider the relation $x = y^2 + 1$.

(a) Complete the table of values and hence sketch the graph of the relation.

(b) Determine the coordinates of the vertex and the equation of the line of symmetry.

x							
y	-3	-2	-1	0	1	2	3

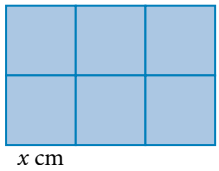
3 Consider the relation $x = (y+1)^2$.

(a) Complete the table of values and hence sketch the graph of the relation.

x							
y	-4	-3	-2	-1	0	1	2

(b) Determine the coordinates of the vertex and the equation of the line of symmetry.

4 The shape given in the diagram is made up of six squares of side length x cm. 8



(a) Construct a formula for x in terms of the area of the shape $A\,\text{cm}^2$.

(b) Sketch a graph of x as a function of A.

(c) Determine the value of x, correct to 2 decimal places, when the area of the shape is $20\,\text{cm}^2$. Confirm the answer from your graph.

5 For a particular type of material, the formula for the diameter D (cm) of rope needed to lift a load of mass m (tonne) is $D = 0.5\sqrt{m}$.

(a) Determine the diameter needed for a mass of 1.5 tonnes. Give your answer correct to 2 decimal places.

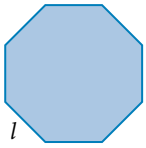
(b) Determine the mass that could be lifted by a rope of diameter 1.5 cm. Give your answer correct to 2 decimal places.

6 An approximate formula for calculating distance to the horizon D (km) in terms of altitude A (m) is $D = 3.57\sqrt{A}$.

(a) Determine the distance to the horizon from an aircraft flying at 10 000 m.

(b) How would the distance to the horizon change if the aircraft dropped down by 3000 m? Give your answer to the nearest kilometre.

7 The side length of a regular octagon in terms of its area is given by $l = \sqrt{\dfrac{A}{4.828}}$.



(a) Determine the side length of an octagon of area $200\,\text{cm}^2$. Give your answer correct to 2 decimal places.

(b) Determine the area of a stop sign of side length 28.7 cm. Give your answer correct to the nearest square centimetre.

8 Consider the relation $x = y^2 + 4$.

(a) Determine the value for x when $y = -2$.

A 8 B 2 C -8 D -2

(b) Explain the common error made by a student who thought the answer was 0.

9 For $y = 2\sqrt{x+1} - 4$, determine the y-value for $x = 3$.

10 For $y = \sqrt{x-2}$, determine the x-value for $y = 2$.

11 For $y = -2\sqrt{x+9} + 4$, the coordinates of the vertex are:

A (-2, -9) B (-9, 4) C (9, 4) D (-2, 4)

Worked Example

12 Determine the coordinates of the y-intercept for $y = -\sqrt{2x+4} - 5$.

9

13 For each of the following functions sketch the graph showing the coordinates of the vertex and any axes intercepts. Include an additional point, if necessary, to show the scale. Identify the domain and range using interval notation.

(a) $f(x) = \sqrt{x} + 2$ (b) $f(x) = \sqrt{x} - 4$ (c) $f(x) = 2\sqrt{x} + 3$

(d) $y = \sqrt{x+2}$ (e) $y = \sqrt{-x-1}$

14 For each of the following functions, sketch the graph showing the vertex and any axis intercepts. Identify the domain and range.

(a) $f(x) = -\sqrt{x}$ (b) $f(x) = -\sqrt{x} + 4$

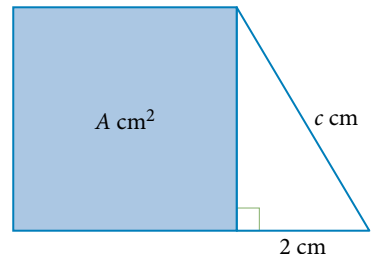
(c) $f(x) = -\sqrt{x} - 1$ (d) $f(x) = -2\sqrt{x} - 4$

15 Sketch the following functions using technology and identify the coordinates of the vertex and axis intercepts.

(a) $f(x) = 3\sqrt{x-1} + 2$ (b) $f(x) = \sqrt{x+9} - 1$

(c) $f(x) = \sqrt{9-x} - 4$ (d) $f(x) = -\sqrt{4-x} + 4$

16 A right-angled triangle has a side length of 2 cm and hypotenuse of c cm. A square of area A cm^2 is drawn on the third side. Create a formula for c as a function of A. Graph the function for areas up to 60 cm^2.



17 Skid marks are often left on the road after a road accident. For a particular car in certain conditions the formula for the speed s (km/h) prior to braking in terms of the length of the skid marks l (m) is given as $s = \sqrt{185l}$.

(a) Determine whether a car leaving skid marks of 12.5 m was travelling above the speed limit in a 60 km/h zone.

(b) A stunt driver travelling at 80 km/h on a similar surface locks the brakes so that the car skids towards a wall 35 m away. Determine how close the car comes to the wall.

18 For each of the following functions, sketch the graph showing the coordinates of the vertex and any axis intercepts. Identify the domain and range.

(a) $f(x) = \sqrt{x+1} - 2$

(b) $f(x) = \sqrt{x-4} + 4$

(c) $f(x) = -\sqrt{x+1} + 3$

(d) $f(x) = -2\sqrt{x-9} - 2$

19 For each of the following functions, sketch the graph showing the vertex and any intercepts. Identify the domain and range.

(a) $f(x) = \sqrt{-(x+1)} - 2$

(b) $f(x) = \sqrt{-(x-4)} + 3$

(c) $f(x) = -\sqrt{-(x+1)} - 2$

(d) $f(x) = -2\sqrt{-(x-4)} + 8$

20 Consider the function $f(x) = -2\sqrt{-2\left(x - \frac{5}{2}\right)} + 8$.

(a) Sketch the graph of the function showing the coordinates of each major point (vertex and intercepts as relevant).

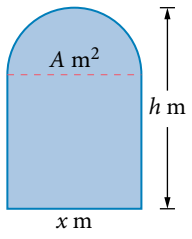
(b) Determine the domain and range of $f(x)$.

21 Consider the function $f(x) = 3\sqrt{\left(\frac{1}{2} + 2x\right)} - 1$.

(a) Sketch the graph of the function showing the coordinates of each major point (vertex and intercepts as relevant).

(b) Determine the domain and range of the function.

22 The shape on the right consists of a semicircle on a square of side x m.

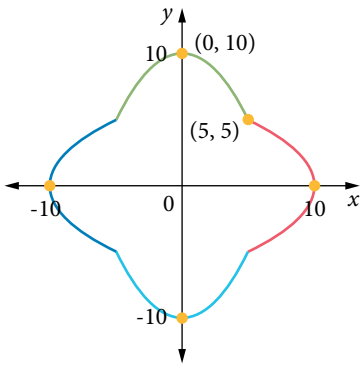


(a) Create a formula for x in terms of the area A (m^2) in the form $x = k\sqrt{A}$ and graph this function. Label the point where $A = 45$ on the graph.

(b) The height of the shape is h (m). Create a formula for h in terms of the area.

(c) Describe the transformation that has occurred to produce the formula for h from the formula for x.

23 The symmetrical diagram below consists of four congruent parabolas. Create an equation for each parabola, and then make two piece-wise functions to replicate the shape.

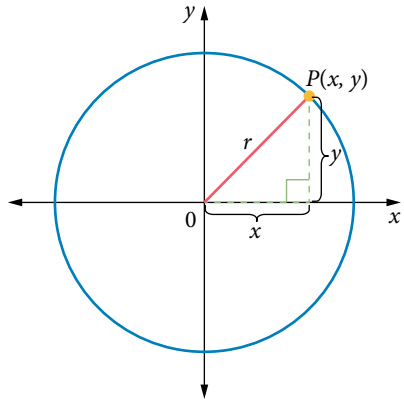


4.3 Circles

Circle with centre at the origin

Consider the point $P(x, y)$ on a circle with its centre at the origin. A vertical line segment drawn from P to the x-axis forms a right-angled triangle. Let the hypotenuse of this triangle be of length r.

From Pythagoras' theorem, the circle with centre at the origin and radius r is the set of ordered pairs (x, y) that satisfies the relation $x^2 + y^2 = r^2$.



The general equation for any circle with its centre at the origin and radius r is $x^2 + y^2 = r^2$.

WARNING

Draw circles with the same scale on both axes. If you don't, the shape will be distorted like an oval and the value of the radius will not be meaningful.

Additional information

The circle formula

Investigate how the equation of a circle is derived.

11 Sketch graphs of circles centred at the origin

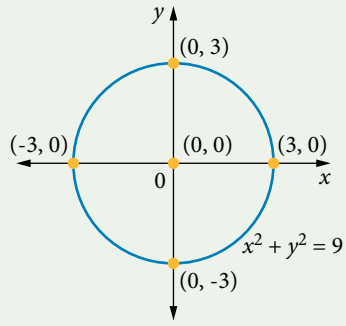
Consider the relation $x^2 + y^2 = 9$.

(a) Identify the centre and radius of the circle.

THINKING	WORKING
1 Write the rule in the form $x^2 + y^2 = r^2$ to identify the value of r.	$x^2 + y^2 = 9$ $x^2 + y^2 = 3^2$
2 Determine the centre and radius.	The centre is at the origin (0, 0). For $r = 3$, the length of the radius of the circle is 3 units.

(b) Sketch the graph of the relation.

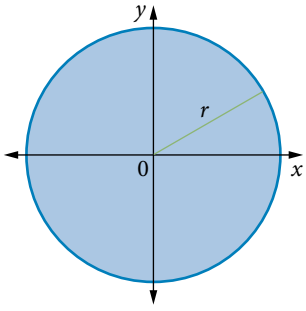
Sketch the graph of the relation showing the centre and x- and y-axis intercepts.



(c) Identify the domain and range of the relation.

Write the domain and range of the relation using interval notation.	The domain of the relation is [-3, 3]. The range of the relation is [-3, 3].

Points within the boundary



Any point lying within or on a circle with centre at the origin and radius r satisfies the inequality $x^2 + y^2 \leq r^2$.

Additional information

Circles with a centre at the origin

Explore writing the equation of circles with centre at the origin.

12 Points lying within the boundary of a circle

Determine whether each point lies inside, outside or on the circle $x^2 + y^2 = 25$.

(a) (4, -3)

THINKING	WORKING
1 Calculate the value of $x^2 + y^2$.	For (4, -3): $x^2 + y^2 = 4^2 + (-3)^2$ $= 16 + 9$ $= 25$
2 Compare $x^2 + y^2$ to r^2.	$r^2 = 25$ $x^2 + y^2 = r^2$ $25 = 25$
3 Interpret the result.	$x^2 + y^2 = r^2$, so the point is on the circle.

(b) (-4.5, 2.5)

1 Calculate the value of $x^2 + y^2$.	For (-4.5, 2.5): $x^2 + y^2 = (-4.5)^2 + 2.5^2$ $= 20.25 + 6.25$ $= 26.5$
2 Compare $x^2 + y^2$ to r^2.	$r^2 = 25$ $x^2 + y^2 > r^2$ $26.5 > 25$
3 Interpret the result.	$x^2 + y^2 > r^2$, so the point is outside the circle.

(c) (-3.5, -1.2)

1 Calculate the value of $x^2 + y^2$.	For (-3.5, -1.2): $x^2 + y^2 = (-3.5)^2 + (-1.2)^2$ $= 12.25 + 1.44$ $= 13.69$
2 Compare $x^2 + y^2$ to r^2.	$r^2 = 25$ $x^2 + y^2 < r^2$ $13.69 < 25$
3 Interpret the result.	$x^2 + y^2 < r^2$, so the point is inside the circle.

Transformations and the graph of the circle with equation $(x - a)^2 + (y - b)^2 = r^2$

If a translation of a units parallel to the x-axis and b units parallel to the y-axis is applied to the relation $x^2 + y^2 = r^2$, the general form of a circle with centre (a, b) and radius r is obtained from Pythagoras' theorem, where $(x - a)^2 + (y - b)^2 = r^2$.

The general form of a circle of radius r is $(x - a)^2 + (y - b)^2 = r^2$, where the relation $x^2 + y^2 = r^2$ has undergone:

- a horizontal translation of a units parallel to the x-axis. If $a > 0$, the translation is right. If $a < 0$, the translation is left
- a vertical translation of b units parallel to the y-axis. If $b > 0$, the translation is up. If $b < 0$, the translation is down.

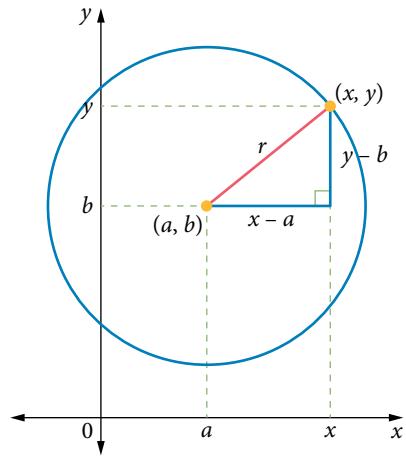
The centre of the circle is at (a, b).

WARNING

The centre of a circle is often displayed on the graph, but it is not a point belonging to the graph of the circle.

Technology worked example

Graphing circles



Making connections

Transforming a circle

Move the sliders to view the effect each parameter has on the relation $(x - a)^2 + (y - b)^2 = r^2$.

13 Sketch circles with centre (a,b)

Consider the relation with equation $(x - 2)^2 + (y + 1)^2 = 4$.

(a) Determine the centre and radius of the circle.

THINKING	WORKING
1 Write the equation in the form $(x - a)^2 + (y - b)^2 = r^2$.	$(x - 2)^2 + (y + 1)^2 = 4$ $(x - 2)^2 + (y + 1)^2 = 2^2$
2 Determine the coordinates of the centre and the length of the radius.	The coordinates of the centre are (2, -1) and the length of the radius is 2 units.

(b) Determine the axes intercepts.

1 Determine the coordinates of the x-intercepts.

x-intercept, let $y = 0$:

$$(x-2)^2 + (1)^2 = 4$$
$$(x-2)^2 = 3$$
$$x - 2 = \pm\sqrt{3}$$
$$x = 2 \pm \sqrt{3}$$

The coordinates of the x-intercepts are $\left(2-\sqrt{3}, 0\right)$ and $\left(2+\sqrt{3}, 0\right)$.

2 Determine the coordinates of the y-intercept(s).

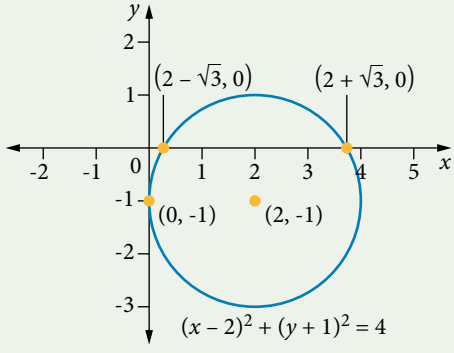
y-intercept, let $x = 0$:

$$(-2)^2 + \left(y+1\right)^2 = 4$$
$$4 + \left(y+1\right)^2 = 4$$
$$\left(y+1\right)^2 = 0$$
$$y + 1 = 0$$
$$y = -1$$

The coordinates of the y-intercept are (0, -1).

(c) Sketch the curve of the relation.

Sketch the graph of the relation showing the centre and axes intercepts.



(d) Identify the domain and range of the relation.

Write the domain and range of the relation using either interval notation or inequalities.

The domain is [0, 4] or $0 \le x \le 4$.

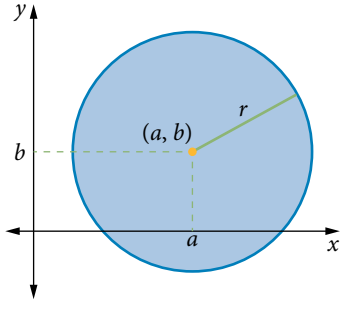
The range is [-3, 1] or $-3 \le y \le 1$.

Additional information

The equation of a circle

Consolidate your understanding of the equation of the circle.

Any point lying within or on the a circle with centre (a, b) and radius r satisfies the inequality $\left(x-a\right)^2 + \left(y-b\right)^2 \le r^2$.



14 Points lying within the boundary of a circle with centre (*a*, *b*)

Determine whether each point lies inside, outside or on the circle $(x+4)^2+(y-2)^2=9$.

(a) (3, 3)

THINKING	WORKING
1 Calculate the value of $(x-a)^2+(y-b)^2$.	For (3, 3): $(x+4)^2+(y-2)^2=(3+4)^2+(3-2)^2$ $=7^2+1^2$ $=50$
2 Identify the value of r^2.	$r^2=9$
3 Interpret the result.	$50>9$ $(x-a)^2+(y-b)^2>r^2$ The point is outside of the circle.

(b) (-3, 4.5)

1 Calculate the value of $(x-a)^2+(y-b)^2$.	For (-3, 4.5): $(x+4)^2+(y-2)^2=(-3+4)^2+(4.5-2)^2$ $=1^2+2.5^2$ $=7.25$
2 Identify the value of r^2.	$r^2=9$
3 Interpret the result.	$7.25<9$ $(x-a)^2+(y-b)^2<r^2$ The point is inside the circle.

Completing the square

Relations containing the terms x^2 and y^2 (or the same multiples of these, such as $3x^2$ and $3y^2$) and terms x and y, as well as a constant term could be formed into the general equation for a circle $(x-a)^2+(y-b)^2=r^2$ by completing the square. If this cannot be done, then there will be no points that satisfy the relation.

15 Use algebra to determine the centre and radius of a circle

By writing the equation $x^2 + 4x + y^2 - 6y = 12$ in the form $(x-a)^2 + (y-b)^2 = r^2$, demonstrate that the relationship is a circle and determine its centre and radius.

THINKING	WORKING
1 Use the process of completing the square on both the group of x-values and the group of y-values. Simplify by writing the constant terms on the RHS.	$x^2 + 4x + y^2 - 6y = 12$ $(x^2 + 4x + 4) - 4 + (y^2 - 6y + 9) - 9 = 12$
2 Simplify the equation by collecting like terms and writing in the form $(x-a)^2 + (y-b)^2 = r^2$.	$(x^2 + 4x + 4) + (y^2 - 6y + 9) - 13 = 12$ $(x+2)^2 + (y-3)^2 = 25$ $(x+2)^2 + (y-3)^2 = 5^2$
3 Determine the coordinates of the centre and the length of the radius.	The coordinates of the centre are (-2, 3) and the length of the radius is 5 units.

Semicircles

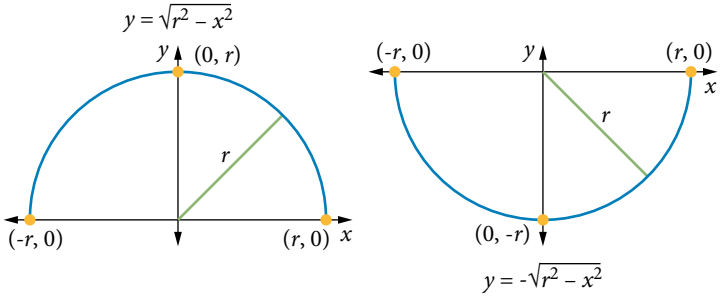
Semicircles centred at the origin

A semicircle is half a circle.

Transposing the general equation of a circle, centred at the origin with radius r, to make y the subject gives:

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

The equation represents two semicircles. The upper semicircle has the equation $y = \sqrt{r^2 - x^2}$ and the lower semicircle has the equation $y = -\sqrt{r^2 - x^2}$.

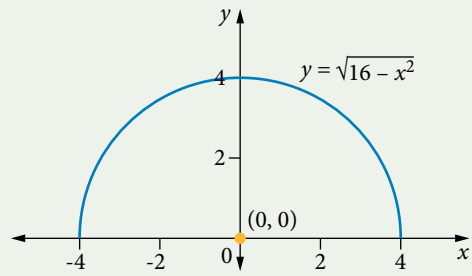


When graphed together, the full circle will be formed.

16 Sketching semicircles centred at the origin

Consider the semicircle with rule $y = \sqrt{16 - x^2}$.

(a) Sketch the graph.

THINKING	WORKING
1 Write the equation in the form $y = \pm\sqrt{r^2 - x^2}$.	$y = \sqrt{16 - x^2}$ $y = \sqrt{4^2 - x^2}$
2 From the equation, identify the value for r and the centre of the circle. Is this the upper or lower semicircle?	The radius is 4. This is the upper semicircle of a circle with centre at the origin.
3 Use the radius to determine the axes intercepts.	The coordinates of the x-intercepts are (-4, 0) and (4, 0). The coordinates of the y-intercept are (0, 4).
4 Sketch the graph of the function showing the centre and axes intercepts.	

(b) Write the domain and range using interval notation.

Write the domain and range of the function using interval notation.	Domain: [-4, 4] Range: [0, 4]

Semicircles centred at (*a*, *b*)

Transposing the general equation of a circle, centred at the (a, b) with radius r, to make y the subject gives:

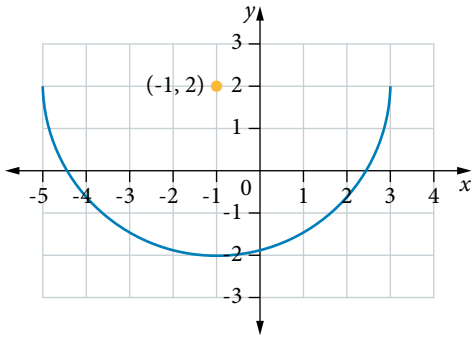
$$(x-a)^2 + (y-b)^2 = r^2$$
$$(y-b)^2 = r^2 - (x-a)^2$$
$$y - b = \pm\sqrt{r^2 - (x-a)^2}$$
$$y = b \pm \sqrt{r^2 - (x-a)^2}$$

The equation represents two semicircles. The upper semicircle has the equation $y = b + \sqrt{r^2 - (x-a)^2}$ and the lower semicircle has the equation $y = b - \sqrt{r^2 - (x-a)^2}$.

Semicircles centred: (0, 0):	Semicircles centred: (a, b):
Upper semicircle: $y = \sqrt{r^2 - x^2}$	Upper semicircle: $y = b + \sqrt{r^2 - (x-a)^2}$
Lower semicircle: $y = -\sqrt{r^2 - x^2}$	Lower semicircle: $y = b - \sqrt{r^2 - (x-a)^2}$

17 Equations of a semicircles with centre (*a*, *b*)

Determine the equation of the semicircle on the right.



THINKING	WORKING
1 Identify the coordinates of the centre. Determine the length of the radius r.	The coordinates of the centre are (-1, 2). The length of the radius is 4 units.
2 Identify which semicircle is shown.	This is the lower semicircle.
3 Write the equation in the form $y = b \pm \sqrt{r^2 - (x-a)^2}$.	$y = 2 - \sqrt{4^2 - (x+1)^2}$ $= 2 - \sqrt{16 - (x+1)^2}$
4 Check the reasonableness of the equation.	A circle with (-1, 2) and $r = 4$: $(x+1)^2 + (y-2)^2 = 4^2$ $(y-2)^2 = 16 - (x+1)^2$ $y - 2 = \pm\sqrt{16 - (x+1)^2}$ $y = 2 \pm \sqrt{16 - (x+1)^2}$ The graph is the lower half of the circle: $y = 2 - \sqrt{16 - (x+1)^2}$

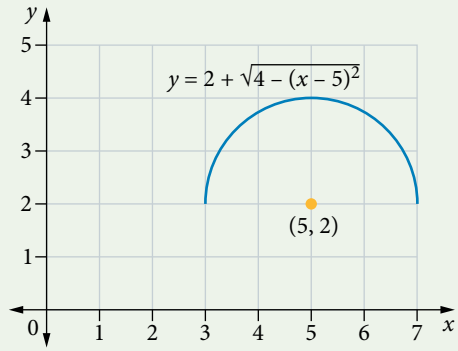
Piece-wise functions with semicircles

Semicircles can form part of a piece-wise function. Even though semicircles have a finite domain, it is usual to specify the domain within the rule for a piece-wise function.

18 Create a continuous piece-wise function involving a semicircle

Determine the values of the parameters so that the piece-wise function is continuous.

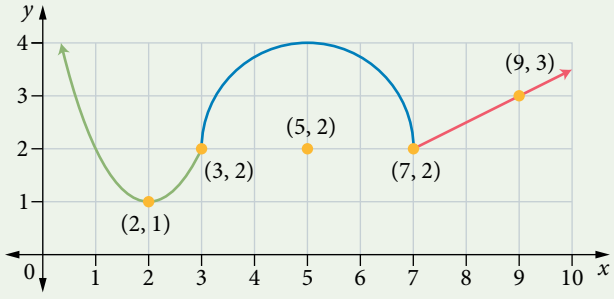
$$y = \begin{cases} a(x-2)^2 + 1, x < 3 \\ 2 + \sqrt{4 - (x-5)^2}, 3 \le x \le 7 \\ \frac{1}{2}x + c, x > 7 \end{cases}$$

THINKING	WORKING
1 Graph the semicircle, because the equation is known. Use $y = b + \sqrt{r^2 - (x-a)^2}$ with centre (a,b) and radius r.	Semicircle: $y = 2 + \sqrt{4-(x-5)^2}, 3 \le x \le 7$ $= 2 + \sqrt{2^2-(x-5)^2}, 3 \le x \le 7$ The coordinates of the centre are (5, 2). The length of the radius is 2 units. The semicircle is the upper half of a circle. 
2 Equate the expression for the quadratic and semicircle. Substitute the required value for x and hence solve for a.	Parabola and semicircle meet at $x = 3$: $a(x-2)^2 + 1 = 2 + \sqrt{4-(x-5)^2}$ $a(3-2)^2 + 1 = 2 + \sqrt{4-(3-5)^2}$ $a + 1 = 2 + 0$ $a = 1$
3 Determine the equation for the parabola.	The equation of the parabola is $y = (x-2)^2 + 1$
4 Specify any points needed to assist a sketch.	Turning point: (2, 1)
5 Equate the expression for the straight line and semicircle. Substitute the required value for x and solve for c.	Line and semicircle meet at $x = 7$: $\frac{1}{2}x + c = 2 + \sqrt{4-(x-5)^2}$ $\frac{1}{2} \times 7 + c = 2 + \sqrt{4-(7-5)^2}$ $\frac{7}{2} + c = 2 + 0$ $c = -\frac{3}{2}$
6 Determine the equation for the line.	The equation of the line is $y = \frac{1}{2}x - \frac{3}{2}$
7 Specify another point to assist a sketch.	For $x = 9$: $y = \frac{1}{2} \times 9 - \frac{3}{2}$ $= 3$ Another point is (9, 3).

8 Write the equation of the piece-wise function.

$$y = \begin{cases} (x-2)^2 + 1, x < 3 \\ 2 + \sqrt{4-(x-5)^2}, 3 \le x \le 7 \\ \frac{1}{2}x - \frac{3}{2}, x > 7 \end{cases}$$

9 Sketch the graph to support your answers.



EXERCISE 4.3 Circles

Worked Example

1 Consider the relation $x^2 + y^2 = 4$ and answer the following questions. 11

(a) Sketch the graph of $x^2 + y^2 = 4$ showing the axes intercepts in coordinate form.

(b) Write the domain and range using interval notation.

2 (a) A point that lies on the circle with equation $(x-5)^2 + (y+2)^2 = 36$ is:

A (5, 2) B (-5, 2) C (5, 4) D (-5, -4)

(b) Explain the common error made by a student who thought the answer was (5, -2).

3 Determine the position of the following points in relation to the circle $x^2 + y^2 = 16$. 12

(a) (-1, 3) (b) (1.8, 2.5) (c) (-3.5, -2.1)

4 Determine the position of the following points in relation to a given circle. 14

(a) $(x-3)^2 + (y-1)^2 = 9$; (2, -1)

(b) $x^2 + (y+4)^2 = 16$; (-3, -3)

(c) $(x+2)^2 + y^2 = 16$; (-2, -4)

(d) $(x+3)^2 + (y-4)^2 = 14$; (-3, 5)

(e) $(x-7)^2 + (y+1)^2 = 21$; (6, 7)

(f) $(x+1)^2 + (y+1)^2 = 10$; (-4, 3)

5 A garden is laid out in a grid system with positions determined by pacing out distances in metres in two directions at right angles from the corner of the garden.
A petunia is growing at position (1, 3). Determine whether the petunia will receive water from a rotating sprinkler with a reach of 3 m, if the sprinkler is placed in each of the following positions.

(a) (6, 6) (b) (4, 5) (c) (2, 2)

6 The equation of the circle with centre (-2, -4) and radius $\sqrt{2}$ is:

A $(x-2)^2 + (y-4)^2 = \sqrt{2}$

B $(x+2)^2 + (y+4)^2 = \sqrt{2}$

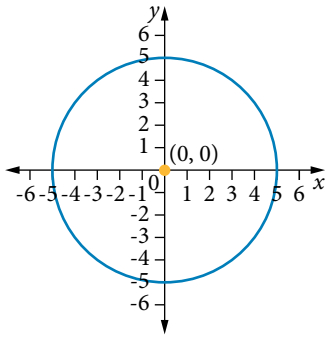
C $(x-2)^2 + (y-4)^2 = 2$

D $(x+2)^2 + (y+4)^2 = 2$

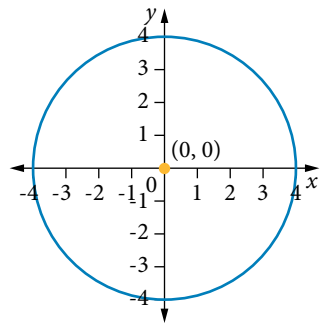
7 Determine the domain and range of the relation $(x+1)^2+(y-2)^2=1$.

8 Determine the equations of the following circles.

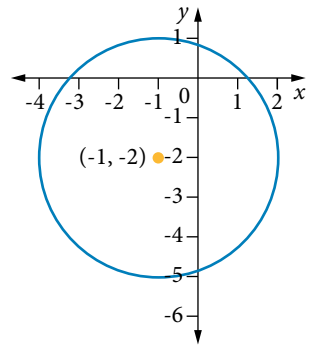
(a)



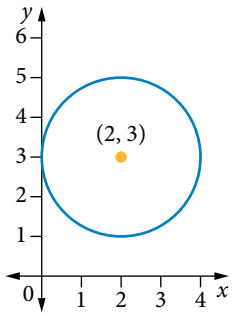
(b)



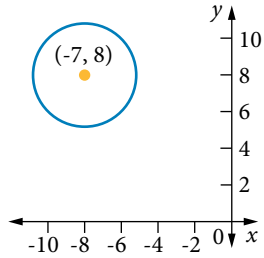
(c)



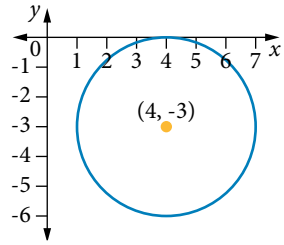
(d)



(e)

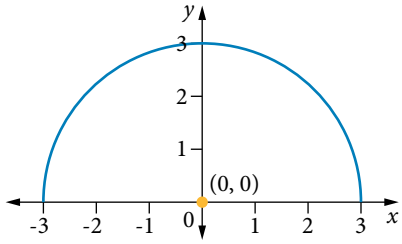


(f)

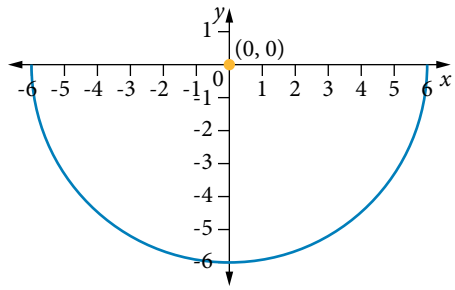


9 Determine the equations of the following semicircles.

(a)



(b)

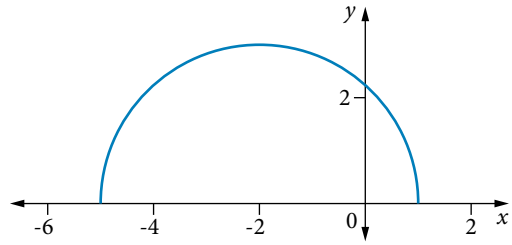


Worked Example

16 10 Sketch the graph of each semicircle. Identify the domain and range of each function.

(a) $y=\sqrt{25-x^2}$ (b) $y=-\sqrt{4-x^2}$ (c) $y=\sqrt{10-x^2}$ (d) $y=-\sqrt{20-x^2}$

17 11 Determine the equation of the semicircle shown.



15 12 For each of the following relations:

(i) write the following relations in the form $(x-a)^2+(y-b)^2=r^2$

(ii) determine the centre and radius of the circle

(iii) write the domain and range of the relation, using interval notation.

(a) $x^2+y^2+2y=0$ (b) $x^2+2x+y^2=3$

(c) $x^2+4x+y^2-2y=-4$ (d) $x^2-8x+y^2+6y=0$

Worked Example

13 Consider the relation with equation $(x+2)^2+(y-3)^2=16$.

(a) Determine the centre and radius of the circle.

(b) Determine, in exact form, the axes intercepts and then sketch the graph.

(c) Write the domain and range, using inequality symbols.

14 Graph the relation with equation $x^2+y^2+2x+4y=31$ showing all relevant features. State the domain and range.

15 Consider the relation with equation $(x-3)^2+(y+1)^2=10$.

(a) Determine the centre and exact radius of the circle.

(b) Determine the axes intercepts and then sketch the graph.

(c) Write the domain and range in exact form, using interval notation.

16 Consider the function with equation $y=-\sqrt{16-(x+1)^2}+2$.

(a) Determine the coordinates of the centre and the length of the radius of the semicircle.

(b) Determine the axes intercepts.

(c) Sketch the curve of the function, showing the coordinates for the endpoints and intercepts.

(d) Write the domain and range of the function using interval notation.

17 Consider the relation with equation $y=\sqrt{9-(x-1)^2}-3$.

(a) Identify the centre and radius of the function.

(b) Determine the axes intercepts.

(c) Sketch the curve of the function, showing the coordinates for the endpoints and intercepts.

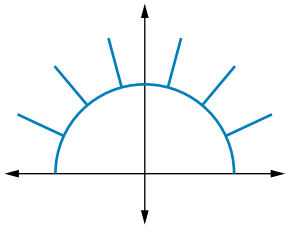
(d) Write the domain and range of the function using interval notation.

18 Determine the rule for the continuous piece-wise function and justify your answer with a graph.

$$f(x)=\begin{cases} a(x+4)^2+3, & x<-2 \\ \sqrt{4-x^2}, & -2\le x\le 2 \\ c-x, & x>2 \end{cases}$$

19 Consider the functions $g(x)=\sqrt{4-(x-2)^2}+2$, $h(x)=x+2$ and $P(x)=2x-6$. Using $g(x)$, $h(x)$ and $P(x)$, construct a continuous piece-wise function, $f(x)$ over the domain $[-4, 8]$.

20 Consider the simplified diagram of a sunset, which can be produced by combining the graphs of several functions. Use your knowledge of piece-wise functions to create this image with equations and restricted domains. Let the radius of the semicircle be 3 units and the length of the rays be 2 units. The rays need not be equally spaced, but make sure your diagram is symmetrical about the vertical axis. Assume each ray is part of a line through the origin.



21 A rectangular piece of fabric, 18 cm by 24 cm, is to be decorated with scalloped edges, with each scallop consisting of a semicircle of diameter 6 cm. Write a list of rules to model the entire boundary of the finished article.

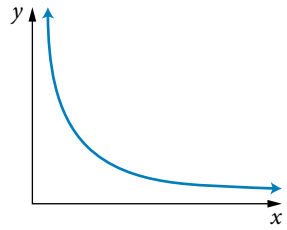
4 Summary

Inverse proportion

The algebraic relationship describing inverse proportion is of the form $y = \frac{k}{x}$, where the variables are x and y with k being a constant.

The graph of $y = \frac{k}{x}$, where $k > 0$, lies entirely within the first quadrant.

The graph approaches, but does not touch, the axes.



Graphs of rectangular hyperbolas

The graph of $y = \frac{a}{x-b} + c$ is called a rectangular hyperbola.

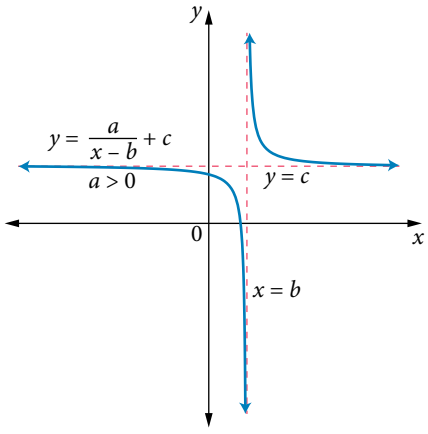
The vertical asymptote is $x = b$.

The horizontal asymptote is $y = c$.

The domain of the hyperbola is $\mathbb{R} \setminus \{b\}$.

The range of the hyperbola is $\mathbb{R} \setminus \{c\}$.

$f(x) = \frac{a}{(x-b)} + c$ can be represented by a linear function divided by another linear function in the form $f(x) = \frac{a + c(x-b)}{x-b}$.

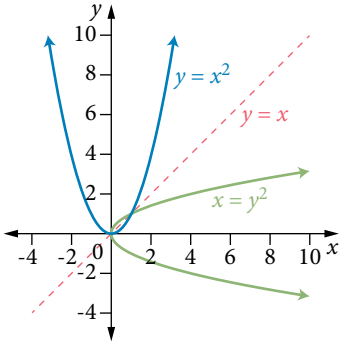


The graph of $x = y^2$

The relation $x = y^2$ is not a function. Its graph can be obtained from the function $y = x^2$ by reflecting in the line $y = x$.

The vertex of $x = y^2$ is (0, 0).

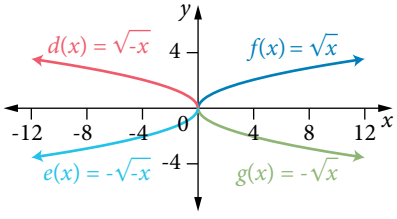
The line of symmetry is the x-axis: $y = 0$.



The function $y = \sqrt{x}$

The function $f(x) = \sqrt{x}$ corresponds to the positive (and zero) part of the relation $x = y^2$. The range is also the set of non-negative numbers, $[0, \infty)$, or simply $y \geq 0$.

The function $g(x) = -\sqrt{x}$ corresponds to the negative (and zero) part of the relation $x = y^2$.
The graph of $g(x) = -\sqrt{x}$ is the reflection of the graph of $f(x) = \sqrt{x}$ in the x-axis.



The graph of the function $d(x) = \sqrt{-x}$ is the reflection of the graph of $f(x)$ in the y-axis and the graph of the function, $e(x) = -\sqrt{-x}$ is the reflection of the graph of $g(x)$ in the y-axis.

Graphs of square roots

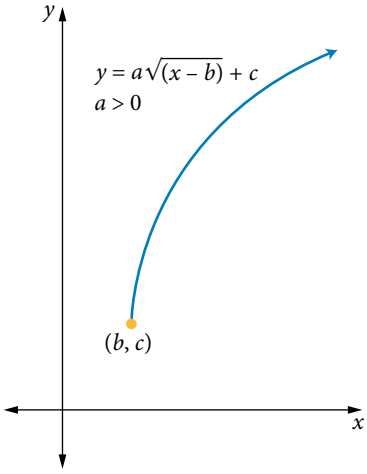
The graph of $y = a\sqrt{(x-b)} + c$ or $y = a\sqrt{-(x-b)} + c$ is called a square root graph.

The vertex is (b, c).

The domain of the square root graph is:

- $[b, \infty)$, where the coefficient of x is a positive number
- $(-\infty, b]$, where the coefficient of x is a negative number.

The range of the square root graph for $a > 0$ is $[c, \infty)$, and for $a < 0$ is $(-\infty, c]$.



Circles

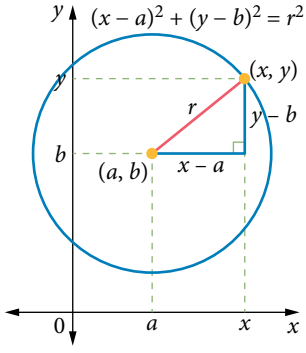
The graph of the relation $x^2 + y^2 = r^2$ is a circle with centre $(0, 0)$ and radius r, where $r > 0$.

The graph of the relation $(x-a)^2 + (y-b)^2 = r^2$ is a circle.

The centre is (a, b) and radius is r.

Domain: $[a - r, a + r]$

Range: $[b - r, b + r]$



Semicircles

The graph of $y = b + \sqrt{r^2 - (x-a)^2}$ is the upper semicircle of the general circle $(x-a)^2 + (y-b)^2 = r^2$:

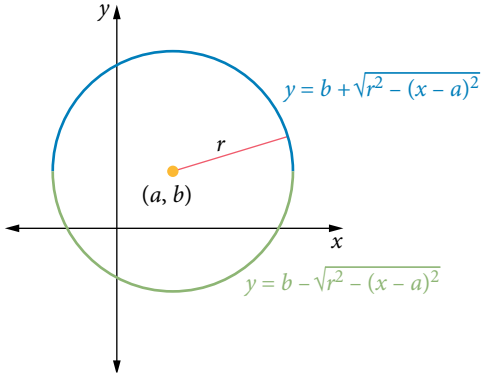
Domain: $[a - r, a + r]$

Range: $[b, b + r]$

The graph of $y = b - \sqrt{r^2 - (x-a)^2}$ is the lower semicircle of the general circle $(x-a)^2 + (y-b)^2 = r^2$:

Domain: $[a - r, a + r]$

Range: $[b - r, b]$



Chapter review 4

Exercise 4.1

1 As a piston is pushed down into a cylinder, the volume decreases and the pressure increases. The pressure P (Pa) inside the cylinder is inversely proportional to the volume V (cm^3). When the volume in the piston is $50\,cm^3$ the pressure is 0.05 Pa. Determine the volume when the pressure is 0.001 Pa.

Exercise 4.1

2 Ash decided to test how long it would take a $5\,cm^3$ ice cube to melt at different room temperatures in his kitchen. The table on the right shows the times t (min) it took to fully melt the ice cube at temperature T°C.

T°C	12	24	30
t (min)	17	8.5	6.8

Estimate the time taken to melt a $5\,cm^3$ ice cube at a room temperature of 7°C. Answer correct to 1 decimal place.

Exercise 4.1

3 The speed S (m/s) at which an object flies for a fixed distance of 120 m varies with the time in seconds according to the formula $S = \frac{120}{t}$.

(a) Sketch the graph for the first 120 seconds of travel.

(b) Determine the speed, given that the object takes 60 seconds to complete its flight.

Exercise 4.1

4 Sketch the graph of $f(x) = \dfrac{2}{x-2} + 2$, clearly showing axes intercepts and asymptotes.

Exercise 4.1

5 Sketch the graph of $f(x) = -\dfrac{4}{x+1} + 8$, clearly showing all intercepts and asymptotes.

Exercise 4.1

6 Write each expression in the form $\dfrac{a}{x-b} + c$.

(a) $\dfrac{3x-1}{x+2}$ **(b)** $\dfrac{2-x}{x+3}$ **(c)** $\dfrac{x}{x+4}$ **(d)** $\dfrac{3x}{x+5}$

Exercise 4.1

7 The general gas equation states that $PV = kT$, where P (pascals) is the pressure of the gas, V (m^3) is the volume of a container, T is the temperature in kelvin and k is a constant representing a fixed amount of gas. Draw a sketch for each of the following situations.

(a) The volume is fixed as a constant and the pressure varies, depending on the temperature.

(b) The temperature is fixed as a constant and the pressure depends on the volume.

(c) The temperature is fixed as a constant and the volume depends on the pressure.

Exercise 4.2

8 Sketch the graph of $f(x) = 2\sqrt{4-x} - 6$ with clearly labelled intercepts and vertex.

Exercise 4.2

9 Sketch the graph of each relation using technology. Hence, or otherwise, label the coordinates of the vertex, the equation of the line of symmetry and an additional point.

(a) $x = \dfrac{1}{5}y^2$ **(b)** $x = y^2 + 1$

Exercise 4.2

10 The speed of tsunami in the open ocean is given by the formula $c = \sqrt{gH}$, where c is the speed in ms^{-1}, $g = 9.8\,ms^{-2}$ and H is the ocean depth in metres.

(a) Sketch a graph of c as a function of H for depths of at least 5000 m.

(b) Determine the speed of a tsunami at the common ocean depth of 4000 m. Write your answer to the nearest whole number. Verify your answer with a graph.

11 The function $f(x) = \sqrt{x}$ is changed according to the following transformations. Write the equation for each. Exercise 4.2

(a) For $g(x)$, $f(x)$ is reflected in the x-axis, translated 2 units left and 1 unit down.

(b) For $h(x)$, $f(x)$ is dilated by factor 2 parallel to the y-axis, reflected in the x-axis, translated 4 units left and 1 unit down.

(c) For $p(x)$, $f(x)$ is dilated by factor 4 parallel to the y-axis, reflected in the x-axis, translated 2 units right and 5 units down.

(d) For q(x), $f(x)$ is dilated by factor 2 parallel to the y-axis, reflected in the x-axis, translated 3 units right and 6 units up.

12 The transformations required to produce the graph of $y = -3\sqrt{x+1} - 2$ from $y = \sqrt{x}$ are: Exercise 4.2

A dilation of factor 3 parallel to the y-axis, reflection in the x-axis, translation 2 units up and 1 unit left

B dilation of factor 3 parallel to the y-axis, reflection in the x-axis, translation 2 units down and 1 unit left

C dilation of factor 3 parallel to the y-axis, translation 2 units down and 1 unit left

D dilation of factor 3 parallel to the y-axis, reflection in the x-axis, translation 1 unit down and 2 units left

13 Consider $(x-2)^2 + (y-4)^2 = 16$. Exercise 4.3

(a) Sketch the graph of the relation, labelling the axes intercepts.

(b) Determine the domain and range.

14 Consider $y + 1 = \sqrt{4 - (x+2)^2}$. Exercise 4.3

(a) Sketch the graph of the relation, labelling the axes intercepts.

(b) Determine the domain and range.

15 Determine whether the following points lie within the circle $x^2 + y^2 = 14$. Exercise 4.3

(a) (2.4, 1.3)

(b) (-2.6, -2.7)

16 Determine whether each point is within or outside the given circle. Exercise 4.3

(a) $(x+5)^2 + (y-4)^2 = 16$; (-7, 5)

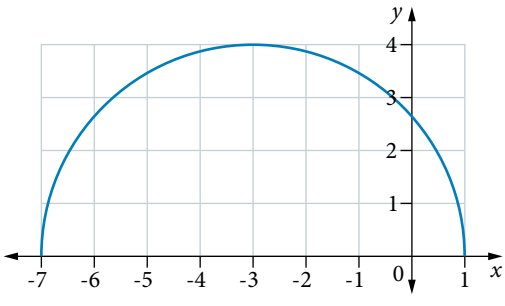
(b) $(x+5)^2 + y^2 = 8$; (-4, -3)

17 Sketch the graph of each semicircle, stating the domain and range of each function. Exercise 4.3

(a) $y = -\sqrt{49 - x^2}$

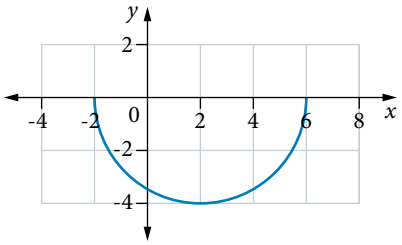
(b) $y = \sqrt{13 - x^2}$

18 Determine the equation of the semicircle shown. Exercise 4.3



Exercise 4.3 **19** Determine the equation of the circle with radius 9 units and centre (-1, 2).

Exercise 4.3 **20** Determine the equation of the semicircle shown.



Exercise 4.1 **21** Consider the graph of $f(x) = -\dfrac{2}{4-x} + 4$.

(a) The coordinates of the x- and y-intercepts are, respectively:

A $\left(3\frac{1}{2}, 0\right), \left(0, 3\frac{1}{2}\right)$

B $\left(3\frac{1}{2}, 0\right), \left(0, 4\frac{1}{2}\right)$

C $\left(4\frac{1}{2}, 0\right), \left(0, 4\frac{1}{2}\right)$

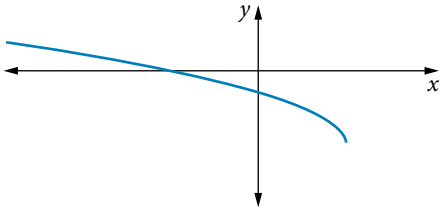
D $\left(4\frac{1}{2}, 0\right), (0, 4)$

(b) Sketch a graph of the function.

Exercise 4.2 **22** Consider the function $f(x) = -\sqrt{2-x} - 2$.

(a) Choose the graph that most accurately represents the function.

A



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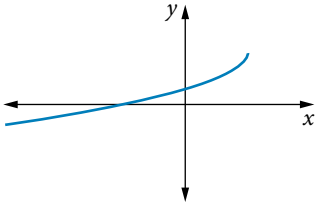

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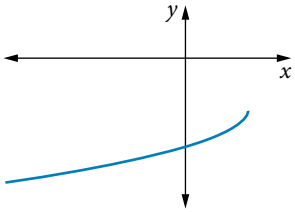

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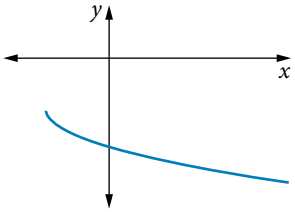

(b) Identify the coordinates of the vertex and determine any intercepts on the axes.

Exercise 4.1 **23** The graph $f(x) = \frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 2 parallel to the y-axis, translated 4 units right and then translated 2 units down to produce $g(x)$.

(a) Determine the equation for $g(x)$.

(b) Sketch the graph of $f(x)$ and $g(x)$ on the same set of axes.

Exercise 4.1 **24** Sketch the graph of $y = \dfrac{2x+3}{x-1}$, showing all key points.

Exercise 4.2 **25** Sketch the graph of $f(x) = 2\sqrt{x+2} - 4$, showing the axes intercepts in exact form.

26 Sketch the piece-wise function $f(x)$ and write the domain and range in interval form. Exercise 4.2

$$f(x)=\begin{cases} 1-x, & x<-4 \\ \sqrt{x+4}+5, & -4\le x<5 \\ x+3, & x\ge 5 \end{cases}$$

27 Sketch the graphs of the following, showing all significant points labelled in exact form. Exercise 4.2

(a) $y=2\sqrt{2x+3}-1$

(b) $y=1-4\sqrt{3-2x}$

28 Consider the relation $x^2-2x+y^2-4y-4=0$. Exercise 4.3

(a) Write the relation in the form $(x-h)^2+(y-k)^2=r^2$ and hence identify the centre and radius of the circle.

(b) Sketch the circle, labelling the intercepts in exact form.

(c) A circle can be split into an upper and lower semicircle. Give the equation of each semicircle, clearly stating which represents the upper semicircle.

(d) If the original circle was translated 2 units right and 3 units down, identify the equation of the circle and determine the axes intercepts in exact form.

29 A graph of the form $y=\frac{ax+b}{cx+d}$ has both domain and range of $\mathbb{R}\setminus\{3\}$ and passes through the point (1, 5.5). Write the equation of the graph in the given form. Sketch the graph, showing exact values of any intercepts on the axes. Exercise 4.1

30 The hyperbola $f(x)$ with asymptotes at $x=1$ and $y=2$ passes through the points (2, 3) and (0, 1). Exercise 4.2

(a) Determine the rule of the graph and sketch the hyperbola, clearly labelling the asymptotes, intercepts and the point (2, 3).

(b) $f(x)$ is dilated by factor 2 from the x-axis and translated 3 units units left and 4 units down to produce $g(x)$. Determine the equation for $g(x)$ and sketch it showing all the major features.

(c) $g(x)$ is reflected in the x-axis to produce $h(x)$. Determine the equation for $h(x)$ and the axes intercepts.

(d) $g(x)$ is reflected in the y-axis to produce $p(x)$. Determine the equation for $p(x)$ and the axes intercepts.

31 The piece-wise function $f(x)$ is defined as follows: Exercise 4.3

$$f(x)=\begin{cases} x+2, & x\le 1 \\ g(x), & 1<x\le 7 \\ h(x), & x>7 \end{cases}$$

If $g(x)$ is a semicircle and $h(x)=c-x$, calculate the value of c and determine the equations of two semicircles that would make $f(x)$ a continuous function. Graph $f(x)$ for both cases.

5

Probability

Recall 286

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5.3 Conditional probability 311

5.4 Independent events 321

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5 Recall

Fraction calculations

1 Calculate the following, expressing your answer in simplest fraction form.

(a) $\frac{1}{2}+\frac{1}{4}$

(b) $1-\frac{2}{3}$

(c) $\frac{2}{3}\times\frac{1}{2}$

Understand the structure of a pack of cards

2 The following questions relate to a standard pack of 52 playing cards.

(a) How many different suits are there?

(b) How many cards are in each suit?

(c) How many cards are picture cards (Kings, Queens or Jacks)?

Identify the set of outcomes of a simple experiment

3 Three coins are tossed, one after another.

(a) Given that order is important, how many different outcomes are possible for the three tosses?

(b) How many of the outcomes include exactly one head?

(c) How many of the outcomes include no heads?

The basics of set theory

4 If $K = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, choose the elements of K that belong to the following sets.

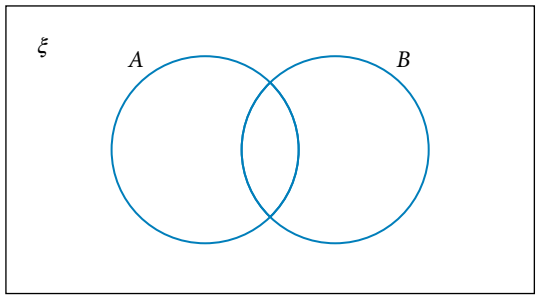
(a) $A = \{\text{even numbers}\}$

(b) $B = \{\text{numbers greater than 9}\}$

(c) $C = \{\text{factors of 10}\}$

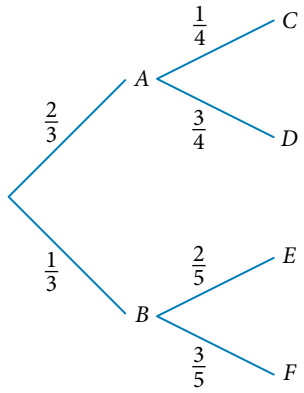
Create a Venn diagram

5 Complete the Venn diagram that represents the following information:
$\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$A = (2, 4, 6\}$
$B = \{1, 2, 3, 4, 5\}$



Determine simple probabilities using a tree diagram

6 Consider the following probability tree diagram.



(a) Determine $P(AC)$ in simplest fraction form.

(b) Determine $P(BF)$ in simplest fraction form.

(c) Determine the total sum $P(AC) + P(AD) + P(BE) + P(BF)$ of the probabilities of the outcomes at the second stage of this experiment.

Calculate probabilities using a two-way table

7 An incomplete two-way table is given below.

	A	A'	
B	0.15		
B'		0.3	
		0.45	

(a) Complete the two-way table.

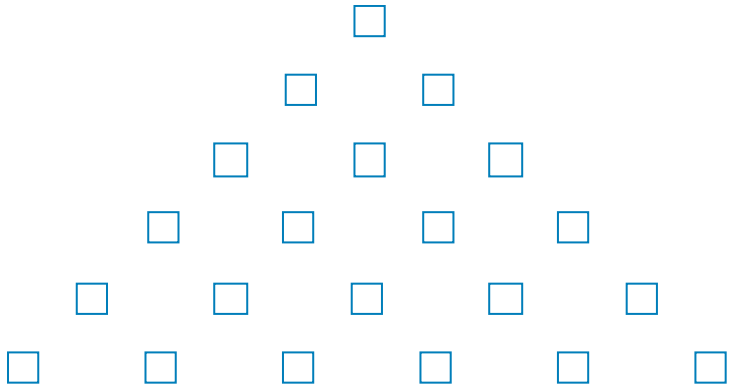
(b) Use the completed two-way table to calculate the following probabilities.

(i) $P(A' \cap B)$

(ii) $P(B')$

Construct Pascal's triangle

8 Complete the following diagram to show Row 0 to Row 5 of Pascal's triangle.

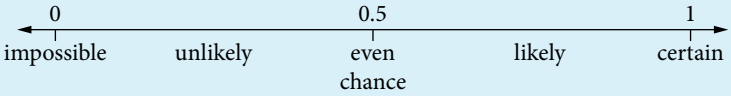


5.1 Probability review

From impossible to certain

Probability is the study of chance. The probability of an event must fall in the range from 0 (impossible) to 1 (certain); that is, all probabilities have a value in the interval [0,1].

For an event A: $0 \le P(A) \le 1$



Additional information

Events

Review the concept of events.

Consider the following example to help define some terms. When you roll a fair six-sided die, you are conducting a trial. The possible outcomes of this trial are 1, 2, 3, 4, 5 and 6.

$P(\text{a number} > 6) = 0$, $P(2) = \frac{1}{6}$, $P(\text{a number from 1 to 6}) = 1$

The set of all the possible outcomes is called the sample space.

Experimental probability

Experimental probability is based on data collected from experiments, surveys or observation.

Experimental probabilities come from collected data, so the probability values are estimates only and are usually expressed as percentages or decimals rounded to 2 decimal places, where neccesary.

$$\text{Probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of trials}}$$

For an event, A:

$$P(A) = \frac{n(A)}{n(\text{total})}$$

1 Use the results of an experiment to predict future outcomes

A six-sided die was rolled 30 times, giving the following results.

Number showing on die	1	2	3	4	5	6
Number of times	3	5	8	4	4	6

Based on these results, determine the probability of obtaining a 1 on the next roll of this die.

THINKING	WORKING
1 Identify the number of successful outcomes.	The number 1 appeared three times.
2 Substitute into the experimental probability formula: $P(A) = \frac{n(A)}{n(\text{total})}$	$P(1) = \frac{3}{30}$
3 Experimental probability is an estimate, round your answer correct to 3 decimal places if necessary.	$P(1) = 0.1$

4 Alternatively, you can use a more formal approach, defining the event.	Let A be the event of rolling a 1. $P(A) = 0.1$
5 Interpret the result.	The probability of obtaining a 1 on the next roll of this die is 0.1.

Theoretical probability

Theoretical probability is based on listing all of the possible outcomes.

Theoretical probability values are exact, so express values as fractions.

$$\text{Probability of an event} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

where it is assumed that the number of possible outcomes is constant.

For an event A: $P(A) = \dfrac{n(A)}{n(\text{total})}$

Complementary events

The complement of an event is the same as the given event 'not' occurring. For example, when a die is rolled, the complement of the event 'rolling a 6' is 'rolling a number that is not 6'.

So, if event A is 'rolling a 6', then the complementary event A' or $\overline{A}$ is 'rolling a number that is not 6'. The probabilities of complementary events add to 1: $P(A) + P(A') = 1$

For both theoretical and experimental data, you may use the probability of an event to determine the complement of that event.

$$P(A') = 1 - P(A)$$

2 Calculate simple theoretical probability

A fair die is rolled once.

(a) Calculate the probability of obtaining an even number.

THINKING	WORKING
1 List the possible outcomes. Identify the favourable outcomes.	Sample space: {1 2, 3, 4, 5, 6} Even numbers: {2, 4, 6}
2 Substitute the values into the rule Probability of an event $= P(A) = \dfrac{n(A)}{n(\text{total})}$	$P(\text{even}) = \dfrac{3}{6}$ $= \dfrac{1}{2}$
3 Interpret the answer.	The probability of rolling an even number is $\dfrac{1}{2}$.

(b) Calculate the probability of not obtaining a six.

THINKING	WORKING
1 Define a convenient, simple event.	Let A be the event of rolling a 6.
2 Determine the probability of the defined event.	$P(A) = \frac{1}{6}$
3 Determine the probability of the complementary event.	$P(\bar{A}) = 1 - P(A)$ $= 1 - \frac{1}{6}$ $= \frac{5}{6}$
4 Interpret the answer.	The probability of not obtaining a six is $\frac{5}{6}$.

Expected number

You can use theoretical and experimental probabilities to determine the expected number of outcomes from multiple trials.

Expected number = probability of an event × number of trials

$E(A) = P(A) \times n$

Additional information

Probable events

Review the concept of the probability of events.

3 Determine the expected value

From previous experience a manufacturer knows that the probability that a box of matches contains fewer than 47 matches is 0.025. In a shipping carton of 5000 boxes, how many boxes would the manufacturer expect to contain fewer than 47 matches?

THINKING	WORKING
1 Identify the probability of the event and the number of trials.	$P(<47) = 0.025$ Number of trials = 5000
2 Apply the rule: Expected number = probability of an event × number of trials	$E(<47) = P(<47) \times n$ $= 0.025 \times 5000$ $= 125$
3 Interpret the answer.	The manufacturer would expect 125 boxes to contain fewer than 47 matches.

Multiple events

A tree diagram or a grid can be used to determine the probabilities of outcomes in multiple events.

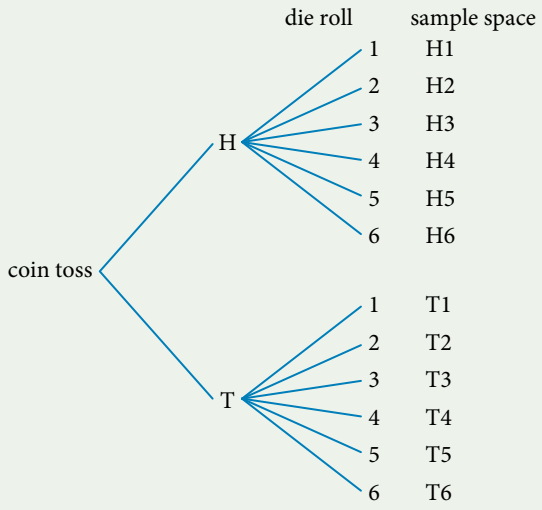
Technology worked example

Simulating events

4 Probability of multiple events

A coin is tossed and a die is rolled. Determine the probability that the upper faces are: 'heads' for the coin and 'a number greater than 2' on the die.

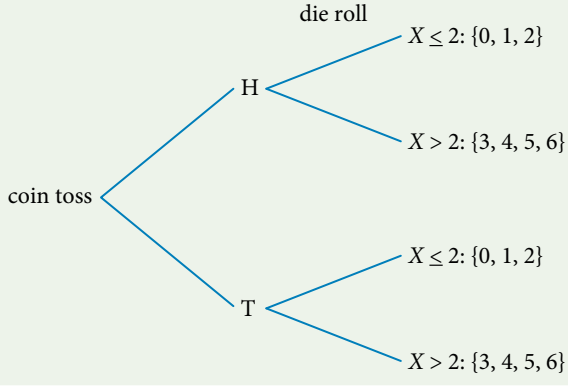
METHOD 1

THINKING	WORKING
1 Construct a tree diagram to identify each possible outcome in the sample space.	
2 Determine the number of successful outcomes.	There are 4 outcomes where the coin is tossed and results in a head (H) and a number greater than 2 (the numbers 3, 4, 5 or 6) is rolled on the die: H3, H4, H5 and H6
3 Determine the probability of the event using the formula: $P(A) = \frac{n(A)}{n(\text{total})}$	$P(\text{heads and number} > 2) = \frac{4}{12} = \frac{1}{3}$

METHOD 2

1 Construct a tree diagram to identify each outcome, but limit the outcomes of each experiment to 2 branches, showing the 'successful' outcome and the 'unsuccessful' outcome.

Let X represent the number obtained when a die is rolled.



2 Determine the probabilities of each event by dividing the number of successful outcomes by the total number of possible outcomes:

$$P(A) = \frac{n(A)}{n(\text{total})}$$

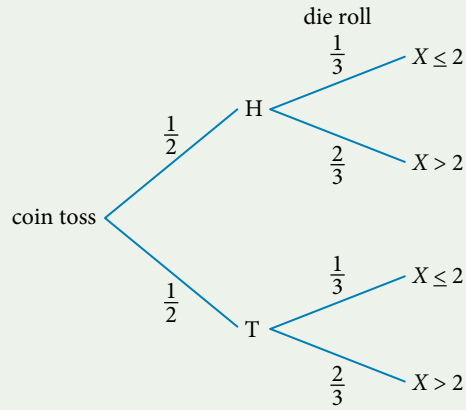
Coin: {H, T}

$$P(\text{H}) = \frac{1}{2}$$

Die: $\{1, 2, 3, 4, 5, 6\}$

$$P(X > 2) = \frac{4}{6} = \frac{2}{3}$$

3 Label the diagram with the known probabilities.



4 Determine the probability of the successful event by reading the diagram.

$\frac{2}{3}$ of $\frac{1}{2}$ of all possible outcomes contain both a head a number greater than 2.

$$P(\text{H}, X > 2) = \frac{1}{2} \times \frac{2}{3} = \frac{2}{6} = \frac{1}{3}$$

5 Evaluate the reasonableness of the answer.

The possible outcomes are:

H1, H2, H3, H4, H5, H6

T1, T2, T3, T4, T5, T6

4 out of the 12 outcomes are the result of tossing a coin resulting in a head (H) and rolling a number greater than 2.

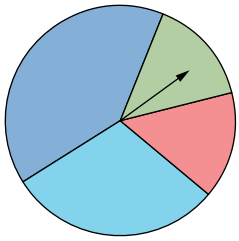
Simulation

Single and multiple events can be simulated using random number generators on your calculator or in a spreadsheet. To simulate the rolling of a die, generate numbers from 1 to 6. To simulate tossing a coin, allocate heads the number 1 and 'tails' the number 0, and then generate numbers from 0 to 1.

EXERCISE 5.1 Probability review

Worked Example

1 A spinner divided into four unequal coloured sections is spun 15 times. 1



Colour	Dark blue	Green	Red	Pale blue
Number recorded	5	3	3	4

Based on these results, calculate the probability (correct to 2 decimal places), that the result of the next spin will be:

(a) dark blue (b) green

2 The 10 letters in the word SMARTPHONE are placed in a box. The letters are mixed and then a single letter is selected at random from the box. Let A be the event 'getting a vowel', B be the event 'getting an A', and C be the event 'getting an A or an N'. 2

Determine the value of the following probabilities in simplest fraction form.

(a) $P(A)$ (b) not getting a vowel (c) $P(B)$

(d) not getting an A (e) $P(C)$ (f) not getting an A or an N

3 At a particular electronics factory, the probability of producing a defective MP3 player is 0.007. If 2859 MP3 players were produced, how many would you expect to be defective? 3

4 The students at Bright Future Secondary College were surveyed about wearing school uniforms. The results are displayed in the table. Determine the probability in simplest fraction form:

	Year 7	Year 8	Year 9	Year 10	Year 11	Year 12
In favour	60	55	40	45	25	15
Against	20	25	35	45	55	60
Not sure	10	10	10	5	5	0

(a) that a Year 7 student selected at random is in favour of wearing a school uniform

(b) that a Year 10 student selected at random is against wearing a school uniform

(c) that a student selected at random will not be sure about wearing a school uniform

(d) that students in Years 11 and 12 combined are in favour of wearing a school uniform, as a percentage.

5 Jackson Brothers sells new and used cars. On a particular day their car yard contains 22 new Fords, 15 used Fords, 12 used Holdens, 5 used Toyotas and 3 used Hondas. If a customer purchases a car at random on this day, determine the probability as a fraction in simplest form that the vehicle is:

(a) a new Ford (b) a Toyota (c) a used car

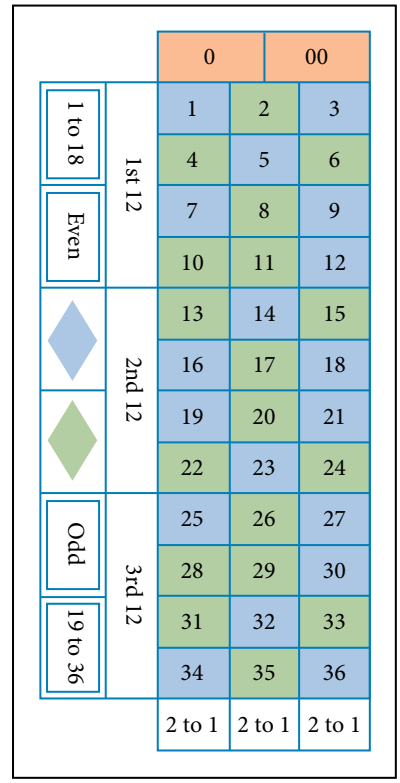
(d) a Ford (e) a used car but not a Ford (f) a Ford or a Holden

6 Antonela has a $\frac{1}{10}$ chance of being late to school on any particular day. In a term of 50 school days, on how many days would you expect Antonela to be late?

A 10 B 5 C 40 D 45

7 Steven observes 60 spins of a roulette wheel that has numbers and colours as shown on the betting table. He records the slot where the ball lands (comes to rest) in each of the spins.

27	2	12	10	31	2	9	34	3	9
23	3	15	14	11	11	28	19	20	12
2	22	29	00	24	14	12	26	27	25
26	28	25	25	32	32	34	1	3	27
17	36	4	21	11	28	0	14	20	18
5	35	30	24	00	26	1	17	30	15



Use the collected data above to calculate the experimental probability that the ball lands:

(a) on an even number (not counting 0 or 00)

(b) on a number in the range 1–18

(c) on green

(d) in the 2nd 12, (13–24)

(e) in the middle column (2, 5, 8 ..., 35).

8 A bag contains 5 red marbles, 6 blue marbles and 9 black marbles. The marbles are mixed and then a single marble is randomly drawn from the bag. Calculate the probability of selecting a marble that is:

(a) red (b) blue (c) red or black (d) white (e) white or black.

9 The Parliament of Austland has the following numbers of parliamentarians, in terms of their gender and political affiliation. Calculate the probability in simplest fraction form, that:

	Male	Female
Conservative	14	11
Socialist	19	8
Independent	1	7

(a) a parliamentarian chosen at random is female

(b) an Independent parliamentarian chosen at random is female

(c) a parliamentarian chosen at random is not a male Socialist.

10 A pack of 20 cards has six cards with a % on them, five with a ⊕, and nine with an e.

(a) What is the probability that a single card drawn at random is not a ⊕?

A $\frac{1}{2}$ B $\frac{1}{4}$ C $\frac{2}{3}$ D $\frac{3}{4}$

(b) Explain the common error made by a student who thought the answer was $\frac{1}{3}$.

Worked Example

4

11 A bag contains some coloured marbles: 7 pink, 4 red and 4 blue.
If two marbles are drawn from the bag in succession, with the first observed and replaced before the second is drawn, determine the following probabilities.

(a) the first is red and the second is blue (b) the marbles are red and blue

(c) both marbles are pink (d) the marbles are the same colour

12 Two bags are placed on a counter. The first contains 3 red, 2 blue and 5 yellow marbles. The second contains 1 red, 6 blue and 4 yellow marbles. If a marble is drawn from each bag, determine the following probabilities.

(a) both marbles are red
(b) both marbles are blue
(c) both marbles are yellow
(d) the marbles are the same colour
(e) the marbles are different colours
(f) the first marble is red and the second is not
(g) the first marble is red and the second is blue
(h) the first marble is blue and the second red
(i) one marble is red and the other is blue

13 A bowl contains ping pong balls numbered 1–6. Two balls are drawn simultaneously from the bowl. Determine the following probabilities.

(a) the total is 4
(b) the numbers are consecutive
(c) both numbers are even

14 Two fair six-sided dice are rolled. Determine the following probabilities.

(a) double 6
(b) any double
(c) 6 on the first and 3 on the second
(d) 3 on the first and 6 on the second
(e) a 3 and a 6
(f) a total of 9

15 If two coins are tossed 100 times, how often would you expect double heads?

16 A red and a green bag are placed on a counter, each containing ping-pong balls with numbers on them. The red bag contains three balls marked 4, two balls marked 5 and one ball marked 8. The green bag contains two balls marked 6 and five balls marked 3. A ball is drawn from each bag. Determine the likelihood that one of the balls is marked:

(a) 6, if the total of the numbers on the two balls is 11
(b) 4, if the product of the numbers on the two balls is 24.

17 At the local show Xavier plays a game that costs \$1 to play. The rules are:
'Two coins are tossed. If they both come up heads you will receive \$2 (the \$1 cost back plus \$1 extra); otherwise you lose the \$1.'

(a) Xavier has studied probability and says that the game is not fair. The game operator then offers \$3 for the two heads (his \$1 back and \$2 extra). Is the new game fair?
(b) If this game were fair, what would you expect the outcome to be at the end of 10 games?
(c) Use two coins (or technology) to play the game according to the new rules. Copy and extend the table to show the outcome of 10 games.

Game number	Result (win/lose)	Profit/loss	Total profit/loss
1			
2			
…			

(d) On the basis of this set of trials, is the game fair?

5.2 Probability and set notation

The elements of a set

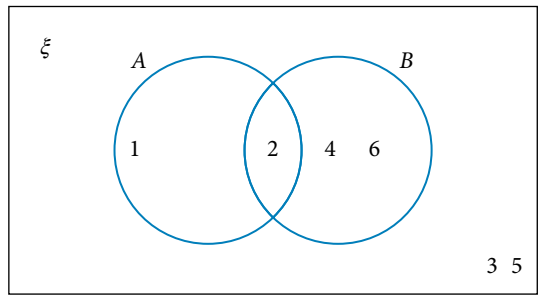
You can use set notation as a form of mathematical shorthand to solve problems involving groups, collections and categories. A set is made up of a collection of elements. The number of elements can be anything from zero to infinity.

Complement, intersection and union

Symbol	Description	Example
ξ (or U or S)	The *universal set* contains all possible objects relevant to a particular situation. This is also called the sample space.	$\xi = (1, 2, 3, 4, 5, 6\}$ $A = \{1, 2\}$ $B = (1, 2, 4, 6\}$ $C = \{\text{negative numbers}\}$
$\in$	An *element* of a set.	$2 \in A$
$\notin$	*Not an element* of a set.	$3 \notin A$
$\subset$	A is a *subset* of B because every element of A is also an element of B.	$A \subset B$
$n(A)$	The number of elements in set A. This is the *cardinal number* of the set.	$n(A) = 2$
A' or $\overline{A}$	The *complement* of a set is all the elements in ξ not in the set.	$A' = \{3, 4, 5, 6\}$
$A \cap B$	The *intersection* of sets A and B is the set containing all elements that are in both A and B.	$A \cap B = \{1, 2\}$
$A \cup B$	The *union* of sets A and B is the set containing all elements that are in either A or B (or both).	$A \cup B = \{1, 2, 3, 4, 6\}$
$\varnothing$ or $\{\}$	The *empty set* (also called the null set). The empty set doesn't contain any elements.	$C = \varnothing$ $n(C) = 0$

Venn diagrams

A Venn diagram can be used to represent sets. A rectangle is used to represent ξ and circles or ellipses represent the sets. If there is an intersection between the sets, then the circles overlap.



Additional information

Sample spaces

Practise writing sample spaces.

If B is a subset of A that is, $B \subset A$, then a Venn diagram shows the circle for B is entirely within the circle for A.

Additional information

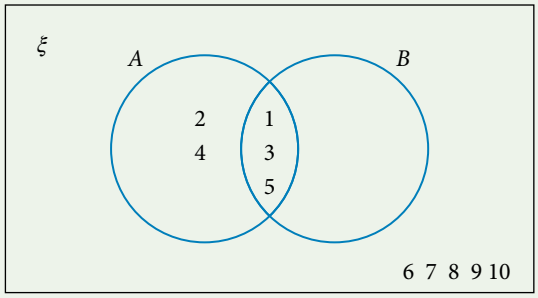
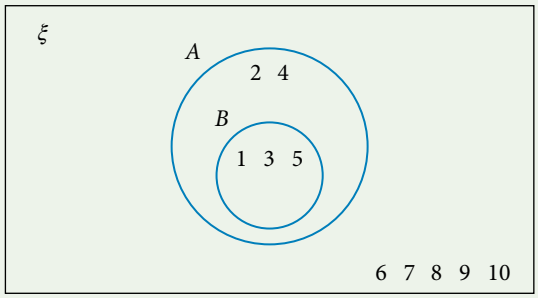
Subsets

Consolidate your understanding of subsets.

5 Draw Venn diagrams

$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$

(a) Draw a Venn diagram to represent the sets.

THINKING	WORKING
1 Draw a rectangle to represent the universal set. Draw overlapping circles to represent sets A and B. Write in each element.	
2 B is a subset of A, so the circle for B can be drawn inside the circle for A.	

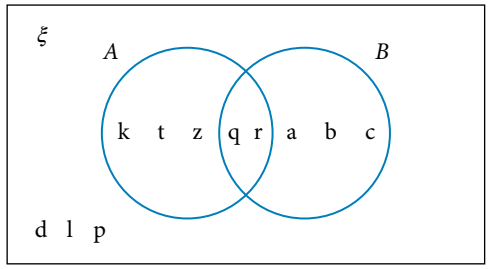
(b) Use your diagram to list the elements of A', B', $A \cap B$, $A \cup B$.

A' means the elements not in A.	$A' = \{6, 7, 8, 9, 10\}$
B' means the elements not in B.	$B' = \{2, 4, 6, 7, 8, 9, 10\}$
$A \cap B$ means elements in both A and B.	$A \cap B = \{1, 3, 5\}$
$A \cup B$ means elements in either A or B.	$A \cup B = \{1, 2, 3, 4, 5\}$

When using a Venn diagram to determine the elements in intersections or unions, it can be useful to show this by shading the parts in different ways. If two sets have different shadings, then their intersection will have both shadings, and their union will have either one shading or both.

6 Use a Venn diagram to list sets

Use the information in this Venn diagram to determine the elements of the described subsets.



(a) Determine $A' \cap B'$.

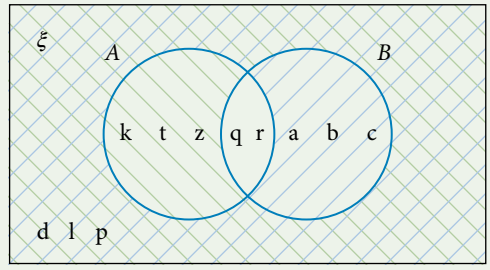
THINKING

Shade A' and B' in different ways. The required intersection is where *both* types of shading exist, because $\cap$ means 'and'.

WORKING

A' is shaded

B' is shaded



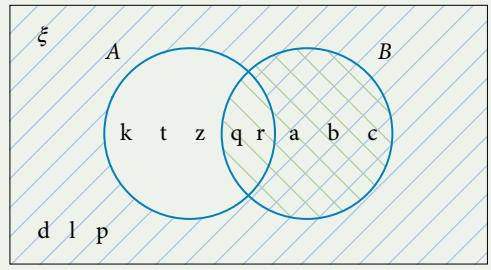
So, $A' \cap B' = \{d, l, p\}$

(b) Determine $A' \cup B$.

Shade A' and B in different ways. The required union is where *any* shading exists, because $\cup$ means 'or'.

A' is shaded

B is shaded

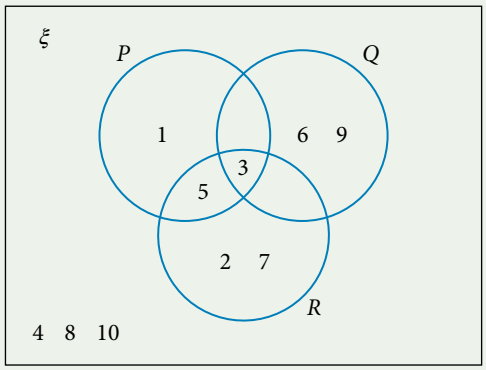


So, $A' \cup B = \{a, b, c, d, l, p, q, r\}$

7 Draw a Venn diagram for three intersecting sets

Draw a Venn diagram for $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $P = \{\text{odd numbers less than 6}\}$, $Q = \{\text{multiples of 3}\}$ and $R = \{\text{prime numbers}\}$. Use the diagram to select the elements in each of the following.

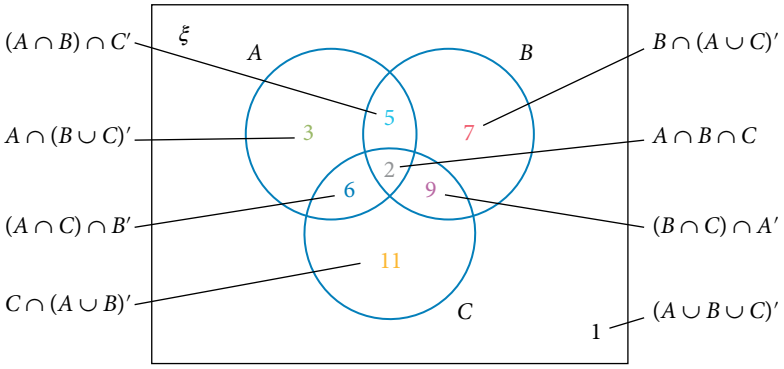
(a) $P' \cap Q'$

THINKING	WORKING
1 Identify the sample space of each set. Here, the universal set is the integers from 1 to 10.	$P = \{1, 3, 5\}$ $Q = \{3, 6, 9\}$ $R = \{2, 3, 5, 7\}$
2 Draw the Venn diagram.	
3 Identify the elements in each component of the required set.	$P' = \{2, 4, 6, 7, 8, 9, 10\}$ $Q' = \{1, 2, 4, 5, 7, 8, 10\}$
4 The required intersection is where elements are in *both* lists.	$P' \cap Q' = \{2, 4, 7, 8, 10\}$

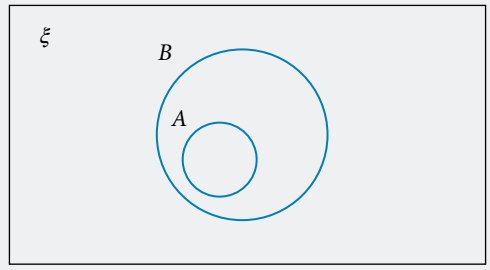
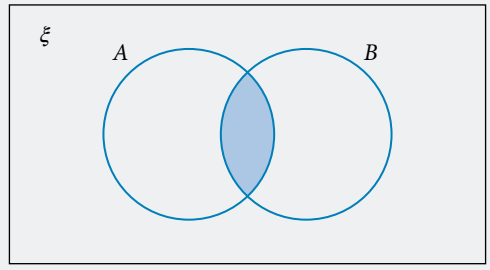
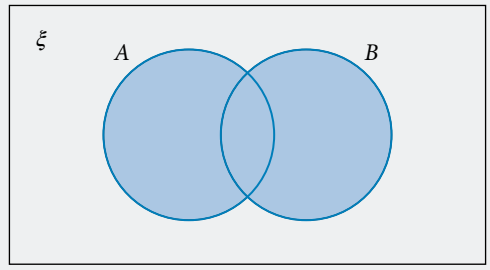
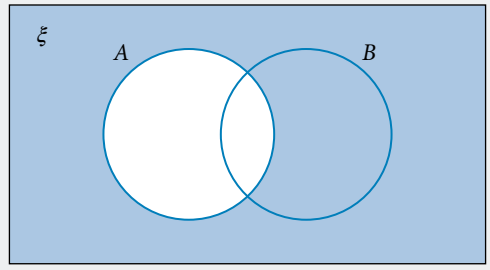
(b) $P \cup R'$

1 Identify the elements in each component of the required set.	$P = \{1, 3, 5\}$ $R' = \{1, 4, 6, 8, 9, 10\}$
2 The union includes elements in one or the other set (or both).	$P \cup R' = \{1, 3, 4, 5, 6, 8, 9, 10\}$

Set notation can be used to identify the elements in the unions and intersections of a Venn diagram.



Summary of set notation and Venn diagrams

$A \subset B$	A is a subset of B	
$A \cap B$	The intersection of sets A and B.	
$A \cup B$	The union of sets A and B.	
A' or $\overline{A}$	The complement of set A.	

Venn diagrams and probability

The probability of a particular outcome can be determined using a Venn diagram.

$$P(A) = \frac{n(A)}{n(\xi)}$$

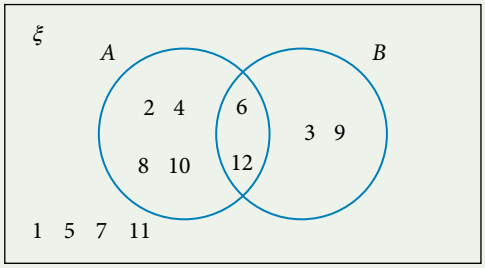
WARNING

Do not confuse an intersection and a union of two sets. For example, 'even numbers that are multiples of 3' represents an intersection, so each number needs to be even *and* a multiple of 3. The expression 'even numbers or multiples of 3' represents a union.

8 Use a Venn diagram to calculate probabilities

A spinner with 12 equal segments numbered 1 to 12 is spun once.

(a) List the elements of set A (even numbers) and set B (multiples of 3) and draw a Venn diagram.

THINKING	WORKING
1 Identify the elements of ξ, A and B.	$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ $A = \{2, 4, 6, 8, 10, 12\}$ $B = \{3, 6, 9, 12\}$
2 Draw the Venn diagram.	

(b) Determine the probability of obtaining each of the following.

(i) an even number

1 List the set A. Count the number of elements.	$A = \{2, 4, 6, 8, 10, 12\}$ $n(A) = 6$ $n(\xi) = 12$
2 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(A) = \dfrac{n(A)}{n(\xi)}$ $= \dfrac{6}{12}$ $= \dfrac{1}{2}$
3 Interpret the answer.	The probability of obtaining an even number is $\frac{1}{2}$.

(ii) an even number or a multiple of 3

1 Determine the set required.	Recognise 'or' so this is the union. $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12\}$
2 Count the number of elements.	$n(A \cup B) = 8$ $n(\xi) = 12$
3 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(A \cup B) = \dfrac{n(A \cup B)}{n(\xi)}$ $= \dfrac{8}{12}$ $= \dfrac{2}{3}$

4 Interpret the answer.	The probability of obtaining an even number or a multiple of 3 is $\frac{2}{3}$.

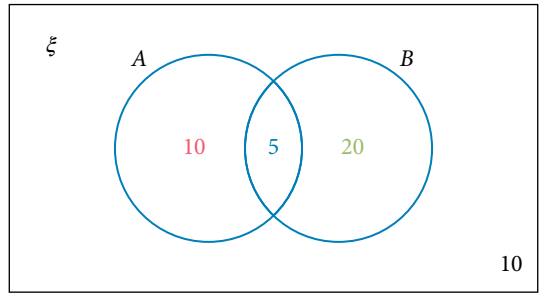
(iii) an even number that is a multiple of 3

1 Determine the set required.	Recognise 'and' so this is an intersection. $A \cap B = \{6, 12\}$
2 Count the number of elements.	$n(A \cap B) = 2$ $n(\xi) = 12$
3 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(A \cap B) = \dfrac{n(A \cap B)}{n(\xi)}$ $= \dfrac{2}{12}$ $= \dfrac{1}{6}$
4 Interpret the answer.	The probability of obtaining an even number that is a multiple of 3 is $\frac{1}{6}$.

(iv) a number that is neither an even number nor a multiple of 3

1 Determine the set required.	Recognise 'nor' so this includes elements that are not in the union. $(A \cup B)' = \{1, 5, 7, 11\}$
2 Count the number of elements.	$n(A \cup B)' = 4$ $n(\xi) = 12$
3 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(A \cup B)' = \dfrac{4}{12}$ $= \dfrac{1}{3}$
4 Interpret the answer.	The probability of obtaining a number that is neither even nor a multiple of 3 is $\frac{1}{3}$.

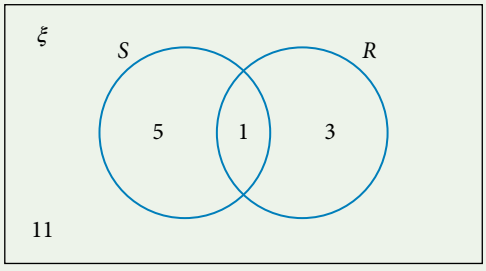
Not all Venn diagrams list every element in every part of the diagram. When there are too many elements, each part of the diagram is labelled with the number of elements in that part. It is usually clear from the diagram whether it shows all the individual elements or only the number of elements in each part.



9 Use Venn diagrams with the number of elements

A box contains 20 toy blocks of various styles and colours. There are 6 square blocks and 4 red blocks. Only 1 of the square blocks is red.

(a) Define the sets and draw the Venn diagram.

THINKING	WORKING
1 Define the sets and determine the number of elements in each set.	$S = \{\text{square blocks}\}$, $n(S) = 6$ $R = \{\text{red blocks}\}$, $n(R) = 4$
2 Identify the number of elements in the intersection, if any.	$n(S \cap R) = 1$
3 Complete the Venn diagram by entering the appropriate values in the parts of the diagram.	

(b) Using the Venn diagram, determine the following probabilities.

(i) $P(\text{block is not red})$

1 Determine the total number of blocks in the sample space.	$n(\text{not red}) = 5 + 11$ $= 16$
2 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(\text{not red}) = \dfrac{n(\text{not red})}{n(\text{blocks})}$ $= \dfrac{16}{20}$ $= \dfrac{4}{5}$
3 Interpret the answer.	$\dfrac{4}{5}$ of the blocks are not red.

(ii) $P(\text{block is square and not red})$

1 Determine the total number of blocks in the sample space.	$n(\text{square and not red}) = 5$
2 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(\text{square and not red}) = \dfrac{n(S \cap R')}{n(\xi)}$ $= \dfrac{5}{20}$ $= \dfrac{1}{4}$
3 Interpret the answer.	$\dfrac{1}{4}$ of the blocks are square, but not red.

(iii) P(block is neither red nor square)

1 Determine the total number of blocks in the sample space.	$n(R \cup S)' = 11$
2 Recall the formula $P(A) = \dfrac{n(A)}{n(\xi)}$ to calculate the probability of the event.	$P(R \cup S)' = \dfrac{n(R \cup S)'}{n(\xi)} = \dfrac{11}{20}$
3 Interpret the answer.	$\dfrac{11}{20}$ of the blocks are neither square nor red.

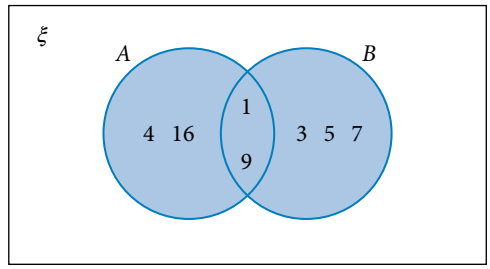
Probability of complementary events

The complement of an event contains all the elements of the sample space that are not in that event. So $n(\xi) = n(A) + n(A')$ and $P(A) + P(A') = 1$.

For example, if $\xi = \{a, b, c, d, e, f, g, h, i, j\}$ and $A = \{\text{vowels}\}$, then $P(A) = \frac{3}{10}$ because the only vowels in the universal set are a, e and i. Without doing any additional counting or checking you can see that $P(A') = 1 - \frac{3}{10} = \frac{7}{10}$ because A and A' together cover all the elements in the universal set.

The addition rule for probability

The Venn diagram is shaded to represent $A \cup B$. You can see that:



$A = \{1, 4, 9, 16\}$, $n(A) = 4$

$B = \{1, 3, 5, 7, 9\}$, $n(B) = 5$

$A \cap B = \{1, 9\}$, $n(A \cap B) = 2$

$A \cup B = \{1, 3, 4, 5, 7, 9, 16\}$, $n(A \cup B) = 7$

In general, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ because the elements of $A \cap B$ are counted twice, once in $n(A)$ and again in $n(B)$.

The *addition rule for probability* is written as:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$n(A) + n(B) = n(A \cup B)$ if the intersection of A and B is the null set, that is A and B are mutually exclusive.

Additional information

Addition rule

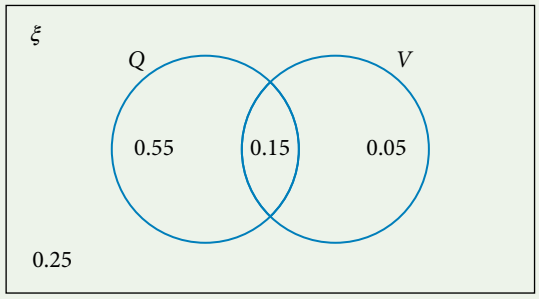
Practise using the addition rule.

10 Use the addition rule to calculate missing values

(a) Given $P(Q) = 0.7$, $P(V) = 0.2$ and $P(Q \cap V) = 0.15$, determine the value of $P(Q \cup V)$.

THINKING	WORKING
1 Recall the addition rule for probability using the appropriate pronumerals, substitute the known values and calculate the answer.	$P(Q \cup V) = P(Q) + P(V) - P(Q \cap V)$ $= 0.7 + 0.2 - 0.15$ $= 0.75$

2 Confirm your answer with a Venn diagram

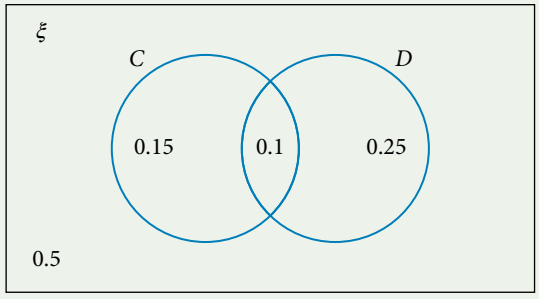


(b) Given $P(C) = 0.25$, $P(D) = 0.35$ and $P(C \cup D) = 0.5$, determine the value of $P(C \cap D)$.

1 Recall the addition rule for probability using the appropriate pronumerals, substitute the known values and calculate the answer.

$$P(C \cup D) = P(C) + P(D) - P(C \cap D)$$
$$P(C \cap D) = P(C) + P(D) - P(C \cup D)$$
$$= 0.25 + 0.35 - 0.5$$
$$= 0.1$$

2 Confirm your answer with a Venn diagram



11 Use the addition rule to solve problems

A die in the shape of a dodecahedron, showing the numbers 1 to 12, is rolled once. Determine the probability that the result is either a prime number or a number less than 9.

THINKING	WORKING
1 Identify the sets involved.	$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ $P = \{\text{prime numbers}\}$ $= \{2, 3, 5, 7, 11\}$ $N = \{\text{numbers less than 9}\}$ $= \{1, 2, 3, 4, 5, 6, 7, 8\}$
2 Calculate (without simplifying) the probability of each event.	$P(P) = \frac{5}{12}$ $P(N) = \frac{8}{12}$
3 Check the wording of the question to identify the operation required.	The word 'or' indicates that a union is required.

4 To use the addition rule for probability determine the unknown term (in this case, the intersection) and its probability.	$P \cap N = \{2, 3, 5, 7\}$ $P(P \cap N) = \frac{4}{12}$
5 Apply the addition rule for probability.	$P(P \cup N) = P(P) + P(N) - P(P \cap N)$ $= \frac{5}{12} + \frac{8}{12} - \frac{4}{12}$ $= \frac{9}{12}$ $= \frac{3}{4}$
6 Interpret the answer.	The probability of obtaining either a prime number or a number less than 9 is $\frac{3}{4}$.

Additional information

Calculating probabilities

Practise solving probability problems.

EXERCISE 5.2 Probability and set notation

Worked Example 5

1 Consider the sets $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4\}$.

(a) Draw the Venn diagram for these sets.

(b) List the elements in:

(i) A' **(ii)** B' **(iii)** $A \cup B$ **(iv)** $A \cap B$

2 Consider the sets $\xi = \{2, 4, 6, 8, 10\}$, $A = \{2, 4\}$, $B = \{2, 8\}$.

(a) Draw the Venn diagram for these sets.

(b) List the elements in:

(i) A' **(ii)** $A \cup B$ **(iii)** $A \cap B$

3 Given $\xi = \{11, 12, 13, 14, 15, 16, 17, 18\}$, $A = \{11, 12, 15\}$ and $B = \{12, 15, 17, 18\}$, answer the following questions.

(a) Determine $n(A \cup B)$.

A 8 **B** 1 **C** 5 **D** 2

(b) Explain the common error made by a student who calculated $n(A \cup B) = 7$.

(c) Which set represents $(A \cap B)'$?

A $\{15\}$ **B** $\{11, 12, 13, 17, 18\}$ **C** $\{11, 13, 14, 16, 17, 18\}$ **D** $\{14, 16\}$

4 Draw a Venn diagram for the following:
$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$, $C = \{3, 5, 6\}$

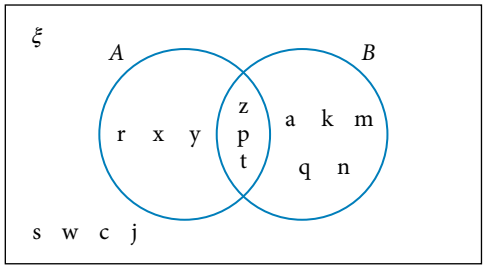
5 Given that $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 20\}$ and $C = \{2, 4\ 6\}$, choose the best symbol from $\varnothing$, $\in$, $\notin$, $\subset$, $\not\subset$, for each of the following statements.

(a) $1 \square A$
(b) $C \square B$
(c) $5 \square B$
(d) even numbers less than $12 \square B$
(e) $A \square \xi$
(f) $A \cap B = \square$

6 For $\xi = \{a, b, c, d, e, f, g, h, i, j, k\}$, $A = \{b, c, d\}$ and $B = \{f, g, h, k\}$, choose True or False for the following statements.

(a) $b \in A$
(b) $n(C) < n(A)$
(c) $c \in A \cap B$
(d) $A \subset B$
(e) $e \notin B$
(f) $A \subseteq C$
(g) $A \subset C'$
(h) $n(B \cap C) = 1$

7 Using set notation, give the simplest expression for each of the following sets. (Note that the answers are not necessarily unique.)



(a) {p, t, z}
(b) {a, k, m, n, p, q, t, z}
(c) {c, j, s, w}
(d) {a, c, j, k, m, n, p, q, s, t, w, z}
(e) {r, x, y}
(f) {a, c, j, k, m, n, p, q, r, s, t, w, x, y, z}

8 A, B and C are sets. Write the following in set notation.

(a) not A
(b) A or B
(c) neither A nor C
(d) not A, B or C
(e) A and C
(f) A or not C
(g) A or not B, and not C
(h) B and not A
(i) C or not B
(j) A and B and C

Worked Example 8

9 A spinner with 24 equal segments numbered 1 to 24 is spun. Determine the probabilities of the following outcomes in simplest fraction form.

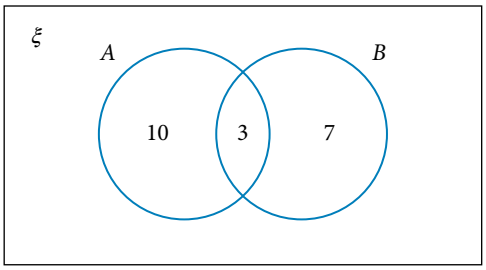
(a) a multiple of 4
(b) a multiple of 6
(c) a multiple of 4 that is not a multiple of 6
(d) a number that is neither a multiple of 4 nor a multiple of 6

9

10 Of the 100 PE students at Orls Port University, 85 play hockey or basketball, 75 play basketball and 15 play hockey. Determine the probability that a student chosen at random:

(a) plays hockey
(b) plays both sports
(c) does not play basketball
(d) does not play either sport.

11 Use the Venn diagram, which shows the number of elements, to determine the following probabilities in simplest fraction form.



(a) $P(A)$
(b) $P(B)$
(c) $P(A \cap B)$
(d) $P(B')$
(e) $P(B' \cap A)$
(f) $P(B' \cap A')$

10

12 Use the given information to calculate the required probability.

(a) Given $P(Z) = 0.6$, $P(T) = 0.3$ and $P(Z \cap T) = 0.15$, find $P(Z \cup T)$.

(b) Given $P(K) = 0.25$, $P(J) = 0.5$ and $P(K \cup J) = 0.6$, find $P(K \cap J)$.

(c) Given $P(Q) = 0.3$, $P(Q \cap R) = 0.1$ and $P(Q \cup R) = 0.65$, find $P(R)$.

11

13 A card is drawn randomly from a standard pack of 52 playing cards. Determine the probability of a:

(a) red or a King

(b) an Ace or a heart

(c) a picture card (Jack, Queen or King) or red

(d) a club or a number card (including the Ace)

14 A die in the shape of a decahedron, with the faces numbered 1 to 10, is rolled once. Determine the probability of the following outcomes.

(a) less than 10 or even

(b) odd or greater than 5

(c) divisible by 4 or 3

(d) prime or a square number

15 For two events T and V, $P(T) = 0.3$, $P(V) = 0.8$ and $P(T \cap V) = 0.25$. Determine the value of $P(T \cup V)'$.

16 For two events Z and Y, $P(Z) = \frac{1}{3}$, $P(Y) = \frac{1}{5}$ and $P(Z \cup Y) = \frac{2}{5}$. $P(Z \cap Y)$ is:

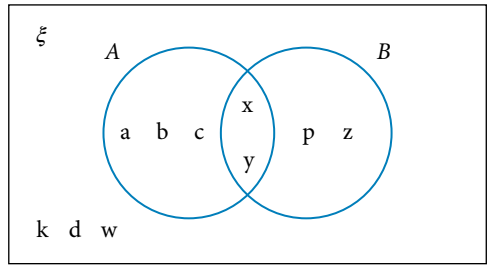
A $\frac{3}{5}$ B $\frac{1}{15}$ C $\frac{14}{15}$ D $\frac{2}{15}$

6

17 For the Venn diagram shown, list the following sets.

(a) $A \cap B'$

(b) $(A \cup B)'$



7

18 For the Venn diagram shown, list the following sets.

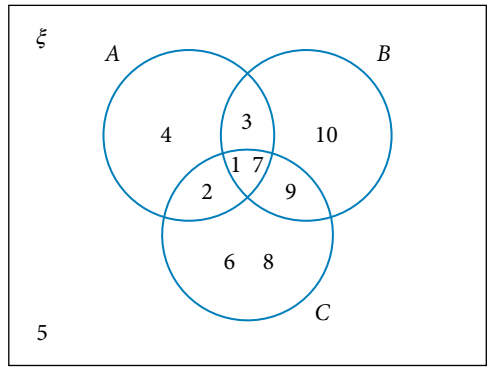
(a) $(A \cap B) \cup C$

(b) $A \cap (B \cup C)$

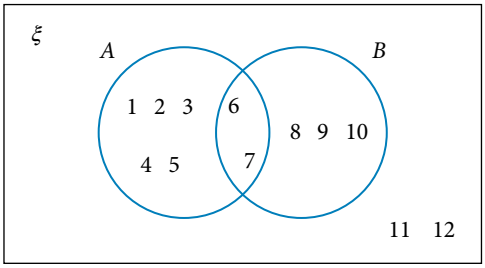
(c) $A' \cup B'$

(d) $C' \cup (A \cap B)$

(e) $A' \cup B' \cup C'$



19 **(a)** List the elements in each set using the information given in the Venn diagram on the right.



(i) $A \cap B$ **(ii)** $A' \cup B'$

(iii) $(A' \cup B')'$ **(iv)** $A \cup B$

(v) $(A \cup B)'$ **(vi)** $A' \cap B'$

(b) What do your answers to **(i)** and **(iii)** tell you about $A \cap B$ and $(A' \cup B')'$? Explain using Venn diagrams.

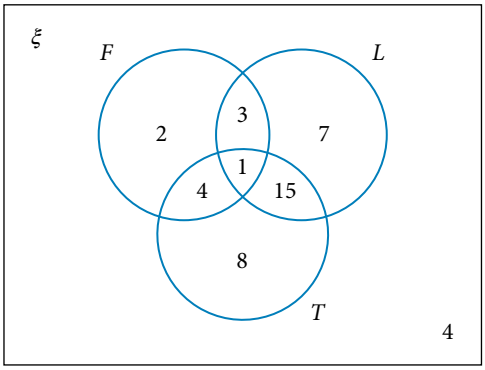
(c) What do your answers to **(v)** and **(vi)** tell you about $(A \cup B)'$ and $A' \cap B'$?

20 For 16 cards, numbered 1 to 16, the following sets are defined:
$A = \{\text{even numbers}\}$, $B = \{\text{numbers less than ten}\}$, $C = \{\text{square numbers}\}$.
Determine the probability, in simplest fraction or whole number form, that the number on a card selected at random will be an element of the following.

(a) $A \cup B$ **(b)** $A' \cap C'$ **(c)** $A \cup (B \cap C)$

(d) $A \cap B \cap C$ **(e)** $(A \cup B \cup C)'$ **(f)** $(A \cup B)' \cap C$

21 The following Venn diagram gives information from a survey about the number of students who are the only child in their family (F), who live more than 2 km from their school (L), and who have travelled interstate at some time in the past (T). Based on this information, determine the probability that a student chosen at random is:



(a) an only child and has travelled interstate

(b) an only child and lives no more than 2 km from the school

(c) a student who has not travelled interstate

(d) a student who lives more than 2 km from the school and has travelled interstate but is not an only child.

22 Of 60 people surveyed about the previous week, 20 had eaten takeaway food (F), 35 had visited a supermarket (S) and 10 had done neither.

(a) Represent this information using a Venn diagram.

(b) Determine the probability that a person chosen at random had in the previous week:

(i) visited a supermarket but not eaten takeaway food

(ii) not eaten takeaway food

(iii) visited a supermarket and eaten takeaway food.

23 A library of 140 books contains 45 science fiction (S) books in total and 75 adventure (A) books in total, and of these 30 books are both science fiction and adventure. Determine the probability that a book selected at random is:

(a) not an adventure book

(b) neither adventure nor science fiction

(c) science fiction only

(d) science fiction or adventure, but not both.

24 Draw a Venn diagram using the following information.
Some students were surveyed about how they get to school. The options were walking (W), riding a bike (R) and taking a bus (B).
Of the 60 students surveyed, 4 use all three methods and 7 use none. 27 students walk, and of these 10 also ride a bike but never take a bus. 23 students take a bus, and 1 of these also walks but does not ride a bike. 6 students ride a bike and also take a bus.

25 Oceania Airlines has a fleet of 60 aircraft, half of which are seaplanes (S) and 17 are single-engined aircraft (E). Long-range radar (R) is installed in 25 aircraft. 18 of the aircraft neither are seaplanes nor have long-range radar, and only 1 of these is single-engined. Four seaplanes have single engines but don't have long-range radar. Determine the probability that an aircraft chosen at random:

(a) is a seaplane with long-range radar

(b) does not have long-range radar

(c) is not a seaplane

(d) is either a seaplane or single-engined

(e) does not have long-range radar nor single-engined

(f) has long-range radar but is not a seaplane

(g) not a single-engined seaplane with long-range radar.

26 Knockout sporting competitions consist of a number of rounds leading up to a final. Only the winners from each round progress to the next round.

(a) The Rangers and the Knights are two of the eight teams playing in a knockout competition. Determine the probability that either the Rangers or the Knights reach the final, assuming each team has an equal chance of winning any particular match.

(b) What is the probability that the Rangers play the Knights in the final of the sixteen-team competition?

(c) Later in the year the Rangers and the Knights play in a sixteen-team knockout competition. Calculate the probability of either the Rangers or the Knights reaching the final, but not both.

Conditional probability

5.3

Reducing the universal set

Consider rolling a fair six-sided die. The probability that a roll results in a 1 is $\frac{1}{6}$. However, if you are told the result is an odd number, then the probability that the roll was a 1 is now $\frac{1}{3}$. The extra information reduced the universal set (or sample space) from {1, 2, 3, 4, 5, 6} to {1, 3, 5}. This is an example of *conditional probability* because a condition is given and the sample space is restricted.

The probability of *A*, given *B*

When sets A and B are defined, the phrase 'the probability of A, given B' is written as $P(A \mid B)$. For example:

$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{\text{obtaining an odd number}\}$

$B = \{\text{obtaining a number less than 6}\}$

$$P(A \mid B) = \frac{\text{probability of obtaining an odd number less than 6}}{\text{probability of obtaining a number less than 6}}$$

The numerator of this fraction is the intersection of A and B.

Additional information

Conditional probability

Practise solving conditional probability problems.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},\ P(B) \neq 0$$

You can transpose this formula to get the multiplication rule for probability:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

12 Simple conditional probability

Given $P(A) = 0.5$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$, determine the following.

(a) $P(A \mid B)$

THINKING	WORKING
1 Recall the conditional probability formula $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$, substitute known values and calculate.	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.7} = \frac{3}{7}$
2 Interpret the answer.	The probability of 'A given B' is $\frac{3}{7}$.

(b) $P(B \mid A)$

1 Recall the conditional probability formula $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$, substitute known values and calculate.	$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.5} = \frac{3}{5}$
2 Interpret the answer.	The probability of 'B given A' is $\frac{3}{5}$.

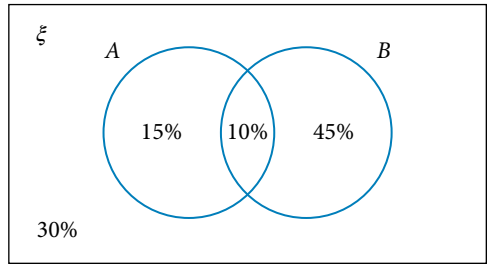
Situations that involve conditional probability usually include the words *given*, *if you know* and *when*. Look for these conditional expressions in probability questions.

WARNING

Do not assume that $P(A \mid B)$ is equal to $P(B \mid A)$. The following worked example shows that this is generally not true.

13 Use a Venn diagram to determine conditional probabilities

From the information given in the Venn diagram, calculate each of the following probabilities.



(a) $P(A \mid B)$

THINKING	WORKING
1 Determine the individual probabilities $P(A)$, $P(B)$ and $P(A \cap B)$ in decimal form.	$A = (15 + 10)\%$ $= 25\%$ $\therefore P(A) = 0.25$ $B = (45 + 10)\%$ $= 55\%$ $\therefore P(B) = 0.55$ $P(A \cap B) = 0.1$
2 Recall the conditional probability formula $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, substitute known values and calculate.	$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.1}{0.55} = \dfrac{2}{11}$
3 Interpret the answer.	The probability of 'A given B' is $\dfrac{2}{11}$.

(b) $P(B \mid A)$

1 Recall the conditional probability formula $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, substitute known values and calculate.	$P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.1}{0.25} = \dfrac{2}{5}$
2 Interpret the answer.	The probability of 'B given A' is $\dfrac{2}{5}$.

Karnaugh maps

A Karnaugh map (also called a two-way table) is useful in conditional probability calculations. The probability of the intersections should add to the total probabilities at the ends of the rows and columns.

	A	A'	
B	$P(A \cap B)$	$P(A' \cap B)$	$P(B)$
B'	$P(A \cap B')$	$P(A' \cap B')$	$P(B')$
	$P(A)$	$P(A')$	1

14 Use a Karnaugh probability map to calculate conditional probabilities

	A	A'	
B			0.4
B'	0.3		
		0.6	1

(a) Complete the Karnaugh probability map.

THINKING

Use the existing values to complete the map. (In this case, the first value that can be filled in is the total for the A column or the B' row.)

WORKING

	A	A'	
B	0.1	0.3	0.4
B'	0.3	0.3	0.6
	0.4	0.6	1

(b) Use the completed Karnaugh probability map to calculate $P(A \mid B)$ and $P(B \mid A)$.

1 Determine the individual probabilities $P(A)$, $P(B)$ and $P(A \cap B)$ in decimal form.

$P(A) = 0.4$

$P(B) = 0.4$

$P(A \cap B) = 0.1$

2 Recall the conditional probability formula $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, substitute known values and calculate.

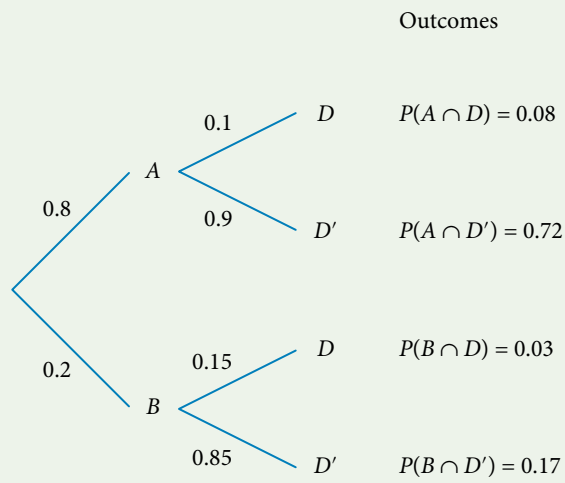
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.4} = \frac{1}{4}$$

If a conditional probability question is complex, then it may be useful to use a tree diagram to establish the outcomes and the associated probabilities.

15 Use a tree diagram to calculate conditional probabilities

Footy Footwear manufactures boots in two separate factories, located at Alphaville and Betatown. The Alphaville factory produces 80% of the total output, but 10% of its output is defective. The Betatown factory produces 20% of the total output, but 15% of its output is defective. Determine the probability that a defective Footy Footwear boot was made at the Betatown factory.

THINKING	WORKING
1 Define the events.	A: produced at Alphaville B: produced at Betatown D: defective D': not defective
2 Substitute into the conditional probability formula $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$.	Express the question as conditional probability. Determine the probability that a Footy Footwear boot was made at the Betatown factory, given that it was defective. $P(B \mid D) = \dfrac{P(B \cap D)}{P(D)}$
3 Determine the denominator for the rule for conditional probability. In this case, $P(D)$.	 $P(D) = 0.08 + 0.03$ $= 0.11$
4 Determine the numerator for the rule for conditional probability.	$P(B \cap D) = 0.03$
5 Substitute into the conditional probability formula $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$.	$P(B \mid D) = \dfrac{P(B \cap D)}{P(D)}$ $= \dfrac{0.03}{0.11}$ $= \dfrac{3}{11}$
6 Interpret the answer.	If you have a defective boot, there is a $\dfrac{3}{11}$ chance that it was produced at Betaville.

The addition rule for probability can also be used in conditional probability calculations.

16 Use the addition rule to calculate conditional probabilities

Calculate $P(A \mid B)$ if $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cup B) = 0.9$.

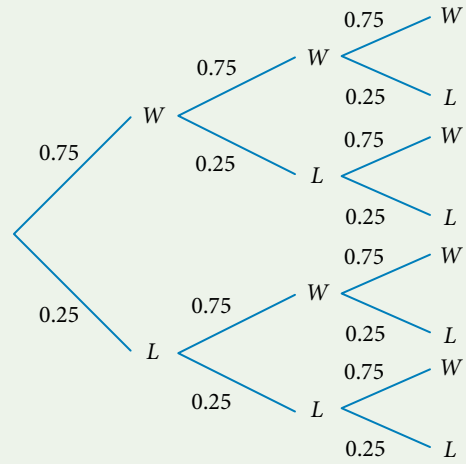
THINKING	WORKING
1 Recall the addition rule for probability $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ and substitute the given information to calculate $P(A \cap B)$.	$P(A \cap B) = P(A) + P(B) - P(A \cup B)$ $= 0.6 + 0.5 - 0.9$ $= 0.2$
2 Recall the rule for conditional probability, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ and substitute the known values to calculate $P(A \mid B)$.	$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ $= \dfrac{0.2}{0.5}$ $= \dfrac{2}{5}$
3 Express the answer as a decimal, as the question was written using decimals.	$P(A \mid B) = 0.4$

17 Conditional probability in a multiple event

Bruno has a 75% chance of winning any particular game of darts. Determine the probability that Bruno wins a set decided by the best of three games, given he wins the first game.

THINKING	WORKING
1 Define the events and assign probabilities.	W: win a game; $P(W) = 0.75$ L: lose a game; $P(L) = 0.25$ S: wins the set (3 games) F: wins the first game; $P(F) = 0.75$
2 Write the events that would represent winning the set with a win in the first game.	There are only two successful events, given that the first game is a win: WW and WLW.

3 Draw a tree diagram.



$P(WW) = 0.5625$

$P(WLW) = 0.140625$

4 Calculate the probability of each successful event and add together.

$$\begin{aligned} P(S \cap F) &= P(WW) + P(WLW) \\ &= 0.5625 + 0.140625 \\ &= 0.703\,125 \end{aligned}$$

Therefore the probability of winning the set with a win in the first game is 0.703 125.

5 Calculate the probability Bruno wins at least one game, given that he wins the first game.

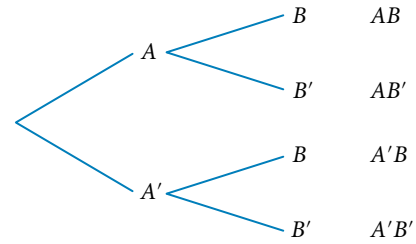
$$\begin{aligned} P(S \mid F) &= \frac{P(S \cap F)}{P(F)} \\ &= \frac{0.703\,125}{0.75} \\ &= 0.9375 \end{aligned}$$

6 Interpret the answer.

The probability that Bruno wins a set, given he wins the first game is 0.9375.

Law of total probability

When you are dealing with a two-event experiment, you can use the multiplication rule for probability and the rule for conditional probability to produce another rule, called the law of total probability.



$$P(AB) = P(A) \times P(B \mid A)$$

$$P(A'B) = P(A') \times P(B \mid A')$$

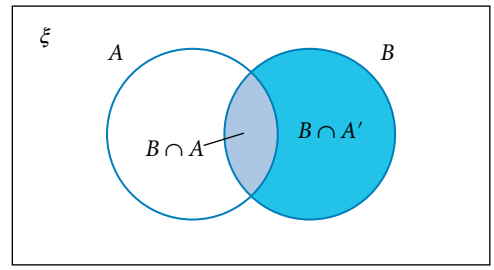
Now, either A or A' must occur, and these are the only ways that B can occur. Therefore:

$$P(B) = P(A) \times P(B \mid A) + P(A') \times P(B \mid A')$$

This can be rewritten as follows:

$$\begin{aligned} P(B) &= \frac{P(B \cap A)}{P(A)} \times P(A) + \frac{P(B \cap A')}{P(A')} \times P(A') \\ &= P(B \cap A) + P(B \cap A') \end{aligned}$$

A Venn diagram helps to understand this.



When the two shaded areas are combined (that is, their union is found), the shaded portion is the whole of B.

$$\begin{aligned} P(B) &= P(A) \times P(B \mid A) + P(A') \times P(B \mid A') \\ &= P(B \cap A) + P(B \cap A') \end{aligned}$$

18 Use the law of total probability

A research laboratory has 400 employees, of whom 220 are scientists. 15% of the scientists and 20% of the non-scientists are managers. Calculate the probability that a randomly selected employee of the company is a manager.

THINKING	WORKING
1 Define the events and assign probabilities.	Let A = scientist $P(A) = \frac{220}{400} = \frac{11}{20}$ Let B = manager $P(B \mid A) = \frac{15}{100}$ $P(B \mid A') = \frac{20}{100}$ A' = non-scientist $P(A') = \frac{9}{20}$
2 Express the probability required using the law of total probability, and determine the answer.	$P(B) = P(B \mid A) \times P(A) + P(B \mid A') \times P(A')$ $= \frac{15}{100} \times \frac{11}{20} + \frac{20}{100} \times \frac{9}{20}$ $= \frac{165 + 180}{2000}$ $= \frac{345}{2000}$ $= \frac{69}{400}$
3 Interpret the answer.	The probability that a randomly selected employee at the company is a manager is $\frac{69}{400}$.

EXERCISE 5.3 Conditional probability

Worked Example

1 Calculate $P(A \mid B)$ and $P(B \mid A)$ in fraction form in each of the following cases. 12

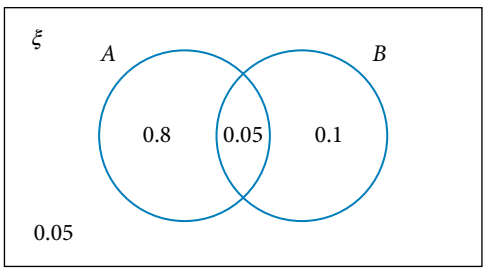
(a) $P(A) = 0.55$, $P(B) = 0.6$, $P(A \cap B) = 0.5$

(b) $P(A \cap B) = 0.2$, $P(A) = 0.3$, $P(B) = 0.4$

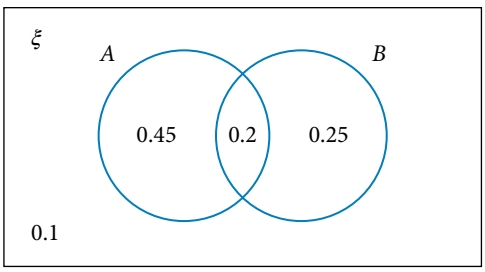
(c) $P(A) = 0.65$, $P(A \cap B) = 0.5$, $P(B) = 0.6$

2 From each Venn diagram, calculate $P(A \mid B)$ and $P(B \mid A)$ in simplest fraction form. 13

(a)

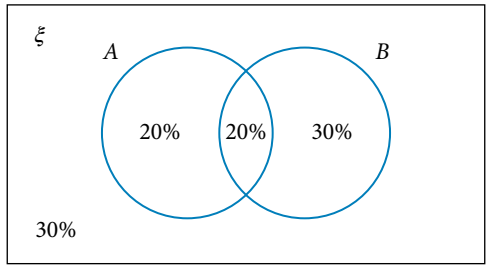


(b)

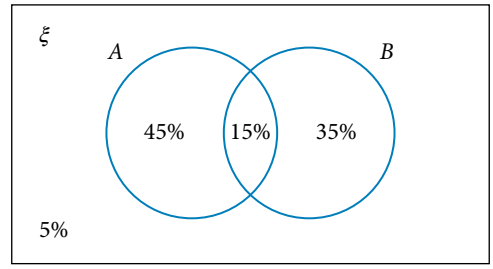


Worked Example

(c)



(d)



14

3 Complete the following Karnaugh probability maps, then use the completed maps to calculate $P(A \mid B)$ and $P(B \mid A)$ in simplest fraction form.

(a)

	A	A'	
B			0.45
B'	0.35		
		0.6	1

(b)

	A	A'	
B	0.2		
B'			0.7
	0.75		1

(c)

	A	A'	
B		0.2	0.35
B'			
	0.55		1

(d)

	A	A'	
B			0.35
B'	0.55		
		0.35	1

15

4 Use the conditional probability rule to answer the following questions.

(a) A coin is tossed three times. Exactly two of the tosses result in heads. What is the probability that the result of the first toss was tails?

(b) A pair of fair six-sided dice are rolled. Determine the probability that the result is two odd numbers, given that the first die shows a number less than 4.

(c) You are to choose a number from 1 to 100. What is the probability that the number selected is divisible by 6, given that you know it is divisible by 9?

16

5 Calculate $P(A \cap B)$ and $P(B \mid A)$ in decimal form, correct to 3 decimal places if necessary.

(a) $P(A) = 0.7$, $P(B) = 0.2$, $P(A \cup B) = 0.85$

(b) $P(A) = 0.85$, $P(B) = 0.65$, $P(A \cup B) = 0.9$

(c) $P(A) = 0.73$, $P(B) = 0.48$, $P(A \cup B) = 0.82$

6 A spinner is divided equally into four segments: two red, one blue and one green. The spinner is spun twice. Determine the probability that exactly one spin results in red, given that the first spin is not red?

A $\frac{1}{8}$ B $\frac{1}{2}$ C $\frac{1}{4}$ D $\frac{1}{16}$

7 The events A and B have probabilities $P(A) = \frac{6}{15}$, $P(B) = \frac{2}{3}$ and $P(A \mid B) = \frac{1}{5}$.

(a) Determine $P(A \cap B)$.

A $\frac{6}{75}$ B $\frac{3}{10}$ C $\frac{7}{15}$ D $\frac{2}{15}$

(b) Explain the common error made by a student who performed the following calculation:

$P(A \cap B) = \frac{1}{5} \times \frac{6}{15}$

8 A card is chosen at random from a standard pack of 52 playing cards. Calculate the probability that the card is:

(a) a heart, given it is a four

(b) a five, given it is a diamond

(c) a picture card, given it is a club

(d) a club, given it is black.

9 The probability that a person owns a dog, given that they also own a cat is $\frac{1}{3}$. If you know that $\frac{3}{4}$ of a particular group of people own a cat, what is the probability that a person chosen at random from the group will own both a dog and a cat?

Worked Example 17

10 In a normal match in the high school volleyball competition, a team must win two out of a possible three sets to win the match. However, in the final a team must win three out of a possible five sets to win the match. Upsen Downs High School's team has a 60% chance of winning any particular set. Determine the probability that the team wins:

(a) two sets in a row

(b) a normal match

(c) a normal match, given that they win the first set

(d) a normal match, given that they lose the first set

(e) the final match, given that they win the first set.

11 In a particular group of people, 20% have blue eyes, 35% are male, and 14% are blue-eyed males.

(a) Construct a two-way probability table to show this information.

(b) Determine the probability that a person chosen at random from this group is:

(i) blue-eyed and female (ii) male and not blue-eyed (iii) female and not blue-eyed.

(c) Given that there are 500 people in the group, how many would be:

(i) female and not blue-eyed? (ii) blue-eyed and male?

12 Use the 60 pairs of randomly generated single-digit numbers and their sums to answer the following questions. Give your answers correct to 3 decimal places, where necessary.

First	8	8	5	0	3	8	3	7	0	1	7	5	0	2	4	3	8	6	5	5
Second	2	1	1	5	4	7	2	3	9	6	9	7	6	9	1	5	4	8	9	4
Sum	10	9	6	5	7	15	5	10	9	7	16	12	6	11	5	8	12	14	14	9

First	5	3	0	6	8	2	5	0	5	1	4	8	5	6	4	7	3	9	1	1
Second	2	7	0	5	8	0	5	2	6	4	7	1	2	3	8	2	1	8	8	2
Sum	7	10	0	11	16	2	10	2	11	5	11	9	7	9	12	9	4	17	9	3

First	6	5	9	9	0	5	5	0	0	4	0	2	8	9	1	9	0	1	0	9
Second	6	2	4	9	5	2	0	9	2	5	3	4	7	2	6	9	2	9	5	1
Sum	12	7	13	18	5	7	5	9	2	9	3	6	15	11	7	18	2	10	5	10

(a) P(sum is less than 10, given that the first is greater than 5)

(b) P(sum is less than 10, given that the first is less than 5)

(c) P(sum is more than 10, given that the first is less than 7)

(d) Now consider *any* pair of randomly generated single-digit numbers. How many different pairs are possible?

(e) For any pair of randomly generated single-digit numbers, find the theoretical P(sum is less than 10, given that the first is greater than 5), giving your answer in decimal form.

(f) For any pair of randomly generated single-digit numbers, find the theoretical P(sum is less than 10, given that the first is less than 5), giving your answer in decimal form.

(g) For any pair of randomly generated single-digit numbers, find the theoretical P(sum is more than 10, given that the first is less than 7), giving your answer in decimal form.

Worked Example

(h) From your answers to the previous questions, does the conditional probability rule predict experimental results well? Explain.

18

13 Many polls are taken during an election campaign. In the country of Oceanalia, where there are only two political parties, a poll showed that 45% of voters supported the Internationalist party, with the rest supporting the Oceanalia First party. Of the Internationalist supporters, 35% were in favour of more funds for education, while 72% of the Oceanalia First party supporters were in favour of this. Calculate the probability that a randomly selected voter in the poll supports more funds for education. Express your answer in decimal form.

14 A ball is chosen at random from a bag of balls. The chosen ball is returned to the bag and an extra ball with the same colour is added to the bag. A ball is then chosen at random from the bag. Calculate the probability that the second ball chosen is red if:

(a) the bag contains five red balls and five green balls

(b) the bag contains five red balls and 10 green balls.

15 Two bags are placed on a counter. The first contains three red balls, four blue and two yellow. The second contains two red balls, one blue and three yellow. At each turn, a single ball is drawn from each bag simultaneously and placed on the counter. The process stops when a yellow ball appears for the first time. Determine the probability that the process stops:

(a) after one turn, with exactly one blue ball on the counter

(b) after two turns, with exactly two blue balls on the counter

(c) after three turns, with exactly three blue balls on the counter

(d) after four turns, with exactly four blue balls on the counter

(e) when the number of turns is the same as the number of blue balls on the counter.

Independent events

5.4

Defining independence

For some experiments $P(A \mid B) = P(A)$. This means that knowing something about event B has no impact on the probability of event A. In these cases, events A and B are said to be independent. With independent events, you would also find that $P(B \mid A) = P(B)$.

If you substitute $P(A)$ for $P(A \mid B)$ in the multiplication rule $P(A \cap B) = P(A \mid B) \times P(B)$ you will obtain $P(A \cap B) = P(A) \times P(B)$. This is the formal definition for independence.

Two events A and B are independent if $P(A \cap B) = P(A) \times P(B)$.

Additional information

Independent events

Investigate the independence of events.

19 Identify independent events

A fair coin is tossed three times. The following events are defined:

A = first toss is heads

B = second toss is tails

C = exactly two tails are tossed in a row

Which pairs of these events are independent, if any?

THINKING	WORKING
1 Determine the sample space.	$\xi = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
2 List the elements in each event.	$A = \{HHH, HHT, HTH, HTT\}$ $B = \{HTH, HTT, TTH, TTT\}$ $C = \{HTT, TTH\}$
3 Calculate the probability for each event.	$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$ $P(C) = \frac{1}{4}$
4 Determine the intersections for each possible pair.	$A \cap B = \{HTH, HTT\}$ $A \cap C = \{HTT\}$ $B \cap C = \{HTT, TTH\}$

5 Calculate the probabilities for each of the intersections.	$P(A \cap B) = \frac{1}{4}$ $P(A \cap C) = \frac{1}{8}$ $P(B \cap C) = \frac{1}{4}$
6 Check for independence: $P(A \cap B) = P(A) \times P(B)$	$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$ so A and B are independent $P(A) \times P(C) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} = P(A \cap B)$ so A and C are independent $P(B) \times P(C) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \neq P(A \cap B)$ so B and C are not independent
7 Check the reasonableness of the answer.	Events B and C are not independent because in order to achieve two tails in a row (C), the second toss must be tails (B).

Probabilities of intersections and unions

The independence of events can be used to assist you in answering a number of questions.

20 Use independent events with the addition rule

Dario and Ali are playing darts. The probability that Dario scores a triple-20 is $\frac{1}{5}$, and for Ali it is $\frac{1}{3}$. Determine the probability that at least one triple-20 will be scored if Dario and Ali have one turn each.

THINKING	WORKING
1 Define the events.	D: Dario scores a triple-20 A: Ali scores a triple-20
2 Convert the question into set notation.	Allowing for either (or both) to hit the target. This is represented by $D \cup A$.
3 Assuming the events are independent, use the definition of independence to find $P(D \cap A)$.	If Dario scores a triple-20, Ali's chance of throwing a triple-20 does not change, so the events are independent. $P(D \cap A) = P(D) \times P(A)$ $= \frac{1}{5} \times \frac{1}{3}$ $= \frac{1}{15}$

4 Use the addition rule to find the required probability.	$P(D \cup A) = P(D) + P(A) - P(D \cap A)$ $= \frac{1}{5} + \frac{1}{3} - \frac{1}{15}$ $= \frac{7}{15}$
5 Interpret the answer.	The probability that at least one triple-20 will be scored is $\frac{7}{15}$.

21 Use independent events with complementary events

If you live in the town of Bad Haven, the probability of your car being stolen on a public street in any year is 0.45 and the probability of your house being burgled is 0.6. If the events are independent, determine the probability that in a year the following will happen.

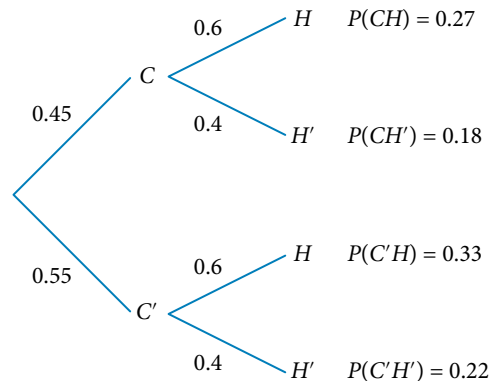
(a) Your car is stolen but your house is not burgled.

THINKING	WORKING
1 Define the events.	C: car is stolen H: house is burgled
2 Determine the probabilities of each described event and its complement.	$P(C) = 0.45$, $P(C') = 0.55$ $P(H) = 0.6$, $P(H') = 0.4$
3 Convert the question into set notation.	$P(\text{only car stolen}) = P(C \cap H')$
4 Recall the formula for independent events $P(A \cap B) = P(A) \times P(B)$, substitute known values and calculate the unknown.	$P(C \cap H') = P(C) \times P(H')$ $= 0.45 \times 0.4$ $= 0.18$
5 Interpret the answer.	The probability that your car is stolen but your house is not burgled is 0.18.

(b) Your house is burgled but your car is not stolen.

1 Recall the formula for independent events $P(A \cap B) = P(A) \times P(B)$, substitute known values and calculate the unknown.	$P(H \text{ only}) = P(H \cap C')$ $= P(H) \times P(C')$ $= 0.6 \times 0.55$ $= 0.33$
2 Interpret the answer.	The probability that your house is burgled but your car is not stolen is 0.33.

The previous worked example could also have been solved using a tree diagram. In the following diagram, C represents 'car stolen' and H represents 'house burgled'.

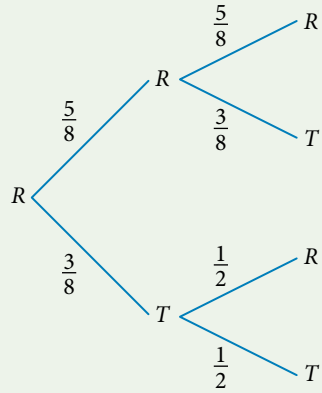


You can check that the same answers can be found using the tree diagram. For example, $P(CH)$ is the equivalent of $P(C \cap H)$, which is the probability of having your car stolen and your house burgled.

22 Use independent events for a sequence of events

An unbiased eight-sided die is rolled. If it shows a 1, 2 or 3, then a fair coin is tossed; otherwise the die is rolled again. If the coin shows heads, it is tossed again; otherwise the die is rolled.

What is the probability that the third event in this sequence is a roll of the die?

THINKING	WORKING
1 Define the events.	R: roll the die T: toss the coin
2 Draw a tree diagram to represent the sequence.	
3 Write the events that would represent success.	The sequences with a roll of the die as the third event are: RRR and RTR
4 Calculate the probability of each successful event.	$P(RRR) = 1 \times \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$ $P(RTR) = 1 \times \frac{3}{8} \times \frac{1}{2} = \frac{3}{16}$

5 Add the probabilities together.	$P(RRR)+P(RTR)$ $=\frac{25}{64}+\frac{3}{16}$ $=\frac{37}{64}$
6 Interpret the answer.	The probability that the third event in this sequence is a roll of the die is $\frac{37}{64}$.

EXERCISE 5.4 Independent events

Worked Example

1 A die in the shape of a tetrahedron has its four faces labelled 1, 2, 3, 4. The result of a roll of this die is the number facing down. The die is rolled twice, and the following events are defined. 19

A = an even number on the first roll
B = at least one even number on the two rolls
C = 3 on the second roll
Determine whether the following pairs of events are dependent or independent.

(a) A and B (b) A and C (c) B and C

2 A set of playing cards contains only the picture cards (King, Queen, Jack). A card is drawn at random from the set and then replaced before the next card is drawn. Three cards are drawn from the set using this process. The following events are defined:
H: the first card drawn is a heart
R: all cards drawn are red
T: exactly two hearts are drawn
Determine whether the following pairs of events are dependent or independent.

(a) H and R (b) H and T (c) R and T

3 Sean and Sura are playing pool. The probability that Sean pots his first shot is $\frac{1}{6}$. The probability that Sura pots her first shot is $\frac{2}{5}$. Calculate the probability that at least one of Sean and Sura pots their first shot. 20

4 Given that D and K are independent events, determine the value of the unknown in each of the following, in decimal form.

(a) $P(D)=0.45$, $P(K)=0.35$, $P(D\cap K)=?$ (b) $P(D)=0.8$, $P(D\cap K)=0.5$, $P(K)=?$
(c) $P(D)=0.55$, $P(K)=0.4$, $P(D\cup K)=?$ (d) $P(D\cup K)=0.8$, $P(D)=0.6$, $P(K)=?$

5 A fair six-sided die is rolled. If a 3 or 4 is rolled, a fair coin is tossed next.

(a) What is the probability that the coin is tossed and the result is a head?

A $\frac{1}{6}$ B $\frac{1}{3}$ C $\frac{1}{2}$ D 1

(b) Explain the common error made by a student who thought the answer was $\frac{5}{6}$.

6 Two independent events Z and Y are such that $P(Z) = z$ and $P(Z \cap Y) = p$. Determine an expression for $P(Y)$.

A pz B $\frac{p}{z}$ C $\frac{z}{p}$ D $z - p$

7 A fair six-sided die is rolled twice. Determine the probability for each of the following.

(a) two even numbers are rolled

(b) the first number rolled is larger than the second

(c) both numbers rolled are divisible by 3

(d) the numbers rolled are the same

8 Box A contains 6 yellow blocks and 4 green blocks. Box B contains 5 yellow blocks and 7 green blocks. A block is taken at random from each box. Determine the probability of each of the following.

(a) both blocks are yellow (b) at least one block is green (c) the blocks are different colours

9 A fair coin is tossed and then a six-sided die is rolled. Calculate the probability for each of the following.

(a) a head and a 4

(b) a tail and an odd number

(c) a head or an even number but not both

10 Bag A contains 5 lemons and 5 limes. Bag B contains 4 lemons and 8 limes. What is your chance of choosing a lemon if you select a bag at random and choose one piece of fruit from it?

Worked Example

21 11 The probability that Ramesh will play hockey for five more years is $\frac{1}{3}$, but for Cassie the probability is $\frac{2}{3}$. Assuming the events are independent, calculate the probability of the following happening five years from now.

(a) only Ramesh is playing hockey

(b) only Cassie is playing hockey

(c) both are playing hockey

(d) neither is playing hockey

12 Three clay target shooters, Ahn, Bella and Chris, have probabilities $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$ respectively of hitting a target. Calculate the probability for each of the following.

(a) Ahn and Bella both hit and Chris misses (b) all three hit

(c) all three miss (d) exactly two of them hit a target

22 13 A coin is tossed. If heads appears, the coin is tossed again. If tails appears, a fair six-sided die is rolled. If a 6 is rolled, the die is rolled again, otherwise the coin is tossed. What is the probability that the coin is tossed as the third event of this sequence?

14 A fair six-sided die is rolled. If a factor of 6 is rolled, a coin is tossed, otherwise the die is rolled again. If heads appears on the coin, the coin is tossed again, otherwise the die is rolled. What is the probability that the coin is tossed as the fourth event of this sequence?

15 Carmen and Ngarla have finished equal first in a golf tournament. In the play-off, the winner will be the first person to win two holes in a row.

(a) If Carmen and Ngarla each have a 40% chance of winning a hole, determine the probability that Carmen wins two holes in a row in the first three holes. Give your answer in decimal form.

(b) If Carmen has a 50% chance of winning a hole and Ngarla has only a 45% chance of winning a hole, determine the probability that Ngarla wins two holes in a row in the first three holes.

(c) If Carmen and Ngarla both have a 45% chance of winning a hole, determine the probability that neither will have won after three extra holes.

16 At an archery tournament, 8 of the 20 competitors are left-handed and the remainder are right-handed.

(a) Two archers are chosen at random. Determine the probability that exactly one is left-handed.

A strong west wind is making conditions difficult for left-handers. The probability of a left-hander hitting the central bullseye on any particular shot is 0.7 and for a right-hander it is 0.9.

(b) Determine the probability that an archer chosen at random is left-handed and hits the bullseye.

(c) Calculate the probability that an archer chosen at random hits the bullseye.

(d) Calculate the probability that an archer who hits the bullseye is right-handed.

(e) Determine the probability that two archers chosen at random are both left-handed or both right-handed, and both hit the bullseye. Give your answer correct to 2 decimal places.

17 A fair coin is tossed four times. The following events are defined:

A = exactly two tosses are heads

B = at least two tosses are tails

C = second toss is a head

Select the option that best describes the following pairs of events (independent or not independent).

(a) A and B **(b)** A and C **(c)** B and C

18 A fair six-sided die numbered 1 to 6 is rolled 3 times in each of 30 sets of rolls.

1, 6, 4	4, 2, 3	2, 6, 6	1, 6, 1	2, 2, 3
4, 3, 4	2, 5, 5	5, 2, 5	1, 4, 3	1, 5, 1
3, 3, 4	5, 1, 4	1, 4, 5	3, 3, 4	2, 4, 6
4, 4, 5	1, 1, 6	1, 4, 2	5, 4, 4	5, 2, 5
3, 4, 1	3, 3, 3	2, 4 , 3	1, 2, 4	1, 4, 5
5, 3, 3	5, 3, 6	4, 5, 2	1, 2, 1	6, 4, 2

The following events are defined:

A: the first roll is even

B: the second roll is odd

C: exactly two odd numbers are rolled in a row

(a) Based on these results, which pairs of events appear to be independent?

(b) In theory, which pairs of events are independent?

5.5 Counting techniques

Displaying the options

The local takeaway has five different hamburgers and three different chicken meals. If you buy one of these for lunch, then there are eight choices. But if you buy one hamburger and one chicken meal, then there are more unique meal options.

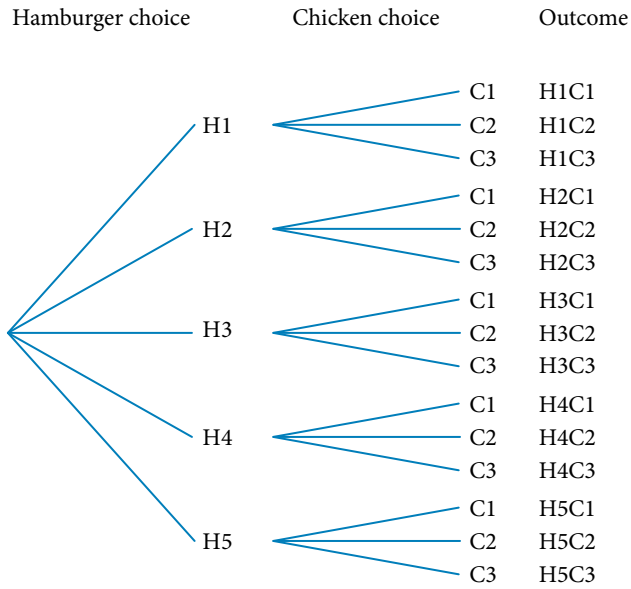
Let the hamburgers be H1, H2, H3, H4 and H5, and the chicken meals C1, C2 and C3.
There are a total of 15 different options.

H1C1 H2C1 H3C1 H4C1 H5C1

H1C2 H2C2 H3C2 H4C2 H5C2

H1C3 H2C3 H3C3 H4C3 H5C3

Listing the sample space in a grid is convenient when the number of outcomes is small and there are two events. For three or more events, you could use a tree diagram to identify the number of possible outcomes if the number of possible outcomes is not too large.



The multiplication principle

You can determine the number of possible outcomes using the *box method*. In this method a box is drawn to represent each option (in the example above these would be *hamburger* and *chicken meal*). This would give:

5	3
Number of hamburgers	Number of chicken meals

The numbers are then multiplied together to give the total number of possible outcomes. This principle, known as the *multiplication principle*, can be extended to any number of choices.

The multiplication principle says that if one event can occur in m ways, and another event can occur in n ways, then the two events together can occur in $(m \times n)$ different ways.

23 The box method

A rowing club plans to send a team of two scullers to a regatta. Four scullers are available for selection for the two different positions in the boat (front sculler and rear sculler).

In how many different ways can the positions be filled?

THINKING	WORKING
1 Identify how many arrangements can be made and use a box to represent this.	There are 2 positions in the arrangement. [ ] [ ]
2 Determine the number of options for the first position.	There are 4 possibilities for the front position, so put a 4 in the first box. [4] [ ]
3 Determine the number of options to fill the next position.	There are 3 possibilities left for the rear position, so put a 3 in the second box. [4] [3]
4 Calculate the total number of arrangements and interpret the answer.	$4 \times 3 = 12$ So there are 12 different arrangements.

You have seen that two events are mutually exclusive if membership of one event excludes membership of the other. For example, the set of odd numbers and the set of even numbers are mutually exclusive, because a number cannot be both odd and even.

The *addition principle* says that if two events are mutually exclusive, then the total number of ways those two events can occur is the sum of the individual number of ways each of the events can occur.

That is, if one event has m ways of occurring and a mutually exclusive event has n ways of occurring, then the two events can occur in a total of $m + n$ ways.

You can explore the multiplication and addition principles in an extension of the previous worked example.

24 The multiplication and addition principles

A rowing club plans to send a women's sculling pair and a men's sculling pair to a regatta. There are four female scullers available to fill the two positions (front sculler and rear sculler) in one boat, and three male scullers available to fill the two positions (front sculler and rear sculler) in the other boat.

(a) Determine the total number of (ordered) pairs, both women's and men's, that can be formed?

THINKING	WORKING
1 Determine the total number of arrangements for the female scullers.	First position can be filled in 4 ways. Second position can be filled in 3 ways. $\boxed{4}\boxed{3}$ $4 \times 3 = 12$ There are 12 different arrangements for the female pair.
2 Determine the total number of arrangements for the male scullers.	First position can be filled in 3 ways. Second position can be filled in 2 ways. $\boxed{3}\boxed{2}$ $3 \times 2 = 6$ There are 6 different arrangements for the male pair.
3 Calculate the number of ordered pairs that can be formed from either the men's or women's pairs.	The events are mutually exclusive, so apply the addition principle: $12 + 6 = 18$ different pairs.

(b) What is the total number of combinations of teams of women's (ordered) pairs and men's (ordered) pairs that can be formed?

1 Using the results from part **(a)**, write in the number of possible ways each team can be formed. Multiply the values together.	There are 12 ways to form the women's pair and 6 ways to form the men's pair to make a team: $\boxed{12}\boxed{6}$ $12 \times 6 = 72$
2 Interpret the answer.	72 different teams can be formed.

The following worked example shows how to extend the multiplication principle to cover more than two events.

25 The multiplication principle extended

A particular type of number plate consists of two letters followed by three digits. Determine how many different number plates can be made in each of the following circumstances.

(a) There are no restrictions.

THINKING	WORKING
1 Draw the number of boxes required.	There are five positions to be filled.
2 Fill in each box with the number of possible characters it can have.	There are 26 letters and 10 digits.
3 Apply the multiplication principle.	$26 \times 26 \times 10 \times 10 \times 10 = 676\,000$
4 Interpret the answer.	There are 676 000 different number plates with no restrictions.

26	26	10	10	10

(b) Both letters are different and all the digits are different.

1 Fill in each box with the number of possible characters it can have. When a letter or digit has been used, there is one fewer to choose from each time.	
2 Apply the multiplication principle.	$26 \times 25 \times 10 \times 9 \times 18 = 468\,000$
3 Interpret the answer.	There are 468 000 different number plates available with all different letters and digits.

26	25	10	9	8

Factorials

If another type of number plate consisted of 6 digits and used only the digits 1–6, and all digits had to be different, there would be $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ different plates possible.

Notice that this example involves multiplying together all the integers from a certain number down to 1. This type of calculation is called a *factorial* calculation, and it is used frequently in probability calculations. $6 \times 5 \times 4 \times 3 \times 2 \times 1$ is referred to as 6 factorial or factorial 6. The shorthand notation for this is 6!.

In the general case you write $n!$

$1! = 1$

$2! = 2 \times 1! = 2 \times 1 = 2$

$3! = 3 \times 2! = 3 \times 2 \times 1 = 6$

and so on, to obtain $n! = n \times (n-1)!$

n factorial:

$n! = n \times (n-1)!$

$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

If the factorial number is relatively small, such as 3 or 5, you can do the calculation by hand or mentally: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. For larger factorial expressions, such as 15!, technology can be used. All calculators from basic scientific to CAS technology have factorial capability.

Note that $0! = 1$ (you can check this result using technology). You will return to this idea later in this chapter.

In calculations involving division, factorials can be cancelled in a similar way to fractions, using common factors in the numerator and denominator.

26 Simplify factorials within fractions

Simplify $\frac{10!}{7!}$

THINKING	WORKING
1 Write the expressions in expanded form.	$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{7} \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
2 Cancel as appropriate and calculate the solution.	$= 10 \times 9 \times 8$ $= 720$

To perform the type of calculation in the previous worked example, it is sometimes useful to write factorials in slightly different ways. Consider the following examples:

$$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 10 \times 9 \times 8 \times 7!$$

$$\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}}$$
$$= 10 \times 9 \times 8$$

These alternative notations can be manipulated in various ways.

For example, you can show that $7 \times 6 \times 5 \times 4 \times 3 = \frac{7!}{2!}$

$$\frac{7!}{2!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$
$$= 7 \times 6 \times 5 \times 4 \times 3$$

Technology worked example
Factorials

27 Factorial manipulation

Express $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$ in fractional form using factorial notation and then determine the value of the expression.

THINKING	WORKING
1 Write the expression in the required format.	$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$ $= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!}$ $= \frac{10!}{3!}$
2 Use technology to determine the value of the expression.	$= 604\,800$

You can also simplify factorial expressions involving a variable.

28 Factorial expressions with variables

Simplify $\dfrac{n!}{(n-1)!}$

THINKING	WORKING
1 Write the expression, using a shorthand notation if possible.	$\dfrac{n!}{(n-1)!} = \dfrac{n \times (n-1)!}{(n-1)!}$
2 Simplify the result.	$= n$

Permutations

You can apply the multiplication principle to determine how many different ways the letters of a word such as HORSE can be arranged if every letter is used.

The answer is $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

These arrangements all contain the same letters; they are just written in a different order. For example, you can have arrangements: SOREH, SHORE and ORSEH. These different arrangements of the letters are called *permutations*. The word 'permutation' comes from the word 'permute' which means 'to change the order'.

However, you might not always want to use all the letters. Consider the following situation.

A president, secretary and treasurer must be chosen from the five members of a student committee. The box method shows the following working.

5	4	3

There are $5 \times 4 \times 3 = 60$ different ways that the three leadership positions can be filled. If the five students were called Ankita, Brian, Cheo, Dominique and Erkan, some of the ways the positions could be filled are shown below.

President	Secretary	Treasurer
Ankita	Brian	Erkan
Ankita	Cheo	Brian
Brian	Ankita	Cheo
Brian	Cheo	Ankita
Cheo	Ankita	Brian
Cheo	Brian	Ankita

In each of these six arrangements, the three students are the same but it is clear that each row of the table produces a different result. In this case, you can say there are 60 permutations of the 5 students taken 3 at a time.

How many arrangements would there be if you had permutations of the 5 students taken 2 at a time? The multiplication principle (or if you prefer, the box method) tells you there are $5 \times 4 = 20$ different arrangements.

Now consider the general case of n objects taken r at a time. If 3 positions must be filled from 10 available objects, you need $10 \times 9 \times 8$. If $n = 10$ and $r = 3$ then $n - r = 7$. The last factor in the product (which is 8) is 1 more than 7. That is, $n - r + 1$.

Order is important when dealing with permutations. For example, ABC is considered to be a different arrangement from BAC.

You can use the multiplication principle to calculate the number of permutations possible when n objects are taken r at a time.

The number of permutations (arrangements) is:

$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$

Additional information

Permutations

Explore permutation problems.

The ${}^nP_r$ formula

The notation ${}^nP_r$ is used to represent n objects arranged in r positions. You can say this as 'N P R'. So, ${}^6P_2$ means 6 objects taken 2 at a time and is read as '6 P 2'. The P stands for permutations.

$${}^nP_r = \frac{n!}{(n-r)!}$$

29 The ${}^nP_r$ definition

Use the definition of ${}^nP_r$ to determine the value of ${}^{11}P_7$, the number of ways 11 students could be picked for the 7 different positions on a netball team.

THINKING	WORKING
1 Write the ${}^nP_r$ rule.	${}^nP_r = \frac{n!}{(n-r)!}$
2 Substitute the values of n and r into the rule.	${}^{11}P_7 = \frac{11!}{(11-7)!}$
3 Simplify the expression.	${}^{11}P_7 = \frac{11!}{4!}$ Notice that there are 7 factors left after 4 are cancelled.
4 Determine the value of the expression.	${}^{11}P_7 = 1663200$
5 Interpret the answer.	There are 1663200 different ways that 11 people could fill the 7 different positions of a netball team.

Technology worked example

Calculating permutations

EXERCISE 5.5 Counting techniques

Worked Example

1 From a team of six archers, a captain and a vice-captain must be selected for the state championships. How many different ways can this be done? 23

2 An opera company has two tenor roles and two soprano roles for *The Magic Flute*. Eight tenors and seven sopranos are competing for the roles, and each singer can fill only one role. 24

(a) In how many ways can the tenor roles be filled?

(b) In how many ways can the soprano roles be filled?

(c) How many different (ordered) pairs of tenors and sopranos can be chosen in total?

(d) Determine the number of ways in which the four positions can be filled.

3 Calculate the value of each of the following expressions. Leave fractional answers in fraction form. 26

(a) $\frac{10!}{6!}$

(b) $\frac{8!}{6!}$

(c) $\frac{4!}{8!}$

(d) $\frac{3!}{5!}$

(e) $6! - 5!$

(f) $4! + 5!$

(g) $12! + 4!$

(h) $10! - 8!$

4 Calculate the value of each of the following expressions.

(a) $\frac{10!}{8!}$

(b) $\frac{100!}{98!}$

(c) $\frac{1000!}{998!}$

(d) $\frac{10\,000!}{9998!}$

5 Convert the following expressions to factorial form and then calculate. 27

(a) $12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$

(b) $11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$

(c) $15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$

(d) $16 \times 15 \times 14 \times 13 \times 12$

(e) $17 \times 16 \times 15 \times 14 \times 13 \times 12$

(f) $13 \times 12 \times 11 \times 10 \times 9$

6 Simplify each of the following factorial expressions. 28

(a) $\frac{(n-1)!}{n!}$

(b) $\frac{n!}{(n+2)!}$

(c) $\frac{(n+2)!}{(n-3)!}$

(d) $\frac{(n-2)!}{(n+3)!}$

29

7 Calculate the value of each of the following, using the ${}^{n}P_{r}$ formula.

(a) ${}^{10}P_{7}$ (b) ${}^{12}P_{6}$ (c) ${}^{15}P_{2}$ (d) ${}^{16}P_{7}$

8 The fixed-price menu at a cafe includes soup, a main meal and a dessert. The cafe offers three soups, five main meals and two desserts for their lunch menu. How many different three-course meals are possible?

9 Write the expression $\frac{n!}{(n-2)!}$ in polynomial form.

10 Solve for n using algebraic techniques in each of the following.

(a) ${}^{n}P_{2} = 132$

(b) ${}^{n}P_{8} = 1814400$

(c) ${}^{n}P_{7} = 32432400$

(d) ${}^{n}P_{3} = 9240$

11 Four awards (Best player, Most improved, Most consistent and Best effort) are to be presented to the players in a softball team. If there are fifteen players in the squad and no player can receive more than one award, in how many ways can the awards be distributed?

12 The finals in a hockey competition with 10 teams are to be contested by the top 4 teams.

(a) In how many different ways can these four positions be filled, given that the order of the four positions is important?

A $\frac{6!}{10!}$ B $\frac{10!}{6!}$ C $4!$ D $6!$

(b) Explain the common error made by a student who thought the answer was $\frac{10!}{4!}$.

13 Select the expression that is equal to ${}^{18}P_{7}$.

A $18! - 7!$ B $\frac{18!}{7!}$ C $11!$ D $\frac{18!}{11!}$

14 The 100 m dash at the athletic sports has eight competitors. Determine the number of ways the first, second and third place ribbons can be awarded.

15 Annie needs to pick two activities for the day: one for the morning and one for the afternoon. The activities available are bushwalking, canoeing, abseiling, rock climbing, aerobics, high ropes course and orienteering.

(a) In how many ways can she make her choice if she can repeat an activity?

(b) In how many ways can she make her choice if she is not able to repeat the same activity?

(c) What percentage of her total number of choices involves doing the same activity twice? Round your answer to the nearest percentage.

Worked Example 25

16 A particular type of number plate consists of two digits followed by three letters. How many different number plates could be made in each of the following circumstances?

(a) There are no restrictions on the letters or digits.

(b) The letters have to start with a T.

(c) No digit may be repeated.

(d) A letter cannot be repeated.

(e) The letters must start with K and no digit may be repeated.

(f) No digit or letter can be repeated in the number plate.

17 Determine the value of the following pairs of expressions. Use your results to draw conclusions about general expressions of the same kind.

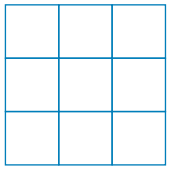
(a) $4! + 5!$ and $9!$

(b) $10! - 7!$ and $3!$

(c) $3! \times 2$ and $6!$

(d) $\frac{12!}{2!}$ and $6!$

18 Noughts and crosses is normally played on a 3×3 grid. The first player to make a line of three noughts or crosses in any direction (across, down or diagonally) wins. In the questions that follow, assume that the first move is a cross.



(a) In how many ways can the players make the first four moves?

(b) In an extended game of noughts and crosses, a 4×4 grid is used. In how many ways can the players make the first four moves?

(c) A 5×5 grid is now used. In how many ways can the players make the first four moves?

(d) An $n \times n$ grid is now used. In how many ways can the players make the first four moves?

19 2317 is an example of a number in which all the digits are different. Using these four digits, determine how many different numbers can be made from the following.

(a) four digits

(b) three digits

(c) any number of digits

20 A young couple intend to continue having children until their family includes both a boy and a girl. They will stop at six, no matter what. Make any assumptions you need to answer the following questions.

(a) Determine the number of different arrangements of gender that are possible for the family.

(b) What is the probability, as a percentage, that they will stop after two children?

(c) What is the probability, as a percentage, that the family will end up with fewer than four children?

21 Four letters are placed side-by-side so that no two adjacent letters are the same. Determine the number of possible arrangements.
Demonstrate two different solutions: determine a straightforward product, and then subtract the number of 'not allowed' arrangements, such as 'doubles' and 'triples', from the number of 'unrestricted' arrangements.

5.6 Combinations

When order does not matter

Imagine that you had four cards with the symbols ♥ ♦ ♠ ♣ and you select two cards at random in any order.

There are 12 possible permutations (arrangements): ♥♦ ♦♥ ♠♥ ♣♥ ♥♠ ♦♠ ♠♦ ♣♦ ♥♣ ♦♣ ♠♣ ♣♠	There are 6 different combinations (selections): ♥♦ ♦♠ ♠♣ ♥♠ ♦♣ ♥♣
There are 12 different *permutations* (arrangements) because the order **is** important: ♥♣ is a different arrangement from ♣♥.	There are 6 different *combinations* (selections) because the order **is not** important: ♥♦ is the same selection as ♦♥.

Note that in this case each combination can be arranged in $2! = 2$ ways.

There is a mathematical connection between the total number of possible combinations and the total number of possible permutations.

Consider choosing three letters from A, B, C, D and E. If the letters chosen are, for example, A, B and C, these can be arranged in $3! = 6$ ways:

3	2	1

$3 \times 2 \times 1 = 6$ unique arrangements of the 3 letters.

ABC, BAC, CAB, ACB, BCA, CBA

So any selection of three letters results in 3! permutations.

The number of combinations is the number of permutations divided by $r!$, where r is the number of objects chosen. This eliminates repeated rearrangements of the same objects (in this case, letters).

Making connections

Permutations and combinations

View the number of combinations and permutations formed.

The number of n objects chosen at r at a time $= \dfrac{{}^nP_r}{r!}$

The notation for combinations is similar to the notation for permutations.

${}^nC_r$ or $\dbinom{n}{r}$ is the number of combinations (selections) of n objects taken r at a time.

$$
\begin{aligned}
{}^nC_r &= \frac{{}^nP_r}{r!} \\
&= \frac{n!}{(n-r)!\,r!}
\end{aligned}
$$

30 The $\binom{n}{r}$ formula

(a) Determine the number of possible selections if five starting players in a basketball team can be selected from a team of eight players.

THINKING	WORKING
1 Express in the form ${}^nC_r = \binom{n}{r}$.	${}^8C_5 = \binom{8}{5}$
2 Substitute the values into the ${}^nC_r$ formula, simplify, and calculate the answer.	$\binom{8}{5} = \frac{8!}{(8-5)!5!}$ $= \frac{8!}{3!5!}$ $= \frac{8\times7\times6\times5!}{3\times2\times1\times5!}$ $= 56$
3 Interpret the answer.	There are 56 selections of a starting 5 from 8 players.

(b) Determine the number of possible selections of three students starting on the bench from a team of eight players.

1 Express in the form ${}^nC_r = \binom{n}{r}$.	${}^8C_3 = \binom{8}{3}$
2 Substitute the values into the ${}^nC_r$ formula, simplify, and calculate the answer.	$\binom{8}{3} = \frac{8!}{(8-3)!3!}$ $= \frac{8!}{5!3!}$ $= \frac{8\times7\times6\times5!}{3\times2\times1\times5!}$ $= 56$
3 Interpret the answer.	There are 56 selections of the 3 students starting on the bench from 8 players.

Technology worked example
Calculating combinations

You can then manipulate the various forms of the ${}^nC_r = \binom{n}{r}$ formula to calculate the value of the unknowns in certain expressions.

31 Connect ${}^nC_r$ and ${}^nP_r$

Determine the value of the unknown in ${}^8C_3 = \dfrac{{}^8P_3}{t!}$.

THINKING	WORKING
1 Write ${}^nC_r$ in terms of ${}^nP_r$.	${}^nC_r = \dfrac{{}^nP_r}{r!}$
2 Substitute the known values into the formula.	${}^8C_3 = \dfrac{{}^8P_3}{3!}$
3 Identify the solution.	$\therefore t! = 3!$ $t = 3$

Technology can be used to determine the number of combinations.

Verify using technology that: ${}^5C_2 = {}^5C_3 = 10$ ${}^8C_3 = {}^8C_5 = 56$ ${}^9C_6 = {}^9C_3 = 84$

This leads to a general result. ${}^nC_r = {}^nC_{n-r}$

If you are working without technology you can determine the value of n, given r and ${}^nC_r$ using either algebra or trial and error. When $r > 2$, algebra is too difficult for finding n, so it would probably be better to use trial and error. However, if you are given n and asked to determine r, you must use trial and error.

Both methods are shown in the following worked example.

32 Determine unknown values in ${}^nC_r$

Determine the unknown value of the following.

(a) Determine n if $\binom{n}{2} = 36$

THINKING	WORKING
1 Recall the factorial form $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$.	$\binom{n}{2} = 36$ $\dfrac{n!}{(n-2)!2!} = 36$
2 Expand, transpose and simplify.	$\dfrac{n(n-1)(n-2)!}{(n-2)!2!} = 36$ $n(n-1) = 36 \times 2!$ $n(n-1) = 72$ $n^2 - n - 72 = 0$

3 Factorise and solve.	$(n-9)(n+8)=0$ $n=9$ or $n=-8$ Only the positive solution makes sense in this situation as n represents the number of objects, so $n=9$.

(b) Determine r if $\binom{7}{r}=21$.

THINKING	WORKING
1 Guess $r=3$.	${}^{7}C_3=35$
2 Guess $r=2$.	${}^{7}C_2=21$, the required value, therefore $r=2$ is a solution.
3 Check for another solution. (When finding the values of r, there will usually be two solutions.)	Pairs of solutions exist, ${}^{n}C_r$ and ${}^{n}C_{n-r}$ so the other solution is: $n-r=7-2$ $=5$
4 Interpret the answer.	For $\binom{7}{r}=21$, $r=2$ or $r=5$

33 Combinations with restrictions

A committee of four is to be selected from a group consisting of six teachers and eight students. For each of the following cases, determine the number of ways in which the committee can be selected.

(a) There are no restrictions.

THINKING	WORKING
Write in ${}^{n}C_r$ form and determine its value.	In this case, $n=14$, $r=4$: ${}^{14}C_4=1001$ ways

(b) The committee is to consist of two teachers and two students.

Identify how to treat the restriction mathematically.	The events (teachers and students) are independent, so the teachers: ($n=6$, $r=2$) must be selected separately from the students: ($n=8$, $r=2$) and the results then multiplied. ${}^{6}C_2\times{}^{8}C_2=15\times28$ $=420$ ways

(c) One of the students, Agnetha, must be on the committee.

Identify how to treat the restriction mathematically.	One position on the committee is already filled, so there are only three places to fill and 13 people available for selection. $n = 13, r = 3$: ${}^{1}C_1 \times {}^{13}C_3 = 286$ ways

Pascal's triangle

Additional information

Combinations

Practise solving a problem involving restricted combinations.

You have already used Pascal's triangle to determine coefficients of $(x + y)^n$ for small values of n. Here, you can use the values in each row to read the number of combinations of r objects taken from n.

In Pascal's triangle the 1 at the top of the triangle is called Row 0, the 1 on the next line is called Row 1, and so on. You should also note that there are $n + 1$ elements in row n.

For example, in Row 5 there are 6 elements, giving the number of combinations of 0, 1, 2, 3, 4 or 5 objects taken from 5.

					1						Row 0
				1		1					Row 1
			1		2		1				Row 2
		1		3		3		1			Row 3
	1		4		6		4		1		Row 4
1		5		10		10		5		1	Row 5

You can rewrite Pascal's triangle using the notation of combinations:

$$
\begin{array}{c}
{}^{0}C_0 \\
{}^{1}C_0 \; {}^{1}C_1 \\
{}^{2}C_0 \; {}^{2}C_1 \; {}^{2}C_2 \\
{}^{3}C_0 \; {}^{3}C_1 \; {}^{3}C_2 \; {}^{3}C_3
\end{array}
$$

Row n of Pascal's triangle gives the number of ways n objects can be selected 0, 1, 2,... n at a time. Remember, the first row is row 0.

The symmetry of each row in Pascal's triangle also helps explain why ${}^{n}C_r = {}^{n}C_{n-r}$.
For example, ${}^{3}C_1$ and ${}^{3}C_2$ are the middle terms of Row 3 and are each equal to 3.
${}^{5}C_2$ and ${}^{5}C_3$ are the middle terms in Row 5 and are each equal to 10.
Notice that for these pairs of values the r-values add to n.
In the first example: $1 + 2 = 3$
In the second example: $2 + 3 = 5$

Earlier in this chapter it was stated that $0! = 1$. In Row 0 of Pascal's triangle you will see it is ${}^{0}C_0$, which by the definition of ${}^{n}C_r$ is equal to $\frac{0!}{(0-0)!0!}$. Now, if this is to be equal to 1 then 0! must be equal to 1.
(Note: ${}^{0}C_0$ represents the number of ways you can select no objects from nothing. There is only one way this is possible, which is to select none of the nothing available.)

34 Use Pascal's triangle to determine the number of possible selections

Use Pascal's triangle to determine the number of ways five objects can be selected in the following manner.

(a) two at a time

THINKING	WORKING
1 Identify the relevant row in Pascal's triangle.	There are five objects, so it is Row 5.
2 Write out the elements of that row.	1 5 10 10 5 1
3 Identify the relevant element in the row and write its value as the answer.	Two at a time, so it is the third element: 10 There are ten ways of selecting two objects from five.

(b) four at a time

1 Identify the relevant row in Pascal's triangle.	There are five objects, so it is Row 5.
2 Write out the elements of that row.	1 5 10 10 5 1
3 Identify the relevant element in the row and write its value as the answer.	Four at a time, so it is the fifth element: 5 There are five ways of selecting four objects from five.

Probability applications of combinations

You can use the basic definition of the probability of an event with counting techniques, including the calculation of the number of arrangements (permutations) and selections (combinations), to calculate the probability of more complex events.

35 The box method and probability

The five vowels A E I O U are arranged randomly in a row. Determine the probability of A being first.

THINKING	WORKING
1 Calculate the total number of possibilities.	$5! = 120$
2 Calculate the number of possibilities for the given restriction.	The first place is fixed, but the others can vary: $1 \times 4 \times 3 \times 2 \times 1 = 24$
3 Calculate the probability using the rule: $P(A) = \dfrac{n(A)}{n(\xi)}$	$P(A \text{ first}) = \dfrac{24}{120}$ $= \dfrac{1}{5}$

36 Restrictions and probability

Car number plates can be made from three letters followed by three digits, which can be repeated. If all possible plates of this type are equally likely, calculate the following probabilities.

(a) The three letters spell the word SUM.

THINKING	WORKING
1 Calculate the total number of outcomes.	Using the box method, there are $26 \times 26 \times 26 \times 10 \times 10 \times 10$ different number plates.
2 Calculate the number of favourable outcomes.	The three letters are specified, so there is only 1 choice for each letter. Using the box method, there are $1 \times 1 \times 1 \times 10 \times 10 \times 10$ ways this event can occur.
3 Substitute values into the basic rule of probability: $P(A) = \dfrac{n(A)}{n(\xi)}$	$P(\text{SUM}) = \dfrac{1 \times 1 \times 1 \times 10 \times 10 \times 10}{26 \times 26 \times 26 \times 10 \times 10 \times 10}$
4 Determine the value of the expression.	$P(\text{SUM}) = \dfrac{1}{26^3}$ $= \dfrac{1}{17\,576}$
5 Interpret the answer.	The probability of the number plate with the three letters SUM is $\dfrac{1}{17\,576}$.

(b) All three digits are even, counting 0 as even.

1 Calculate the number of favourable outcomes.	There are five even digits, including 0. Using the box method, there are $26 \times 26 \times 26 \times 5 \times 5 \times 5$ ways this event can occur.
2 Substitute values into the basic rule of probability: $P(A) = \dfrac{n(A)}{n(\xi)}$	$P(\text{evens}) = \dfrac{26 \times 26 \times 26 \times 5 \times 5 \times 5}{26 \times 26 \times 26 \times 10 \times 10 \times 10}$
3 Determine the value of the expression.	$P(\text{evens}) = \dfrac{1}{8}$
4 Interpret the answer.	The probability of having a number plate with three even digits is $\dfrac{1}{8}$.

(c) The number plate is OOO 000.

1 Calculate the number of favourable outcomes.	Each element of the number plate is specified. Using the box method, there are $1\times1\times1\times1\times1\times1$ ways this event can occur.
2 Substitute values into the basic rule of probability: $P(A)=\dfrac{n(A)}{n(\xi)}$	$P(\text{OOO 000})=\dfrac{1\times1\times1\times1\times1\times1}{26\times26\times26\times10\times10\times10}$
3 Determine the value of the expression.	$P(\text{OOO 000})=\dfrac{1}{260^3}$ $=\dfrac{1}{17\,576\,000}$
4 Interpret the answer.	The probability of a random number plate being OOO 000 is $\dfrac{1}{17\,576\,000}$.

EXERCISE 5.6 Combinations

Worked Example

1 Determine the number of ways the following can be done. 30

(a) selecting 6 objects from 8
(b) selecting 6 objects from 6
(c) selecting 6 objects from 12
(d) selecting 6 objects from 7

2 Determine the value of the unknowns. 31

(a) ${}^6C_2=\dfrac{{}^6P_2}{z!}$
(b) ${}^8C_4=\dfrac{{}^8P_t}{x!}$
(c) ${}^yC_5=\dfrac{{}^8P_5}{w!}$
(d) ${}^kC_v=\dfrac{{}^9P_q}{3!}$
(e) ${}^dC_e=\dfrac{{}^{11}P_6}{6!}$
(f) ${}^bC_7=\dfrac{{}^{10}P_7}{h!}$
(g) ${}^5P_2=n!\,{}^5C_f$
(h) ${}^6P_3=i!\,{}^6C_u$

3 (a) Calculate the value of each of the following pairs.

(i) ${}^6C_2$ and ${}^6C_4$
(ii) ${}^8C_3$ and ${}^8C_5$
(iii) ${}^{10}C_4$ and ${}^{10}C_6$

(b) Given the results above, make a statement about the values of ${}^nC_r$ and ${}^nC_{n-r}$.

4 Determine the value of the pronumerals in the following equations. 32

(a) ${}^6C_4={}^6C_x=y, x\neq4$
(b) ${}^8C_5={}^8C_k=m, k\neq5$
(c) ${}^zC_6={}^zC_3=x$
(d) ${}^sC_7={}^sC_a=8, a\neq7$
(e) ${}^jC_4={}^jC_t=126, t\neq4$
(f) ${}^dC_5={}^{11}C_6=w$

33

5 A committee of 4 is to be selected from 12 members. For each of the following cases, determine the number of different ways this can be done.

(a) There are no restrictions.

(b) Mary must be on the committee.

(c) Mary and Stefan must be on the committee.

(d) Either Mary or Stefan must be on the committee, but not both.

6 Calculate the value of each of the following expressions.

(a) ${}^{5}C_3$ (b) ${}^{6}C_2$ (c) ${}^{8}C_7$ (d) $\binom{8}{1}$

(e) $\binom{9}{2}$ (f) ${}^{9}C_7$ (g) ${}^{15}C_7$ (h) $\binom{15}{8}$

7 In how many ways can three letters be selected from each of the following sets of letters of the alphabet?

(a) A, B, C, D (b) A, B, C, D, E, F (c) all the vowels (d) all the consonants

8 Consider the word VIRTUAL.

(a) In how many ways can five letters be chosen?

A $2!$ B $\dfrac{{}^{7}P_5}{5!}$ C $\dfrac{7!}{5!}$ D $\dfrac{{}^{7}P_5}{2!}$

(b) Explain the common error made by a student who thought the answer was $\dfrac{7!}{2!}$.

9 Determine the value(s) of the pronumeral in each of the following.

(a) ${}^{n}C_2 = 105$ (b) ${}^{m}C_5 = 65\,780$ (c) ${}^{p}C_7 = 22\,481\,940$

(d) ${}^{11}C_r = 462$ (e) ${}^{25}C_s = 4\,457\,400$ (f) ${}^{41}C_t = 1\,121\,099\,408$

10 In how many ways can four letters be selected from each of the following words?

(a) PEACH (b) PARSLEY (c) PARCEL (d) LEXICOGRAPHY

11 In how many ways can five chocolates be selected from ten different milk chocolates and 12 different dark chocolates, if it includes exactly three dark chocolates? Write your answer as the product of two combinations.

12 A pennant team consists of eight players (four men and four women). There are seven men and six women available for selection.

(a) In how many ways can the men be selected?

(b) In how many ways can the women be selected?

(c) How many different teams can be chosen?

(d) How many different teams can be chosen if Phillip must be one of the male players chosen and Erica must be one of the female players chosen?

13 In how many different ways can a committee of six be chosen from eight men and nine women in the following situations?

(a) There are no restrictions.

(b) There must be four women and two men on the committee.

(c) There must be equal numbers of men and women on the committee.

(d) There must be no more than four men on the committee.

14 Two cards are drawn, without replacement, from a set of 8 cards marked with the numbers 1 to 8. Determine the probability, expressed as a fraction in simplest form, for the following events.

(a) Both cards show numbers less than 3.

(b) The first card is even.

(c) The sum of the numbers is greater than 9, given that the first card is marked with a 6.

(d) The sum of the numbers is less than 6.

15 On a particular examination there are seven questions, but only five have to be answered. How many ways can a student choose the five questions if the following conditions apply?

(a) There are no restrictions.

(b) Question 1 must be attempted.

(c) At least two of the first three questions must be attempted.

(d) No more than two of the first three questions may be attempted.

Worked Example 34

16 Use Pascal's triangle to calculate the number of ways the following selections can be made.

(a) four objects, two at a time

(b) six objects, three at a time

(c) seven objects, seven at a time

(d) nine objects, seven at a time

(e) eight objects, six at a time

(f) four objects, one at a time

(g) five objects, three at a time

(h) five objects, five at a time

(i) eight objects, one at a time

(j) five objects, zero at a time

(k) four objects, zero at a time

(l) seven objects, three at a time

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

17 You are told that selecting n objects r at a time can be done in the following number of ways. For each of the following cases, what are all the possible different pairs of values for n and r in a Pascal's triangle up to Row 10?

(a) 20 ways (b) 28 ways (c) 21 ways (d) 9 ways

(e) 126 ways (f) 10 ways (g) 5 ways (h) 6 ways

(i) 45 ways (j) 7 ways (k) 252 ways (l) 84 ways

18 What is the middle number in Row 6 of Pascal's triangle?

19 Determine the number of ways five objects can be chosen three at a time?

20 The first six positive integers, 1, 2, 3, 4, 5 and 6, are arranged at random in a row. Determine the probability, in simplest fraction form, that 3 is the first number. 35

21 You have four cards marked α, β, χ and δ. Calculate the probability, in simplest fraction form, that if two cards are drawn from the pack with replacement, they are as follows.

(a) α and β in any order (b) different cards

(c) the same card (d) χ and δ, in that order

22 Daryl runs a tutorial group for five students. If he must take at least one student to an enrichment session, how many different selections are possible?

23 You are in a class of 20 students. As part of a survey being conducted in a school, a researcher comes to your classroom, rolls a fair six-sided die and randomly chooses the number of students to be surveyed according to the number showing on the die.
In how many different ways, in total, can students be selected from your class, given that you do not yet know what the die will show?

Worked Example 36

24 Some number plates consist of two letters followed by four digits. If all possible plates of this type are equally likely to occur, determine the probabilities, expressed in simplest fraction form of the following.

(a) All four digits are different.
(b) The letters are different.
(c) All digits are even, counting 0 as even.
(d) The number plate is AL1856.

25 Vijay is a member of a cricket team that consists of 14 players. A captain (C) is to be selected at random, and then a vice-captain (VC) is to be selected at random. Calculate the probability, expressed in simplest fraction form, of each of the following.

(a) Vijay is selected as captain.
(b) After another player is selected as captain, Vijay is selected as vice-captain.
(c) Vijay is selected as either captain or vice-captain.

26 A hockey team consists of 11 players. As coach you are required to select the team from seven girls and eight boys. Calculate the probability, expressed as a fraction in simplest form, that the team consists of the following.

(a) six girls and five boys
(b) seven girls and four boys
(c) eight girls and three boys

27 Calculate the probability, expressed as a fraction in simplest form, that a three-digit number formed from the numbers 2, 4, 5, 7, 8 and 9, with no repetitions, is the following.

(a) greater than 700
(b) less than 500
(c) divisible by 5

28 A box contains six red balls, five white balls and two green balls. Three balls are drawn from the box without replacement. Determine the probability that one ball of each colour is drawn.

29 A school debating squad consists of five girls and four boys. If a team consists of three students and is selected randomly from the students in the squad, calculate the probability of each of the following in simplest fraction form.

(a) The first student selected is a girl and the other two are boys.
(b) Only one girl is selected.

30 At an athletics carnival there are 10 events: four running events, three jumping events and three throwing events. Bettina decides to enter three events, which she chooses at random. Calculate the probability that she randomly chooses:

(a) three jumping events

(b) a running event, a jumping event and a throwing event

(c) two running events and a throwing event

(d) three running events.

31 Three numbers are chosen, without replacement, from the set 1, 2, 3, 4, 5. Calculate the following probabilities.

(a) The number 3 is chosen.

(b) All numbers are odd.

(c) One even and two odd numbers are chosen.

(d) Two numbers are odd, including the first number chosen.

32 Three letters are chosen at random from the word CAMPING and arranged in a row. Calculate the probabilities of the following.

(a) A is chosen first.

(b) A is chosen.

(c) Both A and I are chosen.

(d) Neither A nor I is chosen.

33 A box of coloured table tennis balls contains five green, numbered 1 to 5, six blue, numbered 1 to 6 and four yellow, numbered 1 to 4. Five balls are taken out, and it can be seen that two are green, two are blue and one is yellow.
Determine the probabilities.

(a) All numbers are less than 4.

(b) The numbers are 1 to 5.

(c) All the numbers are different.

34 Freya Faroux's fun fair operates several number games, each costing $1 to play. A player selects a certain number of different letters from a particular word, and then a computer selects the same number of different letters at random from the word.
In 'Choose 2' you select two different letters from the word FREYA. If the selected letters match the computer's letters, you win $2 and get your $1 back.
In 'Choose 3' you select three different letters from the word FAROUX. If the selected letters match the computer's letters, you win $3 and get your $1 back.

(a) Which game results in the higher likely profit (or lower likely loss) for a player per game? Explain your answer.

(b) In order to attract more people to the fair, Freya introduces extra 'consolation' prizes. In Choose 2 you now receive $0.50 plus your $1 back if you correctly select only one of the two letters. In Choose 3 you now receive $1 plus your $1 back if you correctly select only two of the three letters. Which game now results in the higher likely profit (or lower likely loss) for a player per game? Explain your answer.

5 Summary

Probability

Probability is the study of chance.

The probability of an event is a measure of the likelihood that it will occur.

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

Probability of event A: $P(A) = \dfrac{n(A)}{n(E)}$

Expected number of event A: $E(A) = P(A) \times n$, where $0 \le P(A) \le 1$ and $\Sigma P(A) = 1$

The complement of A: $P(A') = 1 - P(A)$

Set notation and Venn diagrams

ξ, S or U: the universal set; the list of possibilities

A, B, C etc.: capital letters are used as set names

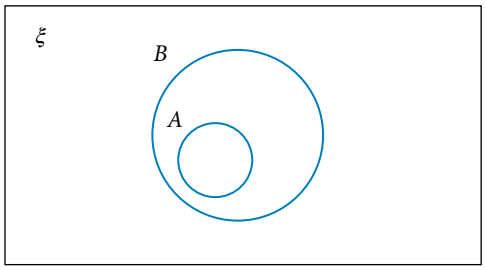
$n(A)$: the number of elements in the set A, also called the cardinal number of the set

$\varnothing$ or $\{\}$: the null set or empty set; a set that contains no elements

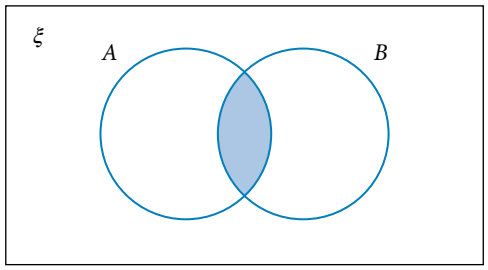
$\in$: is an element of

$\notin$: is not an element of

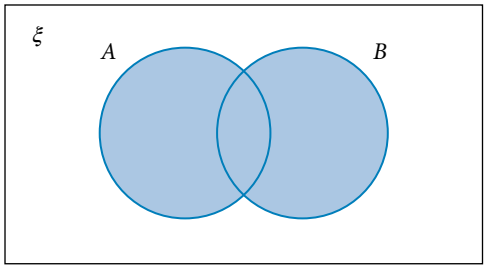
$A \subset B$: A is a subset of B



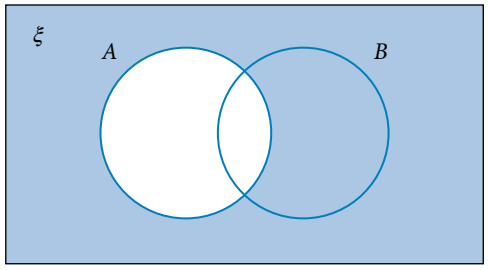
$A \cap B$: intersection of sets A and B



$A \cup B$: union of sets A and B



A' or $\overline{A}$: the complement of set A



Addition rule for probability

The addition rule for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional probability

Conditional probability occurs when you can reduce the sample space because of some extra information. The rule for this is: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$

This can be transposed to give the multiplication rule for probability: $P(A \cap B) = P(A \mid B) \times P(B)$

If $P(A \mid B) = P(A)$, that is, event B has no effect on event A, then the events A and B are independent.

The law of total probability

In a two-event experiment: $P(B) = P(B \mid A) \times P(A) + P(B \mid A') \times P(A')$

Independent events

Two events A and B are independent if $P(A \cap B) = P(A) \times P(B)$.

Counting techniques

Two events are mutually exclusive if the occurrence of one event means that the other other event cannot occur.

The addition principle shows that if one event has m ways of occurring and a mutually exclusive event has n ways of occurring, then the two events can occur in a total of $m + n$ ways.

The multiplication principle shows that if one event can occur in m ways and this is followed by another event that can occur in n ways, then the two-stage event can occur in $m \times n$ different ways.

$n!(n \text{ factorial}) = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ and is the number of ways n objects can be arranged in a row, where $0! = 1$.

Permutations

When you are considering permutations (arrangements), order is important. For example, ABC is different from CBA.

The number of permutations of n objects chosen r at a time can be represented by ${}^nP_r$.

$${}^nP_r = \frac{n!}{(n-r)!}$$

Combinations

When you are considering combinations (selections), order is not important. For example, ABC is the same as CBA.

${}^nC_r = \dbinom{n}{r}$ is the number of selections of n objects taken r at a time.

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

$${}^nC_r = {}^nC_{n-r}$$

Row n of Pascal's triangle gives us the number of ways n objects can be selected 0, 1, 2, ... , n at a time.

Chapter review 5

Exercise 5.1

1 A spinner divided into four unequal coloured segments was spun 20 times.

Colour	blue	orange	purple	pink
Number recorded	7	3	6	4

Based on these results, what is the probability, in simplest fraction form, that the next spin will produce the following results?

(a) orange **(b)** purple **(c)** not pink

Exercise 5.1

2 A fruit bowl contains 3 oranges, 4 bananas, 7 apricots and 2 apples. If a piece of fruit is selected from the bowl at random, determine the probability, in simplest fraction form, that it is the following.

(a) an apricot **(b)** an apple or a banana

(c) not an orange **(d)** neither an apple nor an apricot

Exercise 5.2

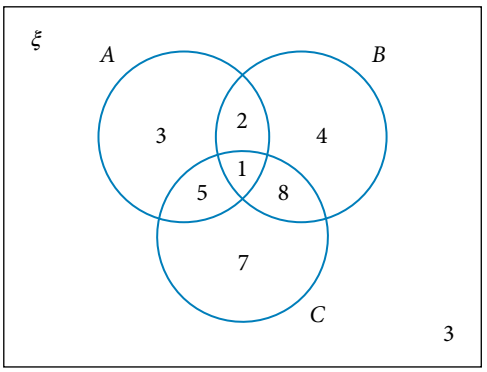
3 Given $\xi = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$, $A = \{3, 11, 31\}$ and $B = \{2, 3, 5, 7\}$, list the following sets.

(a) A' **(b)** $A \cap B$ **(c)** $A \cup B'$ **(d)** $(A \cup B)'$

(e) $A' \cup B'$ **(f)** $A' \cap B'$ **(g)** $(A \cap B)'$ **(h)** $A \cap B'$

Exercise 5.2

4 In the following Venn diagram, the numbers represent the number of elements in each part of the diagram. Use the diagram to determine the following.



(a) $n(A)$ **(b)** $n(A \cap B)$

(c) $n(A \cap B \cap C)$ **(d)** $n(B')$

(e) $n(A \cap C')$ **(f)** $n(A' \cap B')$

Exercise 5.2

5 A 20-sided spinner with the sides numbered 1 to 20 is spun. Determine the probability, in simplest fraction form, for each of the following outcomes.

(a) a multiple of 4

(b) a multiple of 5

(c) a multiple of 5 that is not a multiple of 4

(d) a number that is neither a multiple of 5 nor a multiple of 4

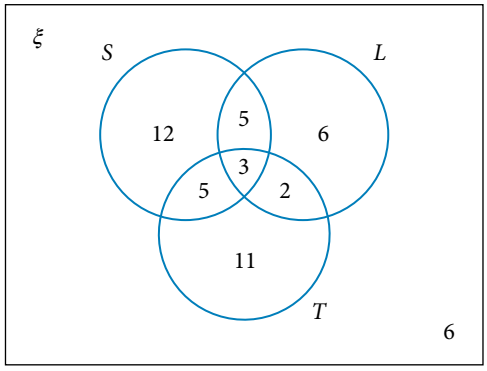
Exercise 5.2

6 Calculate the following probabilities in decimal form.

(a) $P(K \cup L)$, given $P(K) = 0.55$, $P(L) = 0.6$ and $P(K \cap L) = 0.2$

(b) $P(N \cap P)$, given $P(N) = 0.15$, $P(P) = 0.45$ and $P(N \cup P) = 0.55$

7 The following Venn diagram gives information about the number of surveyed students who are active participants in a sport (S), left-handed (L), or more than 165 cm tall (T). Exercise 5.2



Based on this information, determine the probability that a student chosen at random from those surveyed is:

(a) a left-handed active participant in sport

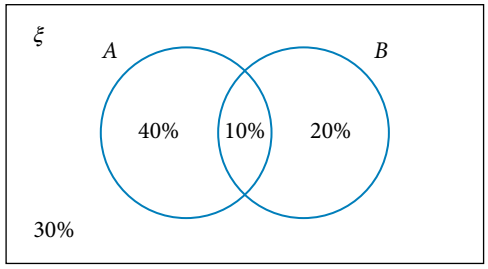
(b) an active participant in sport and taller than 165 cm

(c) not taller than 165 cm

(d) left-handed and taller than 165 cm but not an active participant in sport.

8 If there are two events G and H such that $P(G) = 0.25$, $P(H) = 0.75$ and $P(G \cap H) = 0.2$, determine the value of $P(G \cup H)'$. Exercise 5.2

9 Determine the following probabilities based on the Venn diagram below. Exercise 5.2



(a) $P(A \cap B')$ **(b)** $P(A \mid B)$ **(c)** $P(B \mid A')$

10 A family includes three children. If each child has an equal probability of being a boy or a girl, determine the following probabilities in simplest fraction form. Exercise 5.3

(a) All three children are the same gender.

(b) There are more boys than girls.

(c) The eldest child is a girl and the others are boys.

(d) The youngest child is a boy, given that the other children are girls.

11 Given $P(A) = 0.4$, $P(B \mid A) = 0.2$ and $P(B) = 0.32$, determine the value of $P(A \cup B)$. Exercise 5.3

12 One of the assessment tasks in senior language subjects is an oral exam. For the following classes, determine the probability that, of the first four randomly chosen students, no two consecutive students are the same gender. Give your answer in simplest fraction form. Exercise 5.4

(a) a class of five girls and four boys

(b) a class of four girls and four boys

13 Three students are working independently on a difficult science problem. Their respective probabilities of solving the problem are 0.6, 0.7 and 0.8. What is the probability that at least one of the students solves the problem? Exercise 5.4

Exercise 5.4 **14** A and B are events for which $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.7$. Which one of the following is false?

A $P(A \mid B) = \frac{2}{3}$ **B** $P(A \cap B) = 0.4$

C A and B are independent **D** $P(A' \cap B') = 0.3$

Exercise 5.5 **15** Determine the value of each expression, first writing each in terms of factorials.

(a) ${}^{15}P_6$ **(b)** ${}^{12}P_7$ **(c)** ${}^{11}P_4$ **(d)** ${}^{13}P_5$

Exercise 5.5 **16** Determine the value of each expression.

(a) $7!$ **(b)** $5! - 3!$ **(c)** $4! \times 6!$ **(d)** $9! \div 4!$

Exercise 5.5 **17** You have the letters A, B, C, D, E and F. If you use each letter exactly once, determine the number of ways these can be placed:

(a) in a row **(b)** in a row starting with A **(c)** in a row starting with a vowel.

Exercise 5.5 **18** Which one of the following expressions is equal to ${}^{n}P_3$?

A $\frac{n!}{(n-3)!3!}$ **B** $\frac{n!}{3!}$ **C** $\frac{n!}{(n-3)!}$ **D** ${}^{n}P_{n-3}$

Exercise 5.6 **19** Determine the value of each expression.

(a) ${}^{7}C_5$ **(b)** ${}^{6}C_2$ **(c)** $\binom{10}{4}$ **(d)** $\binom{8}{3}$

Exercise 5.6 **20** In how many ways can four letters be selected from the following sets?

(a) {A, B, C, D, E} **(b)** {A, B, C, D, E, F, G, H, I, J}

Exercise 5.6 **21** Determine the value(s) of the pronumeral in each case.

(a) ${}^{6}C_r = 15$ **(b)** ${}^{n}C_2 = 21$

Exercise 5.6 **22** Use Pascal's triangle to calculate the number of ways the following selections can be made.

(a) six objects three at a time

(b) seven objects five at a time

Exercise 5.6 **23** A sample of 4 items is to be selected at random, without replacement, from a batch of 15 items, 6 of which are known to be defective. Determine the probability, expressed as a fraction in simplest form, that the sample contains:

(a) no defective items **(b)** all defective items

(c) no more than one defective item **(d)** exactly two defective items.

Exercise 5.6 **24** What is the middle number of Row 8 of Pascal's triangle?

Exercise 5.6 **25** A group of three letters is made by drawing letters, without replacement, from the letters of the word LILLIPUT.

(a) What is the probability that the group is LLL?

(b) What is the probability that the group contains at least one vowel?

26 Four golf balls are selected at random from a box containing six Long-Flite balls and eight Ranger balls. Determine the probability that two balls of each brand are selected. Leave your answer in terms of combinations. Exercise 5.6

27 The Abracadabra company makes trick playing cards in two factories. One factory is in Aytown (A) and the other is in Beeville (B). The machinery at Aytown has a 10% chance of being out of order on any particular day, but at Beeville there is only a 5% chance. Determine the probability that: Exercise 5.3

(a) only one factory is operational (as a decimal)

(b) if only one factory is operational, then it is A (as a fraction)

(c) both factories are in the same state, either both operational or both out of order (as a decimal).

28 Many polls are taken during an election campaign. On the island of St Atistician, one particular poll showed that 52% of voters favoured the Union party, with the rest favouring the Intersect party. Of the Union supporters, 40% were in favour of increased spending on health, while 85% of the Intersect party supporters were in favour of this. Determine the probability, in decimal form, that a randomly selected voter in this poll supports increased spending on health. Exercise 5.3

29 There are two events J and K such that $P(J) = 0.7$, $P(K) = 0.4$ and $P(J \cap K) = 0.3$. From the following options, select the statement that is true. Exercise 5.3

A $K \subseteq J$

B $P(J \cup K) < P(J \cap K)$

C J and K are independent

D $P(J \mid K) = \frac{3}{4}$

30 A box contains six green balls and four white balls. A ball is chosen at random from the box. The chosen ball is returned to the box and two other balls of the same colour are added. A second ball is then chosen at random from the box. What is the probability that the second ball chosen is white? Exercise 5.3

A $\frac{2}{5}$ **B** $\frac{6}{25}$ **C** $\frac{3}{5}$ **D** $\frac{1}{5}$

31 A hospital commissioned an independent study to determine if there was a difference between the care provided by hospital-trained nurses and care provided by tertiary-trained nurses. Two hundred nurses were rated as providing superior care (S) or adequate care (A). The results were as follows. Exercise 5.3

	Superior care (S)	Adequate care (A)
Hospital trained (H)	48	12
Tertiary trained (T)	76	64

(a) Represent the information provided in the table on a tree diagram, giving all probabilities in simplest fraction form.

(b) Determine the probability, in simplest fraction form, that a randomly chosen nurse:

(i) is hospital trained

(ii) provides superior care

(iii) is hospital trained and also provides superior care

(iv) is tertiary trained and also provides superior care

(v) is hospital trained and also provides adequate care

(vi) is tertiary trained and is rated as providing adequate service.

(c) Are the events H and S independent? Justify your answer mathematically.

(d) If a patient receives superior care from a nurse, determine the probability that the nurse:

(i) is hospital trained

(ii) is tertiary trained.

Exercise 5.4

32 A biased six-sided die with probabilities shown below is rolled. If the number uppermost is even, then a biased coin $(P(\mathrm{H}) = 3 \times P(\mathrm{T})$ is tossed.

$P(2) = P(4) = P(6) = \frac{1}{12}$, $P(3) = \frac{1}{6}$, $P(5) = \frac{1}{4}$ and $P(1) = \frac{1}{3}$

Determine the probability, in simplest fraction form:

(a) that a number rolled on the die is less than 5

(b) that the coin is tossed

(c) an event occurs that includes the coin landing tails

(d) that the coin lands heads, given that a 2 was rolled on the die.

Exercise 5.4

33 A box contains five white blocks and four blue blocks. Whenever a block is drawn from the box, a block of the other colour is used to replace it. Express each of the following answers in simplest fraction form. Determine the probability that:

(a) the second block drawn is white

(b) the first two blocks drawn are the same colour

(c) the second block drawn is blue, if the first was white

(d) the first block drawn was white, given that the second was blue.

Exercise 5.5

34 A restaurant offers three starters, five entrées, six main courses and three desserts. How many different meals are possible if a meal consists of a least two courses and can include no more than one item from each group?

Exercise 5.6

35 The leadership team for the student council consists of a president, a treasurer, a secretary and three other members. The team is to be selected at random from five Year 11 students and ten Year 12 students. Determine the probability that the leadership team consists of:

(a) three Year 11 and three Year 12 students

(b) more Year 12 than Year 11 students

(c) all Year 11 students

(d) three Year 12 and three Year 11, given that a Year 12 student is selected for the first position

(e) more Year 12 than Year 11, given that a Year 12 student is selected for the first position

(f) three Year 12 and three Year 11, given that two Year 12 students are selected for the first two positions

(g) more Year 12 than Year 11, given that two Year 12 students are selected for the first two positions.

36 A survey at a petrol station revealed that 30% of the customers use diesel (D), 65% use standard unleaded (S) and the remainder use premium unleaded (P). 60% of the customers who use diesel fill their tank, 30% of the standard unleaded customers fill their tank and 50% of the premium unleaded customers fill their tank. Exercises 5.3, 5.4

(a) Determine the probability that the next customer will fill their tank with diesel.

(b) Determine the probability that the next customer does not fill their tank completely.

(c) If the next customer fills their tank, determine the probability that they use diesel.

(d) Determine the probability that the next customer fills their tank or uses standard unleaded, but not both.

(e) If you count groups of n customers at a time, determine the smallest value of n that will make it more than 90% certain that at least one of the customers fills the tank completely.

37 Class A consists of 10 boys and 12 girls. Class B consists of 14 boys and 10 girls. Exercise 5.3

(a) What is the probability that a student randomly selected from Class A is a girl?

(b) What is the probability that a student randomly selected from Class B is a girl?

(c) Now assume you first pick one of the classes at random and then choose the student. Demonstrate that the probability that this student is a girl is $\frac{127}{264}$.

(d) Would you get the same answer if you treated the two classes as one big class?

(e) Explain why this is the case.

Class C contains 10 boys and 10 girls. Class D contains 12 boys and 12 girls.

(f) What is the probability that a student randomly selected from Class C is a girl?

(g) What is the probability that a student randomly selected from Class D is a girl?

(h) Now assume again that you randomly pick the class (C or D) before you select the student. Show that the probability that this student is a girl is $\frac{1}{2}$.

(i) Would the same answer in part **(h)** be obtained if you treated the two classes as one big class?

(j) Explain why this is the case.

38 Russell R cattle station has damaged fences and cattle from other stations have wandered into its yards and it has lost cattle. The manager is taking stock into town to sell and of the 30 cattle in the truck, four belong to other stations. At the stockyards, one-third of the cattle in any lot is inspected. To avoid a fine, the Russell R manager is trying to work out a strategy that has the lowest probability of the cattle from the other stations being detected. Exercise 5.6

(a) For each strategy below, calculate the probability that at least one of the other cattle is detected in the either of the lots.

(i) Strategy A is to sell all the cattle in one lot.

(ii) Strategy B is to sell the cattle in two lots of 15, with two of the other cattle in each lot.

(iii) Strategy C is to sell the cattle in two lots of 15, with three of the other cattle in one lot and one in the other lot.

(iv) Strategy D is to sell the cattle in two lots of 15, with four of the other cattle in one lot and none in the other lot.

(b) Determine which strategy (A, B, C or D) results in the lowest probability of being fined.

6

Index laws

Recall 360

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6 Recall

Create numbers as the product of prime factors

1 Write the following numbers as products of their prime factors.

(a) 12 (b) 36 (c) 162

Round numbers to a given number of decimal places

2 Round the following numbers to 3 decimal places.

(a) 7.1251 (b) 23.3109 (c) 7.699999

Use index notation

3 Express the following using index notation.

(a) $a \times a \times a \times a \times a$ (b) $m \times m \times n \times m \times n$ (c) $\dfrac{5 \times n \times a \times n}{3 \times b \times b \times b}$

Use order of operations

4 Determine the value of each expression without a calculator, using the correct order of operations.

(a) $30 - \left(5 \times 2^3 - 15\right)$ (b) $40\left(1 + \dfrac{1}{2}\right)^2$ (c) $\dfrac{40}{2+3} \times \dfrac{4^2+4}{8 \div 2}$

Substitute numbers into a formula

5 Substitute the indicated variables into each formula to determine the value of the unknown.

(a) $M = A(1+r)^t$, $A = 50$, $r = 0.05$ and $t = 2$

(b) $f(x) = ax^3 - bx^2 + cx + d$, $a = 2$, $b = -3$, $d = 5$, $x = -2$

Use the index laws

6 Simplify the following using the index laws.

(a) $2y \times 3y^3$ (b) $\dfrac{5x^4}{20x}$ (c) $\left(2c^2d\right)^3$ (d) $\left(\dfrac{2m^3}{3h}\right)^4$

Factorise quadratic expressions

7 Factorise the following.

(a) $3x^2 + 9x$ (b) $x^2 - 9$ (c) $x^2 + 7x - 18$

Use technology to solve an equation for a given variable

8 Solve the following equations using technology. Give your answer correct to 2 decimal places, where necessary.

(a) $80 = 2x^2 - 18$ (b) $1000 = 50 \times 0.05 \times m$ (c) $10000 = 500(1+x)^5$

Review of index laws

6.1

Powers

The expression a^m is used in mathematics to refer to the multiplication of m factors of a.

a is called the base and m is called the power, exponent or index.

Index laws	Example	Laws
Multiplication index law When multiplying expressions containing indices, if the bases are the same, keep the base and add the powers.	$2^3 \times 2^2 = (2\times2\times2)\times(2\times2)$ $= 2^5$	$a^m \times a^n = a^{(m+n)}$
Division index law When dividing expressions containing indices, if the bases are the same, keep the base and subtract the powers.	$\frac{2^7}{2^3} = \frac{2\times2\times2\times2\times2\times2\times2}{2\times2\times2}$ $= 2^4$	$\frac{a^m}{a^n} = a^{(m-n)}$
There are two index laws that result from the way division of indices occurs.		
The zero index	$\frac{a^m}{a^m} = 1$ $a^{(m-m)} = 1$ $a^0 = 1$ for $a \neq 0$	$a^0 = 1$ (for $a \neq 0$)
The negative index	$\frac{1}{a^m} = \frac{a^0}{a^m}$ $= a^{(0-m)}$ $= a^{-m}$ for $a \neq 0$	$a^{-m} = \frac{1}{a^m}$
A power change When raising an expression with an index to a power, keep the same base and multiply the powers.	$\left(2^3\right)^4 = 2^3 \times 2^3 \times 2^3 \times 2^3$ $= 2^{(3+3+3+3)}$ $= 2^{12}$	$\left(a^m\right)^n = a^{(m\times n)}$
Products and quotients When raising a product or a quotient to a power, each part is raised to the power.	$(3\times5)^2 = (3\times5)\times(3\times5)$ $= 3\times3\times5\times5$ $= 3^2\times5^2$ $\left(\frac{2}{3}\right)^4 = \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$ $= \frac{2^4}{3^4}$	$(ab)^m = a^m b^m$ $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

1 Multiplication with indices

Simplify the following expressions.

(a) $7a^2b^3 \times 2ab^4$

THINKING	WORKING
1 Rearrange the expression, grouping the coefficients and the same bases.	$7a^2b^3 \times 2ab^4$ $= 7 \times 2 \times a^2 \times a \times b^3 \times b^4$
2 Multiply the coefficients and simplify using: $a^m \times a^n = a^{(m+n)}$	$= 14 \times a^{2+1} \times b^{3+4}$ $= 14 \times a^3 \times b^7$ $= 14a^3b^7$

(b) $\left(3m^2n\right)^3 \times 2m^2$

1 Expand the brackets using $\left(a^m\right)^n = a^{(m \times n)}$ by raising each part of the term inside the brackets to the power.	$\left(3m^2n\right)^3 \times 2m^2$ $= 3^3\, m^6n^3 \times 2m^2$
2 Rearrange the expression, grouping the coefficients and the same bases.	$= 27 \times 2 \times m^6 \times m^2 \times n^3$
3 Multiply the coefficients and simplify using: $a^m \times a^n = a^{(m+n)}$	$= 54m^8n^3$

2 Division with indices

Simplify the following expressions.

(a) $\dfrac{6m^5n^3}{2m^2}$

THINKING	WORKING
1 Rearrange the expression, grouping the coefficients and the same bases.	$\dfrac{6m^5n^3}{2m^2} = \dfrac{6}{2} \times \dfrac{m^5}{m^2} \times \dfrac{n^3}{1}$
2 Simplify the numerical fraction and use $\dfrac{a^m}{a^n} = a^{(m-n)}$ to simplify algebraic factors.	$= 3 \times m^{5-2} \times n^3$ $= 3m^3n^3$

(b) $\left(\dfrac{2p^3q}{pq}\right)^3 \div \dfrac{12p^2q}{\left(pq^5\right)^2}$

1 Change the quotient into a product and multiply by the reciprocal.	$\left(\dfrac{2p^3q}{pq}\right)^3 \div \dfrac{12p^2q}{\left(pq^5\right)^2}$ $= \left(\dfrac{2p^3q}{pq}\right)^3 \times \dfrac{\left(pq^5\right)^2}{12p^2q}$
2 Expand the brackets by raising each part of the term inside the brackets to the power.	$= \dfrac{8p^9q^3}{p^3q^3} \times \dfrac{p^2q^{10}}{12p^2q}$
3 Rearrange the expression, grouping the coefficients and the same bases.	$= \dfrac{8}{12} \times \dfrac{p^9p^2}{p^3p^2} \times \dfrac{q^3q^{10}}{q^3q}$
4 Simplify the numerical fraction and use index laws to simplify algebraic factors.	$= \dfrac{2}{3} \times p^6 \times q^9$ $= \dfrac{2p^6q^9}{3}$

3 Simplify with the zero index

Simplify the following expressions.

(a) $\left(\dfrac{a^2b}{4m}\right)^0$

THINKING	WORKING
By definition, a number raised to the power of zero is equal to 1. $a^0 = 1$ (for $a \neq 0$)	$\left(\dfrac{a^2b}{4m}\right)^0 = 1$

(b) $4m^0$

1 Insert multiplication symbols where multiplication is implied.	$4m^0 = 4 \times m^0$
2 Simplify using index laws. (Remember: $a^0 = 1$)	$= 4 \times 1$ $= 4$

4 Negative indices

Simplify the following, expressing the answer with positive indices.

(a) $\left(\frac{2}{5}\right)^{-3}$

THINKING	WORKING
1 To convert to a positive power, write the reciprocal to the positive power.	$\left(\frac{2}{5}\right)^{-3} = \frac{1}{\left(\frac{2}{5}\right)^{3}}$ $= \left(\frac{5}{2}\right)^{3}$
2 Expand the bracket by writing the powers of the numerator and the denominator.	$= \frac{5^3}{2^3}$

(b) $3a^{-2}b \times 2a^3b^{-5}$

1 Create an expression using positive indices. Simplify the expression, writing the answer using positive indices.	$3a^{-2}b \times 2a^3b^{-5}$ $= 3 \times 2 \times a^{-2} \times a^3 \times b \times b^{-5}$ $= 3 \times 2 \times \frac{1}{a^2} \times a^3 \times b \times \frac{1}{b^5}$ $= \frac{6a^3b}{a^2b^5}$ $= \frac{6a}{b^4}$
2 Verify the answer using index laws.	$3a^{-2}b \times 2a^3b^{-5}$ $= 3 \times 2 \times a^{-2} \times a^3 \times b \times b^{-5}$ $= 6 \times a^{-2+3}b^{1+(-5)}$ $= 6 \times a \times b^{-4}$ $= \frac{6a}{b^4}$

(c) $\frac{\left(x^2y\right)^{-1} \times x^{-3}y^2}{\left(x^{-1}y\right)^{-3}}$

1 Expand the brackets by raising each part of the term inside the brackets to the power using $\left(a^m\right)^n = a^{(m \times n)}$.	$\frac{\left(x^2y\right)^{-1} \times x^{-3}y^2}{\left(x^{-1}y\right)^{-3}}$ $= \frac{x^{2\times-1}y^{1\times-1} \times x^{-3}y^2}{x^{-1\times-3} \times y^{1\times-3}}$ $= \frac{x^{-2}y^{-1} \times x^{-3}y^2}{x^3y^{-3}}$

2 Rearrange so that powers with the same base are together.	$= \dfrac{x^{-2} \times x^{-3}}{x^{3}} \times \dfrac{y^{-1} \times y^{2}}{y^{-3}}$
3 Use index laws to simplify powers with the same base.	$= x^{-2+(-3)-(3)} \times y^{-1+2-(-3)}$ $= x^{-8} y^{4}$
4 Simplify the expression, writing the answer with positive indices.	$= \dfrac{1}{x^{8}} \times y^{4}$ $= \dfrac{y^{4}}{x^{8}}$

Fractional indices

The index rule for powers of powers can be used to show how a cube root can be expressed as a power. For example, the cube root of 8 is 2. So 8 to what power is to equal to 2?

$$8^{x} = 2$$
$$\left(2^{3}\right)^{x} = 2$$
$$\left(2^{3}\right)^{x} = 2^{1}$$

Equating the powers:

$$3x = 1$$
$$x = \frac{1}{3}$$

So $8^{\frac{1}{3}} = 2$ and $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Note that $\sqrt[3]{8}$ is called the radical form and $8^{\frac{1}{3}}$ is called the index form.

When calculating the nth root of a number, raise the number to the power of the reciprocal of n, that is $\frac{1}{n}$.

Additional information

Roots as powers

These two activities share more information about rational indices.

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

5 Convert radicals to index form

Express the following in the simplest index form.

(a) $\sqrt[3]{x^{7}}$

THINKING	WORKING
1 Write the expression as a fractional power.	$\sqrt[3]{x^{7}} = \left(x^{7}\right)^{\frac{1}{3}}$
2 Recall the power rule index law: $\left(a^{m}\right)^{n} = a^{m \times n}$	$= x^{7 \times \frac{1}{3}}$ $= x^{\frac{7}{3}}$

(b) $x^2\sqrt{x}$

1 Write the expression with a fractional power.	$x^2\sqrt{x} = x^2x^{\frac{1}{2}}$
2 Recall the multiplication index law: $a^m \times a^n = a^{m+n}$	$= x^{2+\frac{1}{2}}$ $= x^{\frac{5}{2}}$

Note that in the previous example, $\sqrt{x^5}$ and $x^2\sqrt{x}$ would give the same result.

6 Convert powers with fractional indices to radicals

Express the following in the simplest radical form.

(a) $x^{\frac{2}{5}}$

THINKING	WORKING
1 Expand the fractional power to the form: $a^{m\times n} = \left(a^m\right)^n$	$x^{\frac{2}{5}} = x^{2\times\frac{1}{5}}$ $= \left(x^2\right)^{\frac{1}{5}}$
2 Convert the fractional index to radical form.	$= \sqrt[5]{x^2}$

(b) $x^{\frac{8}{3}}$

1 Convert the improper fraction into a mixed number.	$\frac{8}{3} = 2\frac{2}{3}$
2 Recall the multiplication index law: $a^{m+n} = a^m \times a^n$	$x^{\frac{8}{3}} = x^{2\frac{2}{3}}$ $= x^2x^{\frac{2}{3}}$
3 Simplify using the power rule: $a^{m\times n} = \left(a^m\right)^n$	$= x^2 \times x^{2\times\frac{1}{3}}$ $= x^2 \times \left(x^2\right)^{\frac{1}{3}}$
4 Convert the fractional idex to radical form.	$= x^2\sqrt[3]{x^2}$

Note that in the previous example, $\sqrt[3]{x^8}$ would also be considered simplest form.

Prime factors

When the numbers in an expression can be expressed as products of prime factors, you can sometimes use index laws to simplify the expression.

7 Simplify with prime factors

Simplify the following.

(a) $128^{\frac{3}{7}}$

THINKING	WORKING
1 Write the number as a product of its prime factors: $128 = 2^7$	$128^{\frac{3}{7}} = \left(2^7\right)^{\frac{3}{7}}$
2 Simplify using the index law: $\left(a^m\right)^n = a^{m\times n}$	$= 2^3$

(b) $\dfrac{2^3 \times 12^{\frac{1}{2}}}{\sqrt{3^3}}$

1 Rewrite each base number as the product of its prime factors: $12 = 2^2 \times 3$	$\dfrac{2^3 \times 12^{\frac{1}{2}}}{\sqrt{3^3}} = \dfrac{2^3 \times \left(2^2 \times 3\right)^{\frac{1}{2}}}{\left(3^3\right)^{\frac{1}{2}}}$
2 When raising a power to a power, multiply the indices.	$= \dfrac{2^3 \times 2^{2\times\frac{1}{2}} \times 3^{1\times\frac{1}{2}}}{3^{3\times\frac{1}{2}}}$ $= \dfrac{2^3 \times 2^1 \times 3^{\frac{1}{2}}}{3^{\frac{3}{2}}}$
3 Group together powers with the same bases.	$= 2^3 \times 2^1 \times \dfrac{3^{\frac{1}{2}}}{3^{\frac{3}{2}}}$
4 Recall index laws in order to simplify the expression and write with positive powers.	$= 2^{3+1} \times 3^{\left(\frac{1}{2}-\frac{3}{2}\right)}$ $= 2^4 \times 3^{-1}$ $= \dfrac{2^4}{3}$

8 Simplify with prime factors and algebraic indices

Simplify: $\dfrac{2^x \times 9^{2x}}{6^3}$

THINKING	WORKING
1 Rewrite each base number as the product of its prime factors.	$\dfrac{2^x \times 9^{2x}}{6^3} = \dfrac{2^x \times \left(3^2\right)^{2x}}{(2\times 3)^3}$
2 Simplify the expression using the power rule: $\left(a^m\right)^n = a^{m\times n}$	$= \dfrac{2^x \times 2^{2\times 2x}}{2^{1\times 3} \times 3^{1\times 3}}$ $= \dfrac{2^x \times 3^{4x}}{2^3 \times 3^3}$
3 Use index laws to create a single index for each base.	$= 2^{(x-3)} \times 3^{(4x-3)}$

Factorising, treating powers as single variables

When a power is raised to a power, the indices are multiplied: $\left(a^m\right)^n = a^{mn}$.

So $a^{2x} = \left(a^x\right)^2$.

Note that an expression that includes $2x$ as a power can be thought of as a quadratic equation.

For example, $3^{2x} + 3^x - 6$ is the same as a quadratic expression $\left(3^x\right)^2 + \left(3^x\right) - 6$.

Sometimes, these expressions can be factorised.

9 Simplify expressions that can be factorised

Simplify the following by factorising first.

(a) $\dfrac{3^x + 3^{x-2}}{3^x}$

THINKING	WORKING
1 Use the multiplication index law: $a^{m+n} = a^m \times a^n$	$\dfrac{3^x + 3^{x-2}}{3^x} = \dfrac{3^x + \left(3^x \times 3^{-2}\right)}{3^x}$
2 Factorise.	$= \dfrac{3^x \times 1 + 3^x \times 3^{-2}}{3^x}$ $= \dfrac{3^x\left(1 + 3^{-2}\right)}{3^x}$
3 Cancel the common factors to the numerator and denominator.	$= 1 + 3^{-2}$
4 Simplify the expression.	$= 1 + \dfrac{1}{3^2}$ $= \dfrac{10}{9}$

(b) $\dfrac{x^{2a} - 9}{x^{2a} + x^a - 6}$

1 Replace the power with a single variable.	$\dfrac{x^{2a} - 9}{x^{2a} + x^a - 6} = \dfrac{\left(x^a\right)^2 - 9}{\left(x^a\right)^2 + \left(x^a\right) - 6}$ Let $X = x^a$
2 Factorise and simplify.	$\dfrac{X^2 - 3^2}{X^2 + X - 6} = \dfrac{(X+3)(X-3)}{(X+3)(X-2)}$ $= \dfrac{X-3}{X-2}$
3 Replace the single variable with the power.	$= \dfrac{x^a - 3}{x^a - 2}$

EXERCISE 6.1 Review of index laws

Worked Example

1 Simplify the following. 1

(a) $a^3 \times a^7 \times a$ (b) $2b^2 \times 3b^5$ (c) $5m^2n \times \left(mn^3\right)^2$ (d) $ab^3 \times \left(3a^2b^3\right)^2 \times a^6$

2 Simplify the following. 2

(a) $\dfrac{c^5d^2}{c^2d}$ (b) $6a^3b^6 \div 3b^5$ (c) $\dfrac{4x^5y^2z \times 3^2x^3y^3z^5}{6\left(x^2yz\right)^4}$ (d) $\left(\dfrac{a^3b}{2c^2}\right)^5$

3 Simplify the following.

(a) $\dfrac{x^5y^2}{xy} \times \dfrac{xy^3}{x^2}$ (b) $\dfrac{6\left(a^3b\right)^3}{a^4b^3} \times \dfrac{a^5b^7}{(2a)^2b^6}$ (c) $\dfrac{5c^2d^5}{2cd^3} \div \dfrac{(cd)^3}{4c^7d}$ (d) $\dfrac{5xy^7}{(xy)^3} \div \left(\dfrac{x^2y^3}{3x^3y}\right)^2$

4 Simplify the following. 3

(a) $\left(abc^2\right)^0$ (b) 5^0 (c) $3ab^0$ (d) $y^0 + 5$

(e) $\dfrac{m^5 \times m^2}{m^7}$ (f) $6a^3b^5 \div 3b^5$ (g) $4x^0 + x$ (h) $\left(\dfrac{a^5b^{-2}}{2c^3}\right)^0$

5 Simplify each of the following, expressing the answer with positive indices. 4

(a) x^{-2} (b) $-5m^{-2}$ (c) $\dfrac{1}{m^{-2}}$ (d) $\dfrac{3a}{b^{-3}}$

(e) $x^5 \times x^{-7}$ (f) $s^{-3}t^4 \times s^6t^{-2}$ (g) $\left(4xy^2\right)^{-2} \times \left(3x^2y^3\right)^3$ (h) $\dfrac{c^3d^{-2}}{c^7d}$

(i) $10a^3b^{-5} \div 5b^{-6}$

6 Consider the expression $\dfrac{8x^5y^6}{6x^3y^2} \times \dfrac{3x^4y^7}{2x^5y}$.

(a) Which of the following is equivalent to the original expression?

A $3xy^{10}$ B $\dfrac{xy^{10}}{2}$ C $2xy^{10}$ D $\dfrac{8}{11}xy^{10}$

(b) Explain the common error made by a student who thought the answer was $\dfrac{11}{8}xy^{10}$.

7 Express the following as a single power. 5

(a) $\sqrt[3]{a}$ (b) $\left(\sqrt{x}\right)^7$ (c) $\sqrt[4]{y^5}$

8 Simplify each expression, then write using a radical symbol. 6

(a) $a^{\frac{1}{3}} \times a^{\frac{1}{4}}$ (b) $\dfrac{b}{b^{\frac{1}{2}}}$ (c) $x^{-\frac{1}{2}}$

9 Simplify the following. 7

(a) $16^{\frac{1}{4}}$ (b) $\left(\dfrac{1}{216}\right)^{\frac{1}{3}}$ (c) $32^{\frac{2}{5}}$

(d) $\left(13^4\right)^{-\frac{1}{2}}$ (e) $125^{-\frac{2}{3}}$ (f) $\left(\dfrac{9}{49}\right)^{-\frac{3}{2}}$

8

10 Express the following as prime factors in simplest index form.

(a) $2^2 \times 4^3$ (b) $5^3 \times 15^2 \times 25^{-1} \times 9$ (c) $\dfrac{125^4}{\left(5^3\right)^2}$ (d) $\dfrac{8^2 \times 32}{2^6 \times 16^2}$

11 Determine the difference in value between doubling a number seven times and doubling it eight times.

12 Simplify the following, expressing the answer with positive indices.

(a) $\left(a^3b^{-2}\right)^4 \times \left(a^{-1}b^2\right)^{-2}$ (b) $\left(\dfrac{m^{-2}n^3}{mn^{-1}}\right)^{-1}$ (c) $\dfrac{x^{-3}y^2z \times 3^2x^3y^3z^{-5}}{\left(xy^2z\right)^{-2}}$

(d) $\dfrac{\left(2xy^3z^5\right)^{-2} \times x^3y^{-3}z^5}{\left(x^{-2}yz\right)^4}$ (e) $\left(\dfrac{a^3b}{2ac^2}\right)^{-2} \div \dfrac{ac^3}{a^{-2}b^5c^{-1}}$

13 Which of the following is the simplification of $\left(\dfrac{a^4b^{-2}}{ab^3}\right)^2 \div \left(\dfrac{a^3b}{a^5b^2}\right)^{-1}$?

A $\dfrac{a^4}{b^{11}}$ B $\dfrac{a^8}{b^9}$ C $\dfrac{b^2}{a^7}$ D $\dfrac{a^4}{b^2}$

14 Simplify each expression, then write using radical symbols.

(a) $\dfrac{2xy}{\left(x^{\frac{2}{3}}y\right)^3} \times \left(\dfrac{x^2y^3}{8x^{-1}y}\right)^{\frac{1}{3}}$ (b) $\sqrt{x^2y^5} \times \sqrt[3]{xy^6}$

15 Express each of the following as prime factors in simplest index form.

(a) $2^a \times 10^{a-3}$ (b) $\dfrac{3^b \times 4^{2-b}}{6^{b+1}}$ (c) $\dfrac{2^{2x+1} \times 9^x \times 6^3}{18^{3x} \times 4^{1-x}}$ (d) $\dfrac{2^{2x+1} \times 4^{x+1}}{\sqrt{4^x \times 2^{2x-2}}}$

9

16 Simplify the following by factorising first.

(a) $\dfrac{2^a + 2^{a+1}}{2^a}$ (b) $\dfrac{3^{2n} - 1}{3^n + 1}$ (c) $\dfrac{a^{2m} - 5a^m - 6}{2a^m - 12}$

17 Show that $\dfrac{x^{-2} - 1}{x^{-1} + 1} = \dfrac{1}{x} - 1$.

18 The sum of a number raised to the power of 5 and 9 lots of the reciprocal of the number is the same as six lots of the square of the number. Determine the value of the number.

Scientific notation

6.2

Very large and very small numbers

Scientific notation is used for numbers that are very large or very small. For example, the number of ^{12}C atoms contained in 12 g of carbon-12 is very large. To 3 significant figures, it is usually written as 6.02×10^{23}, rather than writing as 602 000 000 000 000 000 000 000 and then counting all the zeros. It also makes it easier to identify the significant figures. Zeros at the tail of a number indicate the magnitude of the number, not the degree of accuracy or precision of the measurement.

The mass of a single ^{12}C atom can be found by dividing 12 g by 6.02×10^{23}. It is 1.99×10^{-23} g or 0.000 000 000 000 000 000 000 0199 g, which is a very small number.

Scientific notation

In scientific notation, a number is written as the product of a number between 1 and 10 (but excluding 10) and of a power of 10. The general form is $a \times 10^n$.

Additional information

Scientific notation

Strengthen your understanding of scientific notation with this activity.

To convert a number to scientific notation, count how many places the decimal point moves to obtain the number a between 1 and 10.

- If the number is greater than 1, the number of places will be n, the power of 10.
- If the number is less than 1, the number of places will be $|n|$ and n will be negative.

The following examples show conversion of numbers into scientific notation.

The decimal point in 120 000 moves 5 places to the left to obtain a number between 1 and 10. Note that 120 000 is 100 000 times greater than 1.2 and $100\,000 = 10^5$. The original number is greater than 1 so the power is positive (5).

$$\begin{aligned} 120\,000 &= 12\,000 \times 10 \\ &= 1200 \times 10 \times 10 \\ &= 120 \times 10 \times 10 \times 10 \\ &= 12 \times 10 \times 10 \times 10 \times 10 \\ &= 1.2 \times 10 \times 10 \times 10 \times 10 \times 10 \\ &= 1.2 \times 10^5 \end{aligned}$$

The decimal point in 0.002 93 moves 3 places to the right to obtain a number between 1 and 10. Note that 0.002 93 is 1000 times smaller than 2.93 and $1000 = 10^3$. The original number is less than 1 so the power is negative (-3).

$$\begin{aligned} 0.002\,93 &= 0.0293 \times \frac{1}{10} \\ &= 0.293 \times \frac{1}{10} \times \frac{1}{10} \\ &= 2.93 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \\ &= 2.93 \times \frac{1}{10^3} \\ &= 2.93 \times 10^{-3} \end{aligned}$$

10 Convert a number to scientific notation

Express the following numbers in scientific notation and state how many significant figures they contain.

(a) 32 100 000

THINKING	WORKING
1 Convert the number to scientific notation.	$32\,100\,000 = 3.21 \times 10\,000\,000$ $= 3.21 \times 10^7$
2 Count the digits in the decimal number.	The number has 3 significant figures.

(b) 0.000 12

1 Convert the number to scientific notation.	$0.000\,12 = 0.0012 \times \frac{1}{10}$ $= 0.012 \times \frac{1}{10^2}$ $= 0.12 \times \frac{1}{10^3}$ $= 1.2 \times \frac{1}{10^4}$ $= 1.2 \times 10^{-4}$
2 Count the digits in the decimal number.	The number has 2 significant figures.

(c) 58.30

1 Convert the number to scientific notation.	$58.30 = 5.830 \times 10$ $= 5.830 \times 10^1$
2 Count the digits in the decimal number.	The number has 4 significant figures.

11 Convert from scientific notation to an ordinary number

Express the following number, given in scientific notation, as an ordinary number.

(a) 1.792×10^2

THINKING	WORKING
Convert the number by considering the power of 10.	$1.792 \times 10^2 = 1.792 \times 10 \times 10$ $= 17.92 \times 10$ $= 179.2$

(b) 1.07×10^{-3}

Convert the number by considering the power of 10. Fill in any place value digits with zeros.	$1.07 \times 10^{-3} = 1.07 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10}$ $= 0.107 \times \frac{1}{10} \times \frac{1}{10}$ $= 0.0107 \times \frac{1}{10}$ $= 0.00107$

(c) 6.32×10^5

Convert the number by considering the power of 10. Fill in any place value digits with zeros.	$6.32 \times 10^5 = 6.32 \times 10 \times 10 \times 10 \times 10 \times 10$ $= 6.32 \times 100\,000$ $= 632\,000$

Significant figures

Scientific notation is used in science and engineering because it indicates the accuracy of measured values.

Consider the numbers 5.3 and 5.30.

- In 5.30, the zero digit is significant as it informs us that the accuracy of the number extends to the hundredths decimal place. So 5.30 has 3 significant figures
- In 5.3, you do not know anything about the accuracy in the hundredths decimal place. So 5.3 has 2 significant figures.

Significant figures indicate the degree of precision that can be claimed. Measurement, whether by a human or a machine, is always dependent on the accuracy of the instrument.

- Bathroom scales might measure the mass of a 67 kg person to within 1 kg. This is 6.7×10^1 kg in scientific notation and has 2 significant figures. If a person weighed 112 kg, this would be 1.12×10^2 kg and have 3 significant figures.
- Kitchen scales might measure the mass of 250 g of flour to within 10 g. This is 2.5×10^2 g in scientific notation and has 2 significant figures.
- Electronic balances of the type used in laboratories might measure 0.275 g of sodium nitrate to within a single milligram. This is 2.75×10^{-1} g in scientific notation and has 3 significant figures.

Summary of significant figures

All non-zero digits are significant.
All zeros between non-zero digits are significant.
All zeros at the end of decimal numbers are significant.
Zeros at the beginning of a decimal number are not significant.
Zeros at the end of an integer are assumed to be placeholders and not significant, unless the accuracy is specified.
In practical situations, use the degree of accuracy in the least accurate measurement.

Number	Number of significant figures
57.4	3
507.4	4
507.400	6
0.005074	4
574000	Assume 3 (but could be 4, 5 or 6)
574000.0	7

If calculations involve measurements, then the answers must reflect the accuracy of the original measurements. The easiest way to deal with this is to round the final answer to the same number of significant figures as the least accurate measurement in the question.

Additional information

Decimal places and significant figures
Practice rounding numbers to the specified number of significant figures.

12 Calculate the value of expressions with scientific notation

Calculate each of the following, stating your final answer to the appropriate number of significant figures.

(a) $1.2 \times 10^2 \times 9.0 \times 10^8$

THINKING	WORKING
1 Perform the multiplication, use the index laws $a^m \times a^n = a^{m \times n}$ to multiply the powers of 10.	$1.2 \times 10^2 \times 9.0 \times 10^8$ $= 1.2 \times 9.0 \times 10^2 \times 10^8$ $= 10.8 \times 10^{10}$
2 Convert the answer into scientific notation and round the answer to the least number of significant figures (s.f.) in the question.	$= 1.08 \times 10^{11}$ $= 1.1 \times 10^{11}$ (2 s.f.)

(b) $\left(1.75 \times 10^{-3}\right) \div \left(5.0 \times 10^2\right)$

1 Rewrite the expression so that it appears as a fraction or quotient.	$\left(1.75 \times 10^{-3}\right) \div \left(5.0 \times 10^2\right)$ $= \dfrac{1.75 \times 10^{-3}}{5.0 \times 10^2}$
2 Use the index laws to divide the powers of 10. Round the answer to the least number of significant figures in the question.	$= 0.35 \times 10^{-5}$ $= 3.5 \times 10^{-6}$

(c) $6.2 \times 10^5 + 3 \times 10^4$

1 To add up values, they need to have the same powers of 10.	$6.2 \times 10^5 + 3 \times 10^4$
2 Group like terms.	$= 62 \times 10^4 + 3 \times 10^4$ $= 65 \times 10^4$
3 Convert the answer into scientific notation and round the answer to the least number of significant figures in the question.	$= 6.5 \times 10^5$ $= 7 \times 10^5$ (1 s.f.)

WARNING

Be careful when entering powers of 10 on your calculator. Use the correct function or enter 1000 instead of 10^3.

Technology worked example
Evaluating scientific notation

13 Application of scientific notation

The Earth travels in an elliptical orbit around the Sun. At the nearest point of its orbit it is 1.47×10^8 km from the Sun, and at the furthest point it is 1.52×10^8 km away.

Calculate the difference in the time light takes to reach the Earth from the Sun when the Earth is closest and furthest from the Sun. Assume that the speed of light is 2.998×10^8 m s^{-1}.

THINKING	WORKING
1 Calculate the difference between the distances and express the difference in scientific notation.	$1.52 \times 10^8 - 1.47 \times 10^8$ $= 0.05 \times 10^8$ $= 5.0 \times 10^6$ km
2 Make sure that distance is measured in the same unit as speed.	$= 5.0 \times 10^9$ m
3 Recall the equation speed $= \frac{\text{distance}}{\text{time}}$ and transpose to make time the subject. Calculate the time taken.	$\text{time} = \frac{\text{distance}}{\text{speed}}$ $= \frac{5.0 \times 10^9}{2.998 \times 10^8}$ $= 1.667\,77\ldots \times 10^1$ s
4 Round the answer to the least number of significant figures in the question.	1.67×10^1 s (3 s.f.)
5 Interpret the answer.	It takes 16.7 s longer for the light from the Sun to reach Earth at the furthest point.

EXERCISE 6.2 Scientific notation

Worked Example

10 **1** Express each of the following in scientific notation and identify the number of significant figures.

(a) 32 000 (b) 57 200 (c) 1070 (d) 1537

(e) 47 200 000 (f) 52 130 000 (g) 10.27 (h) 5.63

(i) 0.00102 (j) 0.000 056 (k) 0.080 300 (l) 0.003 960

2 (a) As an ordinary number 6.1×10^5 is equivalent to:

A 0.000 061 B 610 000 C 0.00061 D 61 000

(b) Explain the common error made by a student who thought the answer was 6 100 000.

11 **3** Write each of the following as ordinary numbers with the same number of significant figures.

(a) 1.7×10^3 (b) 5.9×10^4 (c) 4.301×10^1 (d) 2.71×10^2

(e) 3.0×10^{-4} (f) 2.06×10^6 (g) 1.732×10^{-3} (h) 6.05×10^{-4}

(i) 5×10^7 (j) 7.98×10^0

4 Convert each of the following distances to the unit indicated. Write all answers in scientific notation.

(a) 3.01×10^{14} m to cm (b) 420 m to mm (c) 1.98×10^{24} m to km

(d) 250 m to cm (e) 88 m to mm (f) 6.127×10^7 m to km

5 Convert each of the following distances to metres. Write all answers in scientific notation.

(a) Length of a virus: 0.000 02 mm (b) Diameter of a bacterium: 0.0006 mm

(c) Circumference of Earth: 40 075 km (d) Distance to the Moon: 384 400 km

(e) Distance from Brisbane to Melbourne via the Pacific Highway: 1798 km

6 Determine the thickness of a ream of paper (500 sheets) if each sheet is 0.1 mm thick. Give your answer in metres, in scientific notation.

7 Determine the number of atoms in four moles of hydrogen gas, given that each mole contains 6.022×10^{23} molecules, and each molecule of H_2 has two atoms.

8 Determine the number of atoms in 21 g of gold, given that each mole contains 6.022×10^{23} atoms of gold and gold has a molar mass of 107.9 g/mole.

12 **9** Calculate the following, and check that your final answer has the appropriate number of significant figures.

(a) $3.0 \times 10^6 \times 4.0 \times 10^3$ (b) $5 \times 10^{15} \times 2.1 \times 10^4$ (c) $4.00 \times 10^{-3} \times 5.1 \times 10^{-2}$

(d) $\left(1.2 \times 10^6\right) \div \left(4.0 \times 10^3\right)$ (e) $\left(7.7 \times 10^3\right) \div \left(1.1 \times 10^{-4}\right)$ (f) $\dfrac{3.6 \times 10^{-4}}{9.0 \times 10^4}$

(g) $4.0 \times 10^5 + 7.2 \times 10^6$ (h) $6.23 \times 10^7 - 9.90 \times 10^6$ (i) $7.12 \times 10^3 - 1.2 \times 10^4$

10 Choose the correct answer to the appropriate number of significant figures.

(a) $4.3 \times 10^2 \times 3.72 \times 10^1$

A 1.6×10^4 B 1.5×10^4 C 1.60×10^4 D 1.5996×10^4

(b) $2.76 \times 10^4 - 8.54 \times 10^3$

A -5.78×10^4 B 1.906×10^4 C 1.91×10^4 D 1.9×10^4

(c) $\dfrac{2.1 \times 10^2 \times 6.3 \times 10^3}{7.0 \times 10^5}$

A 1.89 B 1.9×10^1 C 1.8 D 1.9

11 Calculate the following, giving your final answer with the appropriate number of significant figures.

(a) $4.1 \times 10^3 \times 1.2 \times 10^6$ (b) $2.71 \times 10^2 \times 8.35 \times 10^3$ (c) $7.39 \times 10^{-5} \times 3.7 \times 10^2$

(d) $5.1 \times 10^{-2} \times 2.32 \times 10^{-1}$ (e) $\left(6.5 \times 10^8\right) \div \left(2.1 \times 10^6\right)$ (f) $\left(9.62 \times 10^7\right) \div \left(7.89 \times 10^6\right)$

(g) $\dfrac{1.50 \times 10^5}{7.23 \times 10^3}$ (h) $\dfrac{5.24 \times 10^3}{9.2 \times 10^6}$

12 When Barack Obama was elected in 2012 for his second term as President of the USA, the country had a debt of \$16.4 trillion. The population of the USA at the time was 315 125 000. Note: 1 trillion equals 1 000 000 000 000.

(a) Express 16.4 trillion in scientific notation.

(b) How much debt did the USA have per person at that time? Round your answer to an appropriate number of significant figures.

13 Given $x = 2.00 \times 10^5$, $y = 4.0 \times 10^4$ and $z = 6.0 \times 10^{-3}$, calculate each of the following.

(a) $\dfrac{x+y}{z}$ (b) $\dfrac{x^2 z}{y}$ (c) $\left(\dfrac{xz}{y}\right)^3$ (d) $xz - y$

Worked Example 13

14 The brightest star in the sky is Sirius. It is 8.58 light-years away from Earth. (A light-year is the distance that light can travel in a year.) The speed of light is 2.998×10^8 m s^{-1}.
How many kilometres is Sirius from Earth?

15 Nerve cells transmit some messages via electrical impulses. The message generated by a reflex reaction travels at 99 m s^{-1}. If your fingertips are 83 cm from your brain, how long would it take the message that the stove top is hot to reach your brain?

16 Each molecule of iron(II) sulfate $\left(FeSO_4\right)$ is made up of one iron atom (Fe) and one sulfate ion $\left(SO_4\right)$. Each molecule of iron(III) sulfate $\left(Fe_2\left(SO_4\right)_3\right)$ is made up of two iron atoms and three sulfate ions. A sulfate ion $\left(SO_4\right)$ is made up of one sulfur atom (S) and four oxygen atoms (O). A mole of each substance contains 6.022×10^{23} molecules. A mole of iron (Fe) is 55.845 g, a mole of sulfur (S) is 32.065 g, a mole of oxygen gas $\left(O_2\right)$ with two atoms per molecule is 31.998 g.

Determine which has more molecules, and by how much: 10.0 g of $FeSO_4$ or 10.0 g of $Fe_2\left(SO_4\right)_3$. Give an explanation for the result.

17 Peter and Jing hope to see the Riverfire fireworks in Brisbane from the top of Mt Barney, 130 km away. They want to calculate the time delay between seeing and hearing the burst of an explosion. Peter knows that the speed of light is 2.998×10^8 m s^{-1}, but Jing discovers that speed of sound varies greatly: 340 m s^{-1} at sea level at 15°C and 301 m s^{-1} at 29 000 m above sea level at -48°C. Using this information, calculate the most accurate time delay estimate possible. Explain any assumptions you make.

6.3 Geometric sequences and series

Geometric sequences

A *geometric sequence* is a sequence of numbers with a common ratio between successive terms. For example, the sequence 2, 6, 18, 54, 162, 486, … is a geometric sequence because each term is 3 times larger than the previous term. For this sequence the common ratio r is 3 and the first term t_1 is 2. Any term in this sequence can be generated through (repeated) multiplication of the first term by the common ratio. A geometric sequence is also called a geometric progression.

A geometric sequence is defined by a sequence of terms:

$t_1, t_1 \times r, t_1 \times r^2, t_1 \times r^3, t_1 \times r^4, \ldots\ t_1 \times r^{n-1}$

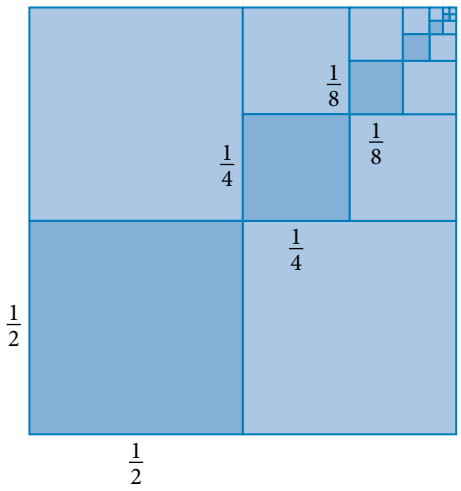
or, if you use a to represent the first term:

$a, a \times r, a \times r^2, a \times r^3, a \times r^4, \ldots\ a \times r^{n-1}$

where n is a natural number (positive integer) and a and r are non-zero numbers.

The following image is an example of a geometric sequence.

With the squares shown, the ratio of successive side lengths is constant. The ratio of successive areas is also constant, but a different value.



The general term of a geometric sequence with the common ratio r for $n \geq 1$ is defined as:

- $t_n = r^{n-1}t_1$, where t_1 is the first term
- $t_n = ar^{n-1}$, where a is the first term.

The recursive definition of a geometric sequence is:

$t_{n+1} = rt_n$ for $n \geq 1$

Additional information

The definition of a geometric sequence

Explore the definition of geometric sequences.

14 Generating terms in a sequence

Generate the first four terms of each geometric sequence, given the first term and the common ratio.

(a) $t_1 = 3, r = 2$

THINKING	WORKING
1 The ratio and first term are given.	$r = 2, t_1 = 3$

2 Multiply each term by the common ratio to determine the following term.	$t_2 = rt_1$ $= 2 \times 3$ $= 6$	$t_3 = rt_2$ $= 2 \times 6$ $= 12$	$t_4 = rt_3$ $= 2 \times 12$ $= 24$
3 Write the first four terms in order.	3, 6, 12, 24		

(b) $t_1 = -2$, $r = -3$

1 The ratio and first term are given.	$r = -3$, $t_1 = -2$		
2 Multiply each term by the common ratio to determine the following term.	$t_2 = rt_1$ $= (-3) \times (-2)$ $= 6$	$t_3 = rt_2$ $= (-3) \times 6$ $= -18$	$t_4 = rt_3$ $= (-3) \times (-18)$ $= 54$
3 Write the first four terms in order.	-2, 6, -18, 54		

(c) $t_1 = 12$, $r = \frac{1}{2}$

1 The ratio and first term are given.	$r = \frac{1}{2}$, $t_1 = 12$		
2 Multiply each term by the common ratio to determine the following term.	$t_2 = rt_1$ $= \frac{1}{2} \times 12$ $= 6$	$t_3 = rt_2$ $= \frac{1}{2} \times 6$ $= 3$	$t_4 = rt_3$ $= \frac{1}{2} \times 3$ $= \frac{3}{2}$
3 Write the first four terms in order.	$12, 6, 3, \frac{3}{2}$		

To prove that a sequence is geometric, you can rearrange the definition to solve for the common ratio r.

$$\frac{t_{n+1}}{t_n} = r \text{ for } n \geq 1$$

15 Test for a geometric sequence

Determine whether the sequence $3, 4, 5\frac{1}{3}, 7\frac{1}{9}, \cdots$ is geometric. If it is geometric, state the value of the common ratio r.

THINKING	WORKING		
1 Apply the formula $\frac{t_{n+1}}{t_n} = r$ to each consecutive pair of terms.	$\frac{t_2}{t_1} = \frac{4}{3}$	$\frac{t_3}{t_2} = \frac{5\frac{1}{3}}{4}$ $= \frac{16}{3} \times \frac{1}{4}$ $= \frac{4}{3}$	$\frac{t_4}{t_3} = \frac{7\frac{1}{9}}{5\frac{1}{3}}$ $= \frac{64}{9} \times \frac{3}{16}$ $= \frac{4}{3}$
2 Explain the solution.	This sequence is geometric with a common ratio $r = \frac{4}{3}$.		

16 Formula for a term

Write a general term that defines the geometric sequence 3, 4, $5\frac{1}{3}$, $7\frac{1}{9}$ … and use the rule to determine the 10th term, correct to 2 decimal places.

THINKING	WORKING
1 The general term of a geometric sequence is $t_n = t_1 r^{n-1}$.	$t_1 = 3, r = \frac{4}{3}$ $t_n = 3\left(\frac{4}{3}\right)^{n-1}$
2 Calculate the 10th term by substituting $n = 10$.	$t_{10} = 3\left(\frac{4}{3}\right)^{10-1}$ $= 3\left(\frac{4}{3}\right)^{9}$ $= 39.95\ (2\text{ d.p.})$
3 Interpret the answer.	The 10th term in the sequence is 39.95, correct to 2 decimal places.

Geometric series

The terms of a geometric sequence can be added together to find the sum S_n:

$$S_n = t_1 + t_1 \times r + t_1 \times r^2 + t_1 \times r^3 + t_1 \times r^4 + \ldots + t_1 \times r^{n-1}$$

Multiplying both sides of the equation by r:

$$rS_n = t_1 r + t_1 \times r^2 + t_1 \times r^3 + t_1 \times r^4 + t_1 \times r^5 + \ldots + t_1 \times r^{n-1} + t_1 \times r^n$$

Subtracting one equation from the other:

$$S_n - rS_n = t_1 - t_1 r^n$$
$$S_n(1-r) = t_1\left(1-r^n\right)$$

The sum of a geometric sequence S_n for $r \neq 1$ is:

- $S_n = \dfrac{t_1\left(1-r^n\right)}{1-r}$, if $r < 1$
- $S_n = \dfrac{t_1\left(r^n-1\right)}{r-1}$, if $r > 1$

In the rule for the geometric series, you can use a for the first term instead of t_1.

$$S_n = \frac{a\left(1-r^n\right)}{1-r} \text{ if } r < 1, \text{ or } S_n = \frac{a\left(r^n-1\right)}{r-1} \text{ if } r > 1.$$

17 Sum of a specified number of terms

For each geometric series, determine the sum of the first 10 terms, correct to 2 decimal places if necessary.

(a) $3+5+\frac{25}{3}+\ldots$

THINKING	WORKING
1 Determine the values of the first term and the common ratio.	$a=t_1=3$ $r=\frac{5}{3}$
2 Recall the formula for the sum, depending on the value of r. $r>1$ so $S_n=\frac{t_1\left(r^n-1\right)}{r-1}$ Substitute for a, r and n to calculate S_{10}, rounding if necessary.	$S_n=\frac{t_1\left(r^n-1\right)}{r-1}$ $S_{10}=\frac{3\left(\left(\frac{5}{3}\right)^{10}-1\right)}{\frac{5}{3}-1}$ $=739.72\ (2\text{ d.p.})$
3 Interpret the answer.	The sum of the first 10 terms in the series is 739.72, correct to 2 decimal places.

(b) $-5+2.5+(-1.25)+\ldots$

1 Determine the values of the first term and the common ratio.	$a=t_1=-5$ $r=-0.5$
2 Recall the formula for the sum, depending on the value of r. $r<1$ so $S_n=\frac{a\left(1-r^n\right)}{1-r}$ Substitute for a, r and n to calculate S_{10}, rounding if necessary.	$S_n=\frac{a\left(1-r^n\right)}{1-r}$ $S_{10}=\frac{-5\left(1-(-0.5)^{10}\right)}{1-(-0.5)}$ $=\frac{-5\left(1-(0.5)^{10}\right)}{1.5}$ $=-3.33\ (2\text{ d.p.})$
3 Interpret the answer.	The sum of the first 10 terms in the series is -3.33, correct to 2 decimal places.

Limiting sum of a geometric series

In an indefinite geometric sequence, the sequence can be written as:

$t_1, t_1\times r, t_1\times r^2, t_1\times r^3, t_1\times r^4, \ldots t_1\times r^{n-1}, \ldots$

The sequence continues indefinitely and contains an infinite number of terms.

For the sum S_n you can say that n 'tends towards' infinity, or $n \to \infty$.

What happens to $S_n = \dfrac{t_1\left(1-r^n\right)}{1-r}$ as $n \to \infty$?

Consider the possibilities:

- If $r > 1$: as $n \to \infty$, $r^n \to \infty$. The sum *diverges*.
- If $r < -1$: as $n \to \infty$, $r^n \to \infty$ for even values of n and $r^n \to -\infty$ for odd values of n. The sum *diverges*.
- If $-1 < r < 1$: as $n \to \infty$, $r^n \to 0$ and $S_n \to \dfrac{t_1(1-0)}{1-r} = \dfrac{t_1}{1-r}$. The sum *converges* to a single value.

Therefore if $-1 < r < 1$, S_n approaches a limiting value of $\dfrac{t_1}{1-r}$.

If $-1 < r < 1$ or $|r| < 1$, you can find the sum of an infinite geometric series:

$S_\infty = \dfrac{t_1}{1-r}$ for $-1 < r < 1$ or $|r| < 1$

This is called the sum-to-infinity of the geometric series.

Explore further

Sum to infinity of a geometric sequence

Explore the graphs demonstrating the sum to infinity.

18 Establish the sum-to-infinity

For the following geometric series, determine whether S_∞ exists by examining the common ratio r. If the sum exists, calculate its value.

(a) $50 + 25 + 12.5 + \ldots$

THINKING	WORKING
1 Determine whether S_∞ exists by examining the common ratio r.	$r = 0.5$ S_∞ exists because $\lvert r\rvert < 1$.
2 Substitute values into the formula $S_\infty = \dfrac{t_1}{1-r}$ to calculate the sum.	$S_\infty = \dfrac{50}{1-0.5}$ $= \dfrac{50}{0.5}$ $= 100$
3 Interpret the answer.	As the number of terms in the series approaches infinity, the sum of the series approaches 100.

(b) $10 + 11 + 12.1 + \ldots$

Determine whether S_∞ exists by examining the common ratio r.	$r = 1.1$ S_∞ does not exist because $\lvert r\rvert > 1$.

(c) $3 - 0.3 + 0.03 - \ldots$

1 Determine whether S_∞ exists by examining the common ratio r.	$r = -0.1$ S_∞ exists because $\lvert r\rvert < 1$.

2 Substitute values into the formula $S_\infty = \frac{t_1}{1-r}$ to calculate the sum.	$S_\infty = \frac{3}{1-(-0.1)}$ $= \frac{3}{1.1}$ $= \frac{30}{11}$ $= 2\frac{8}{11}$
3 Interpret the answer.	As the number of terms in the series approaches infinity, the sum of the series approaches $2\frac{8}{11}$.

(d) $2 + \frac{4}{3} + \frac{8}{9} + \ldots$

1 Determine whether S_∞ exists by examining the common ratio r.	$r = \frac{2}{3}$ S_∞ does exist because $\lvert r \rvert < 1$.
2 Substitute values into the formula $S_\infty = \frac{t_1}{1-r}$ to calculate the sum.	$S_\infty = \frac{2}{1-\frac{2}{3}}$ $= \frac{2}{\frac{1}{3}}$ $= 6$
3 Interpret the answer.	As the number of terms in the series approaches infinity, the sum of the series approaches 6.

19 Express recurring decimals as fractions

Express each of the following recurring decimals as an infinite geometric series. Use the formula for the sum-to-infinity to determine its equivalent value as a simplified fraction.

(a) $0.\dot{7}$

THINKING	WORKING
1 Express the recurring decimal as a geometric series.	$0.\dot{7} = 0.7 + 0.07 + 0.007 + \ldots$
2 Determine the values of t_1 and r.	$t_1 = 0.7, r = 0.1$
3 Calculate the sum by substituting the values into the formula: $S_\infty = \frac{t_1}{1-r}$	$S_\infty = \frac{0.7}{1-0.1}$ $= \frac{0.7}{0.9}$ $= \frac{7}{9}$

(b) $0.\dot{4}\dot{5}$

1 Express the recurring decimal as a geometric series.	$0.\dot{4}\dot{5} = 0.45 + 0.0045 + 0.000045 + \ldots$
2 Determine a and r.	$a = 0.45,\ r = 0.01$
3 Calculate the sum by substituting the values into the formula: $S_\infty = \dfrac{a}{1-r}$	$S_\infty = \dfrac{0.45}{1-0.01} = \dfrac{0.45}{0.99} = \dfrac{45}{99} = \dfrac{5}{11}$

20 Number of terms for the sum to exceed a given amount

Determine the minimum number of terms for the sum of the geometric series $3 + 4 + 5\frac{1}{3} + \ldots$ to exceed 3000.

THINKING	WORKING
1 Determine the values of the first term and the common ratio.	$t_1 = 3,\ r = \dfrac{4}{3}$
2 Recall the formula $S_n = \dfrac{t_1(r^n - 1)}{r-1}$ and substitute the known values to create an inequality and simplify. Use technology to solve for n.	$\dfrac{t_1\left(r^n - 1\right)}{r-1} = S_n$ $\dfrac{3\left(\left(\frac{4}{3}\right)^n - 1\right)}{\frac{4}{3} - 1} > 3000$ $3\left(\left(\dfrac{4}{3}\right)^n - 1\right) > 1000$ $\left(\dfrac{4}{3}\right)^n - 1 > \dfrac{1000}{3}$ $\left(\dfrac{4}{3}\right)^n > \dfrac{1003}{3}$ $n > 20.20\ (2\text{ d.p.})$
3 Interpret the answer.	A sequence of 21 terms are needed for the sum of the series to exceed 3000.

Modelling population growth

Any situation in which a population increases or decreases proportionally, for equal periods of time, can be modelled by a geometric sequence.

21 Population model

The population of a city is approximately 326 000. Urban planners calculate that the city's population is increasing by the same percentage each year, and in 2 years it will be 349 692. If this trend continues, determine:

(a) the city population after 9 years of growth, rounded to the nearest thousand

THINKING	WORKING
1 Determine the given values.	The starting year is $n = 1$: $t_1 = 326\,000$ $t_3 = 349\,692$ So 9 years later, $n = 10$.
2 Use the given values to find the value of r.	$t_3 = 349\,692$ $t_1 \times r^2 = 349\,692$ $326\,000r^2 = 349\,692$ $r^2 = \frac{349\,692}{326\,000}$ $r = \sqrt{\frac{349\,692}{326\,000}}$ $r = 1.0357\ (4\text{ d.p.})$
3 The general term of a geometric sequence is $t_n = t_1 r^{n-1}$.	$t_n = 326\,000(1.0357)^{n-1}$
4 Substitute for n to determine the population after 9 years of growth.	$t_{10} = 326\,000(1.0357)^{10-1}$ $= 447\,000\ (\text{nearest thousand})$
5 Interpret the answer.	The population will be approximately 447 000 people after 9 years.

(b) when the city's population will exceed 500 000 people.

1 The inequality $t_n > 500\,000$ represents the population exceeding 500 000.	$326\,000(1.0357)^{n-1} = t_n$ $326\,000(1.0357)^{n-1} > 500\,000$
2 Use technology or other methods to solve the inequality for n.	$(1.0357)^{n-1} > \frac{500\,000}{326\,000}$ $(1.0357)^{n-1} > 1.5337\ldots$ $n > 13.19\ldots$
3 Interpret the answer.	Rounded to the nearest whole year, $n = 14$. So the population will exceed 500 000 after an additional 13 years of growth.

EXERCISE 6.3 Geometric sequences and series

Worked Example

14 **1** Determine the first four terms of each geometric sequence, given the first term and the common ratio.

(a) $t_1 = 5, r = 2$ (b) $t_1 = -3, r = 4$ (c) $t_1 = 10, r = \frac{1}{5}$

(d) $t_1 = \frac{1}{4}, r = -2$ (e) $t_1 = 6, r = -1$ (f) $t_1 = 20, r = \frac{1}{10}$

15 **2** Determine whether each of the following is a geometric sequence. If the sequence is geometric, state the value of the common ratio r.

(a) 4, 12, 36, 108, … (b) -2, -4, -6, -8, … (c) 3, -9, 27, -81, …

(d) $\pi, \pi w, \pi w^2, \pi w^3, \ldots$ (e) 6, 9, 13.5, 20.25, … (f) $\sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

3 Determine whether each of the following is a geometric sequence. If the sequence is geometric, state the value of the common ratio r.

(a) $A_n = 3x^n$ (b) $T_n = (-1)^n\left(5^n\right)$

16 **4** For the following geometric sequences, devise a general term that defines the sequence. Use this to determine the value of indicated term.

(a) 1, 2, 4, …; 10th term (b) 27, 18, 12, …; 7th term

(c) 1, -10, 100, …; 8th term (d) $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$; 8th term

(e) $xy^2, x^2y^3, x^3y^4, \ldots$; 7th term (f) $\frac{1}{8}, -\frac{1}{12}, \frac{1}{18}, \ldots$; 7th term

(g) $\frac{1}{5}, -\frac{1}{\sqrt{5}}, 1, -\sqrt{5}, \ldots$; 10th term

5 Consider the geometric sequence 48, -32, $21\frac{1}{3}$, $-14\frac{2}{9}$, …

(a) The common ratio and the 6th term are:

A $r = \frac{3}{2}, t_6 = 32$ B $r = -\frac{2}{3}, t_6 = -\frac{512}{81}$ C $r = -\frac{3}{2}, t_6 = 32$ D $r = -\frac{2}{3}, t_6 = \frac{1024}{243}$

(b) Explain the common error made by a student who thought the answer was $r = -\frac{3}{2}, t_6 = -32$.

17 **6** Calculate the sum of the given number of terms for each of the following geometric series.

(a) $1 + 2 + 4 + \ldots$ to 12 terms

(b) $8 - 4 + 2 - \ldots$ to 10 terms (as an improper fraction)

(c) $3 - 9 + 27 - \ldots$ to 8 terms

(d) $5 + 5.5 + 6.05 + \ldots$ to 10 terms (3 d.p.)

(e) $-\frac{2}{5} + \frac{1}{2} - \frac{5}{8} + \ldots$ to 9 terms (3 d.p.)

(f) $4 + 1 + \frac{1}{4} + \ldots$ to 12 terms (3 d.p.)

18 **7** For each of the following geometric series, determine whether the sum-to-infinity, S_∞, exists. Calculate this value, if it exists.

(a) $0.1 + 0.2 + 0.4 + \ldots$ (b) $40 - 32 + 25.6 + \ldots$ (c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ (d) $5 + \frac{3}{2} + \frac{9}{20} + \ldots$

19

8 Express each recurring decimal as an infinite geometric series and use the sum-to-infinity to determine the equivalent value as a fully simplified fraction.

(a) $0.\dot{3}$ (b) $0.\dot{0}\dot{8}$ (c) $0.\dot{1}8\dot{5}$

9 Determine the value of p, given that each of the following are three consecutive terms of a geometric sequence.

(a) $4, p, 9$ (b) $18, 27, p$ (c) $px, 4px^2, 16px^3$

(d) $p, -4\frac{1}{2}, -1$ (e) $2p - 5, p - 4, 10 - 3p$

10 Which term has the value $\frac{32}{9}$ in the geometric sequence $\frac{81}{2}, 27, 18, \ldots$?

A t_7 B t_5 C t_6 D t_8

11 Determine the first three terms of the geometric sequence defined by each of the following formulas.

(a) $t_n = 2\left(5^{n-1}\right)$ (b) $p_n = 4\left(\frac{1}{\sqrt{3}}\right)^{n-1}$

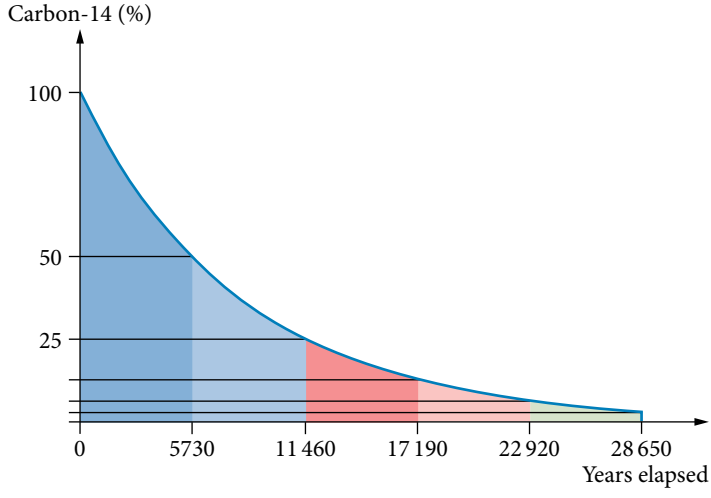
(c) $t_n = (-2)^n\left(6^n\right)$ (d) $f_n = 2xm^{n+2}$

12 Radiocarbon dating (or simply 'carbon dating') is a method used by scientists to calculate the age of organic material such as fossils. This method is based on the radioactive decay of atoms of the carbon-14 isotope (or ^{14}C). Carbon-14 has a half-life of 5730 years. This means that 50% of any sample will have decayed after this length of time. The following graph shows the decay curve of ^{14}C.

(a) A sample of dead wood contains 50 g of ^{14}C. Determine the mass remaining after the following time.

(i) 5730 years (ii) 11 460 years (iii) 17 190 years

(b) If there is 12.5% of ^{14}C remaining in an animal bone, determine how many half-lives this bone has experienced since the animal died.



20

13 Determine the minimum number of terms for the sum of the geometric series $4, 6, 9, 13\frac{1}{2}, \ldots$ to exceed 6000.

14 Determine the number of terms of the series $6 + 3 + \frac{3}{2} + \ldots$ to give a sum of $11\frac{13}{16}$.

21

15 The population of a town is 31 000 and the growth rate is currently 4.56% per annum. If this trend in population growth continues, determine:

(a) the town's population in 10 years (to the nearest person)

(b) the year in which the population of the town reaches 100 000.

16 By expressing the following recurring decimals as infinite geometric series and using the formula for the limiting sum (or sum-to-infinity), find the equivalent values as fully simplified fractions.

(a) $0.08\dot{3}$

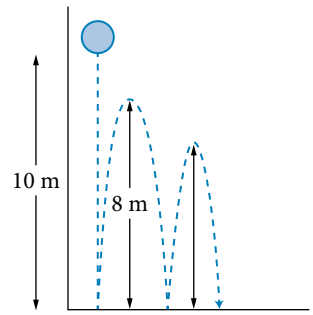
(b) $0.2\dot{3}\dot{7}$

17 A geometric sequence has a common ratio of $\frac{1}{2}$ and the sum of the first five terms is $11\frac{5}{8}$. Find the first three terms of the sequence.

18 If the 4th term of a geometric sequence is 96 and the 5th term is 384, then determine t_{10} and S_{10}.

19 In a geometric sequence, the sum of the first 10 terms is $\frac{341}{64}$ and $r = -\frac{1}{2}$. Determine the value of the first term.

20 A ball is dropped from a height of 10 m and bounces back to a height of 8 m. Each time the ball bounces it reaches 0.8 of its previous height. Show that the total vertical distance travelled by the ball is 90 m. You will need to consider both directions of travel (upward and downward).



21 For each of the following sequences, write an expression for the required terms using index notation, where appropriate.

(a) $2 + 3, 5 + 9, 8 + 27 \ldots$

(i) 12th term

(ii) nth term

(b) $\frac{1}{4} + \frac{1}{125}, \frac{1}{2} + \frac{1}{25}, \frac{3}{4} + \frac{1}{5} \ldots$

(i) 10th term

(ii) nth term

(c) $3 \times 4, 3^3 \times 4^2, 3^5 \times 4^4 \ldots$

(i) 15th term

(ii) nth term

Financial mathematics 6.4

Review of simple interest

Simple interest is calculated directly on the *principal*, which is the initial investment. It is called 'simple' because the interest is constant.

Simple interest

$I = Pin$, where:

I is the amount of interest paid ($) (or charged if you are borrowing money)

P is the principal ($) (the amount of money invested or borrowed)

n is the number of periods for which the interest is paid

i is the interest rate, expressed as a decimal for the same time period as n.

$A = P + I$, where A is the account balance ($).

Compound interest

Compound interest has many more applications in financial situations.

For example, consider $2000 deposited in an account at an interest rate of 6% p.a. compounded annually for 3 years.

Amount after 1 year	Amount after 2 years	Amount after 3 years
$A_1 = 2000(1.06)$ $= \$2120$	$A_2 = 2000(1.06)(1.06)$ $= 2000(1.06)^2$ $= \$2247.20$	$A_3 = 2000(1.06)^2(1.06)$ $= 2000(1.06)^3$ $= \$2382.03$
$A_1 = 2000 \times (1.06)$	$A_2 = 2000 \times (1.06)^2$	$A_3 = 2000 \times (1.06)^3$

The amounts at the *end* of each year form a geometric sequence, because the ratio of successive terms is constant:

So, $A_n = 2000 \times (1.06)^n$, where $a = 2000$ and $r = 1.06$ and the number of terms is $n + 1$.

Interest is usually compounded more often than once a year, so the compound interest formula is written as follows.

Compound interest

$A = Pr^n = P(1+i)^n$, where:
A is the amount or balance
n is the number of times interest is added
P is the principal
r is the growth factor, equivalent to $(1 + i)$, where i is the interest rate per period, expressed as a decimal.

If the total amount of interest is required, then subtract the principal from the final amount: $I = A - P$

Additional information

Compound interest

Investigate the compound interest formula and solve a short problem.

22 Calculate the final value and interest earned

\$10 000 was invested for 20 years at 6% p.a. compounded half-yearly.

(a) Determine the final balance.

THINKING	WORKING
1 Determine the values of P, i and n.	$P = \$10\,000$ $n = 20 \times 2 = 40$ half-years $i = 0.06 \div 2 = 0.03$ half-yearly
2 Substitute the given values into the compound interest formula: $A = P(1+i)^n$	$A = P(1+i)^n$ $= 10\,000\left(1+\frac{0.06}{2}\right)^{2\times 20}$ $= 10\,000(1.03)^{40}$ $= \$32\,620.38$
3 Interpret the answer.	The final value of the investment (account balance) is \$32 620.38.

(b) Calculate the interest earned over the 20 year period.

1 Recall the interest formula $I = A - P$, substitute the known values and calculate I.	$I = A - P$ $= 32\,620.38 - 10\,000$ $= \$22\,620.38$
2 Interpret the answer.	\$22 620.38 interest was earned.

23 Work with compound interest

A principal \$$P$ is invested at 4% p.a. compound interest, compounded annually.

(a) Determine the minimum number of years for the initial amount to double.

THINKING	WORKING
1 Identify the given information.	The unknown value is n. $i = 0.04$

2 Recall the compound interest formula $A = P(1+i)^n$. Use technology to solve for n.	To double: $A = 2P$ $\therefore 2P = P \times (1.04)^n$ $2 = 1.04^n$ $n = 17.67$
3 Interpret the answer.	It will take 18 years for the balance to double.

(b) Determine the minimum number of years for the balance to increase to five times the initial amount.

1 Substitute into $A = P(1+i)^n$. Use technology to solve.	To quintuple: $A = 5P$ $5P = P \times (1.04)^n$ $5 = 1.04^n$ $\log(5) = \log(1.04)^n$ $\log(5) = n\log(1.04)$ $n = \dfrac{\log(5)}{\log(1.04)}$ $n = 41.04$ (2 d.p.)
2 Interpret the answer.	You need to invest for 42 years to increase to 5 times the initial amount.

Inflation

Money loses its buying power over time with increased prices. This is called inflation. If inflation remains at a fairly constant rate, then you can calculate the expected value of goods and services in the future.

24 Compensate for inflation

If inflation remains constant at 3% p.a., then determine each of the following.

(a) Estimate the cost of a \$1500 artwork in 5 years.

THINKING	WORKING
1 Recall the compound interest formula $A = P(1+i)^n$ and calculate A as the future value.	$A = 1500(1.03)^5$ $= \$1739$ (nearest \$)
2 Interpret the answer.	The artwork would cost \$1739 in 5 years.

(b) Calculate the present value of an item if it would cost \$80 000 in 10 years' time.

1 Recall the compound interest formula $A = P(1+i)^n$ and calculate P as the present value.	$80\,000 = P(1.03)^{10}$ $P = \dfrac{80\,000}{(1.03)^{10}}$ $P = \$59\,528$ (nearest \$)
2 Interpret the answer.	The present value is \$59 528.

25 Compare simple and compound interest

Jo wants to invest \$13 000 for 4 years and has two options.

(a) Calculate the interest earned if the money is invested at 8% p.a. simple interest.

THINKING	WORKING
Use the simple interest formula to calculate the interest earned I.	$I = Pin$ $= 13000 \times 0.08 \times 4$ $= \$4160$

(b) Calculate the interest earned if the money is invested at 6.2% p.a. compounding monthly.

1 Determine the values of i and n for the compound interest formula.	Compounded monthly $n = 4 \times 12 = 48$ months $i = \frac{0.062}{12}$ per month
2 Use the compound interest formula to calculate the total amount of the investment.	$A = P(1+i)^n$ $= 13\,000\left(1+\frac{0.062}{12}\right)^{4\times 12}$ $= \$16\,648.35\ (2\text{ d.p.})$
3 Calculate the interest earned so values can be compared.	$I = A - P$ $= 16\,648.35 - 13\,000$ $= \$3648.35$

(c) Compare the options to determine which is the better investment and by how much.

Calculate the difference in interest earned, and interpret the result.	$4160 - 3648.35 = \$511.65$ The simple interest option is the better investment, earning \$511.65 more over the 4 years.

Depreciation

Goods purchased lose value over time due to wear and tear and becoming outdated. The compound interest formula can be used to calculate the value of items that decrease in value by a constant percentage each year.

The annual amounts form a geometric sequence with a common ratio less than 1.

$A_d = Pr^n = P(1-i)^n$, where:
A_d is the depreciated value of the item
P is the principal or the initial value of the item
n is the number of depreciation time periods (usually years)
r is the depreciation factor, equivalent to $(1-i)$, where i is the interest rate per period expressed as a decimal.

26 Calculate depreciation

A boat purchased for \$40 000 depreciates at a rate of 12% per year. How much will it be worth after 5 years?

THINKING	WORKING
1 Using the depreciation formula, substitute the values to calculate the depreciated worth using the formula: $A_d = P(1-i)^n$	$P = \$40\,000$ $n = 5$ years $i = 12\%$ per year $= 0.12$ $A_d = P(1-i)^n$ $= 40\,000(1-0.12)^5$ $= 40\,000(0.88)^5$ $= \$21\,109$ (nearest \$)
2 Interpret the answer.	After 5 years, the boat has depreciated in value to \$21 109.

Annuities

An annuity is a series of equal payments made annually (although any consistent period is allowed) with interest paid annually into the account. By treating each payment separately, you can use the formula for the sum of a geometric sequence to calculate the final value of the investment.

27 Annuity amount using the sum of a geometric sequence

On Lee's first birthday, her parents open a bank account and deposit \$1000. The annual interest rate is 7%, compounded annually and added at the end of each 12-month period. The same amount is deposited into this account every birthday up to (but not including) Lee's 21st birthday. How much will be in this account immediately after her 21st birthday?

THINKING	WORKING
1 Use the formula for calculating final amounts: $A = Pr^n = P(1+i)^n$ Determine the final value of the first few deposits on 21st birthday.	\$1000 deposited 1st birthday: $1000 \times (1.07)^{20}$ \$1000 deposited 2nd birthday: $1000 \times (1.07)^{19}$ \$1000 deposited 3rd birthday: $1000 \times (1.07)^{18}$ The 20th \$1000 is worth: $1000 \times (1.07)^1$
2 Create a geometric series to check for a pattern.	$A_{20} = 1000\left(1.07 + 1.07^2 + 1.07^3 + \ldots + 1.07^{20}\right)$ The terms inside the brackets represent a geometric series of 20 terms with $t_1 = 1.07$ and $r = 1.07$.
3 The terms can be added using the sum formula for geometric series: $S_n = \dfrac{t_1\left(r^n - 1\right)}{r-1}$	$1000 \times S_{20} = 1000\dfrac{(1.07)\left(1.07^{20} - 1\right)}{1.07 - 1}$ $= 43\,865.18$ (2 d.p.)
4 Interpret the answer.	There will be \$43 865.18 in the account.

Ordinary annuities

An ordinary annuity is a series of equal payments at the end of equal time periods.

The amount of an annuity is the sum of all payments, with accrued interest. There is no money in the account during the first period, and the final payment has no time to accrue interest. Using the sum of a geometric sequence you can create a formula for the future value of such an investment.

Future value (FV) of an (ordinary) annuity:

$$\begin{aligned}\text{FV} &= P + Pr^1 + Pr^2 + \ldots + Pr^{n-1}\\ &= \frac{P\left(r^n - 1\right)}{r-1}\\ &= \frac{P(1+i)^n - P}{i}\\ &= P \times \frac{(1+i)^n - 1}{i}\end{aligned}$$

In a similar way you can calculate the present value (PV) of regular payments made at the end of each year, adjusting for inflation.

Present value of an annuity:

$$\begin{aligned}\text{PV} &= \frac{P}{r} + \frac{P}{r^2} + \ldots + \frac{P}{r^n}\\ &= \frac{P(1-r^{-n})}{r-1}\\ &= P \times \frac{1-(1+i)^{-n}}{i}\end{aligned}$$

28 Use the annuity formulas

Use the future value and present value of an annuity formula in each of the following situations.

(a) Determine the value of a lump sum payment in 6 years equivalent to equal payments of \$4500 at the end of each year for the same period if interest is paid at 7% p.a.

THINKING	WORKING
1 Identify the value of the payments P, the number of periods n and the interest rate per period r.	$P = \$4500$ $n = 6$ years $i = 0.07$ per year $r = 1.07$
2 Recall the formula to calculate future value: $\text{FV} = \frac{P\left(r^n - 1\right)}{r-1}$ Substitute the values into the formula.	$\text{FV} = P \times \frac{r^n - 1}{r-1}$ $= 4500 \times \frac{(1.07)^6 - 1}{0.07}$ $= 32\,189.81\ (2\text{ d.p.})$
3 Interpret the answer.	An equivalent lump sum at the end of the 6 years would be \$32 189.81.

4 Check that the answer is reasonable.	$\$4500 \times 6 = \$27\,000$, so an amount is higher than this was to be expected.

(b) \$100 000 is to be paid to you over a period of 10 years in equal instalments at the end of every year. Determine the present value of these payments, given that inflation is estimated to be 3.2% p.a. over that time.

1 Identify the value of the payments P, the number of periods n and the interest rate per period r.	$P = \$100\,000 \div 10 = \$10\,000$ $n = 10$ years $i = 0.032$ per year $r = 1.032$
2 Recall the formula to calculate present value: $\text{PV} = \dfrac{P\left(1 - r^{-n}\right)}{r - 1}$ Substitute the values into the formula.	$\text{PV} = P \times \dfrac{1 - r^{-n}}{r - 1}$ $= 10\,000 \times \dfrac{1 - (1.032)^{-10}}{0.032}$ $= \$84\,437.94\ (2\text{ d.p.})$
3 Interpret the answer.	The payments have a present value of \$84 437.94.
4 Check that the answer is reasonable.	The present value of \$84 437.94 is less than \$100 000, as expected.

EXERCISE 6.4 Financial mathematics

Worked Example 22

1 \$12 000 is invested for 7 years at 8% p.a. compounded quarterly.
   (a) Determine the final balance.
   (b) How much interest was earned during this time?

2 Calculate the final balance after 3 years if \$1200 is invested at 7.5% p.a. compounded at the following frequencies. In each case, give your answer correct to the nearest cent.
   (a) annually
   (b) half-yearly
   (c) quarterly
   (d) monthly
   (e) daily (assuming 365 days in a year)

3 Complete the following table, giving amounts correct to the nearest cent.

Compound interest	Principal	Rate (p.a.)	Compounding period	Time (years)	Final balance
	\$5 000	6%	monthly	4	
		9.5%	daily	3	\$10 000
\$59.60	\$800	7.25%		1	

Worked Example

23 4 A principal $\$P$ is invested at 7% p.a. compounded annually.

(a) How many full years will it take for the balance to double?

(b) How many full years will it take for the balance to be triple the original principal?

(c) How many full years will it take for the balance to be seven times the original principal?

24 5 Assuming that inflation is steady at 2.75% p.a.:

(a) estimate the cost (to the nearest \$) of a \$3600 flat screen TV in 2 years

(b) calculate the present value (to the nearest \$) of \$12 000 to be paid 6 years from now.

6 Complete the following table, giving amounts to the same degree of accuracy as the given values.

Present value	Inflation rate (p.a.)	Time (years)	Future value
\$250	1.6%	5	
	3.4%	3	\$1420
\$610		2	\$700
\$1100	2.9%		\$1233

7 A computer purchased for \$2780 depreciates in value by 15% each year.

(a) The expression that gives the computer's value 3 years after its purchase is:

A $\$2780 - 3 \times 15$ B $\$2780 \times 0.85^3$ C $\$2780 \times 1.15^3$ D $\dfrac{\$2780}{0.85^3}$

(b) Explain the common error made by a student who thought the answer was $\dfrac{\$2780}{1.15^3}$.

8 A teenager is set to inherit \$25 000 when he turns 21 in 4 years' time. The expression that gives the present value of the inheritance, assuming inflation of 3.5% p.a. is:

A $\dfrac{\$25\,000}{1.035^4}$ B $\$25\,000 \times 0.965^3$ C $\$25\,000 \times 1.035^3$ D $\dfrac{\$25\,000}{0.965^3}$

26 9 A laptop computer is purchased for \$1800. If it depreciates at the rate of 15% each year, how much will it be worth after 4 years? Give your answer correct to the nearest dollar.

28 10 Use the future value and present value of an ordinary annuity formula in each of the following situations: $\text{FV} = P \times \dfrac{r^n - 1}{r - 1}$ and $\text{PV} = P \times \dfrac{1 - r^{-n}}{r - 1}$

(a) Determine the value of a lump sum payment in 5 years equivalent to equal payments of \$7200 at the end of each year for the same period if interest is paid at 5.8% p.a.

(b) If \$1 000 000 is to be paid to you over a period of 20 years in equal instalments at the end of every year, determine the present value of these payments, given that inflation is anticipated to be 4.2% p.a. over that time.

Worked Example

25

11 Hanh was given $3000 on his 16th birthday. He would like to deposit it into an account that would give him the best return by his 21st birthday. There are two different investment opportunities he is considering:
Option 1: 8% p.a. simple interest
Option 2: 6.5% p.a. compound interest, compounded half-yearly.
Which is the better investment and by how much?

12 Siobhan inherited $20 000 from a relative. She invests the money at a fixed rate of 5% p.a. for 10 years.

(a) Determine the interest earned, if this was simple interest.

(b) Determine the interest earned, correct to the nearest cent, if this was compound interest, compounded as follows.

(i) yearly (ii) half-yearly (iii) monthly (iv) daily

(c) Calculate the percentage difference in the interest earned (to the nearest percentage) if the interest was simple interest compared to compound interest compounded yearly. Base the percentage on the simple interest amount.

(d) Calculate the percentage difference in the interest earned (to the nearest percentage) if the interest was compounded annually compared to compounded daily. Base the percentage on the annual compounding interest amount.

13 Sound engineer, Benjamin, buys an amplifier for $1350 and plans to claim depreciation of the equipment at 10% of its initial value every year for 10 years. He estimates that the equipment actually loses 15% of its value each year. At what time, in whole years, will the 'paper' value, used for tax purposes, be closest to the actual value? Say what the two values are, and justify your working with a graph.

14 You deposit $200 from your first pay into a high-interest account paying 5.8% p.a. compounding monthly. If you continue to deposit the same amount at the beginning of every month, how much would you expect to have after 5 years?

27

15 You receive 1 cent on the first day of the month in which you were born, then 2 cents on the second day, 4 cents on the third day, and so on until the end of the month.

(a) How much would you receive in total if born in January?

(b) How much would you receive in total if born in April?

(c) If you were born in February, how much more would you receive in a leap year compared to the other years?

16 You deposit $1500 into an account at the beginning of every month.

(a) How many deposits would it take for the end of the month account balance to pass the one million dollar mark, given an interest rate of 6.3% p.a. compounding monthly.

(b) Using just the money from this account, would you be able to purchase a house, currently worth $550 000 if inflation was steady at 3.6% p.a.?

17 A computer game purchased for $45 on 10 May loses 2% of its value every day.

(a) On what date will the game be worth $20? (b) On what date will the game be worth $2?

6 Summary

Index laws

$a^m \times a^n = a^{(m+n)}$ $\quad \dfrac{a^m}{a^n} = a^{(m-n)}$

$a^0 = 1$ $\quad \dfrac{1}{a^m} = a^{-m}$ $\quad \sqrt[n]{a} = a^{\frac{1}{n}}$

$\left(a^m\right)^n = a^{(m\times n)}$ $\quad (ab)^m = a^m b^m$ $\quad \left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Scientific notation and significant figures

Scientific notation, or standard form, expresses a value as the product of a number between 1 and 10 and a power of 10. The precision of a measurement is determined from this form.

The number of significant figures is the same as the number of digits in the first part of a number in standard form. For example, $1730 = 1.73 \times 10^3$ has 3 significant figures and $0.000\,21 = 2.1 \times 10^{-4}$ has 2 significant figures.

Geometric sequences and series

A *geometric sequence* is a sequence that has a common ratio between successive terms.

The general term of a geometric sequence is defined as: $t_n = t_1 r^{n-1}$ for $n \geq 1$, where r is the common ratio and a is the first term, t_1.

The *recursive definition* of a geometric sequence is $t_{n+1} = rt_n$, for $n \geq 1$.

To prove that a sequence is geometric in nature, the common ratio between terms must be constant. Therefore, $\dfrac{t_{n+1}}{t_n} = r$ for $n \geq 1$

To find the sum of n terms in an geometric sequence, use the formulas:

$S_n = \dfrac{t_1\left(1 - r^n\right)}{1 - r}$, $r < 1$ or $S_n = \dfrac{t_1\left(r^n - 1\right)}{r - 1}$, $r > 1$

The sum-to-infinity formula for a geometric series is: $S_\infty = \dfrac{t_1}{1-r}$ for $-1 < r < 1$

Simple interest

$I = Pin$, where:

I is the amount of interest paid ($) (or charged if you are borrowing money)

n is the number of periods for which the interest is paid

P is the principal ($) (the amount of money invested or borrowed)

i is the interest rate expressed as a decimal for the same time period as n.

$A = P + I$, where A is the account balance ($).

Compound interest

$A = Pr^n = P(1+i)^n$, where:
A is the amount or balance at a given time
n is the number of times interest is added
P is the principal
r is the growth factor, as a decimal for the same time period as n
i is the interest rate, expressed as a decimal for the same time period as n.

Depreciation

$A_d = Pr^n = P(1-i)^n$, where:
A_d is the final value
n is the number of depreciation time periods
P is the initial value
r is the depreciation factor as a decimal for the same time period as n
i is the interest rate, expressed as a decimal for the same time period as n.

Annuities

An ordinary annuity is a series of equal payments at the end of equal time periods.

The amount of an annuity is the sum of all payments, with accrued interest.

Future value of an annuity:

$$\begin{aligned}\text{FV} &= P + Pr^1 + Pr^2 + \ldots + Pr^{n-1}\\ &= \frac{P\left(r^n - 1\right)}{r-1}\\ &= \frac{P(1+i)^n - P}{i}\\ &= P \times \frac{(1+i)^n - 1}{i}\end{aligned}$$

Present value of an annuity:

$$\begin{aligned}\text{PV} &= \frac{P}{r} + \frac{P}{r^2} + \ldots + \frac{P}{r^n}\\ &= \frac{P - Pr^{-n}}{r-1}\\ &= P \times \frac{1-(1+i)^{-n}}{i}\end{aligned}$$

Chapter review 6

Exercise 6.1

1 Simplify the following expressions.

(a) $\dfrac{a^3b}{b^4} \times \dfrac{ab^2}{a^5}$ (b) $\left(5a^{\frac{1}{3}}\right)^2$ (c) $\sqrt{36a^3b^5} \div \left(8a^2b^4\right)^{\frac{1}{3}}$

Exercise 6.1

2 Simplify $\dfrac{\left(2x^2y^3\right)^2}{5x^2} \times \dfrac{3xy^2}{4x^3y^7}$.

Exercise 6.1

3 Simplify $\dfrac{3a^4b^2}{2a^2b^3} \div \dfrac{(3ab)^2}{5a^5b^3}$.

Exercise 6.2

4 Express 72100 in scientific notation.

A 72.1×10^3 B 7.21×10^4 C 721×10^3 D 7.21×10^{-4}

Exercise 6.2

5 What is 3.027×10^{-3} equivalent to?

A 0.000302 B 0.0003027 C 3027 D 0.003027

Exercise 6.2

6 If 250 g of pool chlorine is added to a pool containing 4.87×10^7 L of water, determine the increase in concentration of chlorine, in gL^{-1}, in scientific notation correct to 3 significant figures.

Exercise 6.3

7 Write the next three terms of the sequence $\dfrac{1}{5}$, 1, 5, 25, … .

Exercise 6.3

8 Determine a rule for the sequence $\dfrac{1}{10}$, 1, 10, 100, … by expressing t_n in terms of n.

Exercise 6.3

9 Generate the first four terms of each geometric sequence, given the first term and the common ratio.

(a) $t_1 = -6, r = \dfrac{1}{2}$ (b) $t_1 = 0.4, r = 5$

Exercise 6.3

10 Give the first three terms, and the eigth term, of the sequence defined by the general term $a_n = 3(2)^{n-1}$.

Exercise 6.3

11 Determine whether each of the following represents a geometric sequence, and if it is, state the value of the common ratio.

(a) $5x, 5x^2, 5x^3 \ldots$ (b) 3, -9, 27, -81, … (c) $A_n = \dfrac{n}{n+2}$ (d) $\sqrt{2}, 2, 2\sqrt{2}, 4$

Exercise 6.3

12 Determine the sum of the first 10 terms of the series $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots$

Exercise 6.3

13 Determine the sum-to-infinity of the geometric series: $12 + 2 + \dfrac{1}{3} + \dfrac{1}{18} + \ldots$

Exercise 6.3

14 Determine whether each of the following is a geometric sequence. If so, find the common ratio, and the next term, in decimal form.

(a) -8, 4, -2, 1, … (b) 25, 37.5, 56.25, 84.375 (c) 8, 20, 40, 80, …

Exercise 6.4

15 Abeke buys her first car for \$12399. If its value depreciates by 12.5% per year, what will her car be worth, to the nearest dollar, after 5 years?

Exercise 6.4

16 \$8000 is invested in an account with an interest rate of 8% p.a. compounded quarterly. How much interest will be earned in 3 years?

17 Use the future value of an ordinary annuity formula, $\text{FV} = P \times \frac{r^n - 1}{r - 1}$, to determine the value of a lump sum payment in 10 years equivalent to equal payments of \$8600 at the end of each year for the same period if interest is paid at 6.4% p.a. Exercise 6.4

18 What is the result when $\left(\frac{m^3}{n^5}\right)^{-2} \times \frac{3^{-2}m^2n}{mn^3}$ is simplified and expressed with positive powers? Exercise 6.1

A $\frac{9n^8}{m^5}$ **B** $\frac{m^7}{9n^{12}}$ **C** $\frac{n^8}{9m^5}$ **D** $\frac{9m^7}{n^{12}}$

19 What is the result when $\left(\frac{343}{8}\right)^{-\frac{2}{3}}$ is simplified? Exercise 6.1

A $\frac{4}{49}$ **B** $\frac{7}{2}$ **C** $\frac{49}{4}$ **D** $\frac{2}{7}$

20 X, 3, Y are consecutive terms of a geometric sequence. If the sum of these three terms is 13, determine the values of X and Y. Exercise 6.3

21 Express $0.15\dot{7}$ as an infinite geometric series and then use the sum-to-infinity to find the equivalent value as a simplified fraction. Exercise 6.3

22 Determine the difference in the amount of interest earned on an investment of \$10 000 at 5% p.a. for 6 years if the interest is compounded daily compared to annually. Exercise 6.4

23 The value of a new car depreciates 20% as soon as it is purchased, and then by a further 10% every year. What is the value of the car after 3 years, if it was purchased for \$65 000? Exercise 6.4

24 Hans opened a long-term account on 1 February 1990 and deposited \$500. He decided to deposit \$500 on the first day of every month. If interest is held at 9% per year, compounded monthly and paid on the last day of the month, how much will be in his account on 31 January 2020? Exercise 6.4

25 A \$4000 cheque was deposited into an account with an interest rate of 6% p.a. compounded monthly for 3 years. Another account offered 9% p.a. simple interest. Calculate the deposit needed to earn the same amount of interest in this other account over 3 years. Exercise 6.4

26 A \$10 000 large-screen TV depreciates 10% a year after it is purchased. Each year thereafter, its value depreciates by 5% of the previous value. Exercise 6.4

(a) Calculate value of the TV after 5 years.

(b) Determine its value after n years.

(c) After how many years will the TV's value be less than 25% of its purchase price?

27 An ordinary annuity is a series of equal payments at the end of equal time periods. The present value of an annuity of equal payments, P over n periods is given by the formula: Exercise 6.4

$\text{PV} = P \times \frac{1 - r^{-n}}{r - 1}$, where r is the interest rate, as a decimal, for the compounding period.

(a) Create a formula for the present value of a different type of annuity where the payment is made at the beginning (rather than the end) of each period.

(b) Calculate the difference in the present values between the two types of annuity when equal payments of \$3000 are made annually over 15 years at an interest rate of 8.1% p.a.

Exam review

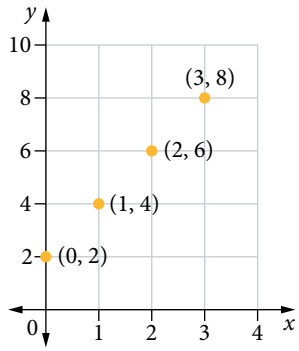
UNIT 1

Paper 1: Technology-free

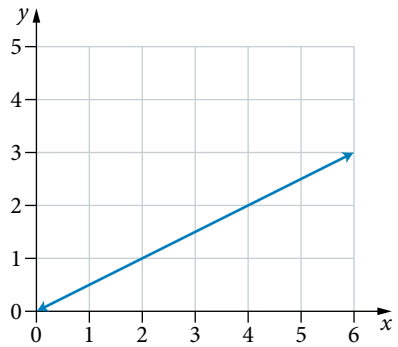
Exercise 1.1

1 Determine whether the following graphs represent direct variation. Determine a rule for each linear function, including the domain.

(a)



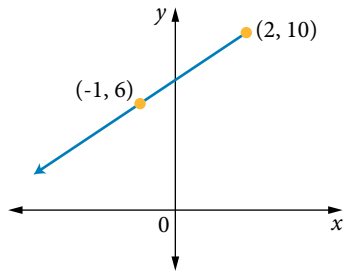
(b)



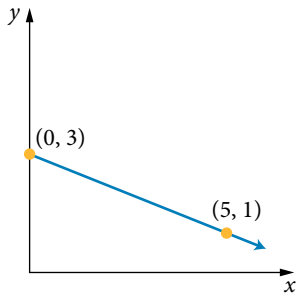
Exercise 1.1

2 Calculate the gradient of each of the following lines.

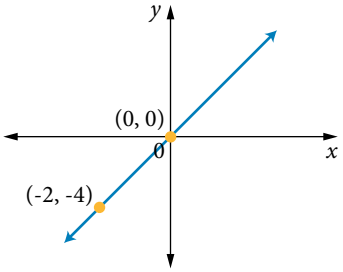
(a)



(b)



(c)



Exercise 2.2

3 Determine the rule of the quadratic function whose graph is obtained from that of the basic quadratic function $y = x^2$ by a translation of 3 units to the right and a translation of 6 units down.

Exercise 3.1

4 Determine the degree of each polynomial and calculate the function values.

(a) $P(x) = (3x-2)^2(x+4)$; $P(3)$

(b) $Q(x) = 4x^3 - 2x^5 + 5x^4 + 3 - 2x$; $Q(-2)$

(c) $R(x) = (2x-1)^3(x+4)^2$; $R(-1)$

(d) $S(x) = 3x^2(2x-1)(x+2)^3$; $S(2)$

(e) $T(x) = (x-2)(x+1)(x-3)(x+4)$; $T(-4)$

(f) $V(x) = (x-4)(x+1)^5(x+2)^2$; $V(-4)$

Exercise 3.1

5 For $P(x) = x^2 - 3x + 2$, $Q(x) = 3x - 1$ and $R(x) = x^2 + 8$ determine each of the following.

(a) $P(5)$

(b) $Q(-4)$

(c) $Q(16)$

(d) $R(-8) - P(-5)$

(e) $R(6) + P(7) - Q(8)$

(f) $P(2x) + Q(2x)$

(g) n if $P(n) + Q(n) = 26$

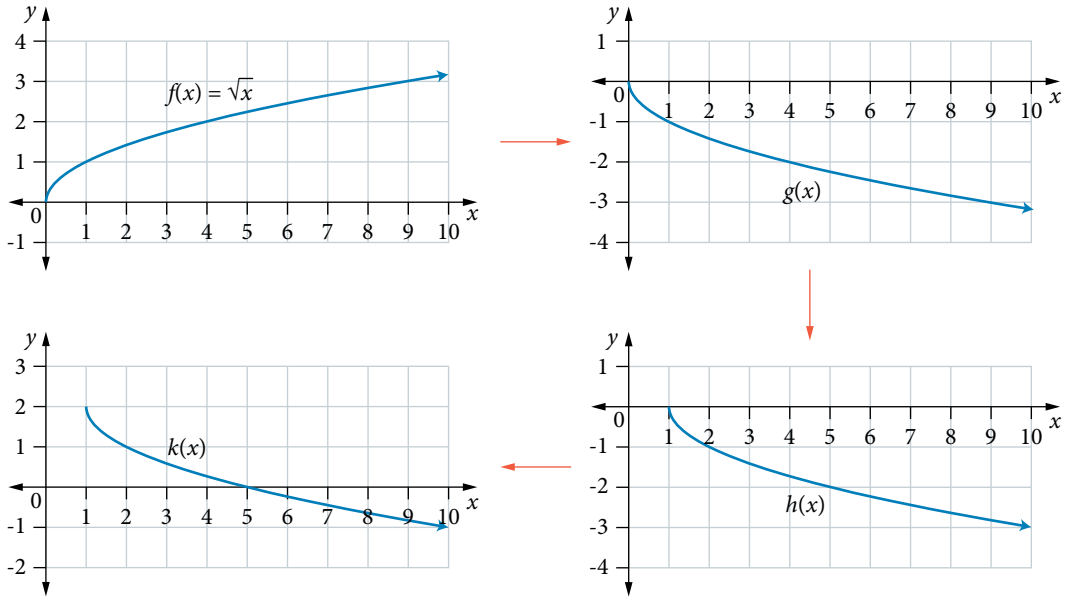
(h) n if $R(n) = 57$

Exercise 3.2

6 If $2x^3 - 3x^2 - 5x + 6 = (x-2)(ax^2 + bx + c)$, then determine the values of a, b and c.

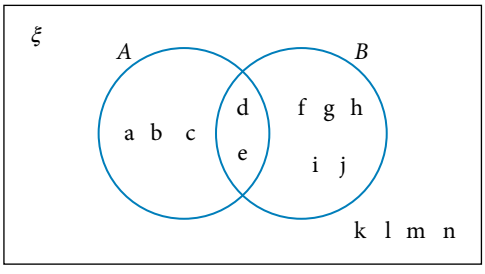
7 The graphs show how $f(x) = \sqrt{x}$ is transformed to the graph of $g(x)$ and then transformed to $h(x)$ and then transformed to $k(x)$. Determine the transformation and the equation for the new function. Exercise 4.2

(a) $f(x)$ to $g(x)$ **(b)** $g(x)$ to $h(x)$ **(c)** $h(x)$ to $k(x)$



8 Sketch the graph of $y = -\sqrt{4-(x+1)^2} + 2$. Determine the centre, endpoints and any intercepts on the axes, in exact form. Exercise 4.3

9 Using the information in the Venn diagram, list the elements in each of the following sets. Exercise 5.2



(a) $A' \cap B$ **(b)** $B' \cap A$

(c) $A \cup B$ **(d)** $A \cup B'$

(e) $(A \cap B)'$ **(f)** $A' \cap B'$

10 Simplify each of the following. Exercise 6.1

(a) $mn^2 \times \left(2m^2n^3\right)^2$ **(b)** $\dfrac{5\left(a^2b\right)^3}{a^3b^2}$ **(c)** $\dfrac{3x^2yz^5}{4x^5} \times \dfrac{2x^6y^2z}{6\left(xy^2\right)^2}$ **(d)** $\dfrac{3ab^6}{(ab)^2} \div \left(\dfrac{a^2b}{2ab}\right)^3$

11 The parabolas $y = 2x^2 - 6x + 1$ and $y = 10 - x^2$ intersect at two places. Construct a continuous piece-wise function over all real numbers, which begins with the concave up parabola, ends with the concave down parabola, and joins the two with the line through their points of intersection. Exercise 2.1

12 The function $f(x) = \frac{1}{x}$ is dilated by a factor of 2 parallel to the y-axis, then reflected in the x-axis and translated 3 units left to produce $g(x)$. Determine the rule for $g(x)$ and show $f(x)$ and $g(x)$ on the same set of axes showing all major points. Exercise 4.1

Paper 2: Technology-active

Exercise 1.2

1 The formula for converting temperature in degrees Celsius (°C) to degrees Fahrenheit (°F) is $F = \frac{9}{5}C + 32$.

(a) Does this formula represent a function? **(b)** Convert 30°C to °F.

(c) Convert 98°F to °C. Give your answer correct to 1 decimal place.

Exercise 2.2

2 Determine the coordinates of the turning point (or vertex) for each function.

(a) $m = (p+8)^2$ **(b)** $f(x) = -\frac{2}{3}(x-3)^2 + 1$ **(c)** $s = 3t^2 - 9$

(d) $y = -9x + 3 - 4x^2$ **(e)** $m = -9.8g^2 + 3.2$ **(f)** $A = \frac{4}{3}r^2$

Exercise 2.3

3 Solve the equation $3 = 7x^2 - 5x + 2$, giving both exact solutions and approximations correct to 2 decimal places.

Exercise 5.1

4 A six-sided die is biased in such a way that $P(1) = P(2) = P(3)$, $P(4) = P(5) = P(6)$ and $P(4) = 2 \times P(1)$. Determine the probability of rolling:

(a) a 3 **(b)** an even number.

Exercise 5.6

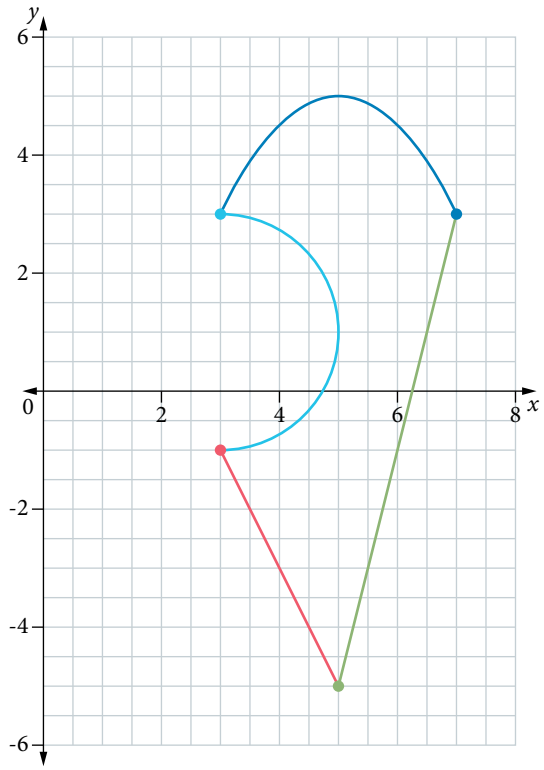
5 A committee of five is to be formed from a group of seven girls and nine boys. If the committee must contain exactly two boys, what is the number of possible committees?

Exercise 3.3

6 Factorise $P(x) = 2x^3 - 5.6x + 1$ by first graphing the function. Round to 2 decimal places where necessary.

Exercise 4.3

7 Determine the equations of the relations required to create the enclosed company logo shown.



8 A game involves tossing two coins: a fair coin and a biased coin with $P(\mathrm{H}) = \frac{2}{3}$. Exercise 5.3
The fair coin is tossed first.
If the first toss is a head, then the biased coin is tossed.
If the first toss is a tail, then the fair coin is tossed.
If the second toss is a head, then the biased is tossed.
If the second toss is a tail, then the fair coin is tossed.
What is the probability that the third toss is a tail?

9 The spaceship Voyager 1 was launched on 5 September 1977. On 9 September 2012 it was 1.823×10^{10} km from Earth. Exercise 6.2

(a) How many days are there between the launch date and 9 September 2012? (There are 9 leap-year days between these days.) Give your answer in scientific notation.

(b) On average, how far did Voyager 1 travel each day? (Assume the distance was given for the same time of day.) Write your answer in scientific notation in kilometres, correct to 4 significant figures.

(c) What is the average speed in kilometres per hour? Write your answer in scientific notation, correct to 4 significant figures.

(d) What is the speed in metres per second? Write your answer in scientific notation, correct to 4 significant figures.

10 Emir and Andrei are training for the Brissie to the Bay fun run. Exercise 1.3

On one weekend Emir plans a 10 km run followed by a 50 km ride. He estimates his running speed to be 12 km/h and his cycling speed to be 40 km/h for the planned route.

Andrei estimates he will be able to average 30 km/h for the ride, and plans to do the same distance, but wants to be dropped off along the track to shorten his run. Given Andrei's running speed is just 9 km/h, determine the distance that needs to be removed from his run so that the friends complete the course at approximately the same time.

Justify your solution with graphs and rules, including domains.

11 Two graphs of cubic polynomials intersect at just two places: where $x = 1$ and where $x = 4$. The first cubic passes through the point (2, 5) and the second passes through (2, 1). Determine possible equations for the equations and support your working with graphs. Exercise 3.1

12 According to legend, an ancient Persian or Indian king was so impressed by the invention of the game of chess that he offered the inventor a reward. The inventor requested one grain of rice for the first square of the chess board, two grains for the second square, four grains for the third square and so on up to the 64th square, doubling the amount each time. How many grains of rice would the inventor have received? Exercise 6.3

7

Exponential and logarithmic functions

Recall 408

7.1 Equations with exponentials 409

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7.3 Indices and logarithms 429

7.4 Modelling with exponentials 438

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Chapter review 448

7 Recall

Solve linear equations

1 Solve the following equations to determine the value of the unknown.

(a) $4x + 1 = 33$ (b) $\frac{x}{3} + 7 = 2$ (c) $2x + 7 = 5x - 8$

Use the index laws

2 Simplify, writing answers with index notation.

(a) $x \times 3x^n$ (b) $\frac{(x^n)^2}{x}$ (c) $x^2\sqrt{x}$

Factorise quadratic expressions

3 Factorise the following.

(a) $x^2 - 1$ (b) $x^2 - 2x - 15$ (c) $3x^2 - 11x + 10$

Use the null factor law

4 Solve the following equations for x.

(a) $0 = (x - 2)(x + 3)$ (b) $x^2 + 5x = 0$ (c) $x^2 + 5x = 6$

Intercepts on the axes

5 Determine the coordinates of the axes intercepts for each of the following parabolas.

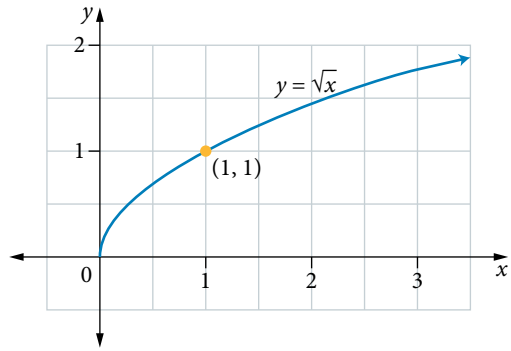
(a) $y = x^2 - 4$ (b) $y = 6 - (x - 2)^2$ (c) $y = (x + 3)^2 + 5$

Reflections and dilations

6 The graph of $y = \sqrt{x}$ is shown. Determine the transformations required for the graph of each of the following equations. List the reflections and dilations and specify the axis in which they occur.

(a) $y = -3\sqrt{x}$

(b) $y = \sqrt{-2x}$

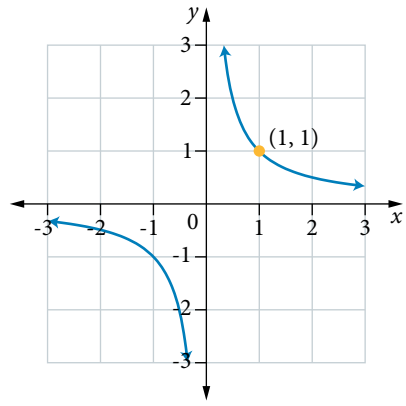


Translations

7 The graph of $y = \frac{1}{x}$ is shown. Determine the translations required for the graph of each of the following equations.

(a) $y = \frac{1}{x} + 2$

(b) $y = \frac{1}{x + 2}$



Equations with exponentials

7.1

Equating the indices

If two index expressions are equal and are written with the same base, then the indices must be equal. For example, if $3^x = 9$, and 9 is equivalent to 3^2, then $3^x = 3^2$ and $x = 2$.

Additional information

Solving exponential equations

This activity explains the process of solving $2^x = 2\sqrt{2}$ and also has practise questions.

1 Solving indicial equations

Solve the following equations for the unknown.

(a) $2^a = 8$

THINKING	WORKING
1 Rewrite the number in index form using the given base.	$2^a = 8$ $2^a = 2^3$
2 Equate the indices.	$a = 3$

(b) $5^b = \sqrt{125}$

1 Convert the root to the fractional power.	$5^b = \sqrt{125}$ $5^b = (125)^{\frac{1}{2}}$
2 Rewrite the number in index form using the given base.	$5^b = \left(5^3\right)^{\frac{1}{2}}$
3 Multiply the indices.	$5^b = 5^{\frac{3}{2}}$
4 Equate the indices.	$b = \frac{3}{2}$

2 Solving indicial equations where the variable appears twice

Solve $3^{2x} \times 9^{x-1} = 81$ for x.

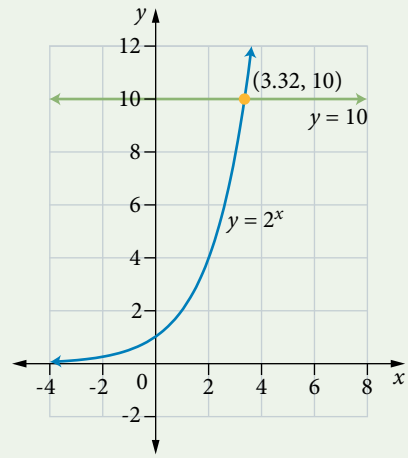
THINKING	WORKING
1 Convert all numbers as powers to the same base.	$3^{2x} \times 9^{x-1} = 81$ $3^{2x} \times \left(3^2\right)^{x-1} = 3^4$

2 Use index laws to simplify each side of the equation as single powers.	$3^{2x} \times 3^{2x-2} = 3^4$ $3^{4x-2} = 3^4$
3 Equate indices.	$4x - 2 = 4$
4 Solve for x.	$4x = 6$ $x = \frac{3}{2}$

Technology can be used to solve equations if the numbers are not readily expressed as exponentials of the same base. For example, $2^x = 64$ is easily solved because 64 can be expressed as 2^4. However, an equation such as $3^x = 64$ cannot be solved in this way. Later in the chapter you will learn techniques for solving such equations with exact answers. For now, you will use technology to determine an approximation to the solution.

3 Solving indicial equations with technology

Solve $2^x = 10$, correct to 2 decimal places.

THINKING	WORKING
1 Estimate the solution.	$2^3 = 8$, $2^4 = 16$ $\therefore 3 < x < 4$
2 Use the solve function to solve for x, correct to 2 decimal places.	$2^x = 10$ $x = 3.32\ (2\text{ d.p.})$
3 Alternatively, graph two functions and read the x-value of the intersection, correct to 2 decimal places.	 $y = 10$ and $y = 2^x$ The coordinates of the point of intersection are (3.32, 10), correct to 2 decimal places
4 Interpret the answer.	The solution to the equation $2^x = 10$ is $x = 3.32$ (2 d.p.).

Quadratics of power functions

When exponentials are raised to a power, the indices are multiplied:
$\left(a^m\right)^n = a^{m\times n}$, so $\left(a^x\right)^2 = a^{2x}$

An expression with the highest power $2x$ can be expressed as a quadratic equation. For example, $7^{2x} + 5\times 7^x - 6$ is the quadratic expression $\left(7^x\right)^2 + 5\times\left(7^x\right) - 6$ or $p^2 + 5p - 6$ where $p = 7^x$. These expressions can be factorised and solved in the same way as other quadratic expressions.

4 Indicial equations that require factorising

Solve the following equations for the unknown.

(a) $3^{2x} - 10\times 3^x + 9 = 0$

THINKING	WORKING
1 Express the expression on the left-hand side as a quadratic.	$3^{2x} - 10\times 3^x + 9 = 0$ $\left(3^x\right)^2 - 10\times\left(3^x\right) + 9 = 0$
2 Substitute another variable for 3^x.	Let $3^x = z$ $z^2 - 10z + 9 = 0$
3 Factorise the quadratic.	$(z-9)(z-1) = 0$
4 Solve using the null factor law.	$z = 9$ or $z = 1$
5 Reintroduce 3^x and solve for x.	$3^x = 9$ or $3^x = 1$ $3^x = 3^2$ $\quad$ $3^x = 3^0$ $x = 2$ $\quad$ $x = 0$

(b) $3^{2m} - 8\times 3^m = 9$

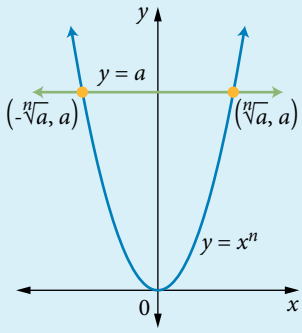
1 Rearrange so that an expression is equal to zero.	$3^{2m} - 8\times 3^m = 9$ $3^{2m} - 8\times 3^m - 9 = 0$
2 Substitute a variable for 3^m.	Let $3^m = g$ $g^2 - 8g - 9 = 0$
3 Factorise the quadratic.	$(g-9)(g+1) = 0$
4 Solve using the null factor law.	$g = 9$ or $g = -1$
5 Reintroduce 3^m and solve for m.	$3^m = 9$ $3^m = 3^2$
6 Identify the solutions and any restrictions.	$m = 2$ 3^m cannot be -1, because powers with positive bases are always positive.

Unknown base

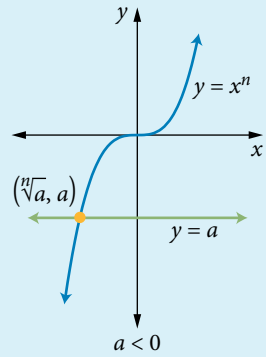
To solve an equation where the base is the unknown, you can use inverse operations or inverse powers. Recall that if $a^n = p$ for $a > 0$, then $a = \sqrt[n]{p}$ or $a = p^{\frac{1}{n}}$.

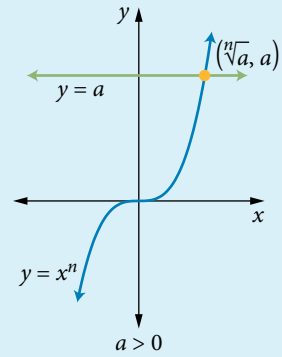
When solving index equations $x^n = a$, where $a > 0$:

If n is even, there will be a positive and a negative solution.



If n is odd, there will be a single solution, the same sign as a.





5 Determine the value of an unknown base

Solve the following equations for the unknown.

(a) $p^3 = 216$

THINKING	WORKING
1 Check to see whether the given number can be expressed as an exponential with the same index.	$p^3 = 216$ $p^3 = 6^3$
2 If the powers match and it is an odd index, then the bases must be equal.	$p = 6$

(b) $x^{\frac{2}{3}} = 9$

Simplify using the index laws and solve for x.

$$x^{\frac{2}{3}} = 9$$
$$x^{\frac{2}{3}} = (\pm 3)^2$$
$$\left(x^{\frac{2}{3}}\right)^3 = \left((\pm 3)^2\right)^3$$
$$x^2 = (\pm 3)^6$$
$$x^2 = \left((\pm 3)^3\right)^2$$
$$x = \pm 3^3$$
$$x = \pm 27$$

The solve function of some calculators may give a single solution when there are in fact two possible solutions. Graphing with a suitable domain and range will demonstrate both solutions.

EXERCISE 7.1 Equations with exponentials

Worked Example

1 Solve the following equations for the unknown. 1

(a) $2^x = 8$ (b) $5^x = 625$ (c) $49^x = 7$
(d) $3^x = \frac{1}{9}$ (e) $6^x = \sqrt{216}$ (f) $5^x = \sqrt{125}$

2 Solve the following equations for the unknown.

(a) $5^{x+3} = \frac{1}{125}$ (b) $7^{-2x} = 343$ (c) $13^{3x} = \frac{1}{169}$ (d) $2^{2x+3} = 128$
(e) $3^{x-1} = 27$ (f) $5^x = \sqrt[3]{25}$ (g) $8^x = \sqrt{32}$ (h) $2^{4x} = \frac{\sqrt{8} \times 16}{32}$

3 Solve the following equations for the unknown. 2

(a) $2^{3x} \times 8^{x+1} = 32$ (b) $3^{2x} \times 27^{x-1} = \frac{1}{3}$ (c) $25^x \div 5^{x+1} = 5^{-3x}$
(d) $4^{x+2} = 8^x$ (e) $3^{4x} = 243^x$ (f) $\frac{9^{x-6}}{27^x} = 3^{2x}$

4 Solve the following equations for the unknown, correct to 2 decimal places. 3

(a) $2^a = 5$ (b) $9^b = 10$ (c) $6^c = 18$
(d) $12^d = 20$ (e) $10^e = 2$ (f) $3^f = 8$

5 (a) Choose the solution to $x^{\frac{3}{2}} = 5$, correct to 2 decimal places.

A 2.92 B -2.92 C 7.50 D 3.33

(b) Explain the common error made by a student who obtained an answer of $x = \pm 2.92$.

6 The solution(s) to the equation $x^{\frac{2}{3}} = 5$, correct to 2 decimal places are:

A 11.18 B ± 11.18 C 7.50 D 3.33

7 Solve the following equations for the unknown. 5

(a) $a^8 = 256$ (b) $b^5 = 243$ (c) $c^{\frac{1}{5}} = 2$
(d) $d^{\frac{3}{4}} = 27$ (e) $m^6 = 8$

8 Solve the following equations for the unknown, first as a surd, then as a decimal, correct to 2 decimal places.

(a) $x^2 = 30$ (b) $x^2 = -30$ (c) $x^3 = 30$ (d) $x^3 = -30$
(e) $x^4 = 30$ (f) $x^4 = -30$ (g) $x^5 = 30$ (h) $x^5 = -30$

9 Solve $9^{x-2} = \frac{1}{27\sqrt{3}}$ for x.

10 Solve the following equations for the unknown, by first factorising. 4

(a) $6^{2a} - 7 \times 6^a + 6 = 0$ (b) $2^{2b} - 6 \times 2^b + 8 = 0$ (c) $7^{2c} + 6 \times 7^c = 7$ (d) $3^{4d} - 3^{2d} - 6 = 0$

11 Explain why the following equations have no real solutions for x.

(a) $5^{2x} + 7 \times 5^x + 10 = 0$ (b) $3^{2x} - 2 \times 3^x + 11 = 0$

12 Determine the value of x if $(x+2)^{\frac{3}{4}} = 27^{\frac{1}{2}} \times 8^{-\frac{1}{4}}$.

13 Determine the value of x if $x^{-\frac{1}{6}} = 8^{-\frac{1}{9}} \times 5^{\frac{1}{6}}$.

14 Solve $2^{(2x+1)} - 17 \times 2^x + 8 = 0$ for x.

15 Solve $8(0.5)^{(x+1)} \leq 2$ for x.

16 Solve $9^x + 3 = 3^{x+1} + 3^x$ for x.

7.2 Graphs of exponential functions

Graphs of $f(x) = a^x$, $a > 0$, $a \neq 1$

Functions in the form of $f(x) = a^x$, where $a > 0$ and $a \neq 1$, are called exponential functions. The graphs of these functions have a horizontal asymptote on the x-axis. The graph approaches the axis but never reaches it.

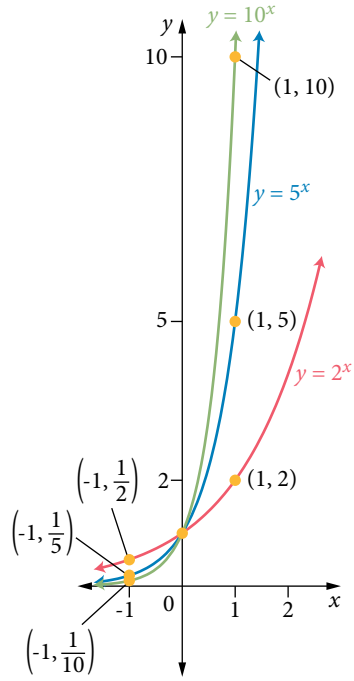
The graphs of $y = 2^x$, $y = 5^x$ and $y = 10^x$ increase as x increases, throughout the domain.

The graph of $y = a^x$ becomes steeper as the value of a increases.

When $x > 0$, $y = 10^x$ is above the graph of $y = 5^x$, which is above the graph of $y = 2^x$ because $10^1 = 10$, $5^1 = 5$ and $2^1 = 2$, and $10^2 = 100$, $5^2 = 25$ and $2^2 = 4$.

When $x < 0$, $y = 10^x$ is below the graph of $y = 5^x$, which is below the graph of $y = 2^x$ because $10^{-1} = \frac{1}{10} = 0.1$, $5^{-1} = \frac{1}{5} = 0.2$ and $2^{-1} = \frac{1}{2} = 0.5$. $2^0 = 5^0 = 10^0 = 1$, so graphs in the form $f(x) = a^x$ cut the y-axis at (0, 1).

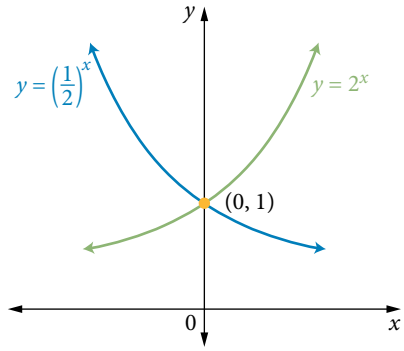
The graphs approach, but never reach $y = 0$. The horizontal asymptote occurs at $a^x = 0$.



The graph of $y = \left(\frac{1}{2}\right)^x$ is the reflection of $y = 2^x$ in the y-axis:

$$\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x$$
$$= 2^{-1 \times x}$$
$$= 2^{-x}$$

$2^0 = \left(\frac{1}{2}\right)^0 = 1$, so both graphs cut the y-axis at (0, 1).



For $y = 2^x$:

As $x \to \infty$, $y \to \infty$

As $x \to -\infty$, $y \to 0$

For $y = \left(\frac{1}{2}\right)^x$:

As $x \to \infty$, $y \to 0$

As $x \to -\infty$, $y \to \infty$

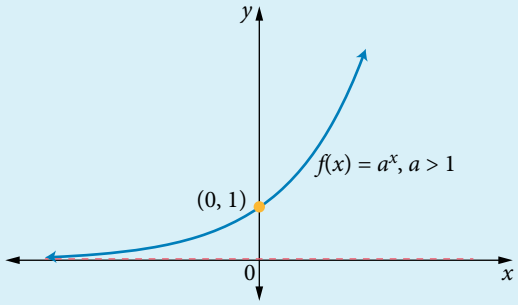
Explore further

Functions in the form $f(x) = a^{(x-b)} + c$

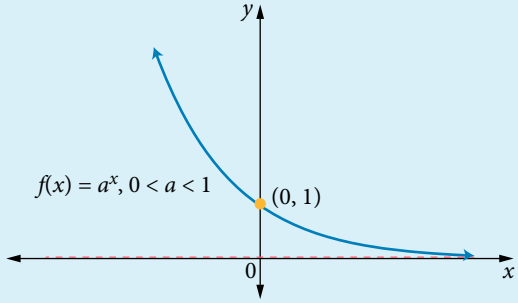
Explore the effect of the parameters a, b and c on the graph of $f(x) = a^{(x-b)} + c$.

The graph of $f(x) = a^x$ is an exponential function.

If $a > 1$, then the graph is an increasing exponential.



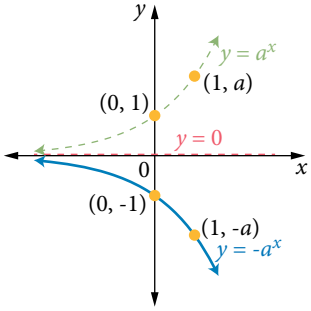
If $0 < a < 1$, then the graph is a decreasing exponential.



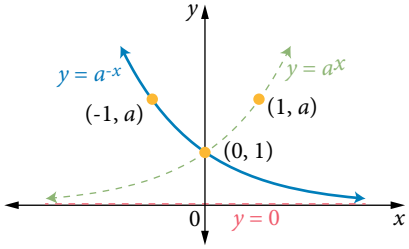
- The coordinates of the y-intercept are (0, 1).
- The equation of the horizontal asymptote is $y = 0$, hence there is no x-intercept.
- The domain is $\mathbb{R}$.
- The range is $\mathbb{R}^+$.

Reflection in the axes

$y = -a^x$ is a reflection of the function $y = a^x$ through the x-axis.



$y = a^{-x} = y = \left(\frac{1}{a}\right)^x$ is a reflection of the graph $y = a^x$ through the y-axis.



Vertical translations

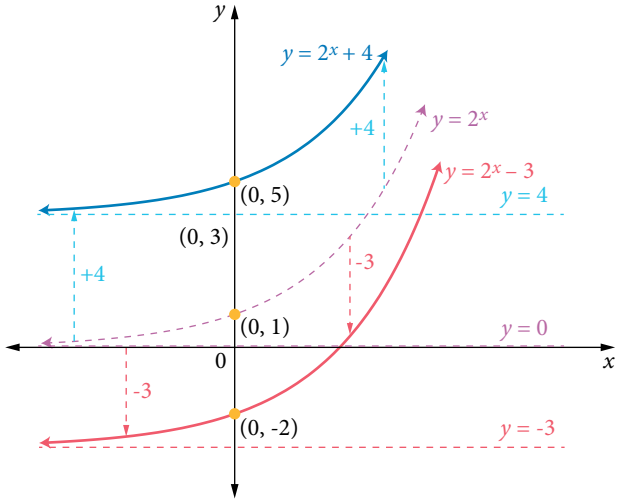
To produce the graph of $f(x) = a^x + c$, the graph of $f(x) = a^x$ is translated:

- up c units, if $c > 0$
- down $|c|$ units, if $c < 0$.

The equation of the asymptote is given by $y = c$.

The y-intercept is at $(0, c + 1)$.

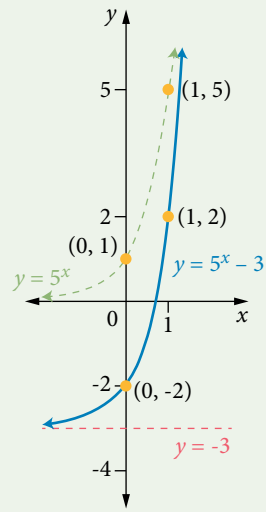
The range is $y > c$.



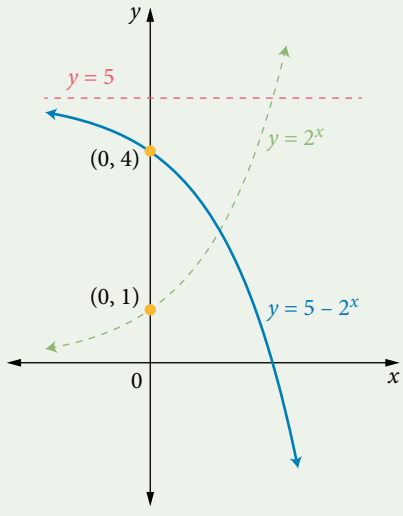
6 Reflections and vertical translations

Sketch the graph of each transformed function and the graph of $f(x) = a^x$ on the same set of axes. List the transformations that have occurred.

(a) $y = 5^x - 3$ and $y = 5^x$

THINKING	WORKING	
1 Determine the coordinates of two points on each graph.	$f(x) = 5^x$: $f(0) = 5^0$ $= 1$ The y-intercept is at (0, 1). $f(1) = 5^1 = 5$ Another point is (1, 5).	$g(x) = 5^x - 3$: $g(0) = 5^0 - 3$ $= 1 - 3$ $= -2$ The y-intercept is at (0, -2). $g(1) = 5^1 - 3 = 2$ Another point is (1, 2).
2 Determine the equation of the asymptotes.	$y = 5^x$: The equation of the asymptote is $y = 0$.	$y = 5^x - 3$ $= 0 - 3$ $= -3$ The equation of the asymptote is $y = -3$.
3 Note the transformations that have occurred.	The graph of $y = 5^x - 3$ is the graph of $y = 5^x$ translated 3 units in the negative direction of the y-axis (down).	
4 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.		

(b) $y = 5 - 2^x$ and $y = 2^x$

1 Determine the coordinates of two points on each graph.	$f(x) = 2^x$: $f(0) = 2^0 = 1$ The y-intercept is at (0, 1). $f(1) = 2^1 = 2$ Another point is (1, 2).	$g(x) = 5 - 2^x$: $g(0) = 5 - 2^0$ $= 5 - 1$ $= 4$ The y-intercept is at (0, 4). $g(1) = 5 - 2^1 = 3$ Another point is (1, 3).
2 Determine the equation of the asymptotes.	$y = 2^x$: The equation of the asymptote is $y = 0$.	$y = 5 - 2^x$: The equation of the asymptote is $y = 5$.
3 Note the transformations that have occurred.	The graph of $y = 5 - 2^x$ is the graph of $y = 2^x$ reflected in the x-axis and then translated 5 units in the positive direction of the y-axis (up).	
4 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.		

Horizontal translations

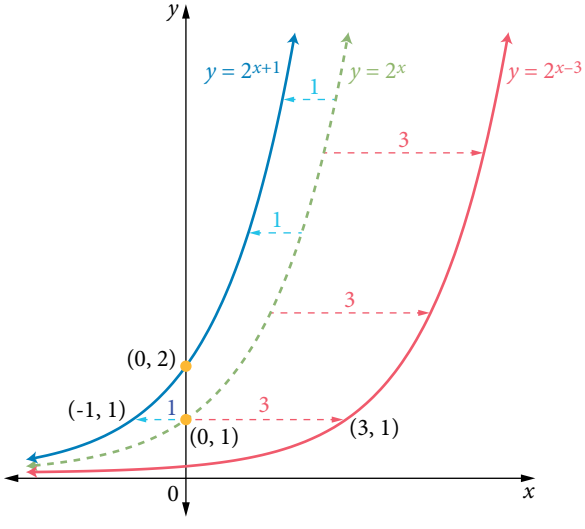
To produce the graph of $y = a^{x+b}$, the graph of $f(x) = a^x$ is translated:

- b units to the left if $b > 0$
- $|b|$ units to the right if $b < 0$

The y-intercept is at $(0, a^b)$.

There is no x-intercept.

The asymptote, domain and range are unchanged.



WARNING

Be careful when translating exponential graphs in the form of $y = a^{x+b}$ horizontally because the left and right movements might seem counter-intuitive.

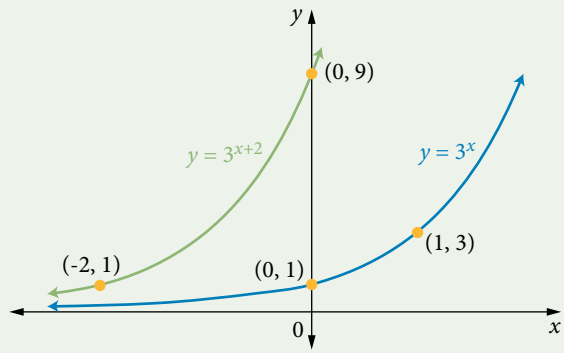
The power $x + 1$ translates the graph to the left 1 unit.

The power $x - 1$ translates the graph 1 unit to the right.

7 Reflections and translations

Sketch the graph of each transformed function and the graph of $f(x) = a^x$ on the same set of axes. List the transformations that have occurred.

(a) $y = 3^{x+2}$ and $y = 3^x$

THINKING	WORKING
1 Determine the coordinates of two points on each graph.	$f(x) = 3^x$: $f(0) = 3^0 = 1$ The coordinates of the y-intercept are (0, 1). $f(1) = 3^1 = 3$ Another point is (1, 3). $g(x) = 3^{x+2}$: $g(0) = 3^{0+2} = 9$ The coordinates of the y-intercept are (0, 9). $g(-2) = 3^{-2+2}$ $= 3^0$ $= 1$ Another point is (-2, 1).
2 Note the transformations that have occurred.	The graph of $y = 3^{x+2}$ is the graph of $y = 3^x$ translated 2 units in the negative direction of the x-axis (left).
3 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.	

(b) $y = 2^{x-2} - 3$ and $y = 2^x$

1 Determine the coordinates of two points on each graph.

$f(x) = 2^x$:

$f(0) = 2^0 = 1$

The coordinates of the y-intercept are (0, 1).

$f(1) = 2^1 = 2$

Another point is (1, 2)

$g(x) = 2^{x-2} - 3$:

$g(0) = 2^{0-2} - 3$

$= \frac{1}{2^2} - 3$

$= -\frac{11}{4}$

The coordinates of the y-intercept are $\left(0, -\frac{11}{4}\right)$.

$g(2) = 2^{2-2} - 3$

$= 1 - 3$

$= -2$

Another point is (2, -2).

2 Determine the equation of the asymptotes.

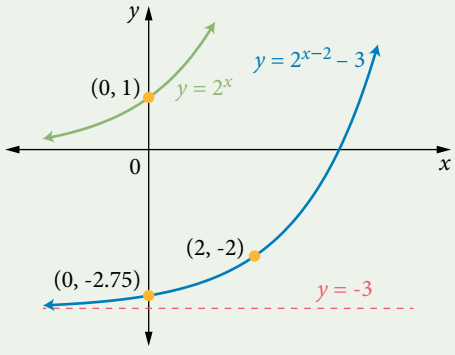
$y = 2^x$:
The equation of the asymptote is $y = 0$.

$y = 2^{x-2} - 3$:
The equation of the asymptote is $y = -3$.

3 Note the transformations that have occurred.

The graph of $y = 2^{x-2} - 3$ is the graph of $y = 2^x$ translated 2 units in the positive direction of the x-axis (right) and 3 units in the negative direction of the y-axis (down).

4 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.



(c) $y = 5^{1-x}$ and $y = 5^x$

1 Determine the coordinates of two points on each graph.

$f(x) = 5^x$:

$f(0) = 5^0 = 1$

The coordinates of the y-intercept are (0, 1).

$f(1) = 5^1 = 5$

Another point is (1, 5).

$g(x) = 5^{1-x}$:

$g(0) = 5^{1-0} = 5$

The coordinates of the y-intercept are (0, 5).

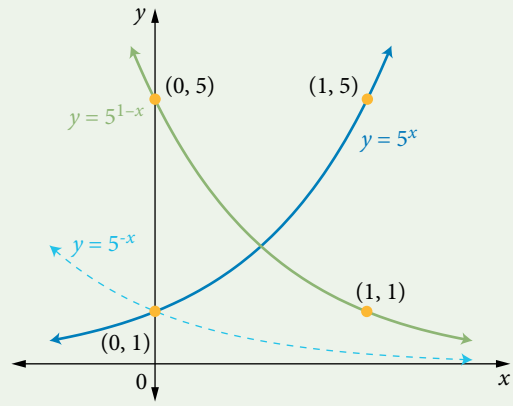
$g(1) = 5^{1-1} = 1$

Another point is (1, 1).

2 Note the transformations that have occurred.

3 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.

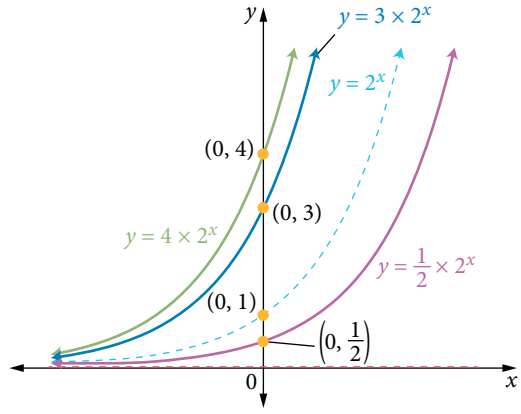
The graph of $y = 5^{1-x}$ is the graph of $y = 5^x$ reflected in the y-axis and then translated 1 unit in the positive direction of the x-axis (right).



Dilation parallel to the y-axis

To produce the graph of $y = n \times a^x$, the graph of $f(x) = a^x$ is dilated by a factor of $|n|$ parallel to the y-axis.

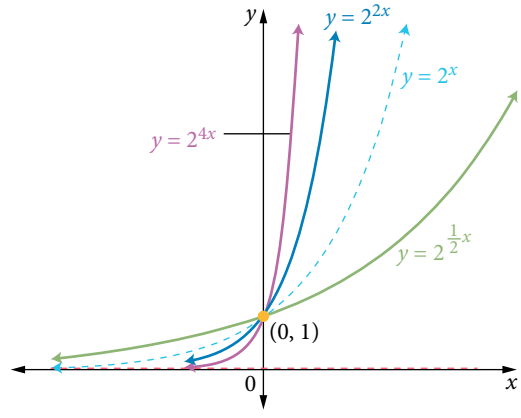
- As the magnitude of n increases, the graph is steeper and stretches further from the x-axis.
- If $n < 0$, then the graph is reflected in the x-axis.
- The y-intercept is at $(0, n)$.
- There is no x-intercept.
- The domain and asymptote are unchanged.



Dilation parallel to the x-axis

To produce the graph of $y = a^{kx}$, $k \neq 0$, the graph of $f(x) = a^x$ is dilated by a factor of $\frac{1}{|k|}$ parallel to the x-axis.

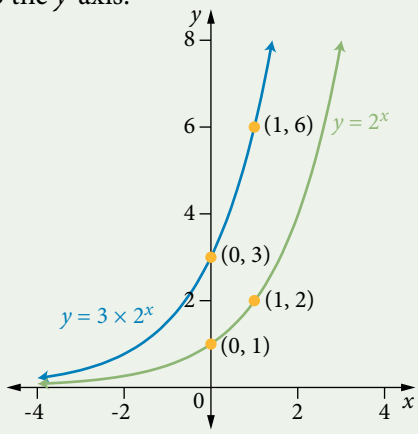
- As the magnitude of k increases, the graph is steeper and compressed towards the y-axis.
- If $k < 0$, then the graph is reflected in the y-axis.
- There is no x-intercept.
- The y-intercept, domain, range and asymptote are unchanged.



8 Dilations

Sketch the graph of each transformed function and the graph of the basic function on the same set of axes. List the transformations that have occurred.

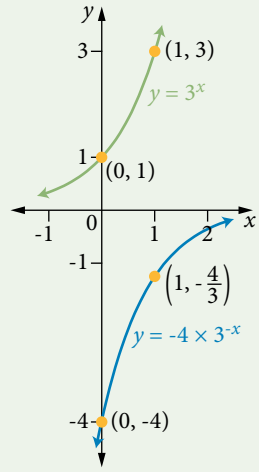
(a) $y = 3 \times 2^x$ and $y = 2^x$

THINKING	WORKING
1 Determine the coordinates of two points on each graph.	$f(x) = 2^x$: $f(0) = 2^0 = 1$ The y-intercept is at (0, 1). $f(1) = 2^1 = 2$ Another point is (1, 2). $g(x) = 3 \times 2^x$: $g(0) = 3 \times 2^0$ $= 3 \times 1$ $= 3$ The y-intercept is at (0, 3). $g(1) = 3 \times 2^1 = 6$ Another point is (1, 6).
2 Note the transformations that have occurred.	The graph of $y = 3 \times 2^x$ is the graph of $y = 3^x$ dilated by a factor of 3 parallel to the y-axis.
3 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.	

(b) $y = -4 \times 3^{-x}$ and $y = 3^x$

1 Determine the coordinates of two points on each graph.	$f(x) = 3^x$: $f(0) = 3^0 = 1$ The coordinates of the y-intercept are (0, 1). $f(1) = 3^1 = 3$ Another point is (1, 3).	For $g(x) = -4 \times 3^{-x}$: $g(0) = -4 \times 3^{-0}$ $= -4 \times 1$ $= -4$ The y-intercept is at (0, -4). $g(1) = -4 \times 3^{-1}$ $= -4 \times \frac{1}{3}$ $= -\frac{4}{3}$ Another point is $\left(1, -\frac{4}{3}\right)$.
2 Note the transformations that have occurred.	The graph of $y = -4 \times 3^{-x}$ is the graph of $y = 3^x$ dilated by a factor of 4 parallel to the y-axis and reflected in both axes.	

3 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.



(c) $y = 5^{2x}$ and $y = 5^x$

1 Determine the coordinates of two points on each graph.

$f(x) = 5^x$:

$f(0) = 5^0 = 1$

The y-intercept is at (0, 1).

$f(1) = 5^1 = 5$

Another point is (1, 5).

$g(x) = 5^{2x}$:

$g(0) = 5^{2\times 0} = 1$

The y-intercept is at (0, 1).

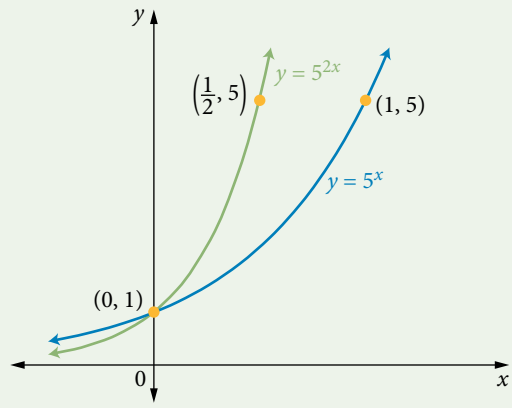
$$g\left(\frac{1}{2}\right) = 5^{2\times\frac{1}{2}}$$
$$= 5^1$$
$$= 5$$

Another point is $\left(\frac{1}{2}, 5\right)$.

2 Note the transformations that have occurred.

The graph of $y = 5^{2x}$ is the graph of $y = 5^x$ dilated by a factor of $\frac{1}{2}$ parallel to the x-axis.

3 Sketch the function in the form $y = a^x$ and then sketch the transformed function. Check your graphs are reasonable using the transformations and calculated coordinates.



WARNING

Transformations can sometimes be described in more than one way. For example:

$3^{x+2} = 9 \times 3^x$; $5^{1-x} = 5 \times 5^{-x}$; $5^{2x} = 25^x$

Additional information

Transforming exponential functions

Strengthen your understanding of the transformations of exponential functions.

Sketching exponential graphs in the form $y = n \times a^x + c$

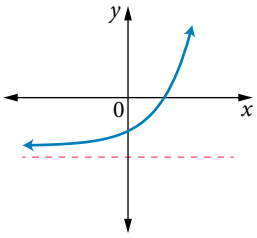
The equation of the horizontal asymptote is $y = c$.

Over an unrestricted domain, there will always be a y-intercept.

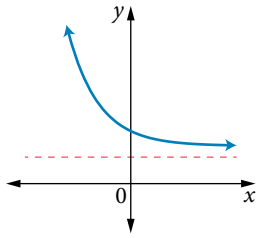
A second point will fix the sign (positive or negative) and orientation of the graph.

Determine the x-intercept, if it exists, or determine another point to position the graph.

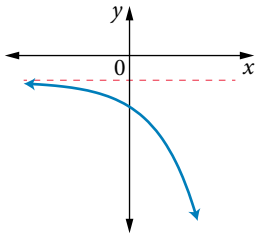
$y = n \times a^x + c$, for $a > 1$, $n > 0$ and $c < 0$



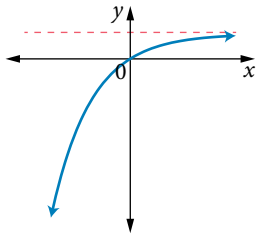
$y = n \times a^x + c$, for $0 < a < 1$, $n > 0$ and $c > 0$



$y = n \times a^x + c$, for $a > 1$, $n < 0$ and $c < 0$



$y = n \times a^x + c$, for $0 < a < 1$, $n < 0$ and $c > 0$



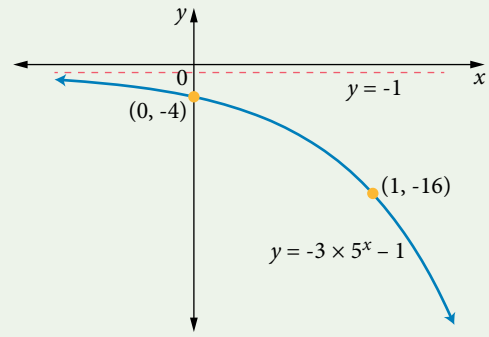
9 Identifying key features from an equation

For the following functions identify the equation of the asymptote, and determine the coordinates of the y-intercept and one other point. Sketch the graph and determine the domain and range.

(a) $f(x) = -3 \times 5^x - 1$

THINKING	WORKING
1 Identify the equation of the asymptote from the constant term.	The equation of the asymptote is $y = -1$.
2 Determine the coordinates of the y-intercept.	For $x = 0$: $f(0) = -3 \times 5^0 - 1$ $= -3 - 1$ $= -4$ The coordinates of the y-intercept are $(0, -4)$.
3 Determine the coordinates of the x-intercept, if possible.	The graph is always below the asymptote, $y = -1$, so it never crosses the x-axis and therefore has no x-intercept.
4 Determine the coordinates of an additional point.	For $x = 1$: $f(1) = -3 \times 5^1 - 1$ $= -15 - 1$ $= -16$ The point $(1, -16)$ is on the graph.

5 Use the intercept and additional point to draw the graph so that it approaches the asymptote.



6 State the domain and range.

Domain: $\mathbb{R}$ Range: $(-\infty, -1)$

(b) $f(x) = 7 \times (0.2)^{1.5x} + 3$

1 Identify the equation of the asymptote from the constant term.

The equation of the asymptote is $y = 3$.

2 Determine the coordinates of the y-intercept.

For $x = 0$:

$$\begin{aligned} f(0) &= 7 \times (0.2)^{1.5 \times 0} + 3 \\ &= 7 + 3 \\ &= 10 \end{aligned}$$

The coordinates of the y-intercept are $(0,\ 10)$.

3 Determine the coordinates of the x-intercept, if possible.

The graph is always above the asymptote, $y = 3$, so it never crosses the x-axis and there is no x-intercept.

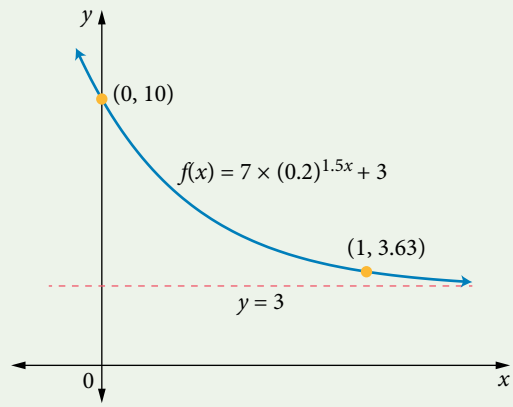
4 Determine the coordinates of an additional point.

For $x = 1$:

$$\begin{aligned} f(1) &= 7 \times (0.2)^{1.5 \times 1} + 3 \\ &= 7 \times (0.2)^{1.5} + 3 \\ &= 3.63 \text{ (2 d.p.)} \end{aligned}$$

The point $(1, 3.63)$ is on the graph.

5 Use the intercept and additional point to draw the graph so that it approaches the asymptote.



6 State the domain and range.

Domain: $\mathbb{R}$ Range: $(3, \infty)$

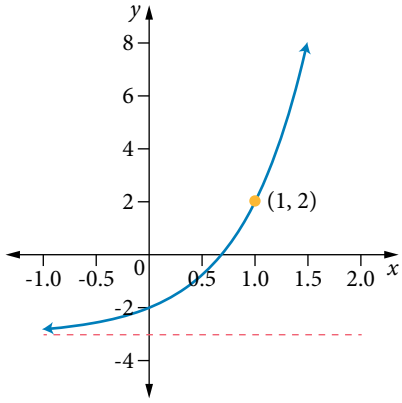
Determining the rule from a graph

The rules for the graphs of exponentials can be determined if enough information is given.

If you have two unknown parameters in a rule, then you need two pieces of information to set up and solve equations to establish the rule.

10 Establishing the rule from the graph

The equation of the graph shown below is of the form $f(x) = a^x + c$. Determine the rule for the graph.



THINKING	WORKING
1 Identify the equation of the asymptote.	The equation of the asymptote is $y = -3$ so $f(x) = a^x - 3$.
2 Use a clearly labelled point to substitute into the equation to determine the value of an unknown.	For $f(1) = 2$: $f(x) = a^x - 3$ $2 = a^1 - 3$ $5 = a$
3 Write the final equation.	$f(x) = 5^x - 3$

EXERCISE 7.2 Graphs of exponential functions

Worked Example

6 **1** Sketch the graph of each transformed function and the graph of the basic function on the same set of axes. List the transformations that have occurred.

(a) $y = 2^x + 1$ and $y = 2^x$

(b) $y = 5^{-x} - 3$ and $y = 5^x$

(c) $y = -10^x$ and $y = 10^x$

(d) $y = -2^x + 6$ and $y = 2^x$

(e) $y = 6^{-x} - 5$ and $y = 6^x$

7 **2** Sketch the graph of each transformed function and the graph of the basic function on the same set of axes. List the transformations that have occurred.

(a) $y = 5^{x-1}$ and $y = 5^x$

(b) $y = -3^{x+1}$ and $y = 3^x$

(c) $y = 3^{x-1} - 3$ and $y = 3^x$

(d) $y = 2^{2-x}$ and $y = 2^x$

(e) $y = 3^{-(x+1)}$ and $y = 3^x$

8 **3** Sketch the graph of each transformed function and the graph of the basic function on the same set of axes. List the transformations that have occurred.

(a) $y = 4 \times 2^x$ and $y = 2^x$

(b) $y = 2 \times 3^{-x}$ and $y = 3^x$

(c) $y = 3^{2x}$ and $y = 3^x$

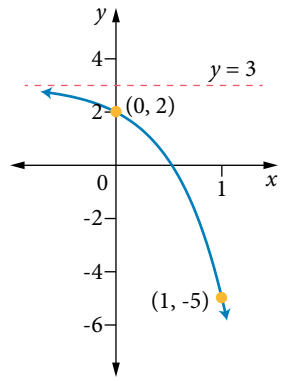
(d) $y = -2 \times 5^{-x}$ and $y = 5^x$

(e) $y = 5^{\frac{x}{2}}$ and $y = 5^x$

4 Sketch the following graphs and complete the table of transformations from the graph of $y = 3^x$.

Rule	Asymptote (equation)	Size of dilation factor (parallel to the y-axis)	Reflection in x-axis (yes or no)	y-intercept (coordinates)
$y = 3^x + 1$	$y = 1$	1		
$y = 2 \times 3^x + 1$				
$y = 3 \times 3^x + 1$				
$y = -3^x + 1$				
$y = -2 \times 3^x + 1$				

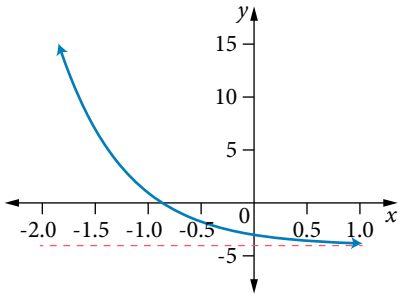
10 **5** Determine the rule for the graph on the right in the form $y = a + b \times 2^{cx}$.



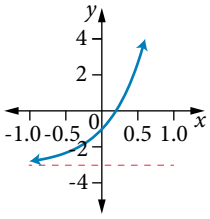
6 Choose the rule for the graph on the right.

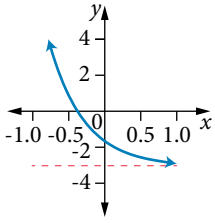
A $y = -5^{x} - 4$ B $y = -5^{-x} - 4$

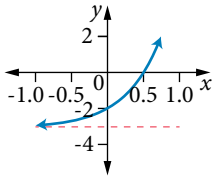
C $y = 2 \times 5^{-x} - 4$ D $y = 5^{-x} - 4$

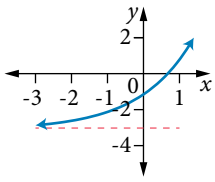


7 (a) Choose the graph of the function $f(x) = 2 \times 2^{3x} - 3$.

A


B


C


D


(b) Explain the error made by a student who thought the x-intercept of the graph in part (a) was (2, 0).

8 The function $f(x) = 4^{x}$ undergoes the following transformations. Write the new rule for each transformation.

(a) Dilated by a factor of 3 parallel to the y-axis.

(b) Dilated by a factor of $\frac{1}{5}$ parallel to the y-axis.

(c) Reflected in the x-axis.

(d) Dilated by a factor of $\frac{1}{2}$ parallel to the x-axis.

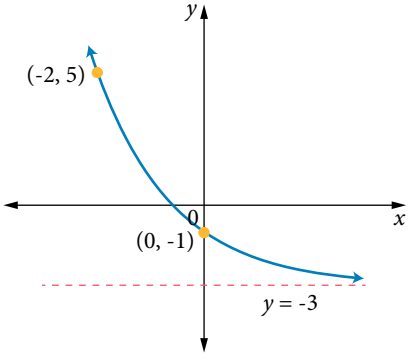
(e) Reflected in the y-axis.

(f) Vertically translated down 7 units.

(g) Horizontally translated 3 units to the right.

(h) Dilated by a factor of 3 parallel to the x-axis.

9 Use the information in the graph on the right to demonstrate that the equation could be $y = 2 \times 2^{-x} - 3$.



Worked Example 9

10 For the following functions identify the equation of the asymptote, and determine the y-intercept and one other point. Sketch the graph and determine the domain and range.

(a) $y = 2 \times 3^{x} - 6$ (b) $f(x) = 5 \times 2^{3-x} + 1$ (c) $f(x) = 3 \times 2^{x-2} - 5$

(d) $f(x) = -2 \times 10^{2x} + 40$ (e) $f(x) = -6^{-3x+2}$ (f) $f(x) = -2 \times 2^{1+x} + 5$

11 The function $f(x) = 2^x$ undergoes several transformations. It is reflected in the y-axis, then dilated by a factor of 3 parallel to the y-axis, and translated 3 units to the right and 5 units up.

(a) Determine the rule of the final function.

(b) Graph the original rule and the transformed rule on the same set of axes and check that each transformation has been successfully represented.

12 The function f is given by the rule $f(x) = 2^x$. The graph of $y = f(x)$ is transformed to the function with equation $y = g(x)$ by reflection in the y-axis, translation of 2 to the right, and translation of 3 down.

(a) Sketch the graph of $y = f(x)$, stating domain and range, equations of asymptote(s) and coordinates of any axis intercepts.

(b) Determine the rule for $y = g(x)$.

(c) Sketch the graph of $g(x)$, stating domain and range, equations of asymptote(s) and coordinates of any axes intercepts. Give the x-intercept correct to 2 decimal places.

13 The formula for the accumulated amount A with compound interest is $A = P(1+r)^n$, where P is the initial investment or principal, r is the interest rate for the compounding period, as a decimal, and n is the number of compounding periods. Sally has \$10 000 to invest and is investigating the offers from a variety of banks.

Bank A offers 6% p.a. compounding quarterly.

Bank B offers 7% p.a. compounding half-yearly.

(a) Formulate equations for the amount after t years for both banks A and B.

(b) Plot the graphs of each function on the same axes for a period of 5 years, labelling the point when $t = 5$ with its coordinates, to the nearest dollar.

(c) Which bank should Sally invest with to get the highest return over a period of 5 years?

14 Sketch the graph of $y = -2f(x-2) + \frac{1}{2}$ where $f(x) = -3^{x+1}$, labelling the asymptote with its equation, and stating the domain and range.

15 A graph of the form $y = 2^{x+c} + b$ has endpoints (-2, -1) and (2, 14).

(a) Determine the rule for the function.

(b) Transform the function so that the endpoints become (0, 0) and (100, 100). Note that the solution is not unique.

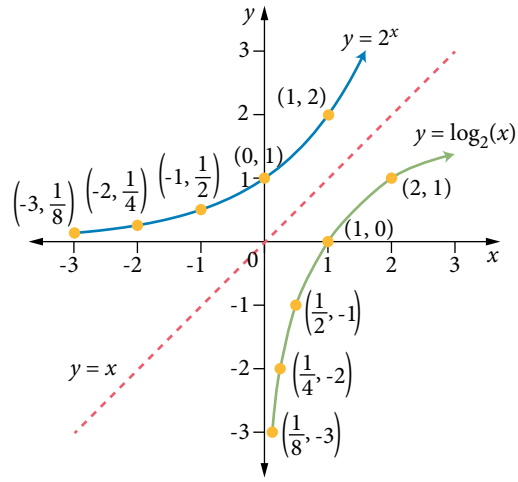
Indices and logarithms

7.3

Inverse functions

Two functions are inverse functions if their graphs are reflected in the line $y = x$.

The inverse of an exponential function is the logarithmic function to the same base. The graph on the right shows that $y = \log_2(x)$ is the reflection of $y = 2^x$ in the line $y = x$.



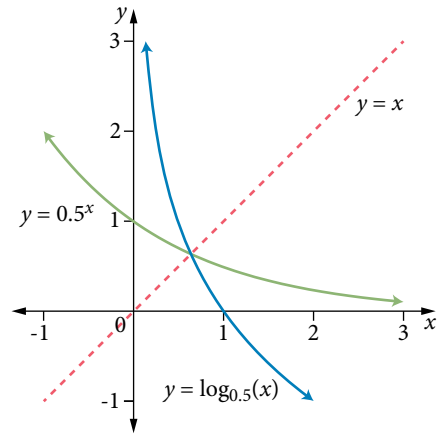
Expanded form	Index form $a^x = b$	Logarithmic form $\log_a(b) = x$
$2 \times 2 \times 2 \times 2 = 16$	$2^4 = 16$	$\log_2(16) = 4$
2	$2^1 = 2$	$\log_2(2) = 1$
1	$2^0 = 1$	$\log_2(1) = 0$
$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$	$2^{-3} = \frac{1}{8}$	$\log_2\left(\frac{1}{8}\right) = -3$

In general, $y = \log_a(x)$ is the inverse of $y = a^x$, where $a > 0$ and $a \neq 1$.

If $0 < a < 1$, then the pair of graphs intersect on $y = x$. The function $y = \log_{0.5}(x)$ is the inverse of $y = 0.5^x$.

Logarithmic functions

Functions of the type $f(x) = \log_a(x)$, where $a > 1$, are called logarithmic functions. Logarithmic functions have a vertical asymptote that the graph approaches but never reaches.



The graph above shows that the logarithms of negative numbers do not exist.

The log of a number is the index when the number is written as an indicial expression to a particular base. For example:

$\log_2(8) = \log_2(2^3) = 3 \times \log_2(2) = 3$

$a^x = b$ is equivalent to $x = \log_a(b)$ provided that $a > 0$ and $a \neq 1$.

Additional information

Logarithms

This video introduces logarithms.

11 Conversion between exponential and logarithmic equations

Rewrite each exponential equation as a logarithmic equation.

(a) $1000 = 10^3$

THINKING	WORKING
1 Identify the base, number and index.	Base is 10, number is 1000 and index is 3
2 Rewrite the number as: $\log_{\text{base}}(\text{number}) = \text{index}$	$\log_{10}(1000) = 3$

(b) $\frac{1}{25} = 5^{-2}$

1 Identify the base, number and index.	Base is 5, number is $\frac{1}{25}$ and index is -2
2 Rewrite the number as: $\log_{\text{base}}(\text{number}) = \text{index}$	$\log_5\left(\frac{1}{25}\right) = -2$

(c) $3 = 9^{\frac{1}{2}}$

1 Identify the base, number and index.	Base is 9, number is 3 and index is $\frac{1}{2}$
2 Rewrite the number as: $\log_{\text{base}}(\text{number}) = \text{index}$	$\log_9(3) = \frac{1}{2}$

Laws of logarithms

Logarithms are the inverse of exponentials, so the index laws can be used to develop the laws for logarithms.

Additional information

Log laws

Watch the video and then simplify the expressions.

Index laws $a^x = b$	Log laws $\log_a(b) = x$
When multiplying indicial expressions with the same base, add the powers. $a^m \times a^n = a^{m+n}$	When adding logs with the same base (which are powers), multiply the numbers. $\log_a(m) + \log_a(n) = \log_a(m \times n)$
When dividing exponential expressions with the same base, subtract the powers. $\frac{a^m}{a^n} = a^{m-n}$	When subtracting logs with the same base (which are powers), divide the numbers. $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
Any base raised to the power of zero is equal to 1. $a^0 = 1$	The log of 1 with any base is zero. $\log_a(1) = 0$
When raising a power to a power, multiply the powers. $(a^m)^n = a^{m \times n}$	The log of a number raised to a power is the product of the power and the log of the number. $\log_a(x^m) = m \times \log_a(x)$
A negative index can be expressed as the reciprocal with a positive index. $a^{-m} = \frac{1}{a^m}$	The log of a fraction is the negative log of the reciprocal. $\log_a\left(\frac{1}{m}\right) = \log_a(m^{-1})$ $= -\log_a(m)$
A surd is the same as a fractional exponential expression. $\sqrt[n]{a} = a^{\frac{1}{n}}$	The log of the nth root of a number is $\frac{1}{n}$ times the log of the number. $\log_a(\sqrt[n]{m}) = \log_a(m^{\frac{1}{n}})$ $= \frac{1}{n}\log_a(m)$
The base raised to the power of 1 will equal the base. $a^1 = a$	The log of any number to the matching base is 1. $\log_a(a) = 1$

12 Simplifying logarithmic expressions using the log laws

Simplify each expression using the log laws and then write as a single term.

(a) $\log_2(72) - \log_2(9)$

1 Recall the law for the subtraction of logs: $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$	$\log_2(72) - \log_2(9)$ $= \log_2\left(\frac{72}{9}\right)$ $= \log_2(8)$
2 Simplify if possible using: $\log_a\left(x^m\right) = m \times \log_a(x)$	$= \log_2\left(2^3\right)$ $= 3\log_2(2)$ $= 3$

(b) $\frac{1}{2}\log_6(81) + 2\log_6(2)$

1 Convert each term to the form: $m \times \log_a(x) = \log_a\left(x^m\right)$	$\frac{1}{2}\log_6(81) + 2\log_6(2)$ $= \log_6\left(\sqrt{81}\right) + \log_6\left(2^2\right)$ $= \log_6(9) + \log_6(4)$
2 Recall the log law for the addition of logs: $\log_a(m) + \log_a(n) = \log_a(m \times n)$	$= \log_6(9 \times 4)$ $= \log_6(36)$
3 Simplify if possible using: $\log_a\left(x^m\right) = m \times \log_a(x)$	$= \log_6\left(6^2\right)$ $= 2\log_6(6)$ $= 2$

(c) $\frac{\log_2(9)}{\log_2(3)}$

1 Express the numbers with the same base.	$\frac{\log_2(9)}{\log_2(3)}$ $= \frac{\log_2\left(3^2\right)}{\log_2(3)}$
2 Simplify the expression using the log law: $\log_a\left(x^m\right) = m \times \log_a x$	$= \frac{2\log_2(3)}{\log_2(3)}$
3 Simplify if possible.	$= \frac{2 \times 1}{1}$ $= 2$

Solving exponential equations using logarithms

Exponentials and logarithms are inverse functions, so exponential equations can sometimes be solved using logarithms.

13 Exact solutions to exponential equations

Determine the exact solution to the following.

(a) $3^x = 7$

THINKING	WORKING
Solve using log laws to convert to logarithm form using: index = $\log_{\text{base}}(\text{number})$	$3^x = 7$ $\log_3(3^x) = \log_3(7)$ $x \times \log_3(3) = \log_3(7)$ $x = \log_3(7)$

(b) $3^{x+3} + 1 = 5$

1 Use algebra and index laws to simplify the equation.	$3^{x+3} + 1 = 5$ $3^{x+3} = 5 - 1$ $3^x \times 3^3 = 4$ $3^x \times 27 = 4$ $3^x = \frac{4}{27}$
2 Solve using: index = $\log_{\text{base}}(\text{number})$	$\log_3(3^x) = \log_3\left(\frac{4}{27}\right)$ $x \times \log_3(3) = \log_3\left(\frac{4}{27}\right)$ $x = \log_3\left(\frac{4}{27}\right)$

Logarithms on calculators

Calculators are set up for logarithms with two different bases: base 10 and base e

If no base is written, then assume it is base 10. For example, $\log(100) = 2$ because the default base is 10 and $100 = 10^2$.

Logarithms to base e are used in calculus and appear as ln where $\ln(x) = \log_e(x)$.

Logarithms of bases other than 10 and e cannot be entered directly in a calculator. To determine these, take the log to base 10 of both sides. For example, $\log_3(7)$ cannot be entered in a calculator, so rewrite as an index.

$$3^x = 7$$
$$\log_{10}(3)^x = \log_{10}(7)$$
$$x\log_{10}(3) = \log_{10}(7)$$
$$x = \frac{\log_{10}(7)}{\log_{10}(3)}$$
$$\therefore \log_3(7) = \frac{\log_{10}(7)}{\log_{10}(3)}$$

Change-of-base rule:

$$\log_a(b) = \frac{\log_x(b)}{\log_x(a)} = \frac{\log_x(\text{number})}{\log_x(\text{base})},$$

where $x > 0$ and $x \neq 1$

Any index could be converted to a different base using this method, so this is written in general form as the change-of-base rule.

Additional information

Change of base

Review the change-of-base rule.

14 Change-of-base rule

Approximate the value of each expression, correct to 2 decimal places.

(a) $\log_5(12)$

THINKING	WORKING
1 Apply the change-of-base rule to change to base 10.	$\log_5(12) = \frac{\log_{10}(12)}{\log_{10}(5)}$
2 Use your calculator to approximate the value.	$\frac{\log_{10}(12)}{\log_{10}(5)} = 1.54\ (2\text{ d.p.})$
3 Check that the answer is reasonable.	$5^1 = 5,\ 5^2 = 25$ $5 < 12 < 25$, so the answer is reasonable.

(b) $\log_{20}(0.1)$

1 Apply the change-of-base rule to change to base 10.	$\log_{20}(0.1) = \frac{\log_{10}(0.1)}{\log_{10}(20)}$
2 Use your calculator to approximate the value.	$\frac{\log_{10}(0.1)}{\log_{10}(20)} = -0.77\ (2\text{ d.p.})$
3 Check that the answer is reasonable.	$20^{-1} = 0.05,\ 20^0 = 1$ $0.05 < 0.1 < 1$, so the answer is reasonable.

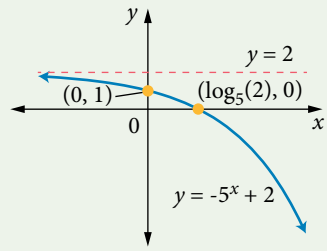
Sketching with exact intercepts

Logarithms are often used to determine the exact x-values in exponential functions.

15 Solve equations to sketch graphs with exact axes intercepts

Sketch the following, labelling all significant points with exact values.

(a) $f(x) = -5^x + 2$

THINKING	WORKING
1 Identify the transformations.	The graph of $y = 5^x$ is reflected in the x-axis and translated up 2 units.
2 Determine the equation of the asymptote.	The equation of the asymptote is $y = 2$.
3 Determine the coordinates of the axes intercepts.	For $x = 0$: $f(0) = -5^0 + 2$ $= -1 + 2$ $= 1$ The coordinates of the y-intercept are $(0, 1)$. For $y = 0$: $0 = -5^x + 2$ $5^x = 2$ $\log_5(5^x) = \log_5(2)$ $x \times \log_5(5) = \log_5(2)$ $x = \log_5(2)$ The coordinates of the x-intercept are $(\log_5(2), 0)$.
4 Sketch the graph of the curve.	

(b) $y = 2^{x-2} - 3$

1 Identify the transformations.	The graph of $y = 2^x$ is translated 2 units to the right and 3 units down.
2 Determine the equation of the asymptote.	The equation of the asymptote is $y = -3$.
3 Determine any the coordinates of the y-intercept.	For $x = 0$: $y = 2^{0-2} - 3 = \frac{-11}{4}$ The coordinates of the y-intercept are $\left(0, -\frac{11}{4}\right)$.

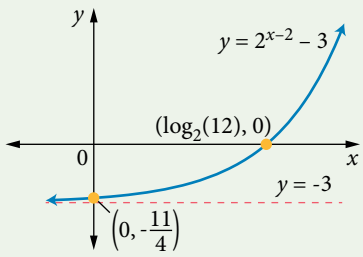
4 Determine the coordinates of the x-intercept.

For $y = 0$:

$$\begin{aligned} 0 &= 2^{x-2} - 3 \\ 2^{x-2} &= 3 \\ \log_2\left(2^{x-2}\right) &= \log_2(3) \\ (x-2)\times\log_2(2) &= \log_2(3) \\ x-2 &= \log_2(3) \\ x &= \log_2(3)+2 \\ &= \log_2(3)+\log_2(2)^2 \\ &= \log_2\left(3\times 2^2\right) \\ &= \log_2(12) \end{aligned}$$

The coordinates of the x-intercept are $\left(\log_2(12), 0\right)$.

5 Sketch the graph of the curve.



Solving exponential equations with factorising

Quadratic functions of terms with powers can be solved using the null factor law and log laws.

16 Determining the exact solutions to exponential equations

Give exact solutions for the following.

(a) $3^{2a} - 9\times 3^a + 14 = 0$

THINKING	WORKING
1 Create an equation in the form $ax^2+bx+c=0$.	$3^{2a}-9\times 3^a+14=0$ $\left(3^a\right)^2-9\left(3^a\right)+14=0$ Let $x=3^a$ $x^2-9x+14=0$
2 Factorise and use null factor law to solve for the introduced and original variable.	$(x-7)(x-2)=0$ $x=7$ or $x=2$ $x=3^a$: $3^a=7$ or $3^a=2$

3 Solve using logarithms as the inverse of exponentials.

$$\log_3(3^a) = \log_3(7) \quad \text{or} \quad \log_3(3^a) = \log_3(2)$$
$$a \times \log_3(3) = \log_3(7) \qquad a \times \log_3(3) = \log_3(2)$$
$$a = \log_3(7) \qquad a = \log_3(2)$$

(b) $2^{2b} - 3 \times 2^b - 10 = 0$

1 Create an equation in the form $ax^2 + bx + c = 0$.

$$2^{2b} - 3 \times 2^b - 10 = 0$$
$$(2^b)^2 - 3(2^b) - 10 = 0$$
Let $x = 2^b$
$$x^2 - 3x - 10 = 0$$

2 Factorise and use null factor law to solve for the introduced and original variable.

$$(x-5)(x+2) = 0$$
$x = 5$ or $x = -2$

$x = 2^b$:

$2^b = 5$ or $2^b = -2$

Given that $2^b > 0$, then the only solution is $2^b = 5$.

3 Solve using logarithms as the inverse of exponentials.

$$\log_2(2^b) = \log_2(5)$$
$$b\log_2(2) = \log_2(2)$$
$$b = \log_2(5)$$

EXERCISE 7.3 Indices and logarithms

Worked Example

11 1 Convert the following exponential equations to logarithmic form.

(a) $25 = 5^2$ (b) $3^4 = 81$ (c) $0.001 = 10^{-3}$

(d) $6^3 = 216$ (e) $17 = 2^x$ (f) $23 = 3^{-x}$

(g) $\sqrt{2} = 8^{\frac{1}{6}}$ (h) $63 = 10^{2x+3}$ (i) $2 = 10^{0.301}$

(j) $50 = 10^{1.699}$

2 Consider the exponential equation $3^x = 45$.

(a) Which is the equivalent log form equation?

A $\log_{45}(3) = x$ B $\log_{45}(x) = 3$ C $x = \log_3(45)$ D $\log_x(3) = 45$

(b) Explain the common error made by a student who thought the answer was $\log_x(45) = 3$.

12 3 Simplify the following expressions using the log laws.

(a) $\log_{15}(3) + \log_{15}(5)$ (b) $\log_a(5) + \log_a\left(\frac{1}{5}\right)$ (c) $\log_3(54) - \log_3(6)$

(d) $\log_{10}(125) + \log_{10}(32) - \log_{10}(4)$ (e) $\frac{1}{2}\log_{10}(16) + 2\log_{10}(5)$ (f) $\dfrac{\log_7(8)}{\log_7(2)}$

4 Solve the following for the exact value of the unknown. 13

(a) $37 = 5^x$ (b) $0 = -6^x + 1$ (c) $12 = 2^{-x}$

(d) $81 = 3^x$ (e) $\frac{1}{15} = 10^x$ (f) $-1 = -5^x + 7$

(g) $6^{b-1} = 15$ (h) $5 - 2^{5c+1} = 0$ (i) $-10^{7-d} = -1$

(j) $3^{2x} + 3 = 7$ (k) $5^m + 3 = 2$

5 Determine the solution to the equation $5^{2x+5} + 1 = 16$.

A $\frac{1}{2}(\log_5(10))$ B $\frac{1}{2}(\log_5(15) + 5)$ C $\log_5(15) - 5$ D $\frac{1}{2}(\log_5(15) - 5)$

6 Change the base to calculate the value of each expression, correct to 2 decimal places. 14

(a) $\log_2(17)$ (b) $\log_5(31)$ (c) $6\log_3(4)$ (d) $\frac{1}{5}\log_2(20)$

7 Determine the exact simultaneous solutions for each of the following pairs of equations. Write your answers as coordinate pairs.

(a) $y = 3,\ y = 5^x$ (b) $y = -1,\ y = -6^x + 1$

8 Sketch each pair of graphs and identify (as a coordinate pair) the simultaneous solution to the equations given.

(a) $y = 2x - 1,\ y = 2^{-x}$ (b) $y = -x,\ y = 3^x$ (c) $y = 2 \times 3^x,\ y = 2^{-x}$

9 Solve $5^x - 130 = 0$ for x, giving the exact answer as well as the decimal approximation, correct to 2 decimal places.

10 Solve for x:

(a) $\log_4(5 - 2x) = 3$ (b) $\log_2(x - 1) + \log_2(x + 1) = 3$

11 Sketch the graphs of the following functions, labelling all significant features in exact form. 15

(a) $f(x) = 10^x - 5$ (b) $f(x) = -3 \times 2^x + 2$ (c) $f(x) = -5^x + 10$ (d) $f(x) = 3^{x+2} - 1$

12 In the following equations, determine the exact value(s) of the unknown, if they exist. 16

(a) $3^{2x} + 2 \times 3^x - 35 = 0$ (b) $5^{2a} - 17 \times 5^a + 60 = 0$ (c) $2^{2m} + 6 \times 2^m = 16$

(d) $7^{2b} = 9$ (e) $9^n - 7 \times 3^n + 12 = 0$ (f) $10^{2x} + 5 \times 10^x + 20 = 0$

13 Solve $2(3)^{x+1} = \frac{6}{3^x} - 5$ for x giving the exact answer as well as the decimal approximation, correct to 2 decimal places.

14 Consider the graphs of $y = 3x - 1$ and $y = 2^x$.

(a) Plot the graphs of $y = 3x - 1$ and $y = 2^x$ on the same set of axes. Determine the solutions to the equation $2^x = 3x - 1$.

(b) For what values of x is $2^x \geq 3x - 1$?

15 If $x = \log_8(225)$ and $y = \log_2(15)$, determine x in terms of y.

16 A formula used to establish how much money is in a term deposit is $A = P\left(1 + \frac{r}{n}\right)^{(n \times T)}$, where A is the balance, P is the original investment, r is the interest rate per annum as a decimal, T is the number of years that the money is invested and n is the number of times per year that the bank compounds the interest.

The bank offers 8% interest per annum, compounding quarterly. Determine and interpret the exact value of T for an investment to double in value.

7.4 Modelling with exponentials

Exponential models

Exponential equations can be used to model many situations in science, ecology, medicine and economics. This modelling is especially useful if a quantity changes by a fixed ratio over time and the initial amount is when time is zero. For example, if you were analysing temperature records that began in 1867, then 1867 represents $t = 0$, and time is measured as years after 1867.

17 Modelling with exponential equations

Bacteria are single-celled microorganisms that reproduce by binary fission. Depending on the temperature and food supply, the number of bacteria can double very quickly. The number of bacteria can be represented by $N(t) = N_0 \times 2^{kt}$, where N_0 is the initial number of bacteria, t is the number of hours after growth begins and k is the number of times each bacteria can reproduce in an hour.

There are 15 salmonella bacteria on a piece of raw chicken when it is taken out of the fridge. The number of bacteria double every half hour at room temperature.

(a) Determine the value of N_0.

THINKING	WORKING
1 When $t = 0$, $N(t) = N_0$.	$N(t) = N_0 \times 2^{kt}$ $15 = N_0 \times 2^{(k \times 0)}$ $15 = N_0$
2 Interpret the answer.	Initially there are 15 bacteria.

(b) Determine the value of k and rewrite the equation by substituting the values of N_0 and a.

The bacteria are known to double every half an hour or twice an hour.	$k = 2$ $N(t) = 15 \times 2^{2t}$

(c) How many bacteria are present after 2 hours? After 10 hours?

1 Substitute $t = 2$ in the equation to determine $N(2)$. Then determine $N(10)$.	$N(2) = 15 \times 2^{2 \times 2}$ $= 15 \times 2^4$ $= 240$	$N(10) = 15 \times 2^{2 \times 10}$ $= 15 \times 2^{20}$ $= 15\,728\,640$
2 Interpret the answer.	After 2 hours there are 240 bacteria. After 10 hours there are 15 728 640 bacteria.	

(d) How long does it take for there to be 750 bacteria present? How about 1 000 000?

1 Substitute for N.

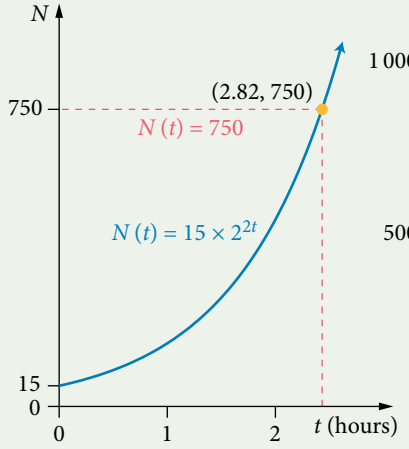
$750 = 15 \times 2^{2t}$

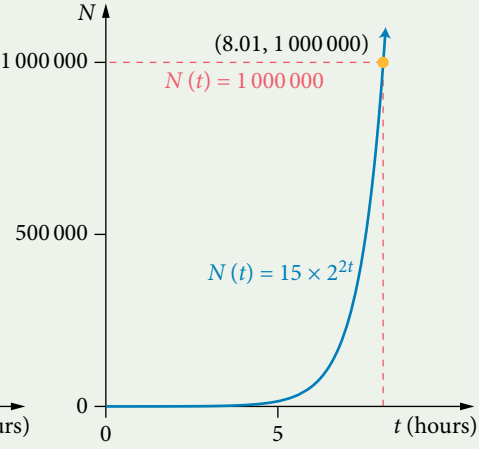
$t = 2.82\ (2\text{ d.p.})$

$1\,000\,000 = 15 \times 2^{2t}$

$t = 8.01\ (2\text{ d.p.})$

2 Graph the functions and the given population in the form $N(t) = k$, where k is a constant.



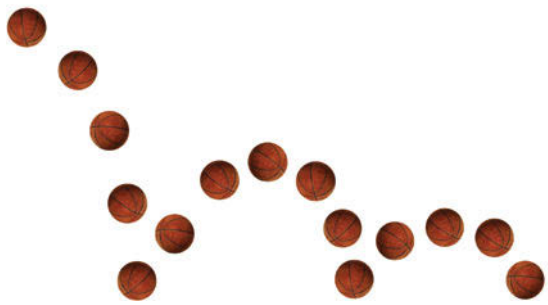


3 Interpret the answer.

The number of bacteria grows to 750 after approximately 2.82 hours, and to 1 000 000 after approximately 8.01 hours.

18 Further modelling

When a ball is dropped, the height that it returns to is 20% less than the original height. This is because some of its energy is lost on impact. The height of each successive bounce can be modelled by the equation $H(n) = H_0 \times (a)^n$, where H is the height in metres and n is the number of bounces. Suppose that the ball is dropped from a second-storey window 4 m above the ground.



(a) Determine the equation for the maximum height of each bounce.

THINKING	WORKING
1 Identify the key information.	The initial height is 4 m. It loses 20% of its height with each bounce: $a = (100 - 20)\%$ $= 80\%$ $= 0.8$
2 Substitute the values into the equation: $H(n) = H_0 \times (a)^n$	$H(n) = 4 \times (0.8)^n$

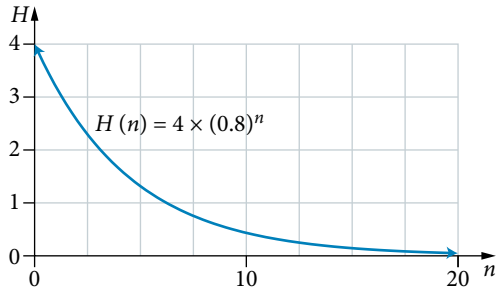
(b) For how many bounces is the ball still achieving a height of at least 2 m?

1 Substitute the given height value and solve the theoretical value for number.	$4 \times (0.8)^n = 2$ $0.8^n = \frac{2}{4}$ $n = \log_{0.8}\left(\frac{1}{2}\right)$ $= \frac{\log_{10}(0.5)}{\log_{10}(0.8)}$ $= 3.11\ (2\text{ d.p.})$
2 Take into consideration the actual limitations of the question (at least 2 m).	For three bounces the ball will achieve at least 2 m.

(c) The height of the bounce is considered to be negligible when it is less than 10 cm above the ground. How many times does the ball bounce before the height of the bounce becomes negligible?

1 Convert the units if necessary.	10 cm is 0.1 m.
2 Substitute 0.1 for H and solve for n.	$0.1 = 4 \times (0.8)^n$ $\frac{0.1}{4} = 0.8^n$ $n = \log_{0.8}(0.025)$ $= \frac{\log_{10}(0.025)}{\log_{10}(0.8)}$ $= 16.53\ (2\text{ d.p.})$
3 Interpret the answer.	The ball will bounce 16 times. The 17th bounce will be less than 10 cm.

The graph of the function from the previous worked example is given with just the whole number values of n valid.



Additional information

Depreciation

This activity explores the use of exponential functions in calculating depreciation.

Notice that after the initial value the graph is entirely in the first quadrant. The values of H decrease towards the asymptote, the horizontal axis. The model is unreliable for very small values of H.

Asymptote in models

An asymptote is a value approached by a function. Theoretically, the function never reaches this value. In real situations you can reach a value to within a reasonable margin of error.

19 Modelling asymptotic behaviour

The temperature on a hot summers day in Cloncurry is 47°. Sally buys a milk drink, which is removed from the refrigerator at 4°C. After 10 minutes the temperature of the milk is 16°C. The warming can be modelled by an exponential equation $T = A \times z^{-t} + B$, where A and B are integers, t is the time in minutes after taking the milk out of the fridge, z is the rate of warming per minute and T is the temperature in degrees Celsius of the milk.

(a) Determine the value of the constant term, given that the locker stays at air temperature.

THINKING	WORKING
1 Identify the limiting value.	The maximum temperature is 47°C.
2 Connect the value to the pronumeral.	$T = A \times z^{-t} + B$: $B = 47$

(b) Calculate A.

1 Rewrite the equation, substituting for B.	$T = A \times z^{-t} + 47$
2 Substitute initial values into the equation. Solve for A.	When $t = 0$, $T = 4$: $4 = A \times z^{0} + 47$ $A = 4 - 47$ $= -43$

(c) Calculate z correct to 2 decimal places.

1 Substitute the value of A into the equation.	$T = -43 \times z^{-t} + 47$
2 Substitute the other known pair of values into the equation. Solve for z.	When $t = 10$, $T = 16$: $16 = -43 \times z^{-10} + 47$ $-31 = -43 \times z^{-10}$ $z^{-10} = \frac{31}{43}$ $z = \left(\frac{31}{43}\right)^{-\frac{1}{10}}$ $= 1.0366\ldots$ $= 1.04\ (2\text{ d.p.})$

(d) Determine the equation for T.

Write out the defined equation by substituting known values.	$T = -43 \times 1.04^{-t} + 47$

(e) Once the temperature of the milk reaches 25°C, it is not safe to drink. How long can Sally leave it in her locker?

1 Substitute for temperature and solve for time t.

$T = 25$:

$$25 = -43 \times 1.04^{-t} + 47$$
$$-22 = -43 \times 1.04^{-t}$$
$$1.04^{-t} = \frac{22}{43}$$
$$-t = \log_{1.04}\left(\frac{22}{43}\right)$$
$$t = -\frac{\log_{10}\left(\frac{22}{43}\right)}{\log_{10}(1.04)}$$
$$= 17.09 \text{ (1 d.p.)}$$

2 Interpret the answer.

It will take approximately 17 minutes for the chocolate flavoured milk to be undrinkable.

The temperature of the milk is increasing, with an initial temperature of 4°C and approaching 47°C.

Additional information

Newton's law of cooling

Explore Newton's law of cooling.

EXERCISE 7.4 Modelling with exponentials

Worked Example 17

1 Iodine-125 is used in brachytherapy for the treatment of cancers. Small amounts of iodine are injected directly into a tumour and the radiation it produces destroys the cancer cells. Iodine-125 decays according to the equation $N(t) = N_0 \times (0.8)^{kt}$, where N is the number of milligrams of iodine present and t is the number of days since the injection.

A patient receives 40.00 mg of iodine to treat prostate cancer. After 10 days, 4.22 mg of iodine has decayed.

(a) Determine the value of N_0 correct to 4 significant figures.

(b) How much iodine is still present after 10 days? Answer to 4 significant figures, including units.

(c) Determine the value of k correct to 2 significant figures.

(d) Write the equation for the model, rounding values to 2 significant figures.

(e) How much iodine is left after 30 days? Round your answer to the nearest whole number.

(f) How long does it take for there to be 20 mg present? Answer to 2 significant figures.

(g) The level of radiation is equivalent to the normal background level after 6 half-lives; that is, the initial amount has been halved six times. It is at this stage that a scan should be taken to see whether the tumour has been successfully treated. How many days after the initial treatment should the patient have the scan? Answer to the nearest day.

2 A wealthy art-lover has learnt that the work of a particular artist is in demand and increasing in value by 12% each year. The art-lover decides to spend \$150 000 to purchase some of the artist's works. The value of the works can be modelled by the equation $V(t) = V_0 \times (a)^t$, where V is the value in dollars and t is the number of years since purchase.

(a) Write an equation to represent $V(t)$, the value of the artwork purchased, after t years. Give your answer to the nearest dollar.

(b) If the art-lover is prepared to sell the work after 5 years, how much profit is likely to have been made? Give your answer to the nearest dollar.

(c) The art-lover believes that the best time to sell is when the artwork has tripled in value. In how many years will that be? Give your answer to 2 decimal places.

3 Housing prices in the city are increasing at a rate of 7% per year. A young family has just purchased a house there for \$378 000 on zero deposit. The mortgage repayments on their 30-year mortgage are \$3320 per month.

(a) Write an equation for the value of the house $V(t)$ after t years.

(b) If the annual percentage increase were to remain the same, what would the theoretical value of the house after 30 years? (Answer to 4 significant figures.)

(c) How much will have been spent purchasing the house over the duration of the mortgage?

(d) Will the house be worth more than was spent on it after 30 years?

4 A large plumbing company lays the pipes for new housing developments. It has just spent \$400 000 on new earthmoving equipment. Each year, the company can claim 18% depreciation of the book value of the equipment as a tax deduction. The formula used for calculating the book value is $V(t) = P\left(1 - \frac{d}{100}\right)^t$, where P is the purchasing price, $V(t)$ is the book value at t years and d is the depreciation rate.

(a) Write a simplified equation to represent the book value $V(t)$ after t years.

(b) How much can be claimed in the first year?

(c) What is the book value of the equipment after 5 years? (Tax claims are to the nearest dollar.)

(d) How much can be claimed in the 6th year?

(e) Once the value of the machine is below \$100 000, the company will replace it with a new one. For how many years will the machine be kept?

5 A model for the quantity remaining over time is given as $Q(t) = 6.33 - 5.26 \times 0.85^t$.

(a) The initial quantity is:

A 0 B 1.07 C 5.26 D 8.5

(b) Explain the common error made by a student who thought the answer was 6.33.

6 The number of bacteria at an infection site is modelled by the equation $N(t) = 6 \times 2^{3t}$, where t is the time in hours after the initial exposure to the bacteria.

Choose the correct statement from the following options.

A The model shows the number of bacteria increasing rapidly at first and then the rate of increase slows down.

B The function has an asymptote that gives a limit to the number of bacteria.

C The model will be quite good as long as $t \geq 0$.

D The equation gives no limit to the number of bacteria so after a certain amount of time the model cannot be accurate.

7 The number of residents in a country town in 2012 was 5897. In 2014 it was 6589. The population has the same percentage increase every 2 years. If the population reaches 10 000, then the shire will be reclassified as a city and additional funds will be available for improving infrastructure.

(a) What was the percentage increase in the population in the 2-year period from 2012 to 2014, correct to 2 decimal places?

(b) Write an exponential equation for the relationship between the town population $P(t)$ and the time t in years since 2012.

(c) How many years will it take for the population to reach 10 000, correct to the nearest year?

Worked Example

19

8 Immediately after filling, the temperature of a hot water bottle is 83°C. As soon as the hot water bottle is filled it begins to cool, rapidly at first, and then more slowly, until it remains at constant temperature of 23°C. The temperature of the hot water bottle (T in Celsius) can be modelled by the exponential equation $T = A(z)^t + B$, where t is the time in minutes after the bottle is filled.

(a) What is the temperature that the hot water bottle will not drop below?

(b) Calculate A.

(c) If the temperature of the hot water bottle after 10 minutes was 55°C, calculate z, to 3 decimal places.

(d) Write the rule relating the temperature of the hot water bottle and the time, in minutes, since the bottle was filled.

(e) What is the temperature of the bottle after 30 minutes, correct to 1 decimal place?

(f) How long does it take the bottle to reach 28°C? Answer in minutes and seconds, correct to the nearest second.

9 Brendan had a fall and cut his knee. It was cleaned and stitched, but an estimated 3 bacteria remained in the wound. He was sent home unaware of the problem. The bacteria doubled in number every 15 minutes. Assume that the number of bacteria at any time t hours after Brendan is released from hospital can be modelled by $N(t) = Aa^{kt}$, where A, a and k are constants.

(a) Use the information provided to determine the values of A, a and k.

(b) After the number of bacteria reaches 10 000, Brendan will begin to notice a redness and increased warmth around the wound. How long is it, in hours and minutes, until this happens?

(c) The next day, his mother takes Brendan back to the hospital. He is suffering a high fever and is in pain. How many bacteria are present after the 24 hours, correct to 4 significant figures?

10 A business in a small town of 6000 residents wants to increase its share of the market. After launching an advertising campaign about buying locally, the number of people passing through the doors increases to 1200 in the first month, and continues to increase, approaching a limit of 3000 per month. The number of customers can be modelled by $N(t) = A(1.1)^{-t} + c$, where A and c are constants and t is the time in months after launching the campaign.

(a) Determine the values of c and A and hence write the rule for the model.

(b) How many people were shopping in the store prior to the advertising campaign?

(c) Over the course of the advertising campaign, what is the percentage increase in customer numbers to the nearest whole percent?

(d) Would you consider that the advertising campaign was successful?

11 Currently there is grave concern for the fate of the Tasmanian devil population in the wild, because of the deadly devil facial tumour disease (DFTD). The disease first began spreading in 1997 and it is believed that the devil population had been reduced by 85% by 2012. Projections in 2012 estimated that the devils would be extinct by 2017. The devil population figures can be modelled by $P(t) = A(b)^t - c$, where A, b and c are numbers, t is the number of years since 1997 and P is the percentage of the initial population. Demonstrate that the points fit the model $P(t) = 142.4 \times (0.9412)^t - 42.41$. Write the domain and range of the model.

12 In the middle of 2015 the population of three countries are as follows:
A: 40 million and increasing uniformly at the rate of 20 million every 5 years.
B: 45 million and increasing by 25% every 5 years.
C: 80 million and decreasing by 10% every 5 years.
Let A, B, C represent population in millions for the respective countries.

(a) Draw a graph of population vs time for the three countries and determine the year in which the population of country A first overtakes the population of country B.

(b) For how many years does country C have the largest population?

(c) The government of country C plans to alter their immigration policy when their population has halved. In what year will this occur?

13 A fish farmer began business on 1 January 2010 with a stock of 100 000 fish. He has a contract to supply 15 400 fish at a price of \$10 per fish to a retailer each year in December. Each year, in the period between January and the harvest in December, the number of fish increases by 10%.

(a) Determine the number of fish just after the second harvest in December 2011.

(b) It can be shown that $f(N)$ the number of fish just after the nth harvest, is given by $f(n) = 154\,000 - 54\,000 \times 1.1^n$, where $n = 1, 2, 3, \ldots$

(i) When will the fish farmer have half the number of fish he started with?

(ii) When will the farmer have sold all of his fish?

(iii) What will his total income be?

(c) At the end of December each year, the retailer offers to buy the farmer's business by paying \$15 per fish for his entire stock. When should the farmer sell to maximise his total income?

14 Consider the set of data below.

x	0	1	2	3	4	5	6	7	8	9	10
y	2.5	2.00	1.80	1.66	1.22	1.18	1.00	0.72	0.68	0.56	0.61

(a) Develop two different exponential models of the form $y = y_0 b^x$ to fit the data as closely as possible. Determine the effectiveness of the two models in terms of closeness of fit.

(b) Determine values given by each model when: $x = 5.5$, $x = 15$. Discuss the similarity and differences of the values for the two models, as well as their reliability.

7 Summary

Exponential equations

If $a^m = a^n$, then $m = n$.

$\left(a^x\right)^2 = a^{2x}$ so equations involving a^{2x} may use quadratic methods for solving.

Equations of the form $a^x = b$ can be solved using the solve function or by finding the intersection of the graphs $y = a^x$ and $y = b$.

Solving for the base in $x^n = a$, where $a > 0$: if n is even, there will be a positive and a negative solution; if n is odd, there will be a single solution, the same sign as a.

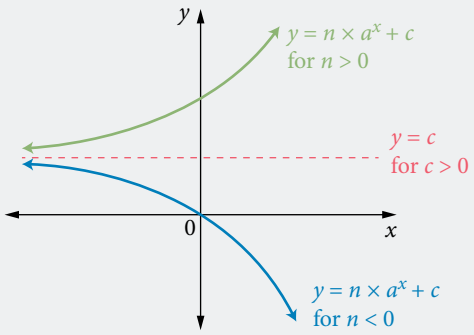
Graphing exponential functions

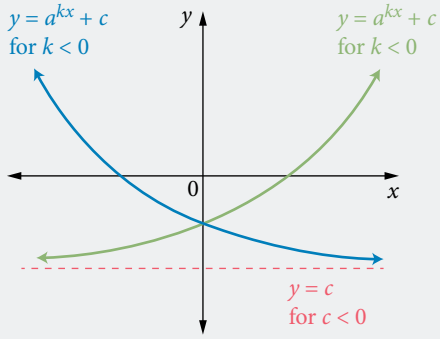
For a graph in the form $y = n \times a^x + c$:

- the equation of the horizontal asymptote is $y = c$, which represents a vertical translation of c units from $y = a^x$
- the coordinates of the y-intercept are $(0, n + c)$.
- the graph is dilated parallel to the y-axis by a factor of $|n|$
- for $n < 0$, the graph is reflected in the x-axis.

For a graph in the form $y = a^{kx} + c$:

- the equation of the horizontal asymptote is $y = c$, which represents a vertical translation of c units from $y = a^x$
- the coordinates of the y-intercept are $(0, 1 + c)$
- for $k < 0$, the graph is reflected in the y-axis, this can also be written in the form $y = \left(\frac{1}{a^{|k|}}\right)^x$.





The domain for all functions in the form $y = n \times a^{kx} + c$ is $\mathbb{R}$.

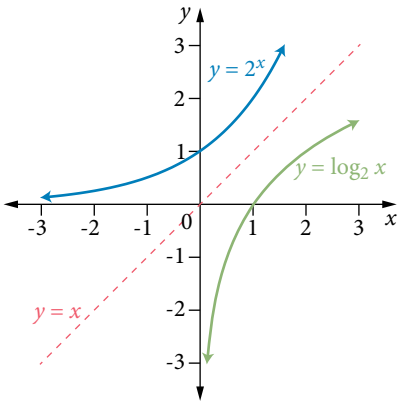
The range:

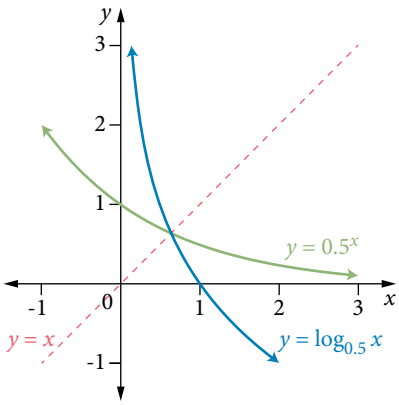
- For $n < 0$ is $(-\infty, c)$.
- For $n > 0$ is (c, ∞).

A graph in the form $y = n \times a^{k(x-b)} + c$ has been translated b units along the horizontal axis.

Logarithms

If the bases are the same, then the logarithm is the inverse of the exponential.





$a^x = y$ is equivalent to $x = \log_a(y)$ as long as $a > 0$ and $a \neq 1$.

Log laws

$$\log_a(b) = \frac{\log_{10}(b)}{\log_{10}(a)}$$

$$\log_a(m) + \log_a(n) = \log_a(m \times n)$$

$$\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$$

$$\log_a(1) = 0$$

$$\log_a(x^m) = m \times \log_a(x)$$

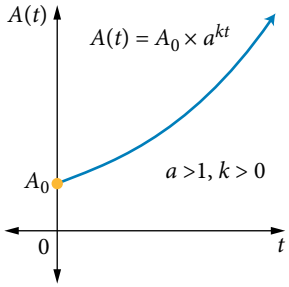
$$\log_a\left(\frac{1}{m}\right) = -\log_a(m)$$

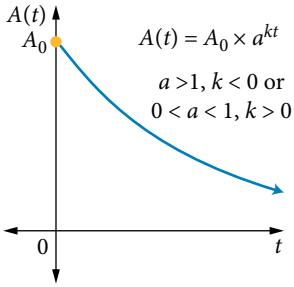
$$\log_a\left(\sqrt[n]{m}\right) = \frac{1}{n}\log_a(m)$$

$$\log_a(a) = 1$$

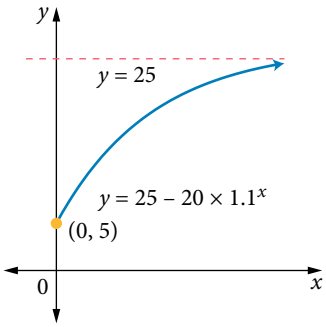
Modelling with exponential functions

For the basic model of a quantity increasing (or decreasing) exponentially over time: $A(t) = A_0 \times a^{kt}$ where A_0 is the initial amount.





Models by exponential equations where the graph has an asymptote other than the horizontal axis must include a constant term.



Chapter review 7

Exercise 7.1 **1** Solve for the pronumeral.

(a) $3^{2x+1} - 27^x = 0$ **(b)** $9^{x-1} = 27$ **(c)** $5 \times 2^{2x-3} = 10 \times 2^{x-1}$

Exercise 7.1 **2** Solve the following equations for the unknown using change of base, correct to 2 decimal places.

(a) $8^a = 23$ **(b)** $44^b = 16$ **(c)** $10^c = 0.04$

Exercise 7.1 **3** Solve the following equations for the unknown.

(a) $a^5 = -32$ **(b)** $b^6 = 64$ **(c)** $c^{\frac{3}{2}} = 125$

Exercise 7.1 **4** Determine the value of x that satisfies each of the following equations.

(a) $9^x = 3$ **(b)** $16^{x-2} = 64$

Exercise 7.1 **5** What value of x, correct to 2 decimal places, satisfies the equation $3^{2x-5} = 18$?

Exercise 7.2 **6** The function $f(x) = 3^x$ undergoes the following transformations: reflection in the y-axis, dilation by a factor of 5 parallel to the y-axis, a horizontal translation of 3 units to the left and a vertical translation of 4 units down. Determine the rule of the transformed function $g(x)$.

Exercise 7.2 **7** Identify the equation of the asymptote, domain and range for the following functions.

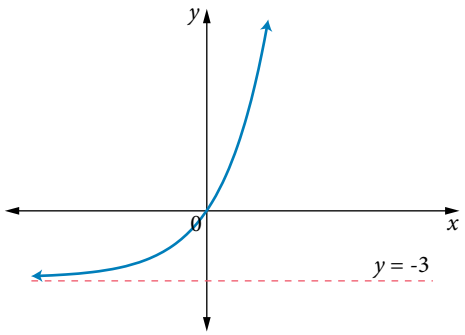
(a) $f(x) = 3^x$ **(b)** $f(x) = -5^x + 2$ **(c)** $f(x) = 2^{-x} - 7$

Exercise 7.2 **8** Sketch the graph of each transformed function and the graph of the basic function on the same set of axes. List the transformations that have occurred.

(a) $y = 4^{-x} + 2$ and $y = 4^x$ **(b)** $y = -5^{x+2}$ and $y = 5^x$ **(c)** $y = 3 \times 7^x$ and $y = 7^x$

Exercise 7.2 **9** Determine the equation of the asymptotes and the coordinates of the y-intercept for the function $f(x) = 3^{-(x-1)} + 2$?

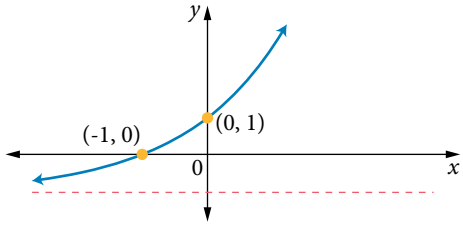
Exercise 7.2 **10** Which rule represents the following graph?



A $f(x) = 3 \times 5^{-x} - 3$ **B** $f(x) = 5^x - 3$

C $f(x) = 3 \times 5^x - 3$ **D** $f(x) = 2 \times 5^x - 3$

11 The graph below can be represented by $y = a \times 2^x - b$. Determine the values of a and b. Exercise 7.2



12 Approximate the value of each expression, correct to 2 decimal places. Exercise 7.3

(a) $\log_8 6$

(b) $\frac{1}{4}\log_3(50)$

13 Convert the following exponential equations to logarithmic form. Exercise 7.3

(a) $2^6 = 64$

(b) $44 = 5^{2-x}$

(c) $\sqrt{46} = 8^{5x}$

14 For the following equations, determine the exact solutions. Exercise 7.3

(a) $3^x = 42$

(b) $2^{x+1} = 15$

15 Simplify the following expressions and write as a single term using the log laws. Exercise 7.3

(a) $\log_3(6) + \log_3(7)$

(b) $\log_2(6) - \log_2(12)$

(c) $\log_a(2a) + \log_a(a) - \log_a(a^2)$

(d) $2\log_2(3) + \log_2(12) - \frac{1}{2}\log_2(16)$

(e) $\dfrac{\log_5(32)}{\log_5(4)}$

16 For the equation $2^x = 5$, choose the incorrect solution. Exercise 7.3

A $x = \log_2(5)$

B $x = \dfrac{\log_{10}(5)}{\log_{10}(2)}$

C $x = \dfrac{\log_7(2)}{\log_7(5)}$

D $x = \dfrac{1}{\log_5(2)}$

17 For $y = 50 \times 2^{3x}$, determine the exact value of x when $y = 750$. Exercise 7.3

18 For the equation $y = a \times 2^{nx}$, when $x = 0$, $y = 200$ and when $x = 4$, $y = 100$. Exercise 7.4

(a) Determine the value of a.

(b) Determine the value of n.

(c) Determine the value of y when $x = 16$.

19 The equation $N(t) = 50 \times 2^{3t}$ gives the number of bacteria $N(t)$ present on an agar plate t hours after the culture has begun. Exercise 7.4

(a) Determine the initial number of bacteria added to the agar plate.

(b) How long does it take, in minutes, for the initial number of bacteria to quadruple?

(c) Correct to 3 significant figures, how many bacteria are there after 10 hours?

(d) In how many hours, correct to 2 decimal places. will all the food on the plate be consumed if there is only enough to support 2×10^{40} bacteria?

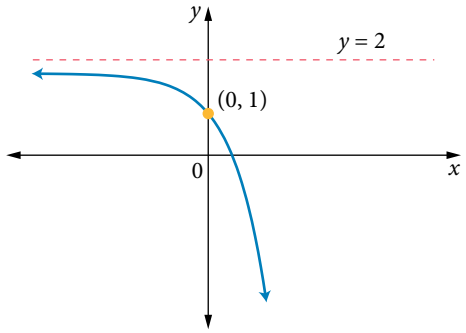
20 Solve for the pronumeral. Exercise 7.1

(a) $5^{2x} + 3 \times 5^x - 4 = 0$

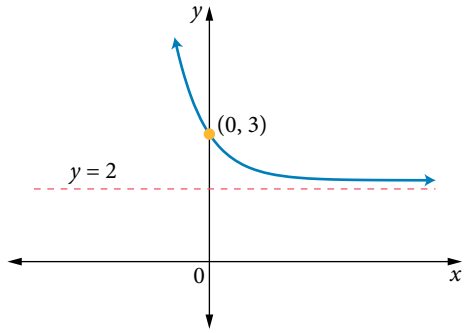
(b) $2^x + \dfrac{8}{2^x} = 9$

Exercise 7.2 **21** Choose the graph for the equation $f(x) = -3^{-2x} + 2$.

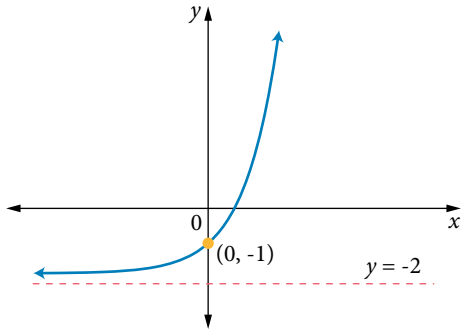
A



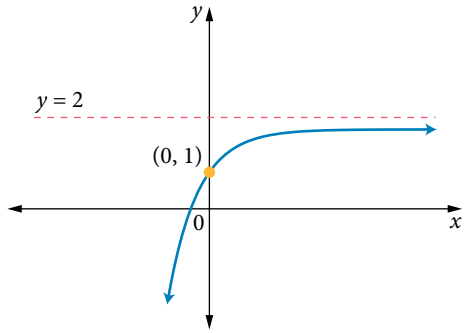
B



C



D



Exercise 7.2 **22** Consider the function $f(x) = 2^x$.

(a) Determine the transformations necessary to obtain the new function $g(x) = 5 \times 2^{-3(x-1)} + 2$.

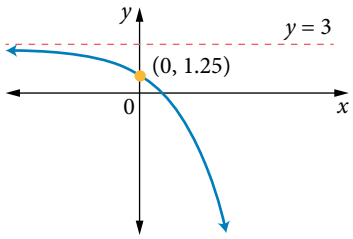
(b) Write the rule for the asymptote for $g(x)$.

(c) Write the domain and range for $g(x)$.

(d) Determine the values of $g(0)$ and $g(1)$.

(e) Plot $y = g(x)$.

Exercise 7.2 **23** The graph of $f(x) = -a \times 2^{nx} + c$ is shown below.



(a) Determine the value of c.

(b) Write the domain and range of the function.

(c) Determine the value of a.

(d) The function also passes through point (1, -11). Determine the value of n.

(e) Determine the coordinates of the x-intercept, correct to 2 decimal places.

Exercise 7.3 **24** Graph the function $f(x) = -2^{2x} + 5$ and clearly label the asymptote and the exact value of any intercepts.

25 Apply index and log laws to answer the following. Exercise 7.3

(a) Solve for x: $\log_3(2x-9)-2\log_3 x=-2$.

(b) Solve the simultaneous equations to determine the values of a and b: $a=2b$ and $\log_2 a+\log_2 b=3$.

(c) Let $f(x)=1+2^x$. Demonstrate that $f(x)\times f(-x)=f(x)+f(-x)$.

26 When $2^{7x}=5^{(x+1)}$, the exact value of x is: Exercise 7.3

A $\dfrac{\log_{10}5}{\log_{10}5-7\log_{10}2}$

B $\dfrac{\log_{10}2-\log_{10}5}{7\log_{10}5}$

C $\dfrac{7\log_{10}2-\log_{10}5}{\log_{10}5}$

D $\dfrac{\log_{10}5}{7\log_{10}2-\log_{10}5}$

27 The annual rate of inflation over the 10 years between 2002 and 2012 was around 3.27%. Exercise 7.4

(a) Write an equation to represent the relationship between the purchase price (P) of an article and its replacement value (V) after t years.

(b) If a bicycle was purchased for \$750 in 2004 and kept for 6 years, what was its replacement value if inflation is the only factor taken into account? Give your answer to the nearest dollar.

(c) If the average yearly wage of a shearer was \$65 000 in 2002 and \$80 000 in 2012, did the wages keep pace with inflation? Quantify your answer with values.

28 The mass of a radioactive element present in the soil after a minor nuclear leak is carefully monitored. It gradually decays according to the equation $C=C_0(5)^{-kt}$ where C is the concentration in grams per cubic metre in the soil after t years. The initial concentration was 270 g/m^3 and after 1 year it is 240 g/m^3. Exercise 7.4

(a) Determine the value of C_0.

(b) Determine the value of k correct to 3 significant figures.

(c) Determine the half-life of the element, to the nearest year.

29 Kangaroos were causing a problem for local farmers, so a nearby reserve was fenced off. A local conservation group thinks that the reserve can only support 200 kangaroos, so the numbers will decrease until the population reaches 200. The kangaroo numbers can be modelled according to $N(t)=A\times 0.9^t+c$, where A and c are constants and t is the number of years since the fence went up. At the end of the 4th year there are 350 kangaroos. Determine the rule for the model and the number of kangaroos trapped initially.

30 Solve the inequality for x: $9^x+2\times 3^2<3^{x+2}$ Exercise 7.1

31 A function f is defined as $f(x)=\log_m(nx^3)$, where $f(4)=0$ and $f(8)=\frac{3}{2}$. Determine f(32). Exercise 7.3

32 A metal ball is heated to 98°C and then allowed to cool to room temperature, 18°C. The temperature T°C after t minutes of cooling is given by the equation $T-18=c\times 2^{kt}$. When the ball has cooled for 5 minutes, the temperature is 48°C. Exercise 7.4

(a) Determine the average rate of cooling in the first 5 minutes, correct to 1 decimal place.

(b) After how many minutes (to the nearest minute) does the cooling rate slow to less than 1°C/minute?

(c) Determine a period of 5 minutes when the average cooling rate was about 2°C/minute.

5
6
7

8

Trigonometric functions

Recall 454

8.1 Circular measure and radian measure 456

8.2 The unit circle 462

8.3 Dilations of trigonometric graphs 478

8.4 Translations of trigonometric graphs 491

8.5 Solving trigonometric equations 503

8.6 Applications of trigonometric functions 510

Summary 517

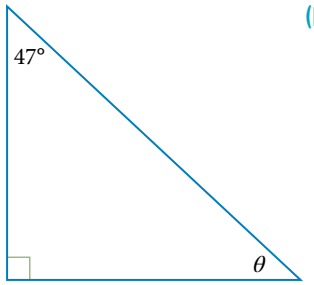
Chapter review 521

8 Recall

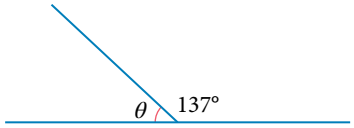
Use degree measurements for angles

1 Calculate the size of the unknown angle in each of the following diagrams.

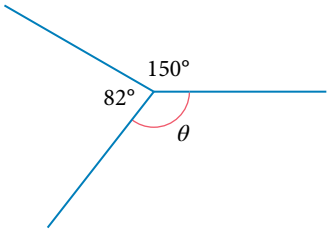
(a)



(b)



(c)



Solve linear equations

2 Solve each of the following linear equations.

(a) $2x - 5 = 6$

(b) $\frac{5x}{7} + 3 = 1$

(c) $3(x - 7) = 5(3x + 4)$

Solve quadratic equations

3 Solve each of the following quadratic equations.

(a) $x^2 - 49 = 0$

(b) $x^2 - 12x + 36 = 0$

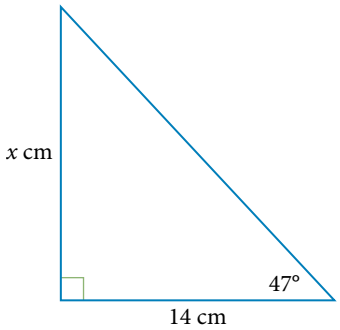
(c) $x^2 - 5x - 14 = 0$

(d) $3x^2 + 4x + 1 = 0$

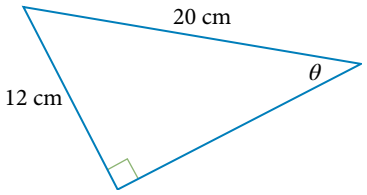
Identify the trigonometric ratios in right-angled triangles

4 Identify the trigonometric equation that would be used to solve for the unknown in each right-angled triangle.

(a)



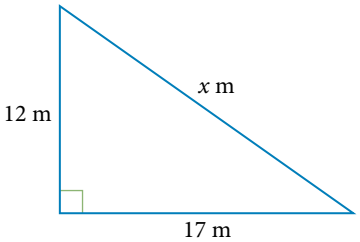
(b)



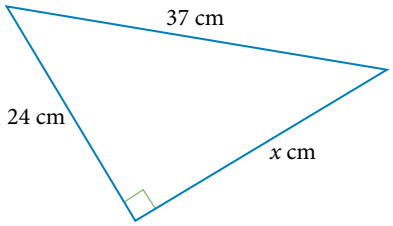
Use Pythagoras' theorem in right-angled triangles

5 Calculate the unknown side lengths in each right-angled triangle, correct to 2 decimal places.

(a)



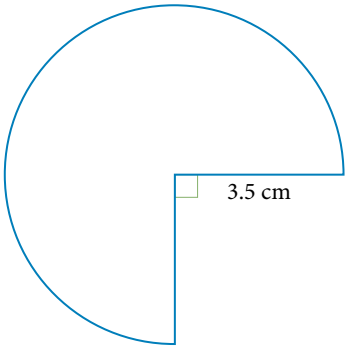
(b)



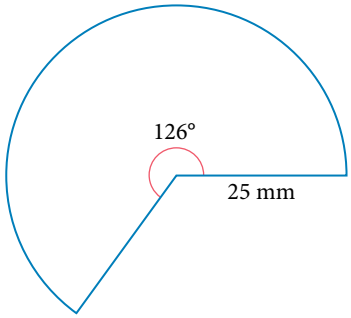
Calculate the perimeter of a sector of a circle

6 Calculate the perimeter of each sector, correct to 2 decimal places.

(a)



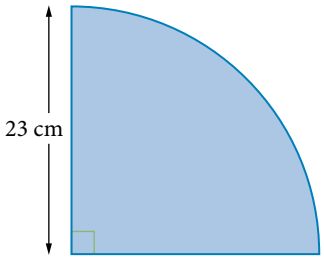
(b)



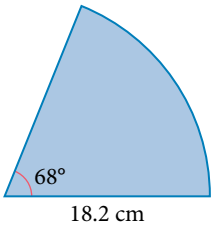
Calculate the area of a sector of a circle

7 Calculate the area of each sector, correct to the nearest square centimetre.

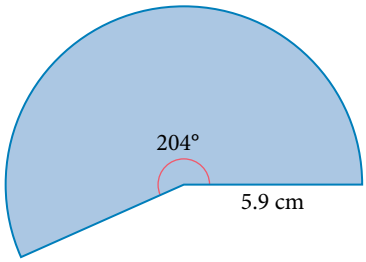
(a)



(b)



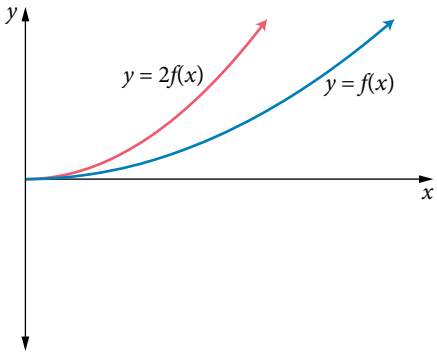
(c)



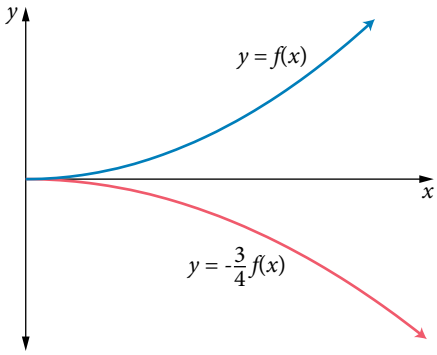
Identify the transformations of basic graphs, including dilations, reflections and translations

8 Determine the transformation that has been applied to the graph of $y = f(x)$ to obtain each function.

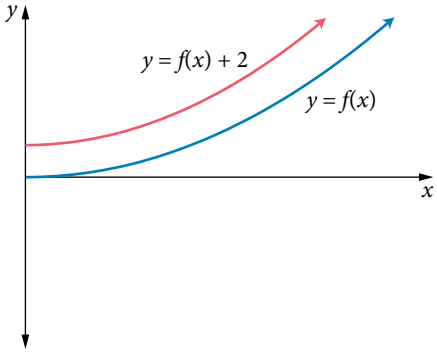
(a) $y = 2f(x)$



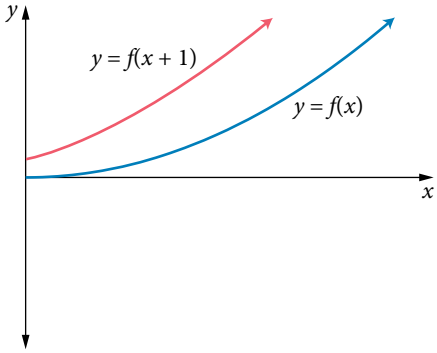
(b) $y = -\frac{3}{4}f(x)$



(c) $y = f(x) + 2$



(d) $y = f(x + 1)$



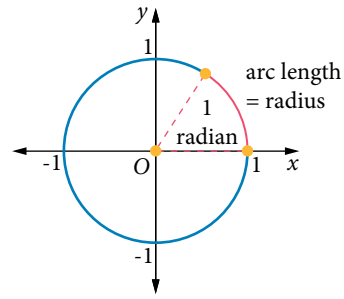
8.1 Circular measure and radian measure

Angles in degrees and radians

Angles can be measured in *degrees* or *radians*. Angles are usually measured in degrees in practical problems involving the trigonometric ratios in triangles. Angles are usually measured in radians when working with periodic functions such as modelling the depth of water during a tidal cycle. A third angle measure, *gradians*, where a right angle is exactly 100 gradians, is used in surveying.

Angles have been measured in degrees since the Babylonians used a sexagesimal number system based on the number 60. The Babylonian year was 360 days in duration (12 months of 30 days), so 360° was used for a full 'revolution'. Each degree is divided into 60 arc minutes, and an arc minute further divided into 60 arc seconds.

One radian is the angle subtended at the centre of a circle by two radii that enclose an arc one radius (r) long on the circumference of the circle. The product of the angle (measured in radians) and the radius of the circle is equal to the length of the arc on the circumference of the circle.



- An angle of 1.6 radians encloses an arc of length $1.6r$.
- An angle of 2π radians encloses an arc of length $2\pi r$.
- The circumference of a circle is $2\pi r$. A unit circle has a radius $r = 1$ unit, so the length of the circumference is 2π units and 2π radians is therefore equal to one revolution of 360°.

The symbol used for degrees is °. The symbol for radians is ^{c}. Radians are the ratio of lengths, so they do not require units. Use the symbols to clarify which angle is specified.

A full revolution: $2\pi^{c} = 360°$
A semicircle: $\pi^{c} = 180°$
A right angle of 90°: $\frac{\pi^{c}}{2} \approx 1.57^{c}$
The Greek letter θ 'theta' is often used to represent an angle.

Making connections

Radians, degrees and the unit circle
This animation demonstrates the relationship between radians, degrees and the unit circle.

Relationship between radians, degrees and the unit circle

Depending on the context, it may be necessary to convert between degrees and radians.

To convert an angle measured in degrees to radians, multiply by $\frac{\pi}{180°}$.

To convert an angle measured in radians to degrees, multiply by $\frac{180°}{\pi}$.

Additional information

Radians
View the following video for an explanation of radian measure.

$$\text{Angles in radians} = \text{angles in degrees} \times \frac{\pi}{180°} \qquad \theta^{c} = \theta° \times \frac{\pi}{180°}$$

$$\text{Angles in degrees} = \text{angles in radians} \times \frac{180°}{\pi} \qquad \theta° = \theta^{c} \times \frac{180°}{\pi}$$

Additional information

Converting between degrees and radians
Practise converting between degrees and radians.

It may be useful to write an angle in terms of π when it is measured in radians.

1 Convert degrees to radians, in terms of π

Convert 50° to radians in terms of π.

THINKING	WORKING
1 Convert the angle using the formula: $\theta^c = \theta° \times \frac{\pi}{180°}$	$\theta^c = \theta° \times \frac{\pi}{180°}$ $= 50° \times \frac{\pi}{180}$
2 Simplify and express in terms of π. Units are not needed.	$= \frac{5\pi}{18}$

2 Convert degrees to radians, as a decimal

Convert 123° to radians, correct to 2 decimal places.

THINKING	WORKING
1 Convert the angle using the formula: $\theta^c = \theta° \times \frac{\pi}{180°}$	$\theta^c = \theta° \times \frac{\pi}{180°}$ $= 123° \times \frac{\pi}{180}$
2 Round, as required, including the symbol for radians.	$= 2.15^c$ (2 d.p.)

3 Convert radians to degrees

Convert 1.78^c to degrees, correct to the nearest degree.

THINKING	WORKING
1 Convert the angle using the formula: $\theta° = \theta^c \times \frac{180°}{\pi}$	$\theta° = \theta^c \times \frac{180°}{\pi}$ $= 1.78^c \times \frac{180°}{\pi}$
2 Calculate the angle, rounding as directed.	$= 102°$ (nearest degree)

4 Convert radians involving π to degrees

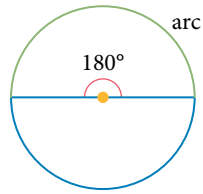
Convert $\frac{5\pi}{12}$ to degrees.

THINKING	WORKING
1 Convert the angle using the formula: $\theta° = \theta^c \times \frac{180°}{\pi}$	$\theta° = \theta^c \times \frac{180°}{\pi}$ $= \frac{5\pi}{12} \times \frac{180°}{\pi}$
2 Cancel and determine the answer.	$= 5 \times 15°$ $= 75°$

Arcs and sectors of circles

An *arc of a circle* is a section of its circumference. The length of an arc is proportional to the angle subtended at the centre of the circle and also to the radius of the circle.

For example, a semicircle subtends an angle of 180° or π^c at the centre of a circle, so the arc length is $\frac{180°}{360°} = \frac{\pi}{2\pi} = \frac{1}{2}$ of the circumference of the circle.



In general, if an arc of length l units subtends an angle of $\theta°$ at the centre of a circle with radius r, then $l = 2\pi r \times \frac{\theta}{360} = \frac{\pi r\theta}{180}$ if θ is measure is degrees.

Alternatively, $l = 2\pi r \times \frac{\theta}{2\pi} = r\theta$ if θ is measured in radians.

The length of an arc in a circle of radius r is given by:

$l = \frac{\pi r\theta°}{180}$, where $\theta°$ is the angle (measured in degrees)

$l = r\theta^c$, where θ^c is the angle (measured in radians).

Additional information

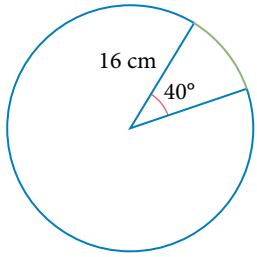
Arc length

Review and calculate the lengths of given arcs and areas of sectors.

5 Determine the length of an arc of a circle

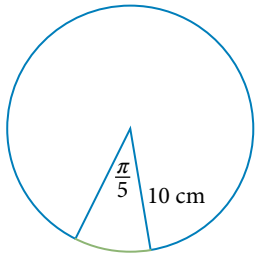
Determine the arc length correct to 2 decimal places.

(a)



THINKING	WORKING
1 Identify the length of the radius and the size of the angle subtended by the arc.	$r = 16$ cm $\theta = 40°$
2 Recall the formula for arc length (in degrees) $l = \frac{\pi r\theta}{180}$, substitute the radius and the angle and calculate the value of the unknown.	$l = \frac{\pi r\theta}{180}$ $l = \frac{\pi \times 16 \times 40}{180}$ $= 11.17$ cm (2 d.p.)
3 Interpret the answer.	The length of the arc is 11.17 cm.

(b)



1 Identify the length of the radius and the size of the angle subtended by the arc.	$r = 10$ cm $\theta = \frac{\pi}{5}$
2 Recall the formula for arc length (in degrees) $l = \frac{\pi r\theta}{180}$, substitute the radius and the angle and calculate the value of the unknown.	$l = r\theta$ $l = 10 \times \frac{\pi}{5}$ $= 2\pi$ $= 6.28$ cm (2 d.p.)
3 Interpret the answer.	The length of the arc is 6.28 cm.

Area of a sector

A *sector of a circle* is a slice of a circle bounded by two radii and an arc joining the radii. The area of a sector will be in proportion to the total area of the circle.

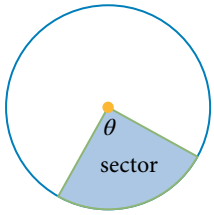
Given that the area of a circle of radius r units is πr^2, then the area of a sector of the circle that subtends an angle of θ at the centre of the circle is given by: $A = \pi r^2 \times \frac{\theta}{360°}$, where θ is measured in degrees.

If θ is measured in radians, then the area of the sector is $A = \pi r^2 \times \frac{\theta}{2\pi}$ or $\frac{r^2\theta}{2}$.

Additional information

Sectors and segments

Review and determine the area of a sector of a circle.



The area of a sector of a circle is given by:

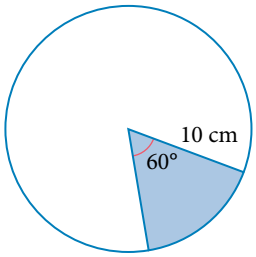
$A = \frac{\pi r^2 \theta°}{360}$, where θ is measured in degrees

$A = \frac{r^2 \theta^c}{2}$, where θ is measured in radians.

6 Find the area of a sector of a circle

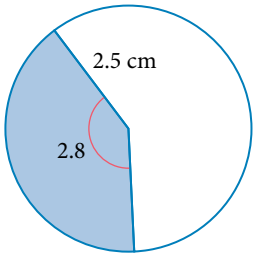
Calculate the area of the sector of a circle, correct to 2 decimal places.

(a)



THINKING	WORKING
1 Identify the length of the radius and whether the angle is in degrees or radians.	$r = 10$ cm $\theta = 60°$
2 Substitute the values of the radius and the angle into the correct formula. If necessary, round the answer to the stated degree of accuracy.	$A = \frac{\pi r^2 \theta°}{360}$ $= \frac{\pi \times 10^2 \times 60}{360}$ $= \frac{50\pi}{3}$ $= 52.36 \text{ cm}^2$ (2 d.p.)
3 Interpret the answer.	The area of the sector is 52.36 cm^2.

(b)



1 Identify the length of the radius and whether the angle is in degrees or radians.	$r = 2.5$ cm $\theta = 2.8$
2 Substitute the values of the radius and the angle into the correct formula. If necessary, round the answer to the stated degree of accuracy.	$A = \frac{r^2 \theta^c}{2}$ $= \frac{2.5^2 \times 2.8}{2}$ $= 8.75 \text{ cm}^2$
3 Interpret the answer.	The area of the sector is 8.75 cm^2.

7 Determine the area of a sector of a circle, given the arc length

Determine the area of the sector of a circle if the diameter is 12 cm and the arc length is 15 cm.

THINKING	WORKING
1 Identify the length of the radius and the length of the arc.	$D = 12$ cm $\therefore r = 6$ cm Arc length: $l = 15$ cm
2 Determine the size of the angle, in radians, that the arc subtends.	$l = r\theta$ $15 = 6 \times \theta$ $\theta = \frac{15}{6}$ $= \frac{5}{2}$
3 Substitute the radius and the calculated angle into the formula for the area of a sector.	θ is in radians, so $A = \frac{r^2\theta}{2}$ $A = \frac{1}{2} \times 6^2 \times \frac{5}{2}$ $= 45 \text{ cm}^2$

EXERCISE 8.1 Circular measure and radian measure

Worked Example

1 1 Convert the following degree measures to radians in terms of π.

(a) 30° (b) 48° (c) 150° (d) 210°
(e) 225° (f) 135° (g) 300° (h) 330°

2 2 Convert the following degree measures to radians, correct to 2 decimal places.

(a) 198° (b) 98° (c) 355° (d) 274°

3 3 Convert the following radian measures to degrees, correct to the nearest degree, where necessary.

(a) 0.224^c (b) 2.436^c (c) 4.678^c (d) $\frac{2\pi}{7}$

4 4 Convert the following radian measures to degrees.

(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{9}$ (c) $\frac{3\pi}{5}$ (d) $\frac{7\pi}{12}$
(e) $\frac{5\pi}{6}$ (f) $\frac{3\pi}{8}$ (g) $\frac{8\pi}{5}$ (h) $\frac{7\pi}{6}$

5 5 Determine the length of the arc of a circle of radius r if the arc subtends an angle of θ at the centre of the circle. Give your answer correct to 2 decimal places.

(a) $r = 12$ cm, $\theta = 50°$ (b) $r = 32$ cm, $\theta = \frac{\pi}{8}$ (c) $r = 45$ mm, $\theta = 124°$
(d) $r = 11$ m, $\theta = \frac{3\pi}{7}$ (e) $r = 33$ cm, $\theta = 213°$ (f) $r = 145$ m, $\theta = 0.468^c$
(g) $r = 1$ m, $\theta = 2.34^c$ (h) $r = 235$ mm, $\theta = 32°$

6 Determine the area of the sector of a circle, correct to 2 decimal places, given a:

(a) radius of 21 cm and the arc subtends an angle of 150°

(b) radius of 45 cm and the arc subtends an angle of 200°.

7 The arc length of a circle of radius 22 cm is 55 cm. Calculate the angle at the centre of the circle subtended by the arc, correct to the nearest degree.

8 The arc length of a circle of diameter 56 mm is 124 mm. Calculate the angle at the centre of the circle subtended by the arc, in radians correct to 2 decimal places.

9 (a) The length of the arc subtended by an angle of $\frac{\pi}{8}$ in a circle with diameter 24 cm is:

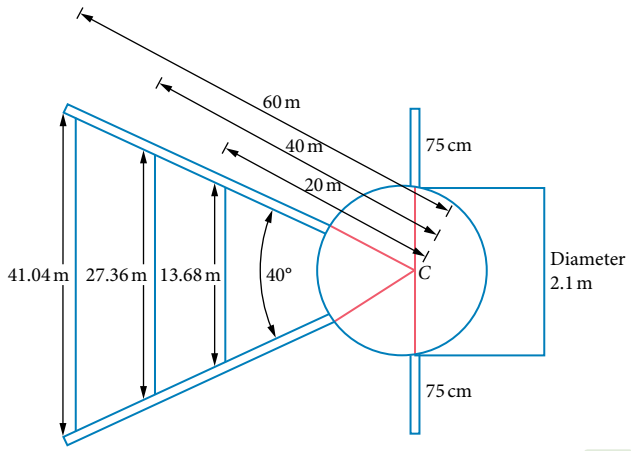
A 3 cm B 24π cm C $\frac{3\pi}{2}$ cm D 9π cm

(b) Explain the common error made by a student who thought the answer was 3π cm.

10 A shot put field is marked on a sports oval. A circle of diameter 2.1 m is painted on the ground and two radii are then marked with an angle of 40° between them at the centre.

(a) The arc between the two radii needs to be painted red. Calculate the length of this arc, correct to the nearest cm.

(b) The sector formed by the two radii is to be painted with blue paint. Calculate the area that needs to be painted, correct to the nearest cm^2.



11 Determine the area of the sector of a circle, correct to 2 decimal places, where necessary, given a:

(a) diameter of 34 cm and arc length of 45 cm

(b) radius of 12 cm and arc length of 35 cm

(c) diameter of 67 mm and arc length of 200 mm.

12 A sector of a circle of radius 15 m has an area of 75π m^2. The sector arc length, in metres, is:

A 10π B $\frac{2\pi}{3}$ C 5π D 150π

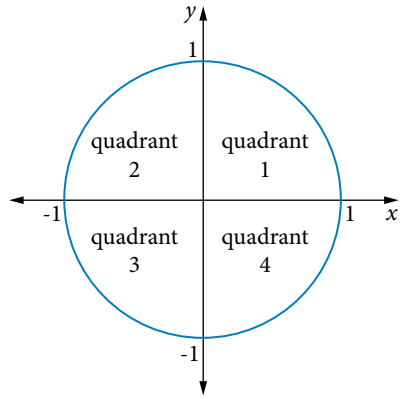
13 Determine the exact area remaining after a sector subtended by an arc length of 16 cm is cut out of a circle of radius 16 cm.

14 A segment is the region bounded by a chord and the circumference. Determine the area of a segment created by the chord joining radii of 6.3 cm with a sector angle of 1.5 radians. Give your answer correct to 2 decimal places.

15 Three sprinklers on a lawn each cover a circular area of radius 5 m. The sprinklers are located at the vertices of an equilateral triangle with side lengths of 10 m. Calculate the area between the circles that are not watered, correct to 2 decimal places.

8.2 The unit circle

The sine, cosine and tangent functions



The unit circle has its centre at the origin (0, 0) and has a radius of 1 unit. Using Pythagoras' theorem, any point (x, y) can be written as $x^2 + y^2 = 1$. The circle itself is the set of all points that satisfy this relation, where x and y are numbers, such that $-1 \le x \le 1$, or $x \in [-1, 1]$ and $-1 \le y \le 1$ or $y \in [-1, 1]$.

The unit circle is divided into four quadrants, which are numbered in an anticlockwise direction. In quadrant 1 both x and y are positive.

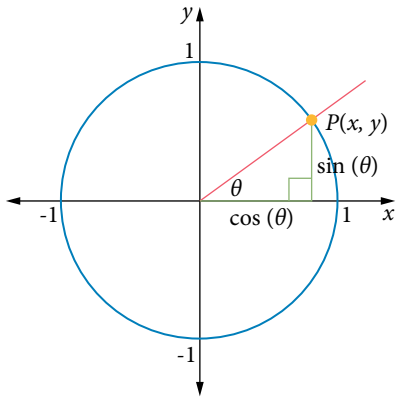
Additional information

The four quadrants of the unit circle

Explore the four quadrants of the unit circle.

A point $P(x, y)$ is located on the circumference of the unit circle, x units to the right and y units above the origin. An angle θ is measured anticlockwise from the x-axis.

A right-angled triangle is drawn as shown, where the hypotenuse extends from the origin to the point $P(x, y)$ and is 1 unit in length.



Using the trigonometric ratios:

$$\cos(\theta) = \frac{A}{H} = \frac{x}{1} = x$$

The x-coordinate of P is defined as $\cos(\theta)$.

$$\sin(\theta) = \frac{O}{H} = \frac{y}{1} = y$$

The y-coordinate of P is defined as $\sin(\theta)$.

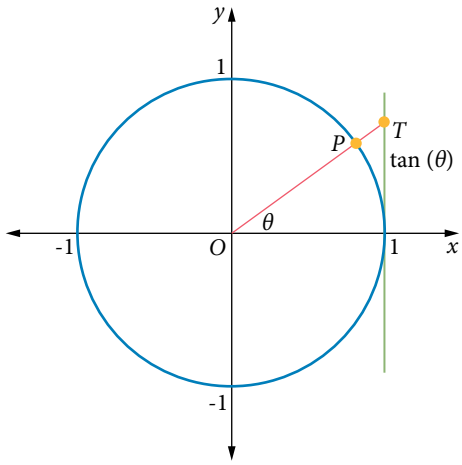
Two functions, sine and cosine, are defined to relate the angle θ to the coordinates x and y:

- x-coordinate of P on the unit circle, $x = \cos(\theta)$, $\theta \in \mathbb{R}$, $x \in [-1, 1]$
- y-coordinate of P on the unit circle, $y = \sin(\theta)$, $\theta \in \mathbb{R}$, $y \in [-1, 1]$.

The equation $\sin^2(\theta) + \cos^2(\theta) = 1$ is called the Pythagorean identity for trigonometric functions.

If a tangent to the unit circle is drawn at the point $A(1, 0)$, and the ray OP is extended until it meets this tangent at T, then the y-coordinate of T is defined as the tangent of angle θ.

By similar triangles, $\tan(\theta) = \dfrac{\tan(\theta)}{1} = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{y}{x}$.

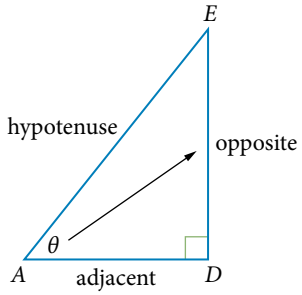


Additional information

Sine, cosine and tangent of an angle

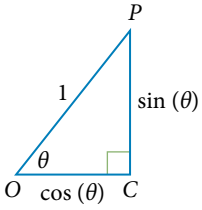
These activities explain the relationship between the trigonometric ratios and the coordinates of a point.

The functions defined on the unit circle can be related to the trigonometric ratios by similar triangles.



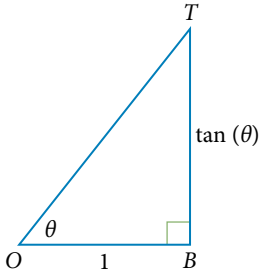
The two triangles ADE and OCP are similar because corresponding angles are equal. So the ratios of corresponding side lengths are equal.

- $\sin(\theta) = \dfrac{\sin(\theta)}{1} = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- $\cos(\theta) = \dfrac{\cos(\theta)}{1} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$



The two triangles ADE and OBT are also similar because corresponding angles are equal. So the ratios of corresponding side lengths are equal.

- $\tan(\theta) = \dfrac{\tan(\theta)}{1} = \dfrac{\text{opposite}}{\text{adjacent}}$

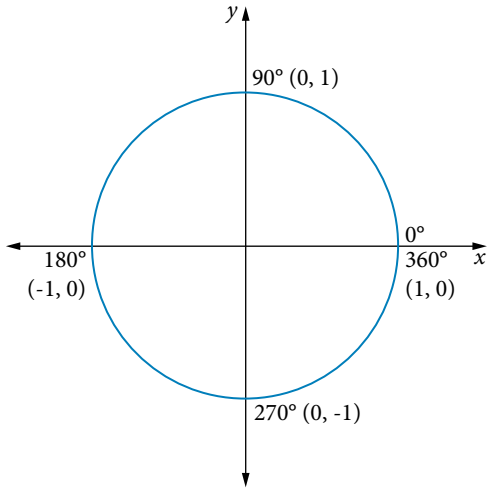


Special angles and exact values

Exact values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for particular values of θ can be readily determined from the unit circle. The values are $\theta = 0°, 90°, 180°, 270°$ and $360°$ in degrees or $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π in radians.

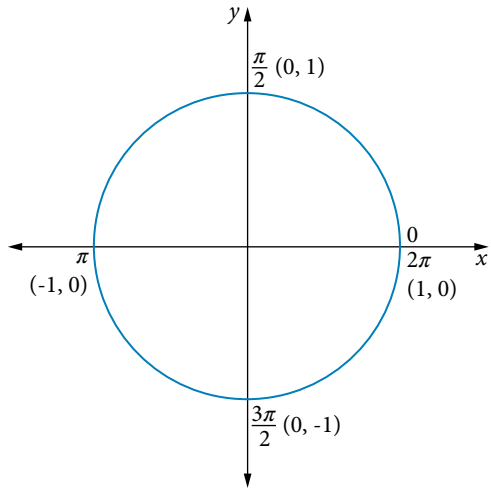
WARNING

$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ is not defined for $\cos(\theta) = 0$, therefore $\tan(\theta)$ is undefined for $90° = \dfrac{\pi}{2}$ and $270° = \dfrac{3\pi}{2}$.



Angles in degrees:

$\sin(0°) = 0$	$\cos(0°) = 1$	$\tan(0°) = 0$
$\sin(90°) = 1$	$\cos(90°) = 0$	$\tan(90°)$ is undefined
$\sin(180°) = 0$	$\cos(180°) = -1$	$\tan(180°) = 0$
$\sin(270°) = -1$	$\cos(270°) = 0$	$\tan(270°)$ is undefined
$\sin(360°) = 0$	$\cos(360°) = 1$	$\tan(360°) = 0$



Angles in radians:

$\sin(0) = 0$	$\cos(0) = 1$	$\tan(0) = 0$
$\sin\left(\frac{\pi}{2}\right) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\tan\left(\frac{\pi}{2}\right)$ is undefined
$\sin(\pi) = 0$	$\cos(\pi) = -1$	$\tan(\pi) = 0$
$\sin\left(\frac{3\pi}{2}\right) = -1$	$\cos\left(\frac{3\pi}{2}\right) = 0$	$\tan\left(\frac{\pi}{2}\right)$ is undefined
$\sin(2\pi) = 0$	$\cos(2\pi) = 1$	$\tan(2\pi) = 0$

A complete revolution of the unit circle corresponds to 360° or 2π.

The sine and cosine functions are derived from the unit circle, so the values repeat as you rotate around the circle. Angles 2π apart will produce the same sine and cosine values because they are at the same position on the unit circle, and the angles are said to be coterminal. Therefore, $\sin(\theta + 2n\pi) = \sin(\theta)$ and $\cos(\theta + 2n\pi) = \cos(\theta)$ for all θ and for every integer n.

Additional information

Angles of rotation

These activities explore the angle of rotation and points on the circle and the tangent ratio.

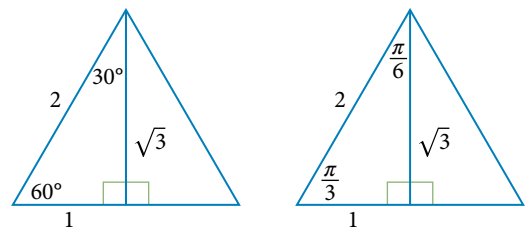
The sine and cosine functions are periodic with period 2π.

The tangent function is a little different in that angles π apart will produce the same tangent values; $\tan(\theta + n\pi) = \tan(\theta)$ for all θ for which tan is defined and for every integer n.

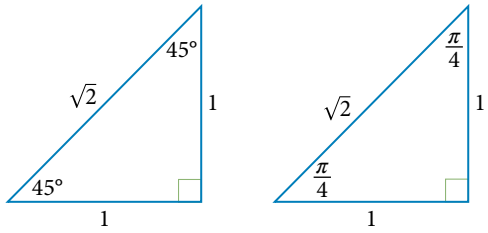
The tangent function is periodic with period π.

The exact values of the sine, cosine and tangent of certain angles are often used in mathematics and are derived from the unit circle and two special triangles.

Consider an equilateral triangle with sides of length two units, and an angle bisected as shown on the right. This triangle will have a perpendicular bisector of length $\sqrt{3}$. This can be seen by using Pythagoras' theorem. The angles can be specified in degrees or radians.



Now consider a unit square cut along its diagonal to produce a right-angled isosceles triangle as shown on the right. This triangle will have a hypotenuse of length $\sqrt{2}$. The angles may be specified in degrees or radians.



The table below summarises the exact and approximate values of sine, cosine and tangent for angles of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$ radians and the equivalent angles in degrees.

θ	$0^c\,(0°)$	$\frac{\pi}{6}(30°)$	$\frac{\pi}{4}(45°)$	$\frac{\pi}{3}(60°)$	$\frac{\pi}{2}(90°)$
$\sin(\theta)$	0	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.71$	$\frac{\sqrt{3}}{2} \approx 0.87$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.71$	$\frac{1}{2} = 0.5$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.58$	1	$\sqrt{3} \approx 1.73$	undefined

8 Find exact values of trigonometric functions

Without using your technology, determine the exact values of:

(a) $\sin(30°)$

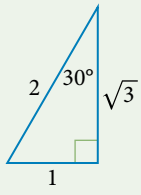
THINKING

Use the appropriate triangle to determine the exact value.

WORKING

$$\sin(\theta) = \frac{O}{H}$$

$$\sin(30°) = \frac{1}{2}$$



(b) $\cos(90°)$

Use the table to determine the exact value.

$$\cos(\theta) = x$$

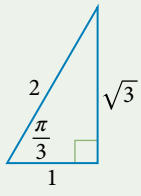
$$\therefore \cos(90°) = 0$$

(c) $\tan\left(\frac{\pi}{3}\right)$

Use the appropriate triangle to determine the exact value.

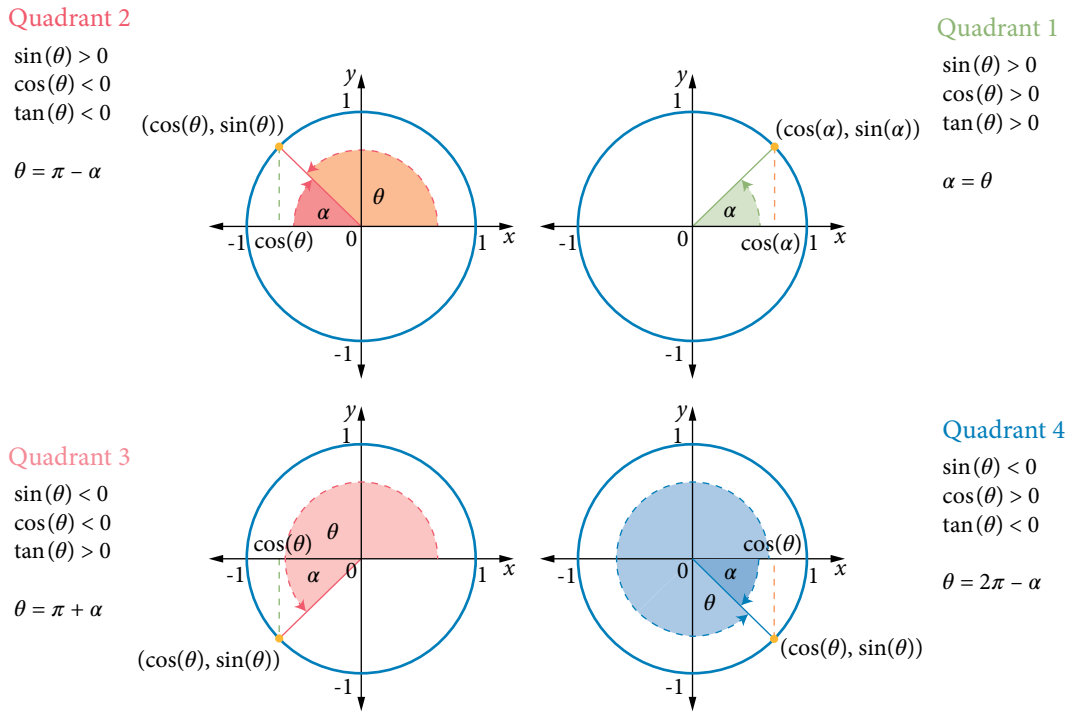
$$\tan(\theta) = \frac{O}{A}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1}$$



Angles in the four quadrants

It is only necessary to know the exact values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ for quadrant 1, as values in other quadrants can be deduced using symmetry of the unit circle. There is an equivalent angle, or reference angle, in quadrant 1 for any other angle over the domain $\mathbb{R}$. Angles in quadrants 2, 3 and 4 can be expressed in terms of the reference angle in quadrant 1, as shown in the diagram below.



The y-coordinates have the same magnitude but are positive above the x-axis and negative below the x-axis, and the x-coordinates have the same magnitude but positive to the right of the y-axis and negative on the left of the y-axis.

Technology worked example

Exact values

Additional information

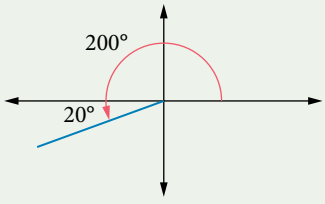
Exact values

Practise finding the exact values of trigonometric ratios for selected angles.

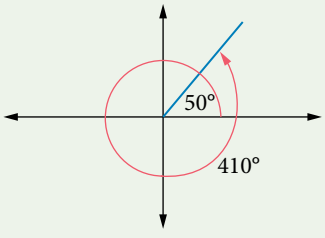
9 Determine values of trigonometric functions from related angles

Determine the value of the trigonometric functions from the given value.

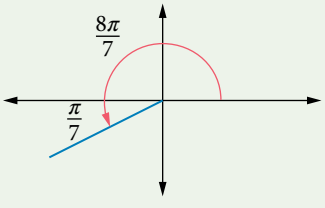
(a) $\sin(200°)$, given $\sin(20°) = 0.34$

THINKING	WORKING
1 Determine the relationship between the angles and sketch the position.	$200° = 180° + 20°$ 
2 Determine the sign of the trigonometric function by the quadrant.	In quadrant 3: $\sin(\pi + \alpha) = -y_1 = -\sin(\alpha)$
3 Determine the required exact value.	$\sin(200°) = -\sin(20°)$ $= -0.34$

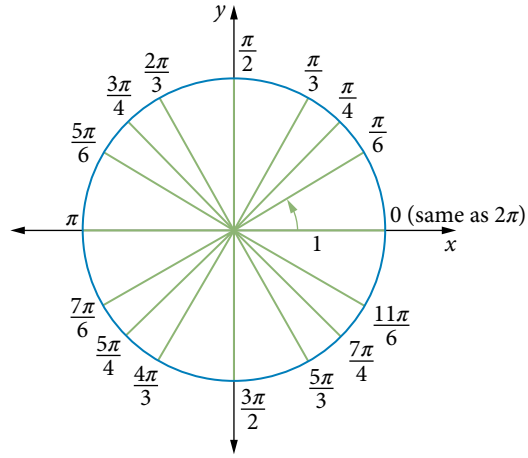
(b) $\cos(410°)$, given $\cos(50°) = 0.64$

1 Determine the relationship between the angles and sketch the position.	$410° = 360° + 50°$ 
2 Determine the sign of the trigonometric function by the quadrant.	Quadrant 1: so same position as 50°.
3 Determine the required exact value.	$\cos(410°) = \cos(50°)$ $= 0.64$

(c) $\cos\left(\frac{8\pi}{7}\right)$, given $\cos\left(\frac{\pi}{7}\right) = 0.9$

1 Determine the relationship between the angles and sketch the position.	$\frac{8\pi}{7} = \pi + \frac{\pi}{7}$ 
2 Determine the sign of the trigonometric function by the quadrant.	In quadrant 3: $\cos(\pi + \alpha) = -x_1 = -\cos(\alpha)$
3 Determine the required exact value.	$\cos\left(\frac{8\pi}{7}\right) = -\cos\left(\frac{\pi}{7}\right)$ $= -0.9$

Together with the symmetry properties of the unit circle, the exact values of sine, cosine and tangent for angles that are integer multiples of $30° = \frac{\pi}{6}$ and $45° = \frac{\pi}{4}$ can be readily determined.



Additional information

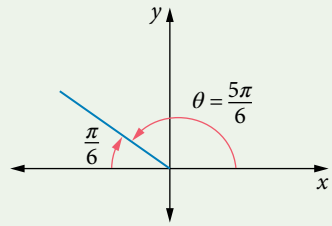
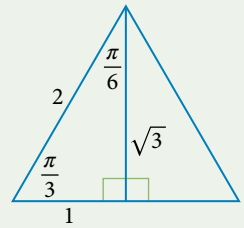
Exact values of sine and cosine angles

This activity explores the sine and cosine of an arbitrary angle.

10 Determine exact values of sine and cosine functions

Calculate the exact value of each of the following.

(a) $\cos\left(\frac{5\pi}{6}\right)$

THINKING	WORKING
1 Identify the quadrant for the angle and the related quadrant 1 angle. Use the appropriate quadrant rule.	The angle is in the second quadrant. Let $\theta = \pi - \alpha$. $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ 
2 Determine whether the value will be positive or negative.	In Quadrant 2, x-values are negative, so $\cos(\theta) = x$ is negative.
3 Determine the exact value of the related angle.	$\cos(\theta) = \frac{A}{H}$ $\therefore \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ 
4 Determine the required exact value.	$\cos\left(\frac{5\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right)$ $= -\cos\left(\frac{\pi}{6}\right)$ $= -\frac{\sqrt{3}}{2}$

(b) $\sin\left(\frac{4\pi}{3}\right)$

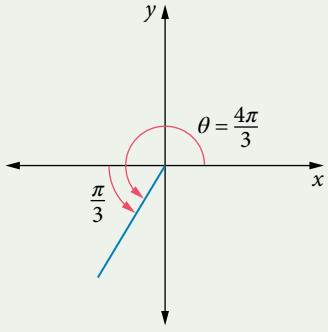
1 Identify the quadrant for the angle and the related quadrant 1 angle.

Use the appropriate quadrant rule.

The angle is in the third quadrant.

Let $\theta = \pi + \alpha$.

$\frac{4\pi}{3} = \pi + \frac{\pi}{3}$



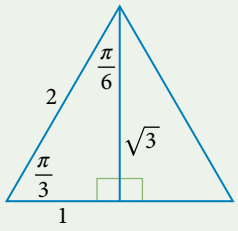
2 Determine whether the value will be positive or negative.

In Quadrant 3, y-values are negative, so $\sin(\theta) = y$ is negative.

3 Determine the exact value of the related angle.

$$\sin(\theta) = \frac{O}{H}$$

$$\therefore \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$



4 Determine the required exact value.

$$\sin\left(\frac{4\pi}{3}\right) = \sin\left(\pi + \frac{\pi}{3}\right)$$

$$= -\sin\left(\frac{\pi}{3}\right)$$

$$= -\frac{\sqrt{3}}{2}$$

(c) $\sin\left(\frac{7\pi}{4}\right)$

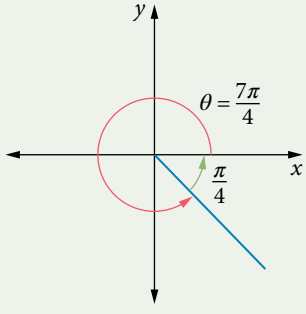
1 Identify the quadrant for the angle and the related quadrant 1 angle.

Use the appropriate quadrant rule.

The angle is in the fourth quadrant.

Let $\theta = 2\pi - \alpha$.

$\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$



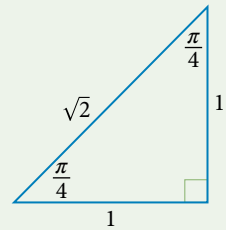
2 Determine whether the value will be positive or negative.

In Quadrant 4, y-values are negative, so $\sin(\theta) = y$ is negative.

3 Determine the exact value of the related angle.

$$\sin(\theta) = \frac{O}{H}$$

$$\therefore \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$



4 Determine the exact value of the actual angle.

$$\sin\left(\frac{7\pi}{4}\right) = \sin\left(2\pi - \frac{\pi}{4}\right)$$
$$= -\sin\left(\frac{\pi}{4}\right)$$
$$= -\frac{1}{\sqrt{2}}$$
$$= -\frac{\sqrt{2}}{2}$$

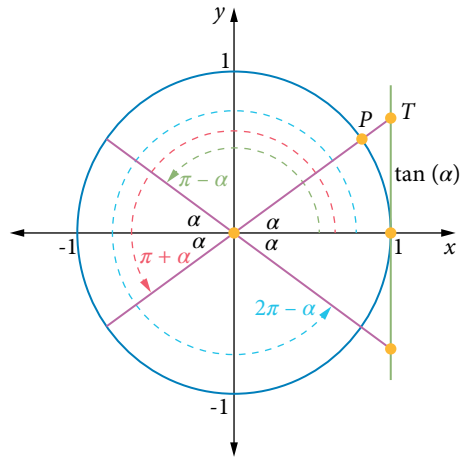
Tangent in four quadrants

$\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{y}{x}$, $\tan(\theta)$ is positive in quadrants 1 and 3, but negative in quadrants 2 and 4.

In quadrant 2, $\tan(\pi - \alpha) = -\tan(\alpha)$.

In quadrant 3, $\tan(\pi + \alpha) = \tan(\alpha)$.

In quadrant 4, $\tan(2\pi - \alpha) = -\tan(\alpha)$.

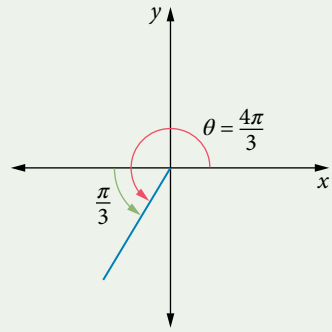


11 Determine exact values of tangent functions

Calculate the exact value of each of the following.

(a) $\tan\left(\dfrac{4\pi}{3}\right)$

THINKING	WORKING
1 Identify the quadrant for the angle and the related quadrant 1 angle. Use the appropriate quadrant rule.	The angle is in the third quadrant. Let $\theta = \pi + \alpha$. $\dfrac{4\pi}{3} = \pi + \dfrac{\pi}{3}$



2 Determine whether the value will be positive or negative.

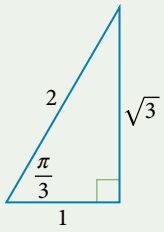
In quadrant 3:

$\tan(\pi + \alpha) = \tan(\alpha)$

3 Determine the exact value of the related angle.

$$\tan(\theta) = \frac{O}{A}$$

$$\therefore \tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1}$$



4 Determine the exact value of the actual angle.

$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right)$$
$$= \tan\left(\frac{\pi}{3}\right)$$
$$= \sqrt{3}$$

(b) $\tan\left(\frac{5\pi}{6}\right)$

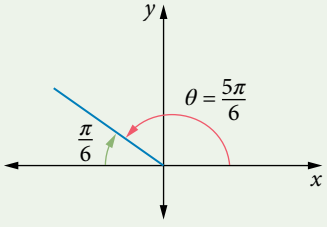
1 Identify the quadrant for the angle and the related quadrant 1 angle.

Use the appropriate quadrant rule.

The angle is in the second quadrant.

Let $\theta = \pi - \alpha$.

$$\frac{5\pi}{6} = \pi - \frac{\pi}{6}$$



2 Determine whether the value will be positive or negative.

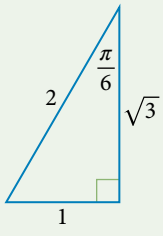
In quadrant 2:

$\tan(\pi - \alpha) = -\tan(\alpha)$

3 Determine the exact value of the related angle.

$$\tan(\theta) = \frac{O}{A}$$

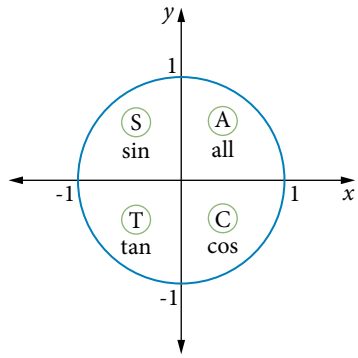
$$\therefore \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$



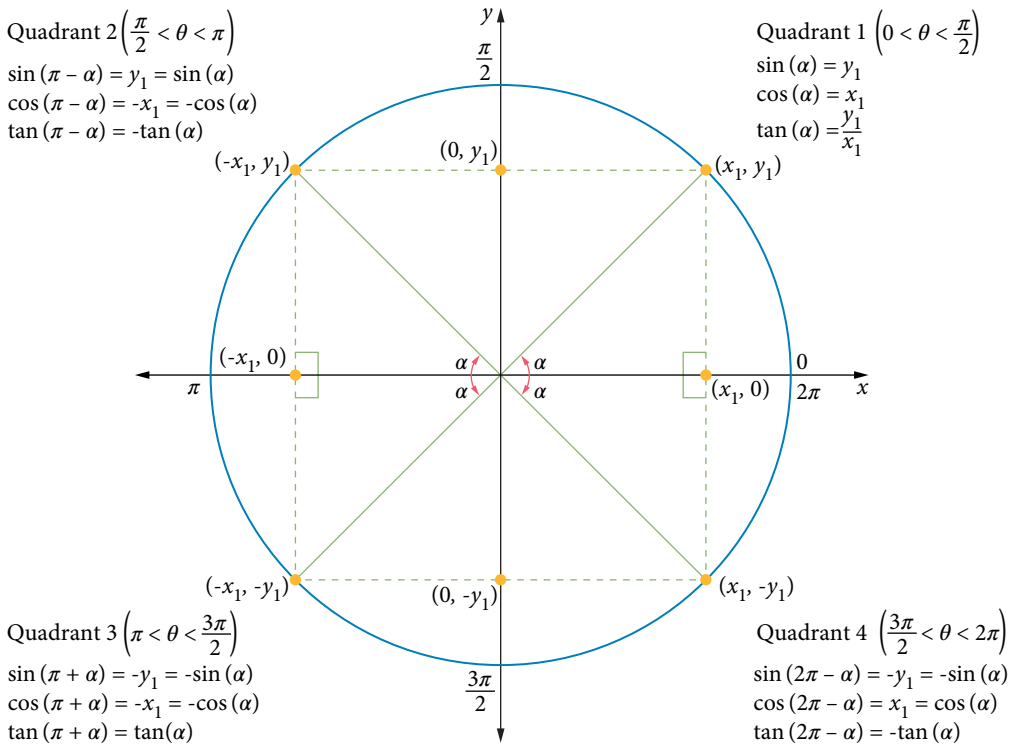
4 Determine the exact value of the actual angle.

$$\tan\left(\frac{5\pi}{6}\right) = \tan\left(\pi - \frac{\pi}{6}\right)$$
$$= -\tan\left(\frac{\pi}{6}\right)$$
$$= -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= -\frac{\sqrt{3}}{3}$$

Rather than try to remember whether $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ are positive or negative, it is easier to remember the quadrants in which they are positive.

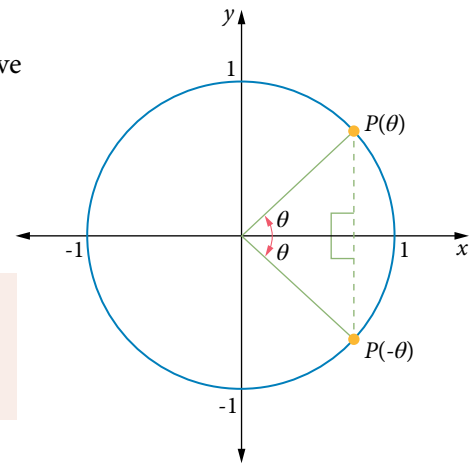


The symmetry of the four quadrants of the unit circle allows you to see the relationships between angles in each of the four quadrants.



Negative angles

A negative angle can be thought of as a movement from the positive x-axis in a clockwise direction. The symmetry properties of the unit circle can be used to determine the sine, cosine or tangent of a negative angle, by writing it in terms of its equivalent positive angle.



By symmetry, it can be seen that:

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\tan(-\theta) = -\tan(\theta)$

WARNING

Although the diagram shows θ as acute, the relationships are true for angles of any size.

Additional information

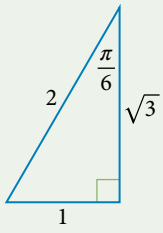
The general angle

This activity further explores trigonometric functions.

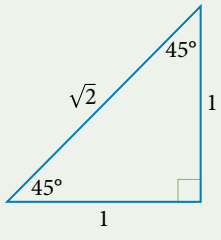
12 Determine exact values of trigonometric functions using symmetry (acute)

Calculate the exact value of each of the following.

(a) $\cos\left(-\frac{\pi}{6}\right)$

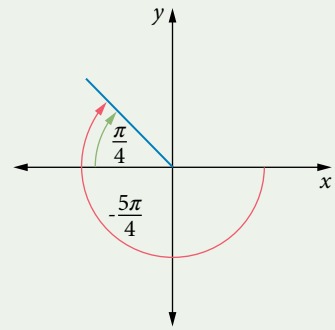
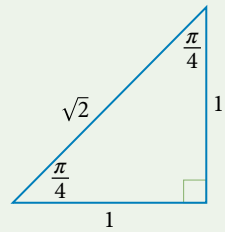
THINKING	WORKING
1 Determine whether the value will be positive or negative.	In quadrant 4, x-values are positive, so $\cos(\theta) = x$ is positive.
2 Determine the exact value of the related angle.	$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ 
3 Determine the required exact value.	$\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$ $= \frac{\sqrt{3}}{2}$

(b) $\tan(-45°)$

1 Determine whether the value will be positive or negative.	In quadrant 4, x-values are positive and y-values are negative, so $\tan(\theta) = \frac{y}{x}$ is negative.
2 Determine the exact value of the related angle.	 $\tan(45°) = 1$
3 Determine the required exact value.	$\tan(-45°) = -\tan(45°)$ $= -1$

13 Determine exact values of trigonometric functions using symmetry (reflex)

Calculate the exact value $\tan\left(-\frac{5\pi}{4}\right)$.

THINKING	WORKING
1 Determine the related angle and sketch the position.	$\frac{5\pi}{4} = \pi + \frac{\pi}{4}$ 
2 Determine whether the value will be positive or negative.	In quadrant 2, x-values are negative and y-values are positive, so $\tan(\theta) = \frac{y}{x}$ is negative.
3 Determine the exact value of the related angle.	$\tan\left(\frac{\pi}{4}\right) = 1$ 
4 Determine the required exact value.	$\tan\left(-\frac{5\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right)$ $= -1$

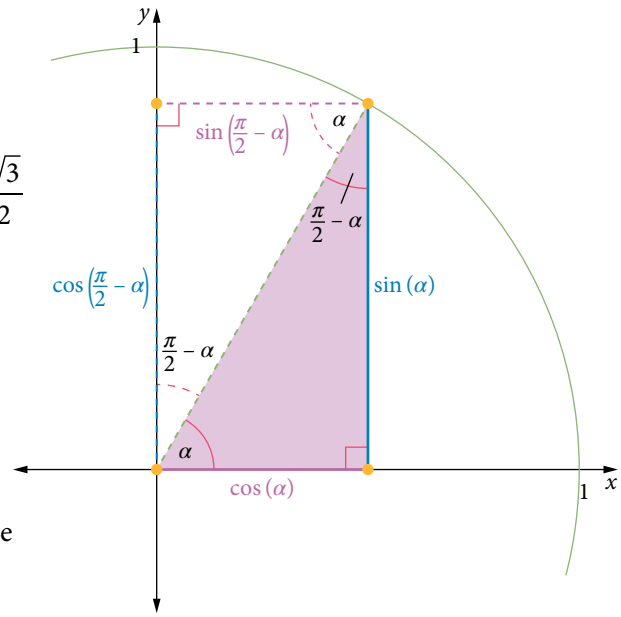
Complementary relationships

You will have noticed that some values appear more than once on the trigonometric summary table.

For example:

$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \Leftrightarrow \cos(30°) = \sin(60°) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \Leftrightarrow \cos(60°) = \sin(30°) = \frac{1}{2}$$



The two triangles in the diagram are congruent because all corresponding side lengths and angles are equal.

It can be seen from these triangles that the blue length represents both $\sin(\alpha)$ and $\cos\left(\frac{\pi}{2} - \alpha\right)$, and the purple length represents both $\cos(\alpha)$ and $\sin\left(\frac{\pi}{2} - \alpha\right)$.

These are called complementary relationships for sine and cosine.

The symmetry properties of the unit circle mean that similar complementary relationships can be found for the other quadrants. The complementary relationships for sine and cosine are summarised in the diagram.

Quadrant 2

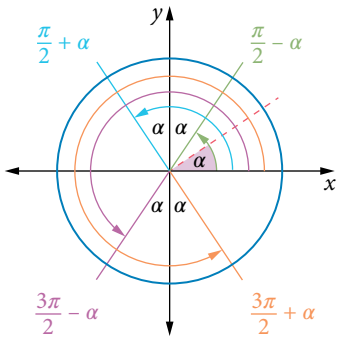
$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha)$

$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos(\alpha)$

Quadrant 1

$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha)$

$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha)$



Quadrant 3

$\cos\left(\frac{3\pi}{2} - \alpha\right) = -\sin(\alpha)$

$\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos(\alpha)$

Quadrant 4

$\cos\left(\frac{3\pi}{2} + \alpha\right) = -\sin(\alpha)$

$\sin\left(\frac{3\pi}{2} + \alpha\right) = -\cos(\alpha)$

WARNING

Do not assume that α has to be an acute angle; the relationships are true for angles of any size.

14 Determine values of trigonometric functions using complementary angles

Given that $\cos(35°) = 0.82$, determine the value of $\sin(125°)$.

THINKING	WORKING
1 Determine the relationship between the angles.	$35° = 90° - 55°$ $125° = 180° - 55°$
2 Write the complementary relationship for first quadrant angles.	$\sin(55°) = \cos(35°)$
3 Determine whether the value will be positive or negative.	In quadrant 2, y-values are positive, so $\sin(\theta) = y$ is positive.
4 Determine the required value.	$\sin(125°) = \sin(55°)$ $= \cos(35°)$ $= 0.82$

EXERCISE 8.2 The unit circle

Worked Example 8

1 Determine the exact values of each trigonometric ratio.

(a) $\sin(60°)$ (b) $\cos(45°)$ (c) $\sin(90°)$ (d) $\tan(45°)$

(e) $\cos\left(\frac{\pi}{6}\right)$ (f) $\tan\left(\frac{\pi}{4}\right)$ (g) $\sin\left(\frac{\pi}{6}\right)$ (h) $\tan\left(\frac{\pi}{2}\right)$

Worked Example

2 Determine the value of the following.

(a) $\sin(90°)$ (b) $\cos\left(-\frac{\pi}{2}\right)$ (c) $\tan(0)$

(d) $\sin(\pi)$ (e) $\cos(180°)$ (f) $\tan\left(\frac{3\pi}{2}\right)$

9

3 Determine the value of the trigonometric functions from the given value, correct to two decimal places.

(a) $\sin(310°)$, given $\sin(50°) = 0.77$ (b) $\cos(460°)$, given $\cos(80°) = 0.17$

(c) $\tan\left(\frac{19\pi}{10}\right)$, given $\tan\left(\frac{\pi}{10}\right) = 0.32$ (d) $\cos(400°)$, given $\cos(40°) = 0.77$

(e) $\tan(260°)$, given $\tan(80°) = 5.67$ (f) $\sin\left(\frac{8\pi}{9}\right)$, given $\sin\left(\frac{\pi}{9}\right) = 0.34$

10

4 Calculate the exact value of each of the following:

(a) $\cos\left(\frac{2\pi}{3}\right)$ (b) $\sin\left(\frac{7\pi}{4}\right)$ (c) $\sin\left(\frac{7\pi}{6}\right)$ (d) $\cos\left(\frac{7\pi}{6}\right)$

(e) $\sin\left(\frac{5\pi}{3}\right)$ (f) $\cos\left(\frac{3\pi}{4}\right)$ (g) $\cos\left(\frac{11\pi}{6}\right)$ (h) $\sin\left(\frac{3\pi}{4}\right)$

12

5 Calculate the exact value of each of the following.

(a) $\sin\left(-\frac{\pi}{3}\right)$ (b) $\tan(-30°)$ (c) $\cos(-45°)$

(d) $\tan(-60°)$ (e) $\cos\left(-\frac{\pi}{4}\right)$ (f) $\sin(-30°)$

6 Calculate the following cos and sin values, correct to 4 decimal places.

(a) $\cos(35°)$ (b) $\sin(55°)$ (c) $\cos\left(\frac{\pi}{5}\right)$ (d) $\sin(2)$

(e) $\cos(50°)$ (f) $\sin(40°)$ (g) $\cos\left(\frac{2\pi}{7}\right)$ (h) $\sin(1.1)$

7 If θ is in the first quadrant and $\sin(\theta) = \frac{\sqrt{2}}{2}$, determine the value of $\cos(\theta)$.

A $\frac{\sqrt{3}}{2}$ B $\frac{\sqrt{3}}{3}$ C $\frac{\sqrt{2}}{2}$ D $\frac{1}{2}$

8 Students run around a circular track at a sports carnival. The first aid equipment is set up at the starting point. The best way to get the equipment to a student requiring assistance is around the track, in either direction. Determine the number of degrees and the direction the first aid equipment should be taken if a student has run:

(a) 300° in a clockwise direction (b) 50° in an anticlockwise direction

(c) 400° in a clockwise direction.

9 Determine the following values from the given information.

(a) If θ is an angle in the first quadrant and $\cos(\theta) = 0.669$, determine the value of $\cos(\pi - \theta)$, $\cos(\pi + \theta)$ and $\cos(4\pi - \theta)$.

(b) If θ is an angle in the first quadrant and $\sin(\theta) = 0.324$, determine the value of $\sin(\pi + \theta)$, $\sin(2\pi + \theta)$ and $\sin(-\theta)$.

(c) If θ is an angle in the first quadrant and $\tan(\theta) = 0.839$, determine the value of $\tan(\pi - \theta)$, $\tan(6\pi + \theta)$ and $\tan(5\pi + \theta)$.

14

10 Determine the value of each trigonometric ratio from the given value.

(a) $\cos(35°)$, given $\sin(55°) = 0.8192$ (b) $\sin(40°)$, given $\cos(50°) = 0.6428$

(c) $\sin(100°)$, given $\cos(10°) = 0.9848$

11 Calculate the following values from the given information.

(a) Given tan(48°) = 1.1106, determine the values of tan(228°) and tan(312°).

(b) Given sin(234°) = -0.8090, determine the values of cos(36°) and cos(144°).

12 Simplify the following expressions using complementary angle properties and symmetry properties of the unit circle, given that $0 \le \theta \le \frac{\pi}{2}$.

(a) $\sin\left(\frac{5\pi}{2} - \theta\right)$ (b) $\sin\left(\frac{7\pi}{2} - \theta\right)$ (c) $\cos\left(\frac{5\pi}{2} + \theta\right)$ (d) $\cos\left(\theta - \frac{\pi}{2}\right)$

13 Calculate the exact value of each of the following.

(a) sin(225°) (b) tan(135°) (c) cos(330°) (d) cos(240°)

Worked Example 11

14 Calculate the exact value of each of the following.

(a) $\tan\left(\frac{7\pi}{6}\right)$ (b) $\tan\left(\frac{3\pi}{4}\right)$ (c) $\tan\left(\frac{2\pi}{3}\right)$ (d) $\tan\left(\frac{11\pi}{6}\right)$

Worked Example 13

15 Calculate the exact value of each of the following.

(a) $\tan\left(-\frac{7\pi}{6}\right)$ (b) $\cos\left(-\frac{3\pi}{4}\right)$ (c) sin(-120°) (d) $\tan\left(-\frac{5\pi}{3}\right)$

(e) cos(-225°) (f) $\sin\left(-\frac{5\pi}{6}\right)$ (g) tan(-300°) (h) sin(-210°)

(i) cos(-135°)

16 Calculate the exact value of each of the following.

(a) sin(390°) (b) tan(495°) (c) cos(-420°) (d) cos(480°)

(e) sin(420°) (f) tan(-390°)

17 Consider the trigonometric ratio cos(-240°).

(a) Determine the exact value of the ratio.

A $\frac{\sqrt{3}}{2}$ B $-\frac{\sqrt{3}}{2}$ C $\frac{\sqrt{2}}{2}$ D $-\frac{1}{2}$

(b) Explain the common error made by a student who thought the answer was $\frac{1}{2}$.

18 If sin(1) = 0.84, determine values of each of the following using $\pi \approx 3.14$.

(a) sin(5.28) (b) cos(2.57) (c) sin(315)

19 The outer radius of the tyre in the diagram is 33.9 cm. The valve cap on a bicycle wheel is 31.1 cm from the centre of the wheel. From the position shown, the wheel starts to roll. Determine the distance of the valve cap above the ground, correct to the nearest millimetre, after the wheel has rotated clockwise by the following angles.

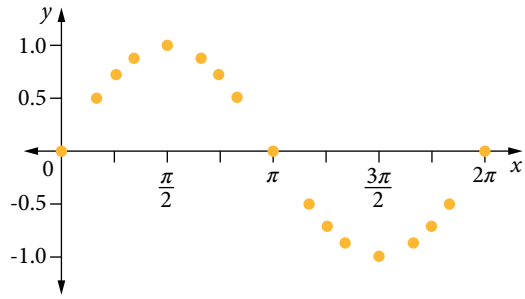
(a) 60° (b) 300°

(c) 390° (d) 420°

8.3 Dilations of trigonometric graphs

Graphs of the sine function $y = \sin(x)$

The graph of $y = \sin(x)$ for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ from 0 to 2π is shown.



Making connections

The unit circle and the sine graph

Move the sliders to view the relationship between the unit circle and the sine graph.

A graph of the sine function over domain $\mathbb{R}$ (all real numbers) forms a smooth and continuous curve that repeats itself at intervals of 2π. The length of one complete cycle is called the period of the graph. The graph of $y = \sin(x)$ has a period of 2π radians.

From 0 to 2π is one cycle, but any interval of 2π also represents one cycle.

The maximum value of $y = \sin(x)$ is 1.

The minimum value $y = \sin(x)$ is -1.

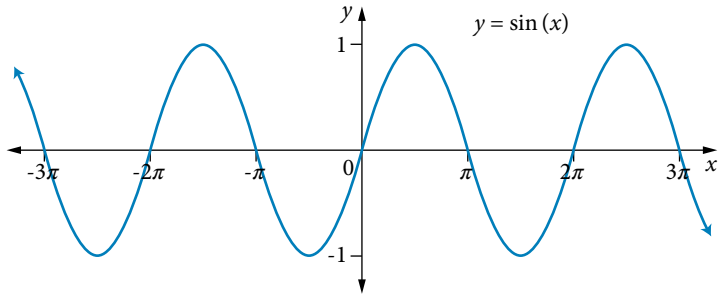
The distance between the median position (the 'horizontal central axis') of the graph and the maximum or minimum position is called the amplitude. The graph of $y = \sin(x)$ has an amplitude of 1.

The domain of $y = \sin(x)$ is $x \in \mathbb{R}$ and the range is $y \in [-1, 1]$.

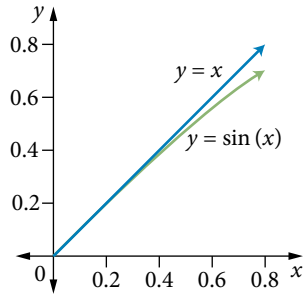
Additional information

The sine function

This activity explores the graph of the sine function.

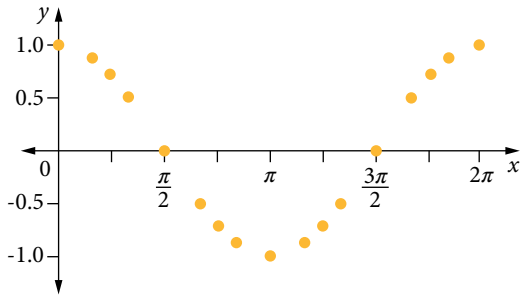


The graphs of $y = \sin(x)$ and $y = x$ are very close for small values of x. For example, $\sin(0.1) = 0.0998$, and $\sin(0.5) = 0.4792$. This leads to the approximation $\sin(x) \approx x$ for small values of x. Note that $\sin(x) < x$ for all positive values of x. This is helpful when checking on expected results or making estimates.



Graphs of the cosine function $y = \cos(x)$

The graph of $y = \cos(x)$ for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ from 0 to 2π is shown below.



Making connections

The unit circle and cosine graph

Move the sliders to view the relationship between the unit circle and the cosine graph.

Additional information

The cosine function

This activity explores the graph of the cosine function.

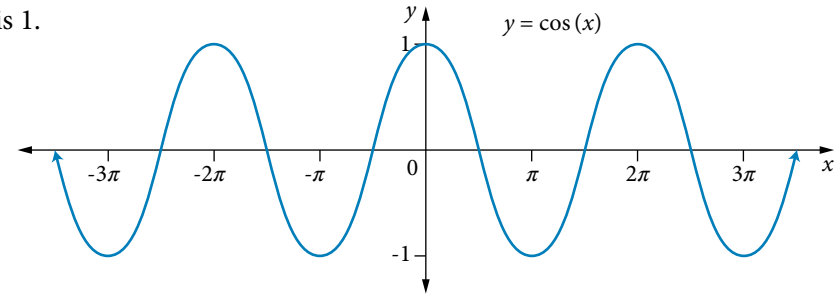
Like the sine function, a graph of the cosine function over domain $\mathbb{R}$ forms a smooth and continuous curve that repeats itself at intervals of 2π. The graph of $y = \cos(x)$ has a period of 2π radians.

The maximum value of $y = \cos(x)$ is 1.

The minimum value is -1.

The graph of $y = \cos(x)$ has an amplitude of 1.

The domain of $y = \cos(x)$ is $x \in \mathbb{R}$ and the range is $y \in [-1, 1]$.



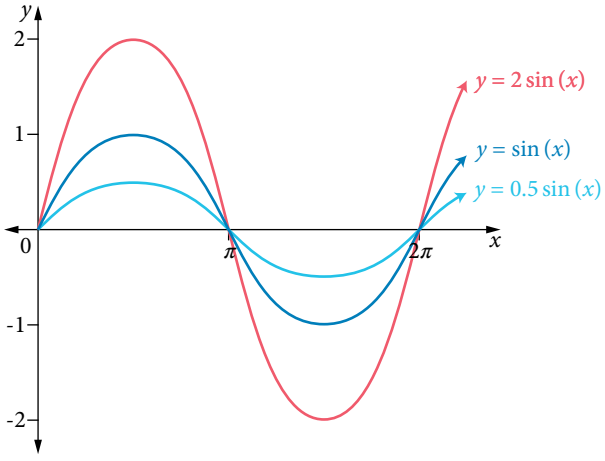
The graph of $y = \cos(x)$ can be obtained from graph of $y = \sin(x)$ by a translation of $\frac{\pi}{2}$ to the left.

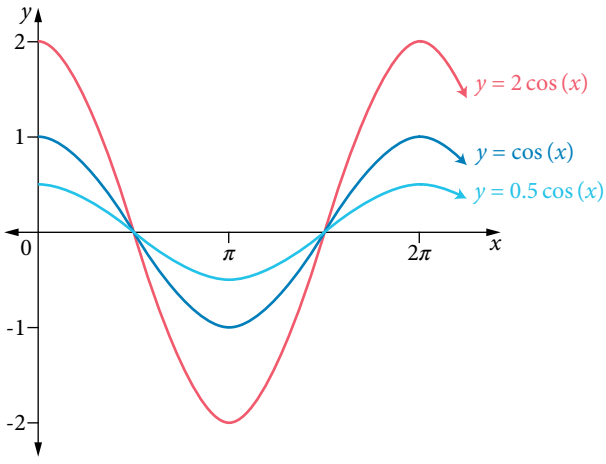
Vertical dilations

The basic functions $y = \sin(x)$ and $y = \cos(x)$, when dilated by a factor of $|A|$ parallel to the y-axis, produce the functions $y = A\sin(x)$ and $y = A\cos(x)$.

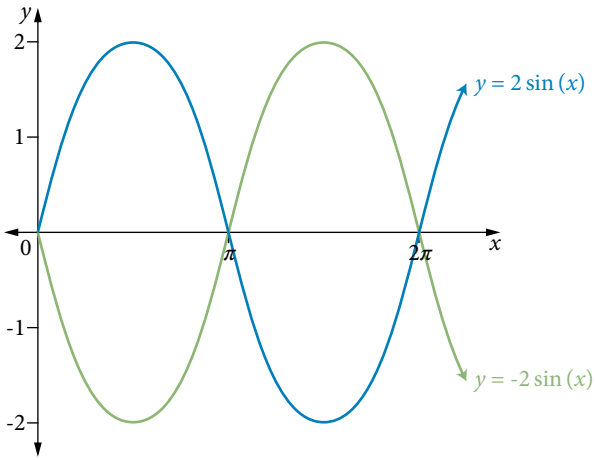
For $|A| > 1$, the graph of the basic function is stretched away from the x-axis.

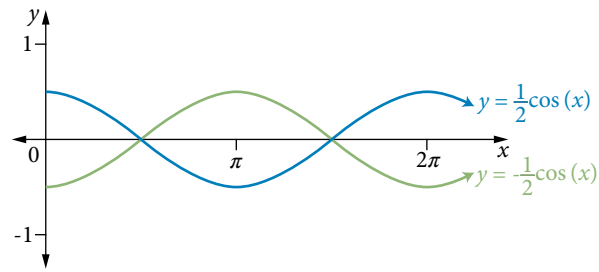
For $0 < |A| < 1$, the graph of the basic function is compressed towards the x-axis.





For $A < 0$, the graph of the basic function is also reflected in the x-axis.





For the functions in the form $y = A\sin(x)$ and $y = A\cos(x)$:

The amplitude of the functions is given by $|A|$.

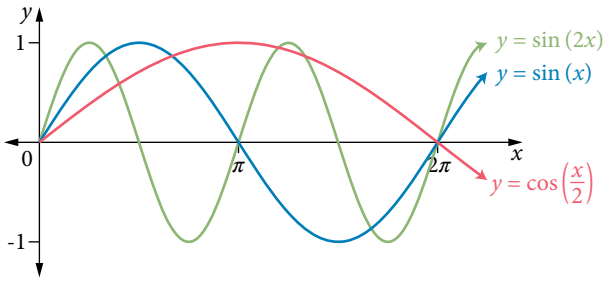
The range of the functions is $[-|A|, |A|]$

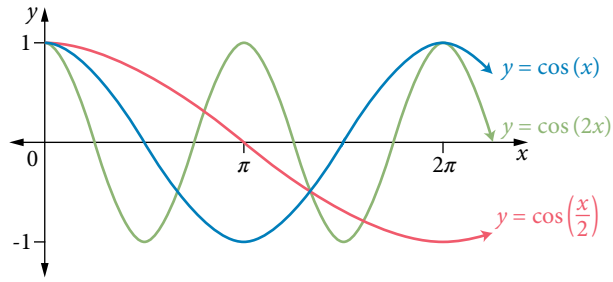
Horizontal dilations

The basic functions $y = \sin(x)$ and $y = \cos(x)$, when dilated by a factor of $\frac{1}{|B|}$ parallel to the x-axis, produce the functions $y = \sin(Bx)$ and $y = \cos(Bx)$.

For $|B| > 1$, the graph of the basic function is compressed towards the y-axis.

For $0 < |B| < 1$, the graph of the basic function is stretched away from the y-axis.





The period of the functions $y = \sin(Bx)$ and $y = \cos(Bx)$ can be calculated using the formula:

$$\text{Period} = \frac{2\pi}{|B|}$$

Additional information

Amplitude and period

Practise finding the period and amplitude.

WARNING

If B is negative, simplify the expression first rather than work with negative values of B.

Remember: $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$

15 Amplitude and period of sine and cosine functions

Determine the amplitude and period of the graphs of the following functions.

(a) $y = 2\sin(x)$

THINKING	WORKING
1 Identify the values of A and B for the transformed function $y = A\sin(Bx)$.	$y = 2\sin(x)$ $A = 2$ and $B = 1$
2 Determine the amplitude from the value of A.	$\lvert A\rvert = \lvert 2\rvert = 2$ The graph extends 2 units above and below the median line.
3 Determine the period from the value of B.	$\frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$ The period or horizontal distance to complete 1 cycle is 2π units.

(b) $y = -\cos(3x)$

1 Identify the values of A and B for the transformed function $y = A\sin(Bx)$.	$y = -\cos(3x)$ $A = -1$ and $B = 3$
2 Determine the amplitude from the value of A.	$\lvert A\rvert = \lvert -1\rvert = 1$ The graph extends 1 unit above and below the median line.
3 Determine the period from the value of B.	$\frac{2\pi}{B} = \frac{2\pi}{3}$ The period or horizontal distance to complete 1 cycle is $\frac{2\pi}{3}$ units.

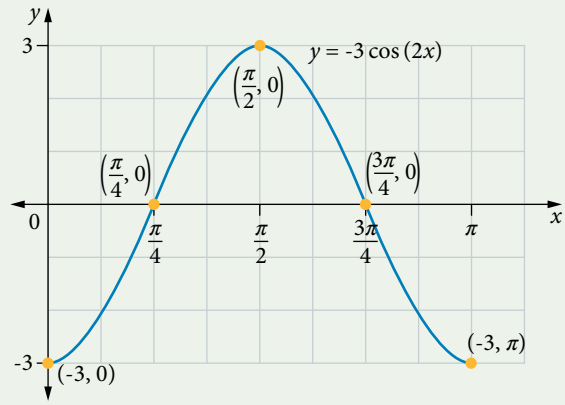
(c) $y = 4\sin(-\pi x)$

1 Recall the formula $\sin(-\theta) = -\sin(\theta)$ and simplify first so the coefficient of x is positive.	$y = 4\sin(-\pi x)$ $= -4\sin(\pi x)$
2 Identify the value of A and B for the transformed function $y = A\sin(Bx)$.	$y = -4\sin(\pi x)$ $A = -4$ and $B = \pi$
3 Determine the amplitude from the value of A.	$\lvert A\rvert = \lvert -4\rvert = 4$ The graph extends 4 units above and below the median line.
4 Determine the period from the value of B.	$\frac{2\pi}{B} = \frac{2\pi}{\pi} = 2$ The period or horizontal distance to complete 1 cycle is 2 units.

16 Sketch one period of the graphs of sine and cosine functions

Sketch one period of the graph of each function from the y-axis.

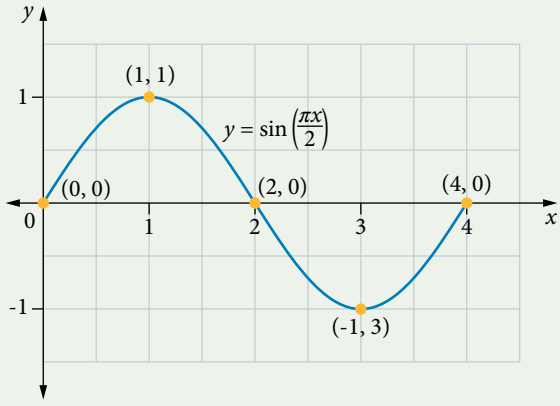
(a) $y = -3\cos(2x)$

THINKING	WORKING
1 Identify the value of A and its effect.	$y = -3\cos(2x)$ $A = -3$ The curve has an amplitude of 3 and is reflected in the x-axis.
2 Identify the value of B and its effect.	$B = 2$ Period $= \dfrac{2\pi}{2} = \pi$
3 Draw appropriate scales on the set of axes. Divide a full period into four equal parts.	Use intervals of $\frac{\pi}{4}$. The minimum y-value is -3. The maximum y-value is 3.
4 Mark in the minimum and maximum y-values and draw the smooth curve.	

(b) $y = \sin\left(\dfrac{\pi x}{2}\right)$

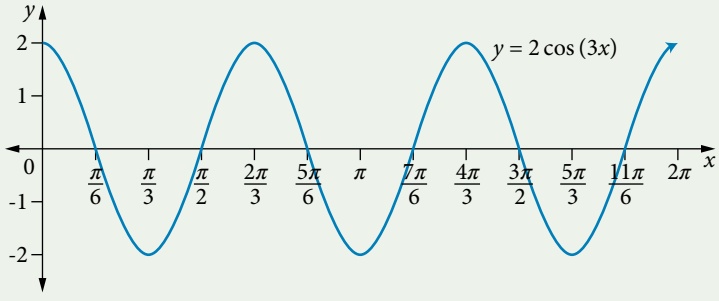
1 Identify the value of A and its effect.	$y = \sin\left(\dfrac{\pi x}{2}\right)$ $A = 1$ The curve has an amplitude of 1 and there is no reflection in the x-axis.
2 Identify the value of B and its effect.	$B = \dfrac{\pi}{2}$ Period $= \dfrac{2\pi}{\frac{\pi}{2}} = 4$
3 Draw appropriate scales on the set of axes. Divide a full period into four equal parts.	Use intervals of 1 The minimum y-value is -1. The maximum y-value is 1.

4 Mark in the maximum and minimum y-values and draw the smooth curve.



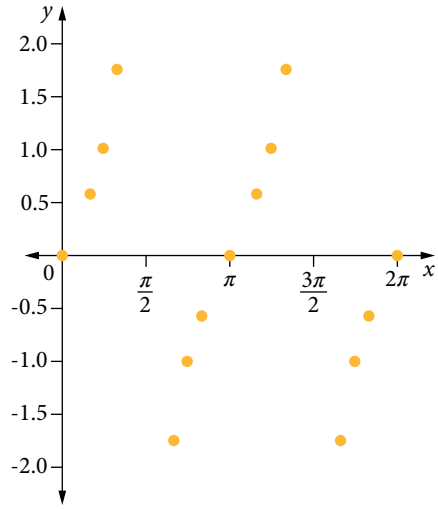
17 Sketch graphs of sine and cosine functions over specified domains

Sketch the graph of $y = 2\cos(3x)$ over the domain $[0, 2\pi]$.

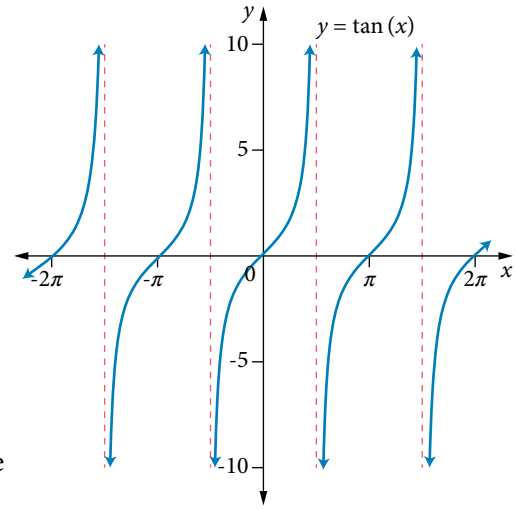
THINKING	WORKING
1 Identify the value of A and its effect.	$y = 2\cos(3x)$ $A = 2$ The curve has an amplitude of 2.
2 Identify the value of B and its effect.	$B = 3$ The graph has a period of $\frac{2\pi}{3}$.
3 Draw appropriate scales on the set of axes. Divide a full period into four equal parts.	The period is $\frac{2\pi}{3}$. $\frac{2\pi}{3} \div 4 = \frac{2\pi}{3} \times \frac{1}{4} = \frac{\pi}{6}$ Use intervals of $\frac{\pi}{6}$. The minimum y-value is -2. The maximum y-value is 2.
4 Mark in the minimum and maximum y-values and draw the smooth curve.	

Graph of the tangent function $y = \tan(x)$

The graph of $y = \tan(x)$ for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ from 0 to 2π is shown below.



A graph of the tangent function extending over domain $\mathbb{R}$ forms a smooth and continuous curve, except at the values for which $\tan(x)$ is undefined. At these values, which are odd integer multiples of $\frac{\pi}{2}$, the graph of the function has vertical asymptotes. The graph repeats itself after an interval of π. The graph of $y = \tan(x)$ has a period of π radians.



The domain of $y = \tan(x)$ is $x \in \mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}\right\}$ where n is an integer, and the range is $y \in \mathbb{R}$.

The lines $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ are the vertical asymptotes.

These vertical asymptotes occur at a spacing of π (which is the period of the tangent graph).

Additional information

The unit circle and the tangent function

This activity explores the relationship between the unit circle and the graph of the tangent function.

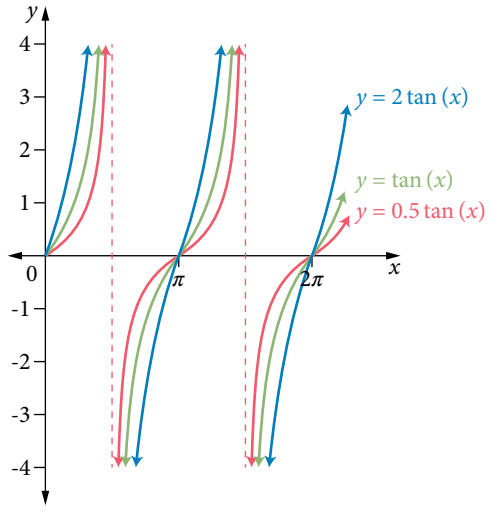
Sketch graphs of $y = A\tan(Bx)$

Vertical dilations

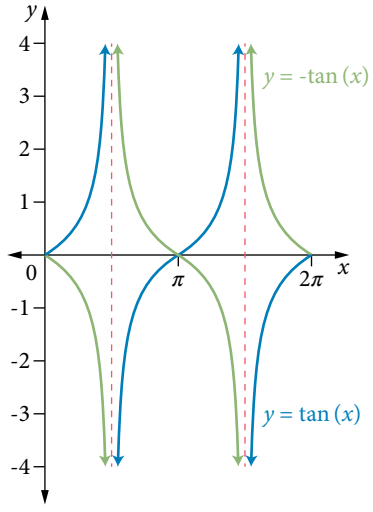
The basic graph of $y = \tan(x)$ is transformed into $y = A\tan(x)$ by a dilation of $|A|$ from the x-axis, parallel to the y-axis.

- The asymptotes and x-intercepts remain unchanged.
- The range is $\mathbb{R}$ so the amplitude is undefined.

- If $A > 0$, the graph is dilated vertically by a factor of A parallel to the y-axis.



- If $A < 0$, the graph is reflected in the x-axis.

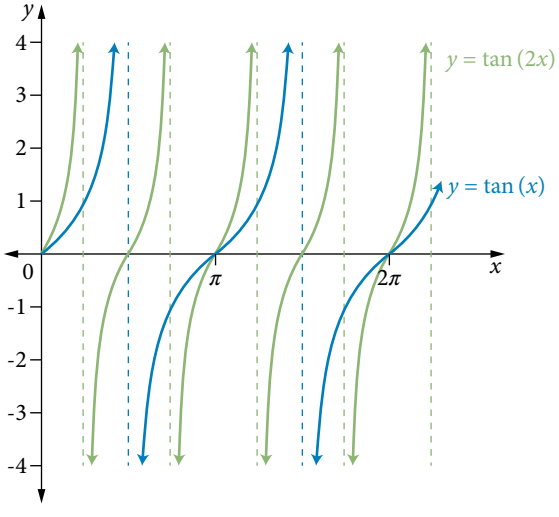


Horizontal dilations

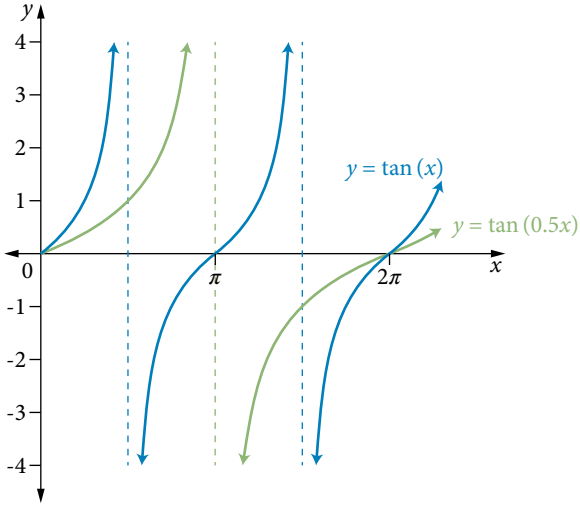
The basic graph of $y = \tan(x)$ is transformed into $y = \tan(Bx)$ by a dilation of $\frac{1}{|B|}$ from the y-axis, parallel to the x-axis.

- The period is $\frac{\pi}{B}$, where $B > 0$.
- The x-intercepts are 1 period apart, starting from the y-axis.
- The first asymptote is half a period from the y-axis and then asymptotes are a period apart at $\pm\frac{\pi}{2B}, \pm\frac{3\pi}{2B}\ldots$

- If $B > 0$, then the graph is squashed horizontally by a factor of $\frac{1}{B}$ parallel to the x-axis.
  $y = \tan(x)$ has a period of π.
  $y = \tan(2x)$ has a period of $\frac{\pi}{2}$.



- If $B < 0$, then the graph is stretched horizontally by a factor of $\frac{1}{B}$ parallel to the x-axis.
  $y = \tan\left(\frac{x}{2}\right)$ has a period of 2π.



Explore further

The function $y = A\tan(Bx)$

Explore the effect of the parameters A and B on the graph $y = A\tan(Bx)$.

18 Asymptotes and x-intercepts of tangent functions

Determine the x-intercepts and equations of the asymptotes for graphs of the following functions.

(a) $y = 3\tan(x)$

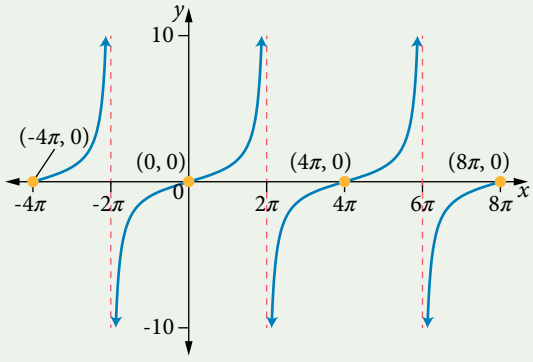
THINKING	WORKING
1 Determine the period $\frac{\pi}{B}$: $y = A\tan(Bx)$	$B = 1$: Period: $\frac{\pi}{B} = \frac{\pi}{1} = \pi$
2 Write the x-intercepts, starting at the origin and then one each period to the left and right.	x-intercepts at $x = 0$, and multiples of π to the left and right: $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ In general, $x = n\pi$, where n is any integer.
3 Write the equations of the asymptotes, starting half-way between two x-intercepts and then one each period to the left and right.	Asymptotes: $x = \frac{\pi}{2}$, and multiples of π to the left and right: $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ In general, $x = \frac{\pi}{2} + n\pi$ or $x = \frac{(2n+1)\pi}{2}$, where n is any integer.

(b) $y = -\tan(2x)$

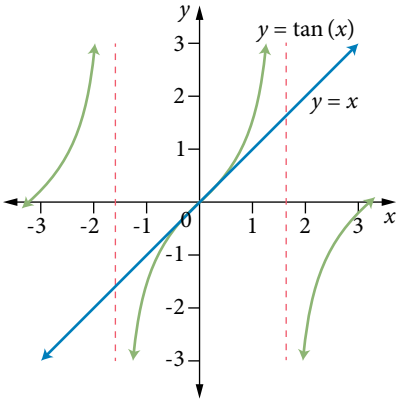
1 Determine the period $\frac{\pi}{B}$: $y = A\tan(Bx)$	$B = 2$: Period: $\frac{\pi}{B} = \frac{\pi}{2}$
2 Write the x-intercepts, starting at the origin and then one each period to the left and right.	x-intercepts at $x = 0$, and multiples of $\frac{\pi}{2}$ to the left and right: $\ldots, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \ldots$ In general, $x = \frac{n\pi}{2}$, where n is any integer.
3 Write the equations of the asymptotes, starting half-way between two x-intercepts and then one each period to the left and right.	Asymptotes: $x = \frac{\pi}{4}$, and multiples of $\frac{\pi}{2}$ to the left and right: $\ldots, -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$ In general, $x = \frac{\pi}{4} + \frac{n\pi}{2}$ or $x = \frac{(2n+1)\pi}{4}$, where n is any integer.

19 Sketch graphs of tangent functions over a specified domain

Sketch the graph of $y = \tan\left(\frac{x}{4}\right)$ over the domain $[-4\pi, 8\pi]$.

THINKING	WORKING
1 Determine the period $\frac{\pi}{B}$: $y = A\tan(Bx)$	$B = \frac{1}{4}$: Period: $\frac{\pi}{B} = \pi \div \frac{1}{4} = \pi \times 4 = 4\pi$
2 Determine the coordinates of the x-intercepts, multiple periods from the origin.	x-intercepts at $x = 0$, and multiples of 4π to the left and right for $[-4\pi, 8\pi]$: $-4\pi, 0, 4\pi, 8\pi$
3 Determine the equations of the asymptotes, starting half-way between two x-intercepts and then one each period to the left and right.	Asymptotes: $x = 2\pi$, and multiples of 4π to the left and right for $[-4\pi, 8\pi]$: $x = -2\pi$, $x = 2\pi$, $x = 6\pi$
4 Draw sections of smooth curves between the asymptotes showing the x-intercepts. Put arrows on the curves at the intercepts.	

When comparing the graph of $y = \tan(x)$ with the graph of $y = x$, the graphs are close to each other near the origin, but the tangent graph increases very quickly as x increases and decreases very quickly as x decreases as it approaches the asymptotes.

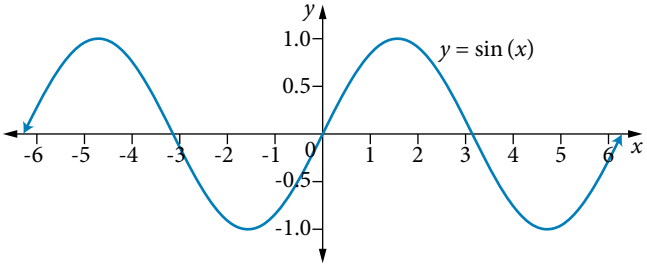


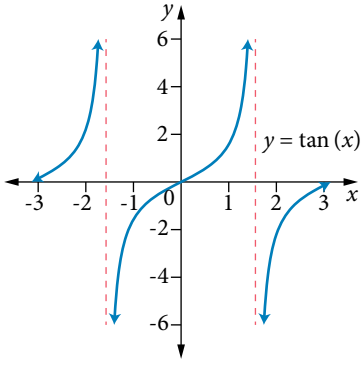
Even and odd trigonometric functions

A function is said to be *odd* if $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is symmetrical with respect to a half-turn rotation about the origin.

The sine and tangent functions are odd functions because:

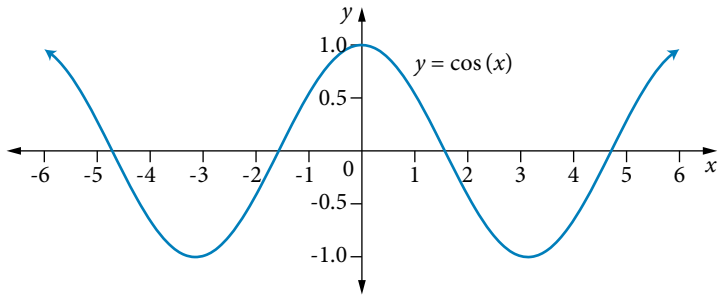
$\sin(-x) = -\sin(x)$ and $\tan(-x) = -\tan(x)$





A function is said to be even if $f(-x) = f(x)$ for all x in its domain. The graph of an even function is symmetrical with respect to reflection in the vertical or y-axis.

The cosine function is an even function because $\cos(-x) = \cos(x)$.



EXERCISE 8.3 Graphs of trigonometric functions, dilations

Worked Example

15

1 Identify the amplitude and period of each of the following trigonometric functions.

(a) $y = 3\cos(x)$ (b) $y = -\sin(4x)$ (c) $y = \frac{5}{2}\sin(2x)$

(d) $y = -\frac{1}{2}\cos\left(\frac{2x}{3}\right)$ (e) $y = -2\cos\left(\frac{4x}{5}\right)$ (f) $y = -\frac{3}{2}\sin\left(\frac{4x}{3}\right)$

(g) $y = 4\cos(2\pi x)$ (h) $y = -3\sin\left(\frac{\pi x}{2}\right)$ (i) $y = \frac{1}{3}\sin(\pi x)$

2 Calculate the period of each of the following trigonometric functions.

(a) $y = 3\tan(x)$ (b) $y = \frac{1}{2}\sin(x)$ (c) $y = -\cos(2x)$ (d) $y = 3\sin(3x)$

(e) $y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$ (f) $y = -4\sin\left(\frac{2x}{3}\right)$ (g) $y = \tan(3x)$ (h) $y = 4\cos\left(\frac{5x}{3}\right)$

(i) $y = -2.1\sin(2.5x)$ (j) $y = -2\tan(\pi x)$ (k) $y = -\cos\left(\frac{\pi x}{2}\right)$ (l) $y = 1.5\sin(0.4\pi x)$

16

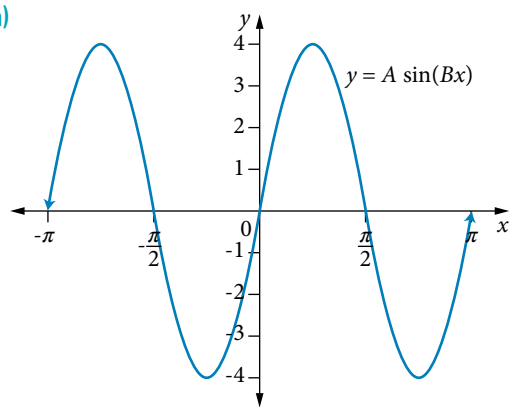
3 Sketch one period of the graph of each function from the y-axis.

(a) $y = 2\cos(3x)$ (b) $y = -\sin(\pi x)$ (c) $y = 4\sin(-x)$

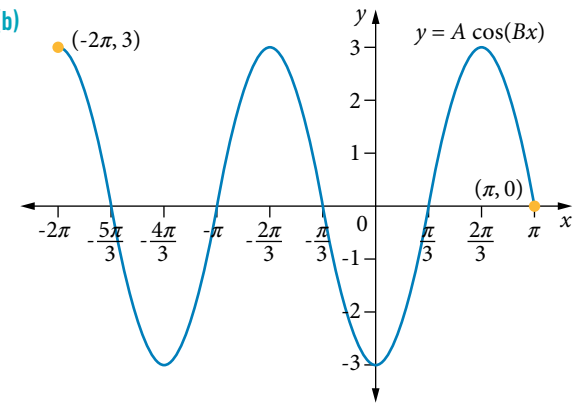
(d) $y = 3\cos\left(\frac{\pi x}{2}\right)$ (e) $y = 20\cos(-x)$ (f) $y = -5\sin\left(\frac{\pi x}{20}\right)$

4 Identify the amplitude and period, and determine the rule for each function in the form specified.

(a)



(b)



5 Determine the values of A and B for each of the following rules.

(a) A function with rule $y = A\cos(Bx)$, where $A, B > 0$, has a range of $[-3, 3]$ and period $\frac{3\pi}{5}$.

(b) A function with rule $y = A\sin(Bx)$, where $A, B > 0$ has a range of $\left[-\frac{5}{2}, \frac{5}{2}\right]$ and period $\frac{15\pi}{2}$.

6 Which of the following functions has values similar to $y = 2x$ near $x = 0$?

A $y = 2\sin(x)$ B $y = 2\cos(x)$ C $y = \sin(2x)$ D $y = \cos(2x)$

7 (a) Which of the following functions has the same values of x as $y = 2\cos(3x)$?

A $y = -2\cos(3x)$ B $y = -2\cos(-3x)$ C $y = 2\cos(-3x)$ D $y = 2\sin(-3x)$

(b) Explain the common error made by a student who thought the answer was $y = 2\sin(3x) - \frac{\pi}{2}$.

Worked Example

8 Determine the x-intercepts and equations of the asymptotes for graphs of the following functions. 18

(a) $y = -4\tan(x)$ (b) $y = 2\tan(3x)$ (c) $y = \tan\left(\frac{x}{3}\right)$

(d) $y = 3\tan(\pi x)$ (e) $y = -2\tan\left(\frac{x}{2}\right)$ (f) $y = 20\tan\left(\frac{3x}{5}\right)$

9 Sketch the graph of each of the following trigonometric functions over the given domain. Label the axes intercepts and the turning points.

(a) $y = 2\cos(x), x \in [0, 2\pi]$ (b) $y = \frac{1}{2}\sin(x), x \in [0, 2\pi]$ (c) $y = -\cos(2x), x \in [-\pi, \pi]$

(d) $y = -4\sin(x), x \in [\pi, 3\pi]$ (e) $y = -\frac{1}{3}\cos(x), x \in [0, 3\pi]$ (f) $y = \sin(2x), x \in [-\pi, \pi]$

(g) $y = 5\sin\left(\frac{x}{2}\right), x \in [-\pi, 3\pi]$

10 Sketch the graphs of each of the following trigonometric functions over the domain $[0, 2\pi]$. For each function, write the coordinates of each turning point and the axes intercepts. 17

(a) $y = 2\cos(3x)$ (b) $y = -3\sin(2x)$ (c) $y = \frac{1}{2}\sin(3x)$

(d) $y = -2\cos\left(\frac{x}{2}\right)$ (e) $y = 5\cos(2x)$ (f) $y = -3\sin\left(\frac{3x}{2}\right)$

(g) $y = 10\sin(4x)$ (h) $y = -4\cos(2x)$ (i) $y = -\frac{1}{2}\cos\left(\frac{x}{4}\right)$

11 Sketch the following graphs over the domain $[0, 2\pi]$. For each function, label the vertical asymptotes and the axes intercepts. 19

(a) $y = \tan\left(\frac{x}{2}\right)$ (b) $y = \tan(3x)$ (c) $y = \tan(2x)$

12 Sketch the graph of each trigonometric function over the given domain. For each function, label the axes intercepts and the vertical asymptotes.

(a) $y = 2\tan(x), x \in [0, 2\pi]$ (b) $y = \tan(2x), x \in [0, 2\pi]$ (c) $y = \tan\left(\frac{x}{4}\right), x \in [-2\pi, 2\pi]$

13 Sketch the following graphs over the given domain.

(a) $y = 2\cos(\pi x), x \in [0, 4]$ (b) $y = 3\sin(2\pi x), x \in [-2, 2]$ (c) $y = 2\tan(\pi x), x \in [-1, 1]$

(d) $y = 2\sin(3\pi x), x \in [-2, 2]$ (e) $y = -3\sin\left(\frac{\pi x}{2}\right), x \in [0, 6]$ (f) $y = -\frac{1}{2}\tan\left(\frac{3\pi x}{2}\right), x \in [-1, 1]$

(g) $y = -2\cos(2\pi x), x \in [-2, 4]$

14 The instantaneous voltage supplied from a household power point can be modelled by the function $V_I = V_S \sin(2f\pi t)$, where V_I is the instantaneous voltage in volts, V_S is the nominal supply voltage in volts, f is the frequency of the supply in hertz and t is time in seconds. For the following electricity supplies, sketch the graph of V_I for $0 \le t \le 0.1$.

(a) $V_S = 230$ volts, $f = 50$ hertz

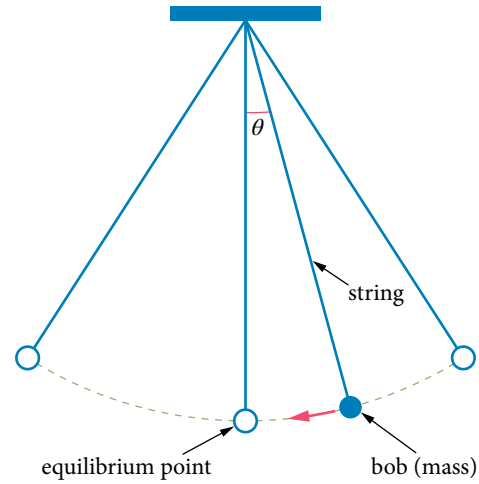
(b) $V_S = 120$ volts, $f = 60$ hertz

15 As a pendulum swings backwards and forwards, its angular displacement θ from its rest position after t seconds is given by the function $\theta = 20\cos(3\pi t)$, where θ is measured in degrees.

(a) Sketch the graph of this function for $0 \le t \le 3$.

(b) What is the maximum angular displacement?

(c) How long does it take for the pendulum to complete one oscillation?

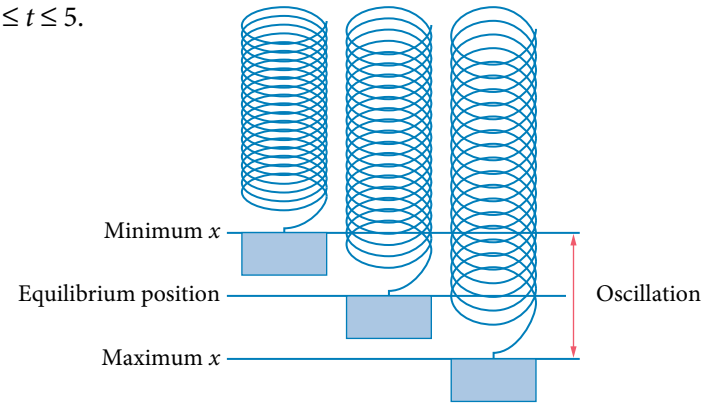


16 A mass attached to a spring oscillates up and down. The displacement of the mass from its equilibrium position after t seconds is given by the function $d = -12.6\cos(2\pi t)$, where d is measured in centimetres.

(a) Sketch the graph of this function for $0 \le t \le 5$.

(b) What is the furthest distance of the mass from its equilibrium position?

(c) How long does it take for the mass to complete one oscillation?



17 The cosine function can be approximated by a quadratic function over a suitable interval. By plotting the graph of $y = \cos(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and a suitable parabola, describe the relationship between the two graphs.

18 Create a quadratic piece-wise rule whose graph will mimic the graph of $y = 3\sin\left(-\frac{\pi x}{2}\right)$ for $0 \le x \le 6$.

Translations of trigonometric graphs

8.4

Vertical translations

The basic trigonometric graphs are transformed into $y = A\sin(Bx) + D$, $y = A\cos(Bx) + D$ and $y = A\tan(Bx) + D$ by a translation of D units parallel to the y-axis.

- If $D > 0$, the graph shifts upwards.
- If $D < 0$, the graph shifts downwards.
- The median line of the graphs is $y = D$.

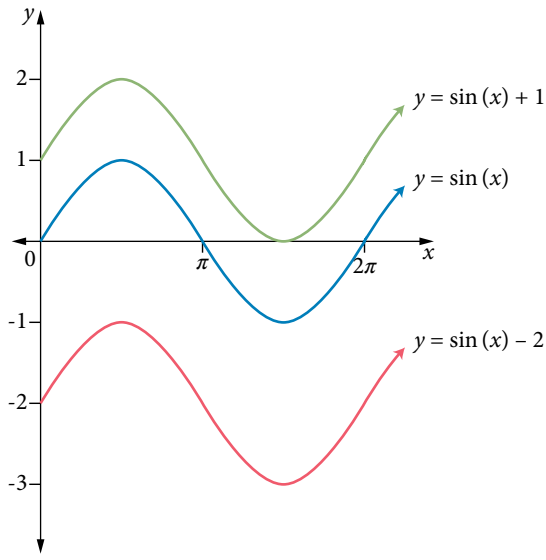
Explore further

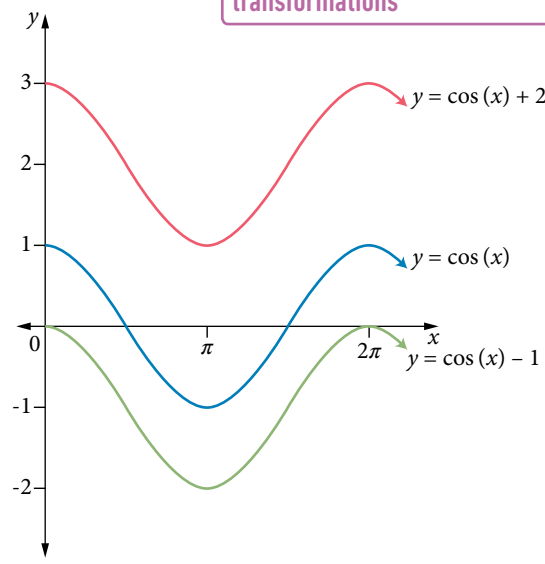
The functions $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

Explore the effect of the parameters A, B, C and D on these graphs.

Technology worked example

Sketching graphs with transformations





The graphs of $y = A\sin(Bx) + D$ and $y = A\cos(Bx) + D$ have amplitude of $|A|$, period of $\frac{2\pi}{B}$ and a median value of D. The range is $[D - |A|, D + |A|]$.

$$D = \frac{\text{maximum} + \text{minimum}}{2}$$

The graph of $y = A\tan(Bx) + D$ has a period of $\frac{\pi}{B}$ and a median value of D.
The amplitude is undefined and the range is $\mathbb{R}$.

20 Determine median and range

Determine the median value and range for each function.

(a) $y = \sin(x) + 5$

THINKING	WORKING
1 Write the value of D and the median: $y = A\sin(Bx) + D$	$D = 5$ The median value of the function is 5.
2 Write the value of A and D, then calculate the range $[D - \|A\|, D + \|A\|]$.	$D = 5$, $A = 1$ $5 - 1 = 4$, $5 + 1 = 6$ The range is $[4, 6]$.

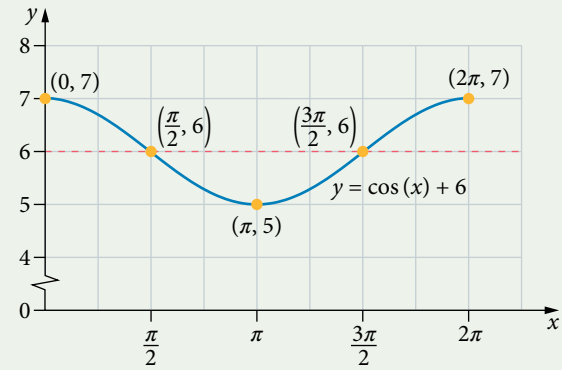
(b) $y = \tan(x) - 1$

1 Identify the value of D and the median: $y = A\sin(Bx) + D$	$D = -1$ The median value of the function is -1.
2 The range for tangent functions is not affected by transformations.	The range is R.

21 Sketch graphs, vertical translations

Sketch the graph of each function in the domain $[0, 2\pi]$.

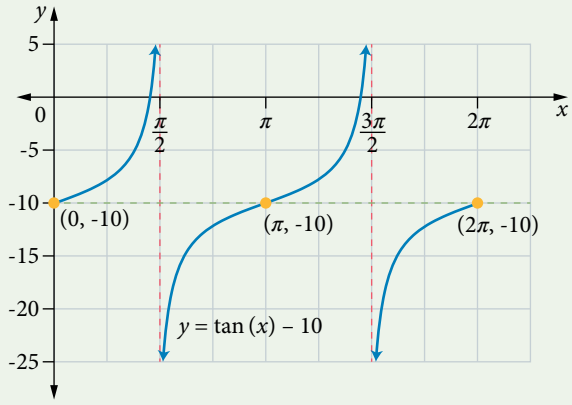
(a) $y = \cos(x) + 6$

THINKING	WORKING
1 Determine key points of the basic graph.	$y = \cos(x)$ for the period $[0, 2\pi]$ has five key points: $(0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)$
2 Describe the translation and determine the key points of the given function.	Translation up 6 units: $(0, 7), \left(\frac{\pi}{2}, 6\right), (\pi, 5), \left(\frac{3\pi}{2}, 6\right), (2\pi, 7)$
3 Note the range. Plot the points and draw a curve in the same shape at the basic graph. Identify the median value.	The range is $[5, 7]$. 

(b) $y = \tan(x) - 10$

1 Determine key features of the basic graph.	$y = \tan(x)$ for the two periods $[0, 2\pi]$ has axis points: $(0, 0), (\pi, 0), (2\pi, 0)$ Vertical asymptotes: $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$
2 Describe the translation and determine the key features of the given function.	Translation down 10 units: The axis points: $(0, -10), (\pi, -10), (2\pi, -10)$ The vertical asymptotes are unchanged.

3 Plot the points and asymptotes then draw the curves in the same shape at the basic graph. Identify the median value.

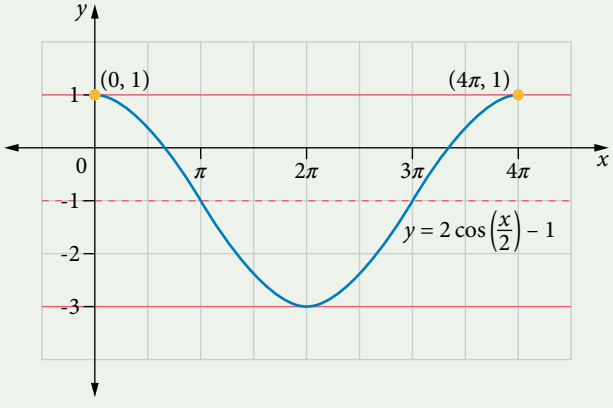


22 Sketching graphs with transformations, including vertical translations

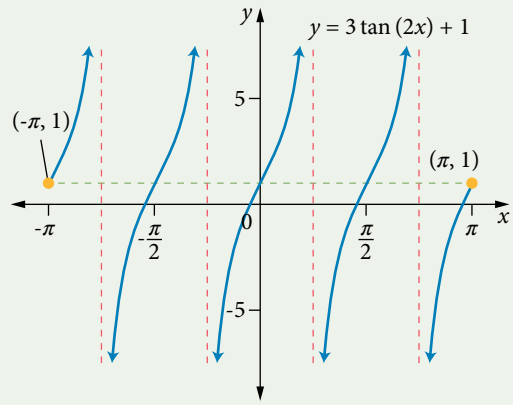
Sketch the following graphs, over the stated domains, without the use of technology. Then use your technology to confirm the graph is correct.

(a) $y = 2\cos\left(\frac{x}{2}\right) - 1$ over the domain $[0, 4\pi]$.

THINKING	WORKING
1 In the form $y = A\cos(Bx) + D$, identify the value of A and its effect.	$A = 2$ The amplitude is 2.
2 Identify the value of B and its effect.	$B = \frac{1}{2}$ Period $= 2\pi \div \frac{1}{2}$ $= 2\pi \times 2$ $= 4\pi$
3 Identify the value of D and the median.	$D = -1$ The median of the graph is -1. Minimum $= -1 - 2 = -3$ Maximum $= -1 + 2 = 1$
4 Draw the scales on the set of axes. Mark in the minimum and maximum values and median line, and draw the smooth curve. The x-intercepts may be too difficult to locate precisely at this stage.	The period is 4π, so one cycle in the domain. The vertical scale needs to extend from -3 to 1.



(b) $y = 3\tan(2x) + 1$ over the domain $[-\pi, \pi]$.

1 In the form $y = A\tan(Bx) + D$, identify the value of A and its effect.	$A = 3$, but has no obvious effect on the general appearance of the graph.
2 Identify the value of B and its effect.	$B = 2$ The period is $\frac{\pi}{2}$. One asymptote is half a period from the y-axis at $\frac{\pi}{4}$. Asymptotes are a period apart: $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}$ and $\frac{3\pi}{4}$. Points of inflection: $x = -\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \pi$.
3 Identify the value of D and its effect.	$D = 1$ The points of inflection are raised to $y = 1$.
4 Draw appropriate scales on the set of axes, and draw the smooth curves between the asymptotes.	The period is $\frac{\pi}{2}$, so there will be four cycles over the given domain. 

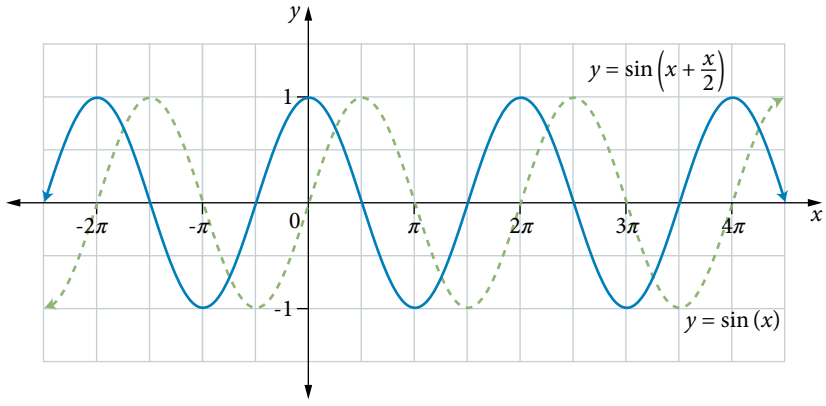
Phase shifts

The basic trigonometric graphs are transformed into $y = \sin(x + C)$, $y = \cos(x + C)$ and $y = \tan(x + C)$, by a translation of C units parallel to the x-axis.

If a constant C is added to x, the resulting graph is translated horizontally by C units in the opposite direction to the sign of C. For periodic functions, this is known as a phase shift. The amount of any phase shift is always less than the period, when written in simplest form.

- If $C > 0$, the graph shifts to the left.
- If $C < 0$, the graph shifts to the right.

For example, for $y = \sin\left(x + \frac{\pi}{2}\right)$, the graph of $y = \sin(x)$ shifts horizontally by $-\frac{\pi}{2}$.



The graphs of $y = A\sin[B(x + C)]$ and $y = A\cos[B(x + C)]$ have an amplitude of $|A|$, a period of $\frac{2\pi}{B}$ and a median value of 0. Their range is $[-|A|, |A|]$.

The graph of $y = A\tan[B(x + C)]$ has a period of $\frac{\pi}{B}$ and a point of inflection at $y = 0$. The amplitude is undefined and the range is R.

The phase shift is $-C$ or C units to the left for all three functions.

23 Determine median, range and phase shift

Determine the median value, range and phase shift from the basic graph for each function.

(a) $y = \sin\left(x - \frac{\pi}{3}\right)$

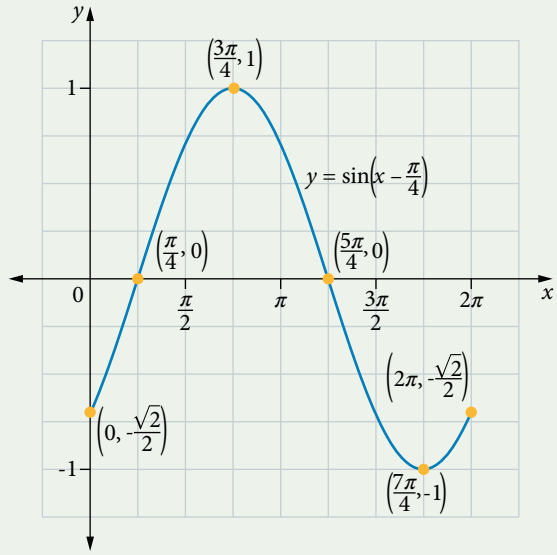
THINKING	WORKING
1 Identify the value of D and its effect on the median: $y = A\sin(Bx + C) + D$	$D = 0$ The median value of the function is unchanged as 0.
2 Use the values of A and D to calculate the range.	$D = 0$, $A = 1$ The range is unchanged as $[-1, 1]$.
3 Identify the value of C and hence the phase shift.	$C = -\frac{\pi}{3}$ so the phase shift is $\frac{\pi}{3}$ to the right.

(b) $y = \cos(x + \pi) - 2$

1 Write the value of D and its effect on the median: $y = A\cos(Bx + C) + D$	$D = -2$ The median value of the function is -2.
2 The range for tangent functions is not affected by transformations.	$D = -2$, $A = 1$ The range is given by $[D - \|A\|, D + \|A\|]$. $-2 - 1 = -3$, $-2 + 1 = -1$ The range is $[-3, -1]$.
3 Identify the value of C and hence the phase shift.	$C = \pi$ so the phase shift is $-\pi$ to the left.

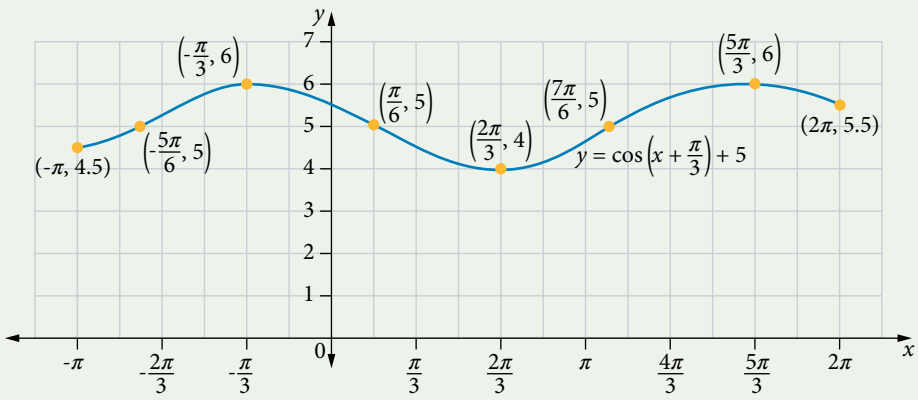
24 Sketch graphs, phase change

Sketch the graph of $y = \sin\left(x - \frac{\pi}{4}\right)$ for $[0, 2\pi]$.

THINKING	WORKING
1 Determine key points of the basic graph.	$y = \sin(x)$ for the period $[0, 2\pi]$ has five key points: $(0, 0), \left(\frac{\pi}{2}, 1\right), (\pi, 0), \left(\frac{3\pi}{2}, -1\right), (2\pi, 0)$
2 Describe the phase shift and determine the key points of the given function.	$y = \sin(x)$ has been translated $\frac{\pi}{4}$ to the right to become $y = \sin\left(x - \frac{\pi}{4}\right)$. The five key points become: $\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 1\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, -1\right), \left(\frac{9\pi}{4}, 0\right)$ The required domain is $[0, 2\pi]$, so the final key point is not included.
3 Determine the endpoints of the domain.	Let $x = 0$: $y = \sin\left(0 - \frac{\pi}{4}\right)$ $= -\sin\left(\frac{\pi}{4}\right)$ $= -\frac{\sqrt{2}}{2}$ So $\left(0, -\frac{\sqrt{2}}{2}\right)$, and one period to the right, $\left(2\pi, -\frac{\sqrt{2}}{2}\right)$
4 Note the range. Plot the points and draw a curve in the same shape at the basic graph. Identify the median value.	The range is $[-1, 1]$. 

25 Sketch graphs, vertical and horizontal translations

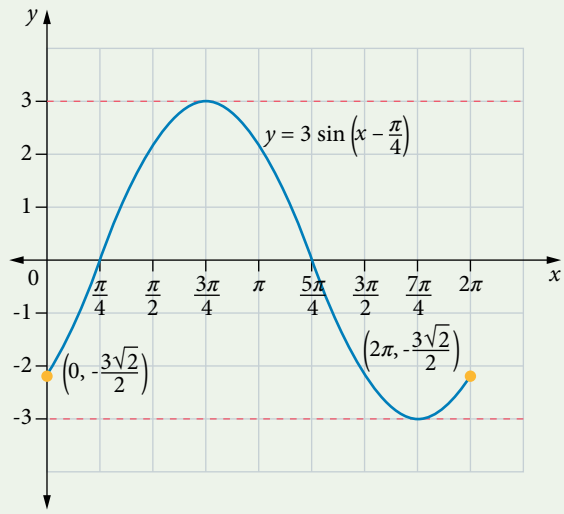
Sketch the graph of $y = \cos\left(x + \frac{\pi}{3}\right) + 5$ for $[-\pi,\ 2\pi]$.

THINKING	WORKING
1 Determine key points of the basic graph.	$y = \cos(x)$ for $[0, 2\pi]$ has key points: $(0, 1), \left(\frac{\pi}{2}, 0\right), (\pi, -1), \left(\frac{3\pi}{2}, 0\right), (2\pi, 1)$
2 Describe the translations and determine the key points of the given function.	$y = \cos(x)$ has been translated up 5 and left $\frac{\pi}{3}$ to $y = \cos\left(x + \frac{\pi}{3}\right) + 5$. The key points become: $\left(-\frac{\pi}{3}, 6\right), \left(\frac{\pi}{6}, 5\right), \left(\frac{2\pi}{3}, 4\right), \left(\frac{7\pi}{6}, 5\right), \left(\frac{5\pi}{3}, 6\right)$ For $[-\pi,\ 2\pi]$ there are one and a half cycles, so add or subtract 2π to determine the additional key point $\left(-\frac{5\pi}{6}, 5\right)$.
3 Determine the endpoints of the domain.	Let $x = -\pi$: $y = \cos\left(-\pi + \frac{\pi}{3}\right) + 5$ $= \cos\left(-\frac{2\pi}{3}\right) + 5$ $= \cos\left(\frac{2\pi}{3}\right) + 5$ $= -\cos\left(\frac{\pi}{3}\right) + 5$ $= -\frac{1}{2} + 5$ $= 4.5$ So $(-\pi, 4.5)$ Let $x = 2\pi$: $y = \cos\left(2\pi + \frac{\pi}{3}\right) + 5$ $= \cos\left(\frac{\pi}{3}\right) + 5$ $= \frac{1}{2} + 5$ $= 5.5$ So $(2\pi,\ 5.5)$
4 Identify the range. Plot the key points and endpoints, then draw the curves in the same shape at the basic graph. The range is $[4, 6]$.	

26 Sketch graphs with transformations, including phase changes

Sketch the following graphs, over the stated domains.

(a) $y = 3\sin\left(x - \frac{\pi}{4}\right)$ over the domain $[0, 2\pi]$

THINKING	WORKING
1 In the form $y = A\sin B((x + C))$, identify the value of A and its effect.	$A = 3$ The amplitude is 3. The median of the graph is 0. Range: [-3, 3]
2 Identify the value of B and its effect.	$B = 1$, so the period is 2π.
3 Identify the value of C and its effect.	$C = -\frac{\pi}{4}$ The graph of $y = \sin(x)$ has been translated horizontally $\frac{\pi}{4}$ units to the right.
4 Determine the endpoints.	Let $x = 0$: $y = 3\sin\left(0 - \frac{\pi}{4}\right)$ $= -3\sin\left(\frac{\pi}{4}\right)$ $= -\frac{3\sqrt{2}}{2}$ The coordinates of the endpoint are $\left(0, -\frac{3\sqrt{2}}{2}\right)$ and one cycle to the right: $\left(2\pi, -\frac{3\sqrt{2}}{2}\right)$
5 Mark in the minimum and maximum values and median line, and draw the smooth curve. Write the exact values of the endpoints.	

(b) $y = \tan\left(2\left(x + \frac{\pi}{3}\right)\right)$ over the domain $\left[-\frac{\pi}{3}, \frac{5\pi}{3}\right]$.

1 In the form $y = A\tan B((x + C))$, identify the value of A and its effect.

$A = 1$, so there is no vertical dilation of the graph.

The points of inflection occur when $y = 0$.

2 Identify the value of B and its effect.

$B = 2$

The period is $\frac{\pi}{2}$.

3 Identify the value of C and its effect.

$C = \frac{\pi}{3}$, so there is a phase shift of $-\frac{\pi}{3}$.

Asymptotes are half a period either side of the point $\left(-\frac{\pi}{3}, 0\right)$.

Half the period is $\frac{\pi}{4}$:

$\frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

The asymptotes are one period apart at:

$x = -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}$.

The points of inflection are one period apart at:

$\left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{6}, 0\right), \left(\frac{2\pi}{3}, 0\right), \left(\frac{7\pi}{6}, 0\right), \left(\frac{5\pi}{3}, 0\right)$.

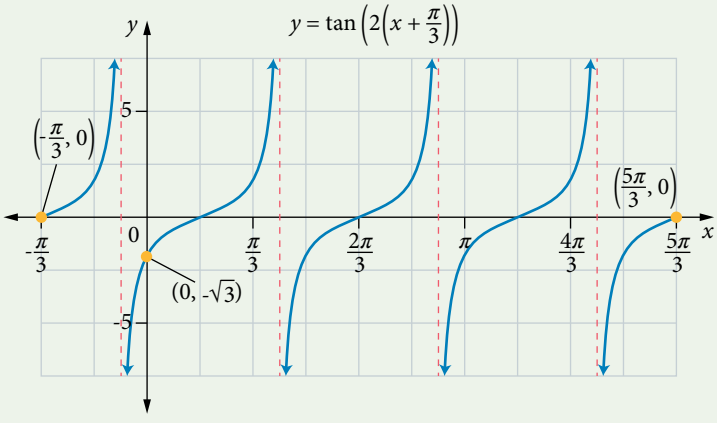
4 Determine the coordinates of the y-intercept.

Let $x = 0$:

$$\begin{aligned} y &= \tan\left(2\left(0 + \frac{\pi}{3}\right)\right) \\ &= \tan\left(\frac{2\pi}{3}\right) \\ &= -\tan\left(\frac{\pi}{3}\right) \\ &= -\sqrt{3} \end{aligned}$$

The coordinates of the y-intercept are $\left(0, -\sqrt{3}\right)$.

5 Draw appropriate scales on the set of axes, then draw smooth curves between the asymptotes.



WARNING

Do not assume that trigonometric graphs are unique.

- the graph of $y = 2\cos\left(x + \frac{\pi}{2}\right)$ is identical to the graph of $y = -2\sin(x)$
- the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ is identical to the graph of $y = \sin(x)$
- the graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is identical to the graph of $y = \cos(x)$

Additional information

Comparing transformations

This activity explores these identities further.

EXERCISE 8.4 Graphs of trigonometric functions, translations

Worked Example

20

1 Determine the median value and range of each of the following functions.

(a) $y = \sin(x) + 2$ (b) $y = \cos(x) - 3$ (c) $y = \tan(x) + 3$

(d) $y = -5\sin(x) - 1$ (e) $y = 4\cos(x) + 3$ (f) $y = -\tan(x) - 7$

21

2 Sketch each of the following trigonometric functions over the given domain.

(a) $y = \cos(x) + 2,\ x \in [0,\ 2\pi]$ (b) $y = \sin(x) - 3,\ x \in [0,\ 2\pi]$

(c) $y = \tan(x) - 2,\ x \in [0,\ 2\pi]$ (d) $y = \cos(x) - 4,\ x \in [-\pi,\ \pi]$

23

3 Determine the median value, range and phase shift from the basic graph for each of the following functions.

(a) $y = \cos\left(x - \frac{\pi}{2}\right)$ (b) $y = \sin\left(x + \frac{\pi}{4}\right)$ (c) $y = \tan(x - \pi) + 1$

(d) $y = \sin\left(x + \frac{\pi}{3}\right) - 1$ (e) $y = \cos(x + \pi) + 2$ (f) $y = 3\sin\left(x - \frac{\pi}{3}\right)$

(g) $y = 10\tan\left(x + \frac{\pi}{3}\right) - 5$ (h) $y = -2\sin\left(x + \frac{\pi}{5}\right) + 2$ (i) $y = 6\cos\left(x + \frac{\pi}{8}\right) - 5$

24

4 Sketch each of the following trigonometric functions over the given domain.

(a) $y = \cos\left(x + \frac{\pi}{2}\right),\ x \in [0,\ 2\pi]$ (b) $y = \sin\left(x - \frac{\pi}{6}\right),\ x \in [0,\ 2\pi]$

(c) $y = \tan\left(x - \frac{\pi}{2}\right),\ x \in [-\pi,\ \pi]$ (d) $y = \sin\left(x + \frac{\pi}{3}\right),\ x \in [-\pi,\ \pi]$

5 Identify the transformations that have been applied to the graphs of $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$.

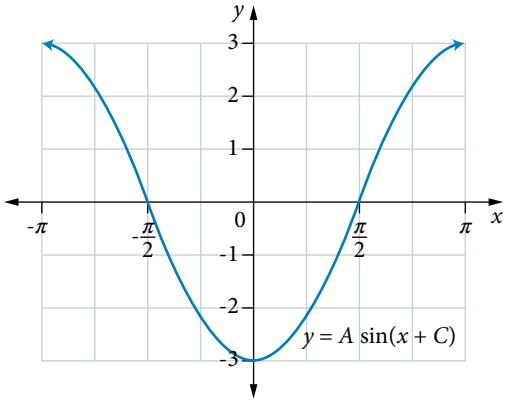
(a) $y = 2\cos\left(x - \frac{\pi}{2}\right) + 1$ (b) $y = \frac{1}{2}\sin(2x) + 3$ (c) $y = -2\tan\left(\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right)$

(d) $y = -3\sin\left(x - \frac{\pi}{4}\right) - 1$ (e) $y = -\frac{5}{2}\sin\left(\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right) + 2$ (f) $y = \frac{1}{2}\cos\left(x - \frac{\pi}{3}\right) - 1$

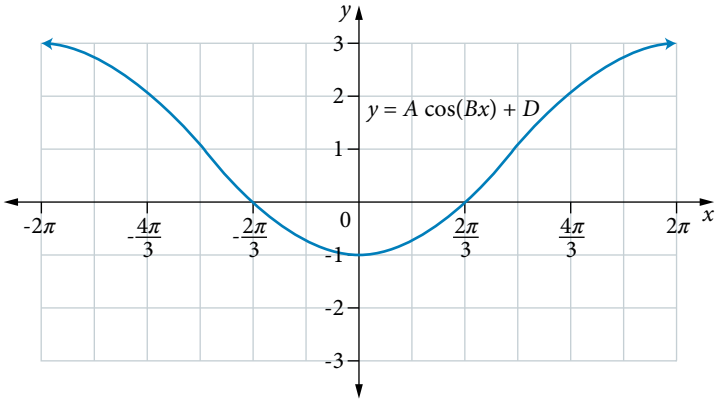
(g) $y = -\frac{2}{5}\tan\left[2\left(x + \frac{\pi}{2}\right)\right] - 1$ (h) $y = -2\cos[3(x + \pi)] - 3$

6 Determine the period and domain of each function and the values of:

(a) A and C, given that $A > 0$ and $C < 0$



(b) A, B and D.



7 Determine the relationship between the graphs of the two functions $y = \sin(x)$ and $y = \cos\left(x - \frac{\pi}{2}\right)$.

A They are exactly the same.

B Translate $y = \cos\left(x - \frac{\pi}{2}\right)$ to the right by $\frac{\pi}{2}$ for $y = \sin(x)$.

C Translate $y = \cos\left(x - \frac{\pi}{2}\right)$ to the left by $\frac{\pi}{2}$ for $y = \sin(x)$.

D Reflect $y = \cos\left(x - \frac{\pi}{2}\right)$ in the horizontal axis for $y = \sin(x)$.

8 **(a)** Determine the correct statement about the graphs of the functions $y = \cos(x)$ and $y = \sin\left(x + \frac{\pi}{2}\right)$.

A They are exactly the same.

B Translate $y = \cos(x)$ to the right by $\frac{\pi}{2}$ for $y = \sin\left(x + \frac{\pi}{2}\right)$.

C Translate $y = \cos(x)$ to the left by $\frac{\pi}{2}$ for $y = \sin\left(x + \frac{\pi}{2}\right)$.

D Dilate $y = \cos(x)$ by a factor of $\frac{\pi}{2}$ for $y = \sin\left(x + \frac{\pi}{2}\right)$.

(b) Explain the common error made by a student who thought that the graph of $y = \cos(x)$ reflected in the horizontal axis would give $y = \sin\left(x + \frac{\pi}{2}\right)$.

9 **(a)** Determine the period, amplitude and the range of $f(x) = a\cos(bx) + c$, where a and b are positive.

(b) Let $g(x) = 3\cos(\pi x) - 1$.

(i) Determine the period, amplitude and range of $g(x)$.

(ii) Sketch the graph of $y = g(x)$ over the domain $[-2, 2]$ and determine the number of solutions to the equation $g(x) = 0$ in this domain.

Worked Example 25

10 Sketch the graph of each trigonometric function over the given domain.

(a) $y = \sin\left(x + \dfrac{\pi}{2}\right) + 1,\ x \in [0, 2\pi]$

(b) $y = \cos\left(x - \dfrac{\pi}{4}\right) - 2,\ x \in [-\pi, \pi]$

(c) $y = \tan\left(x + \dfrac{\pi}{4}\right) + 1,\ x \in [0, 2\pi]$

(d) $y = \cos\left(x + \dfrac{\pi}{3}\right) - 2,\ x \in [-2\pi, 2\pi]$

22

11 Sketch the graph of each function over the given domain. Label the turning points or equations of vertical asymptotes and the y-intercept.

(a) $y = 2\sin\left(\frac{x}{2}\right) + 1,\ x \in [0, 4\pi]$

(b) $y = 2\cos(\pi x) - 1,\ x \in [0, 6]$

(c) $y = 2\tan(2x) - 3,\ x \in [-\pi, \pi]$

(d) $y = 4\sin(x) + 2,\ x \in [0, 2\pi]$

(e) $y = 2\tan(4x) + 1,\ x \in [0, \pi]$

(f) $y = 2\cos(2\pi x) - 1,\ x \in [0, 4]$

(g) $y = -3\cos\left(\frac{x}{2}\right) + 1,\ x \in [-2\pi, 2\pi]$

(h) $y = 2\tan\left(\frac{x}{3}\right) - 1,\ x \in [-\pi, 2\pi]$

(i) $y = -2\sin\left(\frac{\pi x}{2}\right) + 2,\ x \in [-4, 4]$

26

12 Sketch graphs of each function over the given domain. Label the turning points or equations of vertical asymptotes and the axes intercepts, where they exist.

(a) $y = 2\sin\left(x + \frac{\pi}{4}\right),\ x \in [0, 2\pi]$

(b) $y = 3\cos\left(x - \frac{\pi}{2}\right),\ x \in [0, 2\pi]$

(c) $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right),\ x \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$

(d) $y = 4\cos\left(2\left(x - \frac{\pi}{4}\right)\right),\ x \in [0, 2\pi]$

(e) $y = \tan\left(x - \frac{\pi}{2}\right),\ x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

(f) $y = -2\sin\left(x + \frac{\pi}{4}\right),\ x \in [0, 2\pi]$

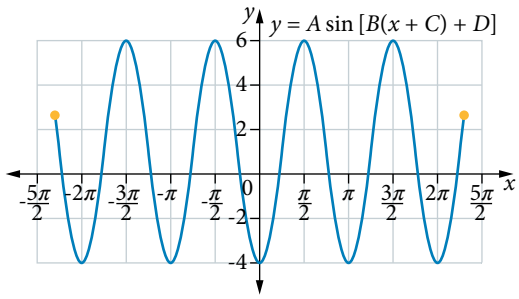
(g) $y = -2\sin\left(x - \frac{\pi}{2}\right),\ x \in [0, 2\pi]$

(h) $y = 2\cos\left(3\left(x + \frac{\pi}{3}\right)\right),\ x \in [-\pi, \pi]$

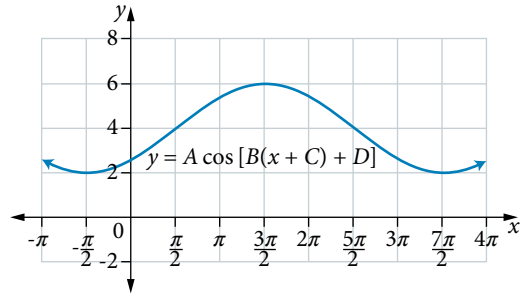
(i) $y = -2\tan\left(x + \frac{\pi}{2}\right),\ x \in [0, 2\pi]$

13 Determine an equation for each graph in the form given. Where C is one of the parameters, use the smallest value of $|C|$ possible. Assume $B > 0$ for all graphs.

(a)



(b)



14 The oscillations in air pressure representing the sound wave for a musical tone can be modelled by the trigonometric function $y(t) = 0.1\sin(500\pi t + 10\pi)$, where $y(t)$ is the sound pressure (measured in pascals, Pa) after t seconds.

(a) Sketch the graph of the sound wave for $0 \le t \le 0.01$.

(b) What is the maximum air pressure of the sound wave?

(c) What is the period of the sound wave?

(d) A moving object displaying simple harmonic motion can be modelled by the function $y = A\sin(\Omega t + \varphi)$ where A, Ω and φ are constants. The constant Ω is the angular frequency and φ is the phase angle of the oscillation. Determine the phase angle for this sound wave.

15 Plot one period of the graph of $y = \frac{1}{2}\sin\left(\frac{3x-\pi}{6}\right) + 1$ from the y-axis.

16 Sketch the graph of $y = x(x-3)(x-6) + 5$. Choosing a suitable domain, mimic this function with a sine graph. Analyse the degree to which the sine graph was able to successfully mimic the cubic function.

Solving trigonometric equations

8.5

Solutions in a fixed domain

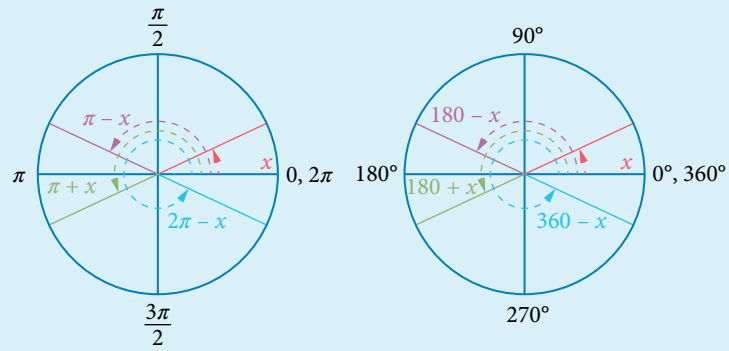
What is the solution to a trigonometric equation such as $\cos(x) = \frac{1}{2}$, where $0 \le x \le 2\pi$?
$x = \frac{\pi}{3}$ is a solution to the equation because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

However, the equation $\cos(x) = \frac{1}{2}$ has another solution in the domain $[0, 2\pi]$.

Related angles to $\frac{\pi}{3}$ are:
$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (quadrant 2),
$\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ (quadrant 3),
and $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ (quadrant 4).
Given that $\cos(x)$ is positive in this equation, the other solution is $\frac{5\pi}{3}$.

Therefore, if $\cos(x) = \frac{1}{2}$, then $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ over the domain $[0, 2\pi]$.

Related angles:



27 Solutions in radians

Determine the exact solutions of $\sin(x) = -\frac{\sqrt{3}}{2}$ over the domain $[0, 2\pi]$.

THINKING	WORKING
1 Determine the reference angle in quadrant 1.	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ $\alpha = \frac{\pi}{3}$ is the reference angle.
2 Recall the quadrants with the required sign.	Sine is negative in quadrants 3 and 4.
3 Determine the related angles in the required quadrants.	Quadrant 3: $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ Quadrant 4: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$
4 State the exact solutions.	$x = \frac{4\pi}{3}, \frac{5\pi}{3}$

28 Solutions in degrees

Determine the exact solutions of $\tan(x) = 1$ over the domain $[-180°, 180°]$.

THINKING	WORKING
1 Determine the reference angle in quadrant 1.	$\tan(45°) = 1$ $\alpha = 45°$ is the reference angle.
2 Recall the quadrants with the required sign.	Tangent is positive in quadrants 1 and 3.

3 Determine the equivalent angles in the required quadrants.	Quadrant 1: 45° Quadrant 3: 180° + 45° = 225°
4 Adjust for the required domain.	225° is the same position as 225° − 360° = -135°
5 State the exact solutions.	$x = -135°, 45°$

If the angle is a function of x, then you need to determine the domain of the angle before solving. For example, if the domain of x is [0°, 180°], then the domain of $2x$ is [0°, 360°].

29 Exact solutions of a trigonometric equation

Determine the exact solutions of the following over the domain $[0, 2\pi]$.

(a) $\cos(2x) = \frac{1}{2}$

THINKING	WORKING
1 Write the domain using inequalities.	$0 \le x \le 2\pi$
Write an equivalent inequality with the argument.	$0 \le 2x \le 4\pi$
2 Determine the reference angle in quadrant 1.	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\alpha = \frac{\pi}{3}$ is the reference angle.
3 Recall the quadrants with the required sign.	Cosine is positive in quadrants 1 and 4.
4 Determine the related angles in the required quadrants.	Quadrant 1: $\frac{\pi}{3}$ Quadrant 4: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$
5 Extend the list to suit the domain.	$\frac{\pi}{3}$ is the same position as $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$ $\frac{5\pi}{3}$ is the same position as $\frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$
6 Write the solutions for the angle expression.	$2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$
7 Solve for the unknown.	$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

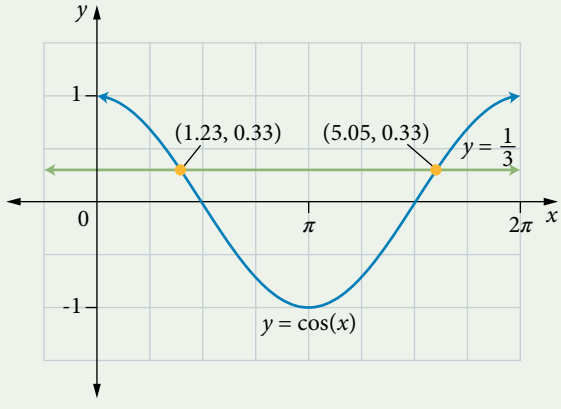
(b) $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

1 Write the domain using inequalities.	$0 \le x \le 2\pi$
Write an equivalent inequality with the expression for the argument.	$-\frac{\pi}{6} \le x - \frac{\pi}{6} \le \frac{11\pi}{6}$
2 Determine the reference angle in quadrant 1.	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ $\alpha = \frac{\pi}{6}$ is the reference angle.
3 Recall the quadrants with the required sign.	Sine is negative in quadrants 3 and 4.
4 Determine the related angles in the required quadrants.	Quadrant 3: $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ Quadrant 4: $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$
5 Extend the list to suit the domain.	$\frac{11\pi}{6}$ is the same position as $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$
6 Write the solutions for the expression.	$x - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
7 Solve for the unknown.	$x = -\frac{\pi}{6} + \frac{\pi}{6}, \frac{7\pi}{6} + \frac{\pi}{6}, \frac{11\pi}{6} + \frac{\pi}{6}$ $x = 0, \frac{4\pi}{3}, 2\pi$

30 Determine approximate solutions of a trigonometric equation using technology

Use technology to determine the solutions of $\cos(x) = \frac{1}{3}$ over the domain $[0, 2\pi]$, correct to 2 decimal places.

METHOD 1

THINKING	WORKING
1 Use technology to draw a graph to represent each side of the equation. Read off the points of intersection in the required domain, ensuring the x-values are to the correct number of decimal places	$\cos(x) = \frac{1}{3}$ for $[0, 2\pi]$:  Points of intersection: (1.23, 0.33) and (5.05, 0.33)
2 Write the solutions.	$x = 1.23$ or $x = 5.05$

METHOD 2

1 Use your calculator to determine the reference angle.	$\cos^{-1}\left(\frac{1}{3}\right) = 1.23$ $\alpha = 1.23$ is the reference angle.
2 Recall the quadrants where the function is the required sign.	Cosine is positive in quadrants 1 and 4.
3 Determine the related angles in the required quadrants.	Quadrant 1: 1.23 Quadrant 4: $2\pi - 1.23 = 5.05$
4 Write the solutions.	$x = 1.23$ or $x = 5.05$

General solutions

If the domain is unrestricted there may be an infinite number of solutions to a trigonometric equation. General solutions can be found using general expressions for angles that are in the same position on the unit circle as angles already identified as solutions.

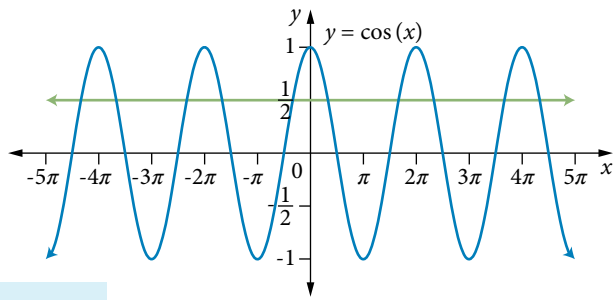
The graph shows the solutions to the equation $\cos(x) = \frac{1}{2}$.

If $x = \frac{\pi}{3}$ is a solution, then $x = \frac{\pi}{3} + 2\pi, \frac{\pi}{3} + 4\pi, \frac{\pi}{3} + 6\pi$ are also solutions.

In the negative direction $\frac{\pi}{3} - 2\pi, \frac{\pi}{3} - 4\pi$ etc. would also be solutions.

Similarly, as $x = \frac{5\pi}{3}$ is a solution, then $x = \frac{5\pi}{3} + 2\pi$, $\frac{5\pi}{3} + 4\pi, \frac{5\pi}{3} + 6\pi, \frac{5\pi}{3} - 2\pi, \frac{5\pi}{3} - 4\pi$ are also solutions.

In each case, the solutions are 2π apart because they are a full revolution of the circle apart. This can be written as $x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{5\pi}{3} + 2n\pi$, where n is an integer.



The general solution of $\sin(x) = k$ is:

$x = 2n\pi + \sin^{-1}(k)$ or
$x = (2n+1)\pi - \sin^{-1}(k)$, where $k \in [-1, 1]$ and $n \in \mathbb{Z}$

The general solution of $\cos(x) = k$ is:

$x = 2n\pi \pm \cos^{-1}(k)$, where $k \in [-1, 1]$ and $n \in \mathbb{Z}$

The general solution of $\tan(x) = k$ is:

$x = n\pi + \tan^{-1}(k)$, where $k \in [-1, 1]$ and $n \in \mathbb{Z}$

Additional information

Solving equations

These activities provide an explanation and practise of basic equation solving.

31 The general solution of a trigonometric equation

Determine the general solution for $\sin(x) = \frac{\sqrt{2}}{2}$.

THINKING	WORKING
1 Determine the quadrant 1 solution.	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\alpha = \frac{\pi}{4}$ is the reference angle.
2 Determine all solutions in the domain $[0, 2\pi]$.	Because $\sin(x)$ is positive, x could be in quadrants 1 or 2. $x = \frac{\pi}{4}$ or $x = \pi - \frac{\pi}{4}$ $= \frac{3\pi}{4}$
3 Write the general solutions.	$x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{3\pi}{4} + 2n\pi$

Equations involving functions of trigonometric functions

In more complex equations you may need to solve for the trigonometric function first. Where squares of the trigonometric function are involved, use methods of solving quadratics such as factorising and applying the null factor law.

32 Solutions of an equation involving a function of trigonometric function

Solve for x such that $1 - 2\sin^2(x) = \sin(x)$, $0 \le x \le 2\pi$.

THINKING	WORKING
1 Replace the trigonometric function with a pronumeral.	$1 - 2\sin^2(x) = \sin(x)$ Let $a = \sin(x)$: $1 - 2a^2 = a$
2 Equate to zero and factorise.	$2a^2 + a - 1 = 0$ $(2a - 1)(a + 1) = 0$
3 Solve using the null factor law.	So, $a = \frac{1}{2}$ or $a = -1$
4 Replace the pronumeral with the trigonometric function.	$\sin(x) = \frac{1}{2}$ or $\sin(x) = -1$
5 Solve for x.	$x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $x = \frac{3\pi}{2}$

EXERCISE 8.5 Solving trigonometric equations

Worked Example

27

1 Determine the exact solutions for each equation over the domain $[0, 2\pi]$.

(a) $\cos(x) = \frac{\sqrt{2}}{2}$ (b) $\sin(x) = \frac{1}{2}$ (c) $\tan(x) = -\frac{\sqrt{3}}{3}$

(d) $\cos(x) = -\frac{\sqrt{3}}{2}$ (e) $\cos(x) - \frac{\sqrt{3}}{2} = 0$ (f) $\sin(x) + \frac{\sqrt{2}}{2} = 0$

(g) $\tan(x) + 1 = 0$ (h) $\sin(x) - \frac{\sqrt{3}}{2} = 0$ (i) $\tan(x) - \sqrt{3} = 0$

2 (a) Which of the following is a solution to the equation $\tan(x) = -\frac{\sqrt{3}}{3}$?

A $\frac{\pi}{6}$ B $\frac{5\pi}{6}$ C $-\frac{\pi}{6}$ D $-\frac{5\pi}{6}$

(b) What common error could have been made by a student who thought the answer was $\frac{7\pi}{6}$?

3 Which option gives the pair of solutions to the equation $\sin(x) = -\frac{\sqrt{3}}{2}$ over the domain $[0, 2\pi]$?

A $\frac{\pi}{3}, \frac{2\pi}{3}$ B $\frac{\pi}{3}, \frac{4\pi}{3}$ C $\frac{4\pi}{3}, \frac{5\pi}{3}$ D $\frac{2\pi}{3}, \frac{5\pi}{3}$

28

4 Determine the exact solutions for each equation over the given domain.

(a) $\cos(x) = \frac{1}{2}$, $[0°, 360°]$ (b) $\sin(x) = \frac{\sqrt{3}}{2}$, $[0°, 360°]$

(c) $\tan(x) = -\sqrt{3}$, $[0°, 360°]$ (d) $\cos(x) = -\frac{\sqrt{2}}{2}$, $[0°, 360°]$

(e) $\cos(x) = \frac{\sqrt{3}}{2}$, $[-180°, 180°]$ (f) $\sin(x) = -\frac{\sqrt{3}}{2}$, $[-180°, 180°]$

(g) $\tan(x) = 1$, $[-180°, 180°]$ (h) $\sin(x) = \frac{1}{2}$, $[-180°, 180°]$

30

5 Determine the solutions, correct to 2 decimal places, to each equation over the domain $[0, 2\pi]$.

(a) $\cos(x) = 0.562$ (b) $\sin(x) = \frac{1}{8}$ (c) $\tan(x) = 0.6$

(d) $\cos(x) = -0.15$ (e) $\sin(x) = -0.55$ (f) $\tan(x) = -2.8$

(g) $\cos(x) = \frac{2}{9}$ (h) $\sin(x) = -\frac{2}{9}$

6 Determine the solutions, to the nearest degree, for each equation over the domain $[0°, 360°]$.

(a) $\sin(x) = \frac{1}{4}$ (b) $\tan(x) = -2$ (c) $\cos(x) = \frac{2}{9}$

(d) $\tan(x) = 0.56$ (e) $\sin(x) = -0.45$ (f) $\cos(x) = -0.06$

7 Solve each equation over the given domains, correct to 2 decimal places.

(a) $2\tan(2x) = 0.45$, domain $[0, 2\pi]$

(b) $5\sin(3\pi x) = 4$, domain $[0, 2\pi]$

(c) $3\cos\left(\frac{2x}{3}\right) + 1 = 0$, domain $[0, 10]$

(d) $\tan\left(3x - \frac{\pi}{4}\right) + 2 = 0$, domain $[0, 3]$

(e) $4\cos(3x + \pi) = 3$, domain $[-2, 2]$

(f) $5\sin(2\pi x - \pi) = 2$, domain $[-1, 1]$

Worked Example 31

8 Solve each of the following equations.

(a) $\sin(x) = \frac{1}{2}$

(b) $\cos(x) = \frac{\sqrt{2}}{2}$

(c) $\tan(x) = \sqrt{3}$

(d) $\sin(x) = -\frac{\sqrt{2}}{2}$

(e) $\cos(x) = -\frac{\sqrt{3}}{2}$

(f) $\tan(x) = -1$

29

9 Determine the exact solutions over the domain $[0, 2\pi]$.

(a) $\sqrt{2}\cos(2x) = 1$

(b) $\sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$

(c) $\tan(3x) - 1 = 0$

(d) $2\cos\left(2x - \frac{\pi}{3}\right) + \sqrt{3} = 0$

(e) $2\sin\left(\frac{x}{2}\right) = \sqrt{2}$

(f) $\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \sqrt{3}$

32

10 Determine the values of x for the given domain.

(a) $\cos(x) = 2\cos^2(x) - 1$, $0 \le x \le 2\pi$

(b) $\sin(x) = 1 - 2\sin^2(x)$, $0 \le x \le 2\pi$

(c) $0 = \tan^3(x) + 2\tan(x)$, $0 \le x \le 2\pi$

(d) $2\cos^2(x) = 1$, $0 \le x \le 2\pi$

11 Determine the exact values of all solutions of the equation $2\cos(2x) + 3 = 2$ for $x \in [-\pi, \pi]$.

12 Solve $4\sin^2(x) - 1 = 0$ over the domain $[0, 2\pi]$.

13 Solve $2\tan^2(x) = \tan(x) + 6$ for $[0, 5]$. Give answers correct to 2 decimal places.

14 Consider the equation $5\sin(x) + 2x = b$, where $b > 0$.

(a) Determine a possible value for b such that the equation has three solutions. Write the equation with the solutions, correct to 2 decimal places.

(b) Determine a possible value for a, where $0 < a < b$, such that $5\sin(x) + 2x = b$ has only one solution. Write the equation with the solution, correct to 2 decimal places.

(c) Determine a possible value for c, where $c > b$, such that $5\sin(x) + 2x = c$ has only one solution. Write the equation with the solution, correct to 2 decimal places.

15 If $\cos(x) = ax^2 + b$, determine possible values for a and b such that the equation has six solutions for each case. Write each equation with the solutions correct to 2 decimal places.

(a) $a > 0$ and $b < 0$

(b) $a > 0$ and $b > 0$

8.6 Applications of trigonometric functions

Time as the independent variable

Trigonometric functions can be used to model periodic phenomena that occur in the physical world. Average monthly temperatures, seasonal climate variations, biorhythms, sound waves, tides, alternating current and the phases of the Moon are examples of quantities that display periodic behaviour and can be modelled by the sine or cosine functions.

In these problems, time is the independent variable. The scale on the horizontal axis will usually have numbers that represent the time elapsed in the situation (hours and minutes) and will not necessarily be represented in units that are exact fractions or multiples of π.

33 Convert time of day to values of t

If t is defined as the number of hours after 6 am, write each of the following times as values of t.

(a) 4 pm

THINKING	WORKING
1 Calculate the number of hours after the defined starting time.	6 am to 4 pm: $6 + 4 = 10$ hours
2 Write the value of t.	$t = 10$

(b) 9.15 am

1 Calculate the number of hours and minutes after the defined starting time.	6 am to 9 am: $9 - 6 = 3$ hours So 6 am to 9.15 am is 3 h 15 min.
2 Convert to hours.	$3\text{ h }15\text{ min} = 3\frac{15}{60}\text{ h}$ $= 3.25\text{ h}$
3 Write the value of t.	$t = 3.25$

34 Convert values of t to time of day

If t is defined as the number of hours after 9 am, write each of the following values of t as time of day.

(a) $t = 8$

THINKING	WORKING
1 Add the value of t to the hours number of the starting time.	$9 + 8 = 17$
2 Interpret the total as a time of day.	$17 - 12 = 5$ The time is 5 pm.

(b) $t = 2.2$

1 Add the whole number part of t to the starting time.	$9 + 2 = 11$
2 Convert the decimal part of t to minutes.	$0.2\text{ h} = (0.2 \times 60)\text{ min}$ $= 12\text{ min}$
3 Write the time of day with hours and minutes.	The time is 11.12 am.

Solving practical problems

To use a sine or cosine function to model periodic behaviour, you need to define a starting point and determine the median value, maximum (or minimum) value and the period for each cycle.

Additional information

Explore trigonometric functions

This activity explores the use of trigonometric functions to model tides.

35 Problems involving trigonometric functions and graphs

A Ferris wheel with a radius of 30 m rotates once every 60 seconds. Passengers get into a cabin 1 m above level ground.

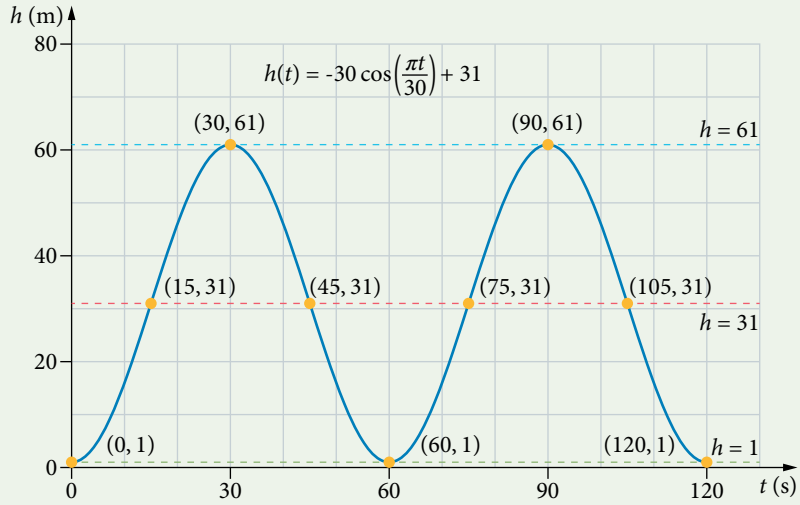
(a) Determine a rule for the height h (m) above the ground in terms of time t (s) after leaving the lowest level.

THINKING	WORKING
1 Choose a sine or cosine function. Avoid phase shift, if possible.	A reflected cosine graph will start at the minimum value, so $h(t) = A\cos(Bt) + D$.
2 List given values to determine the parameters.	Radius: 30 m, so $A = -30$ Period: 60 $\frac{2\pi}{B} = 60$ $B = \frac{2\pi}{60}$ $= \frac{\pi}{30}$ Minimum height: 1 m, so median height is $(1 + 30)$ m $\therefore D = 31$
3 Substitute the parameters to write the rule.	$h(t) = -30\cos\left(\frac{\pi t}{30}\right) + 31$

(b) Plot the graph of $h(t)$ for a 2-minute period after a passenger enters the wheel.

Taking care with the placement of five key points for each period, graph the function.

The period is 60 seconds, so there will be two cycles. Each 15 seconds either a median, maximum or minimum is reached.



(c) Determine the passenger's height, to the nearest metre, above ground after 20 seconds.

1 Determine the height by substituting the value of t into the equation.

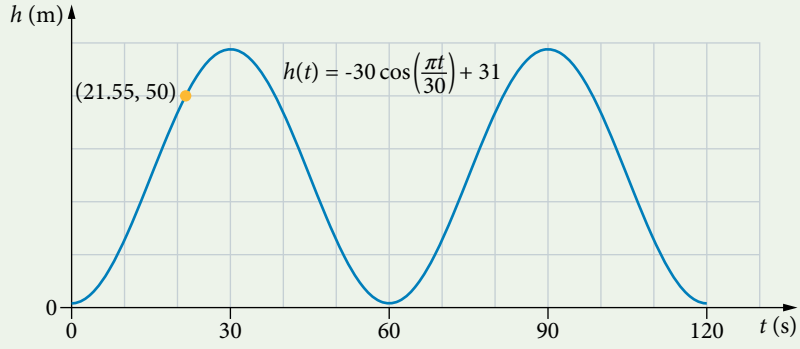
$$\begin{aligned} h(20) &= -30\cos\left(\frac{\pi \times 20}{30}\right) + 31 \\ &= -30\cos\left(\frac{2\pi}{3}\right) + 31 \\ &= -30\left(-\cos\left(\frac{\pi}{3}\right)\right) + 31 \\ &= 30 \times \frac{1}{2} + 31 \\ &= 15 + 31 \\ &= 46 \end{aligned}$$

2 Interpret the answer.

After 20 seconds, the passenger is at a height of 46 m.

(d) When will the cabin be 50 m above the ground for the first time?

Read the value from the graph. A rough approximation is possible from a hand-drawn graph, with greater accuracy available from technology.



The cabin will be 50 m above the ground for the first time between 20 and 25 seconds (21.55 seconds) after the ride begins.

EXERCISE 8.6 Applications of trigonometric functions

Worked Example

1 If t is the number of hours after 10 am, then write each of the following times in terms of t. 33

(a) 12 noon (b) 12 midnight (c) 9 pm

(d) 1.45 pm (e) 11.30 am (f) 6.40 pm

2 If t is the number of months after 1 January 2018, then write each of the following dates in terms of t. Make assumptions, as needed.

(a) 1 June 2018 (b) 1 June 2019 (c) 16 June 2018

(d) 12 March 2018 (e) 25 December 2018 (f) 7 April 2020

3 If t is the number of hours after 7 pm, write each of the following values of t as time of day. 34

(a) $t = 4$ (b) $t = 12$ (c) $t = 10$

(d) $t = 1.5$ (e) $t = 6.35$ (f) $t = 4.75$

4 If t is the number of months after 30 June 2001, write each of the following values of t as a date. Make any assumptions you need to answer the question.

(a) $t = 2$ (b) $t = 8$ (c) $t = 12$

(d) $t = 1.5$ (e) $t = 3.6$ (f) $t = 10.5$

5 A Ferris wheel with a radius of 35 m rotates every 90 seconds. Passengers enter a cabin at the loading point, which is 2 m above level ground. The height $h(t)$ of a cabin above the ground in metres is given by $h(t) = -35\cos\left(\frac{\pi t}{45}\right) + 37$, where t is the time in seconds after the cabin passes the loading point.

(a) Sketch the graph of $h(t)$ for a 3-minute period after a cabin passes the loading point and label the h-intercept.

(b) Determine a cabin's height above the ground in metres after 30 seconds, correct to 1 decimal place.

(c) After how many seconds will the cabin be 40 m above the ground for the first time? Give the answer correct to 2 decimal places.

6 The temperature T°C inside the school classroom during a day in February is given by the equation $T = 15\sin\left(\frac{\pi t}{12}\right) + 20$, where t is the number of hours after 9 am and $0 \le t \le 12$.

(a) Sketch the graph of the temperature T°C inside the school classroom over the 12-hour period.

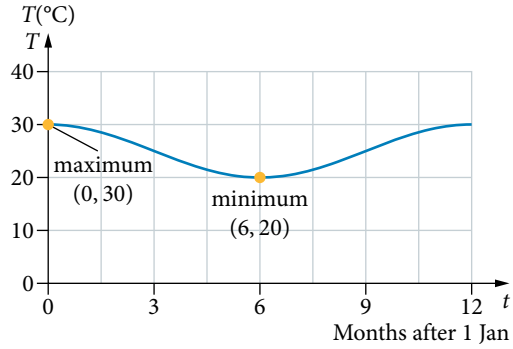
(b) Calculate the temperature at 10 am, correct to the nearest degree.

(c) Calculate the temperature at 12 noon, correct to 1 decimal place.

(d) At what times is the temperature 30°C, to the nearest minute?

(e) What percentage of time between 9 am and 9 pm is the temperature inside the room at least 30°C, correct to the nearest percentage?

7 Write a trigonometric model involving a cosine function in the form $y = A\cos(Bx) + D$ for the monthly average maximum daily temperature over a year in Brisbane as shown below in the temperature T versus time t graph.



8 The voltage, V volts, supplied by an electrical outlet is described by the sine function $V = A\sin(Bt) + D$, where A, B and D are constants and $A > 0$.

(a) If the voltage V oscillates between -240 volts and 240 volts, calculate the values of A and D.

(b) If it has a frequency of 50 cycles per second, determine the value of B.

(c) Write the equation for the voltage (V) involving a transformation of the sine function.

9 Christina programs her sprinkler system according to the equation $p = 20\cos(Bt)$, where p is the water pressure in kilopascals and t is the time in seconds.

(a) Choose the value of B if the system has a period of 1.2 s.

A $\frac{5\pi}{6}$ B $\frac{5\pi}{3}$ C $\frac{\pi}{2}$ D $\frac{3\pi}{5}$

(b) Explain the common error made by a student who thought the answer was $\frac{12\pi}{5}$.

Worked Example

35 10 A Ferris wheel with a radius of 10 m is rotating at a rate of 2 revolutions per minute. When $t = 0$, a chair starts at the lowest point on the wheel, which is 2 m above the ground. Construct a model for the height h (m) of the chair as a function of the time t (s).

11 The height of the water h (m) in Sametime Bay can be modelled by $h(t) = A\cos(Bt) + D$, where A, B and D are constants and t represents the number of hours after midnight. On a particular day a high tide of 4.5 m occurred at 6 am and a low tide of 0.5 m occurred at 12 noon.

(a) Determine the rule for $h(t)$.

(b) When does the second high tide occur?

(c) Draw a graph to show the height of the tide for the first 24 hours after midnight.

12 The largest paddle wheel steamship, the *American Queen*, has a paddle wheel that is 10 m in diameter. When moving at its cruising speed the wheel revolves 12 times per minute. Using this speed and starting from a point at the top of the wheel, write a model for the height h (m) of the paddle relative to the water's surface for time t (s). Assume the paddle is 1 m below the water's surface at its lowest point.

13 Fred was rushed to hospital with a mystery illness that caused his temperature to vary from a low of 37.0°C to a high of 40.2°C. He has been hospitalised for 4 weeks and his temperature has reached the high of 40.2°C every 8 days.

(a) Assuming that the function that describes Fred's temperature T°C after t days can be modelled by a sine function of the form $T = A\sin(Bt) + D$, where A, B and D are constants, determine the rule for $T(t)$.

(b) Sketch a graph of the function for the 4-week period.

(c) According to the model, what was Fred's temperature when he was admitted to hospital?

(d) What was Fred's temperature after 4 days in hospital?

(e) When did Fred's temperature reach the high of 40.2°C for the first time?

(f) What was Fred's temperature after 22 days in hospital?

14 The productivity, P units, of a person at work (on a scale of 0 to 10) is modelled by the cosine function $P = 5\cos\left(\frac{\pi t}{2}\right) + 5$, where t is in hours. If the person starts their 8-hour shift at $t = 0$, which is 8 am, at what times is the worker most productive?

A 12 noon only

B 10 am and 2 pm

C 8 am and 4 pm only

D 8 am, 12 noon and 4 pm

15 A ski hire business has revenue s (in thousands of dollars) throughout the year for slalom skis, which can be modelled using the function $s(t) = -10\cos\left(\frac{\pi t}{6}\right) + 10$, where t is the time in months since the start of the year.

(a) Sketch a graph of the function over a whole year.

(b) What is the period of the revenue function?

(c) Calculate the maximum revenue from sales of skis and when this occurs.

(d) Calculate the minimum revenue from sales of skis and when this occurs.

16 The temperature in a laboratory is regulated and can be represented by an equation of the type $T(t) = A\sin(B(t + C)) + D$, where T is the temperature in degrees Celsius and t is time in hours after midnight.

(a) Determine the rule for $T(t)$ if the temperature falls to a minimum of 8°C once each day at 4 am and the maximum temperature reached is 22°C.

(b) At what time does the laboratory reach its maximum temperature?

17 The depth of water in a canal follows a regular pattern given by $d = 10 - 4\sin\left(\frac{\pi t}{12}\right)$, where d is the depth of the river in metres and t is the time, in hours, after midnight on Sunday.

(a) Determine the maximum and minimum depth of the water in the canal.

(b) When does the depth of the water first reach its minimum?

(c) Sketch a graph of the function for $0 \le t \le 12$.

(d) There is a drain emptying into the canal 2 m below the maximum height of the water. For how many hours a day is the drain exposed?

18 The London Eye has a wheel of diameter 135 m and the platform where people enter the capsules is 2 m from the ground. The height h (m) above the ground of a capsule t minutes after leaving the platform is given by the equation: $h(t) = 69.5 + 67.5\sin\left(\frac{\pi t}{15} - \frac{\pi}{2}\right)$

(a) Rewrite the rule as a cosine function.

(b) How long does the London Eye take to make one complete revolution?

(c) Determine the height of the capsule $7\frac{1}{2}$ minutes after it left the platform.

(d) Plot the graph of h against t for one complete revolution.

(e) When is the capsule at a height of 100 m above the ground? Give your answers correct to the nearest minute.

19 The monthly average daily maximum and minimum temperatures (in °C) for Melbourne are given in the following table. Time t is measured in months, with $t = 1$ representing January. Write a trigonometric model, involving a sine function, that gives each of the following values as a function of t.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Maximum	26°	26°	24°	20°	17°	14°	13°	15°	17°	20°	22°	24°
Minimum	14°	14°	13°	11°	8°	7°	6°	7°	8°	9°	11°	13°

(a) the average daily maximum temperature

(b) the average daily minimum temperature

20 Scientists on Heard Island found that penguin numbers vary between a high of 900 at the start of the year and a low of 700 on 30 June. The population is approximated by $P(t) = A\cos(Bt) + D$, where t is the number of months since the start of the year, and A, B and D are constants.

(a) According to the population function, how many penguins will there be on 31 March, 20 June, and 23 October?

(b) When will there be a population of 850?

21 Landing supplies on Raoul Island, an active volcano in the Pacific Ocean, must be timed carefully because of the large tidal variation. The depth of water, $D(t)$ metres, at the landing point t hours after midnight on a particular day was given by the function $D(t) = A\cos(B(t + C)) + D$, where A, B, C and D are constants. The maximum water depth on that day was 15 m and the minimum depth was 4 m. The first high tide occurred at 12:30 am and the time between successive high tides is 12.5 hours.

Determine the depth of water, correct to the nearest metre, at:

(a) 6:45 am

(b) 1:00 pm

Summary

8

Radian measure

One radian is the angle subtended at the centre of a circle by two radii that enclose an arc one radius (r) long on the circumference of the circle.

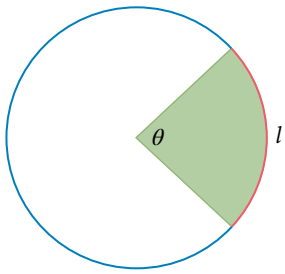
To convert an angle measured in degrees to radians, multiply by $\frac{\pi}{180}$.

To convert an angle measured in radians to degrees, multiply by $\frac{180}{\pi}$.

Arc length

$l = \frac{\pi r \theta}{180}$, where θ is the angle in degrees at the centre of the circle subtending the arc.

$l = r\theta$, where θ is the angle in radians at the centre of the circle subtending the arc.



Area of a sector

$A = \frac{\pi r^2 \theta}{360}$, where θ is the angle in degrees at the centre of the circle.

$A = \frac{r^2 \theta}{2}$, where θ is the angle in radians at the centre of the circle.

Trigonometric ratios in right-angled triangles

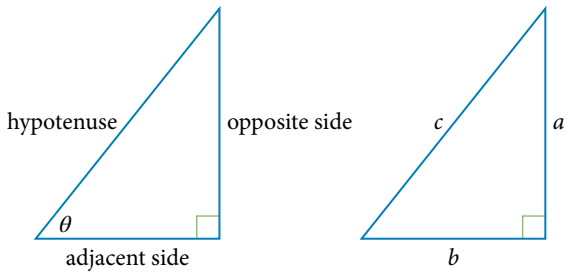
Right-angled triangles can be solved using the trigonometric ratios sine, cosine and tangent (abbreviated to sin, cos and tan respectively), as well as Pythagoras' theorem.

$$\sin(\theta) = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

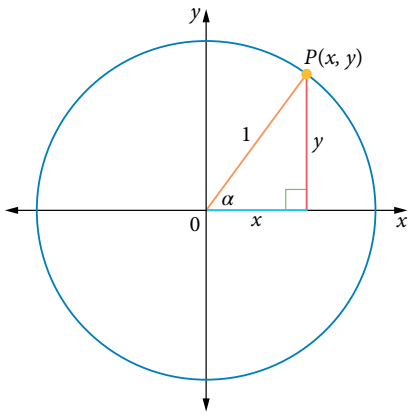
$$\tan(\theta) = \frac{\text{opposite side}}{\text{adjacent side}}$$

Pythagoras' theorem: $c^2 = a^2 + b^2$



The unit circle

The unit circle has its centre at the origin (0, 0) and has a radius of 1 unit. Its equation is $x^2 + y^2 = 1$.



Definitions of trigonometric ratios

$y = \sin(\theta)$, $x = \cos(\theta)$, $\tan\theta = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{y}{x}$, where θ is measured anticlockwise from the positive direction of the x-axis.

$\cos^2(\theta) + \sin^2(\theta) = 1$ is known as the Pythagorean identity.

Related angles

$P(x_1, y_1)$ is the first quadrant point for the reference angle, α.

In the second quadrant:

$\sin(\pi - \alpha) = y_1 = \sin(\alpha)$

$\cos(\pi - \alpha) = -x_1 = -\cos(\alpha)$

$\tan(\pi - \alpha) = \dfrac{y_1}{-x_1} = -\tan(\alpha)$

In the third quadrant:

$\sin(\pi + \alpha) = -y_1 = -\sin(\alpha)$

$\cos(\pi + \alpha) = -x_1 = -\cos(\alpha)$

$\tan(\pi + \alpha) = \dfrac{-y_1}{-x_1} = \tan(\alpha)$

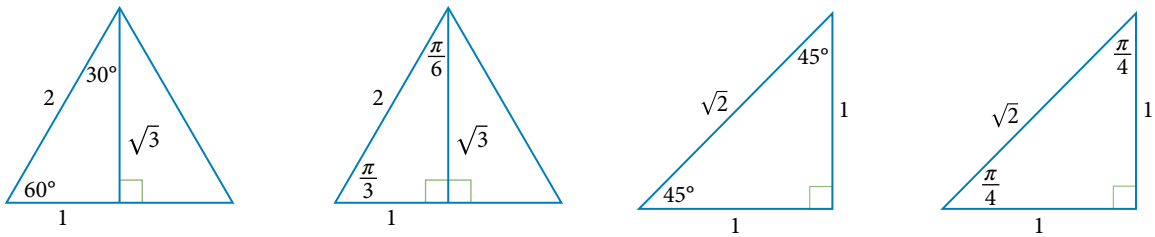
In the fourth quadrant:

$\sin(2\pi - \alpha) = -y_1 = -\sin(\alpha)$

$\cos(2\pi - \alpha) = x_1 = \cos(\alpha)$

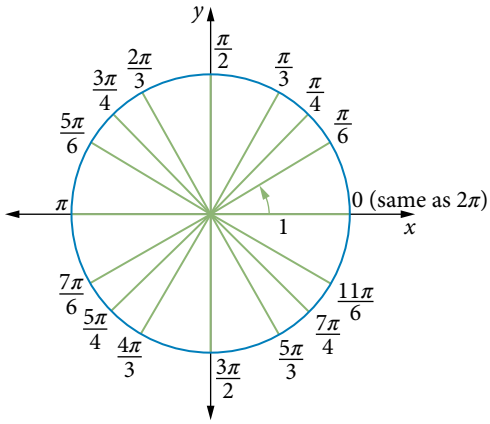
$\tan(2\pi - \alpha) = \dfrac{-y_1}{x_1} = -\tan(\alpha)$

Exact values of trigonometric ratios



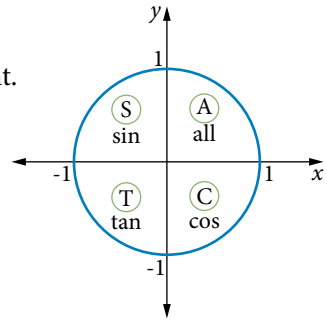
θ	0^c (0°)	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Exact values of angles in other quadrants can be found using related angles.



Sign of trigonometric ratios

The diagram shows the positive ratios for each quadrant.



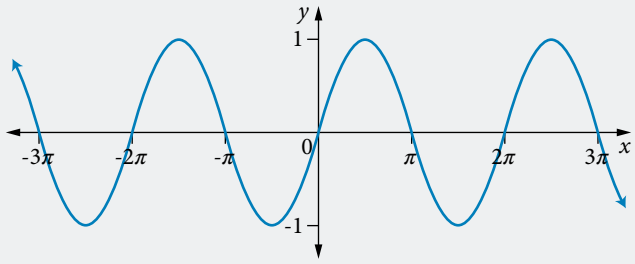
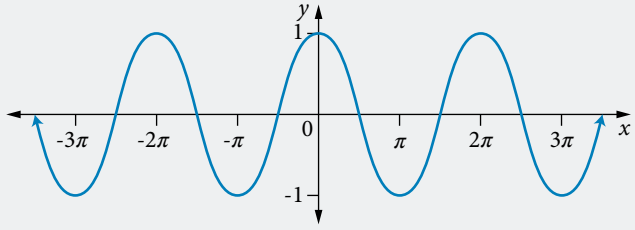
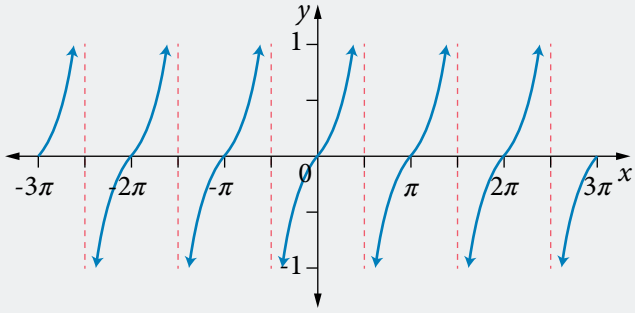
For negative angles:

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\tan(-\theta) = -\tan(\theta)$

Graphs of trigonometric functions

$y = \sin(x)$		Period: 2π Amplitude: 1 Domain: $x \in \mathbb{R}$ Range: $y \in [-1, 1]$
$y = \cos(x)$		Period: 2π Amplitude: 1 Domain: $x \in \mathbb{R}$ Range: $y \in [-1, 1]$
$y = \tan(x)$		Period: π Amplitude: undefined Domain: $x \in \mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$ Range: $y \in \mathbb{R}$ The lines $x = \ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ are vertical asymptotes.

Transformations

The graph of $y = \cos(x)$ is obtained from the graph of $y = \sin(x)$ by translation of $\frac{\pi}{2}$ units to the left.

Dilations

Graphs of the form $y = A\sin(Bx)$, $y = A\cos(Bx)$, where $B > 0$, have amplitude of $|A|$, period of $\frac{2\pi}{B}$, range of $[-|A|, |A|]$.

If A is negative, the graph is reflected in the x-axis.

Vertical translations

Graphs of the form $y = A\sin(Bx) + D$, $y = A\cos(Bx) + D$ are translated vertically by D units.

The median y-value is D and the function values vary from the median by the value of $|A|$.

The range is $[D - |A|, D + |A|]$.

Phase changes

Graphs of the form $y = A\sin(B(x + C)) + D$, $y = A\cos(B(x + C)) + D$ have a horizontal phase shift.

When a number C is added to x, the resulting graph is translated horizontally by $-C$. This is known as a phase shift.

Solving trigonometric equations

The general solution of $\sin(x) = k$ is:

$x = 2n\pi + \sin^{-1}(k)$ or $x = (2n + 1)\pi - \sin^{-1}(k)$, where $k \in [-1, 1]$ and $n \in \mathbb{Z}$

The general solution of $\cos(x) = k$ is:

$x = 2n\pi \pm \cos^{-1}(k)$, where $k \in [-1, 1]$ and $n \in \mathbb{Z}$

The general solution of $\tan(x) = k$ is:

$x = n\pi + \tan^{-1}(k)$, where $k \in [-1, 1]$ and $n \in \mathbb{Z}$

If technology is allowed, graph $y = \sin(x)$ and $y = k$ to solve $\sin(x) = k$.

Modelling

Trigonometric functions can be used to model periodic phenomena that occur in the physical world. Average monthly temperatures, seasonal climate variations, biorhythms, sound waves, variations in tides and the phases of the Moon are examples of quantities that display periodic behaviour and can be modelled by the sine or cosine functions.

Chapter review 8

1 Convert each of the following to radians, in terms of π. Exercise 8.1

(a) $48°$ (b) $250°$ (c) $315°$

2 Convert each of the following to degrees. Exercise 8.1

(a) $\frac{\pi}{3}$ (b) 3π (c) $\frac{5\pi}{18}$

3 For each of the following cases, calculate the exact length, in terms of π, of an arc of a circle of radius r if the arc subtends an angle of θ at the centre of the circle. Exercise 8.1

(a) $r = 12\text{ cm}, \theta = 120°$ (b) $r = 15\text{ cm}, \theta = \frac{\pi}{12}$

4 For the each of the following values of θ, calculate the exact area, in terms of π, of the sector of a circle of diameter 12 cm, given that the arc of the sector subtends an angle of theta at the centre. Exercise 8.1

(a) $\theta = 60°$ (b) $\theta = \frac{5\pi}{6}$

5 What is the length of the arc subtended on a circle of diameter 42 cm by an angle of $\frac{3\pi}{4}$? Exercise 8.1

A $\frac{63\pi}{2}$ cm B $\frac{21\pi}{2}$ cm C $\frac{21\pi}{4}$ cm D $\frac{63\pi}{4}$ cm

6 Determine the exact value of each of the following expressions. Exercise 8.2

(a) $\cos(60°)$ (b) $\tan\left(\frac{\pi}{3}\right)$ (c) $\sin\left(\frac{\pi}{4}\right)$ (d) $\cos(45°)$

7 In what quadrants is $\sin(\theta)$ negative? Exercise 8.2

8 Determine the exact value of each of the following expressions. Exercise 8.2

(a) $\sin(300°)$ (b) $\cos\left(\frac{5\pi}{3}\right)$ (c) $\tan(150°)$ (d) $-\cos\left(\frac{7\pi}{6}\right)$

9 If θ is in the first quadrant and $\sin(\theta) = 0.7234$, determine the value of: Exercise 8.2

(a) $\sin(\pi - \theta)$ (b) $\sin(2\pi - \theta)$ (c) $\sin(2\pi + \theta)$

(d) $\sin(6\pi - \theta)$ (e) $\sin(-\theta)$

10 If θ is in the first quadrant and $\tan(\theta) = 2.658$, determine the value of: Exercise 8.2

(a) $\tan(\pi + \theta)$ (b) $\tan(2\pi - \theta)$ (c) $\tan(2\pi + \theta)$

(d) $\tan(-\theta)$ (e) $\tan(-(5\pi - \theta))$

11 If $\cos(\theta) = \frac{\sqrt{3}}{2}$ and θ is in the first quadrant, what is $\tan(\theta)$? Exercise 8.2

A $\frac{\sqrt{3}}{3}$ B $\frac{1}{2}$ C $\frac{\sqrt{2}}{2}$ D $\sqrt{3}$

12 Determine the amplitude and period of the function $y = -2\cos(3x)$. Exercise 8.3

13 What is the maximum value of $y = 4\sin\left(3\left(x - \frac{\pi}{3}\right)\right) - 2$? Exercise 8.4

14 Determine the equations of the asymptotes for $y = 2\tan(x + \pi) + 2$ in the domain $[0, 2\pi]$. Exercise 8.4

Exercise 8.5 **15** What is the general solution for $\sin(x) = -\frac{1}{2}$?

Exercise 8.5 **16** Solve each of the following equations.

(a) $\sin(x) = -\dfrac{\sqrt{3}}{2}$ over the domain $[-2\pi, 2\pi]$

(b) $\sqrt{3}\tan(x) - 1 = 0$ over the domain $[0, 4\pi]$

Exercise 8.5 **17** Determine the solutions, in radians, to the equation $\tan(x) = 0.397$ over the domain $[0, 2\pi]$.

Exercise 8.5 **18** Solve each equation graphically over the given domain, correct to 2 decimal places.

(a) $10\sin(2\pi x) = 7$, domain $[0, 2]$

(b) $\tan\left(4x + \frac{\pi}{5}\right) - 2.4 = 0$, domain $[0, 3]$

Exercise 8.2 **19** (a) Determine the exact values of: $\sin\left(\frac{2\pi}{3}\right)$, $\cos\left(\frac{2\pi}{3}\right)$, $\sin\left(\frac{4\pi}{3}\right)$ and $\cos\left(\frac{4\pi}{3}\right)$.

(b) Demonstrate that each of the following is correct.

(i) $\sin^2\left(\dfrac{2\pi}{3}\right) + \cos^2\left(\dfrac{2\pi}{3}\right) = 1$

(ii) $\sin^2\left(\dfrac{4\pi}{3}\right) + \cos^2\left(\dfrac{4\pi}{3}\right) = 1$

Exercise 8.2 **20** Simplify the following expressions using complementary angle properties and symmetry properties of the unit circle, where $0 \le \theta \le \frac{\pi}{2}$.

(a) $\sin\left(\dfrac{5\pi}{2} + \theta\right)$

(b) $\cos\left(\theta - \dfrac{3\pi}{2}\right)$

Exercise 8.1 **21** A sector of a circle of radius 15 cm has a sector arc length of 18π cm. What is the area of the sector in square centimetres?

A $\dfrac{375\pi}{2}$ B $\dfrac{6\pi}{5}$ C 270π D 135π

Exercise 8.2 **22** If $\cos(\theta) = \frac{1}{4}$, what are the possible values of $\sin(\theta)$?

Exercise 8.3 **23** (a) Determine a rule for a positive sine graph of period $\frac{\pi}{2}$, with a maximum value of -2 and a minimum value of -5. There is no phase shift.

(b) Determine a rule for a cosine graph with a period of 2π. It has a maximum value of 4, a minimum value of -6 and a y-intercept of -6.

24 A mass attached to a spring oscillates up and down. The length x cm of the spring after t seconds is given by the function $x(t) = 24 - 4\cos(2\pi t)$.

(a) Sketch the graph of this function for $0 \le t \le 5$.

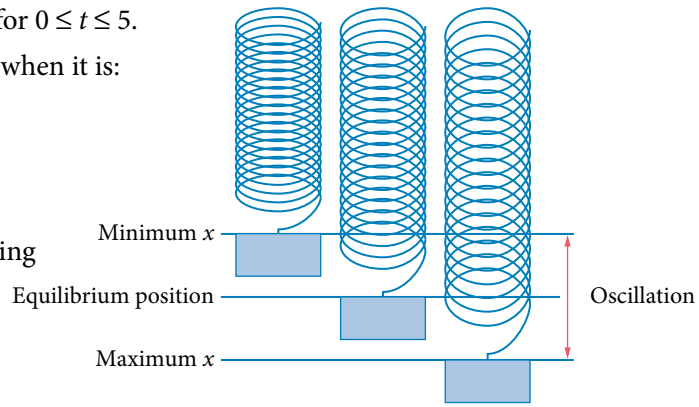
(b) Calculate the length of the spring when it is:

(i) at equilibrium

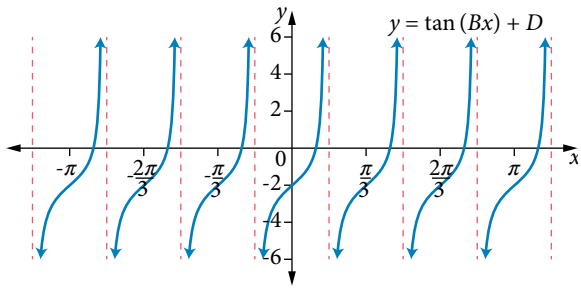
(ii) the shortest

(iii) the longest.

(c) What is the time taken for the spring to complete one oscillation?



25 Determine the equation of the graph shown. Assume $B > 0$.



26 Sketch the graph of each of the following functions for the given domain. Exercise 8.4

(a) Sketch $y = 3\sin(2x) - 2$ for $[0, 2\pi]$.

(b) Sketch $y = -2\cos\left(3\left(x - \frac{\pi}{2}\right)\right)$ for $[-\pi, \pi]$.

(c) Sketch $y = \frac{1}{2}\tan\left(\frac{x}{2}\right) + 1$ for the domain $[0, 2\pi]$, and label the asymptote.

27 A sine function has the equation $y = A\sin(B(x + C)) + D$, where $a > 0$. Exercise 8.4

(a) Determine the rule, given that the median value is 4, the period is 4π, the range is $[-2, 10]$ and the y-intercept is $(0, -2)$.

(b) Sketch the graph of the sine function over the domain $[-2\pi, 2\pi]$.

28 Determine the exact solutions of the following over the domain $[0, 2\pi]$. Exercise 8.5

(a) $\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$

(b) $2\cos\left(3x + \frac{\pi}{2}\right) + \sqrt{2} = 0$

(c) $\tan\left(\frac{x}{3} - \frac{\pi}{5}\right) = 1$

29 Solve the equation $2\cos^2(x) - 1 = 0$ over the domain $[0, 2\pi]$. Exercise 8.5

30 A Ferris wheel with a radius of 30 m makes a full rotation in one minute, and the axle stands 33 m above the ground. A fly lands on the top of the wheel just as it starts to rotate. If the fly stays on the wheel, what is the equation that models the fly's height above the ground $h(t)$ in metres, where t is the time in seconds since the rotation began? Exercise 8.6

A $h(t) = -30\cos\left(\frac{\pi t}{30}\right) + 33$

B $h(t) = 30\sin\left(\frac{\pi t}{30}\right) - 33$

C $h(t) = 30\cos\left(\frac{\pi t}{30}\right) + 33$

D $h(t) = 30\sin\left(\frac{\pi t}{30}\right) + 33$

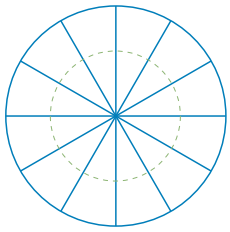
31 The height of a ball on a spring is modelled by $h(t) = 3.6\sin(4\pi t) + 3.6$ where $h(t)$ is in metres and t is in seconds. For the ball's path, determine the: Exercise 8.6

(a) highest point

(b) the lowest point

(c) height of the ball after 2 seconds.

32 Liang has bought a cake that is 40 cm in diameter and 10 cm high. Liang cuts the cake into 12 equal slices for 12 guests. When another 12 people arrive, Liang decides to cut the cake into 24 equal pieces using concentric circles. How far from the centre of the cake should the new circular cut be made so that all 24 guests get an equal amount of cake? Give the answer correct to the nearest centimetre.



33 Given $\sin\left(\frac{\pi}{7}\right) = 0.43$, determine the values of $\cos\left(\frac{5\pi}{14}\right)$ and $\cos\left(\frac{9\pi}{14}\right)$. Exercise 8.2

34 **(a)** If the period of the function is 3, determine the solutions to the equation $\tan\left(m\left(x - \frac{1}{2}\right)\right) = 0$ for $-1 \le x \le 6$ and $m > 0$. Support your working with a graph. Exercise 8.5

(b) If the graph of $y = \tan(mx)$ has an asymptote at $x = 3$, determine all possible values of m. Support your solution with a graph.

35 Solve the equation $4\sin^2 x + 2\sqrt{3}\sin x - 2\sin x - \sqrt{3} = 0$ over the domain $[-2\pi, 2\pi]$. Exercise 8.5

9

Rates of change

Recall 526

9.1 Average and instantaneous rates of change 528

9.2 Functions and rates of change 546

9.3 Derivatives from first principles 561

9.4 Differentiation using the power rule 569

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Chapter review 578

9

Recall

Convert units for rate measurements

1 What operation (multiply or divide by a number) is needed for each of the following conversions?

(a) Convert a speed in km/h to m/h.

(b) Convert a speed in cm/h to km/h

(c) Convert a rate at which water flows from L/h to L/s.

Calculate amounts from a rate

2 In the fuel blend called two-stroke, 200 mL of oil is mixed with every 1 L of petrol.

(a) How much oil should be mixed with 10 L of petrol?

(b) How much petrol would require 1 L of oil?

Convert fractions to decimals

3 Convert each fraction to a decimal number, correct to 2 decimal places.

(a) $\frac{15}{60}$

(b) $\frac{165}{3600}$

(c) $\frac{14}{9}$

Simplify fractions

4 Write each fraction in simplest fraction form.

(a) $\frac{65}{13}$

(b) $\frac{490}{140}$

(c) $\frac{275-110}{55}$

Complete a table of values according to a rule

5 (a) A tank that initially contains 50 L of lemonade is being filled at a rate of 5.3 L/min. Complete the table.

Time (min)	0	1	2	3	4
Volume (L)					

(b) A spa that initially holds 1500 L of water is leaking at a rate of 0.7 L/min. Complete the table.

Time (min)	0	10	20	30	40
Volume (L)					

Determine the gradient of a straight line

6 Determine the value of the gradient m of the line passing through each pair of points.

(a) (2, 5) and (4, 9)

(b) (-1, 2) and (3, -10)

(c) (1.75, -13) and (8.25, -9.75)

Determine the equation of a straight line

7 Determine the equation of the line that passes through each pair of points.

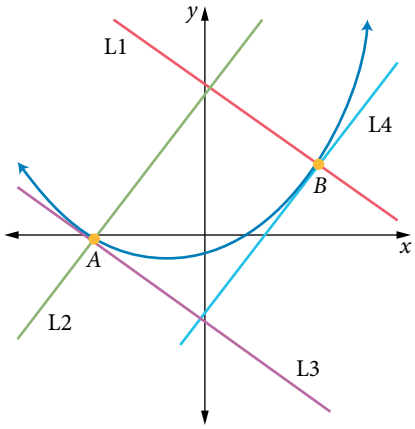
(a) (-1, 0) and (2, 15)

(b) (-1, 10) and (2, -5)

(c) (-2.5, -2) and (0.5, 10)

Identify a tangent to a curve

8 The graph below shows four straight lines that intersect a curve.



(a) Which line is the tangent line through point A?

(b) Which line is the tangent line through point B?

Calculate a function for given values

9 For the function $f(x) = x^3 + 2x^2 + 5x - 1$, calculate each of the following.

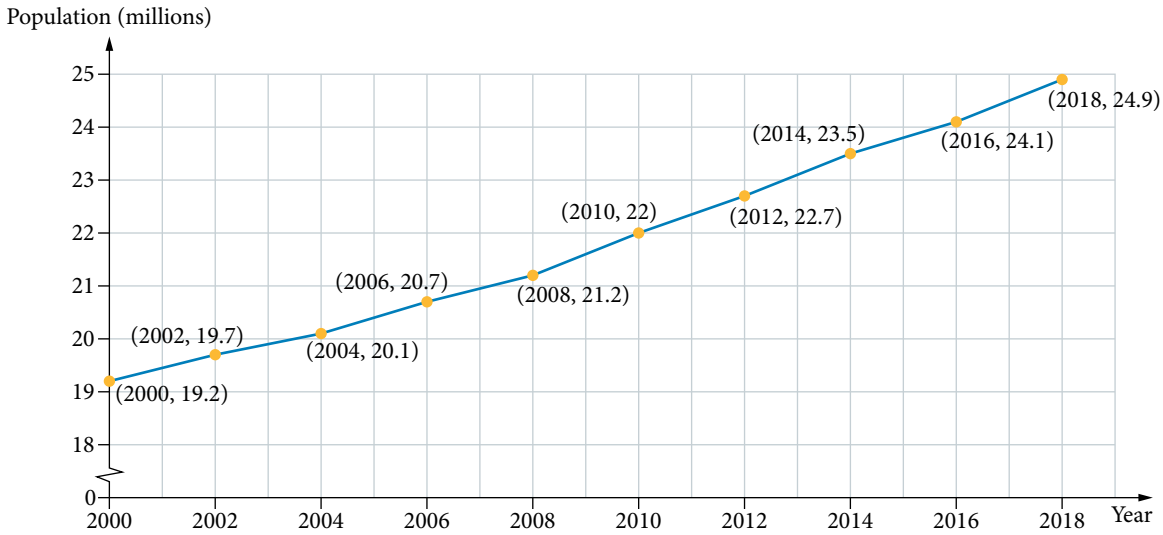
(a) $f(0)$ **(b)** $f(1)$ **(c)** $f(-1)$

9.1 Average and instantaneous rates of change

Calculating rates

In situations where quantities change in some way it is often important to know the rate at which these changes occur. For example, the speed of a car, which is the rate at which a car moves with respect to time, is shown on a speedometer.

The rate of change of the population of a country is important for planning the infrastructure for a larger (or smaller) population. The following graph shows the estimated resident population of Australia from 2000 to 2018.



Source: *Australian Demographic Statistics*, June 2017, cat. no. 3101.0. Australian Bureau of Statistics © Commonwealth of Australia. Licensed under CC by Attribution 2.5 Australia.

Between 2000 and 2010, the average population increased by 2.8 million, from 19.2 million to 22 million. The average growth rate over the 10-year period can be found by dividing 2.8 million by 10, which gives an increasing population by an average of 280 000 people per year.

1 Calculate simple rates

Determine the following rates.

(a) If 24 litres of diesel fuel costs \$35.52, determine the price rate, in dollars per litre, correct to the nearest cent.

THINKING	WORKING
1 Divide the cost of the fuel by its volume.	$\text{Rate} = \dfrac{35.52}{24}$ $= \$1.48/\text{L}$
2 Interpret the answer.	The cost of the fuel is \$1.48 per litre.

(b) A bus travelled 200 kilometres in 5 hours. Calculate the average speed in km/h.

1 Divide the total distance travelled by the total time.	Average speed $= \frac{200}{5}$ $= 40\,\text{km/h}$
2 Interpret the answer.	The bus, on average, travelled 40 km each hour of the 5 hour journey.

(c) A 1260 L water tank is filled in $3\frac{1}{2}$ hours. Calculate the filling rate in L/min.

1 Convert all quantities to the required units.	$3\frac{1}{2}\ \text{h} \times 60\,\text{min/h} = 210\,\text{min}$
2 Divide the volume of water by the time in minutes.	Rate $= \frac{1260}{210}$ $= 6\,\text{L/min}$
3 Interpret the answer.	The tank was filled at an average rate of 6 L per minute until it was full.

Rates compare two or more quantities. You can use a rate to determine the amount of a quantity, if you are given information about the other quantity.

2 Calculate quantities from given rates

The minimum flow rate for fire hydrants is 10 litres of water per second.

(a) At this rate, how many litres of water will flow from the fire hydrant in 5 minutes?

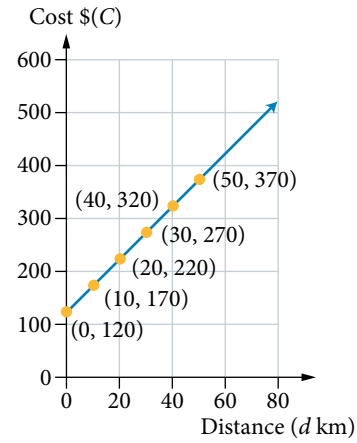
THINKING	WORKING
1 Convert quantities to the units in the rate.	$5\,\text{min} \times 60\,\text{s/min} = 300\,\text{s}$
2 Multiply the flow rate by the time.	Quantity of water: $10\,\text{L/s} \times 300\,\text{s}$ $= \frac{10\,\text{L}}{1\,\text{s}} \times 300\,\text{s}$ $= 3000\,\text{L}$

(b) If 4500 L litres of water was delivered by the fire hydrant at this rate, for how long was the water flowing?

1 To determine the time taken, divide the volume of water by the flow rate.	Time taken: $4500\,\text{L} \div 10\,\text{L/s}$ $= 4500\,\text{L} \times \frac{1\,\text{s}}{10\,\text{L}}$ $= 450\,\text{s}$
2 Convert the time unit in the answer to an appropriate unit (divide the time in seconds to obtain an answer in minutes).	$450\,\text{s} \div 60\ \text{s/min} = 7.5\,\text{min}$

Constant rates of change

A constant rate does not vary. If a transport company charges a fixed fee of \$120 per delivery, plus \$5 per kilometre travelled, the charge rate per kilometre is constant. When data that varies at a constant rate is plotted, the graph will be linear. The gradient of the line represents the constant rate of change.



Additional information

Proportions

Practise writing and using proportions.

Variable and average rates of change

In a Formula 1 race, the speed of each racing car is always changing. A car might reach a top speed of 370 km/h along a straight section of a track but will slow down to travel around sharp turns. The rate at which the distance travelled changes with respect to time, changes throughout the race.

When speed is not constant, the distance–time graph will have various curved or straight sections. A curved line represents a changing rate and a straight line represents a constant rate.

Average speed over an interval

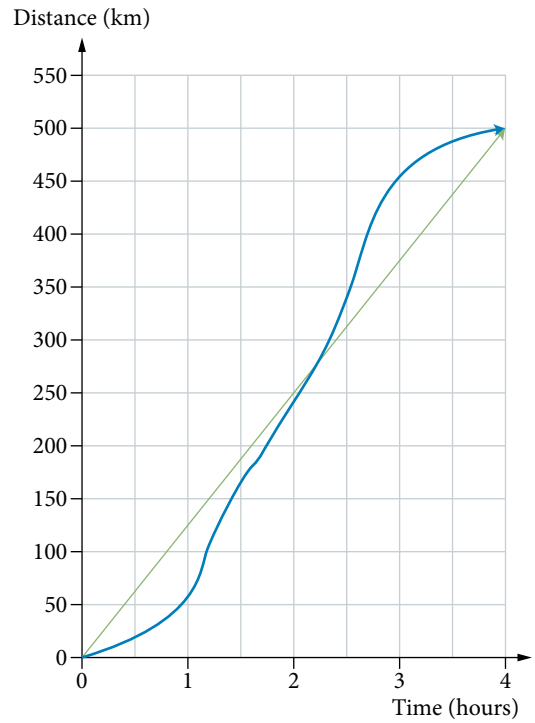
You can determine the average speed using the following formula:

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Consider the distance travelled and time taken by a rally car, recorded on the graph. The curved section shows how the distance of the rally car changes with time over the race. The green line directly connects the starting position to the finishing position. The straight line shows the total distance travelled over a known time. The average speed of the car can be calculated by determining the gradient of the green line.

The average speed of the rally car is:

$$\frac{500}{4} = 125\,\text{km/h}$$



Making connections

Speed of a rally car

Examine the average and instantaneous speed of a race-car at various points around a track.

Additional information

Average temperature

Calculate the average rate of temperature change over different time periods in a day.

3 Working with variable rates

On a hot day, a wet woollen jumper is hung on a clothes line to dry. It is weighed every 10 minutes for half an hour. The mass is shown in the table.

Time (min)	0	10	20	30
Mass (g)	2000	1500	1200	1000

(a) Calculate the average rate of change of mass during the first 10 minutes.

THINKING	WORKING
1 To calculate the average rate of change, divide the change in mass by the amount of time.	From $t = 0$ to $t = 10$ the mass of the jumper changed from 2000 g to 1500 g. $\text{Average rate of change} = \frac{1500 - 2000}{10 - 0} = \frac{-500}{10} = -50\text{ g/min}$
2 Explain the answer.	The rate is negative, so the jumper's mass decreases at an average rate of 50 g/min for the first 10 minutes.

(b) Determine the average rate of change of the jumper's mass over the first 30 minutes. Explain why this rate might vary from the rate for the first 10 minutes.

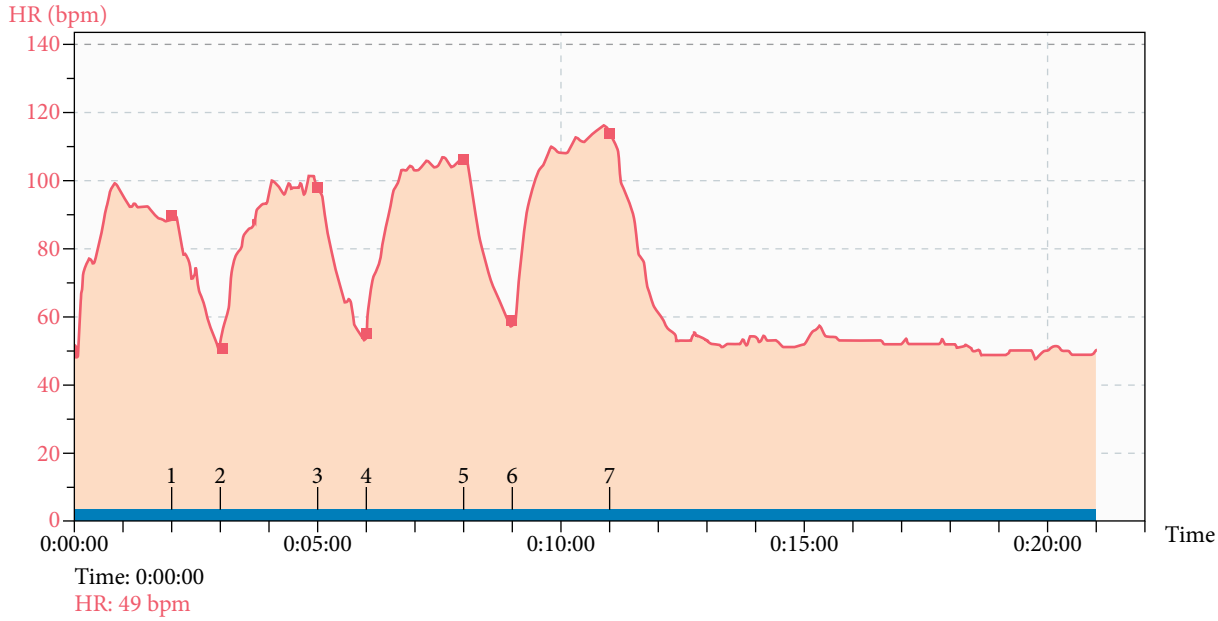
1 To calculate the average rate of change, divide the change of weight by the amount of time.	From $t = 0$ to $t = 30$ the mass of the jumper changed from 2000 g to 1000 g. $\text{Average rate of change} = \frac{1000 - 2000}{30 - 0} = \frac{-1000}{30} = -33\frac{1}{3}\text{ g/min}$
2 Explain the answer.	The rate is negative, so the jumper's mass is decreasing at a rate of $33\frac{1}{3}$ g/min for the first 30 minutes.
3 Compare this rate to the rate for the first 10 minutes.	The rate of drying has slowed from 50 g/min to $33\frac{1}{3}$ g/min. This is because water can evaporate from the jumper quickly in the beginning but as the amount of water in the jumper decreases, the rate at which it is lost also decreases.

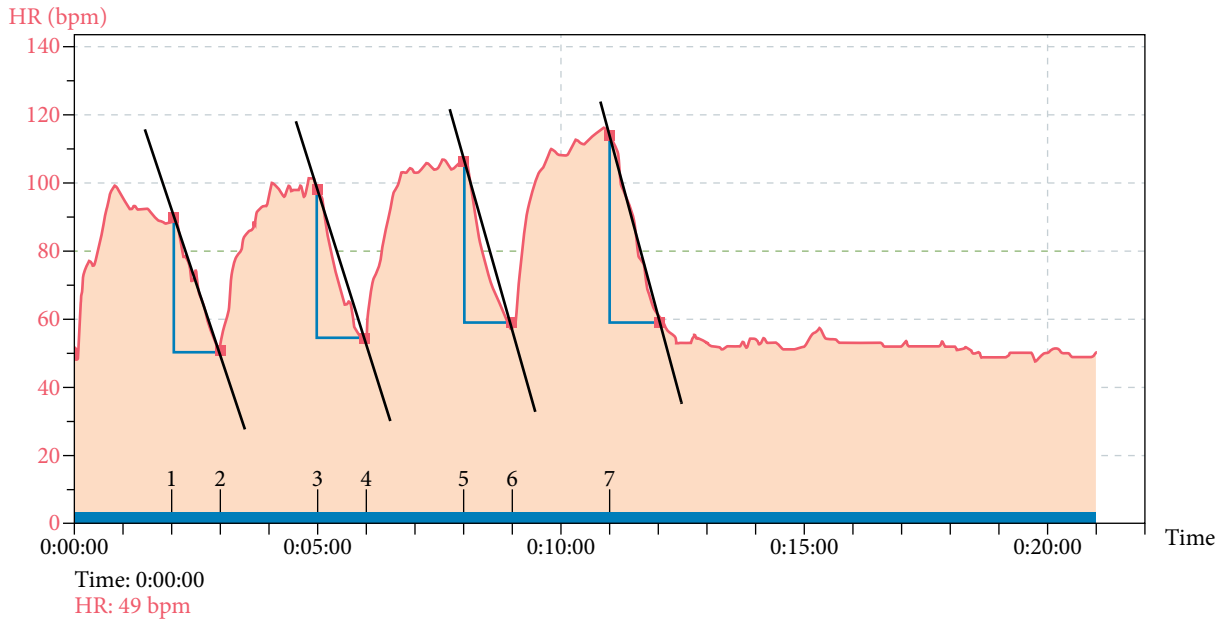
Using graphs to analyse average rates of change

To analyse a graph you need to identify and understand the meaning of key features and interpret the slopes of intervals between points. For example, the ability to measure the physical capability of an athlete can play an important role in training.

The graph below shows the exercise cycle of Mark Shuttleworth, a South African astronaut in training in Star City, Russia, preparing for a space mission. The vertical axis shows his heart rate in beats per minute and the horizontal axis shows the time in minutes. This graph shows his performance for four exercise cycles of 2 minutes, resting for 1 minute between each.

The slope of the curve shows the rate at which his heart beat changes with time; the steeper the curve, the faster his heart beat increases as it rises or decreases as it falls.

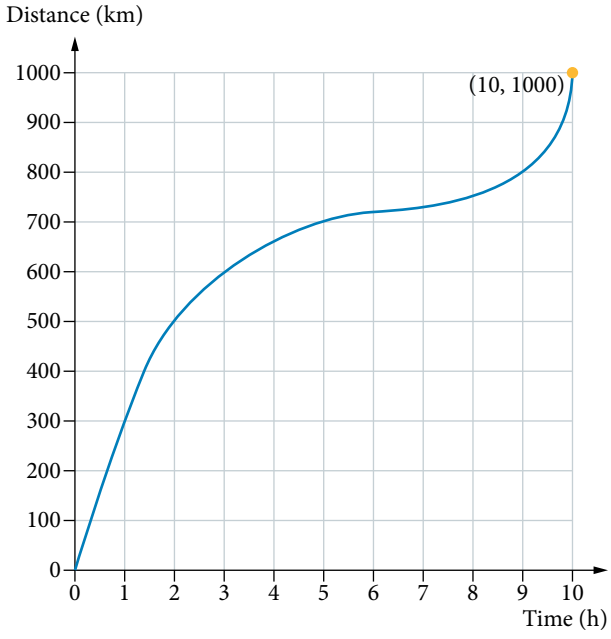




When lines are drawn to approximate the gradient of the curve during the resting stages it can be seen that his heart rate decreases, at a rate of about 40–50 bpm each minute, so there is a rate of change of -50 bpm/min following each of the four cycles.

4 Average rates of change from graphs

The following graph shows the distance travelled by an object.



(a) Use the graph to complete the table of values.

Time (h)	0	1	2	5	9	10
Distance (km)						

THINKING

Use the graph to determine the distance from the start after each hour.

WORKING

Time (h)	0	1	2	5	9	10
Distance (km)	0	300	500	700	800	1000

(b) Calculate the average speed of the object.

1 Identify and interpret the final point on the graph.

The coordinates of the endpoint are (10, 1000).

The total distance travelled by the object is 1000 km and the total time taken is 10 hours.

2 Recall the formula to calculate the average speed $s = \frac{d}{t}$

$$s = \frac{d}{t}$$
$$= \frac{1000}{10}$$
$$= 100 \text{ km/h}$$

3 Interpret the answer.

The average speed of the object was 100 km/h.

(c) Calculate the average speed of the object from the end of the first hour to the end of the ninth hour.

1 Identify the coordinates for the times given.

The coordinates of the endpoints of the specified domain are (1, 300) to (9, 800).

2 Recall the gradient formula, where:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$s = \frac{\Delta d}{\Delta t}$$
$$= \frac{800 - 300}{9 - 1}$$
$$= \frac{500}{8}$$
$$= 62.5 \text{ km/h}$$

3 Interpret the answer.

The average speed from the end of the first hour to the end of the ninth hour is 62.5 km/h.

The average rate of change can be calculated easily from tables of data, but a graph gives a snapshot of how the rate is changing at any point.

5 Working with average rates

On a farm, a biological control is being trialled to reduce the number of foxes.

The results are shown in this table.

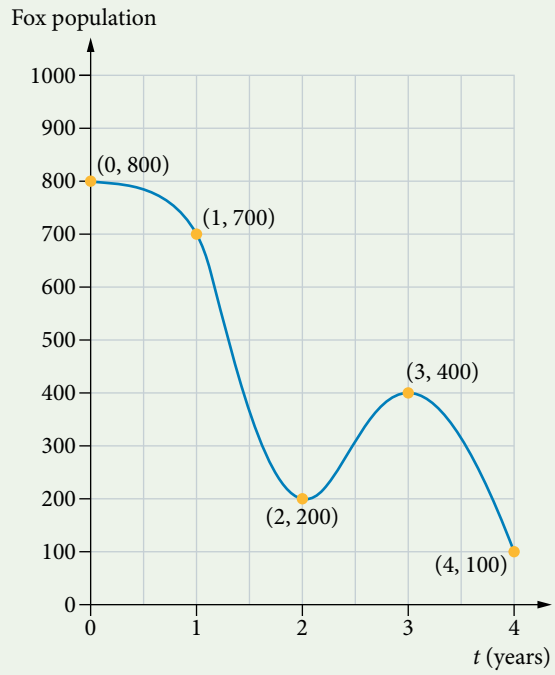
t (years)	0	1	2	3	4
Fox population	800	700	200	400	100

(a) Draw a graph of the fox population from the table of values.

THINKING

Draw and label a set of axes, plot the points on the graph and join them with a smooth curve.

WORKING



(b) Calculate the average decrease in the number of foxes over the first 2 years.

1 Identify the coordinates for the required times. (0, 800) and (2, 200)

2 Apply the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$$m = \frac{200 - 800}{2 - 0}$$
$$= \frac{-600}{2}$$
$$= -300$$

The fox population decreases at an average rate of 300 foxes per year for the first 2 years.

(c) Calculate the average rate of change in the number of foxes over the 4 years.

1 Identify the coordinates for the required times. (0, 800) and (4, 100)

2 Apply the formula: $m = \frac{\Delta \text{ population}}{\Delta \text{ time}}$.

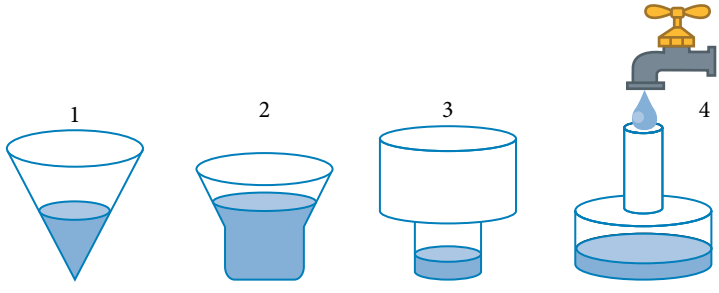
$$m = \frac{100 - 800}{4 - 0}$$
$$= \frac{-700}{4}$$
$$= -175$$

The fox population decreases at an average rate of 175 foxes per year over the 4 years.

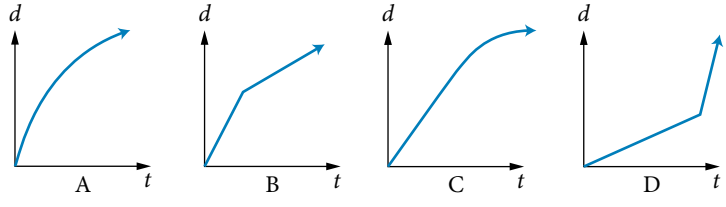
It may be possible to draw the shape of a graph by considering the changing rates over time.

6 Rates of change graphs

Water is poured at a constant rate into the containers.



The graphs show how the depth d of the water varies with time t as they are filled.



(a) Which graph best shows the way the depth changes with time as container 1 is filled?

THINKING	WORKING
Consider the surface area of the water as the container fills.	As the surface area of the water increases, the rate of change of height will decrease as it is filled. Container 1 matches graph A.

(b) Which graph best shows the way the depth changes with time as container 2 is filled?

Consider the surface area of the water as the container fills. Treat the two different shapes separately.	The surface area in the lower part is constant, so the rate of change of height will be constant as it is filled. The surface area in the upper part increases, so the rate of change of height will decrease as it is filled. Container 2 matches graph C.

(c) Which graph best shows the way the depth changes with time as container 3 is filled?

Consider the surface area of the water as the container fills. Treat the two different parts separately.	In the both parts the surface area of the water is constant, so the rate of change of height will be constant as it is filled. The slope of the first line is steeper than the slope of the second line as the container fills more rapidly when the surface area is small. Container 3 matches graph B.

(d) Which graph best shows the way the depth changes with time as container 4 is filled?

Consider the surface area of the water as the container fills. Treat the two different parts separately.	In the both parts the surface area of the water is constant, so the rate of change of height will be constant as it is filled. The slope of the second line is steeper than the slope of the first line as the container fills more rapidly when the surface area is small. Container 4 matches graph D.

Instantaneous rate of change as the gradient of a tangent

Each point on a curve has a different gradient, which represents the instantaneous rate of change at that point. You can estimate the instantaneous rate of change at a point by drawing the tangent to the curve at that point and calculating the gradient of the tangent line. A tangent line is a straight line that just touches a curve, and it has the same gradient as the curve at the point it touches.

Explore further

Average and instantaneous rates of change

Explore the average rate of change as the period decreases.

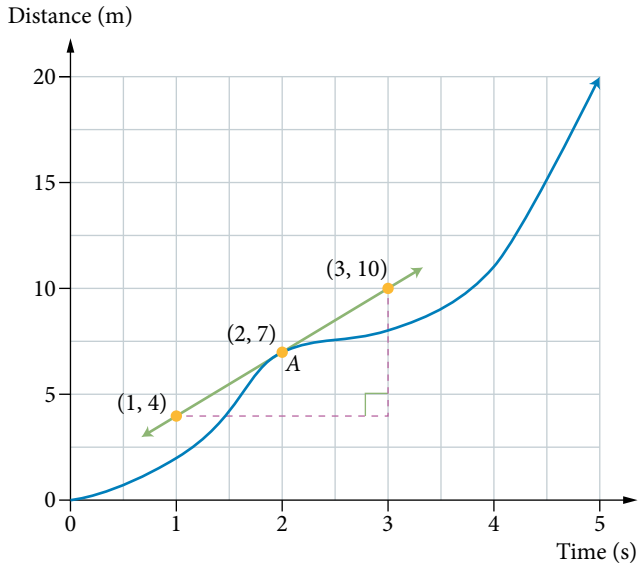
This graph shows the distance that a dog ran in a 5 second sprint. The tangent to the curve at the point $A(2, 7)$ can be used to calculate the dog's speed 2 seconds after it started running.

The speed at $t = 2$ s is the gradient of the tangent to the curve at that point.

$$\begin{aligned}\text{Speed} &= \frac{\Delta d}{\Delta t}\\ &= \frac{d_2 - d_1}{t_2 - t_1}\\ &= \frac{10 - 4}{3 - 1}\\ &= \frac{6}{2}\\ &= 3\,\text{m/s}\end{aligned}$$

The speed of the dog 2 seconds after it started running is approximately 3 m/s.

The distance covered by a dog during a 5 second chase

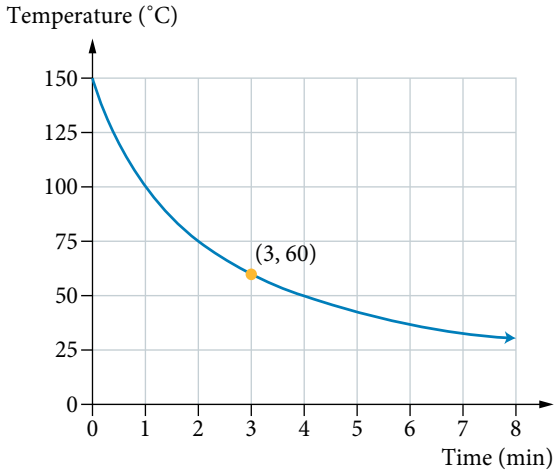


7 Calculate instantaneous rates using graphs

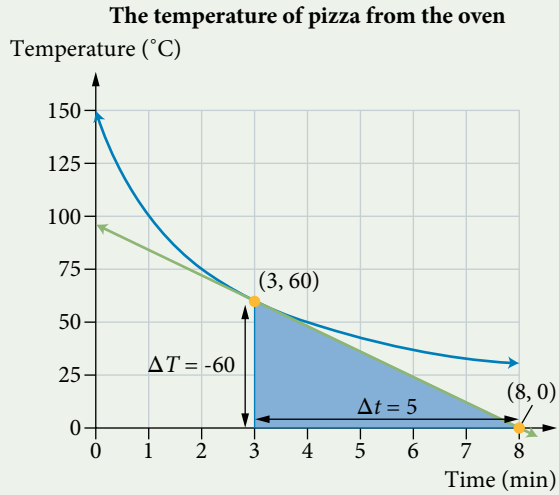
The temperature of a pizza is measured for 8 minutes after it is taken out of a pizza oven. The temperature 3 minutes after it is taken out is 60°C.

The following graph shows the temperature of the pizza as it cools over several minutes.

The temperature of pizza from the oven



Determine the approximate rate at which the pizza is cooling 3 minutes after it is taken out of the oven.

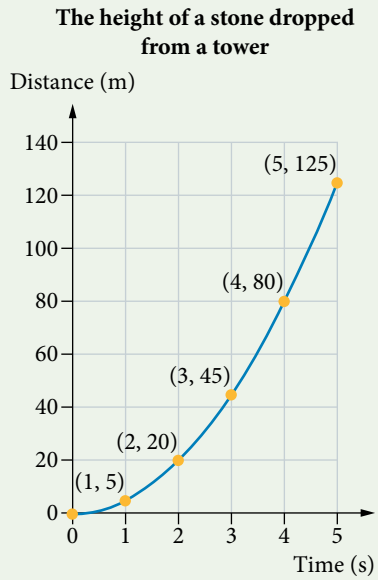
THINKING	WORKING
1 Sketch a tangent to the graph at the specified time. Identify another point that this tangent passes through.	 The tangent passes through (3, 60) and (8, 0).
2 Calculate the gradient of the tangent.	Rate of cooling $= \dfrac{\Delta\text{temperature}}{\Delta\text{time}}$ $= \dfrac{-60}{5}$ $= -12$
3 Interpret the answer.	3 minutes after it is taken out of the oven, the pizza is cooling at an approximate rate of 12°C/min.

8 Calculate instantaneous rates from a table

A stone is dropped from a tower. The table shows the approximate distance it travels during the first 5 seconds.

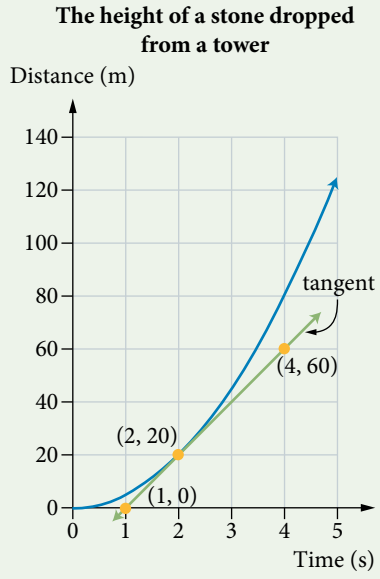
Determine the instantaneous speed of the stone 2 seconds after it is dropped.

Time (s)	0	1	2	3	4	5
Distance (m)	0	5	20	45	80	125

THINKING	WORKING
1 Draw and label a set of axes. Plot the points on the graph and then join them with a smooth curve.	

2 Sketch the tangent at the specified time.

Identify another point that this tangent passes through.



3 Calculate the gradient of the tangent.

For the points (1, 0) and (2, 20):

$$\begin{aligned}\text{Speed} &= \frac{\Delta d}{\Delta t}\\ &= \frac{d_2 - d_1}{t_2 - t_1}\\ &= \frac{20 - 0}{2 - 1}\\ &= 20\end{aligned}$$

4 Interpret the answer.

The instantaneous speed of the stone after 2 seconds is approximately 20 m/s.

EXERCISE 9.1 Average and instantaneous rates of change

Worked Example 1

1 Calculate the following rates.

(a) A car travels 180 km in 3 hours. Determine the average speed.

(b) 4 kg of potatoes cost \$3.80. Calculate the cost per kilogram.

(c) 20 L of paint covers an area of 80 m^2. Calculate the coverage rate.

(d) 32 000 MB of data is downloaded in $2\frac{1}{2}$ hours. Calculate the average download rate.

2 Determine the average speed in km/h to 2 decimal places, to drive between the following cities.

(a) Darwin to Perth, 4250 km in 49 hours

(b) Alice Springs to Melbourne, 2270 km in 24 hours

(c) Sydney to Brisbane, 1010 km in 11 hours

3 Ming's savings account balance changed from \$1120 at the start of March to \$2250 at the end of June. Determine the average monthly rate of change of Ming's account balance.

Worked Example

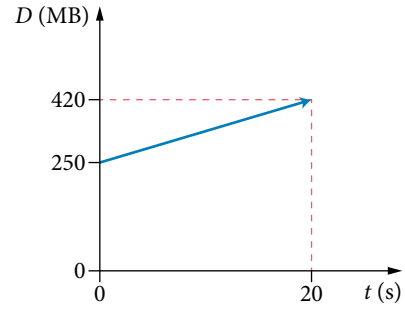
2

4 A tap drips at the rate of 8 drops every 10 seconds. If a drop of water is 0.04 mL, calculate the volume of water that will leak in 10 minutes.

5 The average rate at which hair grows is 1.25 cm per month. At this rate, what length will hair grow in a year?

6 A land-speed record of 1227 km/h was set in 1997 in the Black Rock Desert, Nevada. At this speed, how long, to the nearest minute, would it take to travel around the Earth's equator, a distance of 40 075 km?

7 The amount of data D in megabytes stored on a flash drive after time t seconds is shown on this graph.



(a) What is the rate of data transfer?

A 420 MB/s **B** 250 MB/s

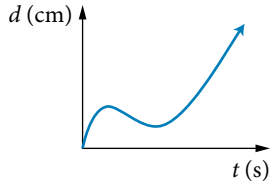
C 12.5 MB/s **D** 8.5 MB/s

(b) Explain the common error made by a student who thought the answer was 21 MB/s.

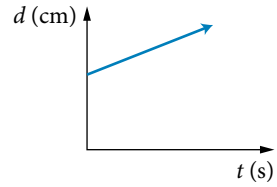
8 Consider the shapes of each set of graphs.

(a) Which of the following graphs show a constant rate of change?

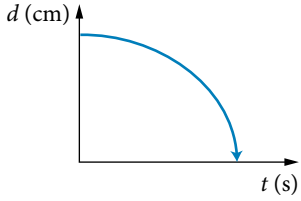
A



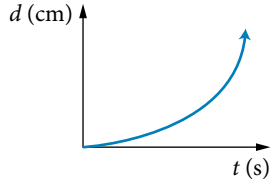
B



C

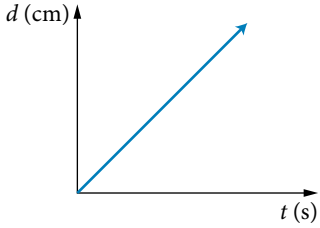


D

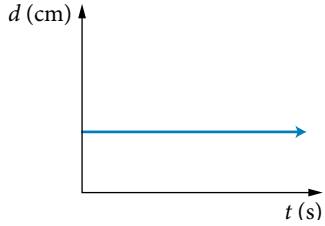


(b) Which of the following graphs show a variable rate of change?

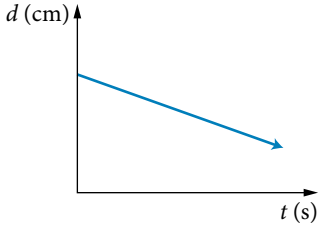
A



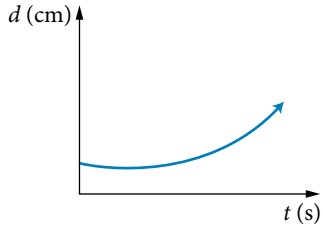
B



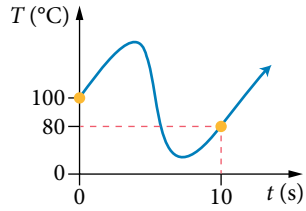
C



D



9 The graph shows the temperature of a piece of metal as it is worked by a blacksmith. Calculate the average rate of change of temperature of the metal in the first 10 seconds.



10 The table shows the value of a share investment. Calculate the average rate of change of the investment for the first 3 years.

Time (years)	0	1	2	3
Investment value ($)	$1000	$1200	$800	$1000

Worked Example

11 A pumpkin growing in a vegetable patch was weighed each week for 3 weeks. 3

Time (weeks)	0	1	2	3
Mass (kg)	0.5	1	1.8	2

(a) Calculate the average change of mass per week of the seedling in the first 2 weeks.

(b) Calculate the average change of mass per week of the seedling in the first 3 weeks.

12 The temperature from midnight to noon on a particular day is recorded in the following table. 5

Time (h)	midnight	1	2	3	4	5	6	7	8	9	10	11	noon
Temperture (°C)	17.7	17.3	16.5	15.6	15.1	14.4	14.4	15.0	16.3	18.3	20.2	23.5	27.4

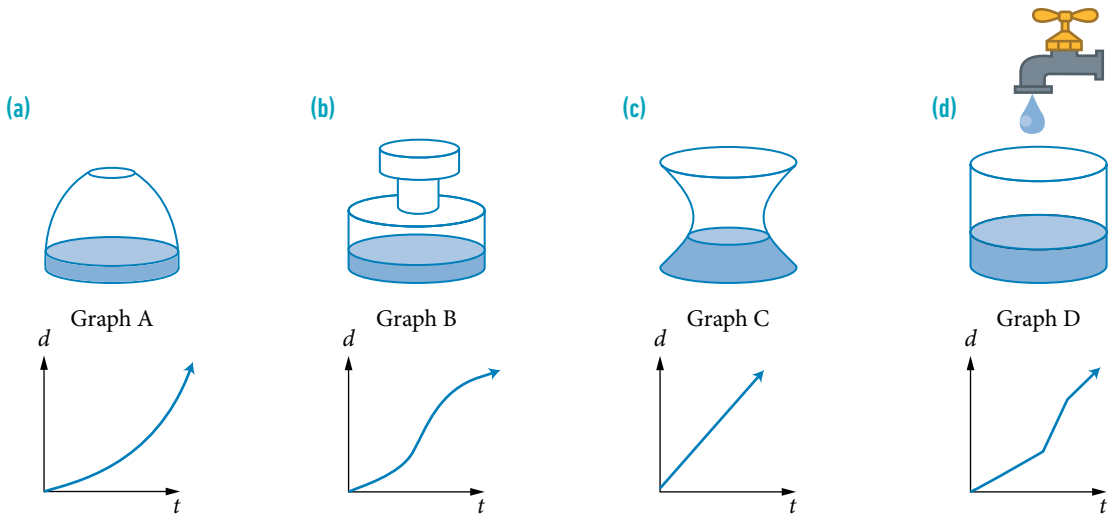
(a) Draw a graph of the temperature over time.

(b) Determine the average hourly temperature change from midnight to 5 am.

(c) Determine the average hourly change in temperature from 5 am to 6 am.

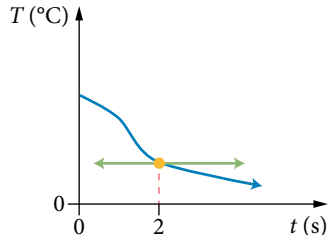
(d) Determine the average temperature change from 6 am to noon. Give your answer correct to 2 decimal places.

13 Water is poured at the same constant rate into four containers. The graphs show how the depth d of the water varies with time t as they are filled. Choose the depth vs time graph for each container. 6

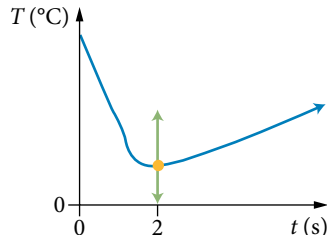


14 Which of the following graphs show the tangent drawn correctly at the point where $t = 2$?

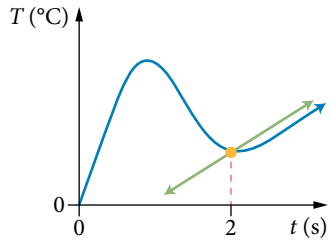
A



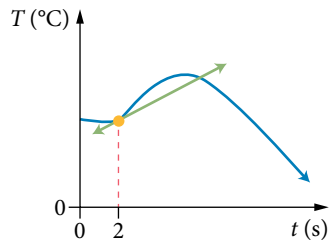
B



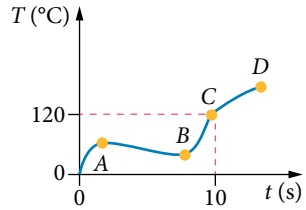
C



D

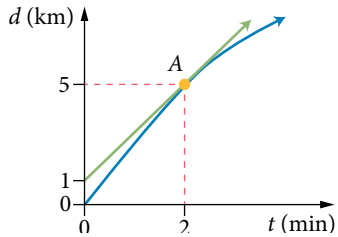


15 The following graph shows the temperature of a pizza oven over a certain period. How could you calculate the rate of change of temperature 10 seconds after the oven is turned on?

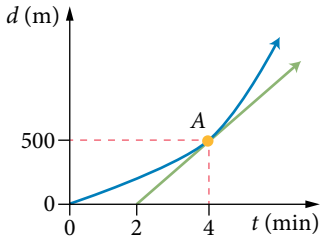


16 For the following distance–time graphs, determine the instantaneous speed at times indicated by point A.

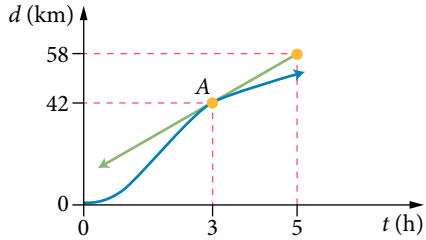
(a)



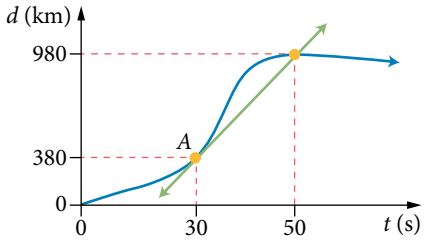
(b)



(c)



(d)



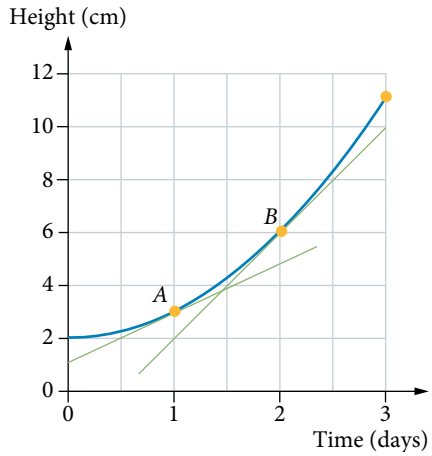
Worked Example

7

17 The height of a fast-growing bamboo plant is recorded over 3 days. The graph on the right shows the bamboo plant's growth with tangents drawn at point A (the end of day 1) and point B (the end of day 2).

(a) Using the tangents, calculate the bamboo plant's instantaneous rate of growth at the end of day 1 and at the end of day 2.

(b) Calculate the bamboo plant's instantaneous rate of growth at the end of day 3, given that the tangent at (3, 11) also passes through (2, 5).



18 In 1990 the median house price in a major city was $141 500 and the average wage per year was $31 200. In 2010 the median house price in the city was $524 500 and the average wage per year was $67 116. Compare the annual rate of change in the median house price with the annual rate of change in the average wage.

19 The local water board sets the following water supply charges. The water usage charge per litre applies to the volume of water used in each step category.

Fixed water service fee	$42.60 every 3 months
Fixed sewerage service charge	$54.29 every 3 months
Water usage charge: Step 1 Up to 440 L per day	$1.7854 per kL
Water usage charge: Step 2 More than 440 L and up to 880 L per day	$2.0950 per kL
Water usage charge: Step 3 More than 880 L per day	$3.0952 per kL

(a) Calculate the total fixed charges set for a year.

(b) A household averages 500 L of water per day. Calculate the total paid by the household for a 3-month period.

(c) On the day of a major water leak, the water usage was 2520 L. Calculate the cost of the water for that day.

20 People pay tax each year according to how much income they earn. The taxation rate increases with the amount of income and steps up in stages, as shown in the following income tax table for the 2017–2018 financial year.

Taxable income	Tax on this income
$0–$18 200	Nil
$18 201–$37 000	19c for each $1 over $18 200
$37 001–$87 000	$3572 plus 32.5c for each $1 over $37 000
$87 001–$180 000	$19 822 plus 37c for each $1 over $87 000
$180 001 and over	$54 232 plus 45c for each $1 over $180 000

(a) Using the tax table, calculate the amount of tax that the following people need to pay in a year, to the nearest cent.

(i) Norman earned $16 250.

(ii) Dhuy earned $29 550.

(iii) Wharminda earned $69 580.

(iv) Carla earned $165 280.

(v) Jennifer earned $250 630.

(b) Calculate the effective rate of tax that each person in (a) pays by first expressing the tax paid as a rate per dollar of income, correct to 2 decimal places, and then converting this to a percentage.

(i) Norman earned \$16 250 and paid no tax.

(ii) Dhuy earned \$29 550 and paid \$2156.50 tax.

(iii) Wharminda earned \$69 580 and paid \$14 160.50 tax.

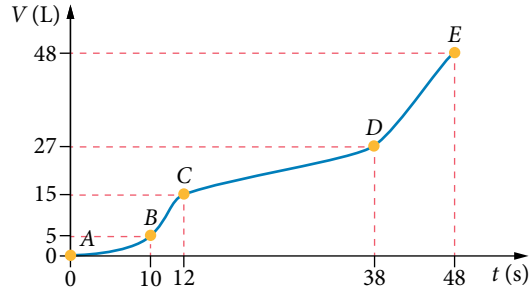
(iv) Carla earned \$165 280 and paid \$49 100.60 tax.

(v) Jennifer earned \$250 630 and paid \$86 330.50 tax.

Worked Example

4

21 The graph on the right shows the volume of water in a bath as it is filled from a tap. The flow of water changes as other water taps in the house are turned on and off.



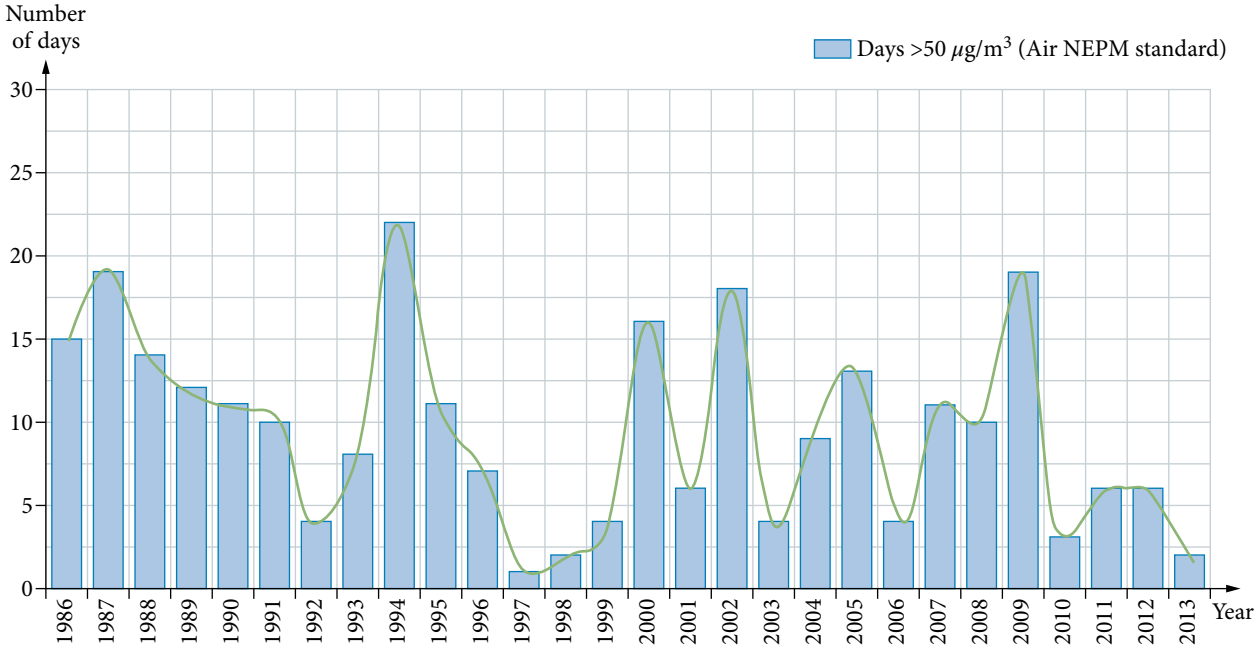
(a) Calculate the average rate of water flowing into the bath during the first:

(i) 10 seconds (ii) 48 seconds

(b) Which section on the graph indicates the bath is filling fastest? Determine the average rate for this period of time.

(c) Which section on the graph indicates the bath is filling slowest? Determine the average rate for this period of time, correct to 2 decimal places.

22 The graph below shows the number of days per year the 24-hour average PM_{10} (airborne particles less than 10 microns) concentration exceeded the specified level in Queensland. PM_{10} particles are generated by a wide range of natural processes and human activities, including wind-blown dust, industrial processes, motor vehicle emissions and fires.



(a) When was the longest period of downtrend of PM_{10} was observed? Calculate the average annual rate of change over this period.

(b) When was the longest period of uptrend of PM_{10} was observed? Calculate the average annual rate of change over this period.

23 The changing price of a volatile share is recorded for the first 4 hours of sharemarket trading.

Time (h)	0	1	2	3	4
Price ($)	4.80	4.20	2.50	3.50	4.50

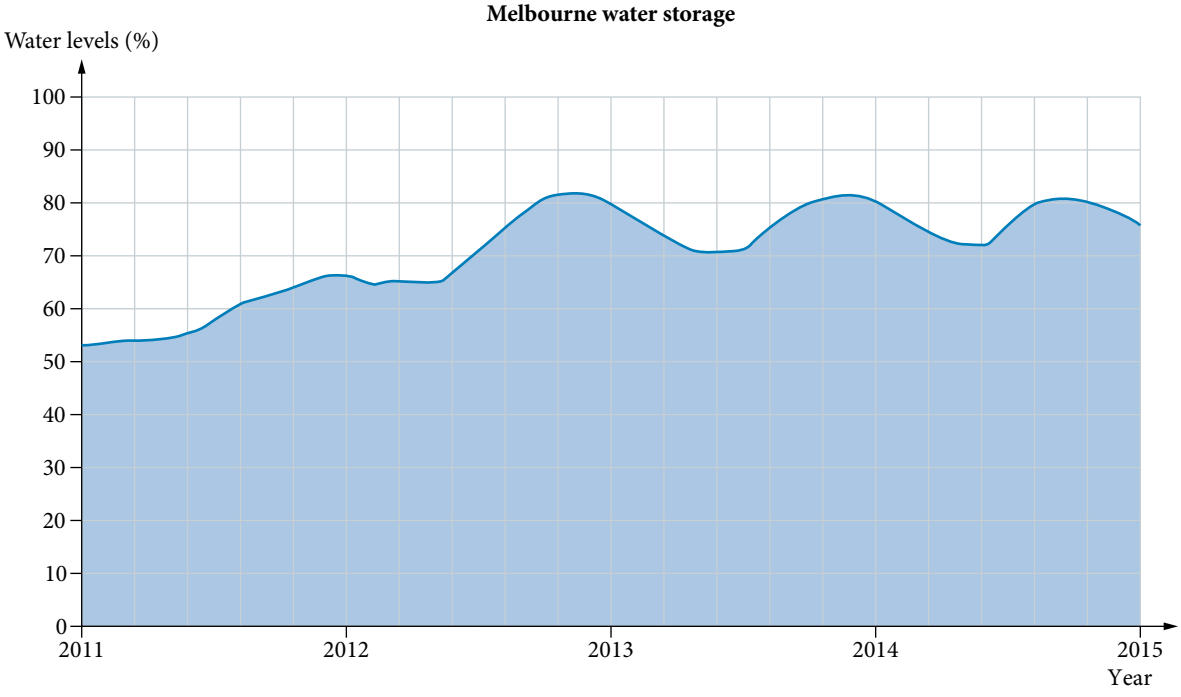
(a) Plot the points and join with a smooth curve. Sketch the graph large enough to identify the x- and y-coordinates.

(b) Sketch your best estimate of the tangent line and use it to estimate the rate of change of the share price at the following times.

(i) 1 hour **(ii)** 2 hours **(iii)** 3 hours

24 The graph below shows the percentage of Melbourne's weekly water storage dam levels. For the period shown in the graph, calculate the:

(a) greatest rate of increase **(b)** greatest rate of decrease **(c)** average rate of change.



25 Hair grows at about 3 mm per week. When you get a haircut, many of the estimated 100 000 hairs on your head will be cut, but not the ones that are too new to need a trim. About 100 hairs fall out (and are replaced) every day. Making any assumptions you need to, estimate the number of hairs cut, to the nearest 100, for each of the following people.

(a) A boy with long hair going for a number 4 buzz cut (1.25 cm) for 80% of his scalp and the rest shaved.

(b) A girl with long hair asking for a 3 cm fringe (assume 10% of her hair) and the rest layered at about 10 cm.

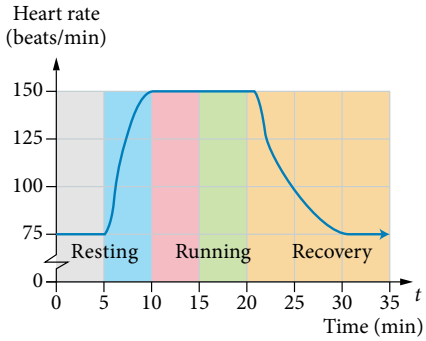
26 Three points (2, 4), (4, 6.5) and (5, 9) lie on a parabola. Calculate the instantaneous rate of change of y in terms of x at the following points on the parabola.

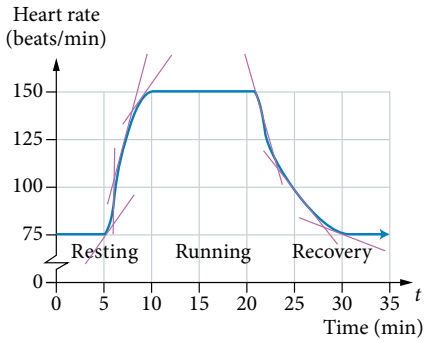
(a) $x = -1$ **(b)** $y = 8$

9.2 Functions and rates of change

Graphs of gradient functions

This graph shows the heart rate of a person before, during and after a 15 minute session on a treadmill.





The resting heart rate is constant at 75 beats per minute (bpm).

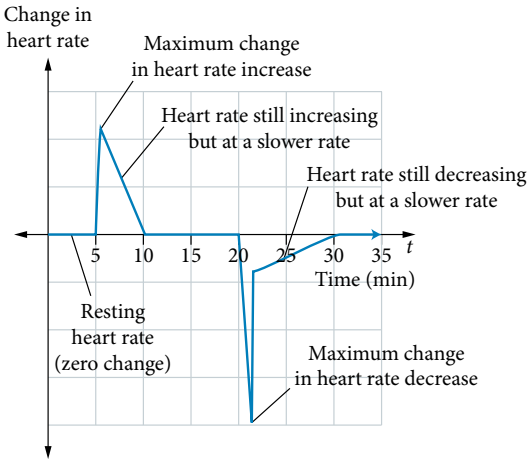
At the 5-minute mark, the heart rate increases quite quickly.

It plateaus at the maximum heart rate of 150 bpm at the 10-minute mark.

The heart rate remains elevated (but unchanged) until the 20-minute mark.

The heart rate then decreases until the resting heart rate of 75 bpm is reached at the 30-minute mark.

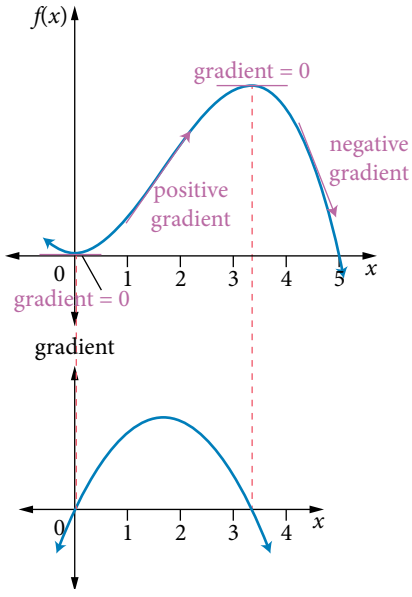
If the gradients at the key points on the graph above were determined, then a gradient graph can be drawn to show the 'rate of change' of the heartbeat. The gradient graph below shows the change in heart rate over the same period of time.



- The horizontal sections of the function, where the heart rate was constant (neither increasing nor decreasing), have zero gradient on the gradient graph.
- The sections on the function that were increasing have positive values on the gradient graph. The steeper the slope, the higher the value on the gradient graph.
- The sections on the function that were decreasing have negative values on the gradient graph.

Gradient graphs of polynomial functions

The combined diagram shows the graph of a cubic function above its gradient function, which is a quadratic function. Gradient functions of polynomials of power n are themselves polynomials of power $n - 1$.



The following table gives information about the original function and the gradient function. The gradient values indicate whether the gradient is negative (–), positive (+) or zero (0).

	Left of y-axis	At y-axis $x = 0$	Between $x = 0$ and $x \approx 3.3$	At $x \approx 3.3$	Right of $x \approx 3.3$
Shape of original function $f(x)$	\	–	/	–	\
Sign of gradient values	–	0	+	0	–

To sketch a gradient function from the original function:

1. Identify stationary points on the original function, where the gradient of the graph is zero and it changes shape (+ to – gradient or – to + gradient). These correspond to where the value of the gradient function is zero and the graph cuts the x-axis.
2. Identify positive gradients on the original function. These correspond to where the gradient function is above the x-axis.
3. Identify negative gradients on the original function. These correspond to where the gradient function is below the x-axis.

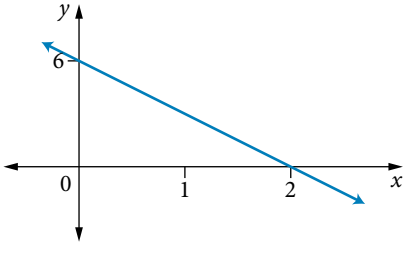
Polynomials of power n have gradient graphs of polynomials with power $n - 1$.

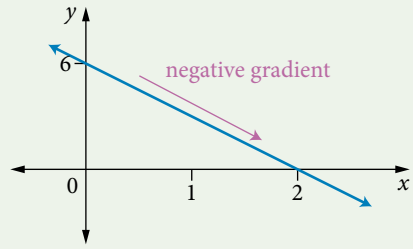
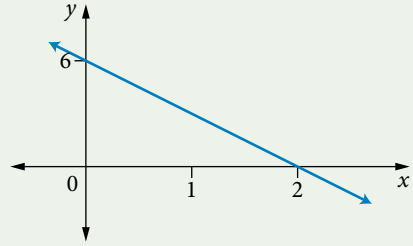
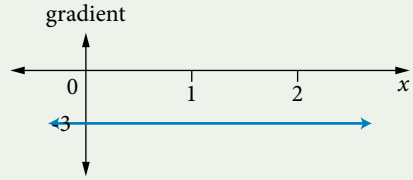
- Positive values on the gradient graph indicate an increase in function values.
- Negative values on the gradient graph indicate a decrease in function values.
- Zero on the gradient graph indicates that the function is unchanging or stationary.

9 Draw the graph of the gradient of a function

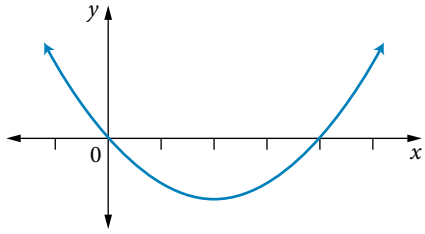
Draw the graph of the gradient for each function, matching key points of change.

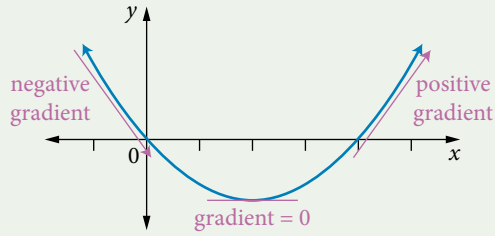
(a)



THINKING	WORKING
1 Analyse the gradient of each section of the original function.	 The line has a constant negative gradient.
2 Calculate the gradient using: $m = \frac{\text{rise}}{\text{run}}$	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{-6}{2}$ $= -3$
3 The gradient is constant throughout. Write the rule of the gradient function.	The rule of the gradient function is: gradient = -3
4 Draw the graph of the gradient over the given domain and show the original function above the gradient function.	 

(b)

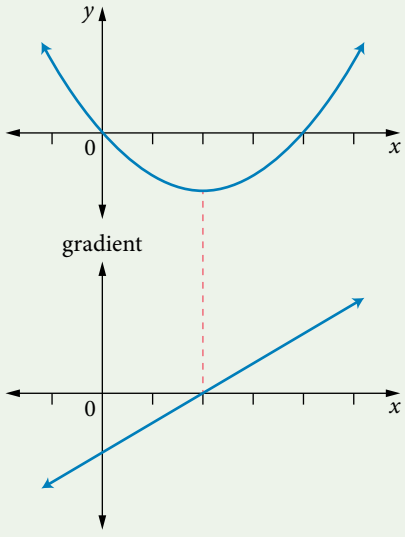


1 Analyse the gradient of each section of the original function.	 The gradient has a value of zero 2 units to the right of the origin.

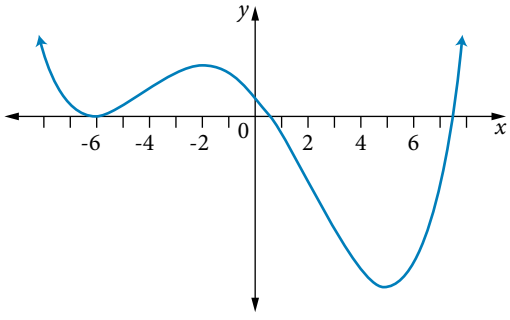
2 The gradient changes from negative to positive.

Shape of the function	\	–	/
Sign of the gradient	–	0	+

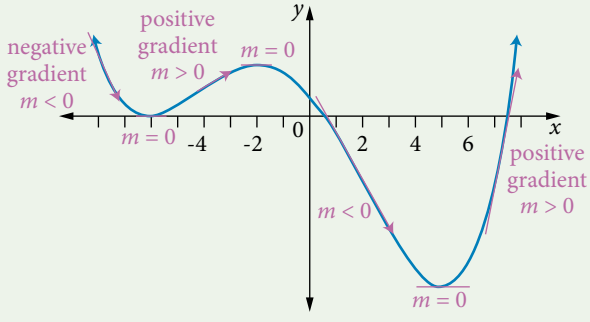
3 Draw the graph of the gradient over the given domain and show the original function above the gradient function.



(c)



1 Analyse the gradient of each section of the original function.

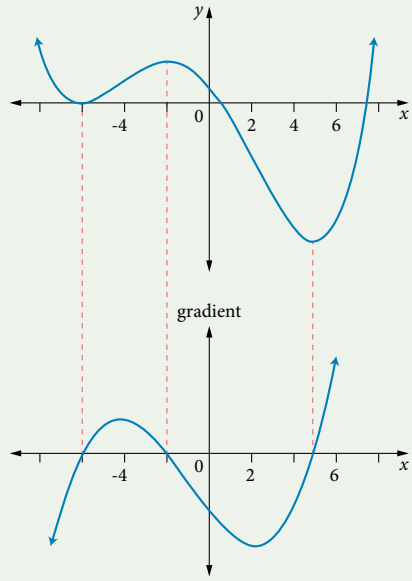


The gradient has a value of zero at $x = -6$, $x = -2$ and $x = 5$.

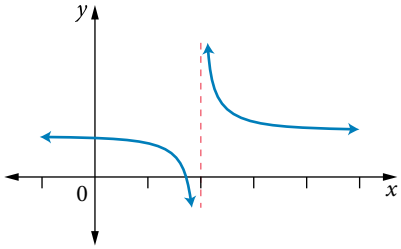
2 The gradient changes from negative to positive to negative to positive.

Shape of the function	\	–	/	_	\	_	/
Sign of the gradient	–	0	+	0	–	0	+

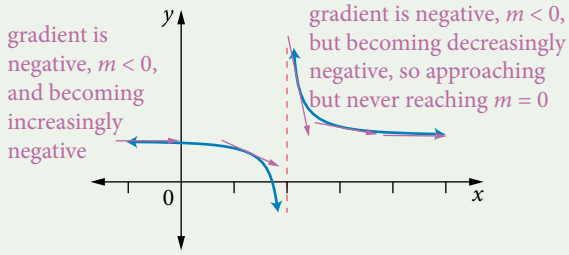
3 Draw the graph of the gradient over the given domain and show the original function above the gradient function.



(d)



1 Analyse the gradient of each section of the original function.

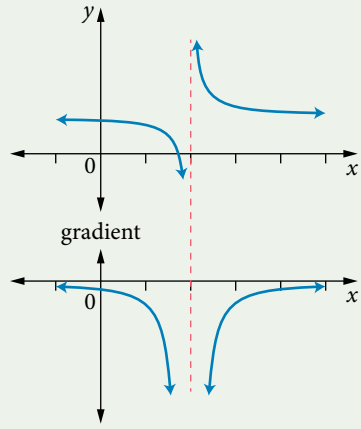


There is an asymptote 2 units to the right of y-axis.

2 The gradient is negative throughout.

Shape of the function	\	asymptote	\
Sign of the gradient	–	asymptote	–

3 Draw the graph of the gradient over the given domain and show the original function above the gradient function.



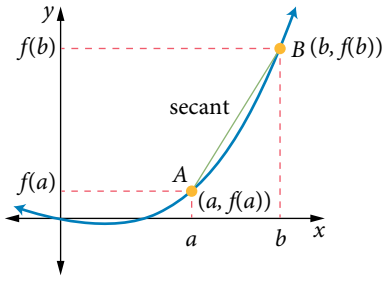
Explore further

Gradient graphs

Explore the relationship between a graph and its gradient.

Rates of change and secants

A *secant* is a straight line that cuts a curve at two or more points.



For the function $y = f(x)$, the point A where $x = a$ has the y-coordinate $f(a)$, and the point B where $x = b$ has the y-coordinate $f(b)$. The line AB is a secant.

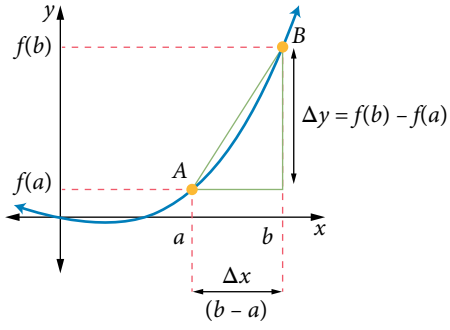
The symbol Δx (called 'delta x') is commonly used to denote a small change in the value of x and Δy is used to denote a small change in the value of y.

Given two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$:

- Δx represents a change in x: $\Delta x = x_2 - x_1$
- Δy represents a change in y: $\Delta y = y_2 - y_1$

The gradient of the secant AB is:

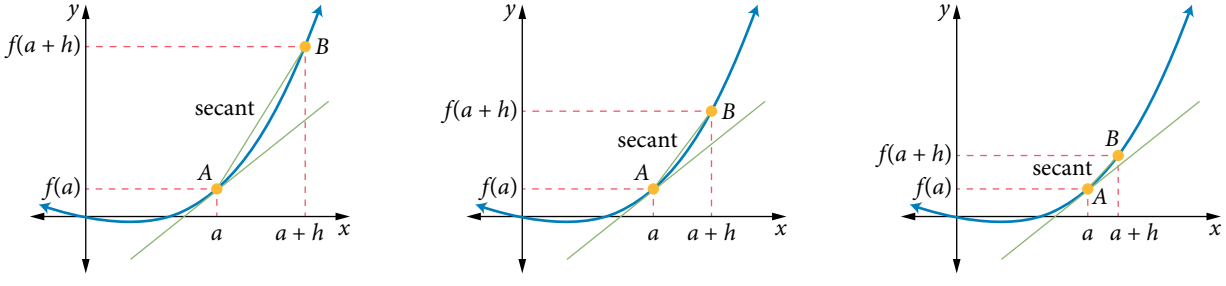
$$\begin{aligned} m_{AB} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$



When investigating instantaneous rates of change, it is useful to replace $x = b$ with $x = a + h$, where h represents a small change in value from $x = a$.

For a function $y = f(x)$, the height of the curve at point A, where $x = a$ is $f(a)$, and the height of the curve at point B, where $x = a + h$ is $f(a + h)$.

The line AB is still a secant. As the value of h becomes smaller and smaller, the gradient of the secant AB becomes closer and closer to the gradient of the tangent at A.



The gradient of the secant AB is:

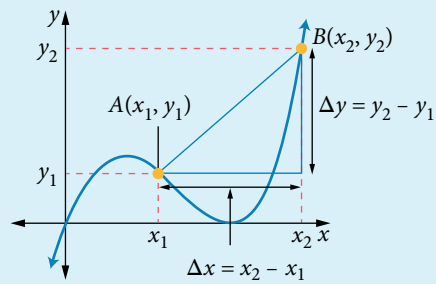
$$\begin{aligned} m_{AB} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

The function, as is, is undefined for $h = 0$; however, you can approximate the gradient as the value of h approaches 0 $\left(\lim_{h \to 0}\right)$ by observing the patterns of decreasing values of h.

The gradient of the secant between points (x_1, y_1) and (x_2, y_2) where $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ is:

$$m = \frac{\Delta y}{\Delta x}$$

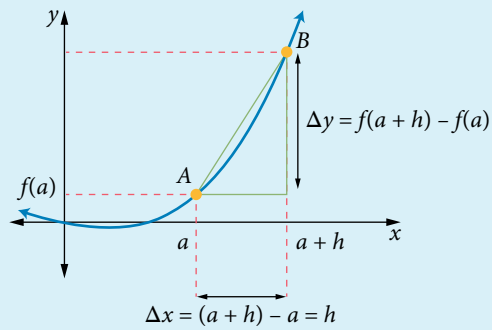
This is the average rate of change of the function between x_1 and x_2.



The gradient of the secant between the points $(x, f(x))$ and $(x+h, f(x+h))$ is:

$$m = \frac{f(x+h) - f(x)}{h}$$

This is the average rate of change of the function between x and $x + h$.



As the value of h becomes smaller, $h \to 0$, the gradient of the secant (line AB) becomes increasingly close to the gradient of the tangent at A.

10 The average rate of change of a function

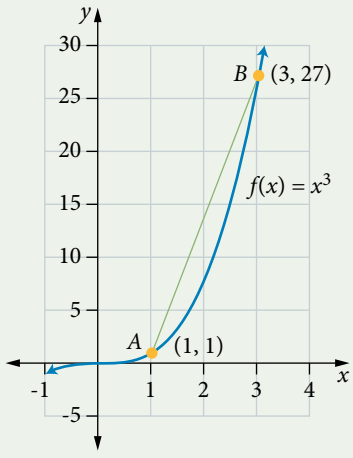
On the curve $f(x) = x^3$, point A has an x-value of 1 and point B has an x-value of 3.

(a) Determine the coordinates of the points A and B.

THINKING	WORKING
1 Determine the y-coordinates $f(1)$ and $f(3)$.	$f(1) = 1^3 = 1$ $\quad$ $f(3) = 3^3 = 27$
2 Interpret the answer.	The coordinates of the points are $A(1, 1)$ and $B(3, 27)$.

(b) Sketch the graph of $f(x) = x^3$ and calculate the gradient of the secant AB.

1 Draw the secant AB on a sketch of $f(x) = x^3$.



2 Substitute values into the formula $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$ to calculate the gradient of the secant AB.

$f(1) = 1$
$f(3) = 27$

$$m_{AB} = \frac{f(b) - f(a)}{b - a}$$

$$m_{AB} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = \frac{26}{2} = 13$$

3 Interpret the answer.

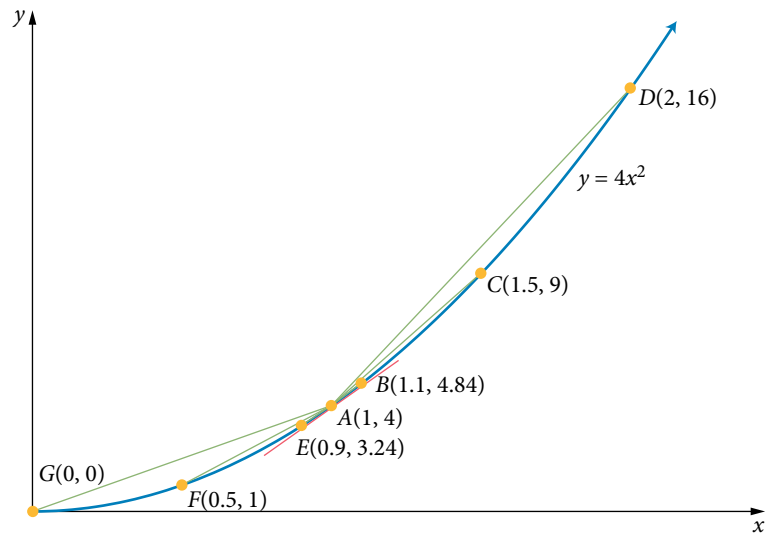
The gradient of the secant AB is 13.

Comparing average and instantaneous rates of change

The average rate of change between two points is the gradient of the secant on the curve.

The instantaneous rate of change at a single point is the gradient of the tangent drawn to the curve.

The graph below illustrates how the distance between two points on curve approaches zero. As the points get closer to point A, the gradient of the secants AG, AF, AE and AD, AC, AB approach the gradient of the tangent at the point A.



The numerical result can be shown as follows.

Secant	Distance between x-values, h	Gradient of secant: $\frac{\Delta y}{h}$
AD	1	12
AC	0.5	10
AB	0.1	8.4
AE	0.1	7.6
AF	0.5	6
AG	1	4

As $h \to 0$, the secants approach the tangent at A and the gradients approach the value of 8. This pattern suggests that the gradient of the tangent or the instantaneous gradient at A is 8. The gradient of a secant AX, where $A(1, 4)$ and $X(1 + h)$, $f(1 + h)$ and h is very small, can be written as:

$$m_{AX} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 8$$

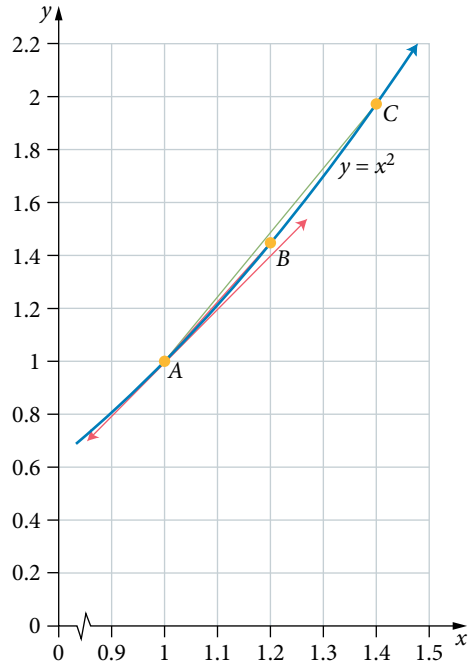
Additional information

Secants and tangents

This video explains the difference between a secant and tangent of a function.

11 Compare average and instantaneous rates of change

Consider the function $f(x) = x^2$.



(a) Determine the gradient of the secant AC by evaluating $\dfrac{f(x+h)-f(x)}{h}$ for $x = 1$ and $h = 0.4$.

THINKING	WORKING
1 Calculate $f(x)$ and $f(x+h)$ for the given values.	$f(1) = 1^2 = 1$ $\quad f(1.4) = 1.4^2 = 1.96$
2 Calculate $m = \dfrac{f(x+h)-f(x)}{h}$.	$m = \dfrac{f(1.4)-f(1)}{0.4} = \dfrac{1.96-1}{0.4} = \dfrac{0.96}{0.4} = 2.4$
3 Interpret the answer.	The gradient of the secant AC is 2.4.

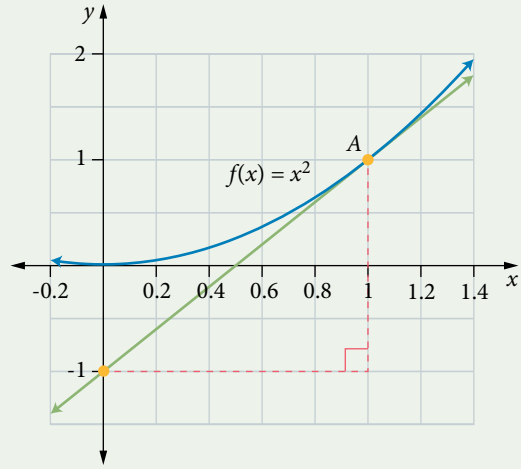
(b) Determine the gradient of the secant AB by evaluating $\dfrac{f(x+h)-f(x)}{h}$ for $x = 1$ and $h = 0.2$.

1 Calculate $f(x)$ and $f(x+h)$ for the given values.	$f(1) = 1^2 = 1$ $\quad f(1.2) = 1.2^2 = 1.44$
2 Calculate $m = \dfrac{f(x+h)-f(x)}{h}$.	$m = \dfrac{f(1.2)-f(1)}{0.2} = \dfrac{1.44-1}{0.2} = \dfrac{0.44}{0.2} = 2.2$
3 Interpret the answer.	The gradient of the secant AB is 2.2.

(c) Determine the gradient of the tangent line at $x = 1$ by sketching your best estimate of the tangent line at this point.

1 Sketch the tangent on a graph of $f(x) = x^2$.

Identify another point that the tangent line passes through.



The tangent line passes through the point (0, -1).

2 Use the two points to calculate the gradient of the tangent.

$m = \frac{2}{1} = 2$

3 Interpret the answer.

The gradient of the tangent at $x = 1$ is 2.

(d) Explain how the three gradients relate to each other.

Consider how the gradients of the secants change as the value of h gets smaller.

The gradient of the secant AC is 2.4.

The gradient of the secant AB is 2.2.

The gradient of the tangent at $x = 1$ is 2.

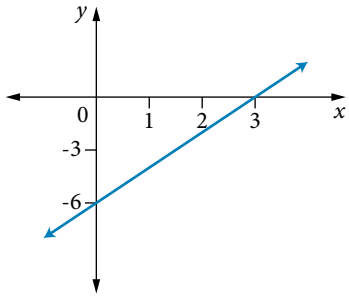
As the value of h gets smaller, the gradient of the secant becomes closer to the value of the gradient of the tangent at the point where $x = 1$.

EXERCISE 9.2 Functions and rates of change

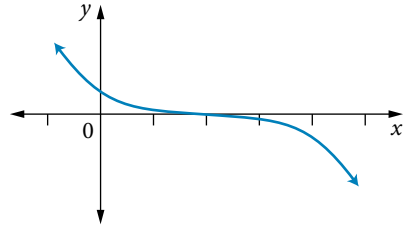
Worked Example 9

1 Draw the graph of the gradient for each function, showing all key features.

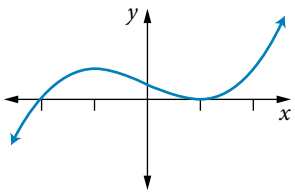
(a)



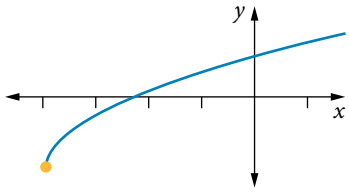
(b)



(c)

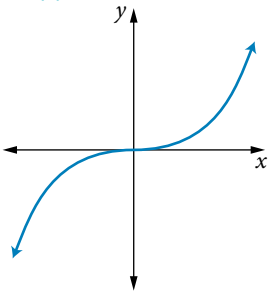


(d)

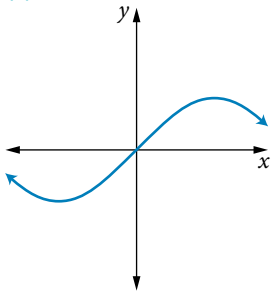


2 Match each of the four graphs with the gradient graphs A, B, C, D.

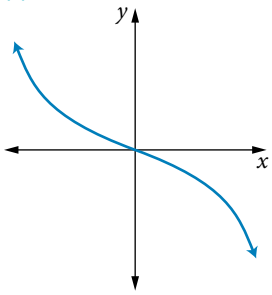
(a)



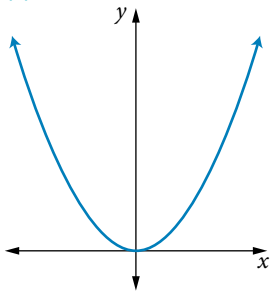
(b)



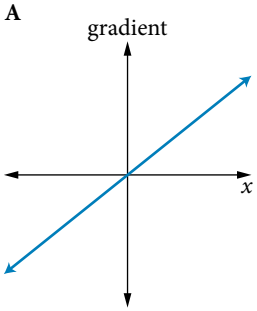
(c)



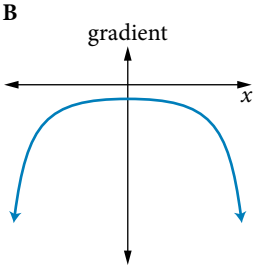
(d)



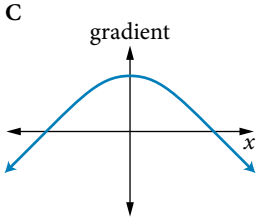
A



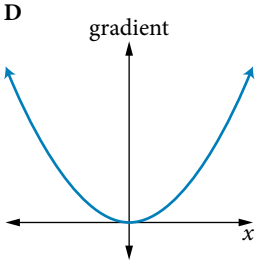
B



C



D



3 (a) The gradient of the line joining $A(4, -1)$ and $B(5, -2)$ is:

A $\frac{1}{2}$ B $-\frac{1}{2}$ C -1 D -2

(b) Explain the common error made by a student who calculated $m = \frac{-1+2}{5-4}$.

4 For $f(x) = x^2$, determine the value of $\frac{f(4) - f(2)}{2}$.

5 For the points $A(2, 3)$ and $B(5, 10)$, what could be the values of Δx and Δy respectively?

A 3 and 7 B 7 and 3

C 2 and 3 D 5 and 3

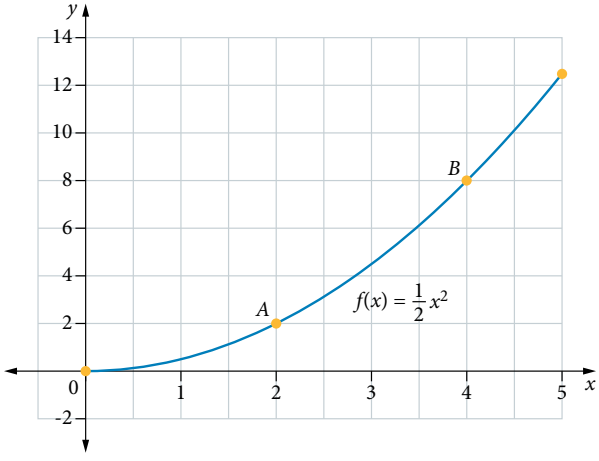
Worked Example 10

6 The graph of the function $f(x) = \frac{1}{2}x^2$ over the domain $[0, 5]$ is shown.

(a) Determine the coordinates of the points A and B.

(b) Calculate the gradient of the secant AB.

(c) Calculate the average rate of change from A to B.



7 On a graph, point A is (2, 6) and point B is (4, 24).

(a) Determine the value of Δy.

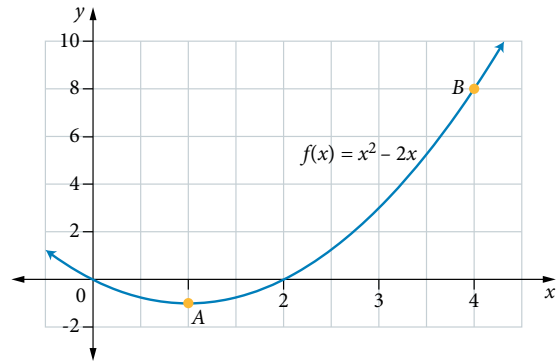
(b) Determine the corresponding value of Δx.

(c) Determine the gradient of the line AB.

8 On the curve $f(x) = x^2 - 2x$, point A has an x-value of 1 and point B has an x-value of 4.

(a) Determine the values of $f(1)$ and $f(4)$.

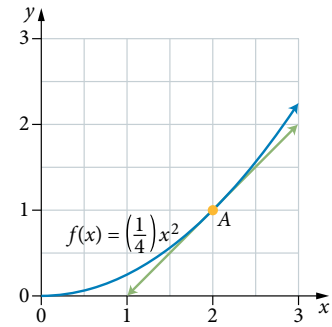
(b) Use the formula $m_{AB} = \frac{f(b) - f(a)}{b - a}$ to calculate the gradient of the secant AB.



9 The graph of the function $f(x) = \frac{1}{4}x^2$ is shown. Point A has an x-value of 2.

(a) Determine the coordinates of point A.

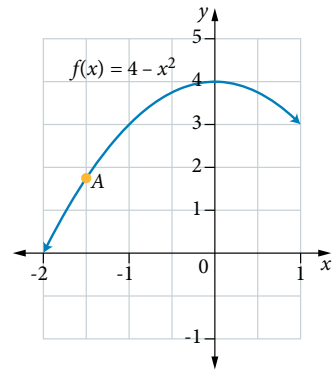
(b) The tangent to the function $f(x) = \frac{1}{4}x^2$ at the point A is shown. Identify a point that the tangent passes through, other than point A, hence calculate an approximation for the instantaneous rate of change at the point A.



10 The graph of the function $f(x) = 4 - x^2$ is shown. Point A has an x-value of -1.5.

(a) Determine the coordinates of point A.

(b) The tangent at A also passes through point $\left(0, 6\frac{1}{4}\right)$. What is the instantaneous rate of change at the point A?



Worked Example

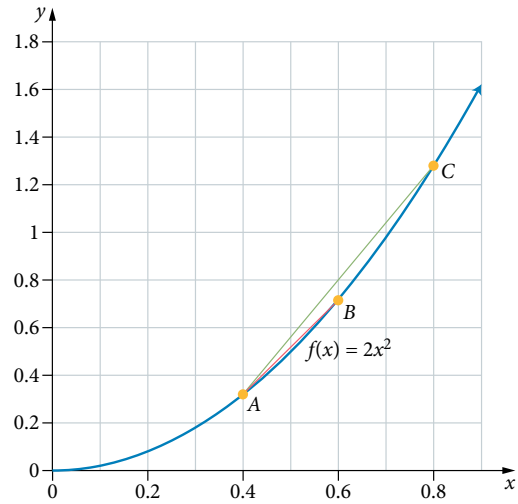
11 11 The graph of the function $f(x) = 2x^2$ is shown.

(a) Calculate $\frac{f(x+h) - f(x)}{h}$ for $x = 0.4$ and $h = 0.4$ to determine the gradient of the secant AC.

(b) Calculate $\frac{f(x+h) - f(x)}{h}$ for $x = 0.4$ and $h = 0.2$ to determine the gradient of the secant AB.

(c) Determine the gradient of the tangent to the function $f(x) = 2x^2$ at the point where $x = 0.4$.

(d) How do the three gradients found in this question relate to each other?



12 Consider the function $f(x) = x^2$.

(a) Draw the graph of $f(x)$ over the domain [0,2] and mark the following points: $A(1, 1)$, $B(1.2, 1.44)$, $C(1.4, 1.96)$, $D(1.6, 2.56)$

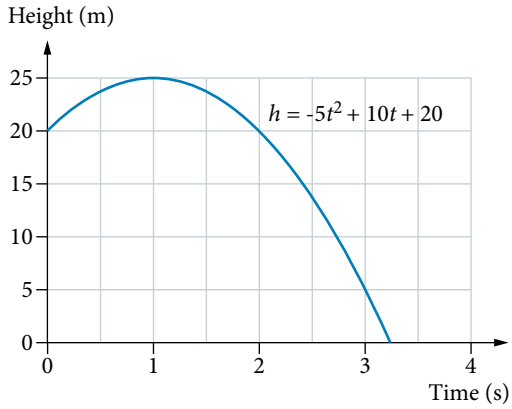
(b) Determine the gradient of the secant:

(i) AD **(ii)** AC **(iii)** AB

(c) Draw a tangent at $A(1, 1)$ over the domain [0,2] and estimate its gradient.

(d) Comment on the gradients of the secants and the tangent. How are they related?

13 An arrow is shot into the air from the top of a building 20 m tall. The height of the arrow h metres above the ground at a time t seconds after it is shot is modelled by the function $h = -5t^2 + 10t + 20$. The graph on the right shows the height of the arrow above the ground.



(a) Use the graph to determine the rate of change of height of the arrow 1 second after it is shot.

(b) A tangent drawn to the curve at point (0, 20) also passes through the point (1, 30). Calculate the initial rate of change of height of the arrow.

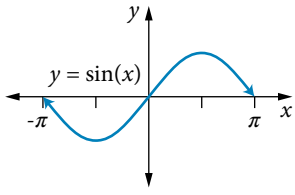
The arrow hits the ground 3.24 seconds after it is shot. The tangent drawn to the curve at the point (3.24, 0) passes through the point (3, 7.78).

(c) Determine the rate of change of height when the arrow hits the ground, correct to 1 decimal place.

14 The graph of $y = \sin(x)$ for $-\pi \le x \le \pi$, where x is the angle in radians is given below.

(a) Draw the graph of the gradient of the function matching key points of change.

(b) The gradient function is given by the equation $y = \cos(x)$. Determine the gradient of the tangent to $y = \sin(x)$ at each of the following x-values.



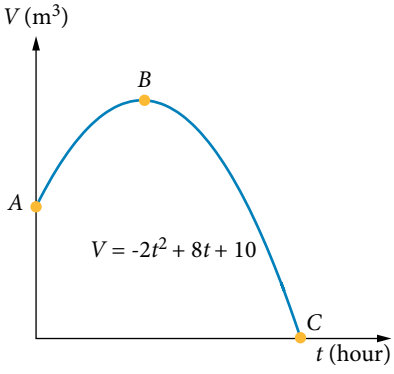
(i) $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$ **(ii)** $x = 0$

(iii) $x = -\dfrac{\pi}{3}$ and $x = \dfrac{\pi}{3}$

15 Water is added to a tank so that the volume of water V m^3 in the tank after t hours is given by $V = -2t^2 + 8t + 10$.

(a) Determine the initial volume of water in the tank.

Water was initially added to the tank and later the tank was emptied as shown in the graph on the right. Point A is the starting point, Point B is the maximum volume of water and Point C is when the tank was empty.



(b) Determine the time when water ceased to be added to the tank.

(c) What was the maximum volume of water held in the tank?

(d) How long did it take for the tank to empty from the time that it held the maximum volume of water?

(e) Calculate the average rate at which water was added to the tank.

(f) Calculate the average rate at which water was lost from the tank.

(g) Calculate the average rate at which the volume of water changed from point A to point C on the curve.

(h) Calculate the gradient of the secant joining points with t-values of 0 and 0.01, and then estimate the rate at which water was initially added to the tank, to the nearest whole number.

(i) Estimate the rate at which water flows out of the tank just before the tank empties, to the nearest whole number.

16 A ball is thrown up from the top of a building so that after t seconds, its height h metres above the ground is given by $h = 147 + 4.9t - 4.9t^2$.

(a) Sketch the graph of the path of the ball.

(b) Identify the height of the building.

(c) When will the ball reach its maximum height, and what is this height?

(d) Determine the average rate at which the ball rises from when it is thrown to when it reaches its maximum height.

(e) Calculate the gradient of the secant joining points with t-values of 0 and 0.01, and then estimate the instantaneous rate of change of height when the ball is initially thrown, correct to 1 decimal place.

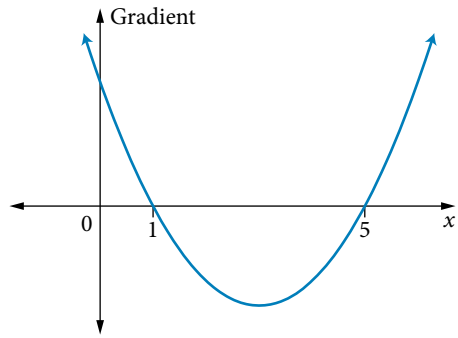
(f) As the ball is falling, what is its average rate of change of height?

(g) Calculate the gradient of the secant joining points with t-values of 5.99 and 6, and then estimate the instantaneous rate of change of height at the point of impact with the ground, correct to 1 decimal place.

17 The gradient graph for a cubic function is shown. Sketch possible graphs for the cubic function that passes though the point:

(a) (1, 10)

(b) (1, 0)



18 The graph of $y = x^3 - 2x^2 - 5x + 15$ has a maximum turning point at B and a minimum turning point at C. (Round answers to an appropriate degree of accuracy in this question.)

(a) Choose a point A, to the left of B, such that AB and AC have positive gradients, and determine the gradients of AB and AC.

(b) Use a secant to estimate the gradient of the tangent at A.

(c) Determine the coordinates of D, to the right of C, such that the gradient of the tangent at D is the same as the gradient of the tangent at A.

(d) Determine the gradients of BD and CD.

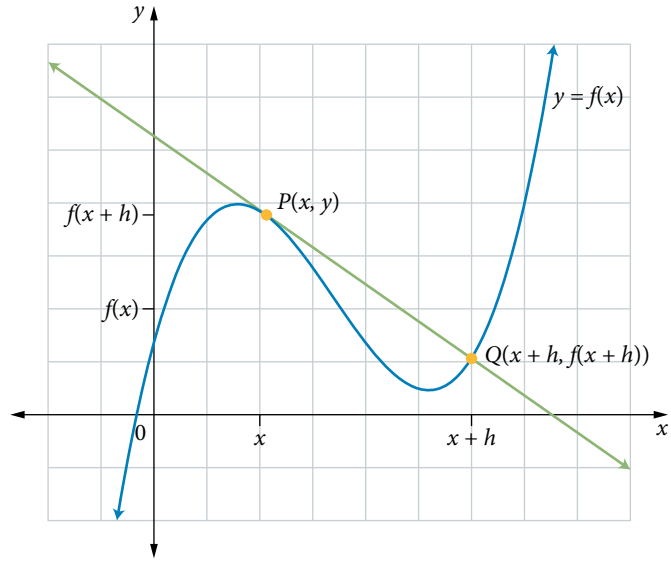
(e) Explain the significance of your answers to parts (a) and (d).

Derivatives from first principles

9.3

Gradient of the graph of a function at a point

A tangent is a straight line that touches a curve at a point. The gradient at a point $P(x, y)$ on the graph of $y = f(x)$ is defined as the gradient of the tangent to the graph at P.



In this module, a method is developed for determining the gradient of the tangent to the graph of $y = f(x)$ at a point $P(x, y)$.

$P(x, f(x))$ is any point on the graph of $y = f(x)$, which is a continuous and smooth curve over the interval $[x, x + h]$. A neighbouring point Q has coordinates $(x + h, f(x + h))$. The point Q is h units horizontally from point P.

The line through P and Q, called a secant, is used to calculate the gradient of the tangent, m_{tangent}, at P.

Using $m = \dfrac{\text{rise}}{\text{run}}$, the gradient of the secant PQ, m_{PQ}, is:

$$m_{PQ} = \frac{f(x+h) - f(x)}{(x+h) - x}$$
$$= \frac{f(x+h) - f(x)}{h}$$

This is known as the difference quotient.

Making connections

Slope of tangent line as a limit of a secant

Move the slider to compare the gradient of the secant and gradient of the tangent.

In the previous module, you saw that the gradient of the secant was a better approximation of the gradient of the tangent as the value of h becomes smaller.

The actual values of the gradient of the tangent can be determined using the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ as $h \to 0$.

If the limit exists, then the gradient of the graph of $y = f(x)$ at $P(x, f(x))$ is equal to the gradient of the tangent at $P(x, f(x))$ and is given by:

$$m_{\text{tangent}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Additional information

Secants and tangents review

This activity reviews secants and tangents of the graph of a function.

12 Calculate small changes in the function

For each function, calculate the expression for $f(x+h)$ and hence the change in the function, $f(x+h)-f(x)$.

(a) $f(x)=5x-1$

THINKING	WORKING
1 Substitute $x+h$ and simplify.	$f(x)=5x-1$ $f(x+h)=5(x+h)-1$ $=5x+5h-1$
2 Determine the expression for $f(x+h)-f(x)$ and simplify.	$f(x+h)-f(x)$ $=(5x+5h-1)-(5x-1)$ $=5h$

(b) $f(x)=x^2+5$

1 Substitute $x+h$ and simplify.	$f(x)=x^2+5$ $f(x+h)=(x+h)^2+5$ $=x^2+2hx+h^2+5$
2 Write the expression for $f(x+h)-f(x)$ and simplify.	$f(x+h)-f(x)$ $=(x^2+2hx+h^2+5)-(x^2+5)$ $=2hx+h^2$

Notice that for both polynomial expressions in the worked example, above the terms in the simplified form of $f(x+h)-f(x)$ had a common factor of h.

13 Use the difference quotient

For the function $f(x)=3x^2$ at $x=2$:

(a) calculate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$

THINKING	WORKING
1 Define $f(x)$ and $f(x+h)$.	$f(x)=3x^2$ $f(x+h)=3(x+h)^2$
2 Use the quotient rule of the given function to write expressions for the function terms.	$\dfrac{f(x+h)-f(x)}{h}$ $=\dfrac{3(2+h)^2-3(2)^2}{h}$

3 Expand the perfect square.	$= \dfrac{3\left(4+4h+h^2\right)-3(2)^2}{h}$
4 Simplify the numerator.	$= \dfrac{12+12h+3h^2-12}{h}$ $= \dfrac{12h+3h^2}{h}$
5 Take out the common factor h on the numerator and simplify.	$= \dfrac{h(12+3h)}{h}$ $= 12+3h$
6 Interpret the quantity that has been found.	The gradient of the secant is $12+3h$.

(b) determine the gradient of the tangent at that point.

1 Calculate the limit of the difference function as $h \to 0$.	$\lim\limits_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h\to 0}(12+3h)$ $= 12+3\times 0$ $= 12$
2 Interpret the answer.	The gradient of the tangent at (2, 12) is 12.

Differentiation from first principles

The limit of the difference quotient as $h \to 0$ can be used to determine the gradient at a point on the graph of a function. This is known as the *gradient function* or *derivative function* and is defined with respect to the same independent variable as the original function (usually x).

The gradient function, also known as the derived function, derivative function or the derivative of $f(x)$, is denoted by $f'(x)$ (read as f dash x) and is given by $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, provided that this limit exists.

The process of determining the rule of the gradient function from the rule of the original function is called differentiation.

Determining $f'(x)$ using the above definition is known as *differentiating from first principles*.

14 Differentiating from first principles

If $f(x) = 5x + 6$, determine an expression for $f'(x)$ by differentiating from first principles.

THINKING	WORKING
1 Recall the general expression for $f'(x)$.	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
2 Substitute for $f(x+h)$ and $f(x)$.	$= \lim_{h \to 0} \frac{5(x+h) + 6 - (5x+6)}{h}$
3 Simplify the quotient expression.	$= \lim_{h \to 0} \frac{5x + 5h + 6 - 5x - 6}{h}$ $= \lim_{h \to 0} \frac{5h}{h}$ $= \lim_{h \to 0} 5$
4 Calculate the limit.	$= 5$
5 Interpret the answer.	If $f(x) = 5x + 6$, then $f'(x) = 5$ The gradient at any point on the graph of $f(x) = 5x + 6$ is constant, and is equal to 5.

As shown in the previous example, the gradient of a linear function is constant.

The gradient of a curve such as the graph of a quadratic function is different for each value of x. The gradient function can be found by *differentiating from first principles.*

15 Differentiating a quadratic function from first principles

Consider $f(x) = x^2 + 3x$.

Calculate $f'(x)$ by differentiating from first principles.

THINKING	WORKING
1 Recall the general expression for $f'(x)$.	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
2 Substitute for $f(x+h)$ and $f(x)$.	$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h}$
3 Expand the brackets.	$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$
4 Simplify the quotient expression.	$= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h}$ $= \lim_{h \to 0} \frac{h(2x + h + 3)}{h}$ $= \lim_{h \to 0} (2x + h + 3)$

5 Calculate the limit.	$= 2x + 3$
6 Interpret the answer.	If $f(x) = x^2 + 3x$, then $f'(x) = 2x + 3$. The gradient at any point on the graph of $y = f(x)$ is $2x + 3$.

Expanding powers can be tedious. Recall that the coefficients match numbers in the rows of Pascal's triangle:

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

16 Differentiating a cubic from first principles

For $f(x) = -2x^3$, calculate $f'(x)$ by differentiating from first principles.

THINKING	WORKING
1 Recall the general expression for $f'(x)$.	$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
2 Substitute for $f(x+h)$ and $f(x)$.	$= \lim\limits_{h \to 0} \dfrac{-(x+h)^3 - (-2x^3)}{h}$
3 Expand the brackets and simplify the numerator.	$= \lim\limits_{h \to 0} \dfrac{-2\left(x^3 + 3x^2h + 3xh^2 + h^3\right) + 2x^3}{h}$ $= \lim\limits_{h \to 0} \dfrac{-2x^3 - 6x^2h - 6xh^2 - 2h^3 + 2x^3}{h}$ $= \lim\limits_{h \to 0} \dfrac{-6x^2h - 6xh^2 - 2h^3}{h}$
4 Factorise the numerator and simplify the quotient expression.	$= \lim\limits_{h \to 0} \dfrac{h(-6x^2 - 6xh - 2h^2)}{h}$ $= \lim\limits_{h \to 0} (-6x^2 - 6xh - 2h^2)$.
5 Calculate the limit.	$= -6x^2$
6 Interpret the answer.	If $f(x) = -2x^3$, then $f'(x) = -6x^2$. The gradient at any point on the graph of $y = f(x)$ is $-6x^2$.

17 Differentiating with a power of x on the denominator

Consider $f(x) = \frac{5}{x^2}$.

Calculate $f'(x)$ by differentiating from first principles.

THINKING	WORKING
1 Recall the general expression for $f'(x)$.	$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
2 Substitute for $f(x+h)$ and $f(x)$.	$= \lim_{h \to 0} \frac{\frac{5}{(x+h)^2} - \frac{5}{x^2}}{h}$
3 Use a common denominator.	$= \lim_{h \to 0} \frac{5x^2 - 5(x+h)^2}{hx^2(x+h)^2}$
4 Expand and simplify the numerator.	$= \lim_{h \to 0} \frac{5x^2 - 5(x^2 + 2xh + h^2)}{hx^2(x+h)^2}$ $= \lim_{h \to 0} \frac{5x^2 - 5x^2 - 10xh - 5h^2}{hx^2(x+h)^2}$ $= \lim_{h \to 0} \frac{-10xh - 5h^2}{hx^2(x+h)^2}$
5 Factorise the numerator and simplify the quotient expression.	$= \lim_{h \to 0} \frac{h(-10x - 5h)}{hx^2(x+h)^2}$ $= \lim_{h \to 0} \frac{-10x - 5h}{x^2(x+h)^2}$
6 Calculate the limit. Simplify, if necessary.	$= \frac{-10x}{x^2x^2}$ $= \frac{-10}{x^3}$
7 Interpret the final answer.	If $f(x) = \frac{5}{x^2}$, then $f'(x) = \frac{-10}{x^3}$. The gradient at any point on the graph of $y = f(x)$ is $\frac{-10}{x^3}$.

Additional information

Differentiating from first principles

Watch the video to see how to calculate the derivative from first principles for $f(x) = \frac{1}{x^2}$ and $f(x) = \sqrt{x}$.

EXERCISE 9.3 Derivatives from first principles

Worked Example

1 For each function, calculate the expression for $f(x+h)$ and hence the change in the function, $f(x+h)-f(x)$. 12

(a) $f(x)=4x-4$ (b) $f(x)=x^2-2$ (c) $f(x)=5x^2+7$

(d) $f(x)=x-x^2$ (e) $f(x)=5x-6x^2$ (f) $f(x)=7x^2+2x+2$

2 Calculate the following limits.

(a) $\lim\limits_{x\to 3}(2x-7)$ (b) $\lim\limits_{b\to -2}\left(b^2-9b+5\right)$ (c) $\lim\limits_{t\to 1}(3-t)$

(d) $\lim\limits_{m\to 0}\dfrac{m^3+4m^2-9m}{m}$ (e) $\lim\limits_{z\to 2}\dfrac{z^2+3z-10}{z-2}$ (f) $\lim\limits_{p\to -1}\dfrac{p^3-3p^2+p+5}{p+1}$

3 For each of the following determine an expression for $f'(x)$ by differentiating from first principles. 14

(a) $f(x)=2x+4$ (b) $f(x)=3-5x$ (c) $f(x)=-x-1$

(d) $f(x)=7x-8$ (e) $f(x)=3x+3$ (f) $f(x)=-2x-3$

4 The gradient of the curve $y=g(x)$ at the point $(c, g(c))$ is given by:

A $\lim\limits_{h\to 0}\dfrac{g(c+h)-c}{h}$ B $\dfrac{g(c+h)-g(c)}{h}$

C $\lim\limits_{h\to 0}\dfrac{g(c+h)-g(h)}{h}$ D $\lim\limits_{h\to 0}\dfrac{g(c+h)-g(c)}{h}$

5 Consider the function $f(x)=2x-3$.

(a) Which expression is not equivalent to the difference quotient?

A $\dfrac{2(x+h)-3-(2x-3)}{h}$ B $\dfrac{2x+2h-3-2x+3}{h}$

C $2h$ D $\dfrac{2h}{h}$

(b) Explain the common error made by a student who thought the difference quotient was $\dfrac{2x+2h-3-2x-3}{h}$.

(c) If you made this common error, how would you become aware that a mistake had been made?

6 Consider the function $f(x)=x^2+2$, where $x=1$. 13

(a) Calculate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

(b) Determine the gradient of the tangent at that point.

7 $P(2, 0)$ lies on the curve $y=x^2-4$. Q is a point on the curve with x-coordinate $2+h$.

(a) Determine the gradient of the secant PQ, expressing the answer in terms of h.

(b) Use limits to calculate the gradient of the tangent at P.

8 Calculate the gradient of the curve from first principles for each of the following:

(a) $y = x^2 - 1$ at $P(2, 3)$ (b) $y = 2x^2 + 3x$ at $P(-1, -1)$

Worked Example

(c) $y = \frac{3}{x}, x \neq 0$ at $P(1, 3)$ (d) $y = -\frac{2}{x}, x \neq 0$ at $P(-2, 1)$

15 9 Differentiate the following from first principles.

(a) $f(x) = 2x^2$ (b) $f(x) = -x^2$ (c) $f(x) = x^2 + 4x - 2$ (d) $f(x) = -x^2 - 7x + 5$

16 10 Differentiate the following cubic functions from first principles.

(a) $f(x) = -x^3$ (b) $f(x) = x^3 + 5$ (c) $g(x) = x^3 - 2x$

11 Differentiate the following from first principles.

(a) $f(x) = (x - 2)^2$ (b) $g(x) = x^3 - 6x + 1$

12 Differentiate the following power functions from first principles.

(a) $f(x) = x^4$ (b) $f(x) = x^5$ (c) $f(x) = x^6$

17 13 Differentiate the following functions with x on the denominator from first principles.

(a) $f(x) = \frac{1}{x}$ (b) $f(x) = -\frac{2}{x}$ (c) $f(x) = \frac{1}{x^2}$ (d) $f(x) = -\frac{3}{x^2}$

14 Differentiate the following from first principles.

(a) $f(x) = \frac{1}{x+3}$ (b) $g(x) = \frac{1}{x} - 4x$

15 $P(2, 10)$ lies on the graph $y = x^3 + x$. Q is a point on the graph with x-coordinate $2 + h$.

(a) Determine the gradient of the secant PQ, expressing the answer in terms of h.

(b) Determine the gradient of the tangent at P.

16 Consider $f(x) = x^2 - 4x + 3$.

(a) Calculate $f'(x)$, using differentiation from first principles.

(b) Calculate the coordinates of the point P on the graph of $y = f(x)$ where the tangent to the graph is parallel to the x-axis.

(c) Determine the gradient of the graph of $y = f(x)$ at P and hence the gradient of the tangent to the graph at P.

(d) Sketch the graph of $y = f(x)$, showing the tangent to the graph at P.

(e) Determine the value(s) of x for which the gradient of the tangent to the graph of $y = f(x)$ is positive.

17 Differentiate the following from first principles.

(a) $f(x) = \frac{1}{x^3}$ (b) $f(x) = \frac{1}{x^4}$ (c) $f(x) = \frac{1}{x^5}$

18 Consider the function $g(x) = -\frac{4}{x}, x \neq 0$.

(a) Calculate $g'(x)$, using differentiation from first principles.

(b) Determine the points on the graph of $y = g(x)$ at which the tangents are perpendicular to the line $y = -x$.

(c) Calculate the exact distance between the points found in part (b).

Differentiation using the power rule

9.4

Function notation for the derivative

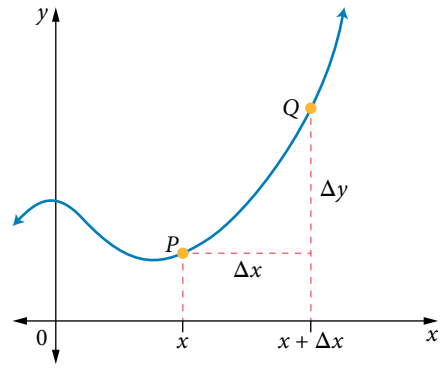
Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Leibniz notation for the derivative

Alternative notation, known as Leibniz notation, can be used to represent the derivative or gradient function.

In work on differentiation, Δx (read as delta x) represents a small change in the value of x. If y is a function of x, then Δy represents a small change in the value of y corresponding to the small change in the value of x, Δx.



If $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ are two points on the graph of $y = f(x)$, then the gradient of the graph, and hence the tangent to the graph, at P is defined as:

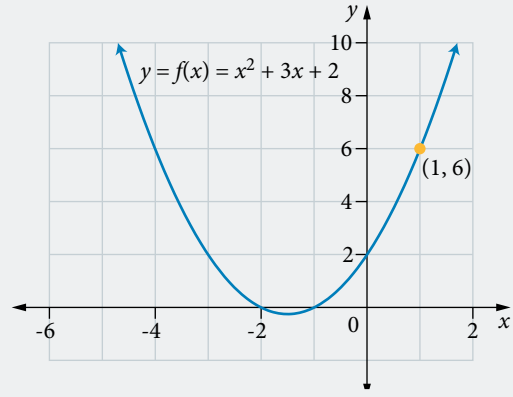
$$\lim_{\Delta x \to 0} \frac{(y + \Delta y) - y}{(x + \Delta x) - x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

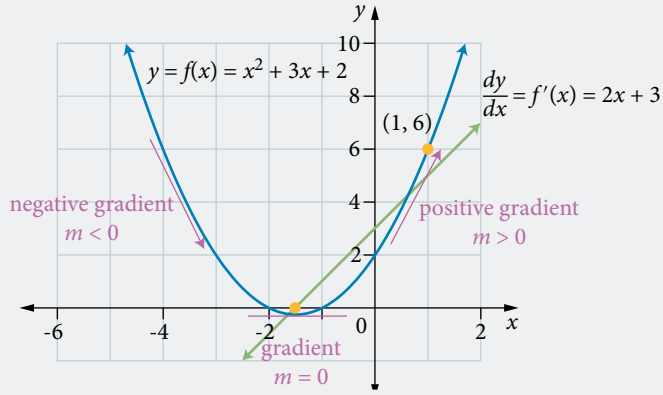
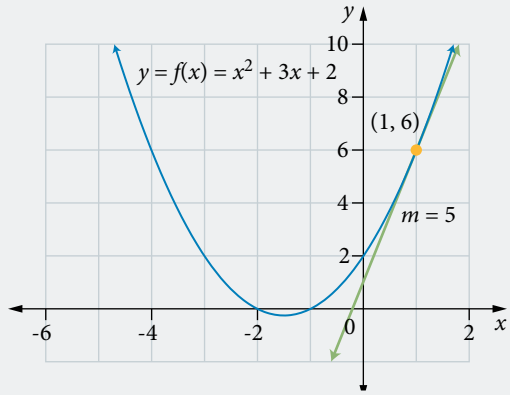
$$= \frac{dy}{dx}$$

The symbol $\frac{dy}{dx}$, pronounced as 'dee y dee x', is the derivative of y with respect to x and is an alternative symbol for the derivative or gradient function. Note that $\frac{dy}{dx}$ is not a fraction; it is to be regarded as a single expression, which is the limit of a quotient.

WARNING

The d in the derivative expression using the Leibniz notation is *not* a pronumeral, so do not try to cancel them.

Function notation	Leibniz notation	Graphical representation
$f(x) = x^2 + 3x + 2$ at $x = 1$ $f(1) = (1)^2 + 3(1) + 2$ $= 1 + 3 + 2$ $= 6$ The coordinates of the point on the graph are (1, 6).	$y = x^2 + 3x + 2$ at $x = 1$ $y = (1)^2 + 3(1) + 2$ $= 1 + 3 + 2$ $= 6$ The coordinates of the point on the graph are (1, 6).	

Function notation	Leibniz notation	Graphical representation
$f'(x) = 2x + 3$ The gradient of a tangent line on the graph of $f(x) = x^2 + 3x + 2$ is $2x + 3$.	$\frac{dy}{dx} = 2x + 3$ The gradient of a tangent line on the graph of $y = x^2 + 3x + 2$ is $2x + 3$.	
At $x = 1$: $f'(1) = 2(1) + 3$ $= 2 + 3$ $= 5$ The gradient of the tangent to the curve at $x = 1$ is 5.	At $x = 1$: $\frac{dy}{dx} = 2(1) + 3$ $= 2 + 3$ $= 5$ The gradient of the tangent to the curve at $x = 1$ is 5.	

Power rule of differentiation

Differentiating from first principles can be a time-consuming process. So a set of rules that help to determine the gradient function or derivative function more readily would be useful.

Functions of the form $f(x) = x^n$ are known as *power functions.*

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, where n is a positive integer. This rule is known as the power rule of differentiation.

The power rule can be extended to rational values of n, where a rational number is any number that can be expressed as the quotient $\frac{p}{q}$, where p and q are integers and $q \neq 0$. The set of all rational numbers is usually denoted by Q.

Function	Derivative
$y = x^2$	$\frac{dy}{dx} = 2x$
$y = x^3$	$\frac{dy}{dx} = 3x^2$
$y = x^4$	$\frac{dy}{dx} = 4x^3$
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$

The derivative of x^n:

If $f(x) = x^n$, $n \in Q$, then $f'(x) = nx^{n-1}$.

Or $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.

This can also be expressed in the form $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ where $\frac{d}{dx}$ is the 'derivative with respect to x'.

Proof of the power rule of differentiation

For this proof, the binomial theorem $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$, where n is a positive integer, is used.

In Chapter 5, $\binom{n}{r}$, read as 'n choose r', was defined as the number of ways to select r objects from a total of n objects.

$\binom{n}{r} = \frac{n!}{(n-r)!r!}$, where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

Note that $\binom{n}{0} = 1$, $\binom{n}{n} = 1$, $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$.

Applying the binomial theorem to the limit definition of the derivative results in:

$$\begin{aligned}
\frac{d}{dx}\left(x^n\right) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\left(\binom{n}{0} x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \ldots + \binom{n}{n-1} xh^{n-1} \binom{n}{n} h^n\right) - x^n}{h} \\
&= \lim_{h \to 0} \frac{\left(x^n + nx^{n-1}h + \frac{1}{2}n(n-1)x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n\right) - x^n}{h} \\
&= \lim_{h \to 0}\left(nx^{n-1}\right) + \lim_{h \to 0}\left(\frac{1}{2}n(n-1)x^{n-2}h\right) + \ldots + \lim_{h \to 0}\left(nxh^{n-2}\right) + \lim_{h \to 0}\left(h^{n-1}\right) \\
&= nx^{n-1} + 0 + \ldots + 0 + 0 \\
&= nx^{n-1}
\end{aligned}$$

So $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

In general, if $f(x) = ax^n$, where a is a constant, then $f'(x) = nax^{n-1}$.

For example, if $f(x) = 3x^4$, then:

$$\begin{aligned}
f'(x) &= 4 \times 3x^{4-1} \\
&= 4 \times 3x^3 \\
&= 12x^3
\end{aligned}$$

Derivative of a constant

The derivative of a constant is zero.

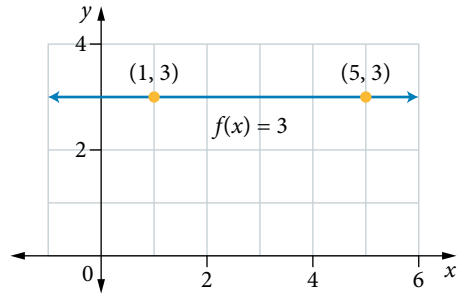
If you think about the graphical representation of a derivative or gradient, the slope of a horizontal line is 0, which comes from two points (x_1, y_1) and (x_2, y_2) where $y_1 = y_2$.

Consider the function $f(x) = 3$.

From the coordinates marked (1, 3) and (5, 3):

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-3}{5-1} \\ &= \frac{0}{4} \\ &= 0 \end{aligned}$$

$\therefore f(x) = 0$



If $f(x) = a$, $a \in \mathbb{R}$, then $f'(x) = 0$.

The function $f(x) = a$ can be expressed as $f(x) = ax^0$ $(x^0 = 1)$.

From the power rule of differentiation, it follows that $f'(x) = 0$ as $\left(a(0)x^{0-1}\right) = 0$.
So the derivative of a constant is zero.

Derivative of the sum (or difference) of terms

Using differentiation from first principles in the previous section, the derivative of $f(x) = x^2 + 3x$ is $f'(x) = 2x + 3$.

In alternative notation, this can be expressed as:

$$\begin{aligned} \frac{d}{dx}\left(x^2 + 3x\right) &= \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(3x) \\ &= 2x^{2-1} + 3x^{1-1} \\ &= 2x + 3 \end{aligned}$$

The derivative of a sum (or difference) of terms is the sum (or difference) of the derivatives of these terms.

This shows that the derivative of a sum of terms is the sum of the derivatives of these terms.

18 Calculate a derivative function

If $f(x) = 2x^3$, calculate $f'(x)$.

THINKING	WORKING
1 Identify the function $f(x)$.	$f(x) = 2x^3$
2 Determine the derivative by multiplying the coefficient by the index then decreasing the index by 1.	$f'(x) = 3 \times 2x^{3-1}$ $= 3 \times 2x^2$ $= 6x^2$

19 Calculate the derivative of a function with two terms

If $g(t) = 4t^3 - t^2$, determine $g'(t)$.

THINKING	WORKING
1 Identify the function $g(t)$.	$g(t) = 4t^3 - t^2$
2 Determine the derivative, $g'(t)$ by multiplying the coefficient by the index then decreasing the index by 1.	$g'(t) = 3 \times 4t^{3-1} - 2 \times t^{2-1}$ $= 3 \times 4t^2 - 2 \times t^1$ $= 12t^2 - 2t$

20 Calculate the derivative of a function with fractional coefficients

If $y = \frac{1}{3}x^6 + \frac{1}{4}x^4$, determine $\frac{dy}{dx}$.

THINKING	WORKING
1 Identify the function y.	$y = \frac{1}{3}x^6 + \frac{1}{4}x^4$
2 Determine the derivative $\frac{dy}{dx}$ by multiplying the coefficient by the index then decreasing the index by 1.	$\frac{dy}{dx} = 6 \times \frac{1}{3}x^{6-1} + 4 \times \frac{1}{4}x^{4-1}$ $= 6 \times \frac{1}{3}x^5 + 4 \times \frac{1}{4}x^3$ $= 2x^5 + x^3$

In subsequent worked examples illustrating the power rule of differentiation, the intermediate steps used in previous examples are omitted.

21 Calculate the derivative of a factorised function

Determine the derivative $\frac{dy}{dx}$ if $y = x(2x - 5)$.

THINKING	WORKING
1 Expand the brackets to express the function as a series of terms in the form ax^n.	$y = x(2x - 5)$ $= 2x^2 - 5x$
2 Determine the equation of the derivative by calculating the derivative of each term.	$\frac{dy}{dx} = \frac{d}{dx}(2x^2) - \frac{d}{dx}(5x)$ $= 4x - 5$

The power rule with negative and fractional powers

The power rule of differentiation can be extended to include negative integers and fractional powers.

22 Calculate the derivative of a function with negative integer powers

Calculate $\frac{da}{dz}$ if $a = \frac{5}{3z^2}$.

THINKING	WORKING
1 Express in the form $y = ax^n$.	$a = \frac{5}{3z^2}$ $= \frac{5}{3}z^{-2}$
2 Use the power rule of differentiation. Express the final answer with positive indices (to match the format in the question).	$\frac{da}{dz} = -2 \times \frac{5}{3}z^{-2-1}$ $= -2 \times \frac{5}{3}z^{-3}$ $= -\frac{10}{3}z^{-3}$ $= -\frac{10}{3z^3}$

23 Calculate the derivative of a function with fractional powers

Calculate $\frac{dm}{dt}$ if $m = 3\sqrt{t}$.

THINKING	WORKING
1 Express in the form $y = ax^n$.	$m = 3\sqrt{t}$ $= 3 \times t^{\frac{1}{2}}$
2 Use the power rule of differentiation.	$\frac{dm}{dt} = \frac{1}{2} \times 3t^{\frac{1}{2}-1}$ $= \frac{1}{2} \times 3t^{-\frac{1}{2}}$ $= \frac{3}{2}t^{-\frac{1}{2}}$
3 Express the final answer in terms of positive indices.	$= \frac{3}{2t^{\frac{1}{2}}}$
4 Convert to surd form (to match the format in the question).	$= \frac{3}{2\sqrt{t}}$

EXERCISE 9.4 Differentiation using the power rule

Worked Example

1 Determine $f'(x)$ if: 18

(a) $f(x) = x^2$ (b) $f(x) = x^3$ (c) $f(x) = x^5$

(d) $f(x) = 4x$ (e) $f(x) = 3x^2$ (f) $f(x) = 3x^6$

(g) $f(x) = \frac{2}{3}x^4$ (h) $f(x) = -5x^3$ (i) $f(x) = -\frac{1}{5}x^5$

2 The derivative of a polynomial of degree n has degree:

A $n-2$ B $n-1$ C n D $n+1$

3 Determine $g'(t)$ if: 19

(a) $g(t) = 5t - 1$ (b) $g(t) = t^2 + 6t$ (c) $g(t) = t^4 - 4t^2$

4 Determine $\frac{dy}{dx}$ if: 20

(a) $y = \frac{1}{3}x^3 + \frac{1}{2}x$ (b) $y = -\frac{2}{5}x^5 + \frac{3}{4}x^4$ (c) $y = \frac{1}{5}x^3 - \frac{2}{3}x^2$

5 Determine $\frac{dy}{dx}$ if:

(a) $y = 8$ (b) $y = -3x^3 + 5x^2 - \sqrt{2}$

6 Consider the function $h(x) = 3x(x-2)$.

(a) $h'(x)$ is equal to:

A $6x$ B $6(x-1)$ C $x^3 - 3x^2$ D $3x$

(b) Explain the common error made by a student who thought the answer was 3.

7 Determine $\frac{dy}{dx}$ if $y = x(x+2)^2$. 21

8 Determine the derivative function if:

(a) $f(x) = x^3 - 9x^2$ (b) $h(x) = 5x(2-x)$ (c) $y = ax^2 + bx + c$

(d) $p = (x-1)^3$ (e) $f(a) = \frac{1}{2}(a-2)^2$ (f) $h = t(t-a)^2$

9 Calculate $f'(-1)$ and $f'(2)$ by substituting $x = -1$ and $x = 2$ into each gradient function.

(a) $f(x) = 2x^3 + 3x$ (b) $f(x) = 3x - x^2$ (c) $f(x) = ax^3 + bx^2 + cx + d$

(d) $f(x) = (2x+1)^2$ (e) $f(x) = x(x+1)$ (f) $f(x) = (x-2)(x-3)$

10 Determine $g'(t)$ if:

(a) $g(t) = \frac{t^2 - 7t}{t}$, $t \neq 0$ (b) $g(t) = \frac{3t^3 + 6t^2 - 8t}{t}$, $t \neq 0$

11 Determine $\frac{dh}{dt}$ if: 22

(a) $h = \frac{1}{t}$ (b) $h = \frac{1}{t^2}$ (c) $h = \frac{2}{5t^3}$

(d) $h = -\frac{3}{4t}$ (e) $h = \frac{1}{t} + \frac{2}{3t^2}$ (f) $h = \frac{t-3}{t}$

Worked Example

23

12 Determine the derivative $\frac{dy}{dx}$ if:

(a) $y = x^{\frac{1}{2}}$ (b) $y = 3x^{\frac{3}{2}}$ (c) $y = 6\sqrt{x}$

(d) $y = \frac{2}{\sqrt{x}}$ (e) $y = \sqrt[3]{x}$ (f) $y = 4x\sqrt{x}$

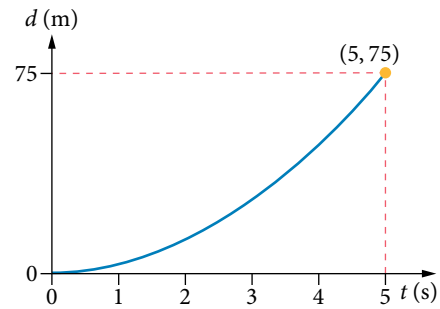
13 Determine the derivative function $f'(x)$ if:

(a) $f(x) = 5x - 3 - \frac{4}{x}$ (b) $f(x) = \frac{1}{2}x^3 + 6x^2 - \frac{1}{x^2}$ (c) $f(x) = 6x - 4\sqrt{x}$

14 Calculate the gradient of the tangent where $x = 1$ for each of the following.

(a) $f(x) = 2x^2 + 2x + \frac{3}{x}$ (b) $f(x) = \frac{1}{2x} - 5x - x^2$ (c) $f(x) = 3x^3 + x\sqrt{x}$

15 The distance time graph on the right is part of a parabola, with its vertex at the origin and is valid for $0 \le t \le 5$ where t is the time in seconds and d is the distance in metres.



(a) Determine the rule of the function.

(b) Sketch the corresponding speed versus time graph (the gradient graph of the distance–time function).

(c) Sketch the acceleration versus time graph (the gradient graph of the speed–time function).

16 If a function f is differentiated, the first derivative f' is generated.

If f' is differentiated, the second derivative f'' is generated.

If f'' is differentiated, the third derivative f''' is generated.

If f''' is differentiated, the fourth derivative f'''' is generated and so on.

This process can be continued and these derivatives are known as higher-order derivatives. A general formula for the nth derivative of a function can be generated.

Consider the function $f(x) = \frac{1}{x}$, $x \neq 0$.

(a) Determine the following:

(i) $f'(x)$ (ii) $f'''(x)$ (iii) $f'''(x)$ (iv) $f''''(x)$

(b) What will the result be after n differentiations?

(c) If a function is a polynomial function of degree n, what will be the result after $n + 1$ differentiations?

17 Given that $\frac{d}{dx}\left(\frac{1}{x-3}\right) = -\frac{1}{(x-3)^2}$, calculate the derivative of:

(a) $\frac{x}{x-3}$ (b) $\frac{x+1}{x-3}$ (c) $\frac{2x+3}{x-3}$

Summary

Rates of change

A rate describes the change in one quantity with respect to the change in another quantity.

When one quantity A changes with respect to another quantity B, the general formula for the rate of change is:

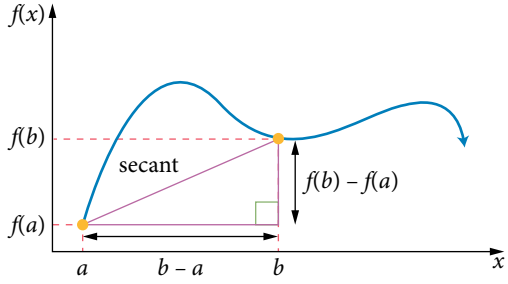
$$\text{rate} = \frac{\text{change in } A}{\text{change in } B} = \frac{\Delta A}{\Delta B}$$

Average rates of change

The average speed is the average rate of change of d with respect to t: $\dfrac{\text{change in } d}{\text{change in } t} = \dfrac{\Delta d}{\Delta t}$

The rate of change of y with respect to x can be written as:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

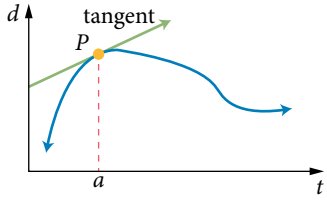


The average rate of change of the function $f(x)$ from $x = a$ to $x = b$ is $\dfrac{f(b) - f(a)}{b - a}$.

With respect to the graph of $f(x)$, the average rate of change between the points $x = a$ and $x = b$ is the gradient of the secant between these two points, namely $\dfrac{f(b) - f(a)}{b - a}$.

Instantaneous rates of change

The instantaneous rate of change is the rate of change at a single instant, between two points infinitely close together. This is the gradient of the tangent line at that point on a graph of the rate of change.



The derivative

A tangent is a straight line that touches a graph at a point $P(x, y)$.

For a function f, the gradient of the graph at $P(a, f(a))$ is the gradient of the tangent at P and is given by $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ provided that the limit exists.

For a function f, the gradient function, also known as the derived function or derivative, is given by $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

If y is expressed in terms of x, then $\dfrac{dy}{dx}$ is the derivative of y with respect to x.

Basic differentiation rules

If $f(x) = c$ where c is a constant, then $f'(x) = 0$.

If $f(x) = ax^n$, $n \in Q$, then $f'(x) = anx^{n-1}$.

The derivative of a sum or difference of terms is equal to the sum or difference of the derivatives of these terms.

Chapter review 9

Exercise 9.1

1 Calculate the rates, giving answers in the units indicated.

(a) A car travels 15 km in 10 minutes. Determine its speed in km/h.

(b) A 2600 L tank empties in 3 hours 15 minutes. Determine the rate at which the tank empties in L/h.

(c) 4 kg of apples costs \$11.20. Determine the cost of the apples in \$ per kg.

Exercise 9.1

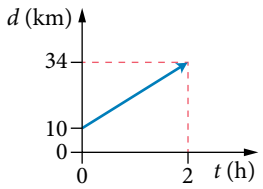
2 Dress fabric costs \$18.50 per metre.

(a) Determine the cost of 3.6 m of fabric.

(b) If dress fabric costing \$51.80 is purchased, determine the length of the fabric.

Exercise 9.1

3 The graph shows the distance of a school bus from a school over a 2-hour period.



(a) Determine the gradient of the line.

(b) Determine the average speed of the school bus in:

(i) km/h **(ii)** m/h **(iii)** km/min

Exercise 9.1

4 An empty swimming pool is filled with water. After 3 minutes the pool contains 2034 L of water.

(a) Determine the average rate at which water is delivered by the hose.

(b) At this rate, how long will it take in days and hours, to the nearest hour, to completely fill the swimming pool, given that the pool has with a capacity of 2 500 000 L?

Exercise 9.1

5 A stone is projected into the air. The following table shows the height of the stone above the ground during the first 4 seconds. Determine the average rate of change of the stone's height in the:

(a) first 2 seconds **(b)** first 4 seconds

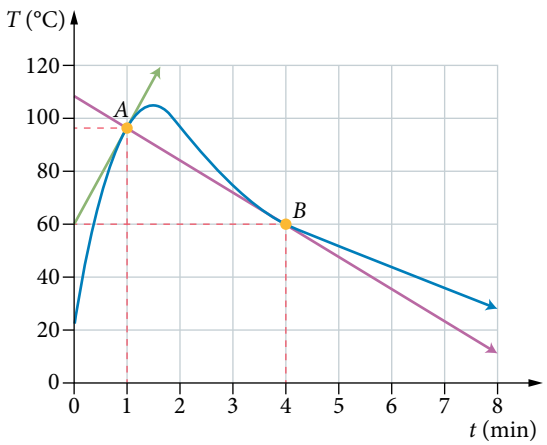
Time (s)	0	1	2	3	4
Height (m)	0	35.1	60.4	75.9	81.6

Exercise 9.1

6 A graph showing the temperature of water in a kettle is shown. Tangents are drawn to the curve at points $A = (1, 96)$ and $B = (4, 60)$. Determine the rate at which the temperature of the water is changing after:

(a) 1 minute

(b) 4 minutes

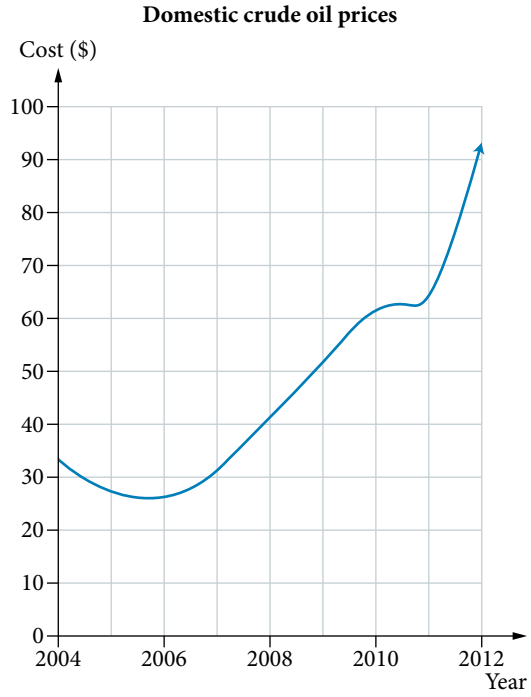


7 A cyclist travels 36 km in 4 hours. Determine the cyclist's average speed in metres per second. Exercise 9.1

A 4 **B** 0.9 **C** 9 **D** 2.5

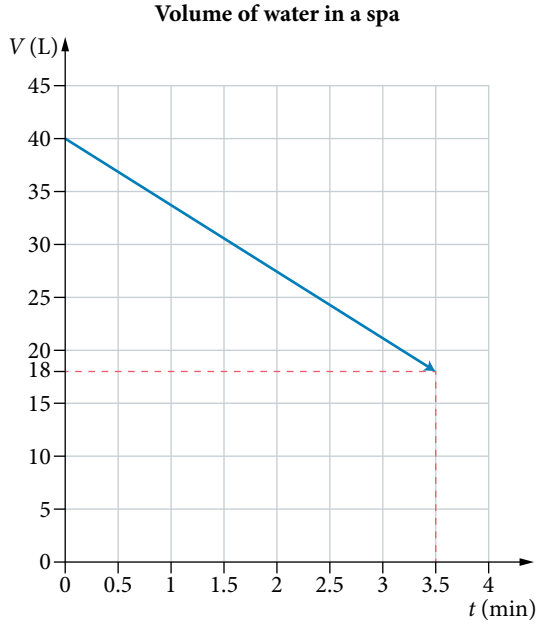
8 The graph shows the average domestic crude oil price over several years. Exercise 9.1

Determine the average annual rise in the crude oil price from 2004 to 2012, to the nearest whole number.



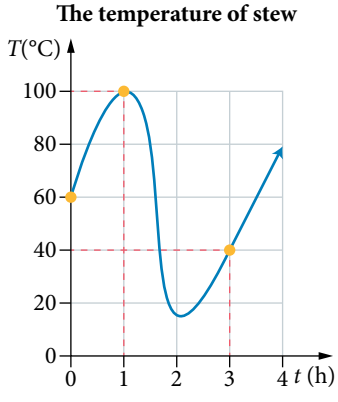
9 The graph shows the volume of water in a spa bath as it empties. Exercise 9.1

At this rate, how much total time does it take for the spa bath to empty, to the nearest half minute?



10 The graph shows the temperature of a stew as it heats up, cools down and is then reheated for dinner. Exercise 9.1

What is the average rate of change of the stew's temperature over 2 hours from the end of the first hour?

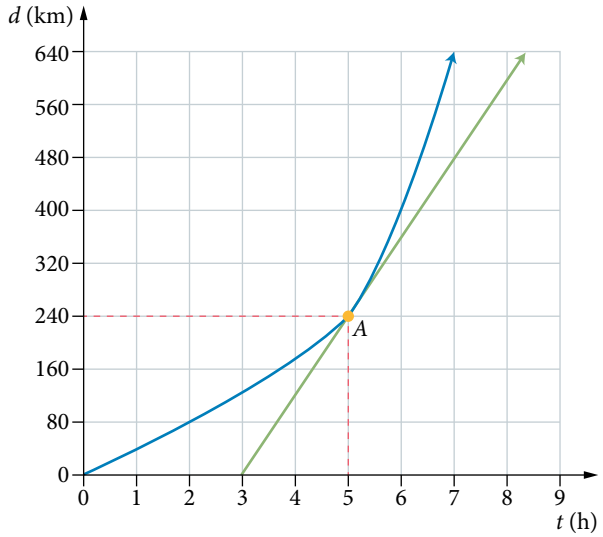


Exercise 9.1

11 The graph shows the distance that an aircraft travels during an air race.

Determine the speed of the aircraft 5 hours after it takes off.

The distance travelled by an aircraft



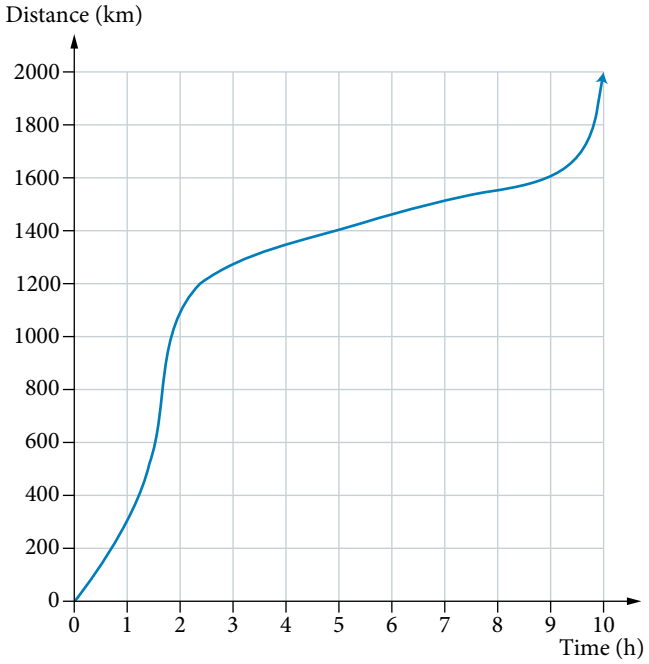
Exercise 9.1

12 The graph shows the distance travelled by an asteroid.

(a) Determine the average speed of the asteroid over the total time of observation.

(b) Determine the average speed from the end of the first hour to the end of the ninth hour of the observation.

The distance travelled by an asteroid



Exercise 9.2

13 On a set of Cartesian axes, point A is (1, 6) and point B is (4, 18).

(a) From A to B determine the value of Δx.

(b) Determine the corresponding value of Δy.

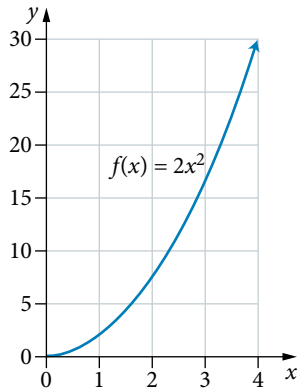
(c) Determine the gradient of the line AB.

Exercise 9.2

14 On the curve $f(x) = 2x^2$, a point A has an x-value of 1 and a point B has an x-value of 3.

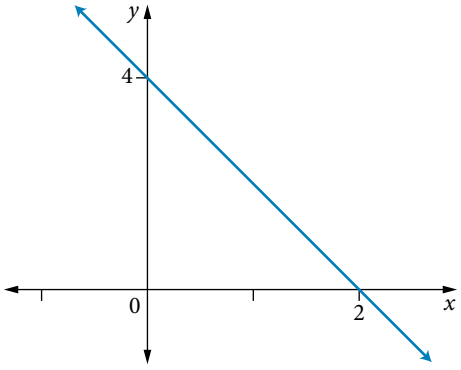
(a) Calculate $f(1)$ and $f(3)$.

(b) Using $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$, what is the gradient of the secant AB?

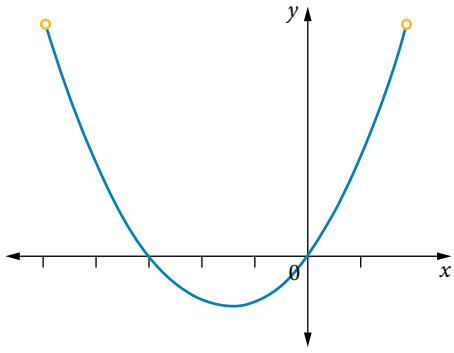


15 Draw the graph of the gradient for each function, matching key points of change. Exercise 9.2

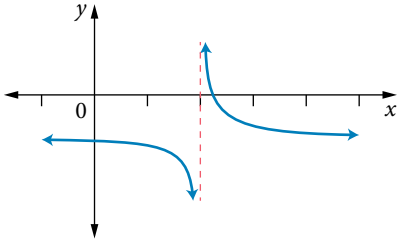
(a)



(b)

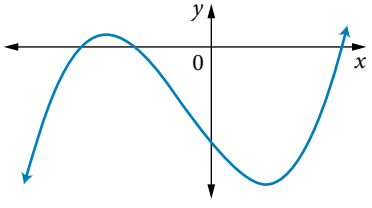


(c)

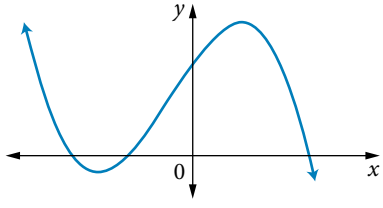


16 Match each of the four graphs below with their matching gradient graph *A*, *B*, *C*, *D*. Exercise 9.2

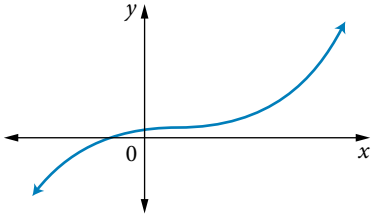
(a)



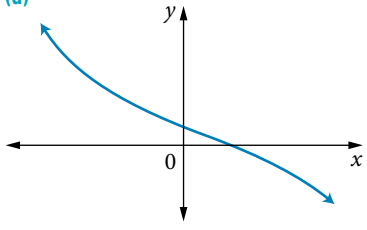
(b)



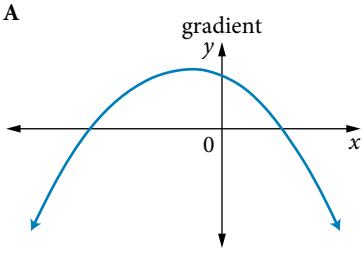
(c)



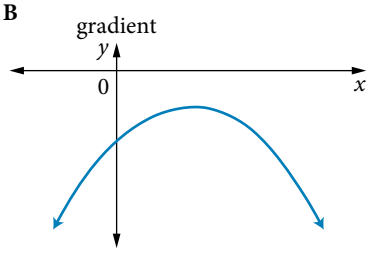
(d)



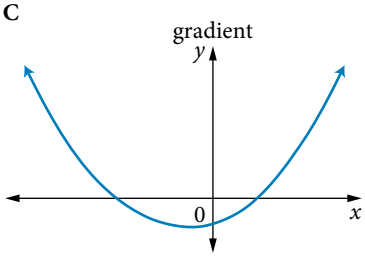
A



B



C



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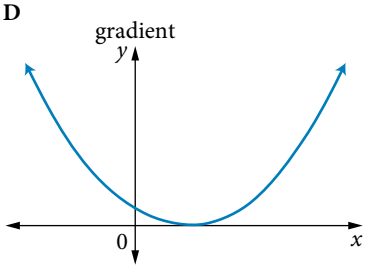
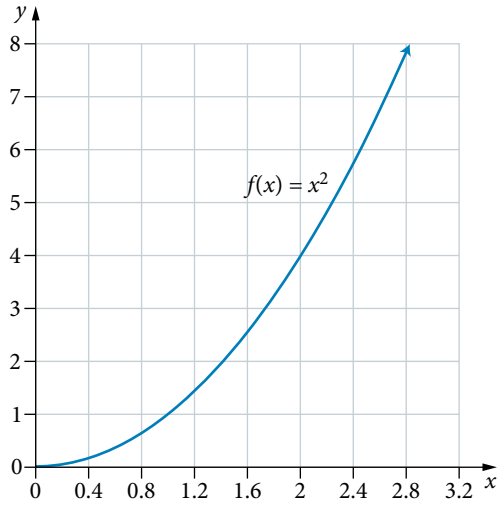


Exercise 9.2 **17** A is the point (-1, 4) and B is the point (0, -2). Going from left to right, what is the value of Δx between the two points?

Exercise 9.2 **18** For $f(x) = 1 - 2x$ determine the value of $\dfrac{f(5) - f(1)}{4}$.

Exercise 9.2 **19** The graph of a function $f(x) = x^2$ is shown.



(a) Copy the graph and mark the following points on your graph.

(i) $A(2, 4)$ (ii) $B(2.2, 4.84)$

(iii) $C(2.4, 5.76)$ (iv) $D(2.8, 7.84)$

(b) Determine the gradient of the secant:

(i) AD (ii) AC (iii) AB

(c) Draw the tangent line at $A(2, 4)$ and estimate its gradient.

(d) Comment on the gradients of the secants and tangent. How are they related?

Exercise 9.4 **20** Determine $f'(x)$ if:

(a) $f(x) = 2x^2 - 5$

(b) $f(x) = -4x^3 + 10x^2 - \dfrac{1}{2}$

(c) $f(x) = (2x - 3)^2$

Exercise 9.4 **21** If $y = 2x(x - 4)$, then determine $\dfrac{dy}{dx}$.

Exercise 9.1 **22** The following table shows the increase in height of a young bamboo plant over 5 days.

Time (days)	0	1	2	3	4	5
Height (cm)	2	4	7	10	12	13

(a) Draw a graph to show the height of the bamboo for the first 5 days of observation.

(b) On your graph, sketch your best estimates for the tangent at $t = 1$, 2 and 3 and use them to estimate the rate of change of the height with respect to time after 1, 2 and 3 days.

Exercise 9.2 **23** A spherical balloon is filled with water so that it's volume v cm^3 in terms of its radius r cm is given by $V = \dfrac{4}{3}\pi r^3$.

(a) Draw the graph of volume against radius for $r \in [0,10]$.

(b) Calculate the average rate of change of the volume of the balloon with respect to its radius as its radius is increased to 10 cm, correct to 1 decimal place.

(c) Calculate the gradient of the secant joining points with r-values of 5 and 5.001, and then estimate the rate at which the volume is changing when the radius is 5 cm.

(d) Calculate the gradient of the secant joining points with r-values of 9.999 and 10, and then estimate the rate at which the volume is changing when the radius is 10 cm.

24 Determine $f'(x)$ by differentiating from first principles. Exercise 9.3

(a) $f(x) = 5x^2$ **(b)** $f(x) = -2x(x+2)$

25 Consider the function $f(x) = x^3 + 3x$. The points $P(2, 14)$ and $Q\left(2+h, (2+h)^3 + 3(x+h)\right)$ are on the graph of the function. Exercise 9.3

(a) Determine the gradient of the secant PQ in terms of h.

(b) Determine the gradient of $y = f(x)$ at P.

26 Calculate $f'(x)$ if: Exercise 9.4

(a) $f(x) = (x-1)(2x+1)^2$ **(b)** $f(x) = \left(x - \frac{3}{x}\right)^2$ **(c)** $f(x) = \sqrt{x} - 9 + \frac{2}{x}$

27 Consider the function $f(x) = 3x^2 - 5x - 1$. Calculate: Exercise 9.4

(a) $f'(-2)$ **(b)** the set of values of x for which $f'(x) > -11$.

28 If $f(x) = -\frac{3}{x^3}$, $x > 0$ then $f'(x)$ is equal to Exercise 9.4

A $\frac{1}{x^4}$ **B** $\frac{3}{x^2}$ **C** $\frac{9}{x^4}$ **D** $-\frac{9}{x^4}$

29 If $f(x) = \frac{5}{\sqrt{x}}$, $x > 0$ then $f'(x)$ is equal to Exercise 9.4

A $-\frac{5}{2}x\sqrt{x}$ **B** $10\sqrt{x}$ **C** $\frac{5}{2\sqrt{x}}$ **D** $-\frac{5}{2x\sqrt{x}}$

30 Consider the graph of the function $y = (x+2)(x-3)(x-4)$. Exercises 9.2, 9.4

Determine the equation of the line joining the two points on the curve where the gradient is $\frac{5}{3}$.

31 Use first principles to determine the derivative of $\frac{2}{3(x-1)^2}$. Exercise 9.3

32 Determine the derivative of $\frac{1}{4x(x+2)}$, given that the derivative of $\frac{1}{x+2}$ is $-\frac{1}{(x+2)^2}$. Exercise 9.4

Mixed review

CHAPTERS 1–9

Exercise 1.3 **1** Plot the graph of the piece-wise function shown on the right.

$$y = \begin{cases} -2x - 2, & x \le -2 \\ 1, & -2 < x \le 1 \\ 2x - 3, & x > 1 \end{cases}$$

Exercise 1.4 **2** Given the second term of an arithmetic sequence is 1406 and the fourth term is 994, determine:

(a) the first five terms **(b)** the first negative term **(c)** the sum of all the positive terms.

Exercise 1.5 **3** A bank offers a simple interest rate of 3.4% p.a. for 2 years. How much money needs to be invested if the final amount is to be at least $4500? Give your answer to the nearest 10 cents.

Exercise 2.2 **4** Determine the rule of the quadratic function whose graph is obtained from the basic quadratic function $y = x^2$ by a horizontal translation of 9 units to the left and a vertical translation of 2 units down.

Exercise 2.3 **5** Determine the value of the discriminant for each of the following quadratic functions and hence state the number of roots for the function.

(a) $y = x^2 - 7x + 9$ **(b)** $y = 4x^2 - 20x + 25$ **(c)** $y = -\frac{1}{4}x^2 + 3x - 3$ **(d)** $y = x^2 + 4$

Exercise 2.2 **6** Determine the rule of the quadratic function whose graph is obtained from the basic quadratic function $y = x^2$ by a dilation parallel to the y-axis with a scale factor $\frac{2}{3}$, and then a translation of 3 units up.

Exercise 4.2 **7** The function $f(x) = \sqrt{x}$ is dilated by a factor 3 parallel to the y-axis, reflected in the x-axis, and then translated 1 unit down to produce $g(x)$. Draw $f(x)$ and $g(x)$ on the same set of axes showing all major points in coordinate form.

Exercise 4.2 **8** List the set of transformations that will move $f(x) = \sqrt{x}$ to $g(x) = -2\sqrt{x+1} + 4$. Draw $f(x)$ and $g(x)$ on the same graph, clearly showing all major points in coordinate form.

Exercise 5.5 **9** Determine the value of n in each of the following.

(a) ${}^nP_2 = 156$ **(b)** ${}^nP_2 = 650$ **(c)** ${}^nP_3 = 4080$

Exercise 5.6 **10** From a carton of 12 eggs a random sample of 2 eggs is chosen. If the order of selection is not important, how many different samples are possible?

Exercise 6.1 **11** Express the following with positive indices.

(a) $2x^{-3}$ **(b)** $\frac{4a^2}{b^{-4}}$ **(c)** $\frac{x^2y^{-3}}{x^6y^2}$

Exercise 6.2 **12** Express the following in scientific notation.

(a) 0.00351 **(b)** 523.7 **(c)** 109 700

Exercise 7.1 **13** Solve the following equations for the unknown pronumeral.

(a) $5^{-3x} = 625$ **(b)** $3^x = \sqrt[3]{81}$ **(c)** $3^x \div 9^{2x} = 3^9$

Exercise 7.3 **14** Calculate each value, correct to 2 decimal places.

(a) $\log_2(15)$ **(b)** $\log_{24}(2)$ **(c)** $\log_6(0.022)$

15 Convert each of the following. Exercise 8.1

(a) $\frac{3\pi}{4}$ to degrees

(b) 1.26° to degrees (2 d.p.)

(c) 124° to radians, in terms of π

(d) 47° to radians (3 d.p.)

16 $P(2, -1)$ lies on the graph of $f(x) = -x^2 + 3$. Q is a point on the graph with x-coordinate $2 + h$. Exercise 9.2

(a) Determine the gradient of the secant PQ, expressing your answer in terms of h.

(b) Use limits to find the gradient of the tangent at P.

17 Determine the solution to the simultaneous equations $5x - 2y + 23 = 0$ and $8x + 9y - 12 = 0$. Exercise 1.2

18 Use interval notation to state the range of each of the following quadratic functions. Exercise 2.3

(a) $y = -3(x + 4)(x - 2)$

(b) $y = \frac{2x^2}{3} - 7x + 1$

(c) $y = -\frac{x^2}{4} + 16$

19 Solve the equations. Exercise 3.6

(a) $x^3 + 3x^2 - 16x = 48$

(b) $x^3 + 16x^2 + 25 = 5x^2 - 35x$

20 List the set of transformations that can move $f(x) = \frac{1}{x}$ to $g(x) = -\frac{2}{(x+3)} - 2$. Show $f(x)$ and $g(x)$ on the same graph clearly showing all major points in coordinate form.

21 Factorise and then simplify the following expressions. Exercise 7.1

(a) $\frac{5^{2n} - 4}{5^n + 2}$

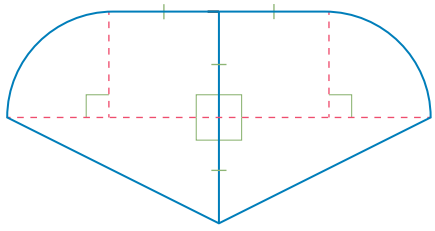
(b) $\frac{3^x - 3^{x-2}}{3^x}$

22 The arc of a sector of a circle with diameter 14 cm is 36 cm in length. Exercise 8.1
Determine the area of the sector.

23 The graph of a cosine function has consecutive maximum turning points at $\left(\frac{\pi}{4}, 5\right)$ and $\left(\frac{7\pi}{4}, 5\right)$, and a minimum value of -1. Determine a possible equation for this function. Exercise 8.4

24 Differentiate $y = \frac{1}{2x^2} + 3x\sqrt{x}$ with respect to x to find $\frac{dy}{dx}$. Exercise 9.4

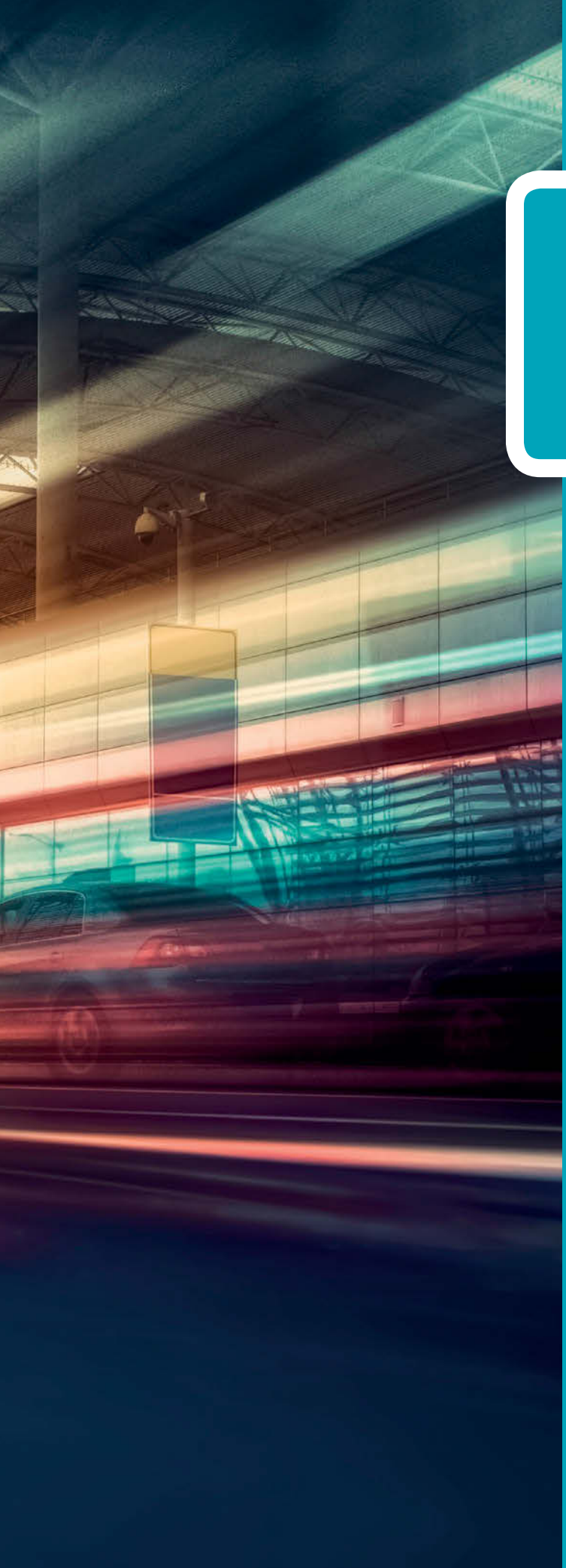
25 The symmetrical shape consists of quadrants, squares and right-angled triangles. Write a rule in terms of the radius r consisting of two piece-wise functions, to model the boundary of the shape. Exercise 4.3



26 The salinity of the oceans is generally 3.5%, or 35 ppt (parts per thousand). The density of pure water is 1000 kg/m^3, but the density of seawater is about 1027 kg/m^3. Use the degree of accuracy in the given values as a guide when writing your answers. Exercise 6.2

(a) Determine the mass of salt that could be extracted from a cubic kilometre of seawater.

(b) The Dead Sea is a hypersaline body of water with a salinity of 342 ppt and density of 1.24 kg/L. Determine the mass of salt that could be extracted from a square kilometre of Dead Sea water taken to a depth of a metre.



10

Differential calculus

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10

Recall

Sketch the graph of a linear function

1 Sketch the following straight lines, showing the coordinates of the x- and y-intercepts.

(a) $y = -x + 3$ (b) $y = \frac{1}{2}x + 1$ (c) $2x - 3y = 9$

Determine the equation of a straight line

2 Determine the equation of the line that is parallel to the line $y = -4x - 1$ and passes through the point (3, -1).

Understand the role of the discriminant

3 Calculate the discriminant and hence determine the number of solutions for the following quadratic equations.

(a) $3x^2 + 4x + 2 = 0$ (b) $2x^2 - 3x - 4 = 0$ (c) $5x^2 + 10x + 5 = 0$

Solve quadratic equations

4 Solve the following quadratic equations for x.

(a) $3x^2 - 48 = 0$ (b) $2x^2 - x - 10 = 0$

(c) $3x^2 - 8x + 4 = 0$ (d) $-2x^2 + 6x + 1 = 0$ (Give answers as exact values.)

Graph quadratic functions specified in various forms

5 Sketch the graph of the following quadratic functions, clearly showing the coordinates of the turning point and the x- and y-intercepts where they exist.

(a) $f(x) = -(x-1)^2 - 2$ (b) $f(x) = (x+4)(x-2)$ (c) $f(x) = x^2 - 2x - 15$

Graph cubic functions specified in various forms

6 Sketch the graph of $f(x) = (x-3)^3 - 1$, clearly showing the coordinates of the stationary point of inflection and axes intercepts.

Expand brackets

7 Write each expression in expanded form.

(a) $(2x-3)^2$ (b) $(3x+4)(3x-4)$ (c) $(x-2)^3$

Solve a system of linear equations

8 Solve the following simultaneous linear equations algebraically.

(a) $x + y = 4$ and $3x + 2y = 11$ (b) $5x - 3y = 6$ and $2x + 4y = 5$ (c) $x - 3y = 2$ and $-2x + y = 3$

Factorise and solve cubic equations algebraically without technology

9 Solve the following cubic equations for x.

(a) $x^3 - 4x^2 + 3x = 0$ (b) $x^3 + 6x^2 + 11x + 6 = 0$ (c) $x^3 - 2x^2 - 4x + 8 = 0$

Gradients of functions and equations of tangents

10.1

Derivatives at specified values of the variable

To determine the derivative of a function $f(x)$ at a specified value $x = a$, determine $f'(x)$ and then substitute $x = a$ into the derivative function. This gives you the value of the gradient of the tangent to the graph of the function f for $x = a$.

1 Find a derivative at a specified x-value

For $f(x) = 2x^3 - 3x^2 + 4x - 1$, calculate $f'(2)$.

THINKING	WORKING
1 Use the power rule of differentiation.	$f(x) = 2x^3 - 3x^2 + 4x - 1$ $f'(x) = 3 \times 2x^{3-1} - 2 \times 3x^{2-1} + 1 \times 4x^{1-1}$ $= 6x^2 - 6x + 4$
2 Substitute the given x-value into $f'(x)$ and calculate the derivative at that point.	$f'(2) = 6(2)^2 - 6(2) + 4$ $= 6 \times 4 - 12 + 4$ $= 24 - 8$ $= 16$

Technology worked example
Derivative at a point

2 Determine gradient properties of a polynomial function

Consider the function $y = x^3 - 12x$.

(a) Determine the expression for $\frac{dy}{dx}$.

THINKING	WORKING
Use the power rule of differentiation.	$y = x^3 - 12x$ $\frac{dy}{dx} = 3x^2 - 12$

(b) Determine the gradient of the graph at $x = -1$.

1 Substitute the given x-value into the derivative equation $\frac{dy}{dx}$ and calculate.

For $x = -1$:

$$\begin{aligned}\frac{dy}{dx} &= 3(-1)^2 - 12\\ &= 3 \times 1 - 12\\ &= 3 - 12\\ &= -9\end{aligned}$$

2 Interpret the answer.

The gradient of the tangent to the curve $y = x^3 - 12x$ at $x = -1$ is -9.

(c) Determine the coordinates of the points at which $\frac{dy}{dx} = 0$.

1 Equate the derivative expression $\frac{dy}{dx}$ to zero.

$$3x^2 - 12 = 0$$

2 Solve for x to determine the value(s) of x such that $\frac{dy}{dx} = 0$.

$$\begin{aligned}3(x^2 - 4) &= 0\\ x^2 - 4 &= 0\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$

3 Substitute these x-values into y to find the corresponding y-values.

For $x = -2$:

$$\begin{aligned}y &= (-2)^3 - 12(-2)\\ &= -8 + 24\\ &= 16\end{aligned}$$

For $x = 2$:

$$\begin{aligned}y &= (2)^3 - 12(2)\\ &= 8 - 24\\ &= -16\end{aligned}$$

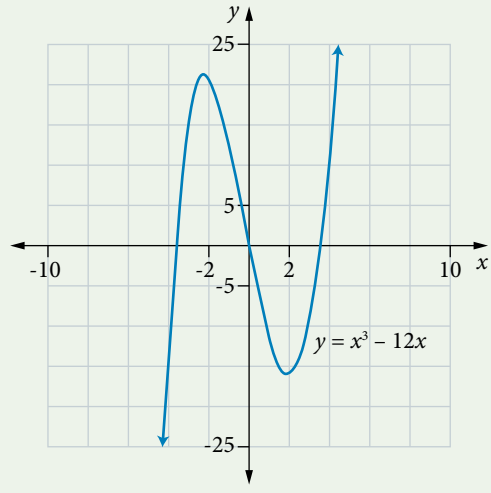
4 Interpret the answer.

The points at which $\frac{dy}{dx} = 0$ are (-2, 16) and (2, -16).

(d) Determine the subdomain for which $\frac{dy}{dx} > 0$.

METHOD 1

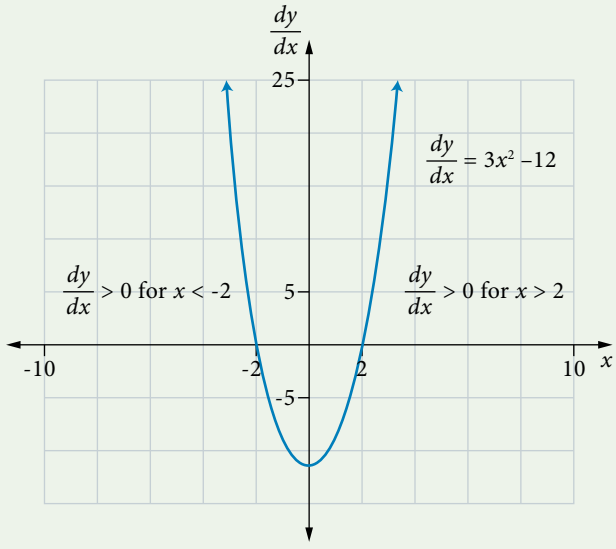
1 Sketch the graph.



2 $\frac{dy}{dx} > 0$ means that the gradient is positive.	From the graph, the gradient is positive for $x < -2$ and $x > 2$. Alternatively: $(-\infty, -2) \cup (2, \infty)$

METHOD 2

To find the set of values of x for which $\frac{dy}{dx} > 0$, sketch a graph of $\frac{dy}{dx}$ and note the values of x for which the graph is above the x-axis.



3 Determine gradient properties of a graph

Let $g(x) = \frac{x-3}{x}$, $x \neq 0$.

(a) Determine the gradient of the graph of g at $P(-1, 4)$.

THINKING	WORKING
1 Express $g(x)$ in a form to which the differentiation rules you have developed can be applied.	$g(x) = \frac{x-3}{x} = \frac{x}{x} - \frac{3}{x} = 1 - \frac{3}{x}$
2 Express terms involving x in the form ax^n.	$g(x) = 1 - 3x^{-1}$
3 Differentiate to find $g'(x)$.	$g'(x) = -1 \times -3x^{-1-1} = 3x^{-2} = \frac{3}{x^2}$
4 Substitute the given x-value into $g'(x)$.	$g'(-1) = \frac{3}{(-1)^2} = 3$
5 Interpret the answer.	The gradient of the graph of $g(x)$ at $P(-1, 4)$ is 3.

(b) Determine the coordinates of the points where the gradient is equal to $\frac{1}{3}$.

1 Create an equation.	$g'(x) = \frac{1}{3}$
2 Solve for x.	$\frac{3}{x^2} = \frac{1}{3}$ $x^2 = 9$ $x = \pm 3$
3 Substitute the given x-values into $g(x)$.	$g(-3) = \frac{-3-3}{-3} = \frac{-6}{-3} = 2$ $\qquad$ $g(3) = \frac{3-3}{3} = \frac{0}{3} = 0$
4 Interpret the answer.	The instantaneous gradient at the points (-3, 2) and (3, 0) is $\frac{1}{3}$.

Finding equations of tangents to graphs

A tangent to a graph at any point is a straight line that touches the graph at the point and has the same gradient as the graph at that point.

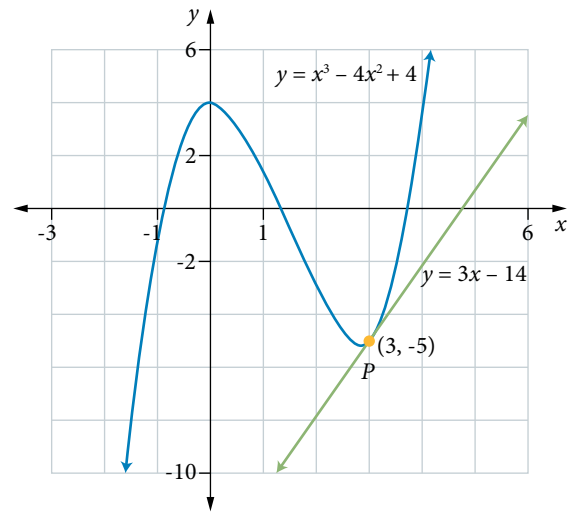
For example, the tangent to the graph of $y = x^3 - 4x^2 + 4$ at the point $P(3, -5)$ is drawn.

At $x = 3$, the gradient of the graph is 3.

The equation of the tangent at $x = 3$ is $y = 3x - 14$. Note that the gradient of the tangent is also equal to 3.

For a function f, the equation of the tangent to its graph at a point $x = a$ is found using the gradient of the function at a which is $f'(a)$, and the coordinates of the point $(a, f(a))$.

The equation of the tangent is given by the general line equation $y = mx + c$, where $m = f'(a)$ and $(x, y) \to (a, f(a))$.



4 Determine the equation of a tangent to a graph

Consider the graph of $y = x^2 - x - 1$.

Determine the equation of the tangent to the graph at $x = 3$.

THINKING	WORKING
1 Determine the equation of the derivative function $\frac{dy}{dx}$.	$y = x^2 - x - 1$ $\frac{dy}{dx} = 2x - 1$

2 Determine the value of $\frac{dy}{dx}$ at the given x-value.	For $x = 3$: $\frac{dy}{dx} = 2(3) - 1$ $= 6 - 1$ $= 5$
3 Determine the value of y at the given x-value.	For $x = 3$: $y = (3)^2 - 3 - 1$ $= 5$
4 Substitute the gradient and the coordinates of the point into $y = mx + c$ and solve for c.	Use $y = mx + c$ with (3, 5) and $m = 5$: $5 = 5(3) + c$ $c = 5 - 15$ $= -10$ $\therefore y = 5x - 10$
5 Interpret the answer.	The equation of the tangent to the graph at $x = 3$ is $y = 5x - 10$.

5 Determine the coordinates of a point of tangency

Determine the coordinates of the points on the graph of $y = 2x^3 - 3x^2$ at which the tangents are parallel to the line $y = 12x - 1$.

THINKING	WORKING
1 Determine $\frac{dy}{dx}$.	$y = 2x^3 - 3x^2$ $\frac{dy}{dx} = 6x^2 - 6x$
2 Substitute the known value of the gradient of the tangents.	For $y = 12x - 1$, $m = 12$ or $\frac{dy}{dx} = 12$
3 Equate the quadratic to zero and solve for x.	$6x^2 - 6x = 12$ $6x^2 - 6x - 12 = 0$ $6(x^2 - x - 2) = 0$ $6(x + 1)(x - 2) = 0$ $x + 1 = 0$ or $x - 2 = 0$ $x = -1$ or $x = 2$
4 Substitute the given x-values to determine the corresponding y-values.	For $x = -1$: $y = 2(-1)^3 - 3(-1)^2$ $= -2 - 3$ $= -5$ For $x = 2$: $y = 2(2)^3 - 3(2)^2$ $= 16 - 12$ $= 4$
5 Express as coordinates and interpret the answer.	The tangents at the points (-1, -5) and (2, 4) are parallel to the line $y = 12x - 1$.

Additional information

Parallel and perpendicular lines

This video explains how to find the tangent to a given parallel or perpendicular line.

EXERCISE 10.1 Gradients of functions and equations of tangents

Worked Example

1 Calculate $g'(2)$ and $g'(-1)$ for each of the following.

(a) $g(x) = 4x^2 + 3$ (b) $g(x) = -x^2 + 5x - 4$ (c) $g(x) = 4x - x^3$

2 Consider the function $y = x^2 - 5x$.

(a) The gradient of the curve at $x = 4$ is:

A -20 B -5 C -4 D 3

(b) Explain the common error made by a student who calculated $(4)^2 - 5 \times 4$.

3 For each cubic function choose the option that gives the subdomain where $\frac{dy}{dx} > 0$.

(a) $y = x^3 - 7x^2 + 7x + 17$

A $(-\infty, -1.08) \cup (4.81, \infty)$ B $(-\infty, 0.57) \cup (4.10, \infty)$

C $x:\{0.57 < x < 4.10\}$ D $x:\{-1.08 < x < 4.81\}$

(b) $y = -x^3 - 2x^2 + 6x + 7$

A $-2.23 < x < 0.90$ B $(-\infty, -2.23) \cup (0.90, \infty)$

C $(-\infty, -3.19) \cup (-1, 2.19)$ D $(-\infty, -3.19) \cup (2.19, \infty)$

4 For the curve $y = x^3 - 3x^2$, determine the:

(a) expression for $\frac{dy}{dx}$ (b) gradient of the curve at $x = -2$

(c) coordinates of the points at which $\frac{dy}{dx} = 0$ (d) the subdomain where $\frac{dy}{dx} > 0$.

5 Determine the coordinates of the points on the curve $h(x) = 2x^3 - 3x^2 - 12x - 1$, where the gradient is zero.

6 Calculate the gradient of the graph $y = x^2 - 7x + 12$ at the point where it crosses the y-axis.

7 Determine the values of x such that the gradient of the graph $y = 2(x + 3)^2$ is greater than -4.

8 Consider $f(x) = \frac{x - 4}{x}, x \neq 0$.

(a) Calculate the gradient of the graph of f at $P(2, -1)$.

(b) Determine the coordinates of the points where the gradient is equal to $\frac{1}{4}$.

9 Determine the coordinates of the point on the graph $y = 2\sqrt{x}$ where $\frac{dy}{dx} = \frac{1}{2}$.

10 Determine the equation of the tangent to the graph $y = x^2 - 4x - 5$ at $x = 2$.

11 Consider $f(x) = x^3 + 2x, x \in R$. Demonstrate that $f'(x) > 0$ for all values of x.

12 Determine the exact coordinates of the points where the curve with equation $y = x^3 - 2x + 1$ has a gradient equal to 7.

13 Consider $f(x) = ax^2 + bx$, where a and b are numbers. If $f(2) = 8$ and $f'(-1) = -8$, determine the values of a and b.

14 For the function $f(x) = x - 6 + \frac{8}{x}$, determine the coordinates of the points where:

(a) the graph crosses the x-axis (b) the gradient is equal to -1.

Worked Example 4

15 Determine the equation of the tangent to the graph:

(a) $y = 2x^2 + 4x + 3$ at $x = -3$ (b) $y = -x^3 + \frac{1}{2}x^2 - x + 3$ at $x = -2$

(c) $y = \frac{3}{x^2} - 12$ at the points where the curve crosses the x-axis.

5

16 Determine the equations of the tangents to the graph $y = x^3 - 3x^2 - 7x + 5$ that are parallel to the line $y = 2x - 1$.

17 Consider the function $g(x) = x^3 - 3x^2 + 4$.

(a) Determine the x-values such that:

(i) $g'(x) = 0$ (ii) $g'(x) < 0$

(b) A tangent perpendicular to the line $y = \frac{x}{3}$ touches the curve at point P. Determine the equation of the tangent to the curve $y = g(x)$ at P.

18 Consider $g(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are numbers.

(a) Calculate x such that $g'(x) = 0$.

(b) List the conditions for which the equation $g'(x) = 0$ has:

(i) exactly one solution for x (ii) two solutions for x (iii) no solutions for x.

19 Demonstrate that the graphs with equations $y = x^2 - 2x$ and $y = \frac{x}{2} - \frac{x^2}{4}$ intersect at right angles.

20 Consider the curve $y = ax^2 + bx + c$. The equations of the tangents at the x-intercepts are $y_1 = mx - 48$ and $y_2 = -mx - 80$ and meet at the point (-1, -64). Determine the rule of the function.

21 Consider the curve $y = x^2 - 6x + 5$.

(a) Determine the equations of the tangents to the curve at the points of intersection with the x-axis. Calculate the coordinates of the point of intersection of these tangents.

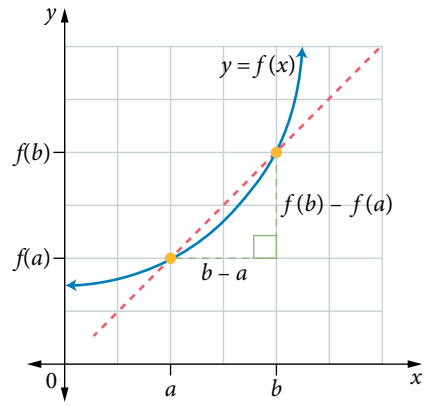
(b) Describe the relationship between the x-coordinate of the point of intersection of these tangents and the coordinates of the points where the curve crosses the x-axis.

22 The curve $y = x^2 - 8x + 15$ intersects the x-axis at points P and Q where Q is further from the origin than P. Determine the point on the curve where the gradient is the square of the gradient at Q.

10.2 Calculus and rates of change

Average rate of change

The average rate of change of a function is the rate at which the function changes from one end of an interval to the other, compared to the size of the interval. Graphically, the gradient of a secant line joining the points $(a, f(a))$ and $(b, f(b))$ on the curve $y = f(x)$ gives the average rate of change of f for the interval $[a, b]$.



Average rate of change

Given $y = f(x)$, the average rate of change of a function f for the interval $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$.

Instantaneous rate of change

A car that travels a distance of 80 km in 1 hour does so at an average speed of 80 km/h, which is read as a rate: the car travels on average 80 kilometres per hour of driving time. Average speed is defined as the average rate of change in distance with respect to time. For the purposes of this discussion, a car's speedometer gives the speed of the car at a particular instant, which is the car's instantaneous speed or instantaneous rate of change with respect to time. As the derivative of a function at a point is given by the limiting value of its average rate of change, the instantaneous rate of change is given by the gradient of the function at that point.

Instantaneous rate of change

Given $y = f(x)$, the (instantaneous) rate of change of a function, $f(x)$, at $x = a$ is $f'(a)$.

This is the derivative of $f(x)$ at $x = a$.

In Leibniz notation, this is the value of $\dfrac{dy}{dx}$ at $x = a$.

Note that the word 'instantaneous' is often omitted and the phrase 'rate of change' is used. When the independent variable is time, the phrase 'with respect to time' is often omitted and so 'rate of change' is used to mean 'instantaneous rate of change with respect to time'.

6 Find the average and instantaneous rate of change of a function

Consider the function $f(x) = x^2 + 6x + 8$.

(a) Determine the average rate of change of f for the interval $0 \le x \le 2$.

THINKING	WORKING
1 Determine the coordinates of the endpoints of the interval over which the average rate of change is to be calculated.	$f(2) = (2)^2 + 6(2) + 8 = 24$ $f(0) = (0)^2 + 6(0) + 8 = 8$

2 Substitute the values for a and b and calculate the average rate of change for the interval $a \le x \le b$:

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{24-8}{2} = 8$$

3 Interpret the answer.

The average rate of change (or the gradient of the secant joining the points (0, 8) and (2, 24)) of f over the interval $0 \le x \le 2$ is 8.

(b) Determine the instantaneous rate of change of f at $x = 2$.

1 Determine $f'(x)$.

$$f'(x) = 2x + 6$$

2 Substitute the given x-value into $f'(x)$ and calculate.

$$f'(2) = 2(2) + 6 = 10$$

3 Interpret the answer.

The instantaneous rate of change (or gradient of the tangent to the curve at $x = 2$) of f at $x = 2$ is 10.

7 Practical applications of average and an instantaneous rate of change

The volume $V(r)$ in cubic metres of a sphere of radius r metres is given by $V(r) = \frac{4}{3}\pi r^3$.

(a) Given that the radius increases from 2 m to 3 m, determine the average rate of change of the sphere's volume with respect to its radius.

THINKING	WORKING
1 Substitute the values for a and b and calculate the average rate of change for the interval $a \le x \le b$: $\frac{V(b)-V(a)}{b-a}$	$\frac{V(b)-V(a)}{b-a} = \frac{V(3)-V(2)}{3-2} = \frac{4\pi}{3}(3^3 - 2^3) = \frac{4\pi}{3}(19) = \frac{76\pi}{3}$
2 Interpret the answer and specify with correct units.	The average rate of change of the sphere's volume is $\frac{76\pi}{3}$ m^3/m, which indicates that the volume of the sphere increases by approximately 80 m^3 for every 1 m increase in radius.

(b) Determine the instantaneous rate of change of the sphere's volume when the radius is 3 m.

1 Determine $V'(r)$.

$$V(r) = \frac{4}{3}\pi r^3$$

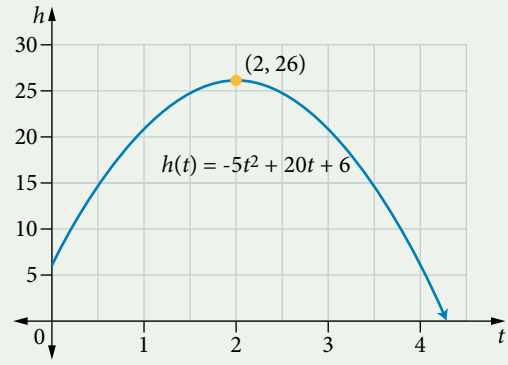
$$V'(r) = 4\pi r^2$$

2 Substitute the given r value into $V'(r)$ and calculate.	$V'(3) = 4\pi(3)^2$ $= 36\pi$
3 Interpret the answer and specify with correct units.	When the radius is 3 m, the sphere's volume is increasing at a rate of 36π m^3/m.

8 Applications of instantaneous rate of change

The height h metres above ground level of a ball thrown upwards at time t seconds, is given by $h = -5t^2 + 20t + 6$, where $t \geq 0$.

Calculate the time and height when the rate of change in height is zero.

THINKING	WORKING
1 Equate the derivative $\frac{dh}{dt}$ to zero.	$\frac{dh}{dt} = -10t + 20, \frac{dh}{dt} = 0$
2 Solve for t.	$-10t + 20 = 0$ $10t = 20$ $t = 2$
3 Substitute the calculated value of t into the formula for h.	$h(2) = -5(2^2) + 20(2) + 6$ $= -20 + 40 + 6$ $= 26$
4 Interpret the answer and specify with correct units.	The rate of change in height is zero after 2 seconds, when the ball reaches a height of 26 m. This is the maximum height reached by the ball.
5 Sketch the graph using technology to verify the answer.	

EXERCISE 10.2 Calculus and rates of change

Worked Example 6

1 Consider the function $f(x) = x^2 - 3x$.

(a) Determine the average rate of change of f for the interval $-2 \leq x \leq 2$.

(b) Determine the instantaneous rate of change of f at $x = 0$.

2 Consider the function $f(x) = x^3 - x + 1$.

(a) Determine the average rate of change of f for the interval $0 \le x \le 3$.

(b) Determine the instantaneous rate of change of f at $x = 2$.

3 Let $f(t) = t^2 + 5t - 2$.

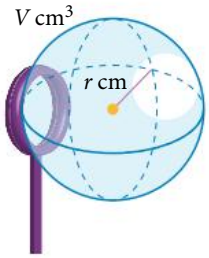
(a) The instantaneous rate of change of f at $t = 3$ is:

A 3 B 4 C 11 D 22

(b) Describe the common error made by a student who thought the answer was 8.

Worked Example 7

4 The volume V cm^3 of a spherical soap bubble of radius r cm is given by $V(r) = \frac{4}{3}\pi r^3$.



(a) Given that the radius increases from 1 cm to 2 cm, calculate the average rate of change of the soap bubble's volume with respect to its radius.

(b) Calculate the exact rate of change of the soap bubble's volume when the radius is 1 cm.

(c) Calculate, correct to 2 decimal places, the rate of change of the soap bubble's volume when the radius is 1.5 cm.

5 The temperature H°C in Mapleton for a 6-hour period on a single day is modelled by $H(t) = -t^2 + 8t + 24$, where t is measured in hours.

(a) Calculate the average rate of change of temperature for the interval $0 \le t \le 4$.

(b) Calculate the rate of change of temperature at $t = 1$.

6 The volume V L of water poured into a large container can be modelled by $V(t) = t^3 + 2t$, where t is measured in minutes.

(a) Calculate the average rate of change of volume for the first 5 minutes.

(b) Calculate the rate of change of volume at $t = 4$.

7 Consider the function $h(x) = 2x^3$.

(a) Calculate $h'(-1)$, the rate of change of h with respect to x at $x = -1$.

(b) Calculate $h'(1)$, the rate of change of h with respect to x at $x = 1$.

(c) Explain the results obtained for $h'(-1)$ and $h'(1)$.

(d) Write an equation relating $h'(x)$ and $h'(-x)$.

8 Given $f(x) = x^3 - \sqrt{x}, x \ge 0$, calculate:

(a) the average rate of change from $x = 0$ to $x = 4$ (b) the instantaneous rate of change at $x = 2$.

9 The height h m of a falling object at time t seconds from when it was released is given by $h = 20 - 5t^2$. 8

(a) Calculate how long it takes the object to reach the ground and determine the speed at impact.

(b) Determine the body's rate of change of height when $t = 1$ and interpret this result.

10 A spherical balloon is being inflated. Its volume V cm^3 when its radius is r cm is given by $V = \frac{4}{3}\pi r^3$. At time t seconds, the radius of the balloon is $3t$ cm. The rate of change of volume 2 seconds after inflation commenced, in cm^3/s is:

A 144π B 216π C 324π D 432π

11 The depth d cm of water in a bathtub t minutes after the bath is turned on is given by $d(t) = \frac{-60}{t} + 80, 1 \le t \le 10$.

(a) Determine the average rate of change of the depth of the water from the start of measuring, $t = 1$, until $t = 10$.

(b) Calculate the instantaneous rate of change of d when $t = 5$.

12 Consider the function $g(x) = -2x^2 + 5x - 3$.

(a) Calculate the average rate of change of g for the interval $2 \le x \le 4$.

(b) Determine, in terms of h, an expression for the average rate of change of g for the interval $2 \le x \le 2 + h$.

(c) Use limits to calculate the instantaneous rate of change of g at $x = 2$.

(d) Determine the instantaneous rate of change of g at $x = 4$.

13 An outlet at the bottom of a tank full of water is opened to allow the tank to empty. The volume, V litres, of water in the tank at time t minutes is given by $V(t) = 8(10 - t)^2$.

(a) Calculate the initial rate at which water flows out of the tank.

(b) Determine the time when the rate at which water flows out of the tank is a maximum.

14 A cylinder of height 15 cm has radius r cm.

(a) Determine an expression for the volume V cm^3 of the cylinder in terms of the radius r.

(b) Calculate the average rate of increase of volume as r changes from 3 cm to 5 cm.

(c) Calculate the instantaneous rate of increase in volume when $r = 2$ cm.

(d) Does the volume of the cylinder increase more rapidly as r increases?

15 The volume V cm^3 of a spherical balloon of radius, r cm, is given by $V = \frac{4}{3}\pi r^3$.

(a) Calculate, correct to 4 decimal places, the average rate of change of r with respect to V for the following intervals.

(i) $0 \le V \le 500$ **(ii)** $500 \le V \le 1000$ **(iii)** $1000 \le V \le 1500$

(b) Is it correct to say that a spherical balloon's radius increases more quickly with respect to volume at the start and then more slowly as further air is blown into it?

(c) Calculate, correct to 4 decimal places, the instantaneous rate of change of r with respect to V when $V = 1000$.

16 Water is flowing from a tank so that the volume of water v m^3 in the tank after time t hours is given by $v(t) = 2t^2 - 10t + 8$.

(a) Determine an expression for the average rate of change of volume in m^3/h for the period $a \le t \le 2a$ and determine the values of a for which the expression is valid.

(b) Calculate the average rate of change for $0.01 \le t \le 0.02$.

(c) Determine whether the value found in part **(b)** is a good approximation for the instantaneous rate as the water commences to leave the tank.

(d) Use average rate of change to improve the approximation found in part **(b)**. Comment on the improvement.

Curve sketching

10.3

Curve sketching using technology

Graphs of polynomial functions can be drawn on your technology, showing the key features such as axes intercepts and asymptotes. All maximum and minimum turning points and points of inflection should be labelled on the graph to the required number of decimal places. Choose a suitable scale so that the shape is clear, including the trends at the extremes.

9 Use technology to sketch the graph of a function

Sketch the graph of $y = -2x^3 - 6x^2 + x - 9$, showing the axes intercepts and turning points correct to 2 decimal places.

THINKING

Draw the graph, adjusting the window so that turning points, intercepts and the overall trend are clear.

Note the trend at the extremes using arrows.

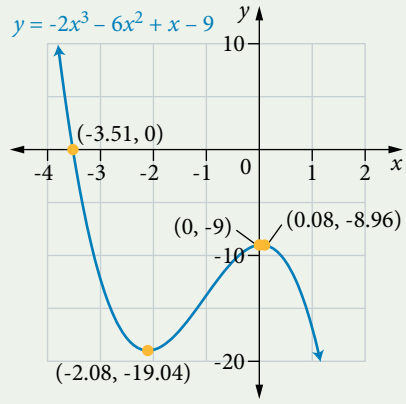
Ensure that you label the y-intercept and roots on the sketch.

Label the coordinates of the minimum and maximum turning points.

WORKING

As $x \to -\infty$, $y \to \infty$.

As $x \to \infty$, $y \to -\infty$.



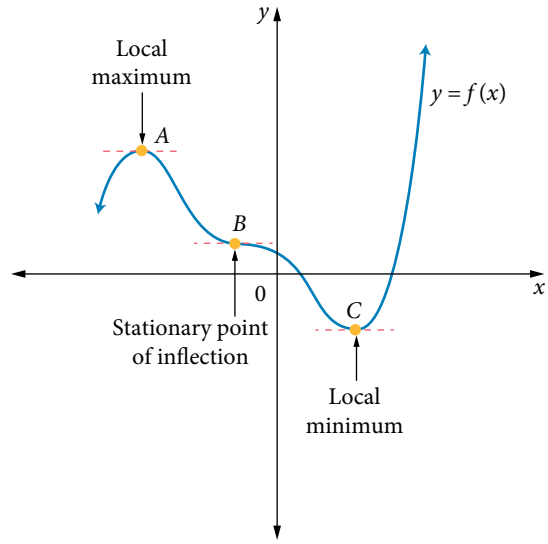
Note that in the previous worked example, as is generally the case, the turning points do not coincide with any axes intercepts. As you learn to identify key features of graphs from your knowledge of calculus, confirm your findings with technology whenever you can.

Stationary points

A stationary point has zero gradient. Turning points and points of inflection that have a gradient of zero are stationary points. Calculus can be used to determine at which points a function is increasing or decreasing, which allows you to determine the set of stationary points for a graph.

Consider the function $y = f(x)$. The point $P(a, f(a))$ on the graph of f is classified as a *stationary point* if the gradient is zero, $f'(a) = 0$. At P, the graph is neither rising nor falling so it is referred to as a stationary point.

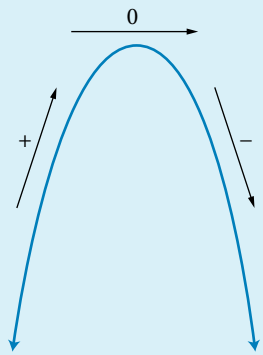
The graph of $y = f(x)$ shown illustrates three types of stationary points.



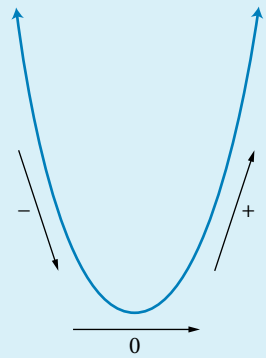
The gradient is zero at points $A(a, f(a))$, $B(b, f(b))$ and $C(c, f(c))$. A tangent to the graph of $y = f(x)$ at points A, B and C has a gradient of zero and is parallel to the x-axis. At these points $f'(x) = 0$.

- The point A is a *local maximum* because the gradient changes from positive to the left of A to negative to the right of A. The graph *turns* through A.
- The point C is a *local minimum* because the gradient changes from negative to the left of C to positive to the right of C. The graph turns through C.
- The point B is a *stationary point of inflection*. The gradient does not change sign and the graph does not turn through B.
- Points A, B and C are stationary points but only points A and C are turning points.

If f' changes sign from positive to negative, then the point is a local maximum and the graph is said to be concave down.



If f' changes sign from negative to positive, then the point is a local minimum and the graph is said to be concave up.



Sketching graphs of polynomial functions

When sketching of polynomial functions, use calculus or your technology to identify the key features:

- axes intercepts
- stationary points and their nature
- intervals for which the function either increases or decreases.

Evaluating a gradient, increasing and decreasing at a point:

- If $f'(a) > 0$, then the gradient is positive at $x = a$ in the domain of a function and the function is increasing at that point.
- If $f'(a) < 0$, then the gradient is negative at $x = a$ in the domain of a function and the function is decreasing at that point.
- If $f'(a) = 0$, then the gradient is zero at $x = a$ in the domain of a function and is a stationary point.

Additional information

Local maximum and local minimum points

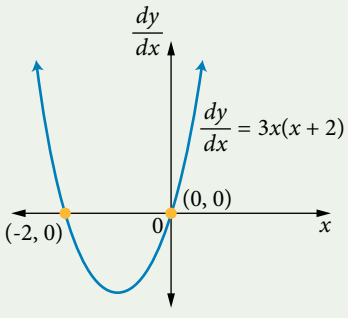
The two videos review the concept of local maximum and local minimum. Select the required video by first clicking on Local Maximum or Local Minimum.

10 Sketch the graph of a cubic function

Consider the function $f(x) = x^3 + 3x^2 - 4$.

(a) Use calculus to locate any stationary points and determine their nature.

THINKING	WORKING
1 Determine $f'(x)$.	$f(x) = x^3 + 3x^2 - 4$ $f'(x) = 3x^2 + 6x$
2 Determine the x-coordinate of the turning point(s) by letting $f'(x) = 0$ and solving for x.	$3x^2 + 6x = 0$ $3x(x + 2) = 0$ $f'(x) = 0$ where $x = 0$ or $x = -2$.
3 Determine the coordinates of each stationary point.	$f(-2) = (-2)^3 + 3(-2)^2 - 4$ $f(-2) = 0$ $f(0) = -4$ The coordinates of the stationary points are (-2, 0) and (0, -4).
4 Classify each stationary point. f is a positive cubic function, so the first stationary point will be a maximum and the second one a minimum. Consider the sign of the gradient function near each stationary point.	(-2, 0) is a local maximum. (0, -4) is a local minimum.



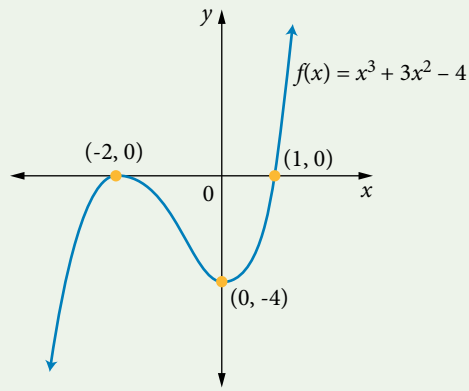
	$x < -2$	$x = -2$	$x > -2$
$f'(x)$	$f'(x) > 0$	$f'(x) = 0$	$f'(x) < 0$
Effect	+	0	–

A local maximum is at $x = -2$.

	$x < 0$	$x = 0$	$x > 0$
$f'(x)$	$f'(x) < 0$	$f'(x) = 0$	$f'(x) > 0$
Effect	–	0	+

A local minimum is at $x = 0$.

(b) Sketch the graph of $y = f(x)$, showing all key features.

1 Determine the y-intercept.	The coordinates of the y-intercept are (0, -4).
2 Factorise the cubic equation using the factor theorem and re-expression or polynomial division.	$f(-2) = 0$ $\therefore (x + 2)$ is a factor. $x^3 + 3x^2 - 4$ $= (x + 2)(x^2 + x - 2)$ $= (x + 2)(x - 1)(x + 2)$ $= (x - 1)(x + 2)^2$
3 Determine the coordinates of the x-intercept(s) by letting $f(x) = 0$ and solving for x.	$(x - 1)(x + 2)^2 = 0$ $x - 1 = 0, x + 2 = 0$ $x = 1, \quad x = -2$ The coordinates of the x-intercepts are (-2, 0) and (1, 0).
4 Comment on the trend at the extremes.	This is a positive cubic: as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$
5 Sketch the graph of $y = f(x)$, showing all key features.	

11 Sketch the graph of a cubic function with one stationary point

Consider the function $f(x) = x^3 - 6x^2 + 12x - 9$.

(a) Use calculus to locate the coordinates of any stationary points and determine their nature.

THINKING	WORKING
1 Determine $f'(x)$.	$f(x) = x^3 - 6x^2 + 12x - 9$ $f'(x) = 3x^2 - 12x + 12$

2 To determine the x-coordinate(s) of the stationary points, find the value(s) of x such that $f'(x) = 0$.

$$3x^2 - 12x + 12 = 0$$
$$3(x^2 - 4x + 4) = 0$$
$$3(x - 2)^2 = 0$$
$$x - 2 = 0$$
$$x = 2$$

A stationary point exists at $x = 2$.

3 Determine the coordinates of the stationary point.

$$f(2) = (2)^3 - 6(2)^2 + 12(2) - 9$$
$$= 8 - 24 + 24 - 9$$
$$= -1$$

The coordinates of the stationary point are (2, -1).

4 Classify the stationary point.

	$x < 2$	$x = 2$	$x > 2$
$f'(x)$	$f'(x) > 0$	$f'(x) = 0$	$f'(x) > 0$
Effect	+	0	+

The gradient does not change sign.

Therefore a stationary point of inflection occurs at $x = 2$.

(b) Sketch the graph of $y = f(x)$, showing all key features.

1 Determine the y-intercept.

The coordinates of the y-intercept are (0, -9).

2 To determine the x-intercept(s), find the value(s) of x such that $f(x) = 0$.

Calculate the discriminant of the quadratic factor Δ to determine whether there are any other roots.

$f(3) = 0$ therefore $(x - 3)$ is a factor.

$x^3 - 6x^2 + 12x - 9 = (x - 3)(x^2 - 3x + 3)$

$\Delta = -3$ so the quadratic factor has no real roots.

$(x - 3)(x^2 - 3x + 3) = 0$ at $x = 3$.

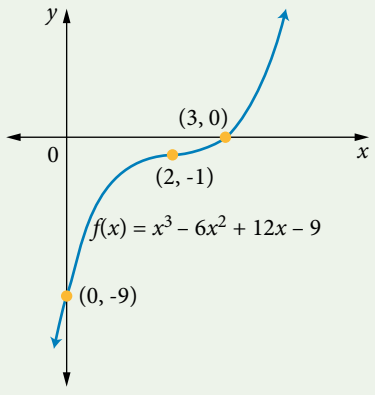
3 Determine the coordinates of the x-intercept.

(3, 0) is the only x-intercept.

4 Comment on the trend at the extremes.

This is a positive cubic:
as $x \to -\infty$, $y \to -\infty$;
as $x \to \infty$, $y \to \infty$

5 Sketch the graph of $y = f(x)$, showing all key features.



To sketch a function, determine:

- the coordinates of the y-intercept ($x = 0$) and x-intercept ($y = 0$)
- the equation of the derivative $f'(x)$
- where $f'(x) = 0$ and then find the coordinates of the stationary points
- where $f'(x) < 0$ and $f'(x) > 0$ to determine the nature of the stationary points
- the equations of asymptotes, coordinates of endpoints and the behaviour of extremities.

Restricted domain

Sometimes a function is specified on a subset of $\mathbb{R}$. In modelling contexts with a specified domain, the graph must indicate the *endpoints* corresponding to the interval over which the function is defined. In these circumstances, the maximum and/or minimum value of a function may occur at the endpoints of an interval.

12 Maximum and minimum values of a function defined over a specified interval

Consider the function $h(x) = x^3 + ax^2 + bx - 2$, where a and b are real constants.

The graph of $y = h(x)$ has a stationary point at (1, 2).

(a) Determine the values of a and b.

THINKING	WORKING
1 Form two equations involving a and b using the coordinates (x, y) and the gradient of the stationary point $h'(x) = 0$.	$h(x) = x^3 + ax^2 + bx - 2$ $h(1) = 2$: $1 + a + b - 2 = 2$ $a + b = 3$ [1] $h'(x) = 3x^2 + 2ax + b$ $h'(1) = 0$: $3 + 2a + b = 0$ $2a + b = -3$ [2]
2 Solve for a and b.	[2] – [1] gives $a = -6$ Substitute $a = -6$ into [1] and solve for b. $-6 + b = 3$ $b = 9$
3 Determine $h(x)$.	$h(x) = x^3 - 6x^2 + 9x - 2$.

(b) Determine the coordinates of all stationary points and determine their nature.

1. Determine $h'(x)$.

$h'(x) = 3x^2 - 12x + 9$

2. To determine the x-coordinate of each stationary point, find the value(s) of x such that $h'(x) = 0$.

$3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x - 1)(x - 3) = 0$

There are stationary points at $x = 1$ and $x = 3$.

3. Determine the coordinates of the stationary points.

$h(1) = (1)^3 - 6(1)^2 + 9(1) - 2$
$= 1 - 6 + 9 - 2$
$= 2$

$h(3) = (3)^3 - 6(3)^2 + 9(3) - 2$
$= 27 - 54 + 27 - 2$
$= -2$

The coordinates of the stationary points are (1, 2) and (3, -2).

4. Classify each stationary point by considering the sign of the gradient near each stationary point.

Alternatively, h is a cubic function with a positive leading coefficient, so (1, 2) is a local maximum and (3, -2) is a local minimum.

	$x < 1$	$x = 1$	$x > 1$
$h'(x)$	$h'(x) > 0$	$h'(x) = 0$	$h'(x) < 0$
Effect	+	0	−

A local maximum occurs at (1, 2).

	$x < 3$	$x = 3$	$x > 3$
$h'(x)$	$h'(x) < 0$	$h'(x) = 0$	$h'(x) > 0$
Effect	−	0	+

A local minimum occurs at (3, -2).

(c) Sketch the graph of h on the interval $0 \le x \le 5$.

1. Determine the y-intercept.

The coordinates of the y-intercept are (0, -2).

2. To find the x-intercept(s), find the value(s) of x such that $h(x) = 0$.

$h(2) = 0$ and $(x - 2)$ is a factor.

$0 = x^3 - 6x^2 + 9x - 2$
$= (x - 2)(x^2 - 4x + 1)$
$= (x - 2)\left(2 + \sqrt{3}\right)\left(2 - \sqrt{3}\right)$

$x = 2$ or $x = 2 \pm \sqrt{3}$

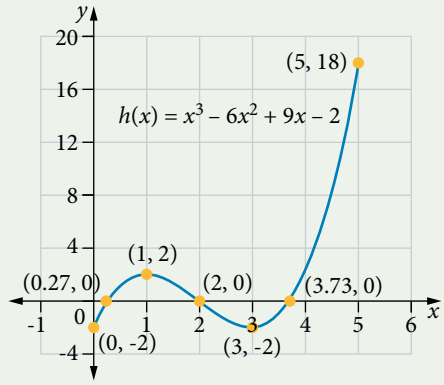
3. Express the coordinates of the last two x-intercepts correct to 2 decimal places.

The coordinates of the x-intercepts are (0.27, 0), (2, 0) and (3.73, 0) correct to 2 decimal places.

4. Calculate $h(x)$ at the endpoints of the interval.

$h(0) = -2$ and $h(5) = 18$

5 Sketch the graph of $y = h(x)$ on the interval $0 \le x \le 5$ showing all key features.



(d) Determine the maximum and minimum values of h on the interval $0 \le x \le 5$ and sketch these points on a graph.

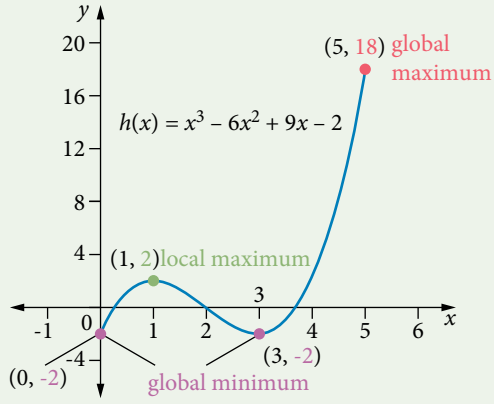
1 Determine the maximum value of h for the given interval.

For $0 \le x \le 5$, an endpoint maximum is at (5, 18).

2 Determine the minimum value of h for the given interval.

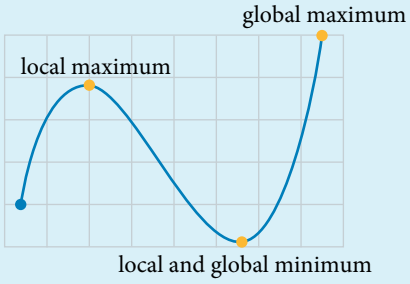
For $0 \le x \le 5$, a local minimum is at (3, -2) and an endpoint minimum is at (0, -2).

3 Sketch the minimum and maximum values.

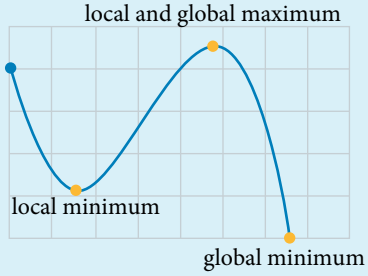


Note that no cubic polynomial function has a global maximum or minimum value over domain $\mathbb{R}$ because the extreme values approach positive or negative infinity. A global maximum or minimum value will only exist for a restricted domain $[a, b]$.

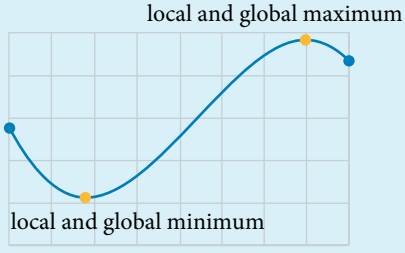
Global minimum is a local minimum and global maximum is the right endpoint.



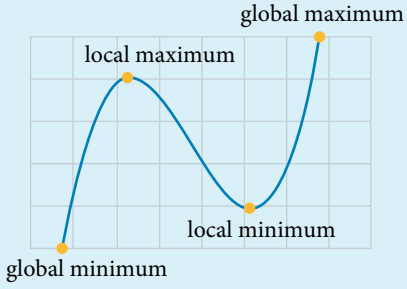
Global minimum is at the right endpoint and global maximum is a local maximum.



Global minimum is at the local minimum and global maximum is at the local maximum.



Global minimum is at the left endpoint and global maximum is at the right endpoint.



EXERCISE 10.3 Curve sketching

Worked Example 9

1 Use technology to sketch the graph of $y = x^3 - x^2 - 5x + 2$. Label axes intercepts, turning points and trends at the extremes. Round values correct to 2 decimal places, where necessary.

2 Consider the function $f(x) = -(x - a)(x - b)(x - c)$, where $a < b < c$.

(a) Identify the true statement about the turning points on the graph of this function.

A Turning points are located at $x = a$, $x = b$ and $x = c$.

B Maximum is located at $x = \frac{a+b}{2}$, minimum is located at $x = \frac{b+c}{2}$.

C There are no turning points.

D Minimum is in the subdomain $a < x < b$ and maximum is within $b < x < c$.

(b) Explain the common error made by a student who thought the minimum was at $x = \frac{a+b}{2}$ and the maximum was at $x = \frac{b+c}{2}$.

3 A polynomial function has the form $y = ax^4 + bx^3 + cx^2 + dx + e$, where $a > 0$.

Which of the following is not possible for this function?

A There is a single minimum turning point.

B There is a minimum, then a point of horizontal inflection.

C There are two points of horizontal inflection.

D There is a minimum, a maximum, and then another minimum.

4 The graph of $y = -x^3 - 5x^2 + cx + 9$ has a local minimum at $x = -3$. Determine the value of c.

5 Sketch the graph of $y = f(x)$, given that $f(2) = 2$, $f'(2) = 0$, $f(0) = 4$, $f'(0) = 0$, $f'(x) > 0$ for $x < 0$ and $x > 2$, and $f'(x) < 0$ for $0 < x < 2$.

6 Consider the function $f(x) = x^3 + 4x - 1$.

(a) Determine the expression for $f'(x)$.

(b) Hence explain why the graph of $y = f(x)$ has no stationary points.

7 Let $y = (x + 2)(x - 1)^2$, $-2.5 \le x \le 2$.

(a) Sketch the graph of y, clearly labelling all key features such as axes intercepts, stationary points and endpoints with their coordinates.

(b) Determine the maximum and minimum values of y over the domain $-2 \le x \le 3$.

Worked Example

10

8 For each of the following functions:

(i) use calculus to locate the coordinates of any stationary points and determine their nature

(ii) sketch the graph, showing all key features.

(a) $f(x) = x^3 - 3x^2 + 4$

(b) $f(x) = 2x^3 - 6x^2$

(c) $f(x) = -x^3 + 12x$

(d) $g(x) = -x^3 + 5x^2 - 8x + 4$

11

9 For each of the following functions:

(i) use calculus to locate the coordinates of any stationary points and determine their nature

(ii) sketch the graph, showing all key features.

(a) $f(x) = x^3 + 6x^2 + 12x + 7$

(b) $f(x) = -3x^3 - 18x^2 - 36x - 23$

(c) $h(x) = x^4 - 4x^3$

(d) $h(x) = x^2(9 - x^2)$

12

10 Let $y = -2x^3 + 3x^2 + 12x - 15$ for $-3 \le x \le 3$.

(a) Use calculus to locate the coordinates of any stationary points and determine their nature.

(b) Sketch the graph of $y = -2x^3 + 3x^2 + 12x - 15$ for $-3 \le x \le 3$, showing all key features, including any x-intercepts correct to 2 decimal places.

(c) Determine the global maximum and minimum values of y for $-3 \le x \le 3$.

11 Consider the function $f(x) = x^3 + ax^2 + bx + 18$ where a and b are real constants. The graph of $y = f(x)$ has a stationary point at (3, 0).

(a) Determine the values of a and b.

(b) Determine the coordinates of all stationary points and determine their nature.

(c) Sketch the graph of $y = f(x)$, showing all key features.

12 Consider the function $f(x) = x^3 - 5x^2 - 8x + 12$.

(a) Demonstrate that $(x + 2)$ is a factor of $f(x)$ and hence find the other factors.

(b) Determine the coordinates of the points where the graph of $y = f(x)$ meets the coordinate axes.

(c) Use calculus to locate the coordinates of any stationary points and determine their nature.

(d) Sketch the graph of $y = f(x)$, showing all key features.

(e) Determine the set of values of x such that $f'(x) < 0$.

13 A curve has equation $y = x^4 - 8x^2 + 60x + 7$.

(a) Determine the expression for $\frac{dy}{dx}$.

(b) Demonstrate that the x-coordinate of any stationary point of the curve satisfies the equation $x^3 - 4x + 15 = 0$.

(c) Use the results above to show that the only stationary point on the curve occurs when $x = -3$. Determine the nature of the turning point.

(d) Sketch the curve showing all relevant features.

14 Consider the function $f(x) = ax^2 + bx + c$ where a, b and c are real numbers.

(a) Determine in terms of a, b and c, the coordinates of the stationary point.

(b) Determine the condition for which the stationary point is a local minimum.

(c) If the curve $y = f(x)$ has two distinct x-intercepts, show that the midpoint of these intercepts gives the x-coordinate of the turning point.

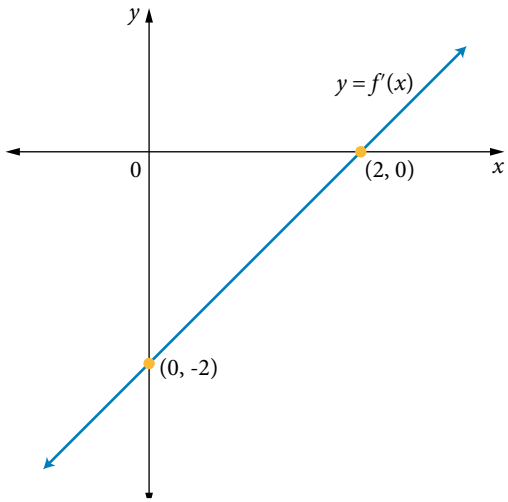
15 Consider a function of the form $f(x) = ax^3 + bx^2 + cx + d$. A graph of this function cuts the y-axis at (0, -1) and the x-axis at (1, 0), with stationary points at $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{3}{2}, -1\right)$.

(a) Determine the rule for this function.

(b) Sketch the graph of $y = f(x)$, showing all key features. Give your answers correct to 2 decimal places where necessary.

(c) Determine the set of values of x such that $f'(x) > 0$.

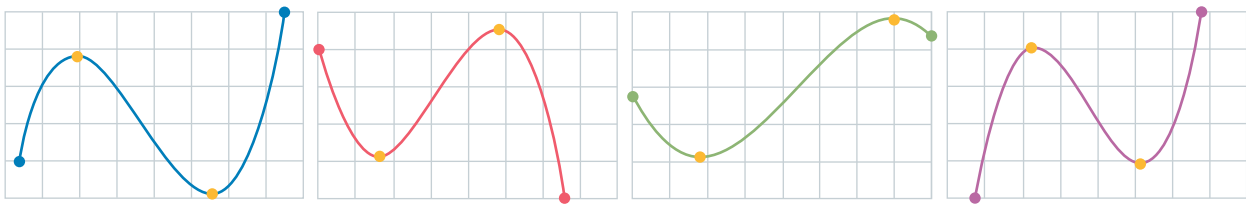
16 The graph of $y = f'(x)$ is shown. Determine the rule and sketch the graph of $f(x)$ if $f(0) = -3$.



10.4 Maximum and minimum problems

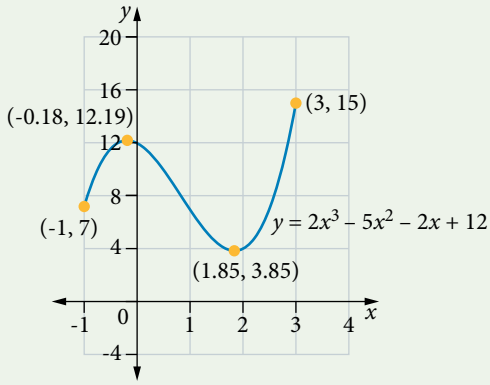
Global maxima and minima within a domain

A maximum or minimum value in a restricted domain of a polynomial function occurs at either a turning point or an endpoint (or both).



13 Use technology to determine global maxima and minima

For the cubic function $y = 2x^3 - 5x^2 - 2x + 12$, use technology to determine the maximum and minimum values for $-1 \le x \le 3$.

THINKING	WORKING
1 Sketch the function in the given domain. Identify the coordinates of any maximum or minimum turning points as well as the endpoints of the domain.	$y = 2x^3 - 5x^2 - 2x + 12$ for $-1 \le x \le 3$ 
2 Interpret the graph.	The maximum value of the function is 15 and occurs at $x = 3$. The minimum value of the function is 3.85 and occurs at $x = 1.85$.

When technology is not available, you can simplify the task by using the fact that maxima and minima values for polynomial functions occur at turning points and/or endpoints.

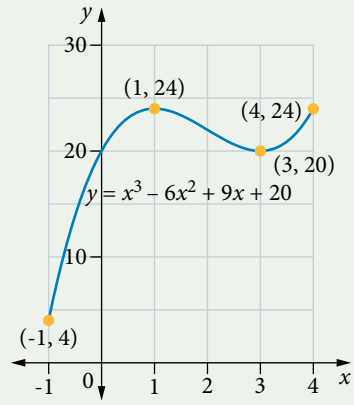
14 Determine the global maxima and minima

For the cubic function $y = x^3 - 6x^2 + 9x + 20$, determine the maximum and minimum values for $-1 \le x \le 4$.

THINKING	WORKING
1 Differentiate the function.	$y = x^3 - 6x^2 + 9x + 20$ $\frac{dy}{dx} = 3x^2 - 12x + 9$
2 Solve for x where $\frac{dy}{dx} = 0$.	$3x^2 - 12x + 9 = 0$ $3(x^2 - 4x + 3) = 0$ $3(x - 1)(x - 3) = 0$ The stationary points occur at $x = 1$ and $x = 3$.
3 Determine the coordinates of the stationary points.	For $x = 1$: $y = (1)^3 - 6(1)^2 + 9(1) + 20$ $= 1 - 6 + 9 + 20$ $= 24$ For $x = 3$: $y = (3)^3 - 6(3)^2 + 9(3) + 20$ $= 27 - 54 + 27 + 20$ $= 20$ The coordinates of the stationary points are (1, 24) and (3, 20).
4 Determine the values of the function at the endpoints and stationary points.	For $x = -1$: $y = (-1)^3 - 6(-1)^2 + 9(-1) + 20$ $= -1 - 6 - 9 + 20$ $= 4$ For $x = 4$: $y = (4)^3 - 6(4)^2 + 9(4) + 20$ $= 64 - 96 + 36 + 20$ $= 24$ The coordinates of the endpoints are (-1, 4) and (4, 24).
5 Write the values of the global maximum and minimum and where they occur.	The maximum value is 24 for $x = 1$ and $x = 4$. The minimum value is 4 for $x = -1$.

6 Verify your answer with a sketch of the function in the given domain, using the points found earlier.

$y = x^3 - 6x^2 + 9x + 20$ for $-1 \le x \le 4$



Solving problems

Many problems in mathematics involve finding the maximum or minimum value of a function *f* over a specified domain. Examples include minimising costs or surface area of packaging and maximising profits or volumes of products. The development of differential calculus in the 17th century was motivated largely by the need to solve *maximum* and *minimum problems*.

Solving these problems involves determining the value of the independent variable for which a maximum or minimum value occurs and then determining the maximum or minimum value of a quantity (dependent variable).

Steps in solving maximum and minimum problems

To solve a maximum/minimum problem:

STEP 1 Define the dependent and independent variables. Draw a labelled diagram.

STEP 2 Express the quantity to be maximised or minimised as a function in terms of a single independent variable. The domain of the function used to model the situation will need to be clearly identified.

STEP 3 Determine the derivative of the function with respect to the independent variable. Equate the derivative to zero and find the value(s) of the independent variable such that the derivative is zero.

STEP 4 Determine whether these value(s) give maximum or minimum values of the function.

STEP 5 Substitute the value into the function and state the maximum or minimum value of the function.

STEP 6 Check whether or not the maximum or minimum value of the function occurs at the endpoints of the modelling function's domain.

15 Express a quantity in terms of a single variable

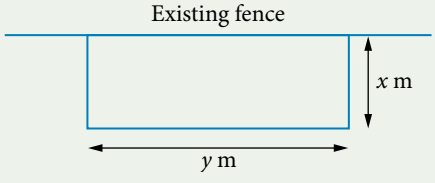
For a rectangle of area 50 m^2, express the perimeter in terms of the length.

THINKING	WORKING
1 Define the variables for the required formulas. $A = lw$ $P = 2l + 2w$	A m^2 is the area of the rectangle. P m is the perimeter. l m the length. w m is the width.
3 Express the variable that is not required in terms of another variable.	$A = lw = 50$ $w = \frac{50}{l}$
4 Substitute for the variable that is not required.	$P = 2l + 2w$ $= 2l + 2 \times \frac{50}{l}$ $= 2l + \frac{100}{l}$
5 Interpret the answer.	The perimeter in terms of the length is $P = 2l + \frac{100}{l}$.

16 Solve a maximum/minimum problem

Novak has 80 m of wire to fence three sides of a rectangular backyard. The fourth side is an existing fence completing the rectangular backyard.

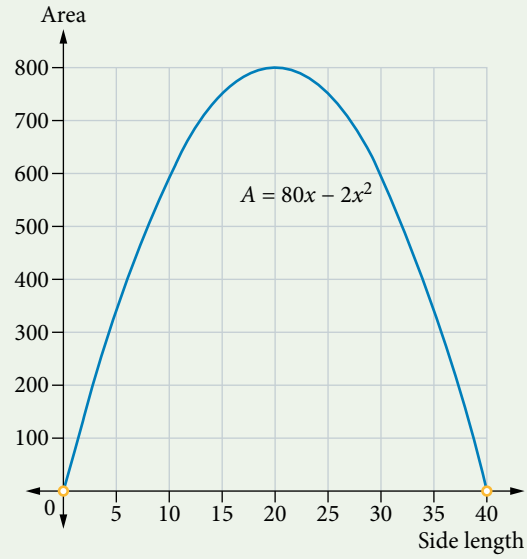
Use calculus to determine the maximum area of the backyard and state its dimensions.

THINKING	WORKING
1 Define the dependent and independent variables and draw a labelled diagram. Define the variables for the required formulas. $A = lw$ $P = 2l + 2w$	Area of the rectangular backyard is A m^2. Side lengths are x m and y m. 
2 Express A as a function of x. Write an equation using the key information provided in the question.	$A = xy$ The total length of the fence in metres is $(2x + y)$. 80 m of wire: $P = 2x + y = 80$
3 Then rearrange the equation to express one variable in terms of the other.	$y = 80 - 2x$ Substituting $y = 80 - 2x$ into $A = xy$: $A = x(80 - 2x)$ $= 80x - 2x^2$

4 Determine the domain of A.

Drawing a graph can often be helpful here.

$x > 0$ and $y > 0$ and $80 - 2x > 0$



$$\begin{aligned} 80 - 2x &> 0 \\ 2x &< 80 \\ x &< 40 \end{aligned}$$

The domain of A is $0 < x < 40$.

5 Let $\dfrac{dA}{dx} = 0$ and solve for x.

$$\frac{dA}{dx} = 80 - 4x$$

$$\begin{aligned} 80 - 4x &= 0 \\ 4x &= 80 \\ x &= 20 \end{aligned}$$

6 Determine whether the given x-value gives the maximum value of A.

The function is not defined at the endpoints of the domain. Both endpoints correspond to an area of zero.

The graph of $A = 80 - 2x^2$ is an inverted parabola, so the stationary point at $x = 20$ is a local maximum.

A is a maximum for $x = 20$.

7 Substitute the given x-value into A and state the maximum value of A.

$A = 20(80 - 40) = 800$

For $x = 20$:

$y = 80 - 2(20) = 40$

Alternatively, completing the square:

$A = -2(x - 20)^2 + 800$

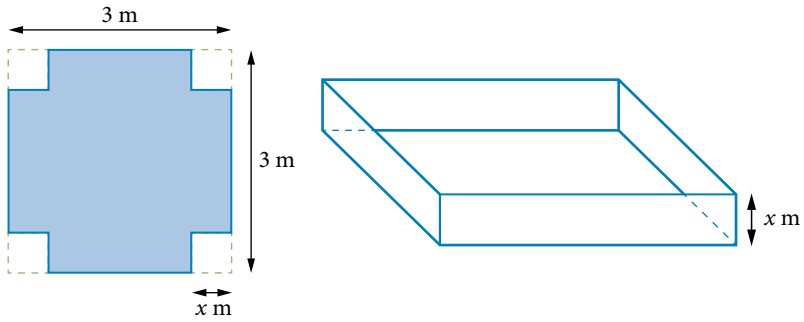
This shows that the maximum value of A is 800 for $x = 20$.

8 Interpret the answer.

To maximise the area of the backyard, the dimensions are 20 m by 40 m and the area is 800 m^2, assuming there exists 40 m of existing fence.

17 Solve a maximum/minimum problem

A square sheet of cardboard has side length of 3 m. A square of side length x m is cut from each corner and the sides are turned up to form an open-topped rectangular box with a square base. Let V cubic metres be the volume of the box.



(a) Express V in terms of x.

THINKING	WORKING
Express V as a function of x.	Height of the box is x m.
$V = lwh$	Side length of square base of box is $(3 - 2x)$.
	$l = 3 - 2x$, $w = 3 - 2x$, $h = x$
	$V = lwh$
	$= x(3 - 2x)^2$

(b) Demonstrate that $V = 4x^3 - 12x^2 + 9x$.

Expand V.	$V = x(3 - 2x)^2$
	$= x(9 - 12x + 4x^2)$
	$= 4x^3 - 12x^2 + 9x$

(c) Determine the domain of V.

Calculate the domain of V.	It is required that $l > 0$, $w > 0$ and $h > 0$.
	$\therefore x > 0$ and $3 - 2x > 0$
	$-2x > -3$
	$x < \frac{3}{2}$
	The domain is $0 < x < \frac{3}{2}$.

(d) Use calculus to determine the value of x for which the box has a maximum volume. Hence state the maximum volume and the dimensions of the box in this case.

1 Determine $\frac{dV}{dx}$, let $\frac{dV}{dx} = 0$ and solve for x.

$$\begin{aligned}\frac{dV}{dx} &= 12x^2 - 24x + 9\\ 0 &= 3(4x^2 - 8x + 3)\\ &= 3(4x^2 - 6x - 2x + 3)\\ &= 3[(4x^2 - 6x) - (2x - 3)]\\ &= 3[2x(2x - 3) - 1(2x - 3)]\\ &= 3(2x - 3)(2x - 1)\end{aligned}$$

$$\begin{aligned}2x - 3 &= 0 & \qquad 2x - 1 &= 0\\ 2x &= 3 & \qquad 2x &= 1\\ x &= \frac{3}{2} & \qquad x &= \frac{1}{2}\end{aligned}$$

$$\therefore x = \frac{1}{2}\left(0 < x < \frac{3}{2}\right)$$

2 Determine whether the given x-value gives the maximum value of V.

The function is not defined at the endpoints of the domain. Both endpoints correspond to a volume of zero.

For $0 < x < \frac{1}{2}$, $\frac{dV}{dx} > 0$

For $\frac{1}{2} < x < \frac{3}{2}$, $\frac{dV}{dx} < 0$

There is a local maximum at $x = \frac{1}{2}$.

3 Substitute the given x-value into V and state the maximum value of V.

For $x = \frac{1}{2}$, $V = 2$

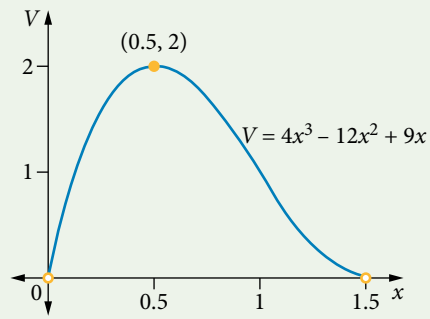
4 Determine the dimensions and volume of the box.

The maximum volume is 2 m^3.

The square base has side length 2 m and the height of the box is $\frac{1}{2}$ m.

(e) Sketch the graph of V against x, showing all key features.

Sketch the graph of V against x over a suitable domain.



The domain $0 < x < \frac{3}{2}$ gives positive values for the volume.

Endpoint maxima/minima

In some of the previous worked examples, the functions for the area of the rectangular backyard and the volume of an open-topped rectangular box with a square base were not defined at the endpoints of the domain. When the endpoints of the domain are included, the solution to the problem may involve an endpoint rather than a stationary point.

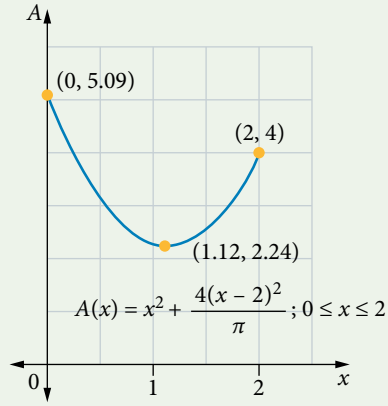
18 Solve an endpoint maxima/minima problem using technology

A square and a circle are made from 8 m of wire. To enclose the maximum area, determine how much of the wire should be used to make the square and the circle.

THINKING	WORKING
1 Define the variables and write the standard formulas to be used. Square: $P = 4x$, $A = x^2$ Circle: $C = 2\pi r$, $A = \pi r^2$	y is the combined area. x is the side length of the square r is the radius of the circle
2 Using the expression for the total perimeter, write the radius of the circle in terms of x.	$P + C = 8$ $4x + 2\pi r = 8$ $2\pi r = 8 - 4x$ $r = \dfrac{4 - 2x}{\pi}$ $r = \dfrac{-2(x - 2)}{\pi}$
3 Express the total area of the square and circle in terms of x. Substitute in the expression for r.	$A(x) = x^2 + \pi r^2$ $= x^2 + \pi\left(\dfrac{-2(x-2)}{\pi}\right)^2$ $= x^2 + \dfrac{4(x-2)^2}{\pi}$
4 Determine the domain of $A(x)$.	The square's perimeter is $4x$; therefore, the domain for a perimeter of 8 m is $0 \le x \le 2$.
5 Graph the function in the required domain. Read endpoints and turning points from your technology, or calculate by substitution.	$A(2) = (2)^2 + \dfrac{4(2-2)^2}{\pi}$ $= 4$ $A(0) = (0)^2 + \dfrac{4(0-2)^2}{\pi}$ $= \dfrac{16}{\pi}$ $= 5.09$ (2 d.p.)

The endpoints are at (2, 4) and $\left(0, \frac{16}{\pi}\right)$.

The minimum point is at (1.12, 2.24).



6 Determine the maximum value and where it occurs.

As expected, the positive quadratic must have a maximum value at an endpoint. The maximum is about 5.09 m^2 for $x = 0$.

Checking both endpoints, the maximum enclosed area of $\frac{16}{\pi}$ m^2, or approximately 5.09 m^2, occurs for $x = 0$.

7 Interpret the answer.

No wire should be used to make the square. The maximum enclosed area occurs when all of the wire is used to make a circle.

EXERCISE 10.4 Maximum and minimum problems

Worked Example

13

1 For each of the cubic functions, use technology to determine the maximum and minimum values for the given domain, correct to 2 decimal places.

(a) $y = 3x^3 - 21x^2 + 8x - 1, -2 \le x \le 5$

(b) $y = 15 + 12x - 6x^2 - x^3, -8 \le x \le 4$

(c) $y = 1.3x^3 + 2.4x^2 - 0.2x + 4, -1.5 \le x \le 1$

14

2 For each of the following functions, determine the maximum and minimum values for the given domain.

(a) $y = x^3 - 6x^2 - 15x + 10, -2 \le x \le 2$

(b) $y = x^2 - 4x + 6, 0 \le x \le 3$

(c) $y = 10 - 8x - 2x^2, 0 \le x \le 5$

15

3 For each of the following express the dependent variable in terms of the specified independent variable.

(a) For a rectangle of perimeter 30 m, express the area in terms of the length.

(b) For a cylinder of volume 3000 cm^3, express the surface area in terms of the radius.

(c) For two numbers whose difference is 40, express the sum of the squares of the numbers in terms of the smaller number.

4 A rectangular field borders a straight stretch of riverbank. It is enclosed on three sides (excluding the length of the river bank) by 100 m of fencing. The width of the field is x m and its length (the side parallel to the river) is y m.

(a) Express y in terms of x.

(b) Use calculus to determine the maximum area of the field.

(c) Determine the dimensions of the field of maximum area.

5 The sum of two positive numbers is 24. Use calculus to determine the smallest possible value of the sum of their squares.

6 A rectangular paddock of dimensions x by y has a perimeter of 48 m.

(a) Express y in terms of x.

(b) Express the area, A square metres, of the rectangular paddock in terms of x.

(c) Determine the domain of A.

(d) Determine the value of x such that $\frac{dA}{dx} = 0$ and show that this value of x corresponds to a maximum.

(e) Determine the maximum area of the paddock and specify its dimensions.

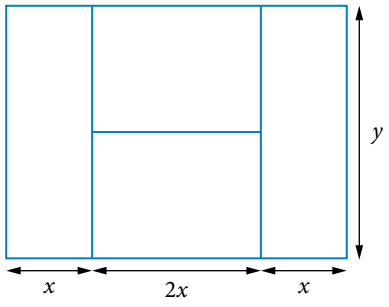
7 The weekly selling price per item, S dollars, is $25 - 0.01x$ where x is the number of items produced and sold. The weekly cost, C dollars, of producing and selling x items is $5x + 5000$. The total weekly income is I dollars.

(a) Determine an expression for I in terms of x.

(b) Determine an expression for the weekly profit, P dollars, which is the difference between income and cost.

(c) To maximise profit, determine how many items should be produced and sold each week.

8 An entire roll of tape, 60 m in length, is used to show the edges and internal lines of a rectangular court of length $4x$ m and width y m, as shown on the right.



(a) The equation for the length of tape used for the court is:

A $60 = 4y + 8x$ B $60 = 4y + 10x$

C $60 = 4y + 12x$ D $60 = 4x \times y$

(b) The area A in terms of x only is:

A $A = 4x + (60 - 10x)$ B $A = 4x(60 - 10x)$

C $A = 4x(30 - 5x)$ D $A = 4x\left(15 - \frac{10}{4}x\right)$

(c) The maximum area of the court, in square metres, is:

A 30 B 60 C 90 D 120

9 The sum of two positive real numbers is 20.

(a) The minimum possible value for the sum of their cubes is:

A 0 B 800 C 2000 D 20 000

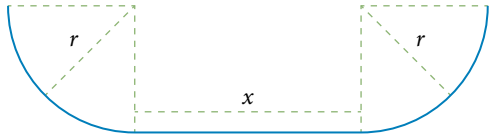
(b) Explain the common error made by a student who gave an answer of 8000.

10 A rectangular piece of metal measures 4 m by 2 m. A square of side length x m is cut from each corner and the resulting piece of metal is then folded and welded to form an open-topped rectangular box. Let V cubic metres be the volume of the box.

(a) Express V in terms of x and the state the domain of V.

(b) Use calculus to determine the value of x for which the box has a maximum volume. Determine the maximum volume. Give both answers correct to 2 decimal places.

11 The diagram shows the cross-section of an open irrigation channel. The flat bottom of the channel is x m across and the curved sides are quarter circles of radius r metres. The total length of the flat bottom section and the two curved sides is 10 m.



(a) Determine the length of the flat bottom section in terms of r.

(b) The cross-sectional area of the irrigation channel is A square metres. Determine a rule for the cross-sectional area in terms of r only.

(c) Use calculus to determine the value of r that maximises the cross-sectional area of the irrigation channel.

(d) Determine the exact maximum cross-sectional area of the irrigation channel and describe the shape.

Worked Example

12 Use calculus to determine the maximum area of a rectangle inscribed in an isosceles right-angled triangle of hypotenuse 20 m with one side of the rectangle lying along the hypotenuse.

17

13 A square sheet of metal measures 6 metres by 6 metres. A square of side length x metres is cut from each corner and the resulting piece of metal is then folded and welded to form an open-topped rectangular box.

Let V cubic metres be the volume of the box.

(a) Express V in terms of x.

(b) Determine the domain of $V(x)$.

(c) Use calculus to find the value of x for which the box has a maximum volume. Determine the maximum volume and the dimensions of the box in this case.

14 A square-based container of side length x metres and height h metres is to be constructed such that the total edge length of the container is fixed at 72 metres.

(a) Determine an expression for h in terms of x.

(b) Write an expression for volume, V, in terms of x.

(c) Determine the value of x such that V is a maximum.

(d) Determine the maximum volume of the container, stating the dimensions and describing the shape.

15 The sum of two non-negative numbers, x and y, is 4.

(a) Determine the maximum value of $x^3 + y^2$.

(b) Determine the minimum value of $x^3 + y^2$.

16 An open crate is to be constructed from a single piece of timber of area 12 m^2. All of the timber is to be used. The crate is to have a square base and sides perpendicular to the base.

(a) Determine the dimensions of the crate so that it has the maximum possible volume, state the limitations of the model.

(b) Calculate the maximum volume.

17 The point A has coordinates (-3, 5) and the point B is on the line $y = 3x + 9$. Determine the minimum value of AB, giving your answer in the form $\frac{1}{2}\sqrt{n}$ where n is an integer.

18 Fizzy Drink sells its soft drink in cylindrical aluminium cans of volume 500 cm^3. Aluminium is expensive, so they want to minimise the amount of aluminium used.

(a) Determine the radius and height of the can that uses the least amount of aluminium. Give answers correct to 2 decimal places.

(b) If the height must be between 7.5 cm and 10.5 cm, calculate the corresponding possibilities for the radius if the cans must hold 500 cm^3. Write your answers correct to 2 decimal places.

19 Metal wire of length 5 m is cut into two pieces. One piece is shaped into a square and the other is shaped into a circle. The circular piece has circumference x metres and area A square metres. The square piece has area B square metres.

(a) Determine an expression for the area enclosed by the circular piece of metal wire in terms of x.

(b) Determine an expression for the area enclosed by the square piece of metal wire in terms of x.

(c) The total area enclosed by the two pieces of metal wire is T square metres. Determine the exact value of x that minimises the total area enclosed by the two pieces of metal wire and determine how the metal wire should be divided to minimise the total enclosed area.

(d) How should the metal wire be divided to maximise the total area enclosed?

20 A square sheet of cardboard measures a by a metres. A square of side length x metres is cut from each corner and the sides are turned up to form an open-topped box with a square base.

(a) For a box of maximum volume, state the relationship that exists between the side length of the original square piece of cardboard and the side length of the square cut out from each corner.

(b) Calculate the maximum volume of the box and state the dimensions that produce this volume.

(c) For a box of maximum volume, show that the area of the square base equals the total area of the four sides of the box.

10.5 Calculus and linear motion

Kinematics

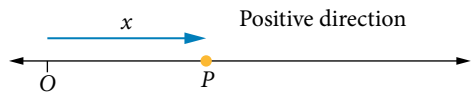
Kinematics involves the study of the motion of a body or bodies without reference to mass or the forces acting on it.

In this section, you will apply your knowledge of calculus to the analysis of a point particle travelling in straight-line (rectilinear) motion.

Position and displacement

For straight-line motion at time t, where $t \geq 0$, the function $x(t)$ denotes the position of the particle relative to a fixed point O, the origin.

A moving particle is represented by a point P, with a position coordinate x measured relative to O.



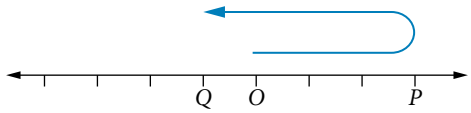
The position coordinate is positive ($x > 0$) if P is to the right of O and negative ($x < 0$) if P is to the left of O.

The *displacement* of a particle is its final position relative to a fixed point (origin) O.

Consider a particle that starts motion at O. If it moves 3 m to the right to P and then 4 m to the left from P to Q, the particle travels a distance of 7 m. The particle's displacement is:

$$3 - 4 = -1 \text{ m}$$

The particle travelled a total distance of 7 m but ended 1 m to the left of O, that is the displacement is -1 m or 1 m to the left of the origin.



Velocity as rate of change of displacement with respect to time

At time t, the position of a particle at P is given by $x(t)$.

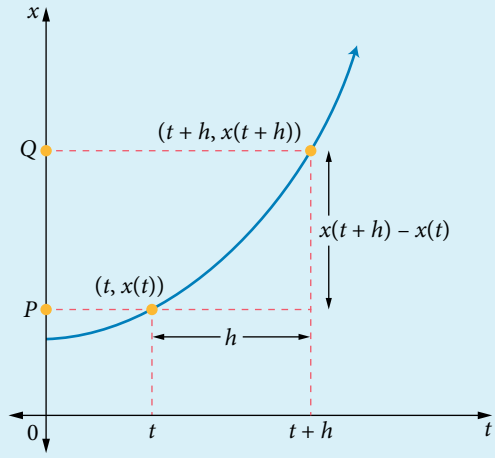
At time $(t + h)$, the position of a particle at Q is given by $x(t + h)$.

In a small time interval h the following quantities are defined:

- The particle's change in position is $x(t + h) - x(t)$.
- The particle's average velocity over the time interval $[t, t + h]$ is $\dfrac{x(t+h) - x(t)}{h}$.
- The particle's instantaneous velocity at time t is $\lim\limits_{h \to 0} \dfrac{x(t+h) - x(t)}{h}$.

The particle's instantaneous velocity at time t is defined by $\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$, where $t > 0$.

The velocity of a particle, denoted by $x'(t)$, $v(t)$ or $\frac{dx}{dt}$, is defined as the rate of change of the particle's displacement with respect to time.



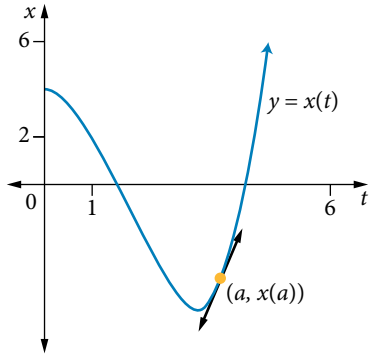
As velocity is a directed quantity, the sign indicates the direction of travel. A positive velocity indicates motion in the positive direction and a negative velocity indicates motion in the negative direction.

Typically, the units for velocity are km/h, m/s or cm/s.

Displacement-time graphs

Consider the displacement–time graph $y = x(t)$.

The gradient of the tangent to the graph at $t = a$ represents velocity. This matches the formal definition of velocity as the rate of change of the particle's position with respect to time.



19 Sketch a displacement–time graph

A particle moves in a horizontal line such that its position x (m) relative to a fixed origin O at time t (s) is given by $x(t) = -3(t-1)^2 + 12$, $t \geq 0$.

(a) Sketch the displacement–time graph showing all key features.

THINKING	WORKING
1 The function $x(t)$ is given in turning point form.	The turning point has coordinates (1, 12). Given that $a < 0$, the displacement–time graph is concave down.
2 Determine the coordinates of the x-axis intercept by letting $t = 0$. (*Note*: in this case the x-axis is the vertical axis.)	For $t = 0$: $x(0) = -3(0-1)^2 + 12$ $= -3 + 12$ $= 9$ The coordinates of the x-intercept are (0, 9). Initially, the particle is 9 m from the origin.

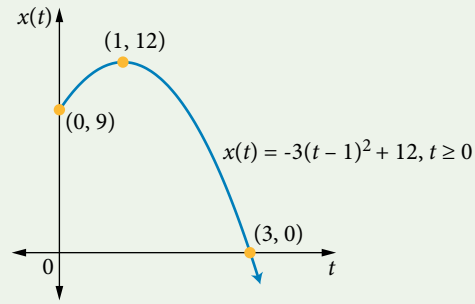
3 Determine the coordinates of the t-axis intercept(s), the value(s) of t such that $x(t) = 0$, for $t \geq 0$.

$$-3(t-1)^2 + 12 = 0$$
$$(t-1)^2 = 4$$
$$t - 1 = \pm 2$$
$$t = -1 \text{ or } t = 3$$

As $t \geq 0$, $t = 3$ is the only solution.

The t-intercept is at (3, 0). 3 seconds after starting its journey, the particle is at the origin.

4 Sketch the graph of $y = x(t)$, showing all key features.



(b) Determine the distance the particle had travelled when it reached the origin and describe the journey taken.

1 All the individual distances travelled in each direction. Use the difference in displacement values.

Travelling away:
(0, 9), (1, 12)
Distance $= |12 - 9|$
$= 3$

Travelling back:
(1, 12), (3, 0)
Distance $= |0 - 12|$
$= 12$

The total distance is $3 + 12 = 15$ m.

2 Describe the journey.

The particle started 9 m to the right the origin and travelled 3 m further right before turning around and travelling left towards the origin, taking 2 seconds to reach O but continuing past.

(c) Determine the particle's maximum velocity.

1 The gradient of a displacement–time graph gives the particle's velocity.

Determine $x'(t)$ to determine the velocity function.

$$x(t) = -3(t-1)^2 + 12$$
$$= -3t^2 + 6t + 9$$
$$x'(t) = -6t + 6$$

2 Interpret the answer.

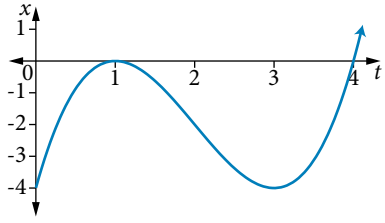
$x'(t)$ is a decreasing linear function and in domain $t \geq 0$ has a maximum velocity of 6 m/s at $t = 0$.

(d) Determine the particle's velocity at $t = 2$. Interpret the result.

1 Determine the value of the derivative at the given time.	$x'(2) = -6(2) + 6$ $= -12 + 6$ $= -6$ At $t = 2$ the derivative is $v(t) = -6$. The velocity is -6 m/s.
2 Interpret the answer in terms of direction of travel.	At 2 s, the particle is 9 m to the right of the origin (by symmetry) and travelling at 6 m/s in the negative direction (to the left).

20 Determine and interpret a particle's velocity

A particle moves in a straight line such that its position x (m) relative to an origin O at time t (s) is given by $x(t) = t^3 - 6t^2 + 9t - 4$, $t \geq 0$.



(a) Determine the time t when the particle is at O.

THINKING	WORKING
1 Solve $x(t) = 0$ for t.	From the graph, $x(t) = 0$ for $t = 1$ and $t = 4$. Checking algebraically: $x(t) = (t-1)^2(t-4)$ $0 = (t-1)^2(t-4)$ $t = 1$ or $t = 4$
2 Interpret the result.	The particle is at O at $t = 1$ s and $t = 4$ s.

(b) Determine the particle's initial velocity.

1 Determine the derivative of $x'(t)$ where $t = 0$. The particle's initial velocity is given by $x'(0)$.	$x'(t) = 3t^2 - 12t + 9$ $x'(0) = 3(0)^2 - 12(0) + 9$ $= 9$
2 Interpret the result and specify the answer with correct units.	The particle's initial velocity is 9 m/s, meaning it is travelling in the positive direction (to the right) at a speed of 9 m/s.

(c) Determine the time t when the particle is instantaneously at rest.

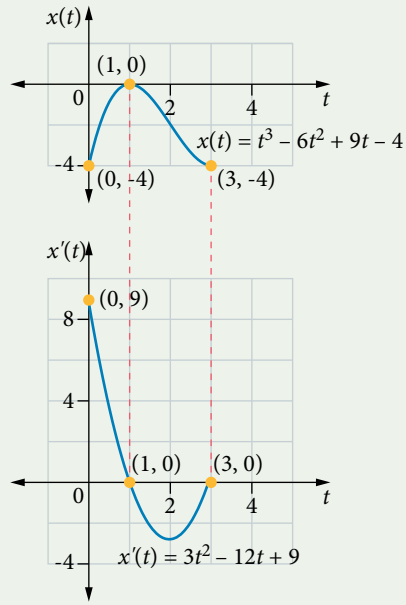
1 Solve $x'(t) = 0$ for t.	$x'(t) = 3(t^2 - 4t + 3)$ $0 = 3(t-1)(t-3)$ $t = 1$ or $t = 3$
2 Interpret the result and specify the answer with correct units.	The particle is at rest at $t = 1$ s and $t = 3$ s, which corresponds to a change in direction.

(d) Describe the particle's maximum and minimum velocity over the interval $0 \le t \le 3$.

1 Sketch the graph of $x(t)$ and $x'(t)$ for $0 \le t \le 3$.

Note that the units of $x(t)$ are in metres, and the units of $x'(t)$ are in metres per second.

Check the endpoints and any local maxima/minima of $x'(t)$.



2 Describe the journey.

At $t = 0$, the particle is 4 m from the origin. It has an initial and maximum velocity of 9 m/s towards the origin.

For $0 \le t < 1$, the particle moves towards the origin with positive velocity, slowing down over this period.

At $t = 1$, the particle reaches the origin, stopping for an instant to turn and travel in the negative direction.

For $1 < t < 3$, the particle moves away from the origin in the negative direction, (giving a negative velocity). It starts at the origin at $t = 1$, then speeds up before slowing to rest at $t = 3$, 4 m from the origin, in the negative direction.

At $t = 2$, the particle has minimum velocity of -3 m/s away from the origin, meaning it is travelling at 3 m/s in the negative direction (to the left).

WARNING

It is interesting to note that the minimum velocity in Worked Example 20 of -3 m/s is not the minimum speed of the particle. The minimum speed is 0 m/s, when the particle is instantaneously at rest. The negative sign in the minimum velocity indicates the maximum speed of the object in the negative direction.

EXERCISE 10.5 Calculus and linear motion

Worked Example 19

1 A particle moves in a line such that its position, x (m), relative to a fixed origin O at time t (s) is given by $x(t) = -2(t-2)^2 + 8$, $t \geq 0$.

(a) Sketch the displacement–time graph, showing all key features.

(b) Determine the particle's maximum velocity.

(c) Determine the distance the particle had travelled when it returned to the origin.

(d) Determine the particle's velocity at $t = 3$ and interpret the result.

2 A ball is projected vertically upwards from the ground. At time t seconds, the position of the ball, h metres above the ground, is given by $h(t) = 20t - 5t^2$, $t \geq 0$.

(a) Sketch the height–time graph, showing all key features.

(b) Determine when the ball is instantaneously at rest and its greatest height.

(c) Determine the ball's velocity at $t = 3$.

3 A particle moves in a straight line such that its position, x metres, relative to an origin O at time t seconds is given by $x(t) = -t^2 + 2t - 8$, $t \geq 0$.

(a) Determine the particle's initial position.

(b) Determine the particle's velocity at $t = 4$.

(c) Determine when the particle's velocity is zero.

4 A particle moves in a straight line such that its position, x metres, from a fixed point O on the line at any time t seconds is given by $x(t) = t(4-t)$, $t \geq 0$.

(a) The distance in metres travelled in the first 3 seconds is:

A 0 B 1 C 4 D 5

(b) Explain the common error made by a student who thought the answer was 3.

5 A particle moves in a straight line such that its position, x (m), from a fixed point O on the line at any time t(s) is given by $x(t) = 10(t-1)(t-3)$, $t \geq 0$. Determine distance in metres travelled in the first 1, 2 and 3 seconds respectively.

6 A particle moves in a straight line such that its position, x (m), from a fixed point O on the line at any time t (s) is given by $x(t) = -0.2(t+5)(t-4)(t-6)$, $t \geq 0$.

Determine the distance in metres, correct to 2 decimal places, where necessary, travelled in the first 2 and 8 seconds respectively.

7 A particle moves in a straight line such that its position, x metres, from a fixed point O on the line at any time t seconds is given by $x = t^2 - 7t + 10$, $t \geq 0$.

(a) Determine the particle's initial velocity.

(b) Determine the distance travelled by the particle in the first 5 seconds of motion.

8 The displacement x (m) from the origin of a body moving in a straight line is given by $x(t) = 2t^3 - 12t^2 + 18t$, where t (s) is time from the start.

(a) Determine the times when the body is instantaneously at rest.

(b) When does the body return to its starting point and what is its velocity at that time?

9 The height of a particle, h metres above the ground, after t seconds, is given by $h(t) = 35 + 25t - 2t^2$, $t \geq 0$. Determine:

(a) the initial height of the particle

(b) when the particle is instantaneously at rest

(c) the greatest height reached by the particle

(d) when the particle reaches the ground, correct to 2 decimal places

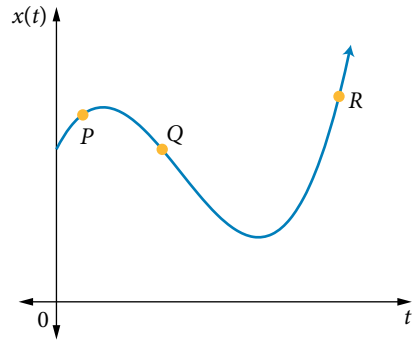
(e) the distance travelled by the particle in the first 4 seconds.

10 A particle moves in a straight line such that its position, x metres, from a fixed point O on the line at any time t seconds as shown on the displacement–time graph.

At which of the points P, Q and R is the particle travelling away from the origin?

A P, Q and R B Q and R

C P only D P and R



Worked Example

20 11 A particle moves in a straight line such that its position x (m), relative to an origin O at time t (s) is given by $x(t) = t^3 + 3t^2 - 4$, $t \geq 0$. Determine the:

(a) time t when the particle is at O (b) particle's initial velocity

(c) time t when the particle is instantaneously at rest.

12 The displacement of a particle x (m) at time t (s) is given by the function $x(t) = t^3 - 8t^2 + 5t + 30$.

(a) Sketch displacement–time and velocity–time graphs.

(b) Describe the particle's position and behaviour at $t = 0$, $t = 2$, $t = 3$ and $t = 6$.

13 The displacement of a particle x (m) at time t (s) is given by the function $x(t) = -t^3 + 10t + 10$.

(a) Describe the position and behaviour of the particle at the start.

(b) Describe where and when the particle was instantaneously at rest, correct to 2 decimal places.

(c) Describe the speed and direction of the particle as it passes through the origin.

14 Particle A starts at a position 3 m to the right of the origin and moves with velocity v m/s is given by $v(t) = 2t - 4$, where time is in seconds. Particle B starts from a position 4 m to the left of the origin and moves with a constant velocity of 2 m/s.

(a) When and where do the particles meet?

(b) Describe the behaviour of each particle when they meet.

15 The velocity of a particle v (m/s) at time t (s) is given by the function $v(t) = 12t^2 - 16t + 1$.

(a) If the particle is 4 m to the right of the origin after 1 s determine the rule for the displacement $x(t)$ of the particle at any time t.

(b) Acceleration $a(t)$ is given by the function $v'(t)$. Describe the position and behaviour of the function when $a(t) = 0$.

Composite functions and their derivatives 10.6

The function $C = \frac{5}{9}(F - 32)$ converts a temperature from Fahrenheit to Celsius, where C is the temperature in degrees Celsius (°C) and F is the temperature in degrees Fahrenheit (°F).

Temperature can also be measured on the absolute or kelvin K scale, where 0 K represents 'absolute zero' and $K = C + 273.15$.

The composite function that combines the two conversion formulas to convert a temperature directly from Fahrenheit to kelvin is $K = \frac{5}{9}(F - 32) + 273.15$.

Convert 356°F to Celsius:	Convert 180°C to kelvin:	Convert 356°F to kelvin:
$C = \frac{5}{9}(F - 32)$ $= \frac{5}{9}(356 - 32)$ $= \frac{5}{9} \times 324$ $= 180°C$	$K = C + 273.15$ $= 180 + 273.15$ $= 453.15K$	$K = \frac{5}{9}(356 - 32) + 273.15$ $= \frac{5}{9} \times 324 + 273.15$ $= 180 + 273.15$ $= 453.15K$

Composite functions

Composite function can be used in currency conversions. For example, a bank offers the exchange rate \$1(AUD) = \$0.90(USD) plus an administration payment of \$5(AUD) for each transaction. Another bank offers \$1(USD) = £0.60(GBP) plus an administration payment of \$3(USD) for each transaction.

How many UK pounds (GBP) would you receive if you first exchange \$100(AUD) into USD in Australia and then into GBP in the USA?

Let t represent the amount in AUD, x represent the amount in USD and y represent the amount in GBP. Set up linear equations to represent the conversions, where y is a function of x and x is a function of t, so express y as a function of t.

AUD to USD:	USD to GBP:	Composite function AUD to GBP:
$x(t) = 0.9(t - 5)$ $= 0.9t - 4.5$	$y(t) = 0.6(x - 3)$ $= 0.6x - 1.8$	$y(t) = y(x(t))$ $= 0.6(x(t)) - 1.8$ $= 0.6(0.9t - 4.5) - 1.8$ $= 0.54t - 4.5$

So $f(t) = 0.54t - 4.5$, where $f(t) = y(x(t))$.

Calculating $f(100)$ gives:

$$f(100) = 0.54 \times 100 - 4.5$$
$$= 49.5$$

In other words, after these two transactions have been performed, \$100(AUD) has been converted to £49.50(GBP).

The function, f, has been 'composed' from two functions, namely, $x(t)$ and $y(x)$.

Additional information

Composite functions

Watch the video and try the activity to check your understanding of composite functions.

Consider the function h defined by $h(x) = \sqrt{x+3}$.

$h(x)$ can be composed using a combination of two functions $g(x)$ and $f(x)$.	$f(g(x))$	
The first function is called the inner function and the second is called outer function.	$g(x) = x + 3$	$f(x) = \sqrt{x}$
In other words, $g(x) = x + 3$ is substituted into the function $f(x) = \sqrt{x}$ to form $h(x)$.	$h(x) = f(g(x))$ $= f(x+3)$ $= \sqrt{x+3}$	

$h(x)$ is obtained by applying $g(x)$ and then applying $f(x)$.

A function that is created from two functions in this way is called a composite function.

So $h(x) = \sqrt{x+3}$ is said to be a composition of $g(x) = x + 3$ followed by $f(x) = \sqrt{x}$.

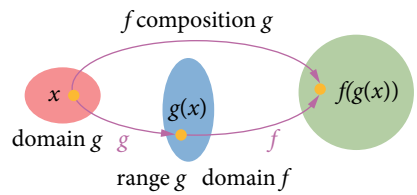
The function h is obtained by substituting the function g into the function f.

In function notation, this is denoted as $f(g(x))$ and is read as 'f of g of x'.

It is important to recognise that in the formation of $f(g(x))$, the function g that is applied first is known as the inner function and the function f that is applied second is known as the outer function.

A composite function can be referred to as a 'function of a function'.

The diagram describes the process of forming f composition g; that is, for forming $f(g(x))$.



- Begin with an input x in the domain of g.
- Apply g to obtain an output $g(x)$ in the range of g.
- $g(x)$ is the new input for f, so it must be in the domain of f.
- Apply f to obtain the final output, f composition g, $f(g(x))$.

Additional information

Composition with linear functions

Watch the video and try the activity to check your understanding of composition with linear functions.

The rule of f composition g, where g is applied first and f is applied second, is denoted by $f(g(x))$.

- The domain of the composite function, with rule $f(g(x))$, is the set of all values of x in the domain of g such that the output of g is in the domain of f.
- For $f(g(x))$ to be defined, the range of $g \subseteq$ domain f.
- If $f(g(x))$ is defined, the domain of the composite function is the same as the domain of g.

The rule of g composition f, where f is applied first and g is applied second, is denoted by $g(f(x))$.

- The domain of the composite function, with rule $g(f(x))$, is the set of all values of x in the domain of f such that the output of f is in the domain of g.
- For $g(f(x))$ to be defined, the range of $f \subseteq$ domain g.
- If $g(f(x))$ is defined, the domain of the composite function is the same as the domain of f.

WARNING

The order of composite functions is not interchangeable. Do not assume that $f(g(x)) = g(f(x))$.

Additional information

Domain of a composite function

Watch the video and try the activity to determine the domain of a composite function.

21 Form a composite function

If $f(x) = x^3$, $x \in \mathbb{R}$ and $g(x) = x - 4$, $x \in \mathbb{R}$, determine expressions for the following.

(a) $f(g(x))$

THINKING	WORKING
Apply g first and f second.	$f(g(x)) = f(x-4)$ $= (x-4)^3$

(b) $g(f(x))$

Apply f first and g second.	$g(f(x)) = g\left(x^3\right)$ $= x^3 - 4$

- If the domain of f and g are both $\mathbb{R}$, then the domain of $f(g(x))$ is $\mathbb{R}$.
- If the domain of g and f are both $\mathbb{R}$, then the domain of $g(f(x))$ is $\mathbb{R}$.

As demonstrated in the previous worked example, the composition of the two functions is usually non-commutative and $f(g(x)) \neq g(f(x))$.

If $f(x) = x^3$ and $g(x) = x - 4$, then $f(g(x)) = (x-4)^3$ and $g(f(x)) = x^3 - 4$, so $f(g(x)) \neq g(f(x))$.

Some pairs of functions may satisfy the equation $f(g(x)) = g(f(x))$ such as when $f(x)$ and $g(x)$ are inverse functions. For example, if $f(x) = 5x$ and $g(x) = \frac{x}{5}$ then:

$f(g(x)) = 5 \times \dfrac{x}{5} = x$ and $g(f(x)) = \dfrac{(5x)}{5} = x$ so $f(g(x)) = g(f(x))$.

Derivative of composite functions

The derivative of some composite functions can be determined by expanding and simplifying the expression. For example, $y = (x^3 - 4)^2$ can be expanded using the perfect square expansion.

$$\begin{aligned} y &= (x^3 - 4)^2 \\ &= (x^3)^2 - 2(4)(x^3) + 4^2 \\ &= x^6 - 8x^3 + 16 \end{aligned}$$

$$\frac{dy}{dx} = 6x^5 - 24x^2$$

Factorising $\frac{dy}{dx}$ allows you to see a connection between y and $\frac{dy}{dx}$.

$6x^5 - 24x^2 = 6x^2(x^3 - 4) = 2(3x^2)(x^3 - 4)^{2-1}$

The explanation below shows the relationship between the expression and its derivative, and demonstrates the application of the chain rule of differentiation.

The index of the outer function appears in the front.

$$\frac{dy}{dx} = 2(3x^2)(x^3 - 4)^{2-1}$$

One is subtracted from the index of the outer function.

The derivative of the inner function appears inside the bracket.

$y = (x^3 - 4)^2$ is a composite function of the form $y = f(g(x))$ and can be represented as a 'chain' of two functions.

$g(x) = x^3 - 4$ is the inner function and is applied first. $f(x) = x^2$ is the outer function and is applied second.	$x \longrightarrow$ Inner function: $x^3 - 4$ $\longrightarrow$ Outer function: $(\ )^2$ $\longrightarrow (x^3 - 4)^2$

Let the inner function be u:

then $y = (x^3 - 4)^2 = u^2$, where $u = x^3 - 4$.

So $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3x^2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2u \times 3x^2 \\ &= 2(3x^2)u \\ &= 2(3x^2)(x^3 - 4) \\ &= 6x^2(x^3 - 4) \end{aligned}$$

This rule, known as the *chain rule of differentiation*, provides an efficient way of finding the derivative of a composite function.

The *chain rule of differentiation* relates the derivative of the composite of two functions to the functions and their derivatives.

If $y = f(u)$ and $u = g(x)$, then, in Leibniz notation, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Alternatively, using function notation, the chain rule states that:

If $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$.

Additional information

The derivative of a composite function

Watch the video and try the activity to determine the derivative of a composite function.

When using the chain rule, work from the outside to the inside. First differentiate the outer function f at the inner function $g(x)$. This is $f'(g(x))$.

Then multiply by the derivative of the inner function. This is $g'(x)$.

It appears that Gottfried Leibniz (1646–1716), who was one of the important contributors to the development of differential calculus, first used the chain rule to calculate the derivative of $\sqrt{a + bz + cz^2}$.

Proof of the chain rule: Leibniz notation	**Proof of the chain rule: function notation**
Δx is an increment in x. Δu is the corresponding increment in u. Δy is the corresponding increment in y. $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x}, \Delta u \neq 0$ As $\Delta x \to 0$: $\Delta u \to 0, \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$ $\frac{\Delta y}{\Delta u} \to \frac{dy}{du}$ and $\frac{\Delta u}{\Delta x} \to \frac{du}{dx}$. So $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.	The difference quotient for $f(g(x))$ is: $\frac{f(g(x+h)) - f(g(x))}{h}$ Let $u = g(x)$ and $u + k = g(x+h)$, where $k = g(x+h) - g(x)$. g is differentiable, so $\lim_{h \to 0} k = 0$. $\frac{f(g(x+h)) - f(g(x))}{h}$ $= \frac{f(u+k) - f(u)}{h}$ $= \frac{f(u+k) - f(u)}{k} \times \frac{k}{h}$ $= \frac{f(u+k) - f(u)}{k} \times \frac{g(x+h) - g(x)}{h}$ $\frac{dy}{dx} = \lim_{h \to 0}\left(\frac{f(u+k) - f(u)}{k} \times \frac{g(x+h) - g(x)}{h}\right)$ As $h \to 0, k \to 0$. $\frac{dy}{dx} = \lim_{k \to 0} \frac{f(u+k) - f(u)}{k} \times \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$ $= f'(u)g'(x)$ $= f'(g(x))g'(x)$

22 Perform chain rule differentiation

Use the chain rule to determine the expression for $\frac{dy}{dx}$ for the following.

(a) $y = (3x^2 - 1)^3$

THINKING	WORKING
1 Let the inner function of the composite function be u and express y in terms of u.	$y = (3x^2 - 1)^3$ Let $u = 3x^2 - 1$, therefore $y = u^3$.
2 Differentiate $u(x)$ and $y(u)$ to find $\frac{du}{dx}$ and $\frac{dy}{du}$.	$\frac{du}{dx} = 6x$ and $\frac{dy}{du} = 3u^2$

3 Calculate $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and simplify.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 3u^2 \times 6x\\ &= 18x \times u^2\\ &= 18x(3x^2 - 1)^2\end{aligned}$$

(b) $y = \sqrt{2x+5}$

1 Let the inner function of the composite function be u and express y in terms of u.

$$\begin{aligned}y &= \sqrt{2x+5}\\ &= (2x+5)^{\frac{1}{2}}\end{aligned}$$

Let $u = 2x + 5$, therefore $y = u^{\frac{1}{2}}$.

2 Differentiate $u(x)$ and $y(u)$ to determine $\frac{du}{dx}$ and $\frac{dy}{du}$.

$\frac{du}{dx} = 2$ and $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

3 Calculate $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and simplify, expressing the answer in the same form as given in the question.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{2}u^{-\frac{1}{2}} \times 2\\ &= \frac{1}{u^{\frac{1}{2}}}\\ &= \frac{1}{\sqrt{u}}\\ &= \frac{1}{\sqrt{2x+5}}\end{aligned}$$

A special case of the chain rule

If $y = (g(x))^n$, $y = f(u) = u^n$ where $u = g(x)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= nu^{n-1}\frac{du}{dx}\\ &= n(g(x))^{n-1}g'(x)\end{aligned}$$

23 Perform chain rule differentiation with function notation

The function h is defined by $h(x) = \sqrt{x^2+3}$.

Calculate the gradient of the curve $y = h(x)$ at the point $(1, 2)$.

THINKING	WORKING
1 Let $h(x) = f(g(x))$ and identify the functions f and g.	$h(x) = \sqrt{x^2+3}$ $= (x^2+3)^{\frac{1}{2}}$ $h(x) = f(g(x))$ Let $g(x) = x^2 + 3$, therefore $f(x) = x^{\frac{1}{2}}$.

2 Differentiate each of the functions f and g to give $f'(g(x))$ and $g'(x)$.	$f'(g(x)) = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}$ and $g'(x) = 2x$
3 Calculate $h'(x) = f'(g(x))g'(x)$ and simplify, expressing your answer in the same form as the question.	$h'(x) = f'(g(x))g'(x)$ $= \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)$ $= \frac{x}{(x^2+3)^{\frac{1}{2}}}$ $= \frac{x}{\sqrt{x^2+3}}$
4 Substitute the x-value of the given point into the $h'(x)$ and calculate.	$h'(1) = \frac{1}{\sqrt{1^2+3}}$ $= \frac{1}{2}$
5 Interpret the answer.	The gradient of the curve at (1, 2) is $\frac{1}{2}$.

Technology worked example
Gradient at a point

24 Determine the gradient of a function using the chain rule

Determine the gradient of $y = (x^2 + 3x + 2)^2$ at the points where its graph meets the x-axis.

THINKING	WORKING
1 Solve $y = 0$ for x to find where the graph meets the x-axis.	$(x^2 + 3x + 2)^2 = 0$ $(x^2 + 3x + 2) = 0$ $(x+2)(x+1) = 0$ $x = -2$ or $x = -1$ The coordinates of the x-intercepts are (-2, 0) and (-1, 0).
2 Let the inner function of the composite function be u and express y in terms of u.	$y = (x^2 + 3x + 2)^2$ Let $u = x^2 + 3x + 2$, therefore $y = u^2$.
3 Differentiate u and y to give $\frac{du}{dx}$ and $\frac{dy}{du}$.	$\frac{du}{dx} = 2x + 3$ and $\frac{dy}{du} = 2u$
4 Calculate $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and simplify.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= 2u \times (2x+3)$ $= 2(2x+3)(x^2+3x+2)$

5 Determine $\frac{dy}{dx}$ at $x = -2$ and $x = -1$.

For $x = -2$:

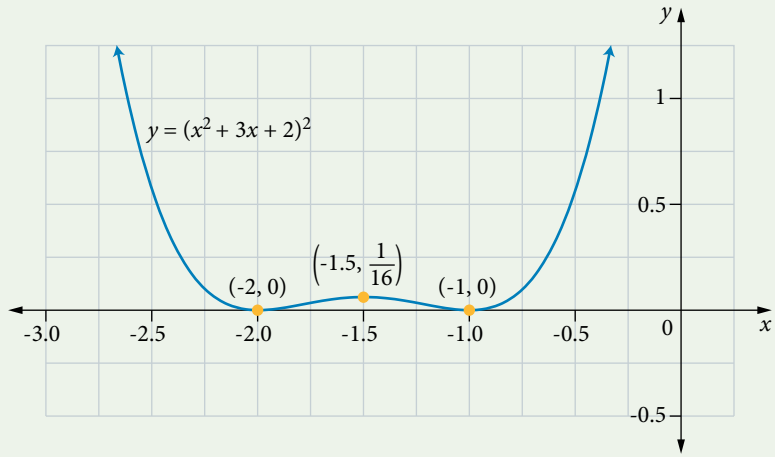
$$\frac{dy}{dx} = 2(2(-2) + 3)((-2)^2 + 3(-2) + 2) = 0$$

For $x = -1$:

$$\frac{dy}{dx} = 2(2(-1) + 3)((-1)^2 + 3(-1) + 2) = 0$$

The gradient where the function meets the x-axis is 0 for both intercepts. This corresponds to stationary points at both x-intercepts.

6 Verify the answer with a graph.



7 Interpret the answer.

The gradient at both $(-2, 0)$ and $(-1, 0)$ is 0.

EXERCISE 10.6 Composite functions and their derivatives

Worked Example

21 1 If $f(x) = x^2$ and $g(x) = 2x - 3$, determine expressions for:

(a) $f(g(x))$ (b) $g(f(x))$

(c) $f(g(x)) = g(f(x))$ and solve the equation for x.

22 2 Determine the derivative function $\frac{dy}{dx}$ for each of the following.

(a) $y = (3x^2 - 1)^3$ (b) $y = (x - 3)^4$ (c) $y = (2 - x^3)^2$ (d) $y = \left(1 + \frac{4}{x}\right)^3$

(e) $y = \frac{1}{(3x^2 - 5)^4}$ (f) $y = \frac{7}{(2x^2 - 1)^4}$ (g) $y = \sqrt{x - 1}$ (h) $y = -2\sqrt{x^3 - 1}$

(i) $y = (x^2 - 3x + 2)^4$ (j) $y = \frac{2}{x^2 - 7x + 1}$ (k) $y = \sqrt[3]{x - 1}$ (l) $y = \sqrt[5]{2x + 4}$

23 3 The function h is defined by $h(x) = \sqrt{4 - x^2}$, $-2 \le x \le 2$. Calculate the gradient of $y = h(x)$ at the point $\left(1, \sqrt{3}\right)$.

4 Calculate the gradient of $y = (x^2 - 2)^3$ at $x = -1$.

5 Given that $h(x) = f(g(x))$ and $g(4) = 8$, $g'(4) = 6$, $f'(4) = 4$ and $f'(8) = 3$, determine the value of $h'(4)$.

A 12 B 18 C 24 D 32

6 Consider $f(x) = \sqrt{x+1}$.

(a) For what values of x is $f'(x)$ defined?

A $-1 \le x \le 1$ B $x \ge 1$ C $x > -1$ D $x > 1$

(b) Explain the common error made by a student who gave an answer of $x \ge -1$ in part (a).

Worked Example

24

7 Determine the derivative of $f\left(\sqrt{1+x^4}\right)$.

8 Determine the gradient of $y = (x^2 - 3x + 2)^2$ at the points where the graph meets the x-axis.

9 Consider the function $y = 3(x+2)^4$. Determine the set of values of x for which the gradient is greater than $\frac{1}{2}$, correct to 2 decimal places.

10 Consider the function f defined by $f(x) = \sqrt{x^2+5}$.

(a) Determine the largest possible domain of f and the range of f.

(b) Determine the exact set of values of x for which the gradient of the curve is less than $\frac{1}{4}$.

11 Consider the functions f and g defined by $f(x) = \sqrt{x+1}$ and $g(x) = x^2$.

(a) What is the largest possible domain of f and the range of f?

(b) What is the largest possible domain of g and the range of g?

(c) Determine the expression for $f(g(x))$.

(d) Determine the expression for $g(f(x))$.

(e) Calculate the derivative of $f(g(x))$ at $x = 0$.

(f) Calculate the exact set of values of x such that $-\frac{1}{3} < (f'(g(x)) < \frac{1}{3}$.

12 Consider the functions with rules $y = \sqrt{2x}$ and $y = \sqrt{-x^2+4x-1}$. Their graphs intersect at $x = k$.

(a) Determine the value of k.

(b) Demonstrate that these graphs have the same gradient at $x = k$.

(c) The function $y = \sqrt{-x^2+4x-1}$ has a point of zero gradient at $x = p$. Determine the value of p.

13 Water spilling from a leaking pipe spreads out in a circular puddle. The radius r of the water puddle is increasing at a rate of 2 cm per hour. The area of the puddle increases at a rate of 10 cm^2 for each centimetre increase in the radius at time $t = 1$ h. The area of the circular puddle is defined by a function $A = f(r)$, where r is the radius of the puddle at any time t defined by the equation $r = g(t)$, where t is time measured in hours.

(a) Write the expressions for each of the rates of change given.

(b) Write an expression for the rate of change of area with respect to time. Hence, calculate how fast the area of the puddle is increasing at time $t = 1$ h.

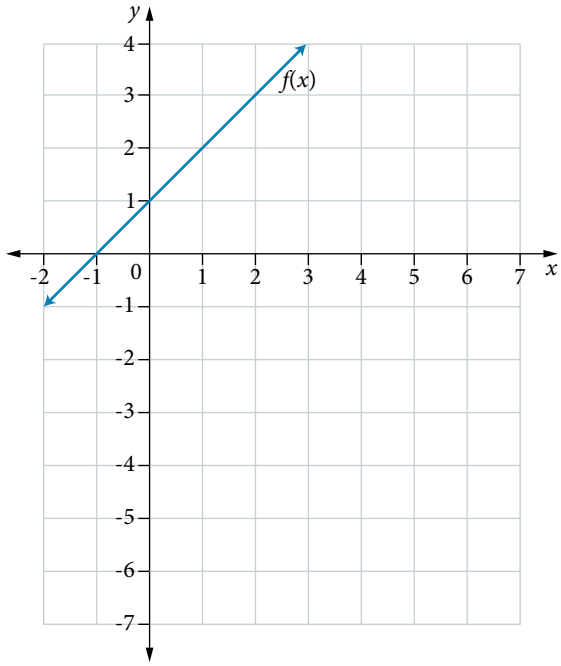
(c) Calculate the rate of change of the area of the puddle with respect to time when the radius of the puddle is 15 cm.

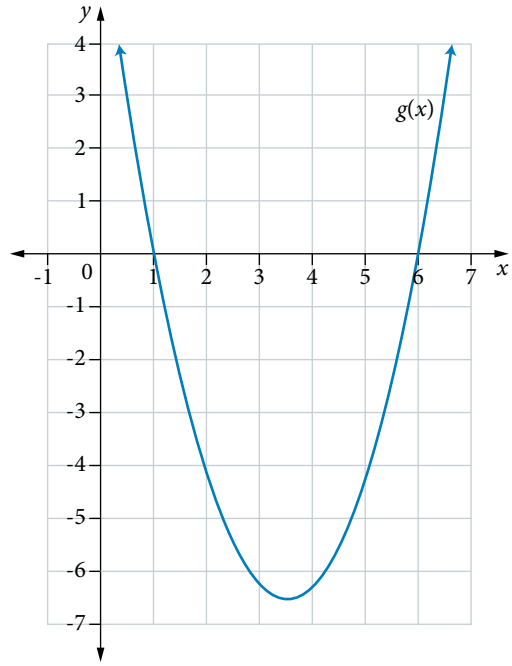
14 Consider the function h defined by $h(x) = \sqrt{-x^2 + x + 6}$.

(a) Determine the largest possible domain of $h(x)$.

(b) Sketch the graph of $y = h(x)$, clearly indicating the point of zero gradient and the axes intercepts.

15 Let f and g be two functions represented by the graphs shown below.





(a) Use the graphs to determine the values of $f(g(x))$ and $g(f(x))$ where $x = 1$.

(b) If $h(x) = f(g(x))$, use transformations to determine the interval over which the gradient function $h'(x)$ is positive. Justify your answer.

(c) Determine the equations of $h(x)$ and $h'(x)$, and validate the answer in part (b).

16 A perfect cube of ice retains its shape as it melts at a constant rate of 3 cm^3/min. The cube of ice has an initial side length of 4 cm.

(a) Write an expression for the rate of change of the side length x of the ice cube at any time t.

(b) How fast is the surface area of the cube decreasing when the side length of the cube is half of its original length?

(c) How fast is the surface area of the cube decreasing when the side length of cube is a quarter of its original length?

(d) Sketch a graph of the rate of change of surface area with respect to time against x. Explain the meaning of the graph as x decreases.

Derivatives of products and quotients 10.7

Derivative of a product of two functions

Using today's function notation, Leibniz initially thought that if $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g'(x)$. He soon realised that this was not the case and promptly corrected this mistake in a note to a colleague.

Consider the functions $f(x) = 3x^4$ and $g(x) = x^2 - 2$ and $h(x) = f(x)g(x)$.

The derivative of $h(x)$:

$$\begin{aligned} h(x) &= 3x^4(x^2 - 2) \\ &= 3x^6 - 6x^4 \\ h'(x) &= 18x^5 - 24x^3 \end{aligned}$$

The product of the derivatives is:

$f'(x) = 12x^3$ and $g'(x) = 2x$

$$\begin{aligned} f'(x)g'(x) &= 12x^3 \times 2x \\ &= 24x^4 \end{aligned}$$

You can see that $h'(x) \neq f'(x)g'(x)$ for this example.

WARNING

If $h(x) = f(x)g(x)$, then $h'(x) \neq f'(x)g'(x)$.

If $h(x) = f(x)g(x)$, then $h'(x)$ is calculated using the product rule of differentiation.

The product rule allows you to determine the derivative of a product of two functions. The derivative is equal to the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function.

If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.

If $u = f(x)$ and $v = g(x)$, then in Leibniz notation, the product rule can be expressed as:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

A proof of the product rule

Let $h(x) = f(x)g(x)$ where f and g are differentiable functions and h is defined for all values of x in the domain.

$$\begin{aligned} h'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \left(f(x+h)\frac{g(x+h)}{h} + \frac{f(x+h) - f(x)}{h} g(x) \right) \\ &= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \lim_{h \to 0} g(x) \\ &= f(x)g'(x) + f'(x)g(x) \end{aligned}$$

So if $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.

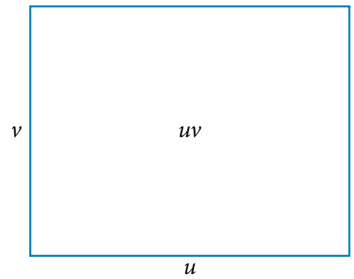
Alternatively, the product rule can be justified, expressed in the form $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, by considering the effect of a small change Δx in x on functions u, v and y.

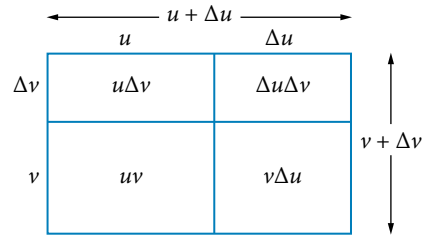
Consider the rectangle with area $y = uv$. Let Δx be a small change in x.

Let Δu, Δv and Δy be the corresponding small changes in u, v and y, caused by this small change in x.

As u and v increase by Δu and Δv, respectively, the area of the larger rectangle below is:

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u\Delta v + v\Delta u + \Delta u\Delta v \end{aligned}$$





So $y = uv$ changes by $\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v$.

Dividing through by Δx gives:

$$\frac{\Delta y}{vx} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

As $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$, $\frac{\Delta v}{\Delta x} \to \frac{dv}{dx}$, $\frac{\Delta u}{\Delta x} \to \frac{du}{dx}$, $\Delta u \to 0$

So $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Additional information

Product rule

Watch the video and complete the questions to check your understanding of the product rule of differentiation.

25 Perform product rule differentiation

Using the product rule, determine an expression for $\frac{dy}{dx}$, given $y = (2x - 3)(x^2 - 6)$.

THINKING	WORKING
1 Define u and v.	$y = (2x - 3)(x^2 - 6)$ Let $u = 2x - 3$ and let $v = x^2 - 6$.
2 Differentiate u and v to find $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 2x$
3 Calculate $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ and simplify.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= (2x - 3)(2x) + (x^2 - 6)(2)$ $= 4x^2 - 6x + 2x^2 - 12$ $= 6x^2 - 6x - 12$
4 Express $\frac{dy}{dx}$ in factorised form.	$\frac{dy}{dx} = 6(x^2 - x - 2)$ $= 6(x + 1)(x - 2)$

26 Perform product rule differentiation involving algebraic fractions

Using the product rule, determine an expression for $\frac{dy}{dx}$, given $y = x\sqrt{3-x}$. Express your answer as a single algebraic fraction.

THINKING	WORKING
1 Define u and v.	$y = x\sqrt{3-x}$ $= x(3-x)^{\frac{1}{2}}$ Let $u = x$ and let $v = (3-x)^{\frac{1}{2}}$.
2 Differentiate u to find $\frac{du}{dx}$. Use the chain rule to differentiate v, to find $\frac{dv}{dx}$.	$\frac{du}{dx} = 1$ Using the chain rule: $\frac{dv}{dx} = \left(\frac{1}{2}\right)(-1)(3-x)^{-\frac{1}{2}}$ $= -\frac{1}{2\sqrt{3-x}}$
3 Calculate $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ and simplify.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $= (x)\left(-\frac{1}{2\sqrt{3-x}}\right) + \left(\sqrt{3-x}\right)(1)$ $= \sqrt{3-x} - \frac{x}{2\sqrt{3-x}}$
4 Express $\frac{dy}{dx}$ as a single algebraic fraction.	$\frac{dy}{dx} = \sqrt{3-x} - \frac{x}{2\sqrt{3-x}}$ $= \frac{\left(2\sqrt{3-x}\right)\left(\sqrt{3-x}\right) - x}{2\sqrt{3-x}}$ $= \frac{2(3-x) - x}{2\sqrt{3-x}}$ $= \frac{6-2x-x}{2\sqrt{3-x}}$ $= \frac{6-3x}{2\sqrt{3-x}}$

27 Perform product rule differentiation with function notation

The function f is defined by $f(x) = (x^2+1)(x^2-2)$, $x \geq 0$.

Determine the coordinates of the points where the gradient of the curve $y = f(x)$ is equal to zero.

THINKING	WORKING
1 Define $u(x)$ and $v(x)$.	$f(x) = (x^2+1)(x^2-2)$ Let $u(x) = x^2+1$ and let $v(x) = x^2-2$.

2 Differentiate $u(x)$ and $v(x)$ to find $u'(x)$ and $v'(x)$.	$u'(x) = 2x$ and $v'(x) = 2x$
3 Calculate $f'(x) = u(x)v'(x) + v(x)u'(x)$ and simplify.	$f'(x) = u(x)v'(x) + v(x)u'(x)$ $= (x^2+1)(2x) + (x^2-2)(2x)$ $= 2x(x^2+1+x^2-2)$ $= 2x(2x^2-1)$
4 Solve $f'(x) = 0$ for x.	$2x(2x^2-1) = 0$ $x = 0$ or $2x^2 - 1 = 0$ $2x^2 = 1$ $x^2 = \frac{1}{2}$ $x = \pm\frac{\sqrt{2}}{2}$ Given that $x \geq 0$, $x = 0$ or $x = \frac{\sqrt{2}}{2}$.
5 Determine the corresponding y-values and state the points of zero gradient in coordinate form.	For $x = 0$: $f(0) = (0+1)(0-2)$ $= -2$ For $x = \frac{\sqrt{2}}{2}$: $f(0) = \left(\left(\frac{\sqrt{2}}{2}\right)^2 + 1\right)\left(\left(\frac{\sqrt{2}}{2}\right)^2 - 2\right)$ $= \left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$ $= -\frac{9}{4}$ The gradient of the curve is zero at $(0, -2)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{9}{4}\right)$.

28 Use the product rule

Consider functions f and g, where $f(x) = x^3\sqrt{g(x)}$.

If $g(2) = 4$ and $g'(2) = 3$, calculate $f'(2)$.

THINKING	WORKING
1 Define $u(x)$ and $v(x)$.	$f(x) = x^3\sqrt{g(x)}$ $= x^3(g(x))^{\frac{1}{2}}$ Let $u(x) = x^3$ and let $v(x) = (g(x))^{\frac{1}{2}}$.

2 Differentiate $u(x)$ and $v(x)$ to find $u'(x)$ and find $v'(x)$.

$$u'(x) = 3x^2$$

$$v'(x) = \frac{1}{2}g'(x)(g(x))^{-\frac{1}{2}}$$
$$= \frac{g'(x)}{2\sqrt{g(x)}}$$

3 Calculate $f'(x) = u(x)v'(x) + v(x)u'(x)$.

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$
$$= (x^3)\left(\frac{g'(x)}{2\sqrt{g(x)}}\right) + \left(\sqrt{g(x)}\right)(3x^2)$$
$$= \frac{x^3 g'(x)}{2\sqrt{g(x)}} + 3x^2\sqrt{g(x)}$$

4 Substitute for x, $g(x)$ and $g'(x)$.

For $x = 2$, $g(2) = 4$ and $g'(2) = 3$:

$$f'(2) = \frac{2^3 g'(2)}{2\sqrt{g(2)}} + 3(2^2)\sqrt{g(2)}$$
$$= \frac{8 \times 3}{2 \times \sqrt{4}} + 12 \times \sqrt{4}$$
$$= 6 + 24$$
$$= 30$$

Derivative of a quotient of two functions

The derivative of a quotient is not simply the quotient of the derivatives. To differentiate a quotient of two functions, use the *quotient rule*.

The quotient rule of differentiation is used to find the derivative when one function is divided by another.

Additional information

Quotient rule

Watch the video and try this activity to check your understanding of the quotient of two functions.

The *quotient rule* relates the derivative of the quotient of two functions to the functions and their derivatives.

- If $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.
- If $u = f(x)$ and $v = g(x)$, then $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $v \neq 0$.

A proof of the quotient rule

Let $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, where f and g are differentiable functions and h is defined for all values of x in the domain.

$$
\begin{aligned}
h'(x) &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}\\
&= \lim_{h\to 0}\frac{g(x)\frac{f(x+h)-f(x)}{h}-f(x)\frac{g(x+h)-g(x)}{h}}{g(x)g(x+h)}\\
&= \frac{\lim\limits_{h\to 0}g(x)\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}-\lim\limits_{h\to 0}f(x)\lim\limits_{h\to 0}\frac{g(x+h)-g(x)}{h}}{g(x)\lim\limits_{h\to 0}g(x+h)}\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{g(x)g(x)}\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}
\end{aligned}
$$

$\therefore$ if $h(x)=\dfrac{f(x)}{g(x)}$, $g(x)\neq 0$, then $h'(x)=\dfrac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$.

Alternatively, if $h(x)=\dfrac{f(x)}{g(x)}$, $g(x)\neq 0$, then by the product rule:

$$h'(x)=f(x)\frac{d}{dx}\left(\frac{1}{g(x)}\right)+\frac{1}{g(x)}f'(x).$$

If $u=\dfrac{1}{g(x)}$, then by using the chain rule you obtain $\dfrac{du}{dx}=-\dfrac{g'(x)}{[g(x)]^2}$.

$$
\begin{aligned}
h'(x) &= \frac{-f(x)g'(x)}{[g(x)]^2}+\frac{f'(x)}{g(x)}\\
&= \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}
\end{aligned}
$$

$\therefore$ If $h(x)=\dfrac{f(x)}{g(x)}$, $g(x)\neq 0$, then $h'(x)=\dfrac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$.

Additional information

Differentiating with the quotient rule

Watch the video and try the questions to check your understanding of the quotient rule of differentiation.

WARNING

Do not assume that all expressions that are quotients must be differentiated using the quotient rule. Always check to see whether the expression can be simplified and then differentiated.

For example:

$$
\begin{aligned}
f(x) &= \frac{x^3-4x^2+5x}{x}\\
&= \frac{x^3}{x}-\frac{4x^2}{x}+\frac{5x}{x}\\
&= x^2-4x+5
\end{aligned}
$$

Then using the power rule:
$f'(x)=2x-4$

The quotient rule, although more complicated for simple expressions can also be used to differentiate the function $f(x) = \dfrac{x^3 - 4x^2 + 5x}{x}$.

Let $u(x) = x^3 - 4x^2 + 5x$ and $v(x) = x$.

Then $u'(x) = 3x^2 - 8x + 5$ and $v'(x) = 1$.

$$\begin{aligned} f'(x) &= \frac{v(x)u'(x) - u(x)v'(x)}{v^2} \\ &= \frac{x \times (3x^2 - 8x + 5) - (x^3 - 4x^2 + 5x) \times 1}{x^2} \\ &= \frac{3x^3 - 8x^2 + 5x - x^3 + 4x^2 - 5x}{x^2} \\ &= \frac{2x^3 - 4x^2}{x^2} \\ &= \frac{2x^3}{x^2} - \frac{4x^2}{x^2} \\ &= 2x - 4 \end{aligned}$$

29 Perform quotient rule differentiation

Using the quotient rule, determine the expression for $\dfrac{dy}{dx}$, given $y = \dfrac{3x}{x^2 - 5x + 6}$.

THINKING	WORKING
1 Define u and v.	$y = \dfrac{3x}{x^2 - 5x + 6}$ Let $u = 3x$ and $v = x^2 - 5x + 6$.
2 Differentiate u and v to find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.	$\dfrac{du}{dx} = 3$ and $\dfrac{dv}{dx} = 2x - 5$
3 Calculate $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ and simplify.	$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \dfrac{(x^2 - 5x + 6)(3) - (3x)(2x - 5)}{(x^2 - 5x + 6)^2}$ $= \dfrac{3x^2 - 15x + 18 - 6x^2 + 15x}{(x^2 - 5x + 6)^2}$ $= \dfrac{-3x^2 + 18}{(x^2 - 5x + 6)^2}$ $= \dfrac{-3(x^2 - 6)}{(x^2 - 5x + 6)^2}$

30 Perform quotient rule differentiation involving algebraic fraction manipulation

Using the quotient rule, determine an expression for $\frac{dy}{dx}$, given $y = \frac{\sqrt{x}}{x+1}$.

THINKING	WORKING
1 Define u and v.	$y = \frac{\sqrt{x}}{x+1}$ Let $u = \sqrt{x}$ and let $v = x + 1$.
2 Differentiate u and v to find $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{dv}{dx} = 1$
3 Calculate $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ and simplify.	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)(1)}{(x+1)^2}$ $= \frac{\frac{x+1}{2\sqrt{x}} - \frac{2x}{2\sqrt{x}}}{(x+1)^2}$ $= \frac{x+1-2x}{2\sqrt{x}(x+1)^2}$ $= \frac{1-x}{2\sqrt{x}(x+1)^2}$

31 Perform quotient rule differentiation with function notation

The function f is defined by $f(x) = \frac{3x-4}{2x^2+3}$.

Calculate the gradient of the curve $y = f(x)$ at $x = -1$.

THINKING	WORKING
1 Define $u(x)$ and $v(x)$.	$f(x) = \frac{3x-4}{2x^2+3}$ Let $u(x) = 3x - 4$ and $v(x) = 2x^2 + 3$.
2 Differentiate $u(x)$ and $v(x)$ to find $u'(x)$ and $v'(x)$.	$u'(x) = 3$ and $v'(x) = 4x$
3 Calculate $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$ and simplify.	$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$ $= \frac{(2x^2+3)(3) - (3x-4)(4x)}{(2x^2+3)^2}$ $= \frac{-6x^2 + 16x + 9}{(2x^2+3)^2}$

4 Determine the value of $f'(-1)$.

$$f'(-1) = \frac{-6(-1)^2 + 16(-1) + 9}{(2(-1)^2 + 3)^2}$$
$$= -\frac{13}{25}$$

5 Interpret the answer.

The gradient of the curve at $x = -1$ is $-\frac{13}{25}$.

EXERCISE 10.7 Derivatives of products and quotients

Worked Example

1 Determine an expression for $\frac{dy}{dx}$ for the following functions. 25

(a) $y = (x+1)(x-3)$ (b) $y = (x-4)(1-x)^2$ (c) $y = x(3-x)^4$

2 Determine an expression for $f'(x)$ for the following functions.

(a) $f(x) = (x+3)(x^2+1)$ (b) $f(x) = (x^2 - 3x + 4)(x-2)$

3 Determine the gradient of $y = x(x-1)^2$ at the point $(-1, -4)$.

4 Consider the function f defined by $f(x) = (x^2 - 1)(x+2)$. 27

(a) Determine an expression for $f'(x)$.

(b) Determine the exact value(s) of x at which the graph $y = f(x)$ has zero gradient.

5 Consider functions f and g where $f(x) = xg(x)$. 28
Calculate $f'(-1)$, given $g'(-1) = 3$ and $g(-1) = -2$.

6 Consider functions f and g where $f(x) = x^2 g(x)$.
Calculate $f'(1)$, given $g(1) = 1$ and $g'(1) = -4$.

7 Determine an expression for $\frac{dy}{dx}$ for the following functions. 29

(a) $y = \frac{x}{x+1}$ (b) $y = \frac{x^2 + 4x}{x-1}$

8 Calculate the gradient of the curve $y = f(x)$ at $x = -3$ for the following functions. 31

(a) $f(x) = \frac{2x+1}{x^2-4}$ (b) $f(x) = \frac{x-1}{(x+1)^2}$

9 Consider the function f defined by $f(x) = \frac{x}{1+x^2}$.

(a) Determine an expression for $f'(x)$.

(b) Determine the value(s) of x at which the graph has zero gradient.

10 Given $f(-1) = 4$, $f'(-1) = -2$, $g(-1) = -4$ and $g'(-1) = 3$.
Calculate $h'(-1)$ if:

(a) $h(x) = f(x)g(x)$

(b) $h(x) = \dfrac{f(x)}{g(x)}$

11 Consider functions f and g where $f(x) = \dfrac{x^3}{g(x)}$, $g(x) \neq 0$.
If $g(1) = 2$ and $g'(1) = -3$, calculate $f'(1)$.

12 Consider the function $h(x) = f(x)g(x)$.

(a) Given that $g(4) = 8$, $g'(4) = 6$, $f'(4) = 4$ and $f(4) = 3$, what is the value of $h'(4)$?

A 0 **B** 12 **C** 48 **D** 50

(b) Explain the common error made by a student who thought the answer was 24.

13 Consider the function $h(x) = \dfrac{f(x)}{g(x)}$.
Given that $g(4) = 8$, $g'(4) = 6$, $f'(4) = 4$ and $f(4) = 3$, what is the value of $h'(4)$?

A $\dfrac{7}{32}$ **B** $\dfrac{2}{3}$ **C** $\dfrac{25}{32}$ **D** 50

Worked Example

26

14 Determine an expression for $\dfrac{dy}{dx}$ as a single algebraic fraction, given $y = \sqrt{x}(x+1)$.

15 Determine an expression for $f'(x)$ as a single algebraic fraction, given $f(x) = x^2\sqrt{x^2+1}$.

16 Consider the function g defined by $g(x) = x\sqrt{3-x}$. Determine:

(a) an expression for $g'(x)$ as a single algebraic fraction

(b) the x-value at which the graph of $y = g(x)$ has zero gradient

(c) the exact set of x-values for which the gradient is less than 3.

30

17 Determine an expression for $\dfrac{dy}{dx}$ as a single fraction for each function.

(a) $y = \dfrac{\sqrt{x}}{2x+3}$

(b) $y = \dfrac{\sqrt{9-x^2}}{2x}$

18 Calculate the gradient of the curve $y = f(x)$ at $x = -1$ for each function.

(a) $f(x) = \dfrac{x}{\sqrt{1-2x}}$

(b) $f(x) = \sqrt{\dfrac{x^2+3}{x^4+3}}$

19 The function f is defined by $f(x) = \dfrac{x^2-2}{x^2+2}$.
Calculate the gradient of the curve $y = f(x)$ at $x = -2$.

20 Consider the function f defined by $f(x) = \dfrac{x}{\sqrt{x}+1}$, $x \geq 0$. Determine:

(a) an expression for $f'(x)$ as a single algebraic fraction

(b) the exact gradient of the graph of $y = f(x)$ at $x = 4$

(c) the number of points of zero gradient for $f(x)$.

21 Consider the function $y = \dfrac{x^2 + 2x}{x^3 - 3x + 4}$.

(a) Determine an expression for $\dfrac{dy}{dx}$ in factorised form.

(b) Calculate, correct to 2 decimal places where necessary, the coordinates of the points on the graph where $\dfrac{dy}{dx} = 0$.

22 Consider the function $y = (x - 4)(x + 1)^2$.

(a) Determine an expression for $\dfrac{dy}{dx}$ in factorised form.

(b) Calculate the gradient of the graph at the points where it crosses the x-axis.

(c) Determine the exact coordinates of the points on the graph where $\dfrac{dy}{dx} = 0$.

(d) Determine the exact set of values of x for which the gradient of the graph is greater than $\dfrac{1}{2}$.

23 Consider the two functions $f(x) = 1 - x$ and $g(x) = \sqrt{1 + x}$. Determine:

(a) an expression for $h_1(x) = f(x) \times g(x)$, its domain, and an expression for $h_1'(x)$

(b) an expression for $h_2(x) = \dfrac{f(x)}{g(x)}$, its domain, and an expression for $h_2'(x)$

(c) the points of zero gradient for both functions $h_1(x)$ and $h_2(x)$

(d) the exact x-value for which the two functions $h_1(x)$ and $h_2(x)$ have the same gradient.

24 Consider the family of functions defined by $f(x) = \dfrac{x^p}{x^2 + 1}$, where p is a positive integer. Determine:

(a) an expression for $f'(x)$ as a single fraction in factorised form

(b) the value(s) of p for which the graph of $y = \dfrac{x^p}{x^2 + 1}$ has two points of zero gradient, and then calculate the coordinates of these points

(c) the value(s) of p for which the graph of $y = \dfrac{x^p}{x^2 + 1}$ has only one point of zero gradient, and then calculate the coordinates of this point.

25 Consider the function g defined by $g(t) = \dfrac{3t}{t^2 - 1}$, $t \neq \pm 1$.

(a) Determine the range of g.

(b) Determine an expression for $g'(t)$ in factorised form.

(c) Calculate the gradient of the graph of $y = g(t)$ at $(0, 0)$.

(d) Make a statement about the sign of the gradient of the graph of $y = g(t)$.

(e) Determine the set of values of k for which the graphs $y = kt$ and $y = g(t)$ have exactly one point of intersection.

10

Summary

Gradients of curves and equations of tangents

Given $y = f(x)$, the quantity $f'(a)$ is the derivative at $x = a$ and is equal to the gradient of the tangent at $x = a$ and the gradient of the graph at $x = a$.

Rates of change

The average rate of change of a function f over the interval $a \le x \le b$ is given by $\dfrac{f(b) - f(a)}{b - a}$.

The instantaneous rate of change of a function f at $x = a$ is given by $f'(a)$.

The instantaneous rate of change of a function f is the gradient of the curve at $P(a, f(a))$.

Curve sketching

A stationary point with coordinates $(a, f(a))$ has $f'(a) = 0$.

If $f'(a) > 0$, then the function f is increasing at $x = a$.

If $f'(a) < 0$, then function f is decreasing at $x = a$.

A stationary point is a local maximum if its gradient changes from positive to negative, a local minimum if its gradient changes from negative to positive, and a stationary point of inflection if its gradient does not change sign.

Maximum and minimum values of a function can occur at an endpoint of an interval.

Maximum and minimum problems

To solve maximum/minimum problems follow these steps:

STEP 1 Define the dependent and independent variables and, if required, draw a labelled diagram.

STEP 2 Express the quantity (dependent variable) to be maximised or minimised as a function in terms of a single independent variable. Determine the domain of this function with respect to the context.

STEP 3 Determine the derivative of the function with respect to the independent variable, set it equal to zero and find the value(s) of the independent variable such that the derivative is zero.

STEP 4 Determine whether the value(s) of the independent variable gives the maximum or minimum value of the function. Check whether or not the maximum or minimum value of the function occurs at the endpoints of the function's domain.

STEP 5 Substitute the value into the function and state the maximum or minimum value of the function.

Calculus and distance-time graphs

For straight-line motion at time t, $x(t)$ denotes the position of a particle relative to a fixed point O in the line and $v(t)$, $x'(t)$ or $\frac{dx}{dt}$ denotes the velocity.

Consider the displacement–time graph $y = x(t)$. The gradient of the tangent to the curve at $t = a$ represents velocity. This matches the definition of velocity as the rate of change of the particle's position with respect to time.

Composite functions and their derivatives

The rule of f composition g, where g is applied first and f is applied second is denoted by $f(g(x))$.

The domain of the composite function, with rule $f(g(x))$, is the set of all values of x in the domain of g such that the output of g is in the domain of f.

For $f(g(x))$ to be defined, the range of $g \subseteq$ domain f.

If $f(g(x))$ is defined, the domain of the composite function is the same as the domain of g.

The chain rule relates the derivative of the composite of two functions to the functions and their derivatives.

If $y = f(u)$ and $u = g(x)$, then, in Leibniz notation, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. Alternatively, using function notation, the chain rule states that if $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$.

Derivatives of products and quotients

The *product rule* relates the derivative of the product of two functions to the functions and their derivatives. If $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.

In Leibniz notation, if $u = f(x)$ and $v = g(x)$, the product rule can be expressed as

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ or as $(uv)' = uv' + vu'$.

The *quotient* rule relates the derivative of the quotient of two functions to the functions and their derivatives. If $h(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

In Leibniz notation, with $u = f(x)$ and $v = g(x)$, the quotient rule can be expressed as

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or as $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ where $v \neq 0$.

Chapter review 10

Exercise 10.1

1 Consider the function $g(x) = (x-3)^2 - 4$.

(a) Calculate the coordinates of the point where the tangent to $y = g(x)$ is parallel to the x-axis.

(b) Determine the equation of the tangent to the graph of g at $x = 2$.

(c) Determine the gradient of $y = g(x)$ at the points of intersection with the x-axis.

Exercise 10.1

2 Calculate the coordinates of the point on the graph of $y = 2x^2 - 3x + 3$ at which the tangent is parallel to the line $x - y = 1$.

Exercise 10.1

3 $P(1, 3)$ lies on the graph of $y = -x^2 + 4$. Q is a point on the graph with x-coordinate $1 + h$. Determine the gradient of the secant PQ.

Exercise 10.1

4 For $f(x) = x^2 + 6x - 2$, determine the subdomain for which the graph of $y = f(x)$ has positive gradient.

Exercise 10.2

5 Let $f(x) = x^3 + 2x$.

(a) Calculate the average rate of change of f over the interval $0 \le x \le 3$.

(b) Calculate the instantaneous rate of change of f with respect to x at $x = 3$.

Exercise 10.2

6 The surface area, S m^2, of a spherical weather balloon of radius r metres is given by $S(r) = 4\pi r^2$. The balloon is inflated with air.

(a) Calculate the exact average rate of change of the balloon's surface area with respect to its radius if the radius increases from 1 metre to 2 metres.

(b) Calculate the exact rate of change of the balloon's surface area with respect to its radius when the radius is 2 metres.

Exercise 10.2

7 The temperature, W°C, on a winter's day in Stanthorpe can be modelled by $W = -\frac{2}{9}t^2 + \frac{8}{3}t + 1$, $0 \le t \le 12$, where t is the time in hours after 6 am.

(a) Determine the temperature at 6 am.

(b) Use calculus to determine the maximum temperature and when it occurs.

(c) Calculate the average rate at which the temperature increases from 8 am to 12 noon.

(d) Determine the rate at which the temperature is increasing at 9 am.

(e) Determine the time at which the temperature is decreasing at 2°C/hour.

(f) Determine when the temperature decreases at 1°C/hour.

Exercise 10.3

8 Consider the function $g(x) = x^3 - 3x^2 + 4x - 2$.

(a) Determine an expression for $g'(x)$.

(b) Determine, by calculating the value of the discriminant, whether the graph of $y = g(x)$ has any stationary points.

9 Which one of the following statements regarding the function $y = 2x^3 + 5x - 7$ is incorrect? Exercise 10.3

A The graph of the function has positive gradient for all values of x.

B The minimum gradient is 5 and occurs at (0, 5).

C The derivative is $\frac{dy}{dx} = 6x^2 + 5$.

D The gradient is 11 when $x = 1$.

10 The following information is known about the function $y = g(x)$. Exercise 10.3

$g(1) = 0$, $g'(1) = 0$, $g'(x) > 0$ for $x < 1$ and $g'(x) < 0$ for $x > 1$

Determine the coordinates of the stationary point on the graph of $y = g(x)$ and determine its nature.

11 A particle moves in a straight line such that its position x metres relative to an origin O at time t seconds is given by $x(t) = 12t - t^3$, $t \geq 0$. Exercise 10.5

(a) Calculate the initial velocity of the particle.

(b) Determine the time and position of the particle when the velocity is zero.

12 A particle moves in a straight line such that its position x metres relative to an origin O at time t seconds is given by $x(t) = t(t - 2)^2$, $t \geq 0$. Exercise 10.5

(a) Determine the initial position of the particle.

(b) Determine the time when the particle is instantaneously at rest.

13 The height h (m) of a body above the ground at time t (s) is $h(t) = -5t^2 + 16t + 1$. Which one of the following statements is incorrect? Exercise 10.5

A The average rate of change of the body's height over the first 3 seconds is 1 m/s.

B The rate of change of the body's height is zero at $t = 1.6$ s.

C The initial rate of change of the body's height is 16 m/s.

D The rate of change of the body's height after three seconds is 4 m/s.

14 If $y = (1 - 3x + x^2)^3$, determine an expression for $\frac{dy}{dx}$. Exercise 10.6

15 Determine an expression for $\frac{dy}{dx}$ for each of the following. Exercise 10.7

(a) $y = (x^2 + 1)(x - 3)^2$

(b) $y = \frac{x^2 + 2}{x - 4}$

(c) $y = \frac{3(x - 5)^4(2x - 7)^5}{3x + 1}$

16 The point on the graph of $y = \frac{1}{2}x^2 - 4x + 3$ at which the tangent is parallel to the line $y = -6x - 1$ is: Exercise 10.1

A (-2, 12)

B (-2, 13)

C (2, -1)

D (2, -13)

17 Consider the function $y = x^2 - (a + b)x + ab$, where a and b are numbers. Exercise 10.1

The tangents to the graph of the function at the points of intersection with the x-axis intersect where:

A $x = \frac{a + b}{2}$

B $x = a + b$

C $x = \frac{a - b}{2}$

D $x = a - b$

Exercise 10.2

18 A biologist studies a population of a species of bird on a remote island.

The number of birds, B, can be modelled by the equation $B = 6(2t + 1)(12 - t)$, $0 \le t \le 6$, where t is the time in months from when the species was first observed.

(a) Use calculus to determine the maximum number of birds and state when this occurs.

(b) Determine the time when the rate of change in the bird population is the same as the average rate of change in the bird population over the sixth month period, and calculate the number of birds present at this time.

Exercise 10.3

19 Consider the function $f(x) = 2x^2 - x^3$.

(a) Use calculus to locate the coordinates of any stationary points and determine their nature.

(b) Sketch the graph of $y = f(x)$, showing all key features.

Exercise 10.4

20 Determine the maximum area of a rectangle inscribed in an isosceles right-angled triangle whose hypotenuse is 10 m in length. Assume that the length of the rectangle lies along the hypotenuse.

Exercise 10.4

21 A piece of wire 100 cm long is to be cut into two pieces, one of which is to be formed into a square and the other into a circle. Let x be the side length of the square.

(a) Determine an expression for S, the sum of the areas of the square and circle, in terms of x.

(b) Calculate how the wire should be cut if the sum of the enclosed areas is a minimum. Give your answer to the nearest centimetre.

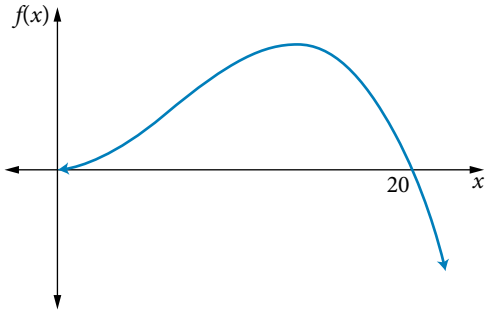
(c) How should the wire be cut if the sum of the enclosed areas is to be a maximum?

Exercise 10.4

22 The equation of a curve is of the form $f(x) = ax^2(x - h)$.

(a) Determine the values of a and h, given that $f'(1) = \frac{185}{64}$ and $f(20) = 0$.

(b) Determine the value of x for which $f(x)$ is a maximum in the domain $0 \le x \le h$ and calculate the maximum value of $f(x)$, correct to 2 decimal places.



Exercise 10.5

23 Two particles move in a straight line such that their displacements x metres from an origin O at time t seconds are given by $x_1(t) = t^2 - 3t + 2$ and $x_2(t) = 2t - t^2$ respectively.

(a) Determine the initial position and velocity of each particle.

(b) Calculate the time(s) when the two particles have the same position.

(c) Calculate the time(s) when the two particles travel with the same velocity.

(d) When in the same position, determine the direction in which each particle travels.

Exercise 10.6

24 For $y = \sqrt{1 - (g(x))^2}$, determine an expression for $\frac{dy}{dx}$ as a single algebraic fraction.

Exercise 10.7

25 Consider the function f defined by $f(x) = x^2\sqrt{2x - 3}$.

(a) Determine an expression for $f'(x)$ as a single algebraic fraction.

(b) Determine the values of x at which the curve $y = f(x)$ has zero gradient.

Exercise 10.7

26 Calculate the gradient of the curve $y = \frac{x + \sqrt{1 + x^2}}{x - 2}$ at $\left(0, -\frac{1}{2}\right)$.

27 For the function $g(x) = (x+4)^2(x-1)$, determine the subdomain for which the gradient of g is negative. Exercise 10.7

28 Write an expression for the derivative of $\dfrac{f(x)g(x)}{h(x)}$, where $h(x) \neq 0$. Exercise 10.7

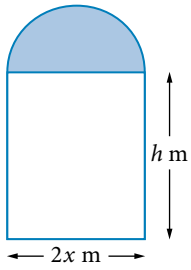
29 For the curve $y = \dfrac{3}{x} + 4$, $x \neq 0$, determine the equation of the tangent parallel to the tangent where the graph crosses the x-axis. Exercise 10.1

30 Consider the function $f(x) = x(x-b)^2$ where b is a number. Exercise 10.3

(a) Calculate the coordinates of the stationary points on the graph of $y = f(x)$.

(b) Demonstrate that the x-value of the middle point of intersection of the graph of $y = f(x)$ and the line joining the stationary points is the average of the x-coordinates of the stationary points.

31 A window in the composite shape of a rectangle and a semicircle is designed to allow in the maximum amount of light. The semicircular section of the window is made from stained glass, which lets in 1 unit of light per square metre. The rectangular section consists of clear glass, which lets in 4 units of light per square metre. The rectangular section has dimensions $2x$ metres by h metres. Exercise 10.4



If the perimeter of the window is 20 metres, determine, correct to 2 decimal places, the values of x and h that need to be used to let in the maximum amount of light.

32 A particle is moving in a straight line so that x, its displacement in metres from the origin at any time t (seconds) is given by: $x(t) = \begin{cases} 6t - t^2, & 0 \le t \le 4 \\ 16 - 2t, & 4 < t \le 8 \end{cases}$ Exercise 10.5

Describe the motion of the particle, supporting your solution with displacement–time and velocity–time graphs.

33 In Australia, the legal maximum blood alcohol concentration is 0.05% for licenced drivers in private vehicles. Peter has taken part in a controlled study into how his blood alcohol concentration changed over a period of 8 hours. It was found that Peter's blood alcohol concentration b% at time t hours after consuming a very strong alcoholic drink can be modelled by $b(t) = \dfrac{0.25t}{t^2+2}$, $0 \le t \le 8$. Exercise 10.7

(a) After consuming the drink, calculate, correct to the nearest minute, the time it would take before it would be illegal for Peter to drive a car.

(b) Determine the time duration for which it would be illegal to drive a car. Give your answer correct to the nearest minute.

(c) Use calculus to find the exact time at which Peter's blood alcohol concentration is a maximum.

(d) Calculate Peter's maximum blood alcohol concentration, correct to 2 decimal places.

(e) Determine the exact instant when Peter's blood alcohol concentration is decreasing most rapidly.

(f) Determine Peter's blood alcohol concentration at the exact instant when it is decreasing most rapidly. Give your answer correct to 2 decimal places.

11

Discrete random variables

Recall 660

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11 Recall

Determine simple probability

1 Calculate the following probabilities by comparing the number of successful outcomes with the total number of outcomes.

(a) A mini-pack of sweets of the same size contained 6 orange, 4 green, 3 blue and 1 black sweet. Determine the probability that a sweet chosen at random is:

(i) orange (ii) not blue.

(b) A multiple-choice test consists of five questions, each of which has three options. Determine the probability that you correctly answer all of the questions if you guess randomly.

(c) A card is drawn at random from a standard pack of playing cards (52 cards). Determine the probability that the card is black or a 3 or both.

Determine conditional probability from the sample space

2 Calculate the following conditional probabilities directly from the sample space.

(a) A coin is tossed three times. Determine the probability that exactly two heads are obtained, given that the first toss resulted in a head.

(b) A fair six-sided die labelled 1–6 is rolled twice. Determine the probability that the sum of the two numbers obtained is greater than 8, given the first number rolled was a 5.

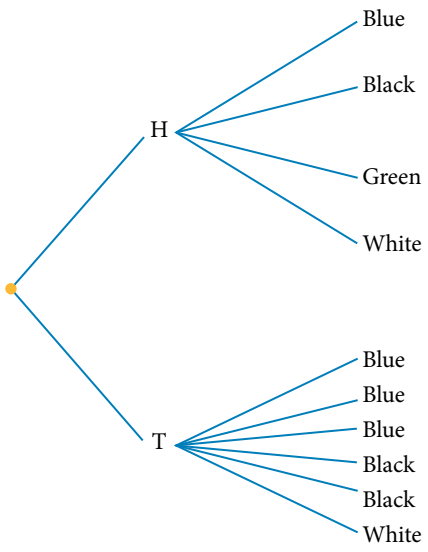
(c) A spinner evenly divided into four sections and labelled 1–4 is spun three times. Determine the probability that the sum of the three numbers is less than 6, given the first number spun was a 2.

Draw a tree diagram to assist in finding the sample space and associated probabilities

3 Spinner A has four equal sections: one blue, one black, one green and one white.

Spinner B has six equal sections: three blue, two black and one white.

A biased coin with $P(\text{heads}) = 2P(\text{tails})$ is tossed. If it lands heads, Spinner A is spun once; otherwise, Spinner B is spun once.



(a) Calculate P(heads) for the biased coin.

(b) Complete the tree diagram by writing the correct probability for each branch.

(c) Complete the sample space for this experiment.

(d) Determine the probability that the spinner will land on blue.

(e) Determine the probability that the spinner will land on green.

Determine the rule for a linear function

4 Determine the rule in the form $y = mx + c$ for each of the following.

(a)

x	5	6	7	8
y	44	47	50	53

(b) Arithmetic progression: 60, 55, 50, …, where x is the term number and y is the term value.

Solve simultaneous equations

5 Solve the following pairs of simultaneous equations.

(a) $3x + 2y = 5$, $2x - 5y = 16$

(b) $3x + 2y = -16$, $5x - y = -18$

(c) $2x - 5y = 16$, $3y = 6 - 4x$

Determine the number of combinations

6 Determine the number of ways the given number of items can be selected from the group of items.

(a) 3 items from a group of 12

(b) 6 items from a group of 9

(c) 12 items from a group of 15

11.1 Random variables and probability

Discrete random variables

When performing a sampling procedure, a number of different outcomes are expected. For example, when rolling a die a large number of times, you can expect to observe each of the values from {1, 2, 3, 4, 5, 6}. The outcome will vary between rolls. The observed outcome X is a *discrete random variable* because the list of possible outcomes is countable.

If the list of possible outcomes is not countable, then the variable is a *continuous random variable*.

WARNING

Do not assume that a discrete random variable is restricted to whole number values. If the list of outcomes is countable, then the variable is discrete. For example, the set {0, 0.2, 0.4, 0.6, 0.8, 1} are not all integers, but the total number of different values can be counted, so the variable is discrete.

Shoe sizing (e.g. $7, 7\frac{1}{2}, 8, 8\frac{1}{2}, 9, 9\frac{1}{2}$) is an example of ordinal (categorical) data converted to a numerical scale using fractions and treated as discrete numerical data. The half sizes can mean an extra width of the whole-number size, so $7\frac{1}{2}$ could mean a wide size 7.

Grade point average (GPA) calculated by universities (e.g. 0, 0.5, 1.5, 2.5, 3.5, 4.0) is an example of ordinal (categorical) data converted to a numerical scale using decimals and treated as discrete numerical data.

Statistical models

Statistical modelling is a process used to predict real-world events. A *statistical model* consists of equations that are based on observed patterns and relationships between the variables. The equations produced by the model can then be tested against some real-world data. It is unlikely that a model will account accurately for 100% of the data, so the model is often modified to make it better fit the data. For example, you might suggest that adults have the same arm span as their height. However, after collecting a large quantity of data, arm span might be observed to be closer to 97% of height.

Notation

Capital letters such as X and Y are used for random variables, and the matching lower case letter such as x and y are used for the values of the random variable. Subscripts distinguish the various possible values of X. As an example, $P(X = x_2)$ is the probability that the variable X has the value x_2.

When tossing 3 coins, there are 8 different possible outcomes, but if you were only interested in the number of heads, then there are only 4 possibilities: 0, 1, 2 and 3 heads. (The number of tails has the same number of possibilities.)

The probability of obtaining 3 heads is the same as the probability of no heads.

Making connections

Three coin toss simulator

Use the simulator to determine the experimental probability of tossing three coins.

The probability of obtaining 1 head is the same as the probability of 2 heads.

In some cases, the variable is not the actual outcome but rather a value assigned to a particular outcome. Consider the experiment in which 3 coins are tossed. Let X stand for the number of heads obtained. There are 8 possible outcomes.

Outcome observed	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
Value of X	3	2	2	2	1	1	1	0

So, X can assume any of the values from {0, 1, 2, 3}. Assuming that each of the 8 observed outcomes are equally likely, which is a reasonable assumption using fair coins, a probability table can be created for the variable X.

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

1 Use the sample space to determine a probability

A fair coin is tossed 4 times. Determine the probability of obtaining exactly 2 heads.

THINKING	WORKING
1 Create the sample space for the experiment. (Two outcomes per coin, $2 \times 2 \times 2 \times 2 = 16$ outcomes.)	{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT}
2 Define the variable to be used.	Let X be the number of heads obtained, so $X = 2$.
3 Identify the outcomes that correspond to the event.	HHTT HTHT HTTH THHT THTH TTHH
4 Calculate the probability for each successful outcome.	$P(\text{HHTT}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ $= \frac{1}{16}$ Each outcome has the same probability of $\frac{1}{16}$.
5 Calculate the probability for all possible outcomes.	x (number of heads): 0, 1, 2, 3, 4 number of outcomes: 1, 4, 6, 4, 1 $P(X=x)$: $\frac{1}{16}$, $\frac{4}{16}=\frac{1}{4}$, $\frac{6}{16}=\frac{3}{8}$, $\frac{4}{16}=\frac{1}{4}$, $\frac{1}{16}$ 6 of the outcomes contain 2 heads:
6 Interpret the answer.	When tossing a fair coin 4 times, the probability of obtaining exactly 2 heads is $\frac{3}{8}$.

When the outcomes are equally likely you can use:

$$P(\text{event}) = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$$

Theoretical probabilities are generally written as fractions because they represent exact values.

Calculate the sample size

Sometimes you need to determine the sample size necessary to meet a particular condition. In many of these questions it is useful to use the complementary event.

For complementary events: $P(A') = 1 - P(A)$.

2 Calculate the sample size

A goal attack in netball scores a goal from 75% of her shots. How many shots at goal would she need to take before she would have at least a 95% chance of scoring at least one goal?

THINKING	WORKING
1 Rethink the question using complementary events.	P(scoring at least one goal) = 1 − P(scoring no goals)
2 What is required in this question?	P(scoring no goals) < 0.05
3 Use trial and error to determine the required value.	Let M be 'misses goal'. $P(\text{M}) = 0.25$ $P(\text{MM}) = 0.25 \times 0.25$ $= 0.0625$ $P(\text{MMM}) = 0.25 \times 0.25 \times 0.25$ $= 0.015\,625$ $0.015\,615 < 0.05$
4 Interpret the answer.	The goal attack would need to take at least three shots to be at least 95% certain of scoring at least one goal.

Discrete probability distributions

A discrete probability distribution lists each of the possible x-values and their probabilities. For example, the table of probabilities when 3 coins are tossed is a discrete probability distribution because x is a discrete variable and the sum of the probabilities adds to 1.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

For a discrete probability distribution:

$0 \le P(X) \le 1$, for all values of x

$\Sigma P(X = x) = 1$, where the symbol Σ is the uppercase Greek letter *sigma* and is used to represent '*the sum of* …'.

If either of these conditions is not met, then it is not a discrete probability distribution.

3 Identify a discrete probability distribution

Does the data in the table represent a discrete probability distribution?

x	1	3	5	7
$P(X=x)$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{5}$

THINKING	WORKING
1 A discrete probability distribution has: $0 \le P(X) \le 1$ Are all probabilities between 0 and 1 inclusive?	Yes
2 For a discrete probability distribution: $\sum P(X=x)=1$ Do the probabilities add up to 1?	$\frac{1}{5}+\frac{1}{10}+\frac{3}{10}+\frac{2}{5}=\frac{2}{10}+\frac{1}{10}+\frac{3}{10}+\frac{4}{10}$ $=\frac{10}{10}$ $=1$
3 Answer the question.	Both conditions have been met, so the table of data represents a discrete probability distribution.

4 Determine the value of an unknown in a discrete probability distribution

Determine the value of k, given that the data in this table represents a discrete probability distribution.

x	3	4	5	6	7
$P(X=x)$	0.14	k	0.36	0.21	0.13

THINKING	WORKING
1 A discrete probability distribution means that $\Sigma P(X=x)=1$.	$\Sigma P(X=x)=0.14+k+0.36+0.21+0.13$ $=k+0.84$ $=1$
2 Solve for k.	$k+0.84=1$ $k=0.16$
3 Interpret the answer.	The conditions for a probability distribution have been met if and only if $k=0.16$.

5 Determine the value of a repeated unknown in a discrete probability distribution

The following discrete probability distribution represents a five-sector spinner with sectors that are not equal.

Determine the value of k.

x	1	2	3	4	5
$P(X=x)$	k	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{4}$	0.25

THINKING

For a discrete probability distribution:

$\sum P(X=x)=1$

Write an equation for the sum of the probabilities and then solve for k.

WORKING

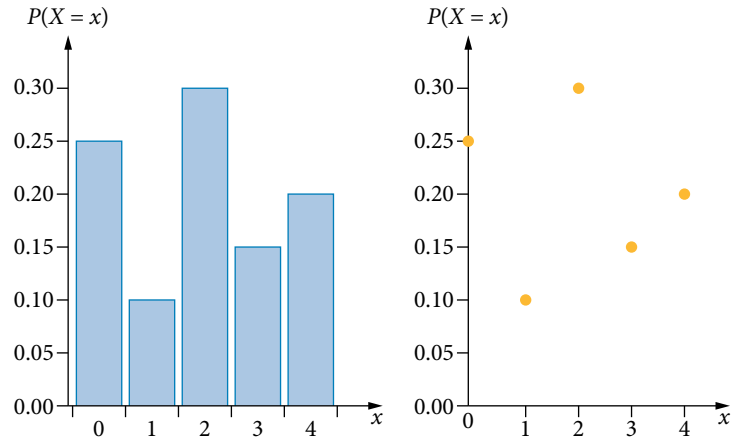
$$k+\frac{k}{2}+\frac{k}{3}+\frac{k}{4}+0.25=1$$
$$\frac{12k}{12}+\frac{6k}{12}+\frac{4k}{12}+\frac{3k}{12}+0.25=1$$
$$\frac{25k}{12}+0.25=1$$
$$\frac{25k}{12}=0.75$$
$$25k=9$$
$$k=\frac{9}{25}$$
$$k=0.36$$

Graphs of probability distributions

Consider the following table of values:

x	0	1	2	3	4
$P(X=x)$	0.25	0.1	0.3	0.15	0.2

The graph of the distribution is shown on the right in two different ways: a column graph on the left and a dot graph on the right.



A discrete probability distribution could also be represented using similar notation as a piece-wise function, if the possible values all have the same probability. For example, the distribution for rolling an unbiased is:

$$P(X=x)=\begin{cases}\frac{1}{6}, & \text{if } x\in\{1,2,3,4,5,6\}\\ 0, & \text{for all other values of } x\end{cases}$$

Here you have specified explicitly the probability attached to every value, not just those with non-negative probability.

EXERCISE 11.1 Random variables and probability

Worked Example

1 Determine the probability of each of the following events. Express your answers as fractions in simplest form. 1

(a) Exactly one head if three fair coins are tossed.

(b) At least one head if three fair coins are tossed.

(c) Exactly two even numbers if two fair dice are rolled.

(d) Exactly one odd number if two fair dice are rolled.

2 Do the following tables represent discrete probability distributions? Justify your decision. 3

(a)

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

(b)

x	2	3	4	5	6
$P(X=x)$	$\frac{1}{18}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$

(c)

x	-1	0	1
$P(X=x)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

(d)

x	1	2	3	4	5
$P(X=x)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$

(e)

x	1	1.5	2	2.5	3
$P(X=x)$	20%	15%	30%	18%	17%

(f)

x	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{3}{4}$
$P(X=x)$	0.24	0.16	0	0.38	0.22

3 For each of the following examples, determine the value of k that enables X to represent a discrete probability distribution. 4

(a)

x	0	1	2	3	4	5
$P(X=x)$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{5}{18}$	k	$\frac{3}{18}$

(b)

x	0	1	2	3	4	5
$P(X=x)$	$\frac{1}{8}$	$\frac{5}{24}$	k	$\frac{7}{24}$	$\frac{1}{24}$	$\frac{1}{6}$

4 Indicate whether the following random variables would be associated with a discrete scale or a continuous scale.

(a) The number of students in each of your classes at school.

(b) The heights of teachers at your school.

(c) The sizes of the shirts worn by each of the students in your mathematics class.

(d) The neck circumference of each of the members of the Australian netball team.

(e) The number of tosses of a fair coin before a head is observed.

(f) The times it takes for a process worker to complete 50 items.

(g) The number of red lights you stop at, per day, on the way to school over the course of a month.

(h) The number of whole lessons missed, per student, this year by members of your English class.

(i) The number of people in the queue when you enter the bank each Friday for a year.

5 The following table represents a discrete probability distribution.

x	3	5	7	9
$P(X=x)$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{3}$	k

The value of k is:

A $\frac{2}{3}$ **B** $-\frac{1}{3}$ **C** $\frac{1}{3}$ **D** 0

6 An apartment complex is built with the units having two, three or four bedrooms in the ratio 3 : 6 : 1. One of the units is chosen at random.

(a) Complete the table for the probability distribution of the number of bedrooms the unit contains. Write the probabilities in decimal form.

(b) Draw a bar graph for the distribution.

x	2	3	4
$P(X=x)$			

7 Two fair dice are rolled. Let X be the total of the two dice.

(a) Calculate the lowest total possible and the highest total possible.

(b) Complete the table for the probability distribution of X. Express the probabilities as fractions in simplest form.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$											

(c) Calculate $P(X \geq 6)$. **(d)** Calculate $P(X < 10)$. **(e)** Calculate $P(4 \leq X \leq 10)$.

8 Two fair coins are tossed. T stands for the number of tails obtained.

(a) Complete the table for the probability distribution of T. Express the probabilities in decimal form.

(b) Draw a dot graph of the distribution.

t	0	1	2
$P(T=t)$			

9 A fair die is rolled and X is the cube of the number showing. Express the probabilities in simplest fraction form.

(a) Determine the smallest value X can take and the largest value it can take.

(b) Use a table for the probability distribution of X.

(c) Calculate $P(X < 100)$.

10 Alexander is the manager of a shoe shop. The shop sells shoes only in the following sizes: 7, 8, 9, 10 and 11. In order to stock the shop according to market demand, Alexander records the sizes of shoes he sells in a month. The table shows his results.

Shoe size	7	8	9	10	11
Number of pairs	24	a	72	a	8

(a) If Alexander sold a total of 200 pairs of shoes and the random variable X represents the shoe size sold, create a discrete probability distribution table (using decimals) for the sale of shoes, given their size, by first calculating the value of a.

(b) Determine the probability that the next customer who buys a pair of shoes will buy a pair of shoes of size less than 10 but greater than 7.

(c) Calculate the probability that the next two pairs of shoes that Alexander sells are a size 10 followed by a size 11. Assume the events are independent.

(d) Alexander has noticed that he made a mistake when he recorded the sales over the given month. In one sale he sold two pairs of shoes whose sizes added up to 18. Mistakenly he recorded two pairs of shoes of size 9, but when he checked the receipt again he realised that the two pairs were of sizes 7 and 11.

(i) Recreate the discrete probability distribution table to display the actual values.

(ii) Calculate the probability that the next two pairs of shoes that Alexander sells are a size 10 followed by a size 11, using the actual values. Assume the events are independent.

Worked Example

11 Wayne is a soccer player who takes the penalty kicks awarded to his team. He has a 60% chance of scoring from the penalty spot. For Wayne to be 95% certain of scoring at least one goal, how many penalty kicks would he need to take? 2

12 For each of the following examples determine the value of k that enables X to represent a discrete probability distribution. 5

(a)

x	5	6	7	8	9
$P(X = x)$	$\frac{1}{9}$	k	$\frac{k}{2}$	$\frac{4}{9}$	$\frac{1}{9}$

(b)

x	8	9	10	11	12
$P(X = x)$	$\frac{1}{6}$	$3k$	$\frac{1}{3}$	k	$\frac{1}{6}$

13 You have 10 cards in a reduced pack. Five of the cards are hearts, three are diamonds and two are spades. You draw two cards, with replacement, from the pack. Let X be the number of diamonds drawn.

(a) List the values that X can take.

x	0	1	2
$P(X=x)$			

(b) Complete the table for the probability distribution of X. Express the probabilities in decimal form.

(c) Determine the probability of drawing exactly one red card from the pack.

(d) Determine the probability of drawing at least one red card from the pack.

14 The following discrete probability distribution represents a six-sector spinner with sectors that are not equal.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{k}{3}$	$\frac{k}{4}$	$\frac{k}{6}$	$\frac{k}{12}$	$\frac{1}{8}$	$\frac{k}{24}$

(a) Which of the following values for k makes this a discrete probability distribution?

A $\frac{5}{12}$ **B** $\frac{1}{3}$ **C** $\frac{1}{5}$ **D** 1

(b) Explain the common error made by a student who thought the answer was $\frac{8}{7}$.

15 Determine the value of k that makes the following a discrete probability distribution table.

x	1	2	3	4
$P(X=x)$	$k+0.1$	$k-0.1$	$k+0.55$	$k-0.15$

16 A board game uses a spinner divided equally into eight sections, numbered 1 to 8. A player must spin a 6, 7 or 8 to start the game.

(a) Determine the probability of spinning a 6, 7 or 8.

(b) Determine the probability of not rolling a 6, 7 or 8 on the first 3 attempts. Give your correct to 3 decimal places.

(c) Determine the probability that a player has not been able to start after the sixth attempt. Give your answer correct to 3 decimal places.

(d) Part of the game design process is to ensure it is not too difficult for a player to start.

Determine the number of attempts a player requires to be at least 97% certain that they have started to play the game.

17 In a game a coin is tossed and a fair die is rolled. If the coin shows tails, then X is the score on the die. If the coin shows heads, then X is the square of the score on the die.

(a) Draw a table for the probability distribution of X. Express the probabilities as fractions in simplest form.

(b) Calculate:

(i) $P(X \leq 12)$ **(ii)** $P(X \geq 16)$ **(iii)** $P(5 < X < 9)$

18 On the way to work one day, Enzo passes through three sets of traffic lights. The probability that he will stop at any particular set of lights is $\frac{1}{2}$.

(a) Assuming the traffic lights are independent of each other, construct a table for the probability distribution of X, the number of traffic lights at which Enzo stops.

(b) Determine the probability that Enzo stops at the following sets of traffic lights:

(i) exactly two

(ii) no more than two

(iii) at least two

(iv) fewer than two

(c) What do you notice about the answers to parts **(iii)** and **(iv)**? Why does this occur?

19 A die is loaded so that the probability of obtaining each number is as shown in the table.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$

The die is rolled twice and Y is the sum of the two results.

(a) Draw a table for the probability distribution of Y. Express the probabilities as fractions in simplest form.

(b) Calculate:

(i) $P(Y \geq 4)$

(ii) $P(Y < 7)$

(iii) $P(3 \leq Y \leq 9)$

(iv) $P(Y \geq 6 | Y \leq 10)$

20 Sharmela is an interior designer who uses up to six colours in her designs. She sometimes chooses colours randomly for a unique effect. The number of colours Sharmela uses in her designs is a discrete random variable X with a probability distribution formula $P(X = x) = \frac{k}{x}$. Draw a dot plot of the distribution, using decimal values for the probabilities.

21 Analyse each situation concerning the number of shots needed to ensure at least a 95% chance of scoring at least one goal.

(a) Determine the number of shots at goal needed in each situation to ensure at least a 95% chance of scoring at least one goal.

(i) Angela plays goal attack in her netball team. She scores a goal with 75% of her shots.

(ii) Emilio has a 71% chance of scoring a goal when he kicks a penalty kick for his soccer team.

(iii) Buddy plays full forward in an AFL team. He has a 68% chance of scoring a goal when taking a kick from 45 m out from goal.

(b) Determine the smallest whole number percentage that your chance of scoring an individual goal needs to be so that you would need more than 3 shots to have a 95% chance of scoring at least once.

11.2 Statistics of discrete probability distributions

Expected value

Previously, you have calculated the mean of a set of values as a measure of central tendency. Another way of describing the mean is to call it the *expected value*. The study of expected value was first related to gambling and being able to decide whether a game is fair. The following example shows how this can be done.

Consider a game in which you roll a die and receive as your prize the number of dollars showing on the die. The following table summarises the outcomes.

Number showing	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
\$ gain	1	2	3	4	5	6

If you were running the game, you would need to decide how much to charge players for a turn. So, you need to know the expected gain for a player. A player can expect to win \$1 on $\frac{1}{6}$ of the rolls, \$2 on $\frac{1}{6}$ of the rolls, etc. For each roll the expected gain is:

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$$
$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6)$$
$$= \$3.50$$

You would expect a player to gain *on average* \$3.50 for each roll of the die. You should note that this average outcome could not happen; there is no result that returns \$3.50 to the player. In the long run:

- if you charge less than \$3.50 to play, you would expect to make a loss
- if you charge \$3.50 to play, you would expect to break even
- if you charge more than \$3.50, you would expect to make a profit.

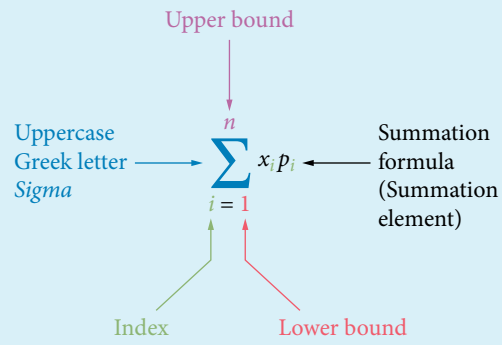
This example leads to the definition for the expected value of a discrete random variable.

The expected value or mean $E(X)$ (read as E of X) of a discrete random variable X is the sum of each possible value multiplied by its respective probability.

Let X be $x_1, x_2, \ldots x_i, \ldots x_n$ with probabilities $p_1, p_2, \ldots p_i, \ldots p_n$.

$$\sum_{i=1}^{n} x_i p_i = x_1 \times p_1 + x_2 \times p_2 + \ldots + x_i \times p_i + \ldots + x_n \times p_n$$

$$\sum_{i=1}^{n} x_i p_i = \sum_{x=1}^{n} xP(X = x)$$



This is similar to the formula used to determine the mean of a data set that has been expressed in a frequency table.

In this case, you have:

$$\text{mean} = \frac{\sum xf}{\sum f} = \frac{105}{30} = 3.5$$

x	f	xf
1	4	4
2	7	14
3	3	9
4	6	24
5	6	30
6	4	24
	$\sum f = 30$	$\sum xf = 105$

6 Determine the expected value of a discrete probability distribution

Consider the following probability distribution table.

Calculate the expected value of the probability distribution.

x	4	8	12	16
$P(X = x)$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

THINKING	WORKING
1 Use the rule $E(X) = \sum x_i p_i$ to calculate the expected value.	$\sum x_i p_i = \left(4 \times \frac{3}{8}\right) + \left(8 \times \frac{1}{4}\right) + \left(12 \times \frac{1}{4}\right) + \left(16 \times \frac{1}{8}\right)$ $= \frac{3}{2} + 2 + 3 + 2$ $= 8\frac{1}{2}$
2 Interpret the answer.	The expected value $E(X)$ is $8\frac{1}{2}$.

7 Use the expected value to calculate unknowns

The following table represents a probability distribution. The expected value $E(X) = 3.4$.

x	1	2	3	4	5	6
$P(X = x)$	0.1	a	0.3	0.2	0.2	b

Determine the values of a and b.

THINKING	WORKING
1 Write an equation using $\Sigma P(X = x) = 1$.	$0.1 + a + 0.3 + 0.2 + 0.2 + b = 1$ $a + b = 0.2$
2 Write an equation using $E(X)$.	$0.1 + 2a + 0.9 + 0.8 + 1 + 6b = 3.4$ $2a + 6b = 0.6$ $a + 3b = 0.3$

3 Solve the equations simultaneously. Alternatively, use your technology to draw the graphs and read the point of intersection.	$a + b = 0.2$ [1] $a + 3b = 0.3$ [2] $2b = 0.1$ [2] − [1] $b = 0.05$ Substitute $b = 0.5$ into [1]. $a + 0.05 = 0.2$ $a = 0.15$
4 Interpret the answer.	For the table to represent a discrete probability distribution, $a = 0.15$ and $b = 0.05$.

Expected value of a function

If you know the expected value for a random variable, then you can determine the expected value for a function of the random variable.

8 Calculate the expected value when a function is applied

The following probability distribution represents the mass of cats, in kilograms, brought into a veterinary clinic. When the cats are vaccinated, they receive 2 mL less than 3 mL/kg.

x	2	3	4	5
$P(X = x)$	0.25	0.15	0.4	0.2

(a) Calculate the expected mass of a cat.

THINKING	WORKING
1 Use the rule $E(X) = \sum x_i p_i$ to calculate the expected value.	$\sum x_i p_i = (2 \times 0.25) + (3 \times 0.15) + (4 \times 0.4) + (5 \times 0.2)$ $= 0.5 + 0.45 + 1.6 + 1$ $= 3.55$
2 Interpret the answer.	The expected (or average) mass of a cat in the clinic is 3.55 kg.

(b) Calculate the expected volume of vaccination given to a cat.

1 Create a linear expression and calculate the volume of vaccination.	$3 \times E(X) - 2 = 3 \times 3.55 - 2$ $= 8.65$
2 Interpret the answer.	The expected (or average) volume of vaccination given to a cat is 8.65 mL.

(c) Determine the expected value for $3X - 2$.

1 Apply the function to each value the variable can take. (This has no effect on the probability of each outcome.)

$$f(x) = 3x - 2$$
$$f(2) = 3(2) - 2 = 4$$
$$f(3) = 3(3) - 2 = 7$$
$$f(4) = 3(4) - 2 = 10$$
$$f(5) = 3(5) - 2 = 13$$

2 Rewrite the probability distribution table, replacing the x-values with the calculated values for $f(x)$.

x	2	3	4	5
$3x - 2$	4	7	10	13
$P(f(X) = f(x))$	0.25	0.15	0.4	0.2

3 Use the rule $E(aX+b) = \sum f(x_i)p_i$ to calculate the expected value.

$$\begin{aligned} E(3X-2) &= \sum (3x-2)P(X=x) \\ &= (4\times0.25)+(7\times0.15)+(10\times0.4)+(13\times0.2) \\ &= 1+1.05+4+2.6 \\ &= 8.65 \end{aligned}$$

(d) Compare the values for $E(3X - 2)$ and $3E(X) - 2$.

Compare the results.

$E(3X - 2) = 3E(X) - 2 = 8.65$

On average a cat will receive 8.65 mL of vaccine.

In the previous worked example, it was demonstrated that $E(3X - 2) = 3E(X) - 2$.

This is true for all linear functions of random variables: $E(aX + b) = aE(X) + b$

If a probability distribution has $E(X) = 2.1$:

$$\begin{aligned} E(3X-5) &= 3(2.1)-5 \\ &= 6.3-5 \\ &= 1.3 \end{aligned}$$

Additional information

Expected values

$E(aX + b) = aE(X) + b$

A proof of the expected value in a non-probability context.

Consider the situation where two spinners are spun.
Spinner X has the following probability distribution:

x	2	3	4	5
$P(X = x)$	0.3	0.25	0.15	0.3

Spinner Y has the following probability distribution:

y	1	2	3	4
$P(Y = y)$	0.2	0.3	0.4	0.1

Demonstrate that $E(X+Y)=E(X)+E(Y)$.

Determine the value of $E(X)$:

$$\begin{aligned}E(X)&=(2\times0.3)+(3\times0.25)+(4\times0.15)+(5\times0.3)\\&=0.6+0.75+0.6+1.5\\&=3.45\end{aligned}$$

Determine the value of $E(Y)$:

$$\begin{aligned}E(Y)&=(1\times0.2)+(2\times0.3)+(3\times0.4)+(4\times0.1)\\&=0.2+0.6+1.2+0.4\\&=2.4\end{aligned}$$

The next step is to add these together:

$$\begin{aligned}E(X)+E(Y)&=3.45+2.4\\&=5.85\end{aligned}$$

Create a table that shows the sample space for $X+Y$ and associated probabilities. For example, if spinner X lands on 2 and Y lands on 1, then $X+Y=3$.

$X+Y$ are independent events so $P(X=2 \text{ and } Y=1)=0.3\times0.2=0.06$. This is written in the table as 3: 0.06.

	$X=2$	$X=3$	$X=4$	$X=5$
$Y=1$	3: 0.06	4: 0.05	5: 0.03	6: 0.06
$Y=2$	4: 0.09	5: 0.075	6: 0.045	7: 0.09
$Y=3$	5: 0.12	6: 0.1	7: 0.06	8: 0.12
$Y=4$	6: 0.03	7: 0.025	8: 0.015	9: 0.03

This table can then be used to create the probability distribution of $X+Y$.

z	3	4	5	6	7	8	9
$P(X+Y=z)$	0.06	0.14	0.225	0.235	0.175	0.135	0.03

Use the values in this probability distribution table to determine $E(X+Y)$:

$$\begin{aligned}E(X+Y)&=(3\times0.06)+(4\times0.14)+(5\times0.225)+(6\times0.235)+(7\times0.175)+(8\times0.135)+(9\times0.03)\\&=0.18+0.56+1.125+1.41+1.225+1.08+0.27\\&=5.85\end{aligned}$$

The values of $E(X+Y)=E(X)+E(Y)$ are equal, therefore:

$E(X+Y)=E(X)+E(Y)$

It is generally easier to add together the two separate expected values than to calculate the distribution for the addition and then determine its expected value.

Expected values of functions of random variables X and Y:

$E(aX+b)=aE(X)+b$

$E(X+Y)=E(X)+E(Y)$

Variance

Statistical analysis includes measures of central tendency (mean, median, mode) and measures of spread (range, interquartile range, variance, standard deviation). You have already seen a measure of central tendency (the expected value or mean, $E(X)$ or μ).

The variance is another measure of spread and is usually denoted by σ^2 (lowercase letter sigma squared) or by Var(X). The variance describes how far the values of a data set are spread. The larger the variance, the more spread out the data.

The variance is defined as the expected value of the square of the difference from the mean, that is $\text{Var}(X) = E(x-\mu)^2$.

For a finite probability distribution, to calculate the variance, subtract the mean μ (mu) from each value of x_i and square the result. Multiply that value by the corresponding probability p_i. The variance is the sum of these products. The further away from the mean the observed values are, the greater the value of the variance.

$$\begin{aligned}\text{Var}(X) &= \sigma^2 \\ &= \sum (x_i-\mu)^2 p_i \text{ where } \mu = E(X)\end{aligned}$$

$E(X) = \sum x_i p_i$, and $(x_i-\mu)^2$ is a function of X:

$$\text{Var}(X) = E(X-\mu)^2$$

It is often more convenient to write this rule in terms of $E(X)$, if it is known.

$$\begin{aligned}E(X-\mu)^2 &= E\left(X^2 - 2X\mu + \mu^2\right) && \text{expansion of perfect square} \\ &= E\left(X^2\right) - E(2X\mu) + E\left(\mu^2\right) && \text{using the property that } E(X+Y) = E(X) + E(Y) \\ &= E\left(X^2\right) - 2\mu E(X) + \mu^2 && \mu \text{ is a constant} \\ &= E\left(X^2\right) - 2\mu^2 + \mu^2 && E(X) = \mu \\ &= E\left(X^2\right) - \mu^2 \\ &= E\left(X^2\right) - [E(X)]^2 && \mu = E(X)\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= \sigma^2 \\ &= E(X-\mu)^2 \\ &= \sum (X-\mu)^2 p_i \\ &= E\left(X^2\right) - [E(X)]^2\end{aligned}$$

Use whichever format is most convenient for the question.

9 Calculate the variance of a discrete probability distribution

Consider the following probability distribution.

x	1	2	3	4	5	6
$P(X=x)$	0.2	0.15	0.35	0.05	0.1	0.15

(a) Calculate the expected value.

THINKING	WORKING
Use: $E(X) = \sum x_i p_i$	$E(X) = (1\times 0.2) + (2\times 0.15) + (3\times 0.35) + (4\times 0.05) + (5\times 0.1) + (6\times 0.15)$ $= 0.2 + 0.3 + 1.05 + 0.2 + 0.5 + 0.9$ $= 3.15$

(b) Calculate the variance.

1 Rewrite the probability distribution table replacing the x-values with the $f(x)=x^2$ values.

x^2	$1^2=1$	$2^2=4$	$3^2=9$	$4^2=16$	$5^2=25$	$6^2=36$
$P(X=x)$	0.2	0.15	0.35	0.05	0.1	0.15

2 Determine $E(X^2)$ by applying the rule $E(f(X))=\sum f(x_i)p_i$.

$$\begin{aligned}E(X^2)&=1\times0.2+4\times0.15+9\times0.35+16\times0.05+25\times0.1+36\times0.15\\&=0.2+0.6+3.15+0.8+2.5+5.4\\&=12.65\end{aligned}$$

3 Calculate $[E(X)]^2$.

$$E(X)=3.15$$
$$\begin{aligned}[E(X)]^2&=(3.15)^2\\&=9.9225\end{aligned}$$

4 Calculate $\text{Var}(X)=E(X^2)-[E(X)]^2$.

$$\begin{aligned}\text{Var}(X)&=E(X^2)-[E(X)]^2\\&=12.65-9.9225\\&=2.7275\end{aligned}$$

Note that with technology it is just as easy to calculate variance directly as the expected value of the difference from the mean, that is $\text{Var}(x)=E(X-\mu)^2$, as it is to use the alternative form $\text{Var}(X)=E(X^2)-[E(X)]^2$. The alternative form is convenient when the variance is being calculated by hand with the assistance of a scientific calculator.

Technology worked example
Expected value, variance and standard deviation

WARNING
$\text{Var}(X)$ must be positive. If you obtain a negative value you have made an arithmetic error.

You could also use a vertical table structure to determine the value of $E(X^2)$. For the probability distribution used in the above example you would have:

x	$P(X=x)$	x^2	$x^2P(X=x)$
1	0.2	1	0.2
2	0.15	4	0.6
3	0.35	9	3.15
4	0.05	16	0.8
5	0.1	25	2.5
6	0.15	36	5.4
			$\sum x^2P(X=x)=12.65$

Standard deviation

The standard deviation is the most common measure of spread. It is measured in the same units as the variable.

The standard deviation of X is written as the Greek letter *sigma* σ:

$$\begin{aligned}\text{The standard deviation of } (X) &= \sqrt{\text{Var}(X)}\\ &= \sqrt{\sigma^2}\\ &= \sigma\end{aligned}$$

Note that you are only interested in the positive square root of the variance, as σ is a measure of spread, which does not involve sign.

10 Determine the standard deviation of a discrete probability distribution

The probability distribution for variable X is shown in the table. Calculate the standard deviation correct to 4 decimal places.

x	-1	0	1	2	3
$P(X=x)$	0.2	0.15	0.25	0.3	0.1

THINKING	WORKING
1 Calculate $E(X)$.	$E(X) = (-1\times 0.2)+(0\times 0.15)+(1\times 0.25)+(2\times 0.3)+(3\times 0.1)$ $= -0.2+0+0.25+0.6+0.3$ $= 0.95$
2 Calculate $[E(X)]^2$.	$[E(X)]^2 = (0.95)^2$ $= 0.9025$
3 Determine X^2.	x: -1, 0, 1, 2, 3 x^2: 1, 0, 1, 4, 9
4 Calculate $E(X^2)$.	$E(X^2) = 1\times 0.2+0\times 0.15+1\times 0.25+4\times 0.3+9\times 0.1$ $= 0.2+0+0.25+1.2+0.9$ $= 2.55$
5 Calculate $\text{Var}(X)=\sigma^2$.	$\text{Var}(X) = E(X^2)-[E(X)]^2$ $= 2.55-0.9025$ $= 1.6475$
6 Calculate σ using $\sigma = \sqrt{\text{Var}(X)}$ to 4 decimal places (if necessary).	$\sigma = \sqrt{\text{Var}(X)}$ $= \sqrt{1.6475}$ $= 1.2835$ (4 d.p.)

Standard deviation of a function

If you know the standard deviation for a random variable, then you can determine the standard deviation for any function of the random variable.

If a probability distribution has $\sigma(X) = 0.55$, then:

$$\begin{aligned}\sigma(3X-5) &= 3(\sigma(X)) \\ &= 3(0.55) \\ &= 1.65\end{aligned}$$

Additional information

Data scaling and standard deviation

Practise using data scaling and standard deviation in this activity.

The connection that exists between the standard deviation of a discrete probability distribution and the standard deviation of a function based on that discrete probability distribution is summarised as:

$\sigma(aX + b) = |a|\ \sigma(X)$

For many variables, about 95% of their values lie within two standard deviations of the mean. The following worked example illustrates this.

11 95% of the data lie within 2 standard deviations of the mean

The following table represents the probability distribution of Y, the number of consultations per hour by a dentist.

y	0	1	2	3	4	5	6	7
$P(Y = y)$	0.015	0.05	0.255	0.321	0.179	0.155	0.016	0.009

For this distribution $E(Y) = 3.173$ and $\sigma = 1.283$.

Determine $P(\mu - 2\sigma \le Y \le \mu + 2\sigma)$ and comment on the result.

THINKING	WORKING
1 Calculate $\mu - 2\sigma$.	$\mu - 2\sigma = 3.173 - 2 \times 1.283$ $= 0.607$
2 Calculate $\mu + 2\sigma$.	$\mu + 2\sigma = 3.173 + 2 \times 1.283$ $= 5.739$
3 Rewrite the probability interval using the values just calculated.	$P(0.607 \le Y \le 5.739)$
4 Calculate the probability from the table. (Remember, these are discrete values.)	$= P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) + P(Y = 5)$ $= 0.05 + 0.255 + 0.321 + 0.179 + 0.155$ $= 0.96$
5 Comment on the result.	About 95% of the values lie within two standard deviations of the expected value, in the interval [0.607, 5.739].

Approximately 95% of the data lies within two standard deviations of the mean: $\mu - 2\sigma \le Y \le \mu + 2\sigma$

EXERCISE 11.2 Statistics of discrete probability distributions

Worked Example

1 Calculate $E(X)$ for each of the following probability distributions. 6

(a)

x	1	3	5	7	9
$P(X=x)$	0.2	0.3	0.25	0.15	0.1

(b)

x	-1	0	1	2	3
$P(X=x)$	0.4	0.15	0.2	0.05	0.2

(c)

x	2	3	4	5	8
$P(X=x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{18}$

(d)

x	-2	-1	1	3	5
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$

2 Use the given value of $E(X)$ to solve for the unknowns. 7

(a) Determine the values of a and b in the following probability distribution, given that $E(X) = 3.8$.

x	2	3	4	5	6
$P(X=x)$	0.4	0.1	a	0.1	b

(b) Determine the values of i and j in the following probability distribution, given that $E(X) = 0.15$.

x	-3	-2	-1	0	1
$P(X=x)$	0.05	i	0.05	j	0.55

(c) Determine the values of i and j in the following probability distribution, given that $E(X) = 6\frac{2}{5}$.

x	-3	4	6	8	10
$P(X=x)$	$\frac{3}{20}$	$\frac{5}{20}$	i	$\frac{7}{20}$	j

(d) Determine the values of a and b in the following probability distribution, given that $E(X) = 1\frac{4}{15}$.

x	-3	-1	1	3	5
$P(X=x)$	a	$\frac{1}{5}$	$\frac{1}{15}$	b	$\frac{1}{5}$

8

3 Consider the following probability distribution.

x	0	1	2	3	4
$P(X=x)$	0.25	0.3	0.2	0.15	0.1

Calculate the following.

(a) $E(X)$ (b) $E(2X-3)$ (c) $E\left(X^2-5\right)$ (d) $E\left(X^2+X-4\right)$

11

4 For the probability distribution below, the mean is 0.03 and the standard deviation is 1.02.

y	-3	-2	-1	0	1	2
$P(Y=y)$	0.02	0.03	0.25	0.35	0.3	0.05

Estimate the value of $P(\mu-2\sigma \le Y \le \mu+2\sigma)$.

5 In each of the following probability distributions, calculate values for k and $E(X)$.

(a)

x	1	2	3	4	5
$P(X=x)$	k	$2k$	$3k$	$12k$	$6k$

(b)

x	-3	-1	1	3	5
$P(X=x)$	$12k$	$2k$	$3k$	k	$2k$

6 For the following probability distribution, calculate each of the following.

w	2	4	6	8
$P(W=w)$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{16}$

(a) $E(W)$ (b) $E(3W+4)$ (c) $E(2W+5)$ (d) $E\left(W^2-7\right)$

7 Variable X has the probability distribution:

x	-2	-1	0	1
$P(X=x)$	0.3	0.4	0.1	0.2

(a) Determine the value of $E\left(X^2-4\right)$.

A -5.4 B -3.36 C -2.2 D -0.8

(b) Explain the common error made by a student who calculated $(-0.8)^2-4$.

8 Two spinners are spun. One spinner X has the following probability distribution.

x	1	2	3	4
$P(X=x)$	0.2	0.2	0.1	0.5

The other spinner, Y, has the following probability distribution.

y	2	3	4	5
$P(Y=y)$	0.2	0.3	0.4	0.1

Calculate $E(X)+E(Y)$ and $E(X+Y)$ separately to demonstrate that $E(X+Y)=E(X)+E(Y)$.

9 A spinner has nine equal sections, five of which are yellow, three are blue and one is red. If the spinner lands on yellow, you receive \$1. If it lands on blue, you receive \$3. If it lands on red, you receive \$5. Let \$$X$ stand for the amount of money you receive.

(a) Draw up a probability distribution table for this game.

(b) What is the expected value of X?

(c) If the game is fair, how much should you pay to play, to the nearest cent?

Worked Example 9

10 For the following probability distributions, determine the expected value and the variance.

(a)

x	1	2	3	4	5	6
$P(X=x)$	0.1	0.3	0.25	0.05	0.15	0.15

(b)

x	5	6	7	8	9
$P(X=x)$	0.15	0.35	0.1	0.25	0.15

11 Determine the standard deviation of the variable X, which has the following probability distribution. Give your answer correct to 4 decimal places.

x	-2	-1	0	1	2	3
$P(X=x)$	0.1	0.15	0.3	0.15	0.2	0.1

12 Variable Y has the probability distribution:

x	0	1	2	3
$P(X=x)$	$\frac{1}{9}$	$\frac{7}{18}$	$\frac{1}{3}$	$\frac{1}{6}$

The value of Var(Y) is:

A $1\frac{5}{9}$ B $\frac{162}{315}$ C $2\frac{5}{38}$ D $\frac{65}{81}$

13 Eric is offered a choice of two salary packages for selling cars. Package 1 has a weekly retainer of \$200 and then \$650 for every car sold. Package 2 has a weekly retainer of \$400, and then \$400 for every car sold. Past sales patterns indicate that the probability distribution for the number of cars sold per week is as follows:

Number of vehicles	0	1	2	3	4	5
Probability	0.45	0.35	0.1	0.05	0.04	0.01

Determine which salary package Eric should take, and calculate the expected advantage of this package.

14 A die is biased in such a way that the following table represents its probability distribution.

d	1	2	3	4	5	6
$P(D=d)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$

(a) Determine the expected (mean) value of the data, stated in mixed number form.

(b) Express the interval $\mu \pm 2\sigma$ in the form a to b, correct to 2 decimal places.

(c) In this case, determine the percentage of values that lie in the interval $\mu \pm 2\sigma$.

15 A random variable T has a probability distribution as shown on the right.

t	$w-3$	$w-2$	$w-1$	w	$w+1$
$P(T=t)$	0.2	0.5	0.1	0.05	0.15

(a) Given that $E(T) = 8.45$, determine the value of w.

(b) Calculate $\text{Var}(T)$.

(c) Calculate the standard deviation of T, correct to 2 decimal places.

(d) Calculate $\text{Var}(2T - 6)$ correct to 2 decimal places.

(e) Calculate $\text{Var}(5 - 3T)$ correct to 4 decimal places.

16 The probability distribution of G is given by:

$$P(G=g) = \begin{cases} k(6-g), & \text{if } g \in \{0,1,2,3,4\} \\ 0, & \text{for all other values of } g \end{cases}$$

Determine the following values.

(a) k

(b) $E(G)$

(c) $\text{Var}(G)$

(d) σ, correct to 4 decimal places

17 The number y appearing on a spinner Y showing the numbers 1, 2, 3, 4, has a probability distribution as follows:

y	1	2	3	4
$P(Y=y)$	0.1	0.2	0.3	0.4

The number x appearing on a spinner X has a probability distribution as follows:

x	0	1	2	3	4
$P(X=x)$	0.6	0.1	0.1	0.1	0.1

(a) Calculate $E(Y)$.

(b) Calculate $E(X)$.

(c) The numbers appearing on each of the 2 spinners are added together to give the variable M.

(i) Draw a table to determine the probability distribution of M.

(ii) Calculate $E(M)$.

(d) Compare the results for $E(X)$, $E(Y)$ and $E(M)$.

18 Enrico runs a game of chance at The Brisbane Exhibition. The player draws a card at random from a normal pack of 52 playing cards. If the card is black, the player gains \$1. If the card is a diamond, the player gains \$5. There is a fee to play this game, and here 'gain' for the player simply means the amount given to them that is above the playing fee.

If Enrico wants to make a profit, what is the minimum whole dollar amount he should charge to play the game?

19 Dubravko and Erina often play a best of three sets match of tennis. From past experience they know that the probability that Dubravko will win a set is $\frac{3}{5}$.

Determine the average number of sets they play in a match.

20 The game of chance requires a player to roll a blue, a red and a white die.
If a 1 shows on the blue die, the player receives \$1.
If a 1 shows on the red die, the player receives \$2.
If a 1 shows on the white die, the player receives \$5.
In all other circumstances, the player receives nothing. A player can receive more than one prize.
How much should the operator charge, to the nearest 5 cents to play if the game is fair?

Uniform distributions

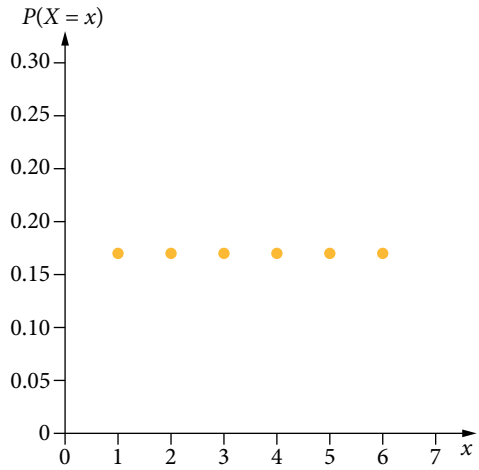
11.3

Uniform distribution of the first *n* natural numbers

In *Uniform* distributions all values are equally likely. For example, all outcomes are equally likely when a fair six-sided die is rolled. If the random variable X represents the number showing on the die, then the probability distribution is uniform.

$$P(X=x)=\begin{cases}\dfrac{1}{6}, & \text{if } x\in\{1,2,3,4,5,6\}\\ 0, & \text{for all other values of } x\end{cases}$$

The graph of this distribution is represented by six discrete points, which all lie on a horizontal line: $(1,\frac{1}{6})$, $(2,\frac{1}{6})$, $(3,\frac{1}{6})$, $(4,\frac{1}{6})$, $(5,\frac{1}{6})$, $(6,\frac{1}{6})$



The expected value is:

$$\begin{aligned}E(X)&=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}\\&=\frac{1}{6}(1+2+3+4+5+6)\\&=\frac{21}{6}\\&=\frac{7}{2}\end{aligned}$$

This can also be determined from the symmetry of the graph of the distribution.

The variance of the distribution is given by $\text{Var}(X)=E(X^2)-[E(X)]^2$:

$$\begin{aligned}\text{Var}(X)&=\frac{1}{6}(1+4+9+16+25+36)-\frac{49}{4}\\&=\frac{35}{12}\end{aligned}$$

A discrete probability distribution is said to be uniform if all values of the random variable are equally likely.

Explore further

Expected value and variance of a uniform distribution

Calculate the expected value and variance of a uniform distribution of the first n natural numbers.

When a uniform distribution is based on the first n natural numbers, some general expressions can be established. The general expression for a discrete uniform distribution with n values is:

$$P(X=x)=\begin{cases}\dfrac{1}{n}, & \text{if } x\in\{1,2,\ldots n\}\\ 0, & \text{for all other values of } x\end{cases}$$

The sum of the first n natural numbers is: $1+2+\ldots+n=\dfrac{n}{2}(n+1)$

The expected value is:

$$\begin{aligned}E(X) &= 1\times\frac{1}{n}+2\times\frac{1}{n}+3\times\frac{1}{n}+\ldots+n\times\frac{1}{n}\\ &= \frac{1}{n}(1+2+3+\ldots+n)\\ &= \frac{1}{n}\times\frac{n}{2}(n+1)\\ &= \frac{n+1}{2}\end{aligned}$$

The variance of X can be obtained by determining $\text{Var}(X) = E\left(X^2\right) - \left[E(X)\right]^2$.

The sum of the squares of the first n natural numbers is: $1^2 + 2^2 + \ldots + n^2 = \frac{n}{6}(n+1)(2n+1)$
The expected value of X^2 is:

$$\begin{aligned}E\left(X^2\right) &= 1^2\times\frac{1}{n}+2^2\times\frac{1}{n}+3^2\times\frac{1}{n}+\ldots+n^2\times\frac{1}{n}\\ &= \frac{1}{n}\left(1^2+2^2+3^2+\ldots+n^2\right)\\ &= \frac{1}{n}\times\frac{n}{6}(n+1)(2n+1)\\ &= \frac{(n+1)(2n+1)}{6}\end{aligned}$$

The variance of X can be obtained by determining $\text{Var}(X)$:

$$\begin{aligned}\text{Var}(X) &= E\left(X^2\right) - \left[E(X)\right]^2\\ &= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2\\ &= \frac{2(n+1)(2n+1)}{12} - \frac{3(n+1)(n+1)}{12}\\ &= \frac{(n+1)\left(2(2n+1)-3(n+1)\right)}{12}\\ &= \frac{(n+1)(4n+2-3n-3)}{12}\\ &= \frac{(n+1)(n-1)}{12}\\ &= \frac{n^2-1}{12}\end{aligned}$$

For a discrete uniform probability distribution $P(X = x) = \frac{1}{n}$ for $x \in \{1, 2, 3, \ldots, n\}$ and zero otherwise:

$$E(X) = \frac{n+1}{2}$$

$$\text{Var}(X) = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

12 The mean and variance of a uniform distribution of the first *n* natural numbers

A roulette wheel has 38 equal-sized spaces showing the numbers 1 to 36 as well as 0 and 00. For the purpose of this question assume that 0 and 00 represent the 37th and 38th possible outcomes.

(a) Determine the mean of the number of the space the ball lands in.

THINKING	WORKING
Use the rule $E(X) = \frac{n+1}{2}$.	$E(X) = \frac{n+1}{2}$ $= \frac{38+1}{2}$ $= \frac{39}{2}$ $= 19\frac{1}{2}$

(b) Determine the variance of the number of the space the ball lands in.

Use the rule $\text{Var}(X) = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$.	$\text{Var}(X) = \frac{n^2-1}{12}$ $= \frac{38^2-1}{12}$ $= \frac{1444-1}{12}$ $= \frac{1443}{12}$ $= 120\frac{1}{4}$

Uniform distribution of any set of consecutive integers

So far the examples presented have involved uniform distributions with x-values from 1 to n. However, you can determine the expected value and variance for other uniform distributions as long as the values of the distribution are consecutive numbers. This is because such a distribution is a translation of the uniform distribution with x-values from 1 to n. If the x-values have all been increased by b, then the expected value will also be increased by b, but the variance will remain unchanged.

For a uniform distribution with x-values from $1 + b$ to $n + b$:

Expected value: $E(X + b) = E(X) + b$

Variance: $\text{Var}(X + b) = \text{Var}(X)$

13 The mean and variance of a uniform discrete distribution of consecutive integers

Consider the probability distribution below.

y	25	26	27	28
$P(Y=y)$	k	k	k	k

(a) Determine the value of $E(Y)$.

THINKING	WORKING
1 Determine the value by inspection.	By symmetry, the average value is 26.5.
2 Solve for k, given that $\sum x_i p_i = 1$.	$4k = 1$ $k = \frac{1}{4}$
3 Determine $E(X)$ for the first four natural numbers.	For $x \in \{1, 2, 3, 4\}$ $E(X) = \frac{n+1}{2}$ $= \frac{4+1}{2}$ $= 2\frac{1}{2}$
4 Write Y in terms of X.	$y = x + 24$ $y \in \{1+24, 2+24, 3+24, 4+24\}$
5 Use $E(X+b) = E(X) + b$	$E(Y) = E(X+24)$ $= E(X) + 24$ $= 2\frac{1}{2} + 24$ $= 26\frac{1}{2}$

(b) Determine the value of $\text{Var}(Y)$.

1 Use the rule for $\text{Var}(X)$.	$\text{Var}(X) = \frac{n^2-1}{12}$ $= \frac{4^2-1}{12}$ $= \frac{15}{12}$ $= \frac{5}{4}$
2 Use $\text{Var}(X+b) = \text{Var}(X)$.	$\text{Var}(Y) = \text{Var}(X+24)$ $= \text{Var}(X)$ $= 1\frac{1}{4}$

Uniform distribution of any arithmetic progression

It is possible to determine the expected value and variance for other uniform distributions using the existing formulas if the values of the distribution form an arithmetic sequence. This may involve changing the uniform distribution where x takes the values 1 to n in the following ways: multiplying each number by a constant factor as well as adding or subtracting a constant term. Recall from the previous module:

Expected value: $E(aX + b) = aE(X) + b$

Standard deviation: $\sigma(aX + b) = |a|(\sigma(X))$

From these you can determine the variance of uniform distributions that form an arithmetic sequence:

Variance: $\text{Var}(aX + b) = a^2(\text{Var}(X))$

14 The mean and variance of a uniform discrete distribution of an arithmetic progression

Consider the probability distribution below.

y	15.5	25.5	35.5	45.5	55.5
$P(Y = y)$	0.2	0.2	0.2	0.2	0.2

(a) Determine the value of $E(Y)$.

THINKING	WORKING
1 Determine the value by inspection.	By symmetry, the average value is 35.5.
2 Calculate $E(X)$ for the first five natural numbers.	For $x \in \{1, 2, 3, 4, 5\}$ $E(X) = \frac{n+1}{2}$ $= \frac{5+1}{2}$ $= 3$
3 Write Y in terms of X.	$25.5 - 15.5 = 10$ $Y = 10X + c$ For $X = 1$, $Y = 15.5$: $15.5 = 10(1) + c$ $c = 5.5$ $Y = 10X + 5.5$
4 Use $E(aX + b) = aE(X) + b$.	$E(Y) = E(10X + 5.5)$ $= 10E(X) + 5.5$ $= 10(3) + 5.5$ $= 35.5$

(b) Determine the value of $\text{Var}(Y)$.

1 Use the rule for $\text{Var}(X)$.

$$\begin{aligned}\text{Var}(X) &= \frac{n^2-1}{12}\\ &= \frac{5^2-1}{12}\\ &= \frac{25-1}{12}\\ &= \frac{24}{12}\\ &= 2\end{aligned}$$

2 Use $\text{Var}(aX+b) = a^2(\text{Var}(X))$.

$$\begin{aligned}\text{Var}(Y) &= \text{Var}(10X+5.5)\\ &= 10^2(\text{Var}(X))\\ &= 100(2)\\ &= 200\end{aligned}$$

Explore further

Summation of the square of natural numbers

Explore the summation of the square of natural numbers using formulas and patterns in a spreadsheet.

EXERCISE 11.3 Uniform distributions

Worked Example

12

1 A number from 1 to 16 is chosen at random. The random variable R represents the value chosen. Calculate the following.

(a) $E(R)$ **(b)** $\text{Var}(R)$

2 A cleaner has nine similar looking keys on a key chain. He tries them in turn, until he finds the one that opens the lock.

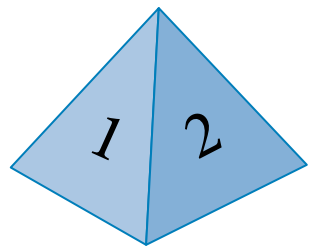
(a) Calculate the expected number of attempts.

(b) What is the variance of the number of attempts?

3 A die in the shape of a tetrahedron (a solid with four triangular faces) is rolled. The four faces are numbered 1–4. Let F be a random variable that represents the value that is face down on the table.

(a) Calculate the expected value $E(F)$.

(b) Calculate the variance $\text{Var}(F)$.



4 A spinner is equally divided into n segments and each segment contains a value from 1 to n.

(a) If the expected value is 11.5, then n is equal to which of the following values?

A 20 B 21 C 22 D 23

(b) If the variance is 14, then n is equal to which of the following values?

A 3 B 9 C 10 D 13

5 Consider the random variable X that has the following probability distribution.

x	1	2	3	4	5
$P(X=x)$	0.2	0.2	0.2	0.2	0.2

Determine the following values.

(a) $E(X)$ (b) $\text{Var}(X)$

(c) Now consider the random variable Y that has the following probability distribution.

y	2	3	4	5	6
$P(Y=y)$	0.2	0.2	0.2	0.2	0.2

Calculate from first principles the following values.

(i) $E(Y)$ (ii) $\text{Var}(Y)$

(d) What can you say about the values of $E(X)$ and $E(Y)$?

(e) What can you say about the values of $\text{Var}(X)$ and $\text{Var}(Y)$?

(f) Now consider the random variable Z that has the following probability distribution.

z	6	7	8	9	10
$P(Z=z)$	0.2	0.2	0.2	0.2	0.2

Without doing any additional calculations determine the following values.

(i) $E(Z)$ (ii) $\text{Var}(Z)$

6 You enter a room that contains a digital clock, such as the one shown. Let T represent the minutes shown on the clock.

(a) Calculate $E(T)$ using symmetry.

(b) Now calculate $\text{Var}(T)$ and $\sigma(T)$, correct to 2 decimal places.

Worked Example 13

7 The faces of a six-sided die are numbered 11 to 16.

(a) Calculate the expected value when the die is rolled once.

(b) Calculate the variance of the values.

8 The four sides of a square spinner are numbered 10, 20, 30 and 40.

(a) Calculate the expected value when the spinner is spun once.

(b) Calculate the variance of the values.

9 Write a formula in the form $Y = aX + b$ to convert each set of numbers to the set of natural numbers $x \in \{1, 2, 3, ..., n\}$. In each case, calculate the value of n.

(a) $y \in \{28, 37, 46, ..., 91\}$

(b) $y \in \{-10, -7, -4, ..., 32\}$

(c) $y \in \{3002.1, 3002.2, 3002.3, ..., 3003\}$

10 The numbers 200, 300, 400, 500 are uniformly distributed.

(a) When a number is drawn at random from the set, what is the variance of the expected number?

A 100 B 10 000 C $\frac{50000}{4}$ D $\frac{300}{4}$

Worked Example

(b) Explain the common error made by a student who thought the answer was $\frac{500}{4}$.

14

11 Consider the probability distribution on the right.

Y	42	53	64	75	86
P	0.2	0.2	0.2	0.2	0.2

(a) Determine the value of $E(Y)$.

(b) Determine the value of $\text{Var}(Y)$.

12 Consider the uniform probability distribution $y \in \{-55, -48, -41, ..., 8\}$.

(a) Determine the value of $E(Y)$.

(b) Determine the value of $\text{Var}(Y)$.

13 Consider the uniform probability distribution $y \in \{1.045, 1.047, 1.049, ... 1.059\}$.

(a) Determine the value of $E(Y)$.

(b) Determine the value of $\text{Var}(Y)$.

14 Group the following data so that uniform distribution formulas can be used to estimate the statistics. Estimate the mean and variance from the grouped data.

120, 122, 126, 135, 138, 138, 144, 147, 149, 151, 153, 156, 160, 160, 162.

15 For a single roll of a fair six-sided die numbered 1 to 6: $E(X) = 3\frac{1}{2}$ and $\text{Var}(X) = 2\frac{11}{12}$. A fair six-sided die numbered 1 to 6 is rolled twice and the values obtained are added together.

(a) Construct a probability distribution table for this event, using S to represent the variable. Calculate the values of $E(S)$ and $\text{Var}(S)$ and determine their relationship with $E(X)$ and $\text{Var}(X)$.

Consider a spinner that has four equal-sized sections labelled 1–4. Let V describe the value the spinner stops on. The spinner is now spun twice. Let T describe the sum of the two values obtained.

(b) Predict the relationship between the values for $E(V)$, $\text{Var}(V)$, $E(T)$ and $\text{Var}(T)$, then check that your predictions were correct.

Now let the four-sided spinner be spun once and the die be rolled once. Let W describe the sum of the two values obtained.

(c) Predict the relationship between the values for $E(V)$, $E(X)$ and $E(W)$ as well as the relationship between $\text{Var}(V)$, $\text{Var}(X)$ and $\text{Var}(W)$, then check whether your predictions were correct or not.

Probability distributions in practical situations

11.4

Expected value and fairness

A game based on random selections is considered fair if the average gain or expected profit is zero.

15 Dice gambling

A game involves rolling a fair six-sided die. If the number rolled is even, then the player wins that number of dollars. If the number is odd, then the played loses that number of dollars. Let Z stand for the number of dollars received by the player.

(a) Construct a probability distribution table for Z.

THINKING	WORKING
1 Decide how to represent wins and losses in the table.	A win will be represented by a positive value and a loss by a negative value.
2 Construct the probability distribution table.	

z	-1	2	-3	4	-5	6
$P(Z=z)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(b) Calculate $E(Z)$.

Calculate $E(Z)=\sum z_i p_i$.

Express the final answer as a fraction in simplest form.

$$\begin{aligned} E(Z) &= -1\times\frac{1}{6}+2\times\frac{1}{6}-3\times\frac{1}{6}+4\times\frac{1}{6}-5\times\frac{1}{6}+6\times\frac{1}{6} \\ &= -\frac{1}{6}+\frac{2}{6}-\frac{3}{6}+\frac{4}{6}-\frac{5}{6}+\frac{6}{6} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

(c) Is the game fair? Explain.

If the expected gain is zero, the game is fair. A positive expected gain for the player means the game is in favour of the player. A negative expected gain means the game is against the player.

The game is not fair. It is actually in favour of the player because the expected return is \$0.50 (greater than the fair game return of \$0).

Expected value and variance of a distribution

Before any statistics can be calculated, a clear distribution of the values and their probabilities should be clearly seen.

16 A dice distribution

Two fair six-sided dice are rolled and the variable X represents the number of sixes obtained.

(a) Calculate the expected number of sixes obtained.

THINKING	WORKING
1 Determine the sample space.	(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)
2 Create a probability distribution.	x: 0, 1, 2 $P(X=x)$: $\frac{25}{36}$, $\frac{10}{36}$, $\frac{1}{36}$
3 Calculate $E(X)=\sum x_i p_i$. Express the final answer as a fraction in simplest form.	$E(X)=0\times\frac{25}{36}+1\times\frac{10}{36}+2\times\frac{1}{36}$ $=0+\frac{10}{36}+\frac{2}{36}$ $=\frac{12}{36}$ $=\frac{1}{3}$
4 Interpret the answer.	The expected number (or average number) of sixes in two rolls of a die is $\frac{1}{3}$.

(b) Determine the variance of the number of sixes obtained.

1 Calculate $E\left(X^2\right)$ from first principles. Express the final answer as a fraction in simplest form.	$E\left(X^2\right)=0^2\times\frac{25}{36}+1^2\times\frac{10}{36}+2^2\times\frac{1}{36}$ $=0+\frac{10}{36}+\frac{4}{36}$ $=\frac{14}{36}$ $=\frac{7}{18}$

2 Use the values of $E\left(X^2\right)$ and $E(X)$ in the formula for variance.	$\begin{aligned}\text{Var}(X) &= E\left(X^2\right)-\left[E(X)\right]^2\\ &= \frac{7}{18}-\left(\frac{1}{3}\right)^2\\ &= \frac{7}{18}-\frac{1}{9}\\ &= \frac{5}{18}\end{aligned}$
3 Interpret the answer.	The variance in the number of sixes in two rolls of a die is $\frac{5}{18}$.

Probability from combinations

To determine the probability of events where more than one item is selected at random from a group containing two different types, use the formulas for theoretical probability and combinations.

$$P(\text{event}) = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$$

$${}^nC_r = \frac{r!(n-r)!}{n!}$$

17 Determine expected values involving combinations

A box contains 10 items, of which four are defective. A sample of two items is selected at random from the box. The random variable Y represents the number of defective items selected. Determine the expected number of defective items.

THINKING	WORKING		
1 Use ${}^nC_r$ to determine the size of the sample space.	${}^{10}C_2 = 45$ There are 45 ways of selecting 2 items from 10.		
2 Use ${}^nC_r$ to calculate the number of ways of selecting each of the possible number of defective items.	0 defectives: ${}^6C_2 \times {}^4C_0$ $= 15 \times 1$ $= 15$	1 defective: ${}^6C_1 \times {}^4C_1$ $= 6 \times 4$ $= 24$	2 defectives: ${}^6C_0 \times {}^4C_2$ $= 1 \times 6$ $= 6$
3 Calculate the probability for each value of the variable.	$P(Y=0) = \frac{15}{45} = \frac{1}{3}$	$P(Y=1) = \frac{24}{45} = \frac{8}{15}$	$P(Y=2) = \frac{6}{45} = \frac{2}{15}$
4 Complete a probability distribution table.			

y	0	1	2
$P(Y=y)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

5 Calculate $E(Y)$ from first principles.	$E(Y)=\sum yp$ $=0\times\frac{1}{3}+1\times\frac{8}{15}+2\times\frac{2}{15}$ $=\frac{4}{5}$ $=0.8$
6 Interpret the answer.	The expected number is 0.8 items. This means that if 2 items are selected from 10 items (with 4 defectives), 'on average' 0.8 items will be defective. The 'best guess' is 1 item.

EXERCISE 11.4 Probability distributions in practical situations

Worked Example 15

1 One of the games at the local sporting club's 'Vegas Night' involves rolling a fair six-sided die. If a non-prime number appears, there is no game charge and the player wins the number of dollars shown on the face of the die. If it is prime, the cost of playing is the number of dollars shown on the face of the die. Let Z stand for the number of dollars received by the player.

z						
$P(Z=z)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(a) Construct a probability distribution table of the variable.

(b) Calculate the value of $E(Z)$.

(c) Comment on the fairness of this game.

2 For the discrete random variable X, the probability distribution is given by:

$$P(X=x)=\begin{cases} kx, & x=1,2,3,4 \\ k(9-x), & x=5,6,7,8 \end{cases}$$

(a) Determine the value of k, expressed in decimal form.

(b) Complete the following table for the probability distribution of X.

x	1	2	3	4	5	6	7	8
$P(X=x)$								

(c) Determine the value of $E(X)$.

(d) Determine the value of $\text{Var}(X)$.

3 Four cards are labelled from 1 to 4. Two cards are dealt at random, without replacement. Let X represent the larger of the two numbers shown on the cards.

(a) How many values can X take?

A 2 **B** 3 **C** 4 **D** 12

(b) Calculate $P(X = 2)$.

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{5}{6}$

4 Tomino has written the following as her answer for the probability distribution of the random variable X:

$$P(X = x) = \begin{cases} \frac{4-x}{5}, & x = 1,2,3,4,5 \\ 0, & \text{otherwise} \end{cases}$$

Explain to Tomino why this cannot be correct.

5 The distribution for the variable X is:

x	a	b	c	d
$P(X = x)$	0.3	0.2	0.1	0.4

The distribution for the variable Y is:

y	$9a + 1$	$9b + 1$	$9c + 1$	$9d + 1$
$P(Y = y)$	0.3	0.2	0.1	0.4

(a) Given that $E(X) = 5$ and $\text{Var}(X) = 4$, then the values of $E(Y)$ and $\sigma(Y)$ are, respectively:

A 46, 18 **B** 45, 19 **C** 46, 19 **D** 45, 36

(b) Explain the common error made by a student who thought the answer was $E(Y) = 46$ and $\sigma(Y) = 324$.

6 A six-sided die has five faces marked with the number 1 and one marked with 6. Calculate the expected value and the standard deviation of the value uppermost on the die when it is rolled once. Give your answers correct to 2 decimal places.

7 A bowl contains 30 counters, five marked with the number 10, six marked with 15 and the rest marked with 20. When a counter is drawn at random from the bowl, what is the expected value and the standard deviation of its value? Give your answers correct to 2 decimal places.

Worked Example 16

8 A fair die, labelled 1–6 is rolled until the total of the scores is 4 or greater. Answer each of the following, giving all answers as exact values, in simplest fraction form.

(a) Determine the probability distribution of the number of rolls, X, required to achieve this total.

x	1	2	3	4
$P(X = x)$				

(b) Determine the expected number of rolls required.

(c) Determine the variance for the number of rolls required, correct to 3 decimal places.

17

9 A factory produces screens for smartphones. A sample of 5 items is selected at random from a batch of 50 screens, 10 of which are defective. Let Y represent the number of defective items selected.

(a) Draw a table for the probability distribution of the variable. Enter the probabilities correct to 6 decimal places.

y	0	1	2	3	4	5
$P(Y=y)$						

(b) Determine the expected number of defective items. Give your answer correct to 2 decimal places.

10 A fair six-sided die is rolled twice.

(a) Let Z be the maximum number in each ordered pair. Draw a table for the probability distribution of Z. Write your answers in fraction form using a denominator of 36.

(b) Now let Y be the minimum number in each ordered pair. Draw a table for the probability distribution of Y. Write your answers in fraction form using a denominator of 36.

(c) How is the distribution of Z related to the distribution of Y?

(d) Calculate the expected value for each of the following.

(i) Z **(ii)** Y

(e) How far from the greatest value of Z is $E(Z)$?

(f) How far from the least value of Y is $E(Y)$?

(g) Explain the connection between your answers to part **(f)** and part **(g)**.

(h) Given $\text{Var}(Z) = 1\frac{926}{1296}$, what can you say about $\text{Var}(Y)$?

11 The discrete random variable X has the probability distribution shown in the following table.

x	1	2	3
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(a) Calculate $E(X)$ and $\text{Var}(X)$.

(b) A second random variable Y has the same values and distribution as X, and the two variables are independent.

Draw a table for the probability distribution of $X + Y$.

(c) Calculate $E(X + Y)$ and $\text{Var}(X + Y)$.

(d) How is the value of $E(X + Y)$ related to the values of $E(X)$ and $E(Y)$?

(e) How is the value of $\text{Var}(X + Y)$ related to the values of $\text{Var}(X)$ and $\text{Var}(Y)$?

12 The discrete random variable X has the following probability distribution.

x	1	2	3	4
$P(X=x)$	0.25	0.1	0.45	0.2

(a) Calculate $E(X)$.

(b) Demonstrate that $E(2X) = 2E(X)$.

(c) Calculate $\text{Var}(X)$.

(d) Calculate the following values.

(i) $\text{Var}(2X)$ **(ii)** $\text{Var}(3X)$

(e) Use your answers to previous parts of the question to determine the relationship between $\text{Var}(X)$ and $\text{Var}(kX)$.

13 Steven has made a game in which the probability of randomly picking a number from 0 to 5 is given by the probability distribution shown in the following table. Answer each of the following, giving all answers correct to 3 decimal places.

x	0	1	2	3	4	5
$P(X=x)$	0.002	0.076	0.293	0.268	a	0.098

(a) Calculate the expected value for this random variable.

(b) Leanne made a game similar to Steven's, but the probability of randomly picking a number from 0 to 5 is given by the following probability distribution. Calculate the expected value for this random variable.

y	0	1	2	3	4	5
$P(Y=y)$	0.005	0.029	0.047	0.219	0.386	0.314

(c) If one value in Steven's game and one value in Leanne's game are chosen at random, calculate the probability that:

(i) they are the same **(ii)** they are different **(iii)** their sum is greater than 8.

14 The ratio of boys to girls in a particular town was found to be 11 : 10, where the gender of one child in the family is independent of the gender of any other child in the family. Write answers correct to 4 decimal places.

(a) What is the probability that a family with three children will have at least one boy?

(b) Calculate the proportion of families in this town with exactly 4 children that will have at least 3 girls.

(c) What proportion of the families with exactly 4 children will have 2 girls and 2 boys?

15 The probability distribution table of a random variable X is shown on the right. Answer each of the following, giving all answers correct to 3 decimal places where necessary.

x	n	$n+1$	$n+2$	$n+3$
$P(X=x)$	0.80	0.12	0.05	0.03

(a) Determine an expression for $E(X)$ in terms of n.

(b) If two independent values of X are chosen at random, calculate the probability of choosing two consecutive values.

(c) If two values of X are chosen independently at random, calculate the probability that the sum of the two values is even.

(d) If four independent values of X are chosen at random, calculate the probability that they are one of each value.

11

Summary

Discrete random variables

For a discrete random variable the number of outcomes is countable.

In a discrete probability distribution:

$0 \le P(X = x) \le 1$ for all values of x and $\Sigma P(X = x) = 1$.

The expected value of the variable X is $E(X) = \sum x_i p_i$.

A game is fair if the cost to play is equal to the expected return.

$E(f(X)) = \sum f(x_i) p_i$ and $E(aX + b) = aE(X) + b$

$E(X + Y) = E(X) + E(Y)$

Variance: $\text{Var}(X) = E(X^2) - [E(X)]^2$

$\text{Var}(aX + b) = a^2 \text{Var}(X)$

Standard deviation: $\sigma = \sqrt{\text{Var}(X)}$

For many distributions, about 95% of results lie within two standard deviations of the mean. This interval can be represented as $\mu \pm 2\sigma$.

Uniform distribution

For a discrete uniform probability distribution, where $X = 1, 2, ..., n$:

$E(X) = \dfrac{n+1}{2}$

$\text{Var}(X) = \dfrac{n^2 - 1}{12}$

Chapter review 11

1 Determine the value of k, given that each set of data represents a discrete probability distribution. In each case express the value in simplest fraction form. Exercise 11.1

(a)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{8}$	$\frac{1}{16}$	k	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{3}{16}$

(b)

x	5	7	7	8	9	10
$P(X=x)$	$\frac{1}{12}$	k	$\frac{1}{6}$	$\frac{1}{3}$	$2k$	$\frac{1}{4}$

(c)

x	(-1)	0	1	2	3
$P(X=x)$	k	$2k$	$3k$	$4k$	$5k$

2 Two fair six-sided dice, numbered 3 to 8, are rolled. Let X be the total from the two dice. Exercise 11.1

(a) Complete the following table that shows the probability distribution of X. Express the probabilities as fractions in simplest form.

x	6	7	8	9	10	11	12	13	14	15	16
$P(X=x)$		$\frac{1}{18}$	$\frac{1}{12}$		$\frac{5}{36}$			$\frac{1}{9}$			$\frac{1}{36}$

(b) Calculate $P(X \geq 11)$. **(c)** Calculate $P(X < 15)$. **(d)** Calculate $P(7 \leq X \leq 11)$.

3 Which one of the following random variables is not associated with a discrete scale? Exercise 11.1

A The number of dim sims sold at the local fish and chip shop per day

B The number of pairs of sneakers owned by students in your maths class

C The time taken to fill your bath each night for one month

D The number of goals scored in each netball match by a team throughout the season

4 A jar contains seven white marbles, three green marbles and two blue marbles. Two marbles are drawn, with replacement, from the jar. Which expression gives the probability of drawing exactly one white marble? Exercise 11.1

A $\frac{7}{12} \times \frac{7}{12}$

B $\left(\frac{7}{12} \times \frac{5}{11}\right) + \left(\frac{5}{12} \times \frac{7}{11}\right)$

C $2\left(\frac{7}{12} + \frac{5}{12}\right)$

D $\frac{7}{12} \times \frac{5}{12}$

5 Mel rolled a die 20 times with the following results: Exercise 11.1

6, 5, 4, 5, 2, 6, 3, 3, 6, 5, 5, 6, 1, 3, 3, 2, 4, 4, 3, 5.

Mel then pairs the values to determine their totals.

Complete a probability distribution table for the totals T giving the probabilities as decimals.

Exercise 11.1

6 Julia kept a record of the number of bus journeys she made each week throughout the year. Complete a probability distribution for the weekly number of bus rides B giving the probabilities as decimals correct to 2 decimal places.

2, 4, 4, 8, 10, 10, 10, 12, 10, 10, 10, 11, 10, 2, 3, 10, 12, 10, 8, 10,

8, 10, 10, 10, 4, 2, 10, 12, 8, 10, 12, 10, 10, 10, 10, 10, 4, 2, 10, 10,

12, 8, 10, 12, 10, 10, 4, 2, 4, 0, 0, 2

Exercise 11.1

7 Use mathematical reasoning to explain why the function given by $P(X = x) = \dfrac{2x-3}{15}$ is a probability distribution for $x \in \{3, 4, 5\}$ but it is not a probability distribution for $x \in \{1, 2, 3, 4, 5\}$.

Exercise 11.2

8 Two fair dice are rolled. Let X be the total shown on the two dice.

(a) Calculate $P(X \geq 10)$.

(b) Calculate $P(5 < X \leq 9)$.

(c) Determine the expected value, $E(X)$.

Exercise 11.2

9 A random variable, X, has the following probability distribution. Give the answers to the following questions correct to 4 decimal places, where necessary.

x	-2	-1	0	1	2
$P(X=x)$	0.15	0.2	0.1	0.35	0.2

(a) $E(X)$

(b) $\text{Var}(X)$

(c) $E(3X-2)$

(d) $E\left(X^2 - 2X\right)$

(e) $\text{Var}(3X-2)$

Exercise 11.2

10 A game at the local festival requires a player to toss two coins. If two heads are obtained, the player receives \$2 and has the cost of the game returned. Otherwise, the player loses the cost of the game. Determine the cost to play for the game to be fair.

Exercise 11.2

11 For the random variable X it is known that $E(X) = 2.3$ and $E\left(X^2\right) = 6.2$. Determine the value of the standard deviation, sigma, correct to 2 decimal places.

Exercise 11.2

12 A variable Y has the probability distribution shown in the following table.

Calculate the variance, $\text{Var}(Y)$.

y	-2	-1	0	1
$P(Y=y)$	0.4	0.3	0.2	0.1

Exercise 11.2

13 The following table, obtained from records over a long period of time, represents the probability distribution of the number of consultations per hour by a dentist Y.

y	0	1	2	3	4
$P(Y=y)$	0.015	0.195	0.305	0.245	0.24

For this distribution it has been calculated that $E(Y) = 2.5$ and $\sigma = 1.1$.

Determine $P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma)$ and comment on the result.

Exercise 11.3

14 Technology is used to select a random number from 1 to 20 inclusive. The variable R represents the value chosen. Calculate the following values.

(a) $E(R)$

(b) $\text{Var}(R)$

Exercise 11.3

15 Each of the numbers from 1 to n appearing on an n-sided spinner has an equal chance of appearing. If the variable S describes this situation, and $\text{Var}(S) = 18\frac{2}{3}$, what is the value of n?

16 A spinner is equally divided into n segments. Each segment contains a different value from 1 to n. Determine the value of n if the expected value is: Exercise 11.3

(a) 26.5 (b) 18 (c) 101

17 A goal attack in netball scores a goal from 80% of her shots. How many shots at goal would she need to take before she would have at least a 99% chance of scoring at least one goal? Exercise 11.1

18 Raduhas developed a website where subscribers can pick up to six motivational videos per day to watch. The number of each video chosen is a discrete random variable X with a probability distribution formula $P(X = x) = k\left(30 - (x-1)^2\right)$, where $x \in \{1, 2, 3, 4, 5, 6\}$. Exercise 11.1

(a) Calculate the value of k.

(b) Display all probabilities as fractions in simplest form in a probability distribution table.

(c) Draw a scatterplot of the distribution.

19 A spinner showing the numbers 1, 2, 3 and 4 is spun twice. Let X be the total of the two numbers obtained. Exercise 11.2

(a) Complete the table for the probability distribution of X.

x	2	3	4	5	6	7	8
$P(X=x)$							

(b) Use your table to determine each of the following values.

(i) $E(X)$ (ii) $\text{Var}(X)$

20 Determine the value of $\text{Var}(X)$ for the following probability distribution. Exercise 11.2

x	0	1	2	3
$P(X=x)$	t	2t	3t	4t

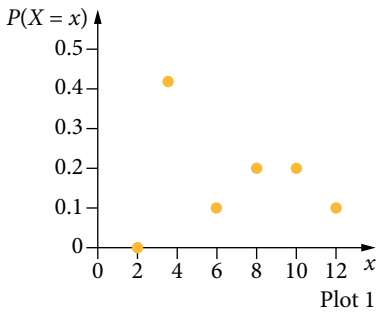
A 2 B 5 C 0.1 D 1

21 Determine the value of $\text{Var}(2X - 3)$ for the following probability distribution. Exercise 11.2

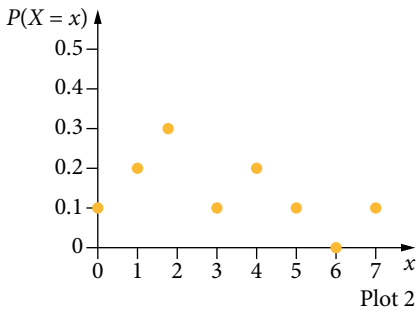
x	1	2	3	4	5
$P(X=x)$	0.2	0.1	0.35	0.05	0.3

A 2.1275 B 4.255 C 7.51 D 8.51

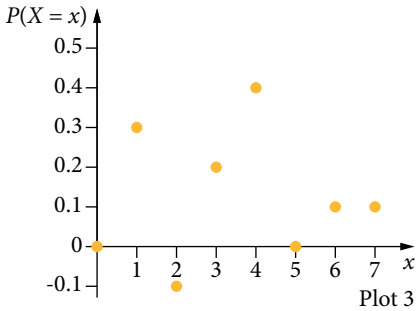
22 Consider the following scatterplots. Exercise 11.2

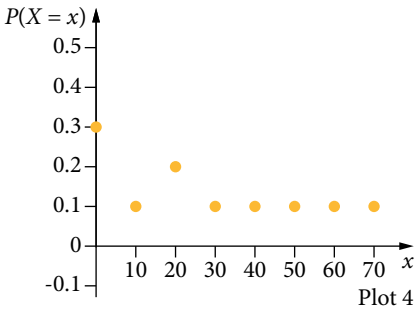


Plot 1



Plot 2





(a) Identify which of the four scatterplots represents a probability distribution function. Explain your answer using mathematical reasoning.

(b) Calculate the mean and variance of the plots that represent probability distribution functions.

Exercise 11.3

23 Many calculators can produce random numbers as whole number values within a specified range.

(a) If the technology produced truly random numbers, what type of distribution would you expect the results to follow?

(b) Listed below are 96 random numbers (as displayed by a calculator) in the range 1 to 6, inclusive. Create a probability distribution table based on these results. To make analysis easier, write the probabilities with a denominator of 96 and use X as the variable.

3	6	2	5	6	2	3	1	6	1	1	4
6	6	2	2	1	1	5	1	3	2	6	1
6	4	2	6	5	2	2	6	6	1	1	5
6	4	2	1	6	2	3	1	2	6	3	4
4	1	1	1	5	5	6	3	1	3	6	5
4	4	6	5	5	4	3	2	4	6	1	3
3	2	5	6	3	3	2	4	4	5	5	3
6	4	3	3	5	6	2	5	3	3	2	1

(c) What do these results suggest about whether the numbers are truly random?

(d) Calculate $E(X)$ and $\text{Var}(X)$ for the sample, correct to 2 decimal places. Compare them to the theoretical values for the underlying distribution.

(e) Now use technology to generate a similar set of 96 numbers in the range 1 to 6. Calculate $E(X)$ and $\text{Var}(X)$ for the sample, correct to 2 decimal places.

(f) Compare the three sets of statistics that you have now calculated.

(g) How would you expect the results to compare to the theoretical values if you generated 5000 random numbers in the range 1 to 6?

Exercise 11.4

24 A sample of 3 items is selected at random from a box containing 10 items, of which 3 are defective. Let Y represent the number of defective items selected.

(a) Complete the table for the probability distribution of the variable. Give the probabilities in simplest fraction form.

y	0	1	2	3
$P(Y = y)$				

(b) Calculate the expected number of defective items.

25 Audrey is playing a popular quest-style game on her games console. It randomly generates whole numbers in the range 3 to 8 each time she hits the *Play* button. She cannot start her quest until the total of the scores is 7 or greater. Exercise 11.4

(a) Determine the probability distribution of the number of times Audrey hits *Play*, given by the variable X, required to achieve this total.

(b) Calculate the expected number of hits of *Play* required.

(c) Calculate the variance, correct to 2 decimal places, for the number of hits of *Play* required.

26 The discrete random variable X can take only the values 1, 2, 3, 4, 5 and 6. The probability distribution of X is described by the following statements: Exercise 11.2

$P(X = 1) = P(X = 3) = P(X = 5) = a$, $P(X = 2) = P(X = 4) = P(X = 6) = b$, $a = 3b$

(a) Determine the values of $E(X)$ and $\text{Var}(X)$.

(b) Determine the probability that the sum of two independent observations from this distribution is greater than 9.

27 Freya throws a fair six-sided die marked 1 to 6. If she does not obtain a 4, her score is the number thrown. If she obtains a 4, she throws the die a second time, and in this case her score is the sum of 4 and the second number obtained. Freya has at most two throws of the die. Let Z be the random variable representing Freya's score. Exercise 11.2

(a) Determine the expected value, $E(Z)$.

(b) What is $P(Z > 6)$?

(c) What is $P(Z < 7)$?

(d) What is $P(Z > 4 | Z \leq 8)$?

28 A fair six-sided die marked 1 to 6 is rolled until an even number shows uppermost or five odd numbers in a row have shown uppermost. Exercise 11.2

(a) Create a probability table showing the number of throws, X and the associated probabilities.

(b) Calculate the expected number of rolls. Round your answer correct to 1 decimal place.

(c) Consider flipping an unbiased coin where you will stop as soon as the coin lands on tails or if five heads in a row appear. Explain how you can easily state the expected number of flips of the coin.

29 Karina and Achim print and sell business cards in packs of 500. They sell a maximum of 4 packs per hour and incur an average of \$16 running costs per hour. The average price for a pack of 500 business cards is \$20. The probability distribution table shown represents the random variable X, the number of packs they sell on any given hour. Exercise 11.4

x	0	1	2	3	4
$P(X = x)$	0.012	0.097	0.138	0.325	0.428

(a) Calculate the number of packs of business cards they can expect to print and sell in any given hour.

(b) Determine how much profit is made, or loss incurred, on average by Karina and Achim in any given hour.

Exam review

UNIT 2

Paper 1: Technology-free

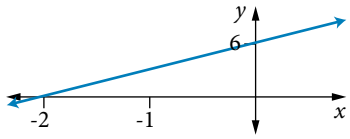
Exercise 7.3

1 Sketch the graph of $y = 5^{x-2} - 3$, showing exact values for the coordinates of the axes intercepts.

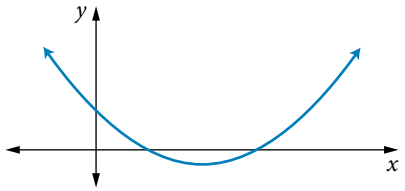
Exercise 9.2

2 Draw the graph of the gradient for each function, matching key points of change.

(a)



(b)



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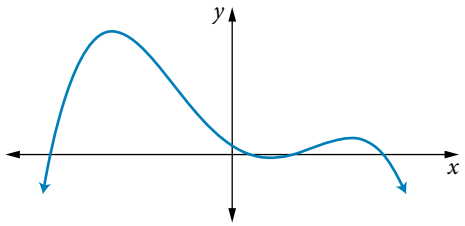

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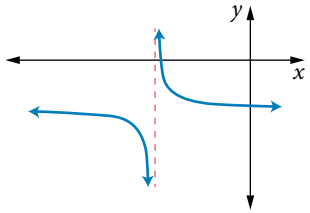


Exercise 10.1

3 Consider the function $f(x) = 4x^2 - 7x + 2$.

(a) Calculate $f'\left(\sqrt{2} + 1\right)$.

(b) Determine the set of values of x for which $f'(x) > 0$.

Exercise 8.5

4 Determine the exact solutions of $2\cos(3x) = 1$ over the domain $[0, \pi]$.

Exercise 9.3

5 Determine the expression for $f'(x)$ by differentiating $f(x) = x^2 - 4x$ from first principles.

Exercise 10.7

6 For each of the following functions, determine an expression for $f'(x)$ as a single algebraic fraction.

(a) $f(x) = x^2\sqrt{2 - x}$

(b) $f(x) = \dfrac{2\sqrt{x}}{x^2 + 4}$

Exercise 10.4

7 A cone with a perpendicular height h (m) is to be carved from a sphere of radius R (m).

(a) Determine a formula for the cone with largest possible volume in terms of R and h, so that the base and tip of the cone touch the surface of the sphere.

(b) Given that $R = 1$ m, determine the value of h for which the volume is the largest, and calculate that volume.

Paper 2: Technology-active

1 Solve the following equations for the unknown, correct to 2 decimal places. Exercise 7.1

(a) $3^x = 6$ (b) $5^x = 10$ (c) $2^x = 1300$

2 Convert the following degree measures to radians, correct to 3 decimal places. Exercise 8.1

(a) 46° (b) 125° (c) 306°

3 Calculate the following sector quantities, correct to 2 decimal places. Exercise 8.1

(a) the length of the arc of a circle of radius 25.43 cm if the arc subtends an angle of 68° at the centre of the circle

(b) the area of the sector of a circle of radius 24 cm, given that the arc of the sector subtends an angle of $\frac{5}{6}\pi$ at the centre

4 The variable X has the following probability distribution. Exercise 11.2

x	4	5	6	7	8	9
$P(X=x)$	0.2	0.15	0.1	0.3	0.2	0.05

Calculate the standard deviation, correct to 4 decimal places.

5 Consider the function $h(x) = 2x^3 + ax^2 + bx + 3$, where a and b are constants. Exercise 10.3

The graph of $y = h(x)$ has a stationary point at (1,-7).

(a) Determine the expression for $h(x)$.

(b) Determine the coordinates and nature of all stationary points.

(c) Sketch the graph of h on the interval $-5 \le x \le 2$ and determine the coordinates of the global maximum and global minimum points over the interval. Give answers correct to 2 decimal places where necessary.

6 The variable X has the following probability distribution. Exercise 11.2

x	1	3	4	6	9
$P(X=x)$	0.25	0.2	0.3	0.2	0.05

Calculate the following values.

(a) $E\left(X^2 + 2X - 3\right)$ (b) $\text{Var}(2X - 3)$

7 The number of cockatoos in a region, previously unfarmed before 2012, was estimated to be 35. After crops were put in, the numbers grew exponentially to 300 cockatoos in 2018. Exercise 7.4

(a) Determine a model for the number of cockatoos in the form $N(t) = N_0.a^t$, where N is the number of birds at any time t (years) after 2012. Use your model to predict the number of birds in 2025, rounding appropriately.

(b) An eminent biologist suggests that with food availability being limited, the number of cockatoos will not go over 500. Create a new model to accommodate the biologist's suggestions as well as using the given data values. Use your new model to predict the number of cockatoos in 2025.

Exam review

UNITS 1&2

Paper 1: Technology-free

Exercise 1.1

1 Determine the gradient and the coordinates of the y-intercept for the graph with equation $8x - 20y = -15$.

Exercise 1.2

2 Determine the solution to the simultaneous equations $10x + y + 21 = 0$ and $4x = 8y$.

Exercise 2.2

3 Write the rule of the quadratic function whose graph is obtained from that of the basic quadratic function by a reflection in the x-axis, a horizontal translation 1 unit to the left and vertical translation 7 units up.

Exercise 2.3

4 Determine the exact solutions for the following equations using the quadratic formula.

(a) $4a^2 = 2 + 7a$ **(b)** $\frac{1}{4}b^2 - 8b = 12$ **(c)** $3c^2 - 9 = 0$

Exercise 2.2

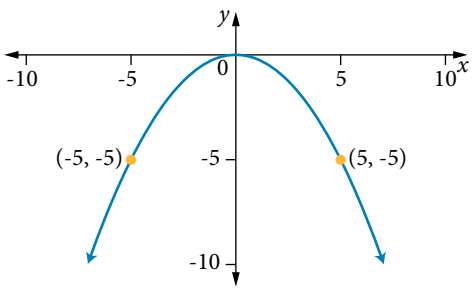
5 Determine the transformation(s) required to obtain each of the following from the basic quadratic function in the form $y = x^2$.

(a) $m = -2p^2 + 1$ **(b)** $s = 3(t+5)^2 - 3$ **(c)** $y = -\frac{3x^2}{4}$

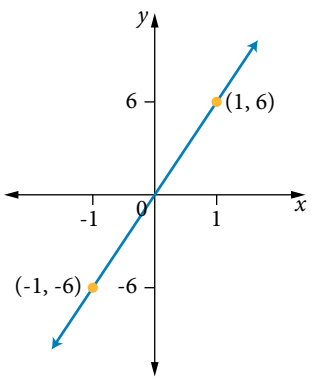
Exercise 3.4

6 For each of the graphs below, determine the rule of a corresponding power function.

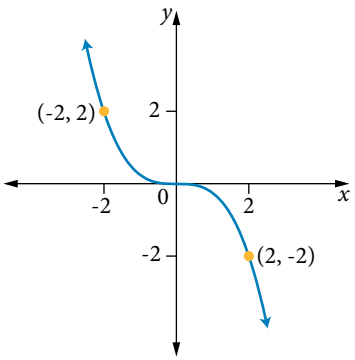
(a)



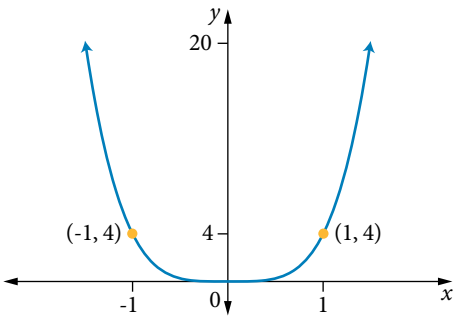
(b)



(c)



(d)



Exercise 3.2

7 Using the binomial theorem, expand and simplify the following expressions.

(a) $(2x - y)^3$ **(b)** $(3m + 2n)^2$ **(c)** $(6x - 2y)^2$ **(d)** $(3x + 1)^3$

8 For $f(x) = \sqrt{x+3}$, sketch the graph, showing the vertex and any axes intercepts. State the domain and range. Exercise 4.2

9 State the radius and centre of the circle with rule $(x+3)^2 + (y-4)^2 = 25$. Draw a graph of the circle, clearly showing the major points in coordinate form. Exercise 4.3

10 If a card is drawn at random from a standard pack of 52 playing cards, determine the probability, in simplest fraction form, that it is: Exercise 5.1

(a) black or a Queen

(b) red or a number card

(c) a spade or a five or a Jack

(d) a club, or a six, a Queen or an Ace

(e) neither red nor a Queen

(f) neither an even number card nor a Jack nor a Queen.

11 Of the 150 students in Year 11 at Tabletop College, 40 have a Learner Driver Permit (L) and 85 have a part-time job (J). Of those with part-time jobs, 30 also have their Learner Driver Permit. Exercise 5.2

(a) Display the information as a Venn diagram.

(b) Determine the probability that a Year 11 student chosen at random:

(i) has a part-time job

(ii) has a part-time job but not a Learner Driver Permit

(iii) has a Learner Driver Permit but not a part-time job

(iv) has neither a part-time job nor a Learner Driver Permit.

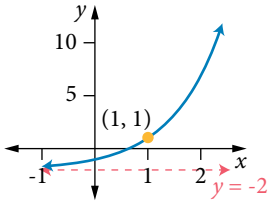
12 Simplify each of the following expressions. Exercise 5.5

(a) $\dfrac{n!}{(n-3)!}$ **(b)** $\dfrac{(n-2)!}{(n-3)!}$ **(c)** $\dfrac{(n-5)!}{(n-7)!}$ **(d)** $\dfrac{(n-4)!}{(n-2)!}$

13 Simplify each of the following. Exercise 6.1

(a) $3(ab)^0$ **(b)** $16^{\frac{3}{4}}$ **(c)** $\sqrt[3]{216^{-2}}$

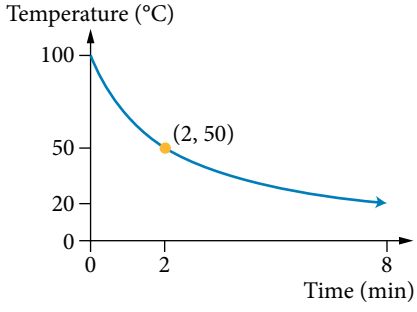
14 The equation of the graph shown is of the form $f(x) = a^x + b$. Identify the rule for the graph. Exercise 7.2



15 Solve the following equations over the domain $[0, 2\pi]$. Exercise 8.2

(a) $\sin(x) = \dfrac{\sqrt{3}}{2}$ **(b)** $\cos(x) = -\dfrac{\sqrt{3}}{2}$ over the domain $[0, 2\pi]$

16 The graph shows the temperature of a cup of coffee for 8 minutes after the coffee is poured into a cup. Determine the rate at which the coffee is cooling 2 minutes after it is poured, correct to 1 decimal place. Exercise 9.1

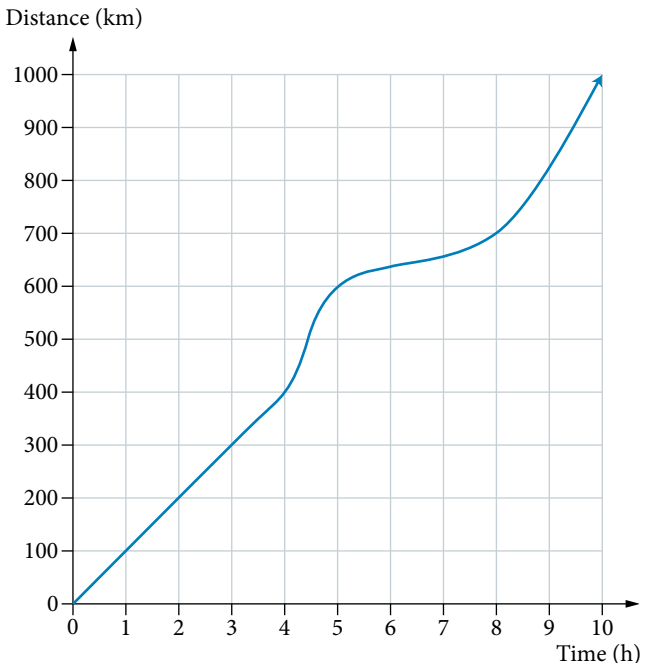


Exercise 9.1

17 The following graph shows the distance travelled by a racing car.
Determine the average speed:

(a) of the racing car for the entire race

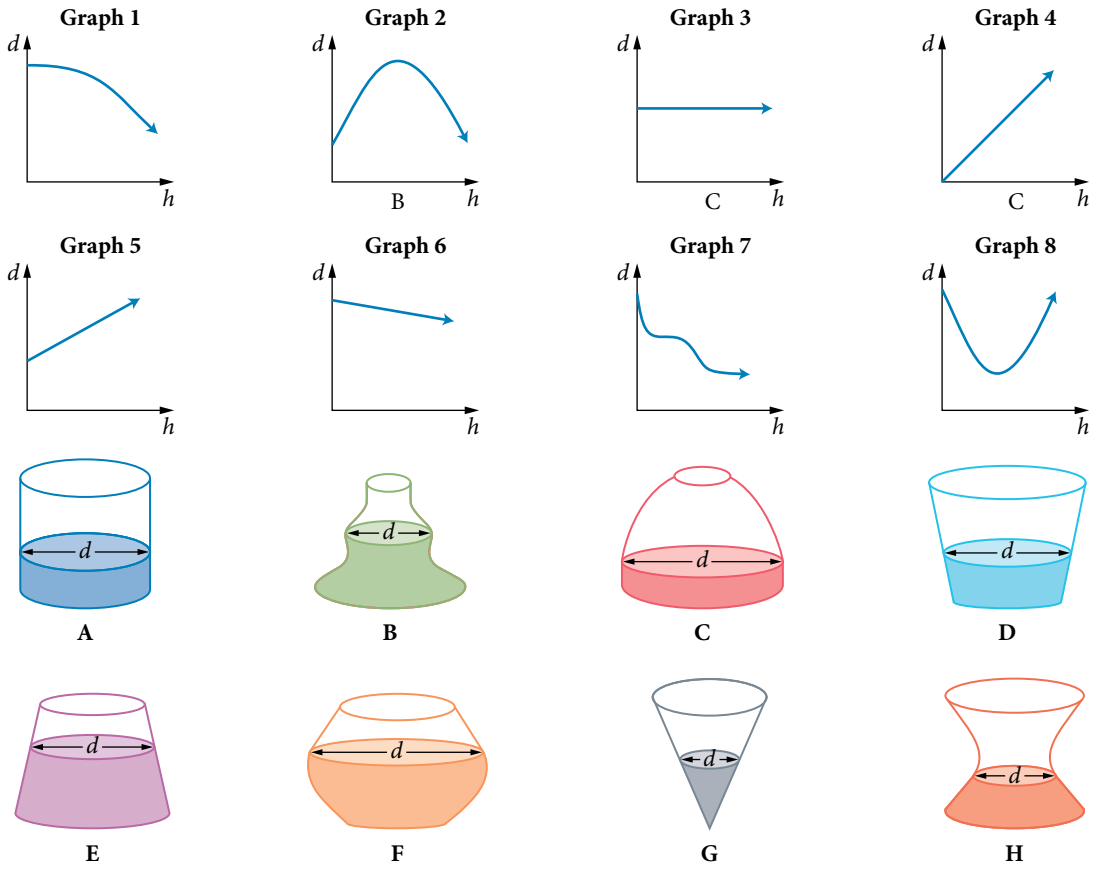
(b) from the end of the second hour to the end of the eighth hour of the race.



Exercise 9.2

18 Water is poured into each of the containers A–H. The graphs 1–8 show how the surface diameter d changes as the height of the liquid h increases.

Match each container to the surface diameter versus height graph.



Exercise 9.3

19 Differentiate the following from first principles; that is, determine $f'(x)$.

(a) $f(x) = -4x^2$

(b) $f(x) = x(x-2)$

20 A liquid is flowing into a storage tank. The volume, V litres, of the liquid in the storage tank after t seconds is given by $V = 2t^3 + 4t$. Exercise 10.2

Calculate the rate of increase in the liquid's volume at time $t = 3$.

21 For each of the following functions, determine an expression for $\frac{dy}{dx}$. Exercise 10.6

(a) $y = \sqrt{5x - 7}$

(b) $y = \left(4x^3 - 5\right)^3$

22 Technology is used to select a random number S from 1 to 25 inclusive. Calculate: Exercise 11.3

(a) $E(S)$

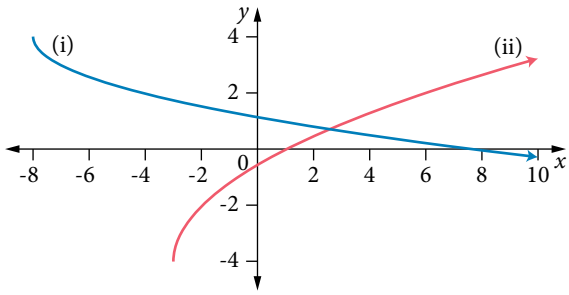
(b) $\text{Var}(S)$

23 Two graphs that are transformations of $f(x) = \sqrt{x}$ are shown. For each graph: Exercise 4.2

(a) state the coordinates of the x-intercept

(b) determine the rule

(c) determine the exact values of the coordinates of the y-intercept.



24 The average time it takes for light from the Sun to reach Earth is about 8 min 20 s. The speed of light is 2.998×10^8 m s^{-1}. What is the average distance of the Earth from the Sun in kilometres? Exercise 6.2

25 Solve $5^{2a} - 26 \times 5^a + 25 = 0$ by first factorising. Exercise 7.1

26 Consider the function $f(x) = x^3 + 6x^2 + 9x + 4$. Exercise 10.3

(a) Use calculus to locate the coordinates of any stationary points and determine their nature.

(b) Sketch the graph of $y = f(x)$, showing all key features.

27 A rectangle is drawn in the first quadrant with sides on the coordinate axes such that one corner is at the origin O and one corner lies on the line $2x + y = 6$. Determine the maximum area of the rectangle. Exercise 10.4

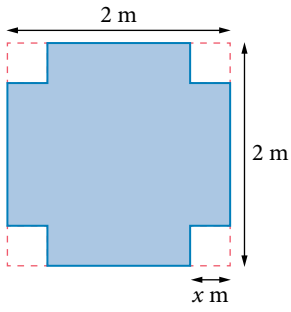
28 A square sheet of cardboard measures 2 m by 2 m. A square of side length x (m) is cut from each corner and the sides turned up to form an open-topped rectangular box with a square base and volume V (m^3). Exercise 10.4

(a) Express V in terms of x in expanded form.

(b) Determine the domain of $V(x)$.

(c) Use calculus to calculate the value of x for which the box has a maximum volume. Hence determine the maximum volume and the dimensions of the box in this case.

(d) Sketch the graph of V against x, showing all key features.



29 Plot one period of the graph of $y = 3\sin\left(\frac{4x + \pi}{8}\right) - 2$ from the y-axis, showing key points and endpoints of the domain. Exercise 8.4

Paper 2: Technology-active

Exercise 4.1

1 The current, I amps, produced by a battery is inversely proportional to the resistance, R ohms, of the circuit to which the battery is connected according to the equation $R = \frac{240}{I}$.

(a) Determine the resistance when a current of 0.25 amps flows through the circuit.

(b) Determine the current that will flow through a circuit with resistance of 800 ohms.

Exercise 5.2

2 $\xi = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$, $A = \{4, 16\}$, $B = \{2, 4, 8, 10\}$. List the elements of each of the following sets.

(a) $A \cap B'$

(b) $A' \cup B$

Exercise 6.2

3 Calculate each of the following, stating your final answer to the appropriate number of significant figures.

(a) $2.1 \times 10^3 \times 1.7 \times 10^2$

(b) $\left(3.7 \times 10^2\right) \div \left(5.1 \times 10^4\right)$

Exercise 7.1

4 Solve $5^x = 4$, correct to 4 significant figures.

Exercise 8.5

5 Solve $\tan(3x) = 1.5$ over the domain $[0, 2\pi]$. Give your answers correct to 2 decimal places.

Exercise 1.4

6 A sequence is defined as $t_{n+1} = t_n + 9$, $t_1 = -20$.

(a) Calculate the first four terms of this sequence

(b) Determine whether this sequence is an arithmetic sequence, providing a reason to support your answer.

(c) How many terms of this sequence are negative?

(d) Starting with the first term, how many terms must be added together for the sum to exceed 5000?

Exercise 3.4

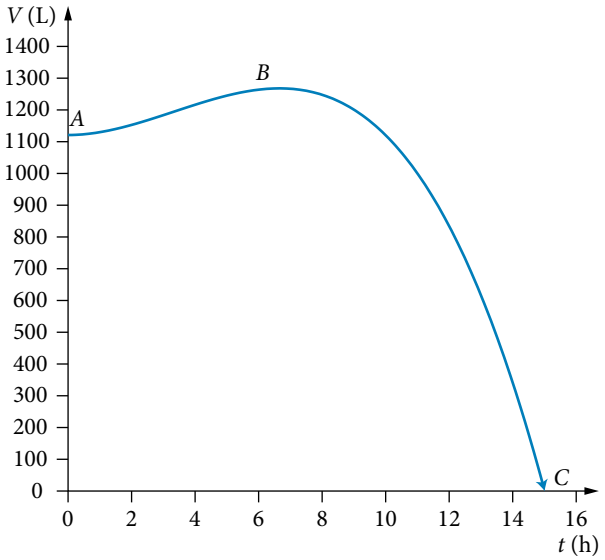
7 Sketch the graph of each cubic function, clearly showing all intercepts and critical points. Round values correct to 2 decimal places where necessary.

(a) $y = 3(x-1)^3 + 2$

(b) $y = -(x+2)^4 + 6$

Exercise 9.2

8 The volume of water, V (L), in a tank after t (h) is given by $V = -t^2(t-10) + 1125$. Water was initially added to the tank (from points A to B) and then the tank was emptied (from points B to C), as shown in the graph. Answer the following questions correct to 1 decimal place.



(a) Determine the average rate at which water was added to the tank.

(b) Determine the average rate at which water was lost from the tank.

(c) By finding the gradient of the secant joining points with t-values of 14.999 and 15, estimate the rate at which water flows out of the tank just before the tank empties.

9 Determine the values of a and b in the following probability distribution, given that $E(X) = 4\frac{7}{16}$. Give your answers as fractions in simplest form. Exercise 11.2

x	2	3	4	5	6
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{4}$	a	b	$\frac{7}{16}$

10 A roulette wheel in Europe usually has 37 equal-sized spaces showing the numbers 1–36 and 0. For the purpose of this question, assume that 0 represents the 37th possible outcome. Exercise 11.3

(a) Calculate the mean of the number of the space the ball lands in.

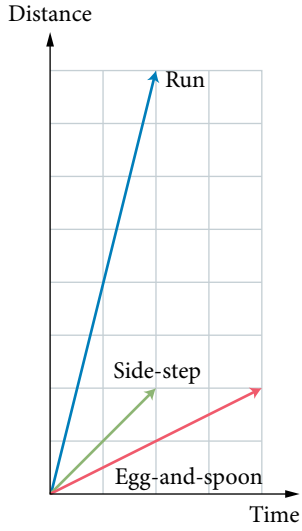
(b) Calculate the variance of the number of the space the ball lands in.

11 Two octahedral dice, with faces each showing the numbers 1–8 are rolled and the variable X represents the number of sixes obtained. Exercise 11.4

(a) Calculate the expected number of sixes obtained.

(b) Calculate the variance of the number of sixes obtained.

12 The three direct variation graphs show the relative distance and time, on average, for a student to complete each of three legs of a novelty race in the sports carnival. The students will cover 120 m in 5 minutes. Exercise 1.3



(a) Write a rule for the piece-wise function of total distance covered d (m) in terms of time elapsed t (s) given that students begin with the egg-and-spoon leg and finish with the run.

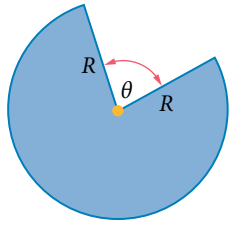
(b) A group of students choose to cover the same course by side-step all the way. Show graphically at what point these students would expect to be overtaken. Write your answer as a distance from the finish.

(c) A group is going to run at the start and then side-step to the end. Determine the distance from the start that they must start to side-step to make it a fair finish.

13 A rectangular children's playground 30 m by 20 m is to be increased by extending the boundary by the same amount on all sides. Create a model for the increase in area $A\left(\text{m}^2\right)$ in terms of the distance between each original boundary and the new boundary, x (m). Use your model to determine the distance each boundary will move to create an increase of 200 m^2 for the playground. Round your answer correct to 2 decimal places. Exercise 2.4

14 Jack, a model boat enthusiast, is designing a nose cone for his latest high-speed destroyer to make it go faster. He removes a sector from a circular sheet of metal of radius R, as shown below. Exercise 10.4

He then joins the edges together to form a cone. What angle θ must he cut out of the circular sheet to generate a maximum volume for his nose cone? Give your answer in radians correct to 2 decimal places, or degrees, to the nearest degree.



Answers

Chapter 1 Linear functions

RECALL

1 $A(10, 6)$, $B(0, 9)$, $C(-2, 5)$, $D(5, 1)$, $E(9, 0)$, $F(5, -2)$, $G(6, -10)$, $H(2, -8)$, $I(-2, -8)$, $J(-9, 0)$, $K(-15, 0)$

2 (a) $\left(-\frac{3}{2}, 0\right), (0, 3)$ (b) $(3, 0), (0, 2)$

(c) The coordinates of the x- and y-axis intercept are $(0, 0)$.

3 (a)

x	-1	0	1	2	3	4
y	1	5	9	13	17	21

(b)

x	-3	-1	1	3	5	7
y	-2	-1	0	1	2	3

(c)

x	-0.5	0	0.5	1	1.5	2
y	-3	-2	-1	0	1	2

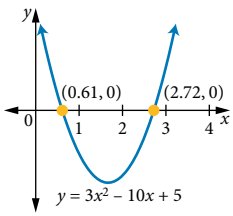
4 (a) positive (b) negative (c) zero (d) undefined

5 (a) A: -3, B: 2, C: undefined, D: $\frac{1}{3}$, E: zero, F: $-\frac{1}{2}$

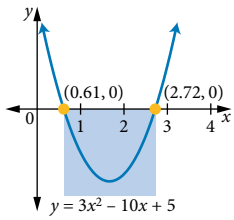
(b) $-\frac{1}{2}$ (c) $\frac{2}{9}$

6 (a) $m = \frac{5}{3}, c = 2$ (b) $m = -3, c = 4$ (c) $m = -\frac{3}{4}, c = -2$

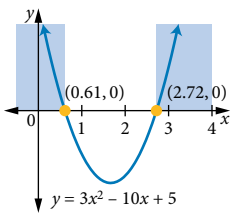
7 (a) $x = 0.61, x = 2.72$



(b) $0.61 < x < 2.72$



(c) $x < 0.61, x > 2.72$



8 (a) $x^2 + 6x + 9$ (b) $x^2 - 12x + 39$ (c) $x^2 + 16x + 44$

9 (a) $(x + 3)(x + 5)$ (b) $(x - 7)^2$ (c) $(x - 9)(x + 2)$

EXERCISE 1.1

1 (a) $\{-7, -5, 3, 4\}$ (b) $\{5, 3, -4, 1\}$

2 (a) $x + y - 3 = 0, -1 \le x \le 2$ or domain: $[-1, 2]$

(b) $3x - y - 1 = 0, x \in \mathbb{R}$ or $-\infty < x < \infty$ or domain: $(-\infty, \infty)$

(c) $y - 3 = 0, x \in \{-2, 0, 1, 4\}$

(d) $x + 3 = 0, x \in \{-3\}$

(e) $2x - y + 1 = 0, -2 < x \le 1$ or domain: $(-2, 1]$

(f) $x + y - 4 = 0, 0 < x < 4$ or domain: $(0, 4)$

(g) $3x - y - 1 = 0, x \in \{-1, 1, 2\}$

(h) $y + 1 = 0, x \le 2$ or domain: $(-\infty, 2]$

3 (a) $y \in \{2, 3, 4, 5\}$ (b) $0 < y < 5$ or range: $(0, 5)$

(c) $y \ge -8$ or range: $[-8, \infty)$ (d) $-1 < y < 5$ or range: $(-1, 5)$

(e) $y \in \{-1, 1, 3, 5\}$ (f) $y \le 6$ or range: $(-\infty, 6]$

(g) $y \in \{1, 2, 3, 4, 5\}$ (h) $-3 < y < 6$ or range: $(-3, 6)$

4 (a) 14 (b) 6 (c) 4

(d) $-\frac{3}{2}$ (e) -3 (f) $-\frac{11}{2}$

5 (a) 1 (b) -1 (c) -5

(d) 3 (e) $\frac{3}{2}$ (f) -12

6 (a) C

(b) The incorrect assumption is the coefficient of x will always give the gradient.

7 B

8 (a) 5 (b) -1.5 or $-\frac{3}{2}$ (c) 0 (d) undefined

9 $\frac{9}{2}$ 10 $(0, -4)$ 11 $\frac{1}{2}$

12 $\left(0, -\frac{3}{2}\right)$ 13 $-\frac{1}{4}$ 14 D

15 (a) $y = 2x - 3$ (b) $y = -3x$ (c) $y = \frac{1}{5}x + \frac{58}{5}$

(d) $y = -\frac{2}{3}x + 9$ (e) $y = 8x + 12$ (f) $y = -\frac{1}{2}x - 1$

(g) $y = \frac{3}{4}x + 2$ (h) $y = \frac{1}{7}x + \frac{2}{7}$

16 (a) $y = 2x + 6$ (b) $y = 4x - 1$ (c) $y = -x + 4$

(d) $y = 0$ (e) $x = 3$

17 $\left(-3\frac{8}{9}, 0\right)$ and $\left(0, 1\frac{2}{3}\right)$

18 $h = -4$ 19 $t = 1$

20 (a) C (b) D

21 (a) $C = 658n, n \in \{1, 2, 3, \ldots\}$, direct variation

(b) $C = 1.42l, l > 0$, direct variation

(c) $C = 80h, h \in \{1, 2, 3, 4, \ldots\}$, direct variation

(d) Not direct variation, no apparent formula, domain: $h \ge 0$ with a suitable upper limit.

(e) Not direct variation, no apparent formula, domain: $n \in \{1, 2, 3, 4, \ldots\}$

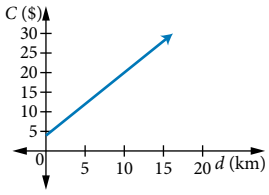
(f) Not direct variation, no apparent formula, domain: (possible) $T > 15$

(g) Not direct variation, no apparent formula, domain: $k \in \{1, 2, 3, \ldots\}$

22 (a) Fare costs \$51.60 (b) $C = 3.90 + 1.59d$

(c)

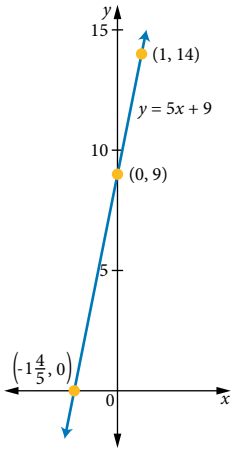
d (km)	C (\$)
5	11.85
10	19.80
15	27.75



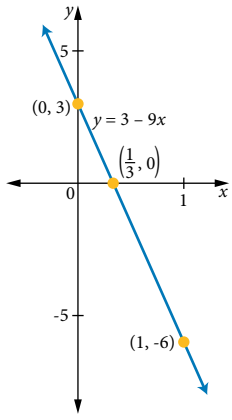
(d) Gradient is 1.59. The gradient represents the cost per kilometre.

(e) 3.90 represents the flagfall.

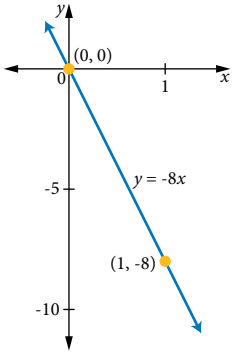
23 (a) $m = 5, c = 9$



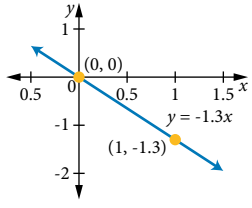
(b) $m = -9, c = 3$



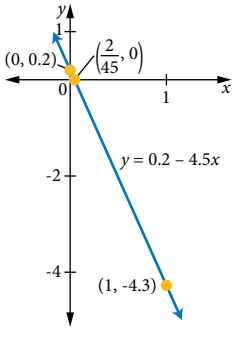
(c) $m = -8, c = 0$



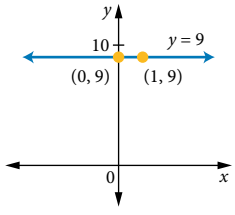
(d) $m = -1.3, c = 0$



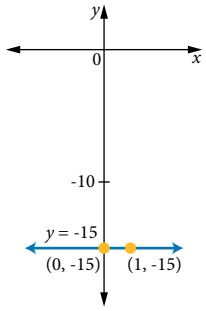
(e) $m = -4.5, c = 0.2$



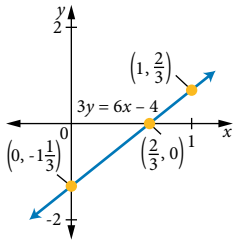
(f) $m = 0, c = 9$



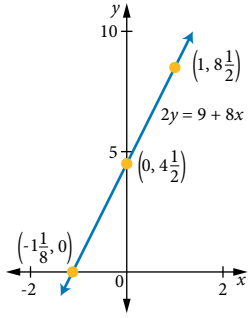
(g) $m = 0, c = -15$



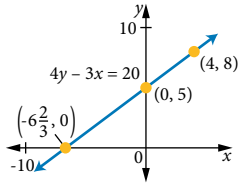
(h) $m = 2, c = -1\frac{1}{3}$



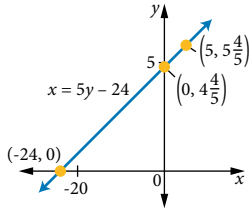
(i) $m = 4, c = 4\frac{1}{2}$



(j) $m = \frac{3}{4}, c = 5$



(k) $m = \frac{1}{5}, c = 4\frac{4}{5}$



24 (a) $y = 2x - 3$ (b) $y = -4x + 3$

(c) $y = \frac{1}{7}x + \frac{9}{7}$ or $x - 7y + 9 = 0$

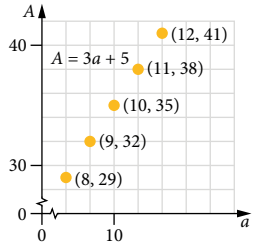
25 (a) $y = x - 7$ (b) $y = 6x - 5$ (c) $y = -x + 2$

(d) $y = -2x + 15$ (e) $y = 2x + 7$

26 $y = 0.5x + 1.35$ 27 $y = 5x - 22.5$ 28 $y = -2x - 7$

29 $p = 10$ 30 416.7 m 31 30 km

32 $A = 3a + 5, a \in \{5, 6, 7, \ldots, 12\}$ restricted to a more likely $a \in \{8, 9, 10, 11, 12\}$.

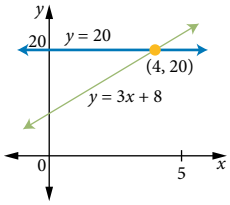


33 $y = -\frac{10}{3}x - \frac{23}{3}$ or $10x + 3y + 23 = 0$

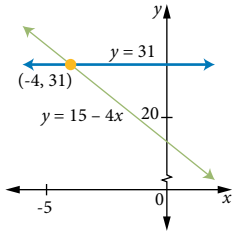
34 $\{263, 264, \ldots, 295\}$

EXERCISE 1.2

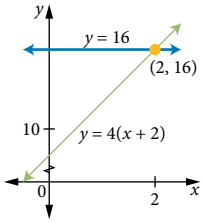
1 (a) $x = 4$



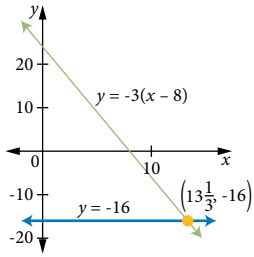
(b) $x = -4$



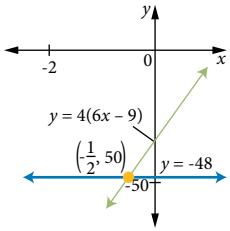
(c) $x = 2$



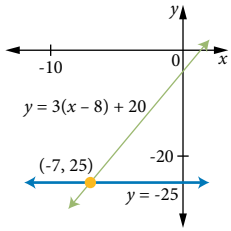
(d) $x = 13\frac{1}{3}$



(e) $x = -\frac{1}{2}$



(f) $x = -7$



2 (a) $x = 20$ (b) $a = 63$ (c) $m = 16\frac{2}{3}$

(d) $h = 18$ (e) $y = 50$

3 (a) $y = 2\frac{1}{2}$ (b) $x = -6$ (c) $x = \frac{5}{8}$ (d) $a = 2\frac{1}{5}$

(e) $b = -3$ (f) $h = 15$ (g) $d = -7$ (h) $x = \frac{2}{3}$

(i) $y = 4$ (j) $a = \frac{1}{8}$ (k) $m = -3$

4 (a) $h = 2.1$ (b) $x = -10\frac{3}{4}$ or $x = -10.75$

(c) $m = -\frac{17}{45}$ (d) $x = 12$

5 (a) $g = -\frac{2}{17}$ (b) $h = 7$ (c) $m = -9$

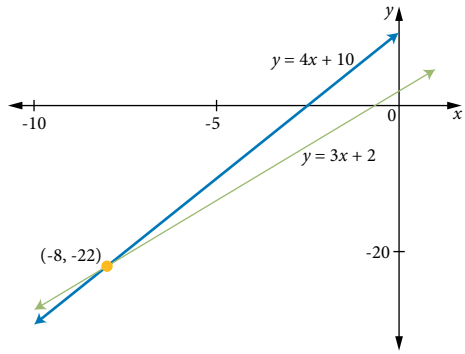
(d) $x = 8$ (e) $k = -\frac{1}{2}$ (f) $y = \frac{1}{4}$

6 C

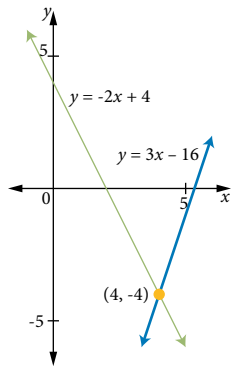
7 (a) $x = -2, y = 3$ (b) $x = 7, y = -1$

(c) $x = -4, y = -4$ (d) $x = 1, y = 3$

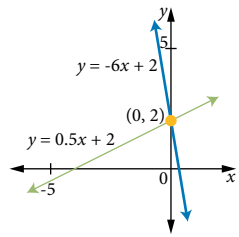
8 (a) $x = -8, y = -22$



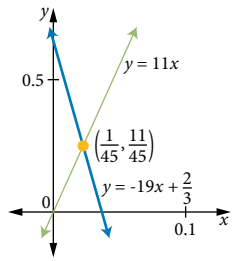
(b) $x = 4, y = -4$



(c) $x = 0, y = 2$



(d) $x = \frac{1}{45}, y = \frac{11}{45}$



9 (a) $x = -3, y = 5$ (b) $x = 8, y = 0$

(c) $x = 6, y = 6$ (d) $x = -1, y = 8$

10 (a) $x = 2, y = 5$ (b) $x = -1, y = -6$

(c) $x = 7, y = \frac{2}{5}$ (d) $x = 0, y = -6$

11 (a) Sharna incorrectly divided the LHS by 4 and the RHS by 2. The correct solution is $x = 6$.

(b) Sharna did not expand the brackets correctly. The correct solution is $m = 12$.

12 (a) $x = 5, y = -3$ (b) $x = 6, y = 8$ (c) $x = -2, y = -5$

(d) $x = 4, y = -2$ (e) $x = 8, y = 4$

13 (a) $x = -2, y = 3$ (b) $x = 4, y = -2$

(c) $x = -4, y = -4$ (d) $x = 1, y = 3$

14 (a) $x = -8.10, y = 1.89$ (b) $x = 16.25, y = -2.10$

(c) $x = 7.83, y = 2.80$ (d) $x = 9.26, y = 26.11$

(e) $x = -2.92, y = 0.69$ (f) $x = -2484.50, y = 23.81$

15 B

16 **(a)** 19 **(b)** 20 **(c)** -3 **(d)** 7 **(e)** 12

17 **(a)** Gerhard is 35 years old and Benny is 7 years old.

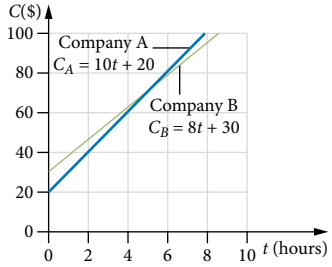
(b) A drink costs \$2.60 and an ice-cream costs \$3.20.

(c) An apple costs \$0.95 and a banana costs \$1.20.

(d) A chair weighs 4.5 kg and a table weighs 85 kg.

18 The equation is $y = \frac{1}{3}x - \frac{4}{3}$ or $x - 3y - 4 = 0$.

19 **(a)**



(b) $t = 5$ hours **(c)** Company B, \$2

20 **(a)** The private charter bus is cheaper for more than 20 students, if one bus is required.

(b) The private charter bus is cheaper for more than 40 students, if two buses are required.

21 **(a)** 10:16 am **(b)** Ashlee: 25 km, Tahnee: 23 km

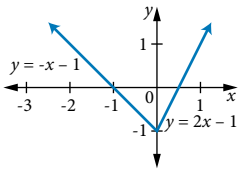
EXERCISE 1.3

1 **(a)** $f(x) = \begin{cases} 5x + 9, & 0 \le x < 7 \\ -3x + 65, & 7 \le x < 12 \\ 0.5x + 23, & x \ge 12 \end{cases}$

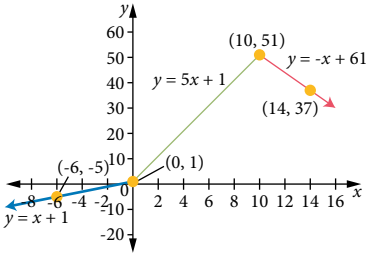
(b) $f(x) = \begin{cases} -0.2x, & x \le 10 \\ 4x - 42, & 10 < x \le 16 \\ 22 & 16 < x \le 20 \end{cases}$

(c) $f(x) = \begin{cases} 4x + 3, & x < 1 \\ 3, & 1 \le x < 7 \\ 10 - x, & 7 \le x < 12 \end{cases}$

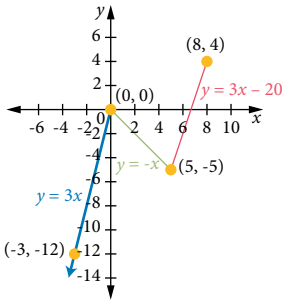
2 (-1,0), (0,-1), $\left(\frac{1}{2}, 0\right)$



3 **(a)**



(b)



4 D **5** $x = 1$

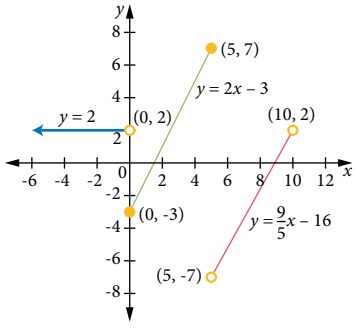
6 **(a)** D

(b) $h(x) = 3x + 2, 0 \le x < 2$ does not include $x = 2$ in the subdomain.

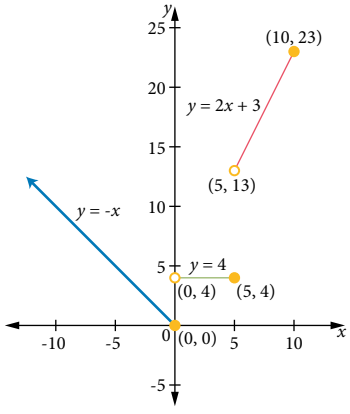
7 Sample answer:

$$y = \begin{cases} x - 3, & x \le 0 \\ -3, & 0 < x \le 2 \\ 3x - 9, & 2 < x \le 4 \end{cases}$$

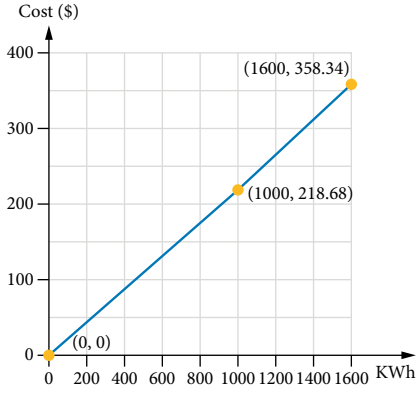
8 **(a)**



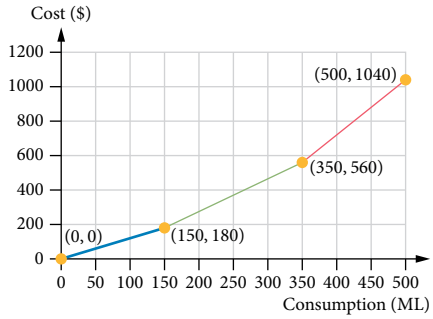
(b)



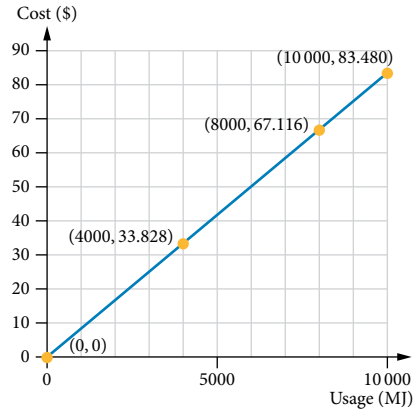
9



10



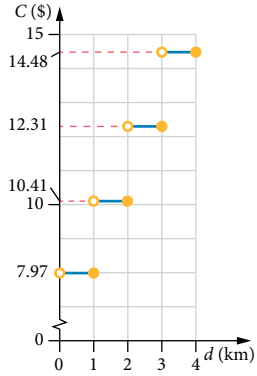
11 **(a)**



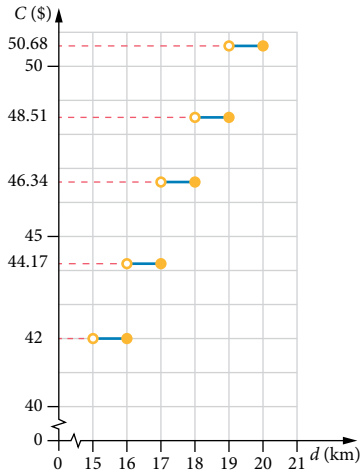
(b) $C = \begin{cases} 0.008457x & 0 < x \le 4000 \\ 0.008322x + 0.54 & 4000 < x \le 8000 \\ 0.008182x + 1.66 & x > 8000 \end{cases}$

(c) \$42 **(d)** \$42.15

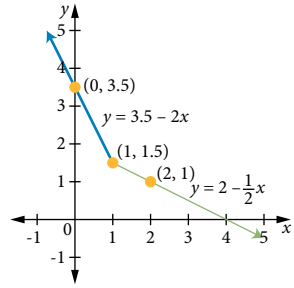
12



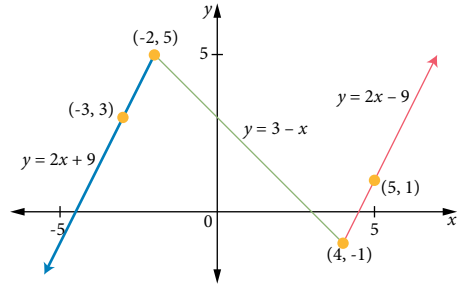
13



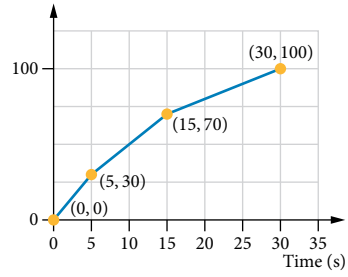
14 $a = 3.5$



15 $a = 9,\ b = -9$

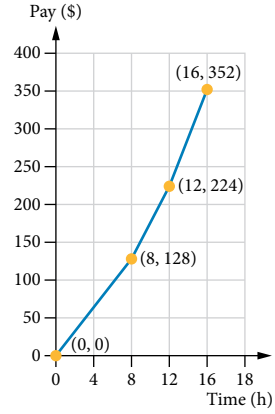


16 **(a)**



(b) $D = \begin{cases} 6t, & 0 \le t < 5 \\ 10 + 4t, & 5 \le t < 15 \\ 40 + 2t, & 15 \le t \le 30 \end{cases}$

17 **(a)**



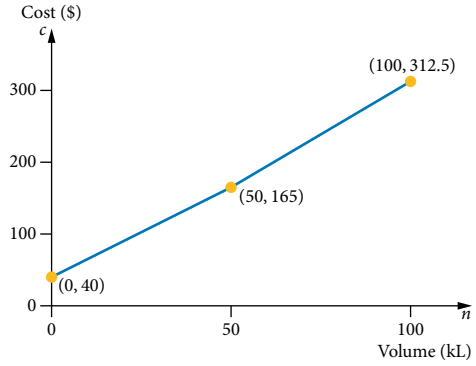
(b) $P = \begin{cases} 16t, & 0 \le t \le 8 \\ 128 + 24t, & 8 < t < 12 \\ 224 + 32t, & 12 < t \le 16 \end{cases}$

18 **(a)** $y = \begin{cases} y = 0.23x, & 0 < x \le 15\,000 \\ y = 3450 + 0.27(x - 15\,000), & 15\,000 < x \le 28\,000 \\ y = 6960 + 0.38(x - 28\,000), & 28\,000 < x \le 55\,000 \end{cases}$

(b) The function is discrete, hence cannot be continuous, however, there are so many points that the graph would appear continuous, since values are equal at the transitions.

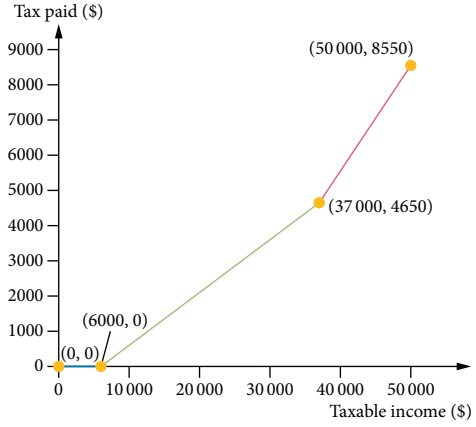
19 (a) $c = \begin{cases} 2.5n + 40, & 0 < n \le 50 \\ 2.95(n - 50) + 165, & n > 50 \end{cases}$

(b)



(c) 62 kL

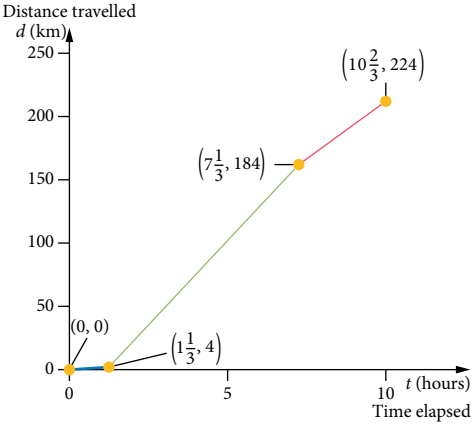
20 (a)



(b) $P = \begin{cases} 0, & 0 \le I \le 6000 \\ 0.15(I - 6000), & 6000 < I \le 37\,000 \\ 4650 + 0.3(I - 6000), & 37\,000 < I \le 80\,000 \end{cases}$

21 (a) $d = \begin{cases} 3t, & 0 \le t \le 1\frac{1}{3} \\ 30t - 36, & 1\frac{1}{3} < t \le 7\frac{1}{3} \\ 12t + 96, & 7\frac{1}{3} < t \le 10\frac{2}{3} \end{cases}$

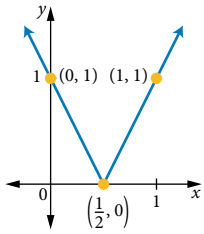
(b)



(c) 114 km (d) 8 hours 40 minutes

22 (a) $y = \begin{cases} 2x - 1, & x \ge \frac{1}{2} \\ 1 - 2x, & x < \frac{1}{2} \end{cases}$

(b)



23 $f(x) = 0, -4 < x < 4,$

$g(x) = \begin{cases} x + 4, & -4 \le x < 0 \\ 4 - x, & 0 \le x \le 4 \end{cases}$ $h(x) = \begin{cases} -x - 2, & -5 \le x < -3 \\ x - 2, & 3 < x \le 5 \end{cases}$

$i(x) = \begin{cases} -x, & -4 \le x < -2 \\ x, & 2 < x \le 4 \end{cases}$ $j(x) = \begin{cases} -x + 2, & -3 \le x < -1 \\ x + 2, & 1 < x \le 3 \end{cases}$

24 Taxpayers with an income less than $61 500 will pay more tax on the proposed system.

Taxpayers with an income of $61 500 will pay the same amount of tax: $12 300.

Taxpayers with an income greater than $61 500 will pay less tax on the proposed system.

EXERCISE 1.4

1 (a) It is arithmetic: $d = 2$ (b) It is not arithmetic.

(c) It is not arithmetic (d) It is arithmetic: $d = 2$

(e) It is arithmetic: $d = 6$ (f) It is arithmetic: $d = \frac{1}{12}$

(g) It is not arithmetic. (h) It is arithmetic: $d = p$

2 C 3 3, 10, 17, 24 4 $n = 12$

5 (a) 27 (b) -57 (c) $8m - 6n$ (d) $6 - 3n$

6 (a) The difference between consecutive terms is constant, -22.

(b) $t_n = 245 - 22n$ (c) -679 (d) $t_{12} = -19$

7 (a) 5, 3, 1, -1 (b) -40, -37, -34, -31 (c) 47, 39, 31, 23

8 $a = 13$, $d = 6$

9 $t_{25} = 189$ 10 19.5 11 $p = 3$

12 (a) 20 (b) 21 (c) 10 (d) 9

13 (a) C

(b) The student has determined the value of the term instead of the sum of terms.

14 4

15 (a) 78 (b) 210 (c) -160

(d) -1680 (e) n^2 (f) $\frac{n}{2}(5n - 19)$

16 2, 7, 12 17 4, 7, 10 18 6, 4.2, 2.4, 0.6

19 (a) $x = 9, d = 2$ (b) $x = 7, d = 3$

(c) $x = -2$ or $x = 3$, $d = 4$ (d) $x = 3, d = \frac{1}{6}$

20 14 21 $a = 1$ or $a = 5$

22 (a) 18 (b) 95 (c) 329 (d) 3

23 (a) 6 km (b) 18th day (c) 5 days (d) 175 km

24 (a) (i) Each expression equals 5.

(ii) Each expression equals 25.

(iii) Each expression equals $\frac{5}{2}$.

(iv) Each expression equals -5.

(b) $j + m = k + l$ (c) $t_1 + t_4 = t_2 + t_3 = 2a + 3d$

25 (a) $y = 2x + 3$ where $x \in \{0, 1, 2, \ldots\}$

(b) $y = x^2 + 2x$, where $x \in \{1, 2, 3, \ldots\}$ or

$y = x^2 + 4x + 3$, where $x \in \{0, 1, 2, \ldots\}$

26 $n = 20$

27 (a) Let the middle term be x and the common difference be d.

The three terms will be $x - d$, x, $x + d$.

Mean: $\frac{x - d + x + x + d}{3} = \frac{3x}{3} = x$

The mean of any three consecutive terms of an arithmetic sequence is the same as the middle term.

(b) The mean of any five consecutive terms of an arithmetic sequence is the same as the middle term.

(c) The mean of the six numbers is the mean of the two middle terms.

EXERCISE 1.5

1 (a) $n = 2t + 30$ (b) $n = 0, 1, 2, \ldots, 485$

2 (a) $n = 300 - 2t$ (b) $n = 0, 1, 2, \ldots, 130$

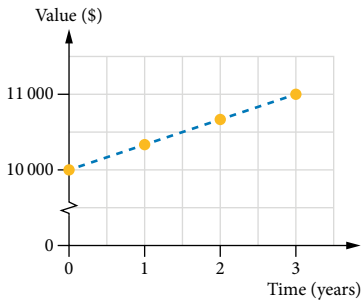
3 (a) \$300

(b) 10 000, 10 300, 10 600, 10 900

(c) \$10 900 is the value of the investment at the end of 3 years.

(d)

Time (years)	0	1	2	3
Value (\$)	10 000	10 300	10 600	10 900



4 (a) D

(b) The student has not distinguished between interest and final balance.

5 (a) \$6000, \$14 000 (b) \$84, \$684

(c) \$192.94, \$2292.94 (d) \$31.26, \$596.26

6 (a) $A_1 = \$10\,011$, $A_2 = \$10\,622$, $A_3 = \$11\,233$

(b) $A_n = 9400 + 611n$, $A_{10} = 15\,510$

(c) 5 years

7 \$1590.50

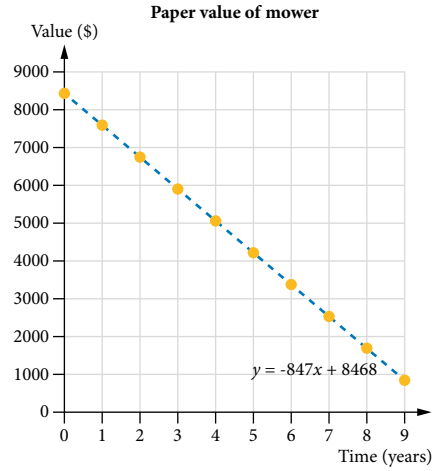
8

Simple interest	Principal	Simple interest rate (p.a.)	Time	Account balance
\$4675	\$8500	11%	5 years	\$13 175
\$157.50	\$2800	4.5%	15 months	\$2957.50
\$180	\$24 000	9%	1 month	\$24 180

9 B

10 (a) \$847 (b) 9 years (c) \$845

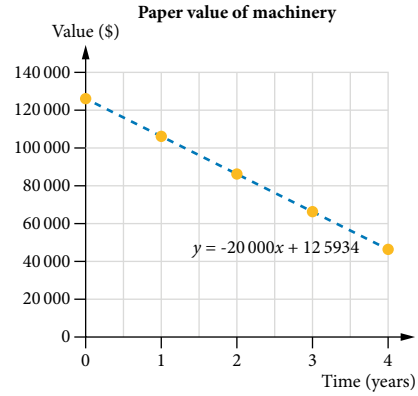
11 $y = 8468 - 847x$, $x \in \{0, 1, 2, \ldots, 9\}$ where y represents value (\$) and x represents time (years).



12 (a) Determine the anticipated number of years that the machinery will be used.

(b) \$45 934

(c) $y = 125\,934 - 20\,000x$, $x \in \{0, 1, 2, 3, 4\}$ where y represents value (\$) and x represents time (years).



13 (a) 23, \$198.89

(b) 8018.89, 7678.89, 7338.89, … , 198.89: 24 terms; final term of 0

(c) 2000, 2340, 2680, … , 9820: 24 terms; final term: 10 018.89.

14 (a) \$40 (b) 16, \$79.98

(c) 719.98, 679.98, 639.98, … , 79.98: 17 terms; final term of 0

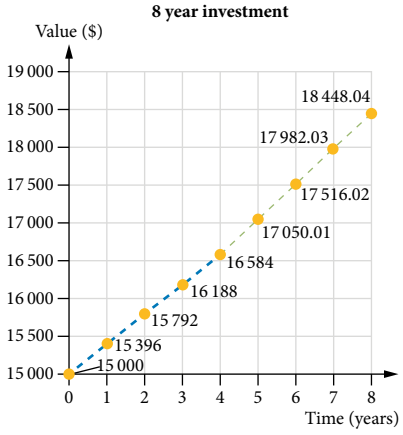
(d) 100, 140, 180, … , 740: 17 terms; final term: 819.98.

15 The 5 year term deposit is the better option by \$39.61.

16 \$38 273.75

17 $y = \begin{cases} 15\,000 + 396x, & x \in \{0,1,2,3,4\} \\ 16\,584 + 466.01(x-4), & x \in \{5,6,7,8\} \end{cases}$

where y represents value ($) and x represents time (years) since the beginning of the initial investment.



18 (a)

Year	Amount at beginning of year ($)	Amount at end of year ($)
1	350 000	376 250
2	353 675	380 201
3	357 389	384 193
4	361 141	388 227
5	364 933	392 303

(b) The balance is increasing at an increasing rate.

(c)

Year	Amount at beginning of year ($)	Amount at end of year ($)
1	350 000	371 000
2	343 175	363 766
3	336 483	356 672
4	329 992	349 717
5	323 488	342 898

(d) The balance is reducing at a reducing rate each year.

19 (a) Initial subdomain: $t \le 1780$, since this number could go back for millions of years.

$$n = \begin{cases} 5900, & t \le 1780 \\ 31\,585 - 14.43t, & 1780 < t \le 1990 \\ 20\,685 - 8.3t, & 1990 < t < 2000 \end{cases}$$

(b) Model: 4002 million hectares, actual: 4033 million hectares, an increase of 31 million hectares more than predicted.

The trend could go either way in the future. More countries could support reforestation, and some that have processes in place may simply maintain what they have achieved rather than continuing to increase the coverage.

(c) Model for Europe: $n = 0.8t - 603$, $1990 \le t \le 2010$

(d) Model for South America: $n = 9105 - 4.1t$, $1990 \le t \le 2010$

CHAPTER REVIEW 1

1 (a) Direct variation (b) Not direct variation

2 (a) -3, (0, 14) (b) -2, (0, 5) (c) $-\frac{5}{7}$, (0, -20) (d) 1, (0, 0)

3 (a) $y = 3x - 2$ (b) $y = \frac{3}{7}x + \frac{1}{2}$ (c) $y = -x + 7$

4 (a) $\frac{3}{4} \le y \le 8\frac{1}{4}$ (b) $-24 \le x \le -10\frac{2}{3}$

5 $y = -\frac{2}{5}x + \frac{28}{5}$ or $2x + 5y - 28 = 0$

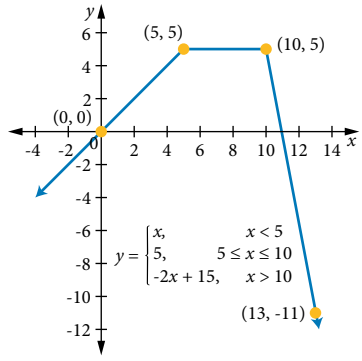
6 $y = \frac{4}{7}x + \frac{8}{7}$ or $4x - 7y + 8 = 0$

7 (a) $x = -2$ (b) $a = \frac{10}{7}$

8 (a) $x = 1.8, y = -1.6$ (b) $x = -2, y = 1$ (c) $x = 3\frac{3}{7}, y = 1\frac{2}{7}$

9 D

10



11 $p = 1$

12 The 23rd term, $t_{23} = -3$

13 D

14 255

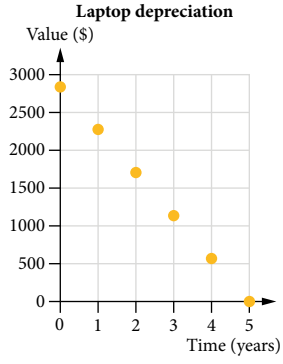
15 (a) 5 cm (b) First rung: 19 cm; tenth rung: 64 cm

16 $66.50

17 (a) $n = 4500 + 825t$ (b) $n = 0, 1, 2, ..., 119$

18 (a) $568, 5 years

(b) $y = 2840 - 568x$, $x \in \{0, 1, 2, 3, 4, 5\}$ where y represents value ($) and x represents time (years).



19 $6200

20 (a) $y = 11x + 33$ (b) $y = -2x + 7$ (c) $y = \frac{5}{2}x + \frac{7}{2}$

21 A

22 $x = 3, y = -7$

23 D

24 12

25 32

26 35 and 18

27 (a) $C = 65h$ (b) $C = 42 + 58h$ (c) $245 (d) 6 hours

28 $n = 6, n = 10$

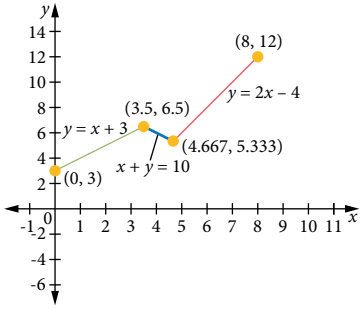
29 (a) $A_n = 9600 + 840n$, $A_6 = \$14\,640$ (b) 12 years

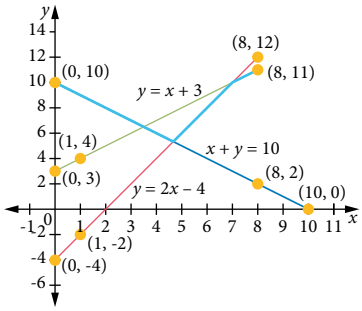
30 **(a)** \$600, \$430.15

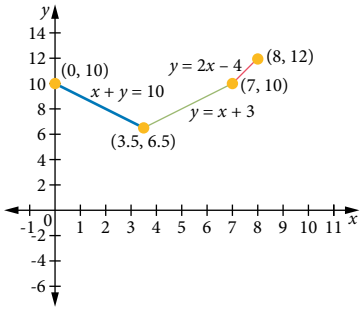
(b) 10 630.15, 10 030.15, 9430.15, ... , 430.15: 18 terms; final term of 0

(c) 6000, 6600, 7200, ... , 16 200: 18 terms; final term: 16 630.15.

31 Any of the following graphs:







32 $a = 5$ or $a = -3$

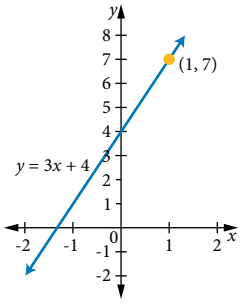
33 $n = 8$

Chapter 2 Quadratic functions

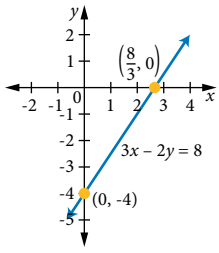
RECALL

1 **(a)** 1 **(b)** 2 **(c)** 1

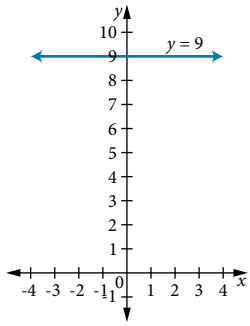
2 **(a)** x-intercept: $\left(-\frac{4}{3}, 0\right)$, y-intercept: (0, 4)



(b) x-intercept: $\left(\frac{8}{3}, 0\right)$, y-intercept: (0, -4)



(c) x-intercept: does not exist, y-intercept: (0, 9)



3 **(a)** no **(b)** yes **(c)** no

4 A, C and E

5 **(a)** $10x - 4y$ **(b)** $8a^2 - 6b^2 + 7ab$ **(c)** $-6xy - 2x^2$

6 **(a)** $5x - 35$ **(b)** $x^2 - 11x$ **(c)** $t^2 - 3t - 10$

7 **(a)** $3y(2y - 1)$ **(b)** $2(a + 3b - 6)$

(c) $4(5 - 2x)$

8 **(a)** $x = 1, y = -2$ **(b)** $x = -\frac{1}{3}, y = -5$ **(c)** $x = 4, y = 19$

EXERCISE 2.1

1 **(a)** function **(b)** function **(c)** not a function

2 B

3 **(a)** D

(b) Incorrect option A. A relation can begin '$x =$' and still be a function.

4 **(a)** Domain: $(-1, \infty)$; Range: $[0, \infty)$

(b) Domain: $(-\infty, 2]$; Range: $(-\infty, 1]$

5 **(a)** $-2 \le x \le 4, 0 \le y \le 8$

(b) $0 \le x < 5, -5 < y \le 0$

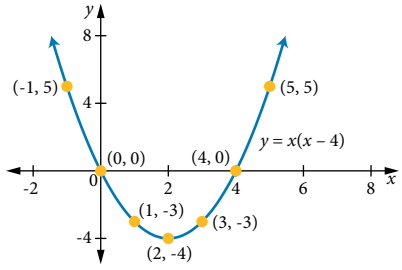
(c) $x \ge -2, y \le 2$

6 **(a)** $x = 5$ **(b)** $x = -3$

7 Domain: $\mathbb{R}^- \cup \{0\}$; Range: $\mathbb{R}^+ \cup \{0\}$

8 **(a)**

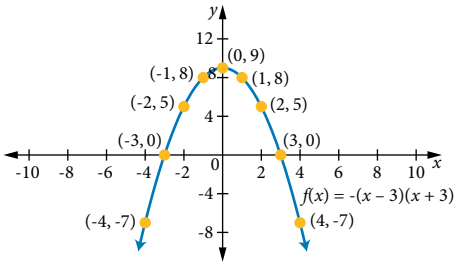
x	-1	0	1	2	3	4	5
y	5	0	-3	-4	-3	0	5



(b) concave up **(c)** (2, -4) **(d)** (0, 0) **(e)** $[-4, \infty)$

9 (a)

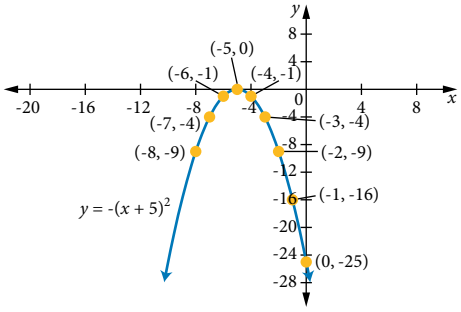
x	-4	-3	-2	-1	0	1	2	3	4
y	-7	0	5	8	9	8	5	0	-7



(b) concave down **(c)** (0, 9) **(d)** (0, 9) **(e)** $y \le 9$

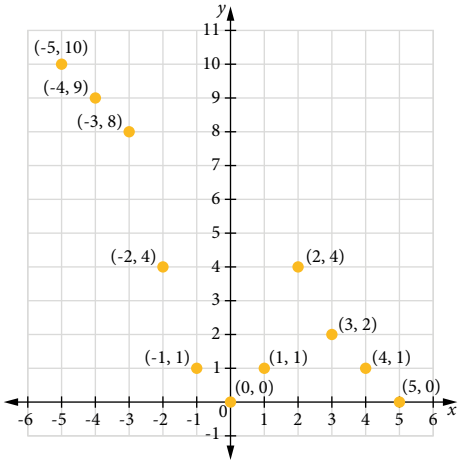
10 (a)

x	-8	-7	-6	-5	-4	-3	-2	-1	0
y	-9	-4	-1	0	-1	-4	-9	-16	-25



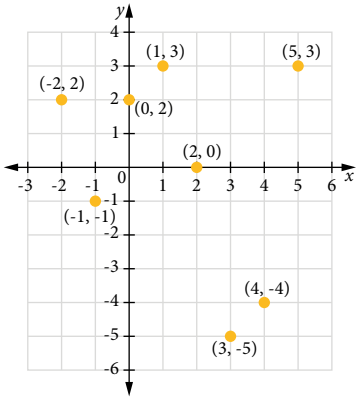
(b) concave down **(c)** (-5, 0) **(d)** (0, -25) **(e)** $(-\infty, 0]$

11 (a)



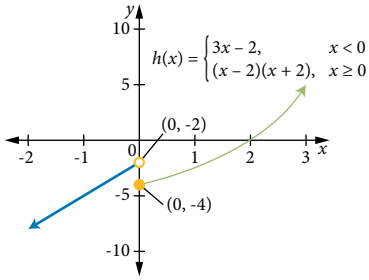
(b) Range: {0, 1, 2, 4, 8, 9, 10}

12 (a)



(b) Range: {-5, -4, -1, 0, 2, 3}

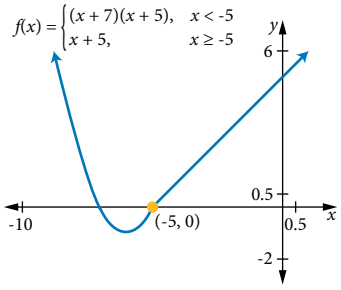
13 (a)



(b) **(i)** 0 **(ii)** -3 **(iii)** -4

(c) $(1-2)(1+2)$

14 (a)



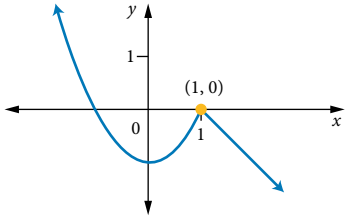
(b) Continuous for all values of x

(c) Not smooth when $x = -5$.

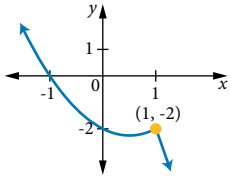
15 (a) $f(x) = x^2, -2 \le x \le 2$ **(b)** $[0, 25)$

(c) All correct possibilities must include [-4, 0] or [0, 4] with x-values not being less than -4 nor greater than 4.

16 $a = \frac{1}{2}, f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -\frac{1}{2}x + \frac{1}{2}, & x > 1 \end{cases}$



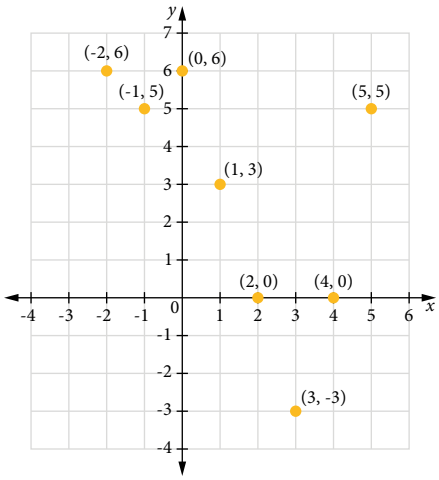
17 $a = 1, f(x) = \begin{cases} (x-2)(x+1), & x \le 1 \\ (x+1)(1-2x), & x > 1 \end{cases}$



18 $a = 4, f(x) = \begin{cases} x + 4, & -4 \le x < 1 \\ (x-3)^2 + 1, & 1 \le x \le 5 \\ 10 - x, & 5 < x \le 10 \end{cases}$

19 $a = 18,\ f(x) = \begin{cases} (x+6)^2, & -10.24 \le x < -3 \\ 18 - x^2, & -3 \le x \le 3 \\ (x-6)^2, & 3 < x \le 10.24 \end{cases}$

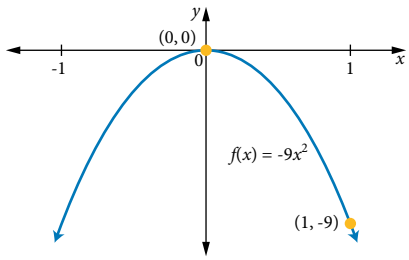
20 $a = 6,\ b = 4,\ f(x) = \begin{cases} 4 - x, & x \in \{-2, -1\} \\ 6 - 3x, & x \in \{0, 1, 2, 3\} \\ 5x - 20, & x \in \{4, 5\} \end{cases}$



EXERCISE 2.2

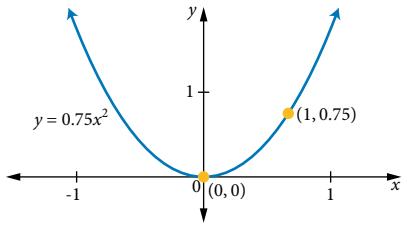
1 (a) Dilated by a factor of 9 parallel to the y-axis and reflected in the x-axis.

y-intercept: (0, 0); Range: $(-\infty, 0]$



(b) Dilated by a factor of 0.75 parallel to the y-axis.

y-intercept: (0, 0); Range: $[0, \infty)$



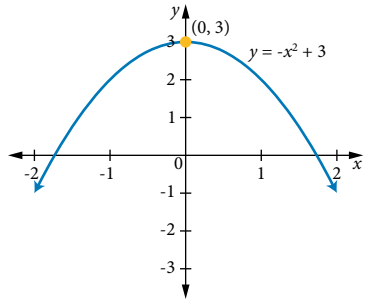
2 (a) 14 units up (b) $x = 0$

(c) Turning point: (0, 14); Range: $(-\infty, 14]$

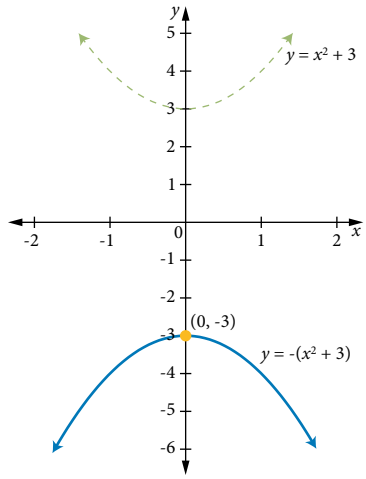
3 (a) 3.9 units down (b) $t = 0$

(c) Turning point: (0, -3.9); Range: $[-3.9, \infty)$

4 (a)



(b) The incorrect transformation results in the graph of $y = -x^2 - 3$.



(c) For a graph in the form $y = ax^2 + c$ determine whether the graph is concave up or down from a and check the coordinates of the turning point $(0, c)$.

5 (a) concave down (b) 3 units left

(c) Turning point: (-3, 0), y-intercept: (0, -18); Range: $(-\infty, 0]$

6 (a) concave up (b) 5 units right

(c) Turning point: (5, 0), y-intercept: (0, 10); Range: $[0, \infty)$ or $\mathbb{R}^+$

7 Translation up 21 units

8 $f(x) = (x+6)^2$

9 Translation left 18 units

10 Translation right 20 units

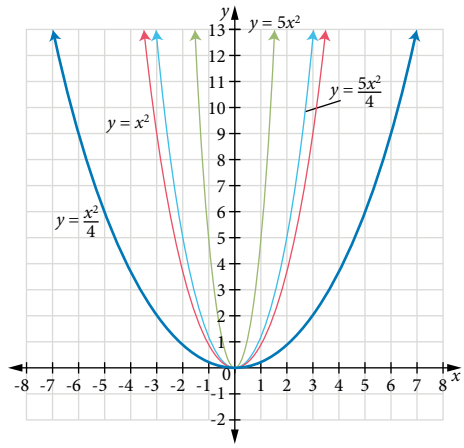
11 $f(x) = 3x^2$

12 $m = \frac{1}{2}p^2$

13 A

14 (a) (i) $y = 5x^2$ (ii) $y = \frac{x^2}{4}$ (iii) $y = \frac{5x^2}{4}$

(b)



15 (a) $y = (x+5)^2 + 2$ (b) $y = (x-10)^2 + 3$

(c) $y = (x-12)^2 - 9$ (d) $y = 9x^2$

(e) $y = \frac{1}{10}x^2$ (f) $y = -2x^2$ (g) $y = -\frac{2}{7}x^2$

16 (a) Translation of 0.5 units to the right.

(b) Translation of 0.12 units down.

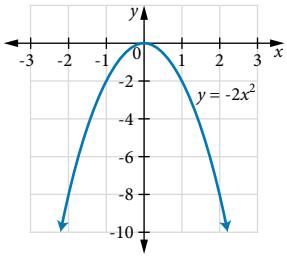
(c) A translation of 10 units left and 12 units down.

(d) A reflection in the horizontal axis and a vertical translation of 16 units up.

(e) A translation of 6 units left and 18 units down.

(f) A reflection in the horizontal axis and a translation of 24 units up.

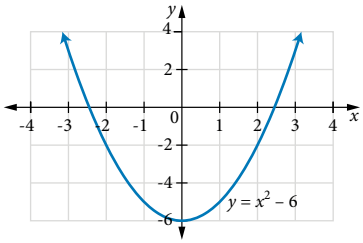
17 (a)



(i) A reflection in the x-axis and a dilation parallel to the y-axis by a factor of 2.

(ii) $x = 0$ (iii) (0, 0) (iv) (0, 0) (v) $(-\infty, 0]$

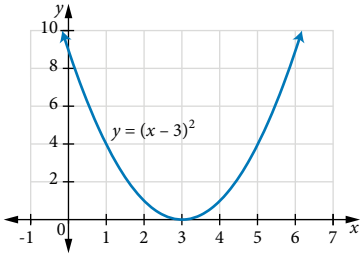
(b)



(i) Translated down 6 units

(ii) $x = 0$ (iii) (0, -6) (iv) (0, -6) (v) $[-6, \infty)$

(c)



(i) translated 3 units right

(ii) $x = 3$ (iii) (3, 0) (iv) (0, 9) (v) $[0, \infty)$

18 (a) $y = 4(x+5)^2 - 9$ (b) $y = \frac{2}{5}(x-3)^2 + 16$

19 (a) A dilation factor of 3 parallel to the y-axis and a translation of 6 units down.

(b) A dilation factor of 4 parallel to the y-axis, translations of 5 units right and 14 units down.

(c) A reflection in the p-axis, translations of 8 units left and 9 units down.

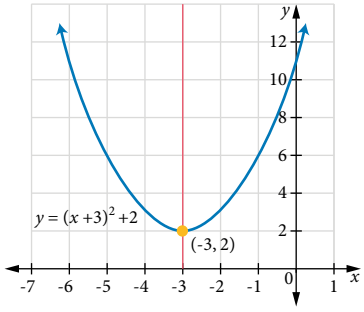
(d) A reflection in the t-axis, a dilation factor of $\frac{3}{5}$ parallel to the s-axis, translations of 5 units right and 2 units up.

20 (a) C

(b) The common error is to translate in the same direction as the constant term in the brackets. The correct translation is in the opposite direction of the constant, so $(x + 12)$ means a translation to the left.

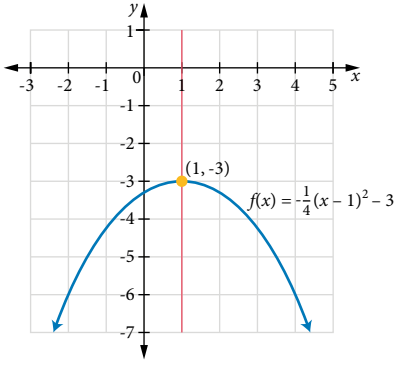
21 (a) Translations of 3 units left and 2 units up

(b) Equation of line of symmetry: $x = -3$
Turning point: (-3, 2); Range: $[2, \infty)$



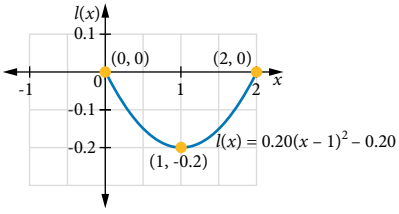
22 (a) Translations of 1 unit right, 3 units down, reflection in the x-axis and a dilation by a factor of $\frac{1}{4}$ parallel to the y-axis.

(b) Equation of line of symmetry: $x = 1$, turning point: (1, -3); Range: $(-\infty, -3]$



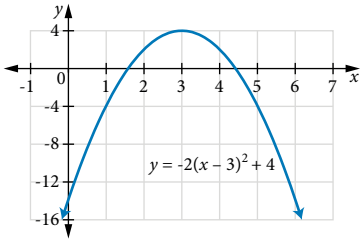
23 (a) $l(x) = 0.20(x-1)^2 - 0.20, 0 \le x \le 2$

(b) Range: [-0.2, 0]



24 $p(x) = 0.1567(x - 0.875)^2 - 0.12$

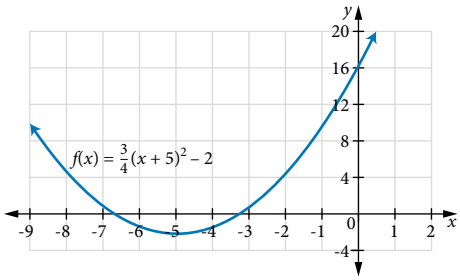
25 (a)



(i) A reflection in the x-axis, dilation by a factor of 2 parallel to the y-axis, horizontal translation 3 units right and vertical translation 4 units up.

(ii) $x = 3$ (iii) $(3, 4)$ (iv) $(0, -14)$ (v) $(-\infty, 4]$

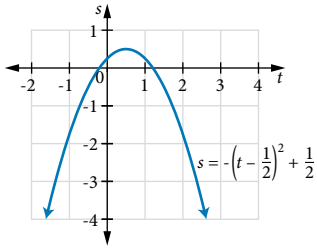
(b)



(i) The function shows a dilation factor parallel to the y-axis of $\frac{3}{4}$, a horizontal translation of 5 units left and a vertical translation of 2 units down.

(ii) $x = -5$ (iii) $(-5, -2)$ (iv) $\left(0, \frac{67}{4}\right)$ (v) $[-2, \infty)$

(c)



(i) A reflection in the t-axis, a translation of $\frac{1}{2}$ a unit right and a translation of $\frac{1}{2}$ a unit up.

(ii) $t = \frac{1}{2}$ (iii) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (iv) $\left(0, \frac{1}{4}\right)$ (v) $\left(-\infty, \frac{1}{2}\right]$

26 25 mm

27 (a) (1.5, 2.25) The maximum height of the water is 2.25 m and occurs 1.5 m from the base.

(b) 3 m

28 $y = -\frac{1}{4}x^2 + 1, -4 \le x \le 2$

29 The rock hits the ground 40 m out and 40 m below the platform, assuming the point from which it was thrown is the same location referred to as the platform in the question.

30 The net is 1.93 m high, assuming the ball was directly over the net at its greatest height.

EXERCISE 2.3

1 (a) $3y(2y-1)$ (b) $(a+4)(a-3)$

(c) $(b-6)(b-9)$ (d) $(2x+3)(x-5)$

(e) $(2x+5)(2x+3)$ (f) $(m+4)(m-4)$

2 (a) $a = -6$ (b) $a = -7$ or $a = 5$

(c) $a = 2$ or $a = 8$ (d) $a = -8$ or $a = 6$

(e) $a = -4$ or $a = \frac{3}{2}$ (f) $a = 4$ or $a = \frac{2}{3}$

(g) $a = -5$ or $a = 2$ (h) $a = 1$ or $a = -\frac{1}{3}$

3 (a) B

(b) For $y = 3(x-1)(x-5)$, turning point: (3, -12), y-intercept: (0, 15)

4 (a) $\Delta = 96$, 2 solutions (b) $\Delta = 241$, 2 solutions

(c) $\Delta = 0$, 1 solution (d) $\Delta = 0$, 1 solution

(e) $\Delta = -20$, no solutions

5 (a) $x = -5$ or $x = 4$ (b) $x = -8$ or $x = -6$ (c) $x = -2$ or $x = \frac{3}{2}$

(d) $a = -1$ or $a = \frac{1}{3}$ (e) $x = -10$ or $x = 2$

6 C

7 $x = 2.303$ or $x = -1.303$

8 (a) no solutions (b) $x = 10.158$ or $x = -2.658$

(c) $x = -2.551$ or $x = 2.551$ (d) $x = -5.408$ or $x = -0.092$

(e) $x = -2.582$ or $x = 2.582$ (f) $x = -2$ or $x = 3$

(g) $x = 0$ or $x = 17$

9 $x = -4$ or $x = 3$

10 $x = -3.45$ (2 d.p.) or $x = 1.45$ (2 d.p.)

11 (a) $a = \frac{9 \pm 4\sqrt{6}}{3}$ (b) $b = \frac{15 \pm \sqrt{321}}{4}$

(c) $c = \frac{28 \pm \sqrt{910}}{2}$ (d) $x = \frac{45 \pm \sqrt{1345}}{34}$

12 $t = \frac{7 \pm \sqrt{89}}{4}$

13 Line 5 is incorrect.

From line 5 onwards the solution should be:

$$\begin{aligned} x + 3 &= \pm\sqrt{8} \\ x &= -3 \pm \sqrt{8} \\ &= -3 \pm 2\sqrt{2} \end{aligned}$$

14

$$\begin{aligned} -8 \div 2 &= -4 \\ (-4)^2 &= 16 \\ \left(x^2 - 8x + 4^2\right) - (-4)^2 + 3 &= 0 \\ (x-4)^2 &= 13 \\ x - 4 &= \pm\sqrt{13} \\ x &= 4 \pm \sqrt{13} \end{aligned}$$

15 $x = 2 \pm \sqrt{10}$

16 (a) $a = -3 \pm \sqrt{26}$ (b) $a = -5 \pm \sqrt{39}$ (c) $a = \frac{-5 \pm \sqrt{61}}{2}$

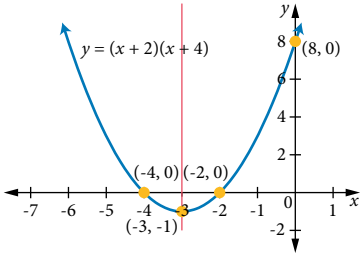
(d) $a = \frac{-9 \pm \sqrt{57}}{2}$ (e) $a = \frac{7 \pm \sqrt{5}}{2}$

(f) $a = 2 \pm \sqrt{\frac{17}{2}}$ or $2 \pm \frac{\sqrt{34}}{2}$ (g) $a = \frac{-3 \pm \sqrt{3}}{2}$

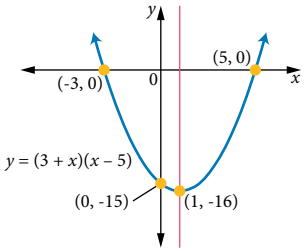
17 $x = 2 \pm \sqrt{5}$

18 (a) $m = \frac{3 \pm 3\sqrt{5}}{2}$ (b) $x = 5 \pm \sqrt{11}$ (c) $t = \frac{-9 \pm \sqrt{173}}{2}$

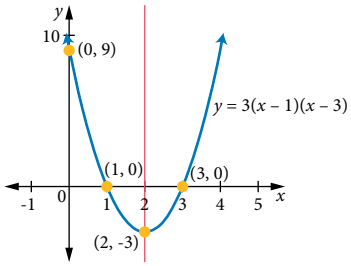
19 **(a)**



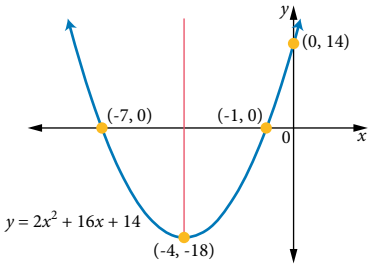
(b)



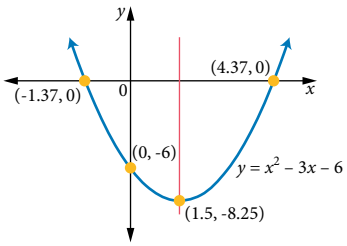
(c)



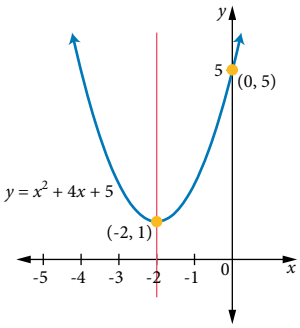
(d)



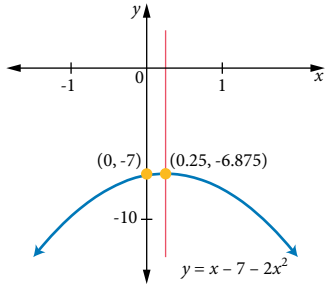
20 **(a)**



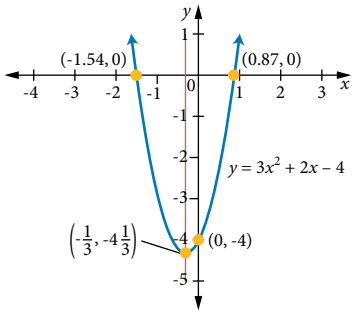
(b)



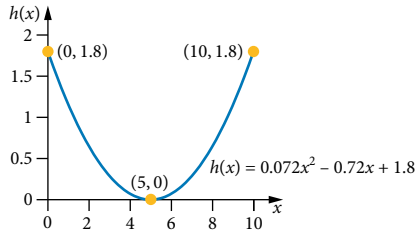
(c)



(d)

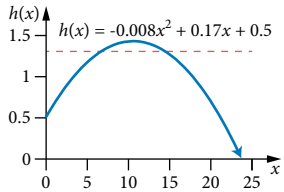


21 **(a)**



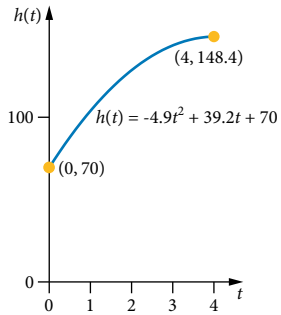
(b) 1.8 m **(c)** 10 m

22



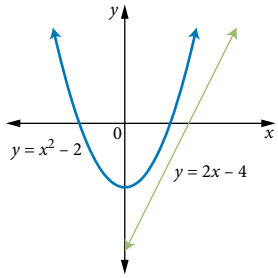
You should stand 14.2 m from the ball-launching machine.

23 **(a)** $h(t) = -4.9t^2 + 39.2t + 70,\ 0 \le t \le 4,\ 70 \le h \le 148.4$



(b) 0.86 s **(c)** 9.50 s

24 **(a)** There are no solutions.



(b) Sample solutions:

$$y = \begin{cases} x^2 - 2, & x \le 1 \\ 2x - 4, & x > 1 \end{cases} \text{ or } y = \begin{cases} 2x - 4, & x \le 1 \\ x^2 - 2, & x > 1 \end{cases}$$

25 For $-2\sqrt{15} < k < 2\sqrt{15}$, $\Delta < 0$ and there are no real solutions.

For $k = \pm 2\sqrt{15}$, $\Delta = 0$ and is one real solution.

For $k < -2\sqrt{15}$ or $k > 2\sqrt{15}$, there are two solutions.

EXERCISE 2.4

1 **(a)** $y = 3(x+3)(x-1)$ **(b)** $y = -2(x+1)(x-2)$

(c) $y = \frac{1}{2}(x+2)(x+1)$

2 **(a)** $y = 2(x-1)^2 + 3$ **(b)** $y = -3(x+1)^2 - 2$

(c) $y = -(x-3)^2 - 1$

3 **(a)** $y = -3x^2 - 5x + 7$ **(b)** $y = 4x^2 + 6x - 13$

4 **(a)** $y = 3x^2 + 6x - 24$ **(b)** $y = -2x^2 - 16x - 35$

(c) $y = -2x^2 - 4x + 30$ **(d)** $y = 5x^2 - 30x + 43$

5 **(a)** $y = \frac{2}{5}x^2 + \frac{1}{5}x - \frac{6}{5}$ **(b)** $y = -\frac{2}{3}x^2 + 4x - 12$

(c) $y = \frac{10}{7}x^2 - \frac{33}{7}x - \frac{54}{7}$ **(d)** $y = \frac{20}{7}x^2 - \frac{20}{7}x + \frac{19}{7}$

6 **(a)** $y = 3(x-4)(x+2)$ **(b)** $y = -2(x^2 - 2x + 3)$

(c) $y = -4(x-5)(x-3)$ **(d)** $y = -3(x-2)(x+6)$

7 **(a)** D **(b)** A

8 B

9 **(a)** y-intercept **(b)** turning point **(c)** x-intercepts

10 **(a)** $y = \frac{18}{25}(x-5)^2$ **(b)** $y = -4x^2 + 8$

11 **(a)** $y = 2x^2 + x - 3$ **(b)** $y = -3x^2 - x + 4$

(c) $y = x^2 - 5x + 2$ **(d)** $y = -\frac{1}{2}x^2 + x + 3$

12 **(a)** $y = \frac{2}{3}(x+3)(x-4)$ **(b)** $y = \frac{4}{5}(x+3)(x-5)$

(c) $y = -\frac{2}{5}(x+2)(x-6)$

13 **(a)** $y = -2(x-1)^2 - 1$ **(b)** $y = 3(x-1)^2 - 48$

(c) $y = 5\left(x - \frac{2}{5}\right)^2 + \frac{26}{5}$ **(d)** $y = -2(x+4)^2 + 8$

14 **(a)** $y = -2\left(x - \frac{1}{3}\right)^2 - \frac{25}{9}$ **(b)** $y = \frac{3}{2}\left(x - \frac{1}{4}\right)^2 - \frac{243}{32}$

15 **(a)** **(i)** no x-intercepts **(ii)** (0, 6)

(iii) (2, 2) **(iv)** $x = 2$

(b) **(i)** $\left(\frac{12+\sqrt{6}}{3}, 0\right)$ and $\left(\frac{12-\sqrt{6}}{3}, 0\right)$

(ii) (0, -46) **(iii)** (4, 2) **(iv)** $x = 4$

(c) **(i)** (-4, 0) and (3, 0) **(ii)** (0, 24)

(iii) $\left(-\frac{1}{2}, \frac{49}{2}\right)$ **(iv)** $x = -\frac{1}{2}$

(d) **(i)** $\left(\frac{1+\sqrt{13}}{3}, 0\right)$ and $\left(\frac{1-\sqrt{13}}{3}, 0\right)$ **(ii)** (0, 4)

(iii) $\left(\frac{1}{3}, \frac{13}{3}\right)$ **(iv)** $x = \frac{1}{3}$

(e) **(i)** $(5+\sqrt{3}, 0)$ and $(5-\sqrt{3}, 0)$ **(ii)** $\left(0, \frac{22}{3}\right)$

(iii) (5, -1) **(iv)** $x = 5$

(f) **(i)** $\left(\frac{5}{2}, 0\right)$ or (-3, 0) **(ii)** $\left(0, -\frac{15}{2}\right)$

(iii) $\left(-\frac{1}{4}, \frac{121}{16}\right)$ **(iv)** $x = -\frac{1}{4}$

16 **(a)** $y = -\frac{1}{60}x^2 + x$

(b) Maximum height: 15 m, 30 m horizontally from the launch position.

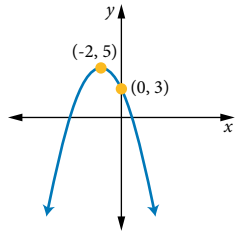
(c) 60 m

17 **(a)** 0.99 s **(b)** 20.41 m, 2.04 s **(c)** 4.08 s

18 **(a)** $y = -2.775x^2 + 11.605x - 7.2$

(b) 4.93 m **(c)** 0.76 m, 3.42 m

19 **(a)**



(b) $y = -\frac{1}{2}(x+2)^2 + 5$ **(c)** $-2 - \sqrt{10} \le x \le -2 + \sqrt{10}$

20 **(a)** A 5 cm increase in height gave a 9 cm increase in throw distance and a 5 cm decrease in height gave a 9 cm decrease in throw distance.

(b) 2.79 m, 2.84 m, 2.89 m

21 $$y = \begin{cases} 80, & -120 \le x \le 0 \\ -0.032x^2 + 80, & 0 < x \le 50 \\ -0.032(x - 93.30)^2 + 60, & 50 < x \le 136.60 \\ -0.032(x - 174.10)^2 + 45, & -136.60 \le x \le 174.10 \end{cases}$$

Height: 45 cm, distance from table: 174.1 cm

22 (a) $n(t) = 3.5635t^2 + 31.227t + 1528.2$ where $n(t)$ is the number of 98-year-old women in Australia, t (years) is the time after 2000. The quadratic model appears to be suitable.

(b) The model predicts 4536, 292 fewer than the ABS prediction. The ABS model is unlikely to be similar.

(c) The model predicts 38 812. The model is unsuitable for long-term predictions.

CHAPTER REVIEW 2

1 $y = x^2 + 3x - 10$ is in the form $y = f(x)$ so each value substituted for x will give one value for y.

2 (a) $x = 0$ and (0, -8) (b) $x = 12$ and (12, 0)
(c) $x = -7$ and (-7, -11) (d) $x = -20$ and (-20, 0)
(e) $x = 0$ and (0, 16) (f) $x = 2$ and (2, -13.9)

3 (a) 9 units up (b) $x = 0$ (c) (0, 9) (d) (0, 9)

4 (a) minimum (b) maximum (c) minimum
(d) maximum (e) minimum (f) minimum

5 (a) $y = x^2 + 3$; Range: $[3, \infty)$
(b) $y = (x+4)^2$; Range: $[0, \infty)$
(c) $y = \frac{2}{3}x^2$; Range: $[0, \infty)$
(d) $y = -\frac{3}{5}(x-3)^2$; Range: $(-\infty, 0]$

6 $x = -9$ 7 (7, 0)

8 translated 9 units right and 3 units up

9 C 10 $y = 4(x+12)^2 - 6$ 11 (-4, -2)

12 (a) $x = -6$ or $x = 9$ (b) $x = -2.5$ or $x = 3$
(c) $x = -\frac{4}{3}$ or $x = \frac{9}{5}$

13 (a) $x = 0.13$, $x = 2.54$ (b) $x = -1.82$, $x = 3.48$
(c) $x = -21.71$, $x = -8.29$

14 $x = \frac{-8 \pm \sqrt{6}}{2}$ 15 $x = 4$ or $x = -8$ 16 $x = \pm 3$

17 (a) $y = -\frac{1}{4}(x+2)(x-6)$ (b) $y = \frac{1}{4}(x+2)^2 + 3$
(c) $y = x^2 - 2x + 4$

18 D 19 $y = x^2 + \frac{5}{2}x - 6$

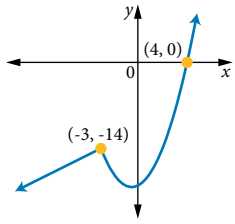
20 (a) $x = -1$ (b) (-1, -25) (c) $x = -6$ or $x = 4$

21 (a) $x = 8$ (b) (8, -29) (c) $x = 8 \pm \sqrt{58}$

22 (a) $y = x^2 + x - 12$ (b) $y = \frac{1}{2}x^2 - 2x + 2$
(c) $y = -x^2 - 4x + 5$

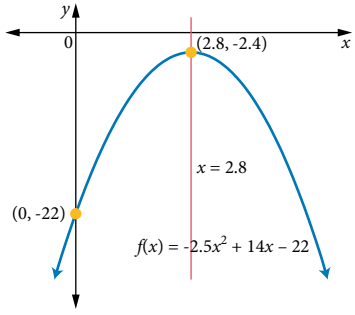
23 B

24 A



25 (a) $x = 4 \pm \sqrt{11}$ (b) $x = -5 \pm \sqrt{23}$
(c) no solutions

26 Axis of symmetry: $x = 2.8$, turning point: (2.8, -2.4), y-intercept: (0, -22), no x-intercepts.



27 $k < -\sqrt{56}$ or $k > \sqrt{56}$ so if k is an integer: $k \in \{\ldots, -10, -9, -8\}$ or $k \in \{8, 9, 10 \ldots\}$

28 $y = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$

29 (a) 2.25 cm (b) 2.75 cm

30 (a) 1251.2 m (b) 247.6 m

31 (a) $y = -4.9t^2 + 12t + 7$ (b) 14.35 m
(c) Domain: $-0.16 \le t \le 2.94$.

Filming didn't start until the object had been in the air for 0.16 s.

The ball was in the air for 3.10 s.

(d) $y = -4.9t^2 + 13.57t + 5$, $0 \le t \le 3.10$

32 $y = \begin{cases} 10 - x, & 4 \le x \le 8 \\ -\frac{1}{4}(x-12)^2, & 8 < x \le 16 \end{cases}$

Range: [-4, 6]

33 $y = \begin{cases} 2(x-3)^2 - 10, & 0 \le x \le 3 \\ 4.5x - 23.5, & 3 < x < 7 \\ -2(x-7)^2 + 8, & 7 \le x \le 10 \end{cases}$

Chapter 3 Polynomial functions

RECALL

1 (a) $y = 2$ (b) $y = 19$ (c) -26

2 (a) $y = \frac{3x-9}{2}$ (b) $y = \frac{4x^2+4}{3}$ (c) $y = \pm\sqrt{\frac{x-4}{2}}$

3 (a) (0, -4) (b) $\left(0, \frac{10}{3}\right)$ (c) (0, 33)

4 (a) $x(x-2)$ (b) $(x+2)^2$ (c) $(x-3)(x-4)$

5 (a) $2x+8$ (b) $-\frac{x}{3}+\frac{4}{3}$ (c) $3x^2-18x$

6 (a) x^2+2x-8 (b) $2x^2-8x-10$ (c) $-4x^2+24x-36$

7 (a) Translation 3 units up. (b) Translation 1 unit right.

(c) Reflection in the x-axis, translation 4 units up.

8 (a) Domain: $\mathbb{R}$; Range: $\mathbb{R}$

(b) Domain: $\mathbb{R}$; Range: $[-\infty, -1)$

(c) Domain: $\mathbb{R}$; Range: $[-9, \infty)$

EXERCISE 3.1

1 (a) Yes, this is a polynomial expression.

(b) Yes, this is a polynomial expression.

(c) No, because one term has a fractional degree.

(d) No, because one term contains a reciprocal (negative power).

(e) Yes, this is a polynomial expression.

(f) No, because there is a radical sign linked to a variable.

(g) Yes, this is a polynomial expression.

2 (a) $P(x) = -2x^2+6x+4$, leading term: $-2x^2$, leading coefficient: -2, degree: 2, constant term: 4

(b) $Q(x) = 8x^4-5x^2+\frac{5x}{6}-6$, leading term: $8x^4$, leading coefficient: 8, degree: 4, constant term: -6

(c) $R(x) = 10$, leading term: 10, leading coefficient: 10, degree: 0, constant term: 10

(d) $S(x) = 2x^3+8$, leading term: $2x^3$, leading coefficient: 2, degree: 3, constant term: 8

(e) $T(x) = x^4+4x^2-3x+\frac{2}{3}$, leading term: x^4, leading coefficient: 1, degree: 4, constant term: $\frac{2}{3}$

(f) $U(x) = -\frac{3x^2}{4}-7x+1$, leading term $-\frac{3x^2}{4}$, leading coefficient: $-\frac{3}{4}$, degree: 2, constant term: 1

3 (a) 4 (b) 63 (c) 9

(d) $3x^4+2x^3-x^2-x+11$ (e) $-3x^4-2x^3-x^2+x+9$

(f) 207

4 B

5 (a) Degree: 3, $P(1) = 4$, $P(-2) = -29$

(b) Degree: 4, $P(1) = 14$, $P(-2) = 146$

(c) Degree: 3, $P(1) = 16$, $P(-2) = 76$

(d) Degree: 9, $P(1) = 9$, $P(-2) = -4083$

(e) Degree: 3, $P(1) = 2$, $P(-2) = -31$

(f) Degree: 7, $P(1) = 7$, $P(-2) = 748$

(g) Degree: 8, $P(1) = -4$, $P(-2) = 2015$

(h) Degree: 0, $P(1) = 8$, $P(-2) = 8$

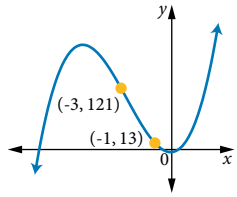
6 (a) $5t^4+2t^2-8t+16$ (b) $-6z^3-2z^2+z-5$

(c) $-5p^4-24p^3+10p^2+4p+6$ (d) $6z^2-3z+1$

7 $c = 1$

8 $n = 3$

9 $a = 3$, $b = 24$, $P(x) = 3x^3+24x^2+3x-5$



10 (a) C

(b) The student may have confused values of x with coefficients.

11 $a = -2$ $b = 4$ 12 $a = 1$, $b = 7$

13 (a) Sample answer: $P(x) = 2x^2+4$

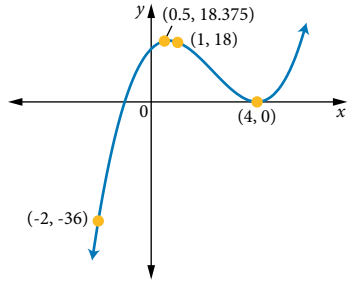
(b) Sample answer: $Q(x) = x^3-10x-5$ or $Q(x) = x^3+\frac{10}{3}x^2-5$

(c) Sample answer: $R(x) = -\frac{1}{8}x^4+x^3+x^2$

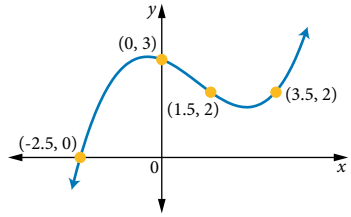
14 (a) $y = 2x^3+4x^2-7x$ (b) $y = x^3-13x+12$

(c) $y = x^3-2x^2-5x+6$

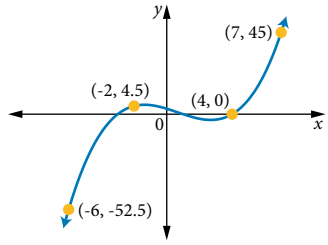
15 (a) $y = x^3-7x^2+8x-16$



(b) $y = 0.110x^3-0.357x^2-0.377x+3$



(c) $y = 0.25x^3-0.5x^2-2.75x+3$



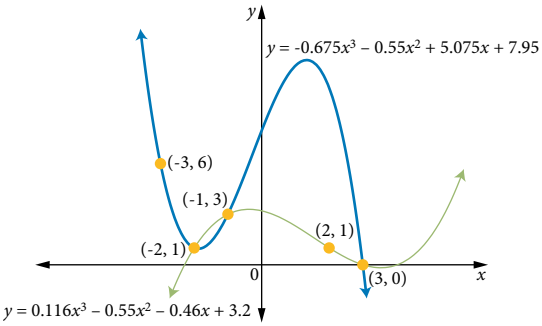
16 (a) $R(x) = 4x^3 - 8x + 13$

(b) (0, 1), (1, 0) and (2, 5) with images (0, 13), (1, 9) and (2, 29)

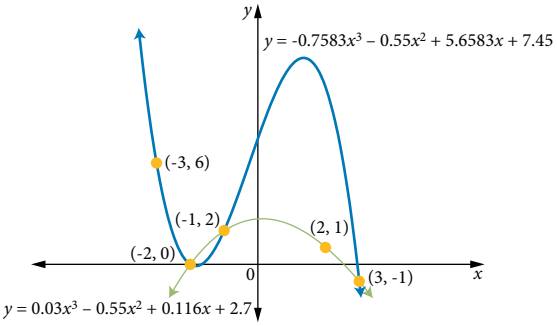
(c) $R(x) = 4Q(x) + 9$

Dilation, factor 4, parallel to the y-axis; translation up 9 units

17 Possible solution: $y = -0.675x^3 - 0.55x^2 + 5.075x + 7.95$ and $y = 0.11\dot{6}x^3 - 0.55x^2 - 0.46\dot{6}x + 3.2$



Second possible solution:
$y = -0.758\dot{3}x^3 - 0.55x^2 + 5.658\dot{3}x + 7.45$ and
$y = 0.0\dot{3}x^3 - 0.55x^2 + 0.11\dot{6}x + 2.7$



EXERCISE 3.2

1 (a) $5x^3 - 15$ (b) $-2x^3 - 8$ (c) $6x^3 + 27x$

(d) $-3x^3 - 4x$ (e) $-x^3 + 9x$ (f) $7x^3 - 28x$

2 (a) $x^2 - 2x - 3$ (b) $y^2 + 3y - 4$ (c) $-m^2 + m + 42$

(d) $100 - p^2$ (e) $2n^2 + 2n - 24$ (f) $3t^2 + 3t - 60$

3 (a) $c^2 + 2cd + d^2$ (b) $m^2 - 2mn + n^2$

(c) $k^3 + 3k^2l + 3kl^2 + l^3$ (d) $s^3 - 3s^2t + 3st^2 - t^3$

(e) $z^3 + 6z^2 + 12z + 8$ (f) $a^3 - 15a^2 + 75a - 125$

(g) $4x^2 + 4x + 1$ (h) $9y^2 - 12y + 4$

(i) $64z^3 + 48z^2 + 12z + 1$ (j) $125a^3 - 225a^2 + 135a - 27$

(k) $x^2 + 8x + 16$ (l) $36 - 12b + b^2$

(m) $y^3 - 9y^2 + 27y - 27$

4 (a) $(a+b)^3 = (a+b)(a+b)^2$
$= (a+b)(a^2 + 2ab + b^2)$
$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$
$= a^3 + 3a^2b + 3ab^2 + b^3$

(b) $(a-b)^3 = (a-b)(a-b)^2$
$= (a-b)(a^2 - 2ab + b^2)$
$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$
$= a^3 - 3a^2b + 3ab^2 - b^3$

5 $343 - 441x + 189x^2 - 27x^3$

6 (a) B (b) D

(c) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ not $a^3 - b^3$

7 (a) $3x^3 - 9x^2 + 9x + 3$ (b) $-3x^3 - 45x^2 - 225x - 375$

(c) $4x^3 + 12x^2 - 15x + 4$ (d) $x^3 + 5x^2 + 3x - 9$

(e) $2x^3 - 20x^2 + 50x$ (f) $-20x^3 - 60x^2 - 45x$

8 C

9 (a) $x^3 - 10x^2 + 17x + 2$ (b) $y^3 + 3y^2 - 10y - 24$

(c) $-p^3 + 2p^2 + 13p + 10$ (d) $-m^3 - 2m^2 + 9m + 18$

10 (a) $3x^3 + 18x^2 + 9x - 30$ (b) $-2x^3 + 12x^2 - 22x + 12$

(c) $-x^3 - 6x^2 - 11x - 6$ (d) $6x^3 + 12x^2 - 30x - 36$

11 $x^3y^3 - 2x^2y + 4xy^2 + 6y^3$

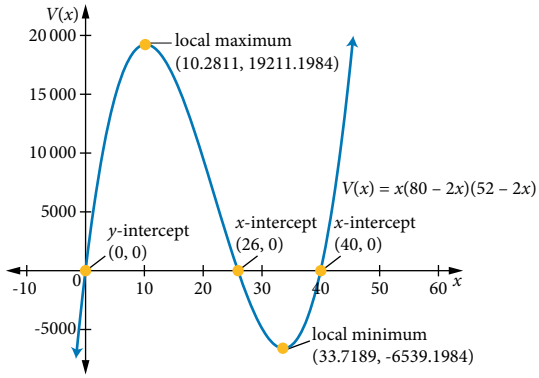
12 $m = 3$ 13 $A = 5, B = -2, C = 6$ 14 $x = 50$

15 (a) $a = -2, b = 3$ and $a = 3, b = -2$ (b) $a^2 + b^2 = 13$

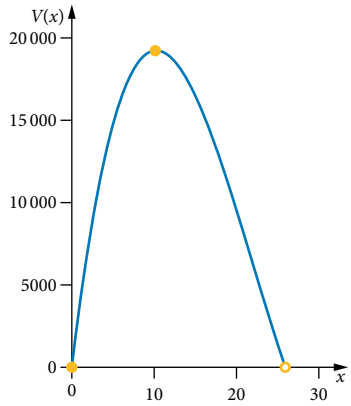
16 (a) $V(x) = x(80 - 2x)(52 - 2x)$

(b) $x = 0, x = 26$ and $x = 40$

(c)



(d) $0 < x < 26$



(e) $x = 10.28$ cm, $V(10.28) = 19\,211\,\text{cm}^3$

EXERCISE 3.3

1 (a) $x\left(3x^2+5\right)$ (b) $-4x\left(x^2+3x-6\right)$ (c) $x(x+5)(x+6)$

(d) $2x(x+5)(x-5)$ (e) $-x(4x-1)(x-1)$ (f) $4x\left(x^2+x+3\right)$

2 (a) $P(x)=(2x+5)\left(x^2-x+1\right)$, $Q(x)=x^2-x+1$, $R=0$

(b) $P(x)=(x-4)\left(x^2+2x+13\right)+50$, $Q(x)=x^2+2x+13$, $R=50$

(c) $P(x)=(x+4)\left(x^2+5x+6\right)$, $Q(x)=x^2+5x+6$, $R=0$

(d) $P(x)=(x-2)\left(x^3-11x^2-22x-44\right)-52$, $Q(x)=x^3-11x^2-22x-44$, $R=-52$

3 (a) $P(x)=(x-3)\left(x^2+5x+14\right)+48$, $Q(x)=x^2+5x+14$, $R=48$

(b) $P(x)=(x+3)\left(x^2-4x+2\right)+2$, $Q(x)=x^2-4x+2$, $R=2$

(c) $P(x)=(x-1)\left(3x^2+2x+1\right)$, $Q(x)=3x^2+2x+1$, $R=0$

(d) $P(x)=(x-1)\left(x^3+x^2-11x-11\right)+24$, $Q(x)=x^3+x^2-11x-11$, $R=24$

4 (a) $P(1)=8$, $P(-1)=8$, $P(2)=20$, $P\left(-\frac{1}{2}\right)=6\frac{7}{8}$

(b) $P(1)=2$, $P(-1)=-10$, $P(2)=8$, $P\left(-\frac{1}{2}\right)=-5\frac{1}{8}$

(c) $P(1)=60$, $P(-1)=6$, $P(2)=120$, $P\left(-\frac{1}{2}\right)=13\frac{1}{8}$

(d) $P(1)=-2$, $P(-1)=16$, $P(2)=-8$, $P\left(-\frac{1}{2}\right)=12\frac{5}{8}$

(e) $P(1)=0$, $P(-1)=-4$, $P(2)=17$, $P\left(-\frac{1}{2}\right)=-\frac{9}{8}$

(f) $P(1)=7$, $P(-1)=9$, $P(2)=27$, $P\left(-\frac{1}{2}\right)=7$

(g) $P(1)=24$, $P(-1)=24$, $P(2)=3$, $P\left(-\frac{1}{2}\right)=\frac{513}{16}$

(h) $P(1)=24$, $P(-1)=50$, $P(2)=-52$, $P\left(-\frac{1}{2}\right)=\frac{603}{16}$

5 (a) 156 (b) 0 (c) 0 (d) $6\frac{5}{8}$

(e) 64 (f) -862 (g) 16 (h) $2\frac{4}{9}$

6 B

7 (a) Yes, it is a factor. (b) No, it is not a factor

(c) No, it is not a factor. (d) Yes, it is a factor.

8 (a) Yes, it is a factor. (b) No, it is not a factor.

(c) No, it is not a factor. (d) Yes, it is a factor.

9 (a) A

(b) If $P(x)=x^3+a^3$, then $P(-a)=0$, so $(x+a)$ is a factor of x^3+a^3.

If $P(x)=x^2+a^2$, $P(x)=0$ is not possible (unless $a=0$), so x^2+a^2 cannot be factorised.

10 (a) $(x+2)\left(x^2-2x+4\right)$ (b) $(1-n)\left(1+n+n^2\right)$

(c) $2(c+2)\left(c^2-2c+4\right)$ (d) $(2e+3f)\left(4e^2-6ef+9f^2\right)$

(e) $(4-y)\left(16+4y+y^2\right)$ (f) $5(p-2q)\left(p^2+2pq+4q^2\right)$

(g) $2\left(m^2+n^2\right)$ (h) $4(p-q)\left(p^2+pq+q^2\right)$

11 (a) $(x+y)^3$ (b) $(p+q)^3$

(c) $2(c+d)^3$ (d) $8(e+f)^3$

(e) $(x-y)^3$ (f) $(c-d)^3$

(g) $2(m-n)^3$ (h) $4(p-q)^3$

(i) $(2x+3)^3$ (j) $(3x-1)^3$

12 (a) $P(x)=(x+4)(x+2)(x-2)$

(b) $P(x)=3(x+7)(x+5)(x+1)$

(c) $P(x)=2(x+2.59)(x+0.84)(x-0.92)$

(d) $P(x)=-5(x+1.68)(x+0.57)(x-1.05)$

13 (a) $(x+1)(x-3)(x-2)$ (b) $(x+1)(x+2)(x-1)$

(c) $(x-2)(x+3)^2$ (d) $(x-2)(x+3)(x+2)$

(e) $(x-1)(x-2)(2x+1)$ (f) $(x+2)(x-6)(3x-1)$

(g) $(x+2)(x-3)(2x-1)$ (h) $(x-3)(x+3)(2x+1)$

14 $p=6$, $q=-3.5$, $r=7.5$ 15 $p=-1$, $q=-5$ 16 $k=2$

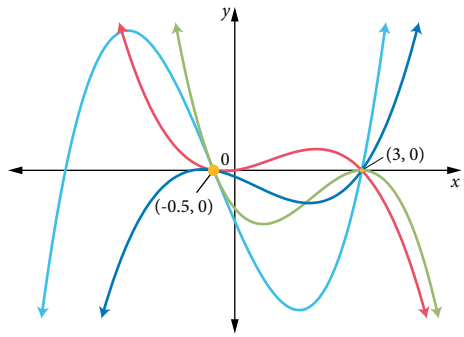
17 (a) Sample answer:

$y=(2x+1)(x-3)(x+1)=2x^3-3x^2-8x-3$

$y=-2(2x+1)(x-3)(x-3)=-4x^3+22x^2-24x-18$

$y=2(2x+1)(x-3)(x+4)=4x^3+6x^2-46x-24$

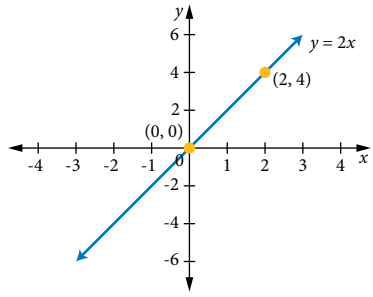
$y=-x(x-3)(2x+1)=-2x^3+5x^2+3x$



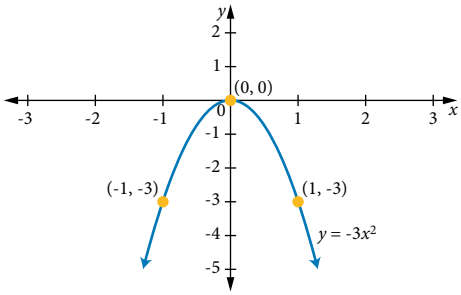
(b) All graphs pass through (-0.5, 0) and (3, 0) so they will all go up, down, up or down, up, down. They may or may not cut the x-axis at another place, and this extra intercept could be anywhere on the x-axis.

EXERCISE 3.4

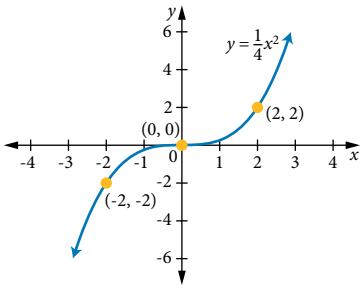
1 (a) As $x\to+\infty$, $y\to+\infty$. As $x\to-\infty$, $y\to-\infty$.



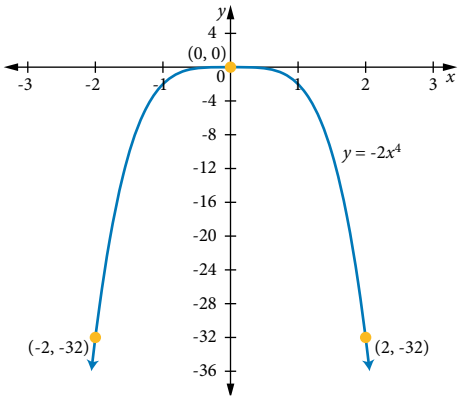
(b) As $x \to +\infty$, $y \to -\infty$. As $x \to -\infty$, $y \to -\infty$.



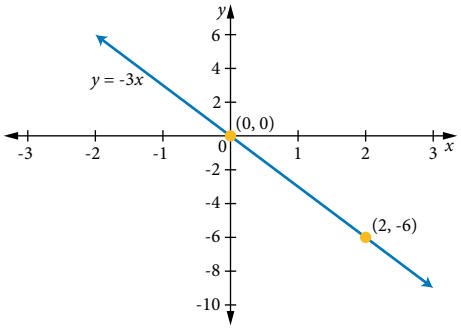
(c) As $x \to +\infty$, $y \to +\infty$. As $x \to -\infty$, $y \to -\infty$.



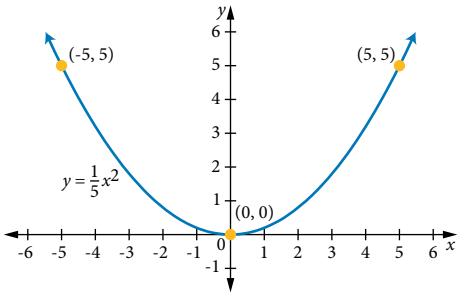
(d) As $x \to +\infty$, $y \to -\infty$. As $x \to -\infty$, $y \to -\infty$.



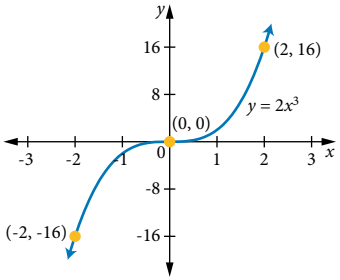
(e) As $x \to +\infty$, $y \to -\infty$. As $x \to -\infty$, $y \to +\infty$.



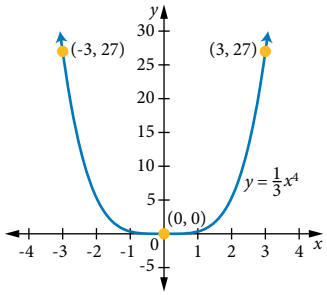
(f) As $x \to +\infty$, $y \to +\infty$. As $x \to -\infty$, $y \to +\infty$.



(g) As $x \to +\infty$, $y \to +\infty$. As $x \to -\infty$, $y \to -\infty$.



(h) As $x \to +\infty$, $y \to +\infty$. As $x \to -\infty$, $y \to +\infty$.



2 (a) $y = -3x$ (b) $y = 2x^2$ (c) $y = -2x^3$
(d) $y = \frac{1}{3}x$ (e) $y = 3x^4$ (f) $y = -4x^2$

3 B

4 (a) C
(b) Only even powers of negative values are positive.

5 (a) $g(0) = 0$, $g(-1) = g(1) = -\frac{1}{4}$ (b) $x = 0$
(c) As $x \to \infty$, $y \to -\infty$; as $x \to -\infty$, $y \to -\infty$

6 (a) $y = 3x^3$ (b) $y = -3(x+3)^3$
(c) $y = 3(x+3)^3$

7 (a) $y = 2x^3$ (b) $y = -4x^3$
(c) $y = 2x^3 + 1$ (d) $y = -2x^3 - 3$
(e) $y = (x+3)^3 - 2$ (f) $y = -(x-2)^3 + 1$
(g) $y = -\frac{1}{2}x^3 + 2$ (h) $y = -\frac{1}{2}(x-2)^3$

8 (a) (0, 1), (-1, 0) As $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$
(b) (0, 16), (2, 0) As $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$
(c) Both intercepts are (0, 0). As $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$

9 Translation right 3 units, up 1 unit

10 Dilate graph parallel to y-axis scale by factor 2, translate right 3 units, up 1 unit.

11 $a = -8$, $b = -1$, $c = 3$

12 (a) $y = x^3 + 3$ (b) $y = -x^3 + 3$
(c) $y = (x-3)^3$ (d) $y = -(x+3)^3$

13 (a) Dilation parallel to the y-axis by a factor of 2 and reflection in the x-axis.
(b) Dilation parallel to the y-axis by a factor of 3 and then a translation 5 units up.

(c) Dilation parallel to the y-axis by a factor of 0.5 and translation 2 units down.

(d) Translation 2 units right 5 units down.

(e) Reflection in the x-axis, translation 4 units left and 1 unit down.

(f) Dilation parallel to the y-axis of scale factor 3, reflection in the x-axis, translation 3 units and 2 units down.

14 $a = 8,\ b = -1,\ c = 3$

15 (a) $y = (x-3)^4 + 4$ (b) $y = -x^4 + 7$

(c) $y = (x+4)^4 - 5$

16 (-8, -4)

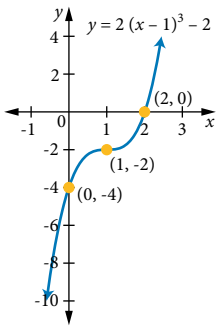
17 (a) odd (b) even (c) odd (d) even

18 (a) A: $y = a\left(x^2+5\right)$, B: $y = x^2+5$, C: $y = (bx)^2 + 5$

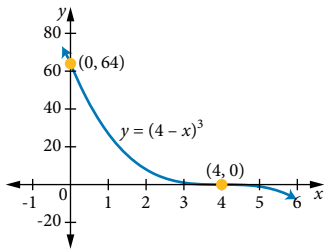
(b) $a = 2,\ b = \dfrac{1}{2}$

(c) a represents dilation parallel to the y-axis by a factor of 2; b represents a translation of 5 units up.

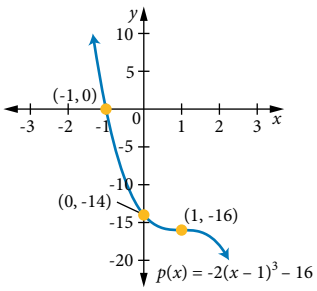
19 (a)



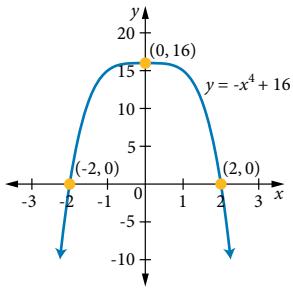
(b)



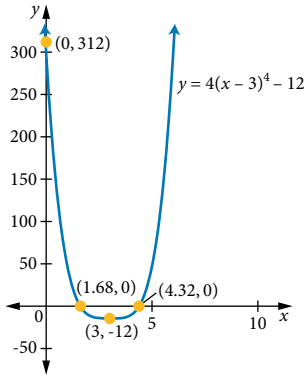
(c)



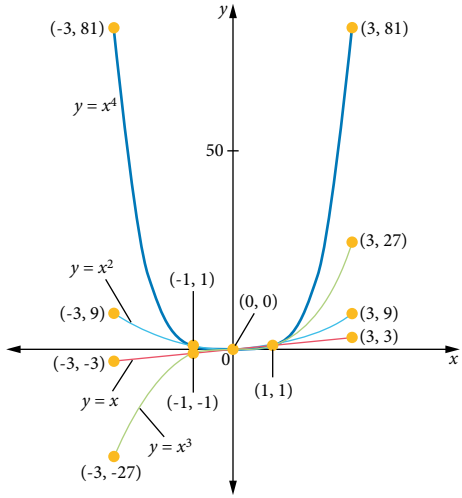
20 (a)



(b)

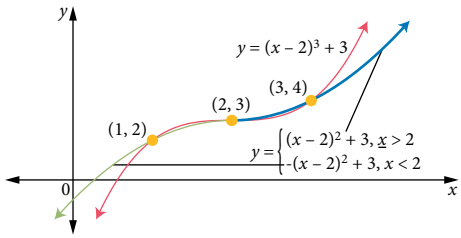


21

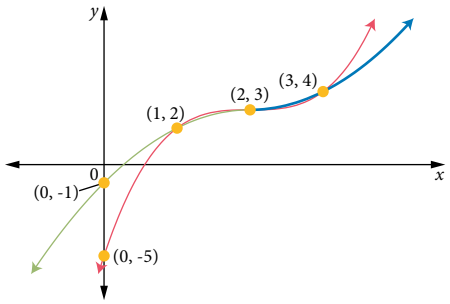


22 $y = \dfrac{d-k}{h^3}(x+h)^3 + k$, dilation factor $\dfrac{k-d}{h^3}$

23 (a) Possible solution: $y = \begin{cases} (x-2)^2 + 3, & x \geq 2 \\ -(x-2)^2 + 3, & x < 2 \end{cases}$



(b)



(c) Continuous and smooth throughout

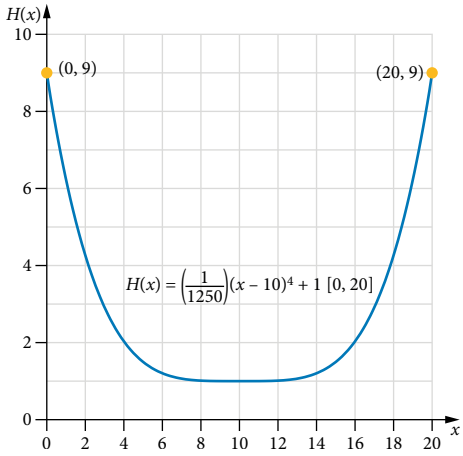
24 (a) $a = -1,\ b = 4,\ c = 8,\ f(x) = -(x-4)^3 + 8$

(b) $g(x) = (x+4)^3 + 8$

(c) $2\left(-\sqrt[3]{2} + 4\right)$ cm, 5.48 cm

25 (a) $H(x) = 0.0008(x-10)^4 + 1$

(b)

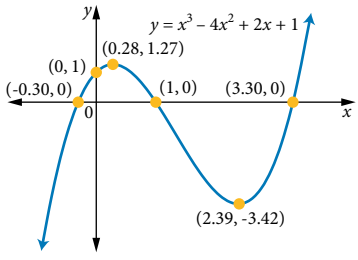


(c) $0 \le x \le 20$ (d) 1.5 m

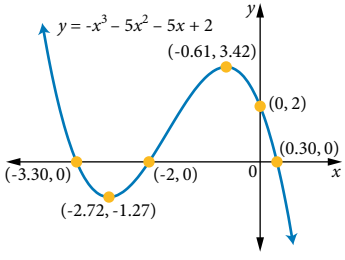
(e) $G(x) = 0.0008(x-10)^4 + 3$ (f) $3 \le G(x) \le 12$

EXERCISE 3.5

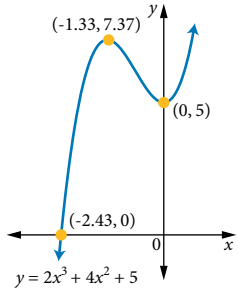
1 (a)



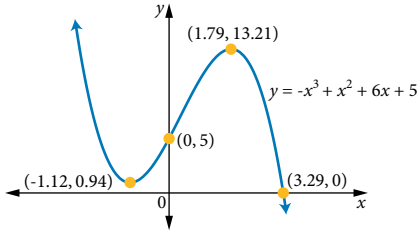
(b)



(c)



(d)



2 (a) D

(b) Null factor law $x - a = 0 \Rightarrow x = a$ has not been applied correctly.

3 B

4 (a) $y = (x+3)(x-1)(x-3)$

(b) $y = -(x+5)(x+3)(x-2)$

5 (a) (0, 16); (-1, 0), (2, 0), (4, 0); as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$

(b) (0, -16); (-2, 0), (-4, 0); as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$

6 (a) $y = k(x+1)(x-2)(x-3), k \ne 0$

(b) $y = kx(x+3)(x-4), k \ne 0$

(c) $y = k(2x+1)(3x+1)(x-2), k \ne 0$

(d) $y = k(2x-1)(3x-1)(x+1), k \ne 0$

(e) $y = kx(x+3)(3x-2), k \ne 0$

(f) $y = -2(3x-1)(5x-2)(x-2)$

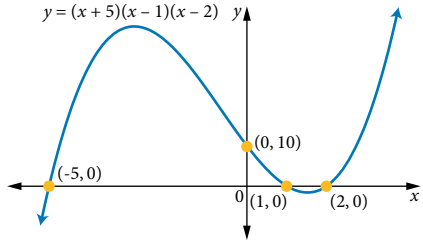
(g) $y = kx(x-1)^2, k \ne 0$

(h) $y = 2(x-2)^3$

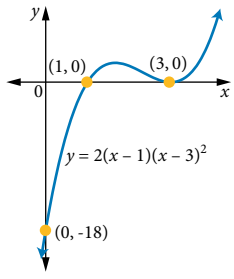
7 (a) $y = (x+2)^2(x-3$ (b) $y = (x-2)^2(x-3)$

(c) $y = -(x-3)^2(x+2)$ (d) $y = -\frac{1}{2}(x-3)^2(x-2)$

8 (a)



(b)



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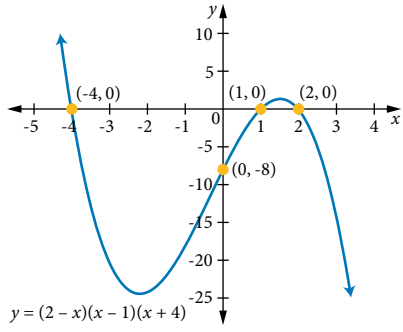
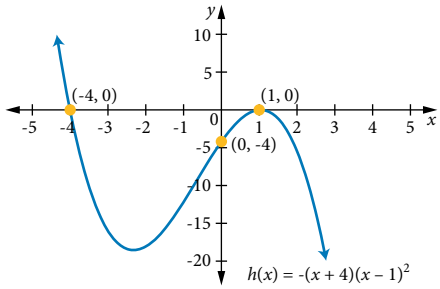


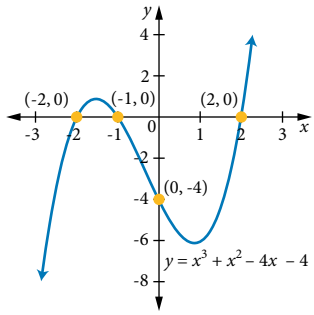
(d)



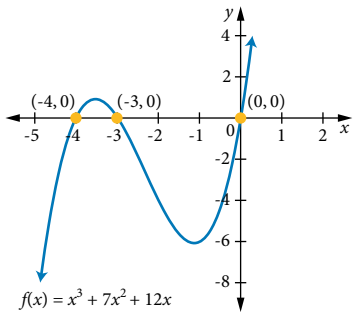
9 $a = -6, b = -1,\ f(x) = x^3 - 5x^2 + 7x - 3$

10 $a = 3, b = 2;\ f(x) = x^3 - x^2 + x + 3,$
$g(x) = -x^3 + 5x^2 - 4x - 10$

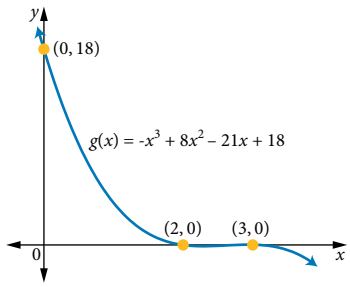
11 (a)



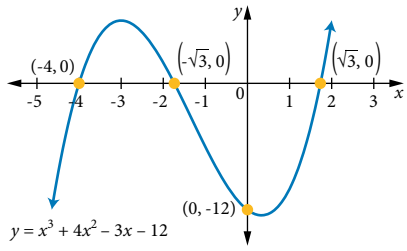
(b)



(c)

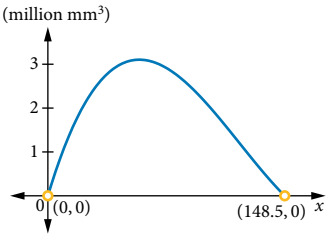


(d)



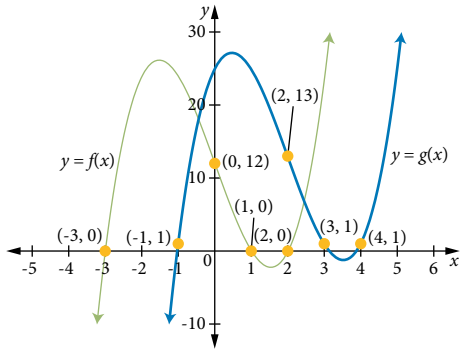
12 (a) $V(x) = x(420 - 2x)(297 - 2x)$

(b)

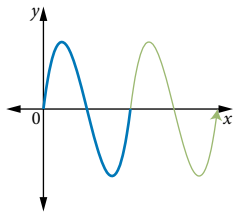


(c) 3 190 000 mm^3, $x = 57$ mm

13 $g(x) = 2(x + 1)(x - 3)(x - 4) + 1$



14 $y = \begin{cases} x(x-2)(x-4), & 0 \le x \le 4 \\ (x-4)(x-6)(x-8), & 4 < x \le 8 \end{cases}$



The sections of the cubic function are not symmetrical about the turning points, so a quadratic or quartic with four pieces may have been a better choice.

15 145 000 cm^3

EXERCISE 3.6

1 (a) $x = -5.36$

(b) $x = -4.69$, $x = -1.36$ or $x = 2.04$

2 (a) $x = -0.76$, $x = 1.33$ or $x = 4.43$

(b) $x = -0.85$, $x = 1.48$ or $x = 4.37$

3 (a) no solutions (b) $x = -3.50$ or $x = -0.50$

4 (a) $x = 2.88$

(b) $x = -21.78$, $x = -0.01$, $x = 3.05$

(c) $x = -3.12$, $x = 1.43$, $x = 2.69$

(d) $x = -1.80$, $x = -0.45$, $x = 1.25$

5 (a) $x = -1.71$ or $x = 1.33$ (b) $x = -1.23$ or $x = 0.19$

6 (a) $x = -2$ is a solution. (b) $x = 4$ is not a solution.

(c) $x = -1$ is not a solution. (d) $x = 2$ is a solution.

7 A

8 (a) C

(b) The steepness of the quartic graph at the extremes has not been taken into account.

9 (a) $f(x) = (x-6)\left(6x^2 - x - 1\right)$

(b) $f(x) = (x-6)(2x-1)(3x+1)$

(c) $x = -\frac{1}{3}, x = \frac{1}{2}, x = 6$

10 (a) $x = 0, x = 1, x = -5$ (b) $x = 0, x = -\frac{3}{2}, x = 7$

(c) $x = 0, x = -2, x = 2$ (d) $x = -5, x = 0, x = 5$

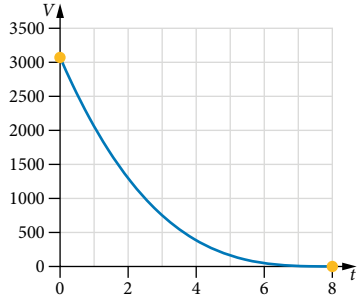
(e) $x = 0, x = 9$ (f) $x = -15, x = 0$

(g) $x = -15, x = 0, x = 2$ (h) $x = -7, x = -3, x = 7$

(i) $x = 5$ (j) $x = -2, x = 1$

11 (a) 3072 L, 750 L, 6 L

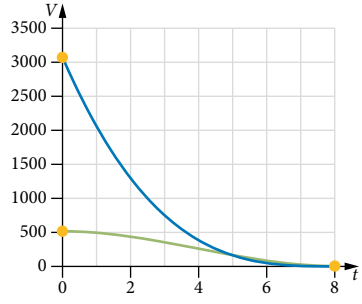
(b)



(c) 2 minutes, 4 minutes, 5 minutes

(d) 512 L, 350 L, 22 L

(e)



(f) 1 minute, 5 minutes, 6 minutes

(g) 5 minutes and 8 minutes

12 (a) $x = 1, x = 2$ (b) $x = -1, x = -\frac{2}{3}, x = 3$

(c) $x = 1$ (d) $x = 1$

(e) $x = 1, x = 2, x = 3$ (f) $x = -1, x = 1, x = 2$

(g) $x = -2, x = \pm\sqrt{3}$

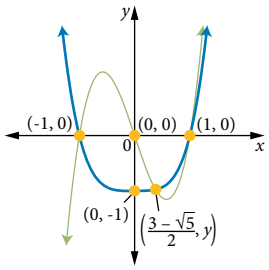
13 $a = 6, b = -24$

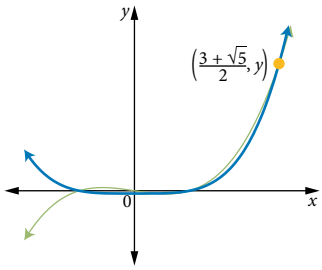
14 (-1, -2), (-3, -8), (-5, -14)

15 (-1, -2), (1, 4), $\left(\frac{5}{2}, \frac{17}{2}\right)$

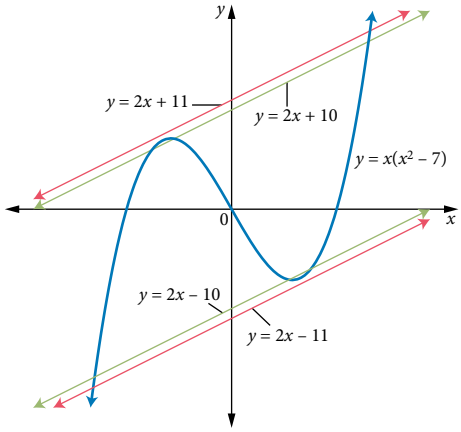
16 (a) $x = 0, x = 2, x = 4$ (b) (0, 3), (2, -1), (4, -5)

17 $x = \pm 1$ or $x = \frac{3 \pm \sqrt{5}}{2}$

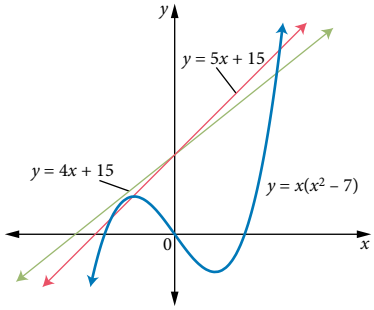




18 (a) $10 < c < 11, -11 < c < -10$



(b) $4 < m < 5$



19 Possible solution: $P(x) = 4(x+3)(x+1)(x-2) + 10$; $x = -2.15$, $x = -0.81$, $x = 2.36$, $x = 4.60$

CHAPTER REVIEW 3

1 (a) Degree: 3; $P(-1) = 12$, $P(2) = 3$

(b) Degree: 4; $P(-1) = -2\frac{1}{2}$, $P(2) = -22$

(c) Degree: 5; $P(-1) = \frac{3}{2}$, $P(2) = -6$

(d) Degree: 3; $P(-1) = 0$, $P(2) = 24$

(e) Degree: 0; $P(-1) = -6$, $P(2) = -6$

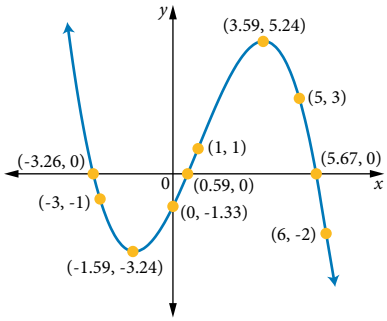
(f) Degree: 6; $P(-1) = 1$, $P(2) = 187$

2 156

3 $a = 1, b = 25$

4 (a) $y = -2x^3 - 3x^2 + x + 4$ (b) $y = -3x^3 - 2x^2 + 6x - 8$

5 $y = -0.12x^3 + 0.37x^2 + 2.09x - 1.33$



6 (a) $3x^3 + 12x$ (b) $2y^3 + 14y^2 - y - 7$

(c) $-2x^3 - 20x^2 - 50x$ (d) $z^3 - 2z^2 + z$

(e) $x^3 + 2x^2 - 5x - 6$ (f) $y^3 - 4y^2 - 3y + 18$

(g) $-g^3 - 4g^2 + 17g + 60$ (h) $-a^3 + 2a^2 + a - 2$

7 C

8 $(2x + 5y)(4x^2 - 10xy + 25y^2)$

9 (a) $A(x)$ is not a factor. (b) $A(x)$ is a factor.

(c) $A(x)$ is a factor. (d) $A(x)$ is not a factor.

(e) $A(x)$ is a factor. (f) $A(x)$ is a factor.

10 (a) 96 (b) $2x^2 + 4x + 1$

11 $P(x) = -2(x + 1.79)(x + 0.14)(x - 1.94)$

12 (a) $y = -\frac{1}{2}x$ (b) $y = 4x$ (c) $y = 3x^2$

(d) $y = -\frac{1}{2}x^2$ (e) $y = -3x^3$ (f) $y = -\frac{1}{3}x^3$

(g) $y = 2x^4$ (h) $y = -\frac{1}{4}x^4$

13 $a = 1, b = 2, c = -4$

14 (a) $y = -x^3 - 4$ (b) $y = -x^3 + 4$

(c) $y = (x - 3)^3 + 4$ (d) $y = 2x^3 - 4$

15 D

16 Dilation parallel to y-axis of scale factor 4, reflection in x-axis, horizontal translation 1 unit right, vertical translation 2 units up.

17 $y = -3(x + 2)^4 + 6$

18 (a) Horizontal translation 3 units to the right and vertical translation 4 units up. $y = (x - 3)^2 + 4$

(b) Reflection in the x-axis, vertical translation 7 units up; $y = -x^2 + 7$

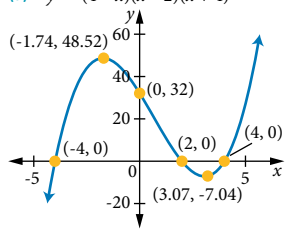
(c) Reflection in x-axis, horizontal translation 3 units left; $y = -(x + 3)^3$

(d) Horizontal translation 4 units left, vertical translation 5 units down; $y = (x + 4)^3 - 5$

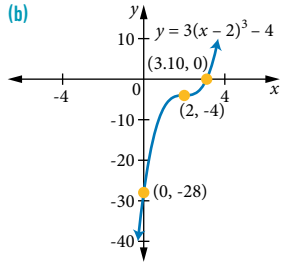
(e) Vertical translation 2 units down; $y = x^4 - 2$

(f) Horizontal translation 2 units left, vertical translation 1 unit down; $y = (x + 2)^4 - 1$

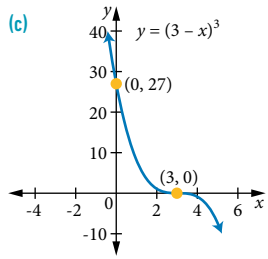
19 (a) $y = -(4 - x)(x - 2)(x + 4)$



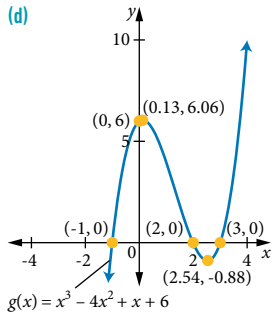
(b)



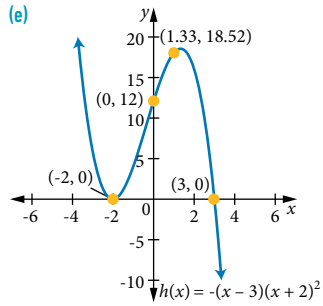
(c)



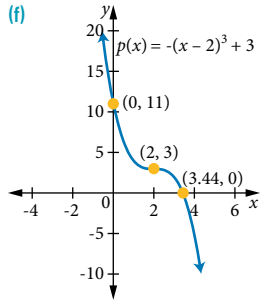
(d)



(e)



(f)



20 (a) $y = -1.5(x + 4)(x - 1)(x - 2.5)$ or $y = -0.75(x + 4)(x - 1)(2x - 5)$

(b) $y = 2(x - 1)^2(x - 4)$ (c) $y = -\frac{1}{9}(x + 3)^3$

21 (a) (-4, 0), (-2, 0), (4, 0); (0, 32)

(b) (-4, 0), (-2, 0), (2, 0); (0, -16)

22 $y = (x + 1)(x - 1)(x - 3)$

23 (a) $y = -\frac{1}{2}(x + 2)(x + 1)(x - 3)$

(b) Minimum turning point: (-1.53, -0.56)
Maximum turning point: (1.53, 6.56)

24 $y = \frac{1}{3}(x + 3)(x - 1)(x - 5)$

25 (a) $x = -3.57, x = -0.46, x = 1.53$

(b) $x = 0.16, x = 4.54$

26 $x = 4$; $l = 4$ m, $w = 21$ m, $h = 9$ m

27 $m = 20.75, n = -8.75$

28 A 29 $a = 4, b = -1$ 30 C

31 (a) $x(2x + 3)(3x - 5)$ (b) $(x + 2)(x^2 + 2x + 2)$

(c) $(x - 2)(x - 5)(x + 2)$

32 (a) $H(x) = -\frac{2}{75}(x - 15)^2 + 6$, domain: [0, 30], range: [0, 6]

(b) 2.75 m

33 (-5, -14), (-3, -8), (-1, -2)

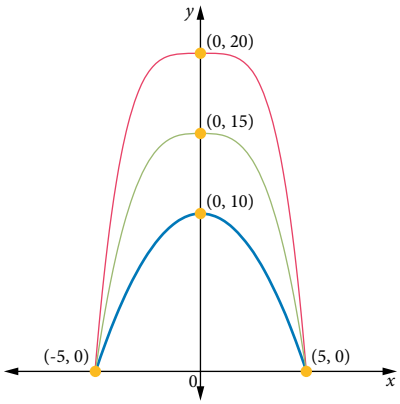
34 $x = 6$; base: 8 m × 7 m, height: 6 m

35 (a) $x = -4, x = -1, x = 2$ (b) $x = -6, x = 1, x = 3$
(c) $x = -2$ (d) $x = -4, x = -1, x = \frac{1}{2}$

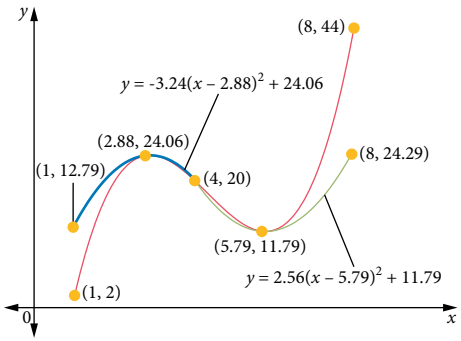
36 (a) $x = 0, x = \frac{1}{2}, x = 5$ (b) $x = -3$
(c) $x = -3, x = -2$

37 $x = 3, y = 5, z = 14$

38 $y = -0.032x^4 + 20, -5 \le x \le 5,\ y = \begin{cases} 0.12x^3 + 15, -5 \le x \le 0 \\ 0.12x^3 + 15, 0 < x \le 5 \end{cases},$
$y = -0.4x^2 + 10, -5 \le x \le 5$



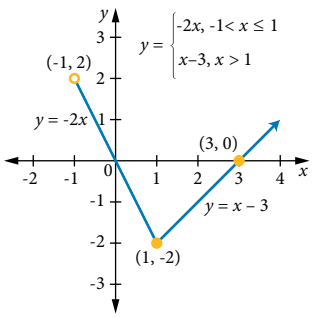
39



Mixed review: Chapters 1–3

1 (a) function
(b) function

2



3 (a) $t_{15} = 53$ (b) $t_{n+1} = t_n + 4,\ t_1 = -3$
(c) $t_n = 4n - 7$ (d) $S_{10} = 150$

4 (a) $x = 0$ (b) $x = 4$ (c) $t = -6$

5 (-4, 0)

6 (0, -5)

7 (a) $x = 0, x = -5$ (b) $m = -8, m = 4$ (c) $t = -\frac{2}{3}, t = 4$

8 (a) polynomial function
(b) polynomial function
(c) not a polynomial function
(d) not a polynomial function
(e) not a polynomial function

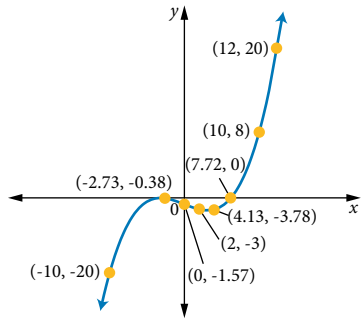
9 (a) $12x^2 - 51xy + 36x + 12y^2 - 9y$
(b) $26x - 36y$
(c) $-3b^2 + 7ab - 4a^2$
(d) $9a^2b - 20ab^2$

10 $y = 2(x + 4)^3 + 2$

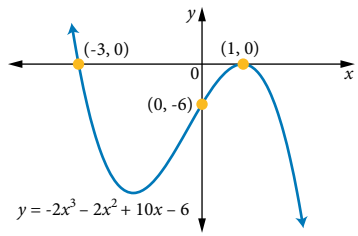
11 $x = 4, y = 0$

12 Domain: $0 \le t \le 6.05$; Range: $0 \le h \le 45.6$; 2.14 s

13 $y = 0.02x^3 - 0.04x^2 - 0.71x - 1.57$



14



15 $y = \begin{cases} 1200 + 300x, & x \in \{0, 1, 2, \ldots, 11\} \\ 4857 + 300(x - 12), & x \in \{12, 13, 14, \ldots, 18\} \end{cases}$, where x is the number of months from the time of the initial balance, and y is the account balance ($)

16 $A(x) = \frac{1}{2}x(80 - x), 0 \le x \le 80$, 40 m of the existing fence with 20 m on the sides at right angles to the existing fence.

Chapter 4 Advanced functions and relations

RECALL

1 (a) $y = \frac{1}{2}$ (b) $y = 5$ (c) $y = \pm 4$

2 (a) $3\sqrt{2}$ (b) $\sqrt{5}$ (c) $8\sqrt{2}$ (d) $-3\sqrt{5}$

3 (a) $\frac{2\sqrt{5}}{5}$ (b) $\frac{\sqrt{6}}{2}$ (c) $\frac{3\sqrt{42}}{14}$

4 A: (0, 3), B: (3, 0), C: (0, -3), D: (-3, 0)

5 Graph A: $y = 5x + 2$; Graph B: $y = -3x + 2$;
Graph C: $y = \frac{1}{2}x - 3$; Graph D: $y = -2x - 3$

6 (a) $-5 \le x < 4, [-5, 4)$ (b) $-4 < x < \infty, (-4, \infty)$
(c) $-\infty < x < \infty, (-\infty, \infty)$

7

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$

8 (a) $x = 4$ (b) $x = -3$ (c) $x = -2$

9 (a) (0, -6) (b) (0, 3) (c) (0, 1)

EXERCISE 4.1

1 (a) (i) 12 hours (ii) 6 hours (iii) 4 hours (iv) 3 hours
(b) 4 painters

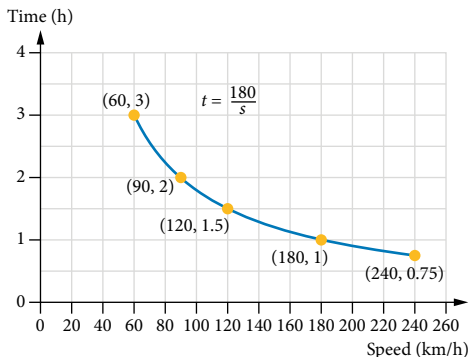
2 (a) 2 cm^3
(b) (i) 4 cm^3 (ii) 8 cm^3 (iii) 200 cm^3

3 1500 hertz

4 (a)

Speed (km/)h	60	90	120	180	240
Time (h)	3	2	1.5	1	0.75

(b) As the speed of the cars increases, the time taken to travel 180 km decreases and as the speed of the cars decreases, the longer it takes to travel the 180 km.



5 $t = \frac{960}{T}$ where $T > 0$

6 (a) $y = 8.2$ (b) $x = 2.8$

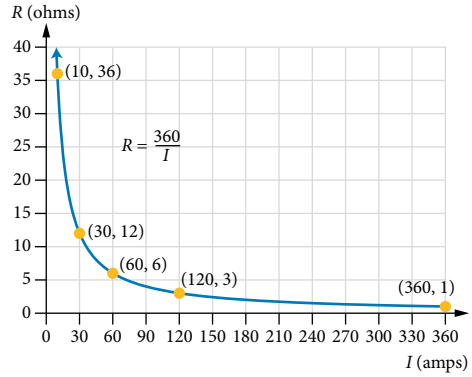
7 (a) $y = 0.3$ (b) $x = 0.0016$

8 $y = 0$ 9 0.135 Pa

10 (a)

Current, I (amps)	10	30	60	120	360
Resistance, R (ohms)	36	12	6	3	1

(b)



(c) As the current increases, the resistance decreases towards zero. As the current gets smaller, and gets closer to zero, the resistance increases very quickly. Neither the resistance nor the current reaches zero.

11 D

12 (a) A
(b) The vertical asymptote $x = 1$ has been used as the x-intercept. The constant term has been assumed to give the y-intercept.

13 A

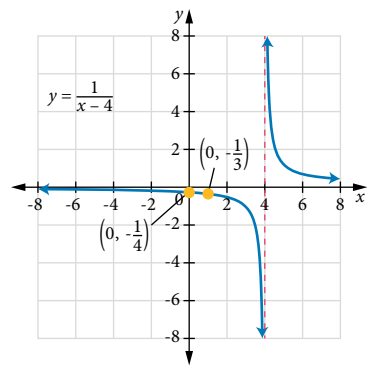
14 (a) $y = \frac{1}{x}$ (b) $y = -\frac{2}{x}$
(c) $y = \frac{1}{x-2} + 1$ (d) $y = -\frac{2}{x+1}$

15 (a) $\frac{9}{x-5} + 1$ (b) $\frac{-3}{x+2} + 1$ (c) $\frac{-3}{x+3} + 1$
(d) $\frac{1}{x+1} + 2$ (e) $\frac{9}{x+4} - 1$

16 (a)

x	1	2	3	4	5	6	7
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	Undefined	1	$\frac{1}{2}$	$\frac{1}{3}$

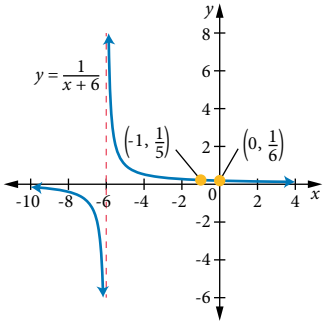
Equations of asymptotes: $x = 4$, $y = 0$



(b)

x	-9	-8	-7	-6	-5	-4	-3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	Undefined	1	$\frac{1}{2}$	$\frac{1}{3}$

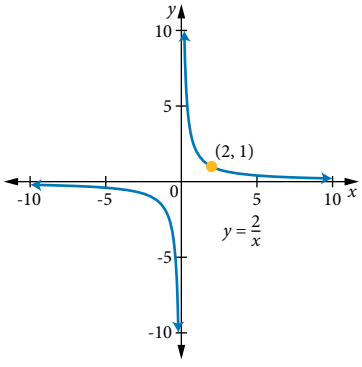
Equations of asymptotes: $x = -6$, $y = 0$



17 (a)

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	-1	-2	-4	4	2	1

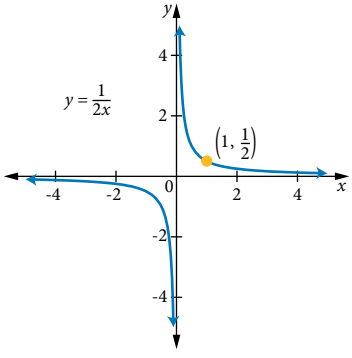
Equation of asymptotes: $x = 0$, $y = 0$



(b)

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	1	$\frac{1}{2}$	$\frac{1}{4}$

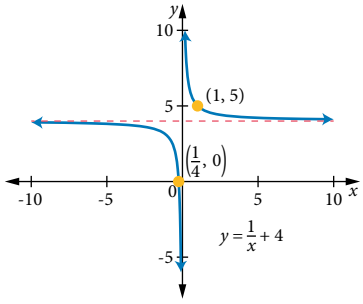
Equation of asymptotes: $x = 0$, $y = 0$



(c)

x	-3	-2	-1	0	1	2	3
y	$3\frac{2}{3}$	$3\frac{1}{2}$	3	Undefined	5	$4\frac{1}{2}$	$4\frac{1}{3}$

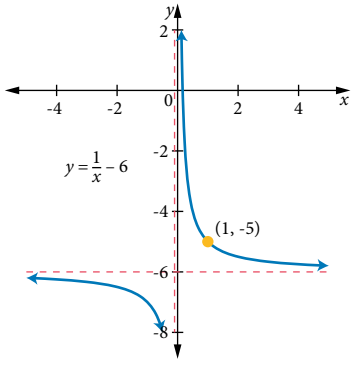
Equation of asymptotes: $x = 0$, $y = 4$



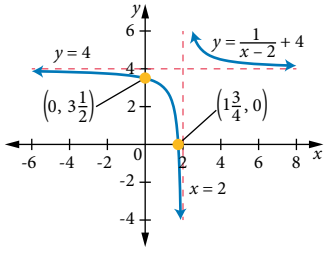
(d)

x	-3	-2	-1	0	1	2	3
y	$-6\frac{1}{3}$	$-6\frac{1}{2}$	-7	Undefined	-5	$-5\frac{1}{2}$	$-5\frac{2}{3}$

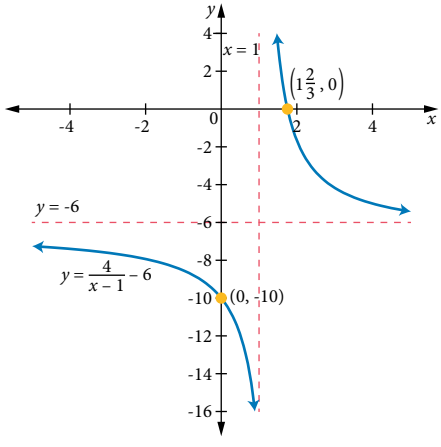
Equation of asymptotes: $x = 0$, $y = -6$



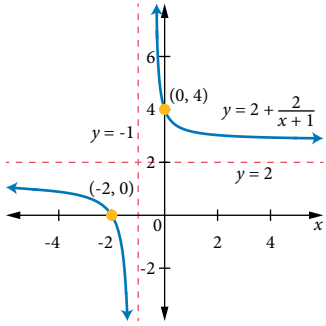
18 (a)



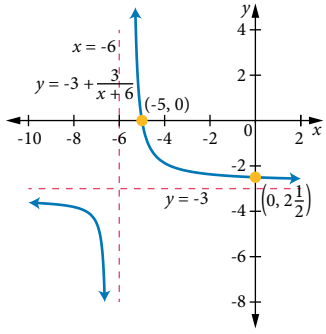
(b)



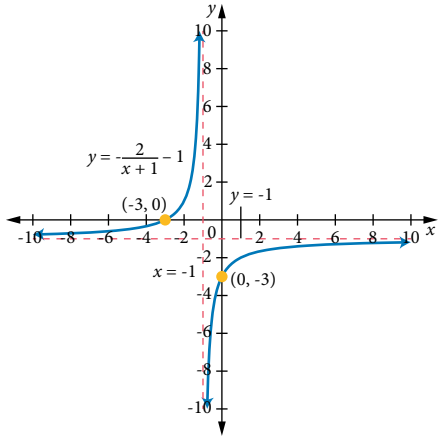
(c)


(0, 4)
$y = 2 + \frac{2}{x+1}$
$y = -1$
$y = 2$
(-2, 0)

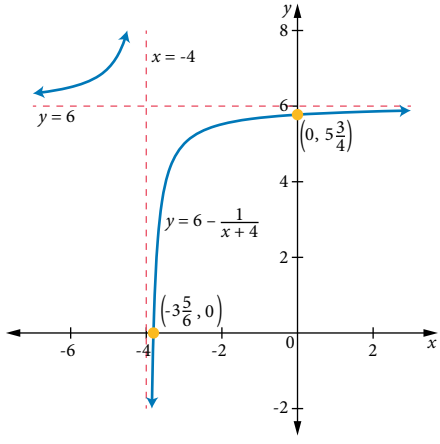
(d)


$x = -6$
$y = -3 + \frac{3}{x+6}$
(-5, 0)
$y = -3$
$\left(0, 2\frac{1}{2}\right)$

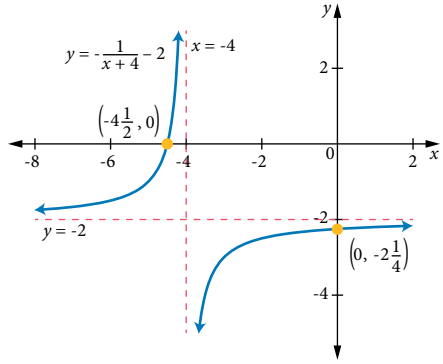
(e)


$y = -\frac{2}{x+1} - 1$
(-3, 0)
$y = -1$
$x = -1$
(0, -3)

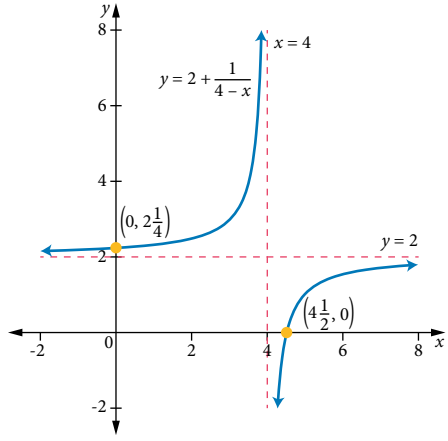
(f)


$x = -4$
$y = 6$
$\left(0, 5\frac{3}{4}\right)$
$y = 6 - \frac{1}{x+4}$
$\left(-3\frac{5}{6}, 0\right)$

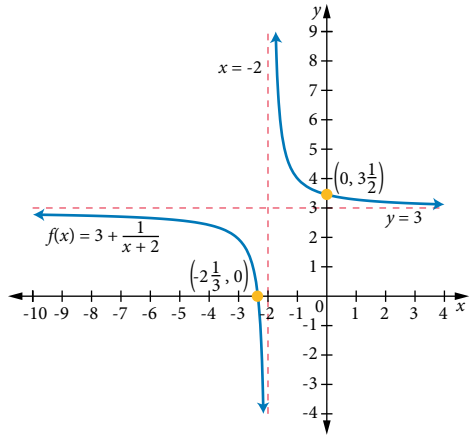
(g)


$y = -\frac{1}{x+4} - 2$
$x = -4$
$\left(-4\frac{1}{2}, 0\right)$
$y = -2$
$\left(0, -2\frac{1}{4}\right)$

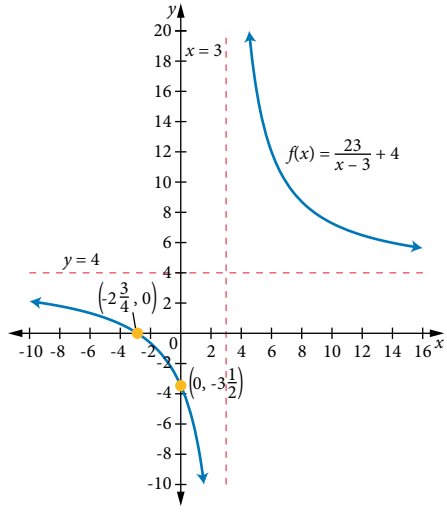
(h)


$x = 4$
$y = 2 + \frac{1}{4-x}$
$\left(0, 2\frac{1}{4}\right)$
$y = 2$
$\left(4\frac{1}{2}, 0\right)$

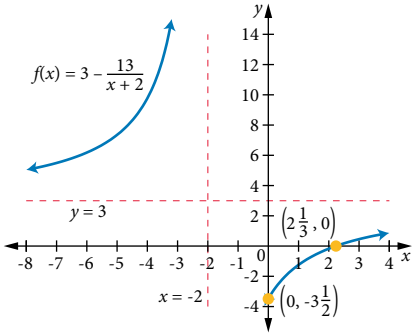
(i)


$x = -2$
$\left(0, 3\frac{1}{2}\right)$
$y = 3$
$f(x) = 3 + \frac{1}{x+2}$
$\left(-2\frac{1}{3}, 0\right)$

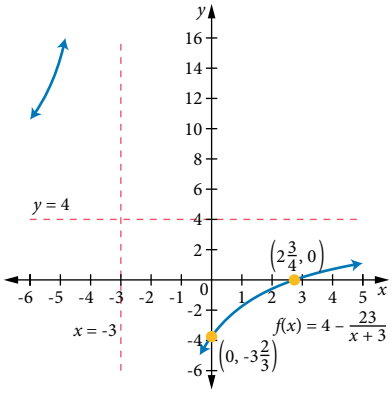
(j)


$x = 3$
$f(x) = \frac{23}{x-3} + 4$
$y = 4$
$\left(-2\frac{3}{4}, 0\right)$
$\left(0, -3\frac{1}{2}\right)$

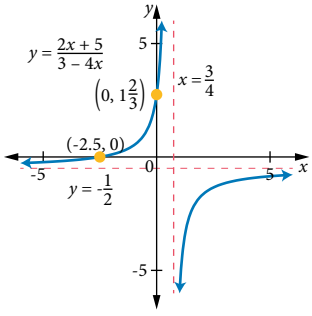
(k)



(l)



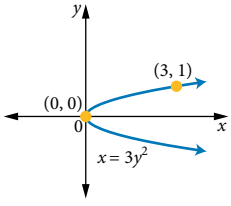
19 Asymptotes: $x = \frac{3}{4}$, $y = -\frac{1}{2}$; intercepts: $(-2.5, 0)$, $\left(0, 1\frac{2}{3}\right)$



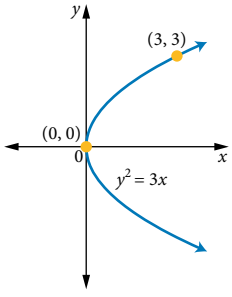
20 $x < -0.15$, $1 < x < 2.15$, $x > 3$ or $(-\infty, -0.15)$, $(1, 2.15)$, $(3, \infty)$

EXERCISE 4.2

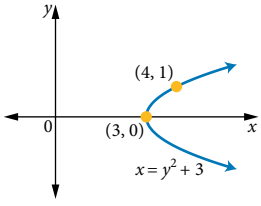
1 (a)



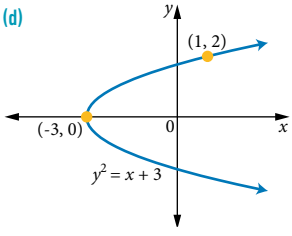
(b)



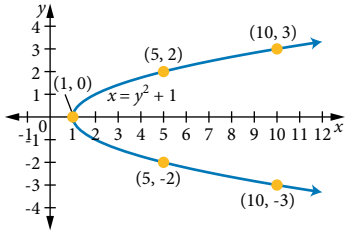
(c)



(d)

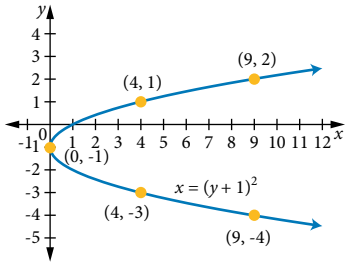


2 (a)



(b) $(1, 0)$, $y = 0$

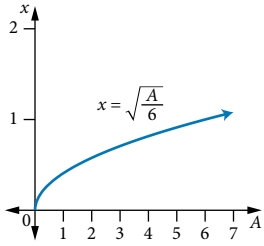
3 (a)



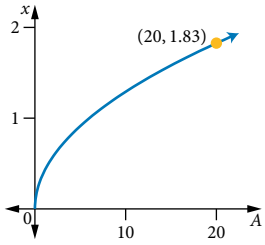
(b) $(0, -1)$, $y = -1$

4 (a) $x = \sqrt{\frac{A}{6}}$

(b)



(c) 1.83 cm



5 (a) 0.61 cm **(b)** 9 tonnes

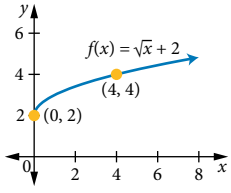
6 (a) 357 km **(b)** 58 km less

7 (a) 6.44 cm **(b)** 3977 cm^2

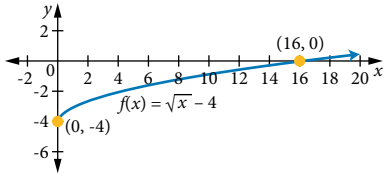
8 (a) A **(b)** $(-2)^2 \neq -2^2$

9 $y = 0$ **10** $x = 6$ **11** B **12** $(0, -7)$

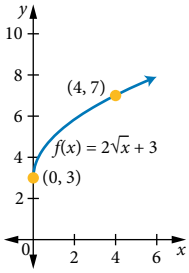
13 (a) Domain: $[0, \infty)$; Range: $[2, \infty)$



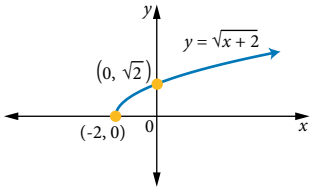
(b) Domain: $[0, \infty)$; Range: $[-4, \infty)$



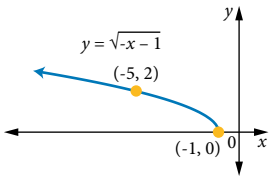
(c) Domain: $[0, \infty)$; Range: $[3, \infty)$



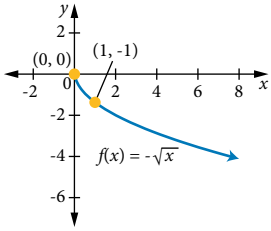
(d) Domain: $[-2, \infty)$; Range: $[0, \infty)$



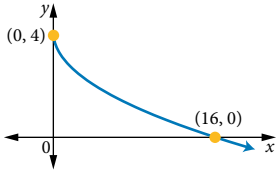
(e) Domain: $(-\infty, -1]$; Range: $[0, \infty)$



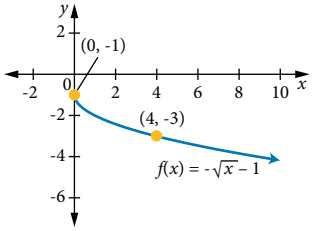
14 (a) Domain: $[0, \infty)$; Range: $(-\infty, 0]$



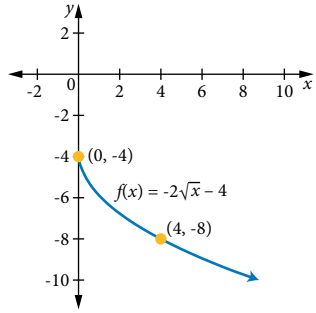
(b) Domain: $[0, \infty)$; Range: $(-\infty, 4]$



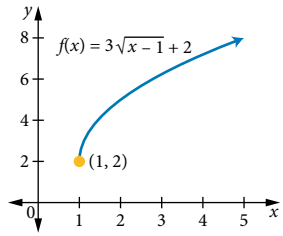
(c) Domain: $[0, \infty)$; Range: $(-\infty, -1]$



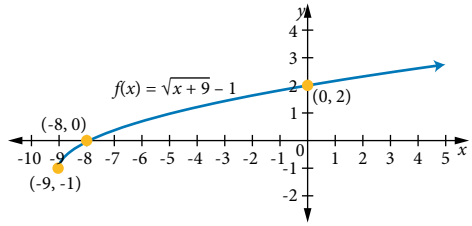
(d) Domain: $[0, \infty)$; Range: $(-\infty, -4]$



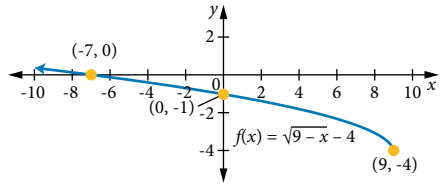
15 (a) Vertex: (1, 2); no axis intercepts.



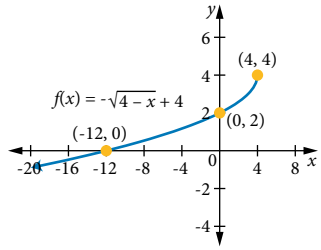
(b) Vertex: (-9, -1); x-intercept: (-8, 0); y-intercept: (0, 2)



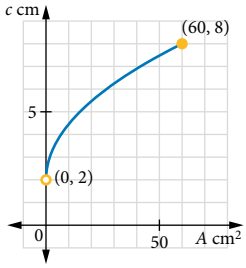
(c) Vertex: (9, -4); x-intercept: (-7, 0); y-intercept: (0, -1)



(d) Vertex: (4, 4); x-intercept: (-12, 0); y-intercept: (0, 2)

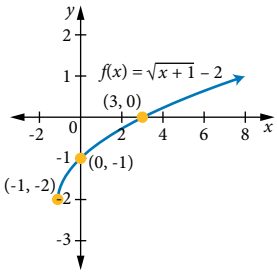


16 $c = \sqrt{A + 4}$



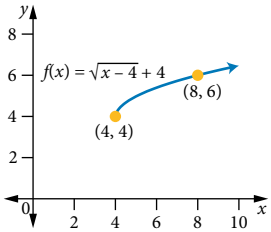
17 **(a)** No, only 48 km/h **(b)** 41 cm

18 **(a)**



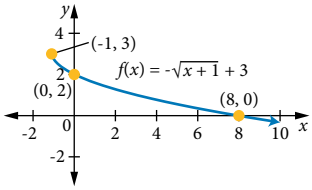
Domain: $[-1, \infty)$; Range: $[-2, \infty)$

(b)



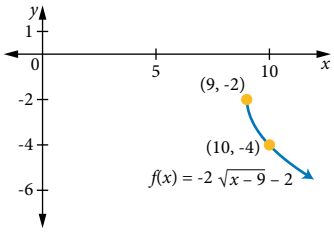
Domain: $[4, \infty)$; Range: $[4, \infty)$

(c)



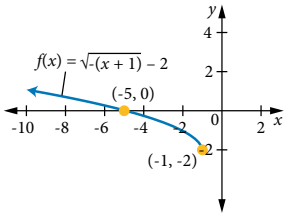
Domain: $[-1, \infty)$; Range: $(-\infty, 3]$

(d)



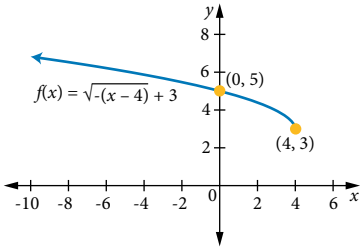
Domain: $[9, \infty)$; Range: $(-\infty, -2]$

19 **(a)**



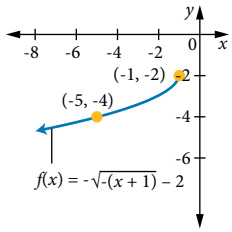
Domain: $(-\infty, -1]$; Range: $[-2, \infty)$

(b)



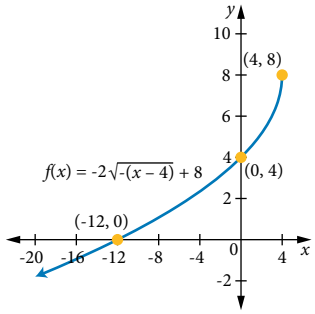
Domain: $(-\infty, 4]$; Range: $[3, \infty)$

(c)



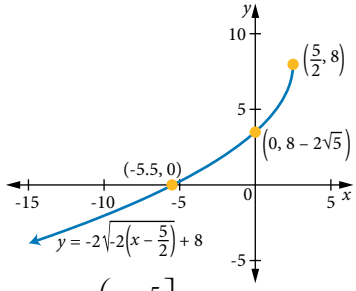
Domain: $(-\infty, -1]$; Range: $(-\infty, -2]$

(d)



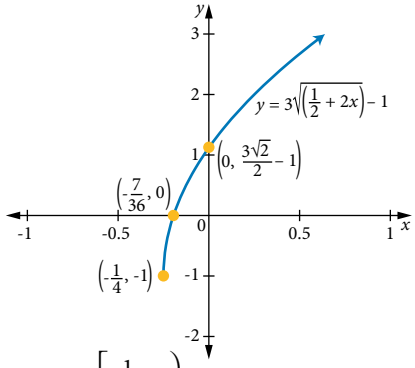
Domain: $(-\infty, 4]$; Range: $(-\infty, 8]$

20 **(a)**



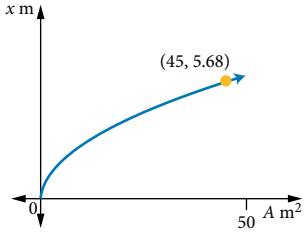
(b) Domain: $\left(-\infty, \frac{5}{2}\right]$; Range: $(-\infty, 8]$

21 **(a)**



(b) Domain: $\left[-\frac{1}{4}, \infty\right)$; Range: $[-1, \infty)$

22 **(a)** $x = \sqrt{\frac{8}{8+\pi}}\sqrt{A}$ or $x = 0.847\sqrt{A}$



(b) $h = 1.27\sqrt{A}$ or $h = \sqrt{\frac{18}{8+\pi}}\sqrt{A}$

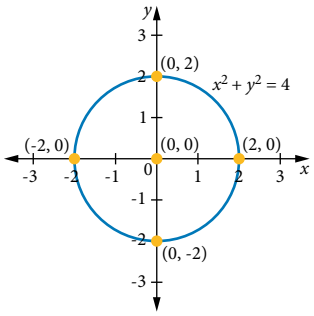
(c) Dilation of factor 1.5 parallel to the vertical axis, from the horizontal A-axis.

23 Top: $y = \begin{cases} \sqrt{5(x+10)}, & -10 \le x < -5 \\ 10 - 0.2x^2, & -5 \le x \le 5; \\ \sqrt{5(10-x)}, & 5 < x \le 10 \end{cases}$

Bottom: $y = \begin{cases} -\sqrt{5(x+10)}, & -10 < x < -5 \\ 0.2x^2 - 10, & -5 \le x \le 5 \\ -\sqrt{5(10-x)}, & 5 < x < 10 \end{cases}$

EXERCISE 4.3

1 (a)



(b) Domain: [-2, 2]; Range: [-2, 2]

2 (a) C

(b) The centre at (5, -2) is not a point on the circle.

3 (a) inside (b) inside (c) outside

4 (a) inside (b) inside (c) on

(d) inside (e) outside (f) outside

5 (a) No water received. (b) No water received.

(c) The petunia will be watered.

6 D

7 Domain: [-2, 0]; Range: [1, 3]

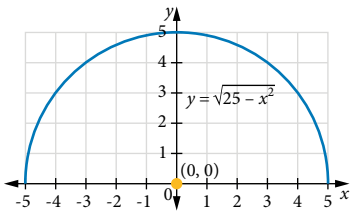
8 (a) $y^2 + x^2 = 25$ (b) $x^2 + y^2 = 16$

(c) $(x+1)^2 + (y+2)^2 = 9$ (d) $(x-2)^2 + (y-3)^2 = 4$

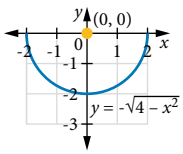
(e) $(x+7)^2 + (y-8)^2 = 4$ (f) $(x-4)^2 + (y+3)^2 = 9$

9 (a) $y = \sqrt{9 - x^2}$ (b) $y = -\sqrt{36 - x^2}$

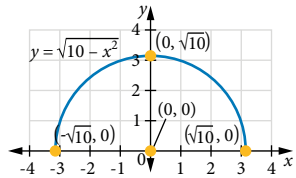
10 (a) Domain: [-5, 5]; Range: [0, 5]



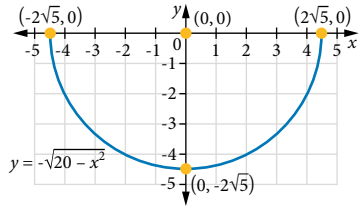
(b) Domain: [-2, 2]; Range: [-2, 0]



(c) Domain: $\left[-\sqrt{10}, \sqrt{10}\right]$; Range: $\left[0, \sqrt{10}\right]$



(d) Domain: $\left[-2\sqrt{5}, 2\sqrt{5}\right]$; Range: $\left[-2\sqrt{5}, 0\right]$



11 $y = \sqrt{9 - (x+2)^2}$

12 (a) (i) $x^2 + (y+1)^2 = 1^2$

(ii) Centre: (0, -1); Radius: 1

(iii) Domain: [-1, 1]; Range: [-2, 0]

(b) (i) $(x+1)^2 + y^2 = 2^2$

(ii) Centre: (-1, 0); Radius: 2

(iii) Domain: [-3, 1]; Range: [-2, 2]

(c) (i) $(x+2)^2 + (y-1)^2 = 1^2$

(ii) Centre: (-2, 1); Radius: 1

(iii) Domain: [-3, -1]; Range: [0, 2]

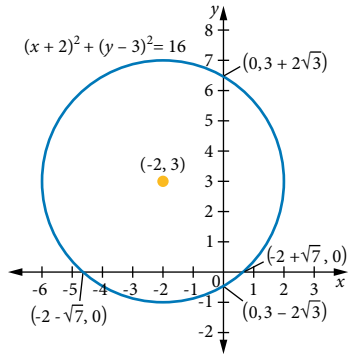
(d) (i) $(x-4)^2 + (y+3)^2 = 5^2$

(ii) Centre: (4, -3); Radius: 5

(iii) Domain: [-1, 9]; Range: [-8, 2]

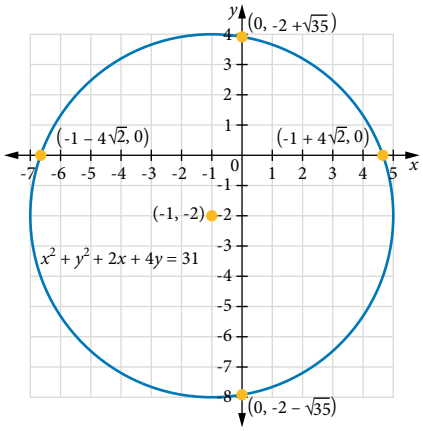
13 (a) Centre: (-2, 3); Radius: 4

(b)



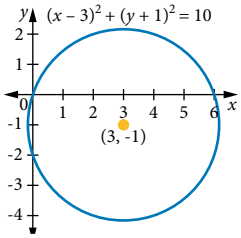
(c) Domain: $-6 \le x \le 2$; Range: $-1 \le y \le 7$

14 Domain: [-7, 5]; Range: [-8, 4]



15 **(a)** Centre: (3, -1); Radius: $\sqrt{10}$

(b) x-intercepts: (0, 0), (6, 0); y-intercepts: (0, -2), (0, 0)



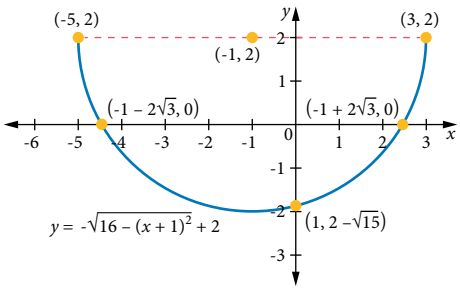
(c) Domain: $\left[3-\sqrt{10}, 3+\sqrt{10}\right]$; Range: $\left[-1-\sqrt{10}, -1+\sqrt{10}\right]$

16 **(a)** Centre: (-1, 2); Radius: 4

(b) y-intercept: $\left(0, 2-\sqrt{15}\right)$

x-intercepts: $\left(-1-2\sqrt{3}, 0\right)$ and $\left(-1+2\sqrt{3}, 0\right)$

(c)

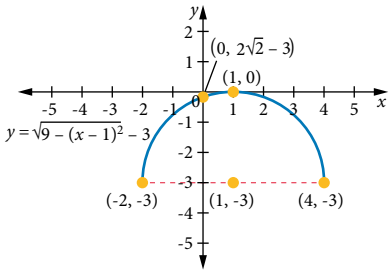


(d) Domain: [-5, 3]; Range: [-2, 2]

17 **(a)** Centre: (1, -3); Radius: 3

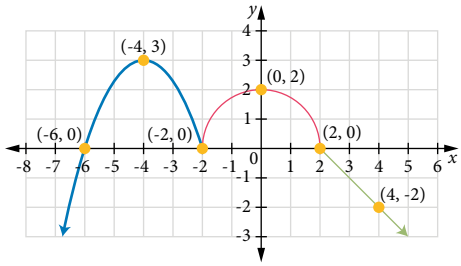
(b) y-intercept: $\left(0, 2\sqrt{2}-3\right)$; x-intercept: (1, 0)

(c)



(d) Domain: [-2, 4]; Range: [-3, 0]

18 $a=-\frac{3}{4}$, c = 2, $f(x)=\begin{cases} -\frac{3}{4}(x+4)^2+3, & x<-2 \\ \sqrt{4-x^2}, & -2\le x\le 2 \\ 2-x, & x>2 \end{cases}$



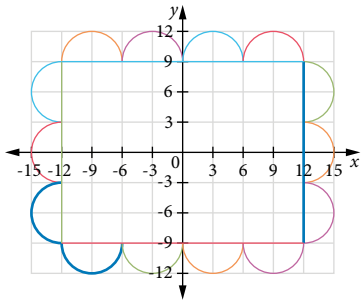
19 $f(x)=\begin{cases} x+2 & -4\le x<0 \\ \sqrt{4-(x-2)^2}+2 & 0\le x<4 \\ 2x-6 & 4\le x\le 8 \end{cases}$

or $f(x)=\begin{cases} x+2 & -4\le x<2 \\ \sqrt{4-(x-2)^2}+2 & 2\le x<4 \\ 2x-6 & 4\le x\le 8 \end{cases}$

20 Possible functions:

$$y=\begin{cases} \sqrt{9-x^2}, & -3\le x\le 3 \\ -\frac{1}{3}x, & -4.74\le x\le -2.85 \\ \frac{1}{3}x, & 2.85\le x\le 4.74 \\ -x, & -3.54\le x\le -2.12 \\ x, & 2.12\le x\le 3.54 \\ -3x, & -1.58\le x\le -0.95 \\ 3x, & 0.95\le x\le 1.58 \end{cases}$$

21 Sample answer for the diagram.



Top row: $y=9+\sqrt{9-(x+9)^2}$, $y=9+\sqrt{9-(x+3)^2}$,

$y=9+\sqrt{9-(x-3)^2}$, $y=9+\sqrt{9-(x+3)^2}$

Bottom row: $y=-9-\sqrt{9-(x+9)^2}$, $y=-9-\sqrt{9-(x+3)^2}$,

$y=-9-\sqrt{9-(x-3)^2}$, $y=-9-\sqrt{9-(x+3)^2}$

Left side: $y=6\pm\sqrt{9-(x+12)^2}$, $x\le -12$,

$y=\pm\sqrt{9-(x+12)^2}$, $x\le -12$,

$y=-6\pm\sqrt{9-(x+12)^2}$, $x\le -12$

Right side: $y=6\pm\sqrt{9-(x-12)^2}$, $x\ge 12$,

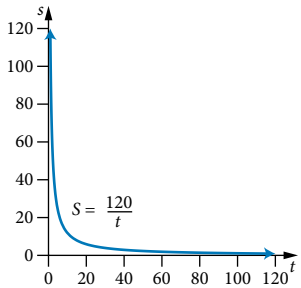
$y=\pm\sqrt{9-(x-12)^2}$, $x\ge 12$, $y=-6\pm\sqrt{9-(x-12)^2}$, $x\ge 12$

CHAPTER REVIEW 4

1 $V = 2500\,\text{cm}^3$

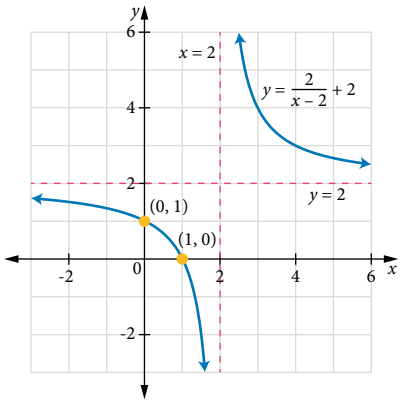
2 Assuming the trend continues, 29.1 min.

3 (a)



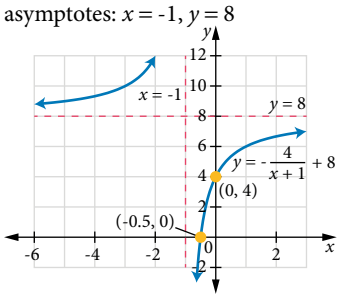
(b) $2\ \text{ms}^{-1}$

4 x-intercept: (1, 0); y-intercept: (0, 1); asymptotes: $x = 2$, $y = 2$



5 x-intercept: $\left(-\frac{1}{2}, 0\right)$; y-intercept: (0, 4)

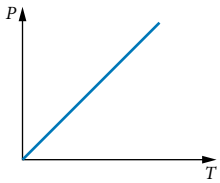
asymptotes: $x = -1$, $y = 8$



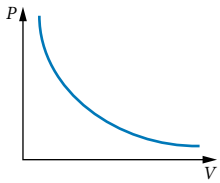
6 (a) $\frac{-7}{x+2} + 3$ (b) $\frac{5}{x+3} - 1$

(c) $\frac{-4}{x+4} + 1$ (d) $\frac{-15}{x+5} + 3$

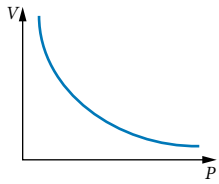
7 (a)



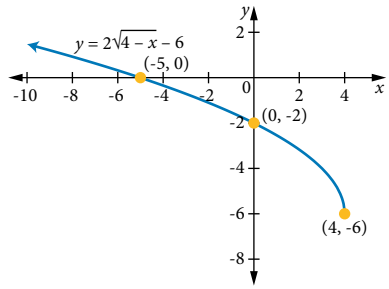
(b)



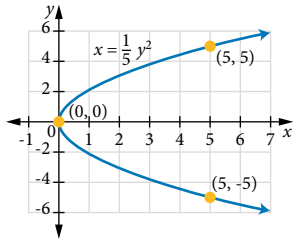
(c)



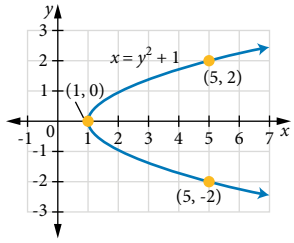
8 x-intercept: (-5, 0); y-intercept: (0, -2); vertex: (4, -6)



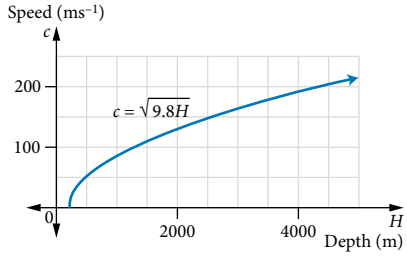
9 (a)



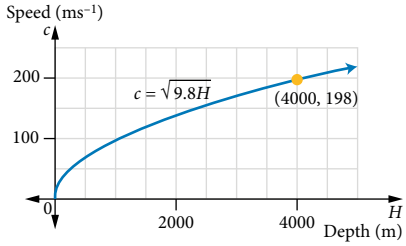
(b)



10 (a)



(b) $198\ \text{ms}^{-1}$ or 713 km/h



11 (a) $g(x) = -\sqrt{x+2} - 1$

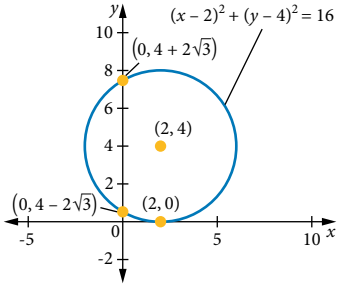
(b) $h(x) = -2\sqrt{x+4} - 1$

(c) $p(x) = -4\sqrt{x-2} - 5$

(d) $q(x) = -2\sqrt{x-3} + 6$

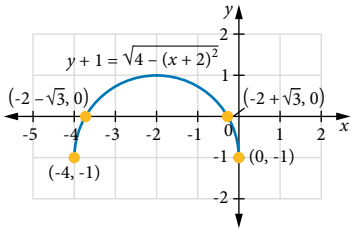
12 B

13 **(a)**



(b) Domain: [-2, 6]; Range: [0, 8]

14 **(a)**

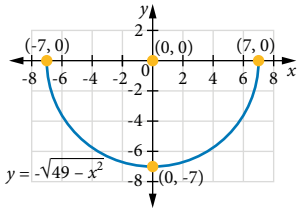


(b) Domain: [-4, 0]; Range: [-1, 1]

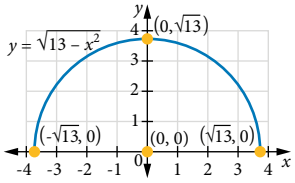
15 **(a)** inside **(b)** outside

16 **(a)** inside **(b)** outside

17 **(a)** Domain: [-7, 7]; Range: [-7, 0]



(b) Domain: $\left[-\sqrt{13}, \sqrt{13}\right]$; Range: $\left[0, \sqrt{13}\right]$

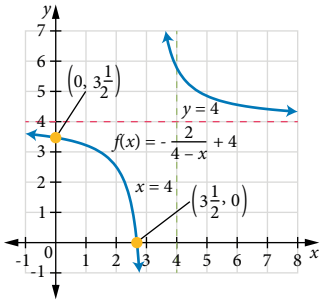


18 $y=\sqrt{16-(x+3)^2}$

19 $(x+1)^2+(y-2)^2=81$

20 $y=-\sqrt{16-(x-2)^2}$

21 **(a)** A **(b)**

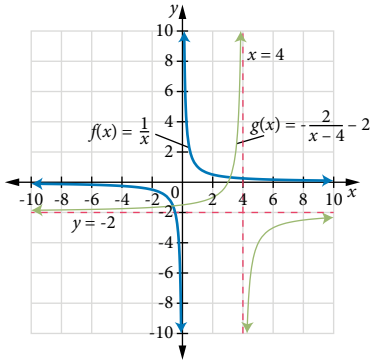


22 **(a)** C

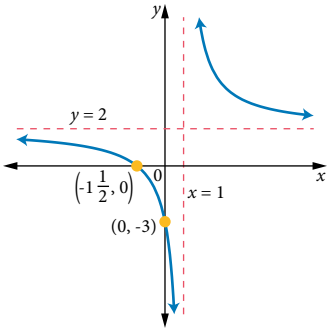
(b) Vertex: (2, -2); y-intercept: $\left(0, -\left(2+\sqrt{2}\right)\right)$

23 **(a)** $g(x)=\dfrac{-2}{x-4}-2$

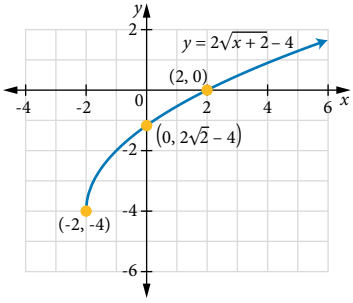
(b)



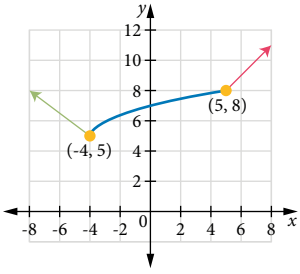
24



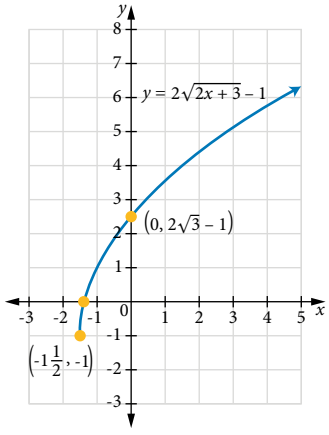
25



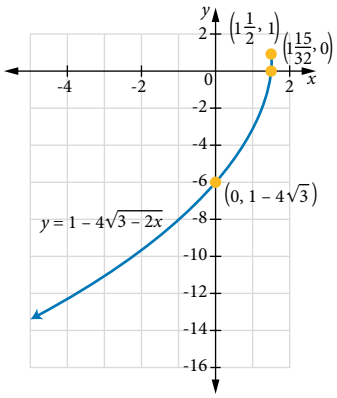
26 Domain: $(-\infty, \infty)$; Range: $[5, \infty)$



27 **(a)**

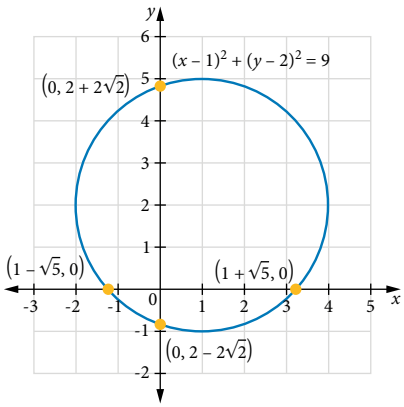


(b)



28 **(a)** Centre: (1, 2); radius: 3

(b)



(c) Upper: $y = \sqrt{9-(x-1)^2} + 2$;

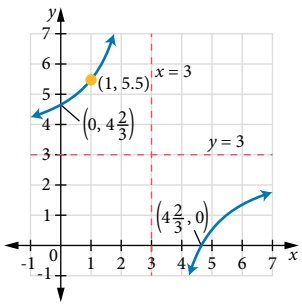
lower: $y = -\sqrt{9-(x-1)^2} + 2$

(d) $(x-3)^2 + (y+1)^2 = 9$

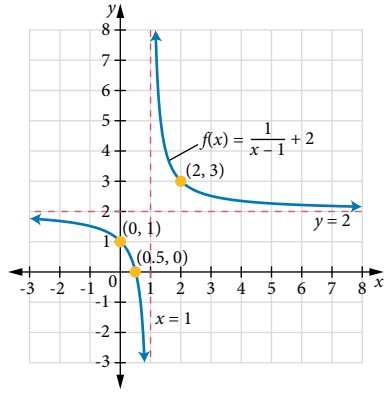
x-intercepts: $\left(3 - 2\sqrt{2}, 0\right), \left(3 + 2\sqrt{2}, 0\right)$

y-intercept: (0, -1)

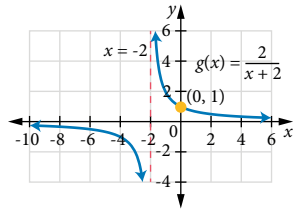
29 $y = \dfrac{3x - 14}{x - 3}$



30 **(a)**



(b)

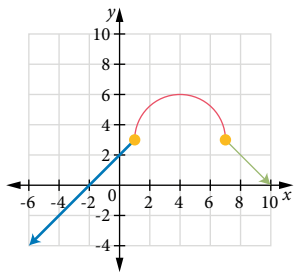


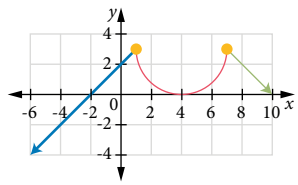
$$g(x) = \frac{2}{x+2}$$

(c) $h(x) = -\dfrac{2}{x+2}$, no x-intercepts; y-intercept is (0, -1)

(d) $p(x) = -\dfrac{2}{x-2}$, no x-intercepts, y-intercept is (0, 1)

31 $c = 10$, $g(x) = 3 + \sqrt{9-(x-4)^2}$, $g(x) = 3 - \sqrt{9-(x-4)^2}$





Chapter 5 Probability

RECALL

1 **(a)** $\dfrac{3}{4}$ **(b)** $\dfrac{1}{3}$ **(c)** $\dfrac{1}{3}$

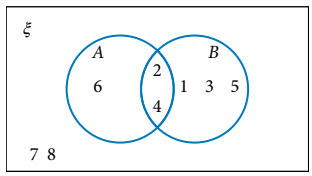
2 **(a)** 4 **(b)** 13 **(c)** 12

3 **(a)** 8 **(b)** 3 **(c)** 1

4 **(a)** $A = \{2, 4, 6, 8, 10\}$ **(b)** $B = \{10\}$

(c) $C = \{1, 2, 5, 10\}$

5



6 (a) $\frac{1}{6}$ (b) $\frac{1}{5}$ (c) 1

7 (a)

	A	A'	
B	0.15	0.15	0.3
B'	0.4	0.3	0.7
	0.55	0.45	1

(b) (i) 0.15 (ii) 0.7

8

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

EXERCISE 5.1

1 (a) 0.33 (b) 0.2

2 (a) $\frac{3}{10}$ (b) $\frac{7}{10}$ (c) $\frac{1}{10}$ (d) $\frac{9}{10}$ (e) $\frac{1}{5}$ (f) $\frac{4}{5}$

3 20

4 (a) $\frac{2}{3}$ (b) $\frac{9}{19}$ (c) $\frac{1}{13}$ (d) 25%

5 (a) $\frac{22}{57}$ (b) $\frac{5}{57}$ (c) $\frac{35}{57}$ (d) $\frac{37}{57}$ (e) $\frac{20}{57}$ (f) $\frac{49}{57}$

6 B

7 (a) 0.5 (b) 0.45 (c) $\frac{13}{30}$ (d) 0.27 (e) 0.37

8 (a) $\frac{1}{4}$ (b) $\frac{3}{10}$ (c) $\frac{7}{10}$ (d) 0 (e) $\frac{9}{20}$

9 (a) $\frac{13}{30}$ (b) $\frac{7}{8}$ (c) $\frac{41}{60}$

10 (a) D

(b) The student has not subtracted from 1.

11 (a) $\frac{16}{225}$ (b) $\frac{32}{225}$ (c) $\frac{49}{225}$ (d) $\frac{81}{225}$

12 (a) $\frac{3}{110}$ (b) $\frac{6}{55}$ (c) $\frac{2}{11}$ (d) $\frac{7}{22}$ (e) $\frac{15}{22}$ (f) $\frac{3}{11}$ (g) $\frac{9}{55}$ (h) $\frac{1}{55}$ (i) $\frac{2}{11}$

13 (a) $\frac{1}{15}$ (b) $\frac{1}{3}$ (c) $\frac{1}{5}$

14 (a) $\frac{1}{36}$ (b) $\frac{1}{6}$ (c) $\frac{1}{36}$ (d) $\frac{1}{36}$ (e) $\frac{1}{18}$ (f) $\frac{1}{9}$

15 25

16 (a) $\frac{4}{9}$ (b) $\frac{6}{11}$

17 (a) No. The game is still not fair. Xavier will lose, on average, \$1 every four games.

(b) If the game was fair, at the end of 10 games there would be a total profit/loss of \$0.

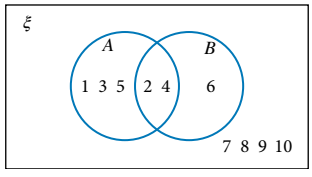
(c)

Game number	Result win/loss	Profit/loss (+\$2 or -\$1)	Total profit/loss
1	L	-\$1	-\$1
2	W	+\$2	+\$1
3	L	-\$1	\$0
4	L	-\$1	-\$1
5	L	-\$1	-\$2
6	L	-\$1	-\$3

(d) You will have your own answer but, on the basis of the trials above, the game is biased against Xavier.

EXERCISE 5.2

1 (a)



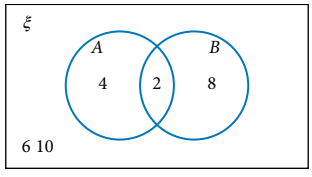
(b) (i) $A' = \{6, 7, 8, 9, 10\}$

(ii) $B' = \{1, 3, 5, 7, 8, 9, 10\}$

(iii) $A \cup B = \{1, 2, 3, 4, 5, 6\}$

(iv) $A \cap B = \{2, 4\}$

2 (a)

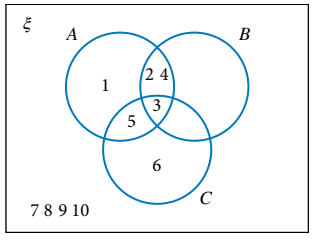


(b) (i) $A' = \{6, 8, 10\}$ (ii) $A \cup B = \{2, 4, 8\}$

(iii) $A \cap B = \{2\}$

3 (a) C (b) Need to subtract $n(A \cap B)$. (c) C

4



5 (a) $\in$ (b) $\subset$ (c) $\notin$ (d) $\subseteq$ (e) $\subset$ (f) $\varnothing$

6 (a) true (b) false (c) true (d) true (e) false (f) false (g) true (h) true

7 (a) $A \cap B$ (b) B (c) $(A \cup B)'$
(d) $A' \cup B$ (e) $A \cap B'$ (f) ξ

8 (a) A' (b) $A \cup B$ (c) $(A \cup C)'$
(d) $(A \cup B \cup C)'$ (e) $A \cap C$ (f) $A \cup C'$
(g) $(A \cup B') \cap C'$ (h) $B \cap A'$ (i) $C \cup B'$
(j) $A \cap B \cap C$

9 (a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{6}$ (d) $\frac{2}{3}$

10 (a) $\frac{3}{20}$ (b) $\frac{1}{20}$ (c) $\frac{1}{4}$ (d) $\frac{3}{20}$

11 (a) $\frac{13}{20}$ (b) $\frac{1}{2}$ (c) $\frac{3}{20}$
(d) $\frac{1}{2}$ (e) $\frac{1}{2}$ (f) 0

12 (a) 0.75 (b) $P(K \text{ and } J) = 0.15$
(c) $P(R) = 0.45$

13 (a) $\frac{7}{13}$ (b) $\frac{4}{13}$ (c) $\frac{8}{13}$ (d) $\frac{43}{52}$

14 (a) 1 (b) 0.8 (c) 0.5 (d) 0.7

15 0.15 16 D

17 (a) {a, b, c} (b) {d, k, w}

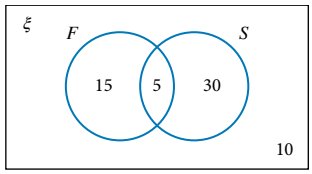
18 (a) {1, 2, 3, 6, 7, 8, 9} (b) {1, 2, 3, 7}
(c) {2, 4, 5, 6, 8, 9, 10} (d) {1, 3, 4, 5, 7, 10}
(e) {2, 3, 4, 5, 6, 8, 9, 10}

19 (a) (i) {6, 7} (ii) {1, 2, 3, 4, 5, 8, 9, 10, 11, 12}
(iii) {6, 7} (iv) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
(v) {11, 12} (vi) {11, 12}
(b) $A \cap B$ and $(A' \cup B')'$ are the same.
(c) $(A \cup B)'$ and $A' \cap B'$ are the same.

20 (a) $\frac{13}{16}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$
(d) $\frac{1}{16}$ (e) $\frac{3}{16}$ (f) 0

21 (a) $\frac{5}{44}$ (b) $\frac{3}{22}$ (c) $\frac{4}{11}$ (d) $\frac{15}{44}$

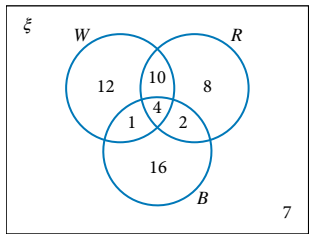
22 (a)



(b) (i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{12}$

23 (a) $\frac{13}{28}$ (b) $\frac{5}{14}$ (c) $\frac{3}{10}$ (d) $\frac{3}{7}$

24



25 (a) $\frac{1}{3}$ (b) $\frac{7}{12}$ (c) $\frac{1}{2}$ (d) $\frac{7}{10}$
(e) $\frac{2}{5}$ (f) $\frac{1}{12}$ (g) $\frac{59}{60}$

26 (a) $\frac{13}{28}$ (b) $\frac{1}{120}$ (c) $\frac{7}{30}$

EXERCISE 5.3

1 (a) $P(A \mid B) = \frac{5}{6}$, $P(B \mid A) = \frac{10}{11}$
(b) $P(A \mid B) = \frac{1}{2}$, $P(B \mid A) = \frac{2}{3}$
(c) $P(A \mid B) = \frac{5}{6}$, $P(B \mid A) = \frac{10}{13}$

2 (a) $P(A \mid B) = \frac{1}{3}$, $P(B \mid A) = \frac{1}{17}$
(b) $P(A \mid B) = \frac{4}{9}$, $P(B \mid A) = \frac{4}{13}$
(c) $P(A \mid B) = \frac{2}{5}$, $P(B \mid A) = \frac{1}{2}$
(d) $P(A \mid B) = \frac{3}{10}$, $P(B \mid A) = \frac{1}{4}$

3 (a)

	A	A'	
B	0.05	0.4	0.45
B'	0.35	0.2	0.55
	0.4	0.6	1

$P(A \mid B) = \frac{1}{9}$, $P(B \mid A) = \frac{1}{8}$

(b)

	A	A'	
B	0.2	0.1	0.3
B'	0.55	0.15	0.7
	0.75	0.25	1

$P(A \mid B) = \frac{2}{3}$, $P(B \mid A) = \frac{4}{15}$

(c)

	A	A'	
B	0.15	0.2	0.35
B'	0.4	0.25	0.65
	0.55	0.45	1

$P(A \mid B) = \frac{3}{7}$, $P(B \mid A) = \frac{3}{11}$

(d)

	A	A'	
B	0.1	0.25	0.35
B'	0.55	0.1	0.65
	0.65	0.35	1

$P(A \mid B) = \frac{2}{7}$, $P(B \mid A) = \frac{2}{13}$

4 (a) $\frac{1}{3}$ (b) $\frac{1}{3}$ (c) $\frac{5}{11}$

5 (a) $P(A \cup B) = 0.05$, $P(B \mid A) = 0.071$

(b) $P(A \cup B) = 0.6$, $P(B \mid A) = 0.706$

(c) $P(A \cup B) = 0.39$, $P(B \mid A) = 0.534$

6 B

7 (a) D

(b) The error was reading $P(A \mid B)$ as $P(B \mid A)$.

8 (a) $\frac{1}{4}$ (b) $\frac{1}{13}$ (c) $\frac{3}{13}$ (d) $\frac{1}{2}$

9 $\frac{1}{4}$

10 (a) $P(\text{WW}) = 0.36$

(b) $P(\text{wins match}) = 0.648$

(c) $P(\text{wins match} \mid \text{wins first set}) = 0.84$

(d) $P(\text{wins match} \mid \text{loses first set}) = 0.36$

(e) $P(\text{wins match} \mid \text{wins first set}) = 0.8208$

11 (a)

	male	female	
blue-eyed	0.14	0.06	0.2
not blue-eyed	0.21	0.59	0.8
	0.35	0.65	1

(b) (i) 0.06 (ii) 0.21 (iii) 0.59

(c) (i) 295 (ii) 70

12 (a) 0.2 (b) 0.828 (c) 0.205 (d) 100

(e) 0.25 (f) 0.8 (g) 0.214

(h) Yes. The experimental and theoretical values are very similar, so the conditional probability rule predicts experimental results fairly well. A close match would be expected because of the reasonably large sample size of 60.

13 0.5535

14 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$

15 (a) $\frac{8}{27}$ (b) $\frac{5}{54}$ (c) $\frac{13}{420}$ (d) $\frac{11}{2520}$ (e) $\frac{1069}{2520}$

EXERCISE 5.4

1 (a) dependent (b) independent

(c) dependent

2 (a) dependent (b) dependent

(c) dependent

3 $\frac{1}{2}$

4 (a) $P(D \cap K) = 0.1575$ (b) $P(K) = 0.625$

(c) $P(D \cup K) = 0.73$ (d) 0.5

5 (a) A

(b) The student has added the probabilities instead of multiplying them.

6 B

7 (a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{9}$ (d) $\frac{1}{6}$

8 (a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{31}{60}$

9 (a) $\frac{1}{12}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$

10 $\frac{5}{12}$

11 (a) $\frac{1}{9}$ (b) $\frac{4}{9}$ (c) $\frac{2}{9}$ (d) $\frac{2}{9}$

12 (a) $\frac{1}{10}$ (b) $\frac{2}{5}$ (c) $\frac{1}{60}$ (d) $\frac{13}{30}$

13 $\frac{2}{3}$

14 $\frac{31}{54}$

15 (a) 0.256 (b) 0.314 (c) 0.37225

16 (a) $\frac{48}{95}$ (b) $\frac{7}{25}$ (c) $\frac{41}{50}$ (d) $\frac{27}{41}$ (e) 0.35

17 (a) not independent (b) independent

(c) not independent

18 (a) A and B appear to be independent.

(b) In theory, events A and B are independent and events A and C are independent.

EXERCISE 5.5

1 30

2 (a) 56 (b) 42 (c) 98 (d) 2352

3 (a) 5040 (b) 56

(c) $\frac{1}{1680}$ (d) $\frac{1}{20}$

(e) 600 (f) 144

(g) 479 001 624 (h) 3 588 480

4 (a) 90 (b) 9900

(c) 999 000 (d) 99 990 000

5 (a) $\frac{12!}{4!} = 19\,958\,400$ (b) $\frac{11!}{3!} = 6\,652\,800$

(c) $\frac{15!}{4!} = 54\,486\,432$ (d) $\frac{16!}{11!} = 524\,160$

(e) $\frac{17!}{11!} = 8\,910\,720$ (f) $\frac{13!}{8!} = 154\,440$

6 (a) $\frac{1}{n}$ (b) $\frac{1}{(n+2)(n+1)}$

(c) $(n+2)(n+1)n(n-1)(n-2)$

(d) $\frac{1}{(n+3)(n+2)(n+1)n(n-1)}$

7 (a) $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 604\,800$

(b) $12 \times 11 \times 10 \times 9 \times 8 \times 7 = 665\,280$

(c) $15 \times 14 = 210$

(d) $16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 = 57\,657\,600$

8 30

9 $n^2 - n$

10 (a) $n = 12$ (b) $n = 10$ (c) $n = 15$ (d) $n = 22$

11 32 760

12 (a) B (b) ${}^nP_r \neq \frac{n!}{r!}$

13 D 14 336

15 (a) 49 (b) 42 (c) 14%

16 (a) 1 757 600 (b) 67 600 (c) 1 581 840

(d) 1 560 000 (e) 60 840 (f) 1 404 000

17 (a) 144 and 362 880; $a! + b! \neq (a+b)!$

(b) 3 623 760 and 6; $a! - b! \neq (a-b)!$

(c) 12 and 720; $a! \times b! \neq (a \times b)!$

(d) 239 500 800 and 720; $a! \div b! \neq (a \div b)!$

18 (a) 3024 (b) 43 680

(c) 303 600 (d) $n^2(n^2-1)(n^2-2)(n^2-3)$

19 (a) 24 (b) 24 (c) 64

20 (a) 12 (b) 50% (c) 75%

21 406 250

EXERCISE 5.6

1 (a) 28 (b) 1 (c) 924 (d) 7

2 (a) $z = 2$ (b) $t = 4, x = 4$

(c) $y = 8, w = 5$ (d) $k = 9, v = 3, q = 3$

(e) $d = 11, e = 6$ (f) $b = 10, h = 7$

(g) $f = 2, n = 2$ (h) $i = 3, u = 3$

3 (a) (i) both 15 (ii) both 56 (iii) both 210

(b) The values of ${}^nC_r$ and ${}^nC_{n-r}$ are always the same.

4 (a) $x = 2, y = 15$ (b) $k = 3, m = 56$

(c) $z = 9, x = 84$ (d) $s = 8, a = 1$

(e) $j = 9, t = 5$ (f) $d = 11, w = 462$

5 (a) 495 (b) 165 (c) 45 (d) 240

6 (a) 10 (b) 15 (c) 8 (d) 8

(e) 36 (f) 36 (g) 6435 (h) 6435

7 (a) 4 (b) 20 (c) 10 (d) 1330

8 (a) B

(b) This is ${}^7P_5$ (arrangements), not ${}^7C_5$ (selections).

9 (a) $n = 15$ (b) $m = 26$

(c) $p = 41$ (d) $r = 5$ or $r = 6$

(e) $s = 11$ or $s = 14$ (f) $t = 10$ or $t = 31$

10 (a) 5 (b) 35 (c) 15 (d) 495

11 $\binom{10}{2} \times \binom{12}{3}$

12 (a) 35 (b) 15 (c) 525 (d) 200

13 (a) 12 376 (b) 3528 (c) 4704 (d) 11 844

14 (a) $\frac{1}{28}$ (b) $\frac{1}{2}$ (c) $\frac{4}{7}$ (d) $\frac{1}{7}$

15 (a) 21 (b) 15 (c) 18 (d) 15

16 (a) 6 (b) 20 (c) 1 (d) 36

(e) 28 (f) 4 (g) 10 (h) 1

(i) 8 (j) 1 (k) 1 (l) 35

17 (a) $n = 6, r = 3$ (b) $n = 8, r = 2$ or $r = 6$

(c) $n = 7, r = 2$ or $r = 5$ (d) $n = 9, r = 1$ or $r = 8$

(e) $n = 9, r = 4$ or $r = 5$ (f) $n = 5, r = 2$ or $r = 3$

(g) $n = 5, r = 1$ or $r = 4$

(h) $n = 4, r = 2$; $n = 6, r = 1$; $n = 6, r = 5$

(i) $n = 10, r = 2$ or $r = 8$ (j) $n = 7, r = 1$ or $r = 6$

(k) $n = 10, r = 5$ (l) $n = 9, r = 3$ or $r = 6$

18 20 19 10 20 $\frac{1}{6}$

21 (a) $\frac{1}{8}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

22 31 23 60 459

24 (a) $\frac{63}{125}$ (b) $\frac{25}{26}$ (c) $\frac{1}{16}$ (d) $\frac{1}{6\,760\,000}$

25 (a) $\frac{1}{14}$ (b) $\frac{1}{13}$ (c) $\frac{1}{7}$

26 (a) $\frac{56}{195}$ (b) $\frac{2}{39}$ (c) 0

27 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$

28 $\frac{30}{143}$

29 (a) $\frac{5}{42}$ (b) $\frac{5}{14}$

30 (a) $\frac{1}{120}$ (b) $\frac{3}{10}$ (c) $\frac{3}{20}$ (d) $\frac{1}{30}$

31 (a) $\frac{3}{5}$ (b) $\frac{1}{10}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$

32 (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{1}{7}$ (d) $\frac{2}{7}$

33 (a) $\frac{9}{200}$ (b) $\frac{1}{25}$ (c) $\frac{3}{25}$

34 (a) Choose 2 results in the lower likely loss. Choose 2 is likely to give an expected loss of \$0.70 per game. Choose 3 is likely to give a loss of \$0.80 per game. Neither game make a profit in the long run for the person choosing to play.

(b) Choose 2 is likely to give a loss of \$0.80 per game. Choose 3 is likely to give a loss of \$0.90 per game. You are likely to lose the least amount of money in Choose 2.

CHAPTER REVIEW 5

1 (a) $\frac{3}{20}$ (b) $\frac{3}{10}$ (c) $\frac{4}{5}$

2 (a) $\frac{7}{16}$ (b) $\frac{3}{8}$ (c) $\frac{13}{16}$ (d) $\frac{7}{16}$

3 (a) {2, 5, 7, 13, 17, 19, 23, 29, 37}

(b) {3}

(c) {3, 11, 13, 17, 19, 23, 29, 31, 37}

(d) {13, 17, 19, 23, 29, 39}

(e) {2, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}

(f) {13, 17, 19, 23, 29, 37}

(g) {2, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}

(h) {11, 31}

4 (a) 11 (b) 3 (c) 1

(d) 18 (e) 27 (f) 10

5 (a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{3}{20}$ (d) $\frac{3}{5}$

6 (a) 0.95 (b) 0.05

7 (a) $\frac{4}{25}$ (b) $\frac{4}{25}$ (c) $\frac{29}{50}$ (d) $\frac{1}{25}$

8 0.2

9 (a) 0.4 (b) $\frac{1}{3}$ (c) $\frac{2}{5}$

10 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

11 0.64

12 (a) $\frac{10}{63}$ (b) $\frac{6}{35}$

13 0.976

14 C

15 (a) 3 603 600 (b) 3 991 680 (c) 7920 (d) 154 440

16 (a) 5040 (b) 114 (c) 17 280 (d) 15 120

17 (a) 720 (b) 120 (c) 240

18 C

19 (a) 21 (b) 15 (c) 210 (d) 56

20 (a) 5 (b) 210

21 (a) $r = 2$ or $r = 4$ (b) $n = 7$

22 (a) 20 (b) 21

23 (a) $\frac{6}{65}$ (b) $\frac{1}{91}$ (c) $\frac{6}{13}$ (d) $\frac{36}{91}$

24 70

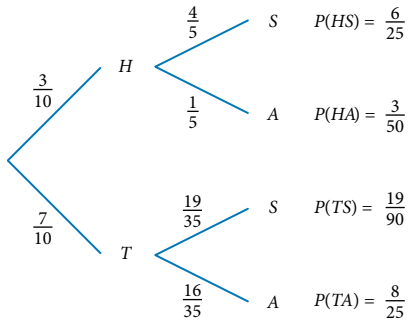
25 (a) $\frac{1}{56}$ (b) $\frac{23}{28}$

26 $\frac{{}^6C_2 \times {}^8C_2}{{}^{14}C_4}$

27 (a) 0.14 (b) $\frac{9}{28}$ (c) 0.86

28 0.616 29 D 30 A

31 (a)



(b) (i) $\frac{3}{10}$ (ii) $\frac{31}{50}$ (iii) $\frac{6}{25}$

(iv) $\frac{19}{50}$ (v) $\frac{3}{50}$ (vi) $\frac{8}{25}$

(c) no

(d) (i) $\frac{4}{5}$ (ii) $\frac{19}{35}$

32 (a) $\frac{2}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{16}$ (d) $\frac{3}{4}$

33 (a) $\frac{44}{81}$ (b) $\frac{32}{81}$ (c) $\frac{5}{9}$ (d) $\frac{25}{37}$

34 654

35 (a) $\frac{240}{1001}$ (b) $\frac{47}{1001}$ (c) $\frac{6}{143}$ (d) $\frac{360}{1001}$

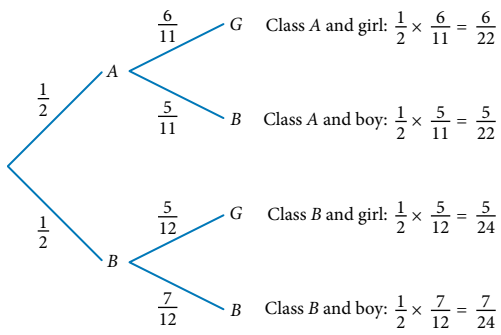
(e) $\frac{6}{11}$ (f) $\frac{72}{143}$ (g) $\frac{29}{143}$

36 (a) 0.18 (b) 0.6 (c) $\frac{9}{20}$ (d) 0.66

(e) At least 5 customers.

37 (a) $\frac{6}{11}$ (b) $\frac{5}{12}$ (c) $P(\text{girl}) = \frac{6}{22} + \frac{5}{24}$

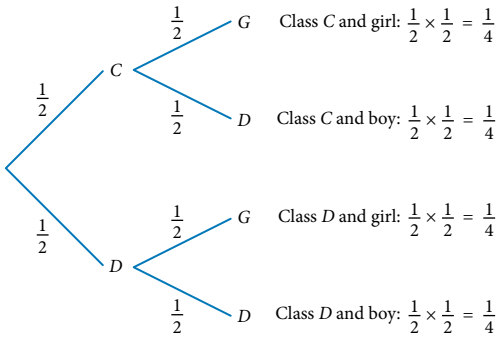
$= \frac{127}{264}$



(d) $\frac{11}{23} \neq \frac{127}{264}$ The answer would be different.

(e) The chances are different in each class, which affects the overall probability.

(f) $\frac{1}{2}$ (g) $\frac{1}{2}$ (h) $P(\text{girl}) = \frac{1}{4} + \frac{1}{4}$
$= \frac{1}{2}$



(i) yes

(j) The two classes have equal probabilities, so the overall probability will be the same.

38 (a) (i) $\frac{1504}{1827}$ (ii) $\frac{40}{49}$ (iii) $\frac{75}{91}$ (iv) $\frac{11}{13}$

(b) Strategy B: 2 lots of 15 cattle with 2 other cattle in each lot.

Chapter 6 Index laws

RECALL

1 (a) $12 = 2^2 \times 3$ (b) $36 = 2^2 \times 3^2$ (c) $162 = 2 \times 3^4$

2 (a) 7.125 (3 d.p.) (b) 23.311 (3 d.p.) (c) 7.700 (3 d.p.)

3 (a) a^5 (b) m^3n^2 (c) $\frac{5an^2}{3b^3}$

4 (a) 5 (b) 90 (c) 40

5 (a) 55.125 (b) 3

6 (a) $6y^4$ (b) $\frac{x^3}{4}$ (c) $8c^6d^3$ (d) $\frac{16m^{12}}{81h^4}$

7 (a) $3x(x+3)$ (b) $(x+3)(x-3)$ (c) $(x+9)(x-2)$

8 (a) $x = 7$ or $x = -7$ (b) $m = 400$ (c) $x = 0.82$

EXERCISE 6.1

1 (a) a^{11} (b) $6b^7$ (c) $5m^4n^7$ (d) $9a^{11}b^9$

2 (a) c^3d (b) $2a^3b$ (c) $6yz^2$ (d) $\frac{a^{15}b^5}{32c^{10}}$

3 (a) x^3y^4 (b) $\frac{3a^8b}{2}$ (c) $10c^5$ (d) 45

4 (a) 1 (b) 1 (c) $3a$ (d) 6
(e) 1 (f) $2a^3$ (g) $4 + x$ (h) 1

5 (a) $\frac{1}{x^2}$ (b) $-\frac{5}{m^2}$ (c) m^2 (d) $3ab^3$
(e) $\frac{1}{x^2}$ (f) s^3t^2 (g) $\frac{27x^4y^5}{16}$ (h) $\frac{1}{c^4d^3}$
(i) $2a^3b$

6 (a) C
(b) The student has added numbers that are factors. Only indices of powers with the same base should be added.

7 (a) $a^{\frac{1}{3}}$ (b) $x^{\frac{7}{2}}$ (c) $y^{\frac{5}{4}}$

8 (a) $\sqrt[12]{a^7}$ (b) $\sqrt{b}$ (c) $\frac{1}{\sqrt{x}}$

9 (a) 2 (b) $\frac{1}{6}$ (c) 4 (d) $\frac{1}{169}$
(e) $\frac{1}{25}$ (f) $\frac{343}{27}$

10 (a) 2^8 (b) $3^4 \times 5^3$ (c) 5^6 (d) $\frac{1}{2^3}$

11 128 lots of the original number

12 (a) $\frac{a^{14}}{b^{12}}$ (b) $\frac{m^3}{n^4}$ (c) $\frac{9x^2y^9}{z^2}$
(d) $\frac{x^9}{4y^{13}z^9}$ (e) $\frac{4b^3}{a^7}$

13 A

14 (a) $\frac{1}{\sqrt[3]{y^4}}$ (b) $\sqrt[3]{x^4}\sqrt{y^9}$

15 (a) $2^{2a-3} \times 5^{a-3}$ (b) $\frac{2^{3-3b}}{3}$ or $\frac{1}{2^{(3b-3)\times 3}}$
(c) $2^{x+2} \times 3^{3-4}$ or $\frac{2^{x+2}}{3^{4x-3}}$ (d) 2^{2x+4}

16 (a) 3 (b) $3^n - 1$ (c) $\frac{a^m + 1}{2}$

17 $\frac{x^{-2} - 1}{x^{-1} + 1} = \frac{\frac{1}{x^2} - 1}{\frac{1}{x} + 1}$
$= \frac{\left(\frac{1}{x} - 1\right)\left(\frac{1}{x} + 1\right)}{\left(\frac{1}{x} + 1\right)}$
$= \frac{1}{x} - 1$

18 $\sqrt[3]{3}$

EXERCISE 6.2

1 (a) 3.2×10^4; 2 (b) 5.72×10^4; 3
(c) 1.07×10^3; 3 (d) 1.537×10^3; 4
(e) 4.72×10^7; 3 (f) 5.213×10^7; 4
(g) 1.027×10^1; 4 (h) 5.63×10^0; 3
(i) 1.02×10^{-3}; 3 (j) 5.6×10^{-5}; 2
(k) 8.0300×10^{-2}; 5 (l) 3.960×10^{-3}; 4

2 (a) B
(b) The power of 10 only matches the number of zeros when the standard form number has a single digit.

3 (a) 1700 (b) 59 000 (c) 43.01 (d) 271
(e) 0.00030 (f) 2060 000 (g) 0.001732 (h) 0.000605
(i) 50 000 000 (j) 7.98

4 (a) 3.01×10^{16} cm (b) 4.2×10^5 mm (2 s.f.)
(c) 1.98×10^{21} km (d) 2.5×10^4 cm (2 s.f.)

(e) 8.8×10^4 mm (f) 6.127×10^4 km

5 (a) 2×10^{-8} m (b) 6×10^{-7} m (c) 4.0075×10^7 m

(d) 3.844×10^8 m, (4 s.f.) (e) 1.798×10^6 m

6 5×10^{-2} m 7 4.818×10^{24}

8 1.2×10^{23}, assuming pure gold

9 (a) 1.2×10^{10} (b) 1×10^{20} (c) 2.0×10^{-4} (d) 3.0×10^2

(e) 7.0×10^7 (f) 4.0×10^{-9} (g) 7.6×10^6 (h) 5.24×10^7

(i) -4.9×10^3

10 (a) A (b) C (c) D

11 (a) 4.9×10^9 (b) 2.26×10^6 (c) 2.7×10^{-2} (d) 1.2×10^{-2}

(e) 3.1×10^2 (f) 1.22×10^{-2} (g) 2.07×10^1 (h) 5.7×10^{-4}

12 (a) 1.64×10^{13} (b) 5.20×10^4 or \$52 000

13 (a) 4.0×10^7 (b) 6.0×10^3 (c) 2.7×10^{-5} (d) -3.9×10^4

14 8.12×10^{13} km or 81.2 trillion km

15 8.4×10^{-3} s or 8.4 ms

16 $FeSO_4$ has 2.46×10^2 more molecules. Each molecule of $FeSO_4$ has fewer atoms and hence less mass.

17 400 s (to the nearest 100 s) or 6–7 min

EXERCISE 6.3

1 (a) 5, 10, 20, 40 (b) -3, -12, -48, -192

(c) 10, 2, $\frac{2}{5}$, $\frac{2}{25}$ (d) $\frac{1}{4}$, $-\frac{1}{2}$, 1, -2

(e) 6, -6, 6, -6 (f) 20, 2, $\frac{1}{5}$, $\frac{1}{50}$

2 (a) Geometric sequence, $r = 3$

(b) Not a geometric sequence

(c) Geometric sequence, $r = -3$

(d) Geometric sequence, $r = w$

(e) Geometric sequence, $r = 1.5$

(f) Geometric sequence, $r = \sqrt{2}$

3 (a) Geometric sequence, $r = x$

(b) Geometric sequence, $r = -5$

4 (a) $t_n = 2^{n-1}$; $t_{10} = 512$

(b) $t_n = 27\left(\frac{2}{3}\right)^{n-1}$; $t_7 = \frac{64}{27}$

(c) $t_n = (-10)^{n-1}$; $t_8 = -10\,000\,000$

(d) $t_n = 3^{n-5}$; $t_8 = 27$

(e) $t_n = x^n y^{n+1}$; $t_7 = x^7 y^8$

(f) $t_n = \frac{2^{n-4}}{(-3)^{n-1}}$; $t_7 = \frac{8}{729}$

(g) $t_n = \left(v\sqrt{5}\right)^{n-3}$; $t_{10} = -125\sqrt{5}$

5 (a) B (b) $t_6 = t_1 r^5$ not $t_1 r^6$

6 (a) $S_{12} = 4095$ (b) $S_{10} = \frac{341}{64}$

(c) $S_8 = -4920$ (d) $S_{10} = 79.687$

(e) $S_9 = -1.502$ (f) $S_{12} = 5.333$

7 (a) S_∞ does not exist. (b) $S_\infty = \frac{200}{9}$

(c) $S_\infty = 1$ (d) $\frac{50}{7}$

8 (a) $\frac{1}{3}$ (b) $\frac{8}{99}$ (c) $\frac{5}{27}$

9 (a) $p = \pm 6$ (b) $p = 40.5$

(c) p can take any non-zero value

(d) $p = -\frac{81}{4}$ (e) $p = 3$ or $p = \frac{22}{7}$

10 A

11 (a) 2, 10, 50 (b) 4, $\frac{4\sqrt{3}}{3}$, $\frac{4}{3}$

(c) -12, 144, -1728 (d) $2xm^3$, $2xm^4$, $2xm^5$

12 (a) (i) 25 g (ii) 12.5 g (iii) 6.25 g

(b) 3 half-lives

13 17

14 6

15 (a) 48 419 (b) During the 27th year

16 (a) $\frac{1}{12}$ (b) $\frac{47}{198}$

17 6, 3, 1.5

18 $t_{10} = 393\,216$, $S_{10} = 524\,287.5$

19 $t_1 = 8$

20 90 m

21 (a) (i) $35 + 3^{12}$ (ii) $(3n-1) + 3^n$

(b) (i) $\frac{5}{2} + 5^6$ (ii) $\frac{n}{4} + 5^{n-4}$

(c) (i) $3^{29} \times 4^{2^{14}}$ (ii) $3^{2n-1} \times 4^{2^{n-1}}$

EXERCISE 6.4

1 (a) \$20 892.29 (b) \$8892.29

2 (a) \$1490.76 (b) \$1496.61 (c) \$1499.66

(d) \$1501.74 (e) \$1502.75

3

Compound interest	Principal	Rate (p.a.)	Compounding period	Time (years)	Final balance
\$1352.45	\$5000	6%	Monthly	4	\$6352.45
\$2479.58	\$7520.42	9.5%	Daily	3	\$10 000
\$59.60	\$800	7.25%	Quarterly	1	\$859.60

4 (a) 11 years (b) 17 years (c) 29 years

5 (a) \$3801 (b) \$10 197

6

Present value	Inflation rate (p.a.)	Time (years)	Future value
\$250	1.6%	5	\$271
\$1284	3.4%	3	\$1420
\$610	7.1%	2	\$850
\$1100	2.9%	4	\$1233

7 (a) B

(b) Dividing by the power expression is used when the original value is not known.

8 A

9 \$940

10 (a) \$40 425.31 (b) \$667 639.17

11 Investment Option 1 is the better option by \$69.32.

12 (a) \$10 000

(b) (i) \$12 577.89 (ii) \$12 772.33

(iii) \$12 940.19 (iv) \$12 973.30

(c) Compound interest is 26% more.

(d) Compounding daily gives 3% more.

13 7 years; paper value: \$405; actual value: \$433

14 \$13 949.62

15 (a) \$21 474 836.47 (b) \$10 737 418.23 (c) \$2 684 354.56

16 (a) 287 (b) No; shortfall: \$277 884

17 (a) 9 June (b) 8 October

CHAPTER REVIEW 6

1 (a) $\frac{1}{ab}$ (b) $25a^{\frac{2}{3}}$ (c) $3a^{\frac{5}{6}}b^{\frac{7}{6}}$

2 $\frac{3y}{5}$ 3 $\frac{5a^5}{6}$ 4 B 5 D

6 $5.13 \times 10^{-6}\ \text{gL}^{-1}$ 7 125, 625, 3125 8 $t_n = 10^{n-2}$

9 (a) -6, -3, $-\frac{3}{2}$, $-\frac{3}{4}$ (b) 0.4, 2, 10, 50

10 3, 6, 12; $a_8 = 384$

11 (a) Yes, the common ratio is x.

(b) Yes, the common ratio is -3.

(c) No, there is no common ratio.

(d) Yes, the common ratio is $\sqrt{2}$.

12 $S_{10} = \frac{29\,524}{19\,683}$ 13 $14\frac{2}{5}$

14 (a) Common ratio: -0.5 ; next term -0.5

(b) Common ratio: 1.5 ; next term: 126.5625

(c) The sequence is not geometric.

15 \$6360 16 \$2145.93 17 \$115 506.88

18 C 19 A

20 $X = 1$, $Y = 9$ or $X = 9$, $Y = 1$ 21 $\frac{71}{450}$

22 Daily compounding gives an increase of \$97.35.

23 \$37 908 24 \$922 237.03 25 $P = \$3769.07$

26 (a) \$7330.56 (b) $A_d = 9000(0.95)^{n-1}$

(c) 26 years

27 (a) $PV = \frac{P(1+r)\left(1-(1+r)^{-n}\right)}{r}$ (b) \$2067.32

Unit 1 Exam review: Chapters 1–6

PAPER 1: TECHNOLOGY-FREE

1 (a) Not direct variation, $y = 2x + 2$, $x \in \{0, 1, 2, 3\}$

(b) Direct variation, $y = \frac{1}{2}x$, $x \in \mathbb{R}$

2 (a) $\frac{4}{3}$ (b) $-\frac{2}{5}$ (c) 2

3 $y = (x-3)^2 - 6$

4 (a) Degree: 3; $P(3) = 343$ (b) Degree: 5; $Q(-2) = 119$

(c) Degree: 5; $R(-1) = -243$ (d) Degree: 6; $S(2) = 2304$

(e) Degree: 4; $T(-4) = 0$ (f) Degree: 8; $V(-4) = 7776$

5 (a) 12 (b) -13 (c) 47 (d) 30

(e) 51 (f) $4x^2 + 1$ (g) $n = \pm 5$ (h) $n = \pm 7$

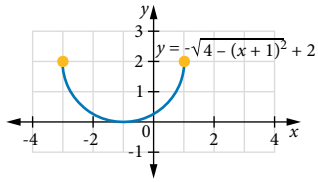
6 $a = 2$, $b = 1$ and $c = -3$

7 (a) Reflection in the x-axis; $g(x) = -\sqrt{x}$

(b) Translation: 1 unit to the right; $h(x) = -\sqrt{x-1}$

(c) Translation: 2 units up; $k(x) = -\sqrt{x-1} + 2$

8 Centre: (-1, 2); x-intercept: (-1, 0); y-intercept: $\left(0, 2-\sqrt{3}\right)$; endpoints: (-3, 2), (1, 2)



9 (a) {f, g, h, i, j} (b) {a, b, c}

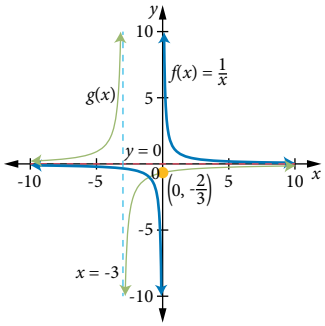
(c) {a, b, c, d, e, f, g, h, i, j} (d) {a, b, c, d, e, k, l, m, n}

(e) {a, b, c, f, g, h, i, j, k, l, m, n} (f) {k, l, m, n}

10 (a) $4m^5n^8$ (b) $5a^3b$ (c) $\frac{xz^6}{4y}$ (d) $\frac{24b^4}{a^4}$

11 $y = \begin{cases} 2x^2 - 6x + 1, & x \le -1 \\ -2x + 7, & -1 < x \le 3 \\ 10 - x^2, & x > 3 \end{cases}$

12 $g(x) = -\dfrac{2}{x+3}$



PAPER 2: TECHNOLOGY-ACTIVE

1 (a) The formula does represent a function.

(b) 86°F (c) 36.7°C

2 (a) (-8, 0) (b) (3, 1) (c) (0, -9)

(d) $\left(-\dfrac{9}{8}, \dfrac{129}{16}\right)$ (e) (0, 3.2) (f) (0, 0)

3 $x = \dfrac{5-\sqrt{53}}{14}$ or $x = \dfrac{5+\sqrt{53}}{14}$

$x = -0.16, x = 0.88$

4 (a) $\dfrac{1}{9}$ (b) $\dfrac{5}{9}$

5 1260 6 $P(x) = 2(x+1.76)(x-0.18)(x-1.58)$

7 $$y = \begin{cases} -\frac{1}{2}(x-5)^2 + 5 & 3 < x \le 7 \\ 1 + \sqrt{4-(x-3)^2} & 3 \le x < 5 \\ 1 - \sqrt{4-(x-3)^2} & 3 < x \le 5 \\ -2x + 5 & 3 \le x < 5 \\ 4x - 25 & 5 \le x < 7 \end{cases}$$

8 $\dfrac{29}{72}$

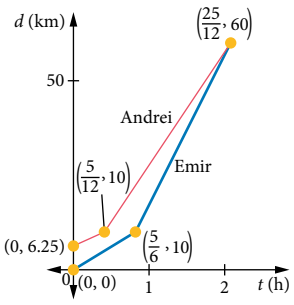
9 (a) 1.2788×10^4 days (b) 1.426×10^6 km/day

(c) 5.940×10^4 km/h (d) 1.650×10^4 m/s

10 6.25 km

$$d_{\text{Emir}} = \begin{cases} 12t, & 0 \le t \le \frac{5}{6} \\ 40t - 23\frac{1}{3}, & \frac{5}{6} < t \le \frac{25}{12} \end{cases}$$

$$d_{\text{Andrei}} = \begin{cases} 6.25 + 9t, & 0 \le t \le \frac{5}{12} \\ 30t - 2.5, & \frac{5}{12} < t \le \frac{25}{12} \end{cases}$$

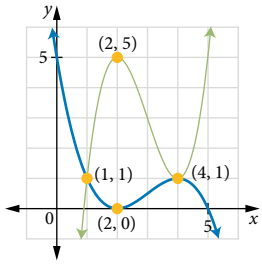


11 Possible solution:

intersection at (1, 1) and (4, 1):

$y = (x-1)(x-4)^2 + 1$

$y = -0.25(x-1)(x-4)^2 + 1$



12 18 446 744 073 709 551 616 or about 1.8445×10^{19}

Chapter 7 Exponential and logarithmic functions

RECALL

1 (a) $x = 8$ (b) $x = -15$ (c) $x = 5$

2 (a) $3x^{n+1}$ (b) x^{2n-1} (c) $x^{\frac{5}{2}}$

3 (a) $(x+1)(x-1)$ (b) $(x+3)(x-5)$ (c) $(3x-5)(x-2)$

4 (a) $x = -3, x = 2$ (b) $x = -5, x = 0$ (c) $x = -6, x = 1$

5 (a) (-2, 0), (2, 0); (0, -4) (b) $\left(2-\sqrt{6}, 0\right), \left(2+\sqrt{6}, 0\right)$; (0, 2)

(c) No x-intercepts; (0, 14)

6 (a) Dilation by factor 3 parallel to the y-axis. Reflection in the x-axis.

(b) Dilation by factor $\dfrac{1}{2}$ parallel to the x-axis. Reflection in the y-axis.

7 (a) up 2 (b) left 2

EXERCISE 7.1

1 (a) $x = 3$ (b) $x = 4$ (c) $x = \dfrac{1}{2}$ (d) $x = -2$

(e) $x = \dfrac{3}{2}$ (f) $x = \dfrac{3}{2}$

2 (a) -6 (b) $x = -\dfrac{3}{2}$ (c) $x = -\dfrac{2}{3}$ (d) $x = 2$

(e) $x = 4$ (f) $x = \dfrac{2}{3}$ (g) $x = \dfrac{5}{6}$ (h) $x = \dfrac{1}{8}$

3 (a) $x = \dfrac{1}{3}$ (b) $x = \dfrac{2}{5}$ (c) $x = \dfrac{1}{4}$ (d) $x = 4$

(e) $x = 0$ (f) $x = -4$

4 (a) $a = 2.32$ (b) $b = 1.05$ (c) $c = 1.61$ (d) $d = 1.21$

(e) $e = 0.30$ (f) $f = 1.89$

5 (a) A

(b) $x^{\frac{3}{2}} = \sqrt{x^3}$ and the square root of a negative is not defined. There would be two solutions if the even number was on the numerator.

6 B

7 (a) $a = \pm 2$ (b) $b = 3$ (c) $c = 32$ (d) $d = 81$

(e) $m = \pm\sqrt{2}$

8 (a) $x = \pm\sqrt{30}, x = \pm 5.48$ (b) no solution

(c) $x = \sqrt[3]{30}$, $x = 3.11$ (d) $x = -\sqrt[3]{30}$, $x = -3.11$

(e) $x = \pm\sqrt[4]{30}$, $x = \pm 2.34$ (f) no solution

(g) $x = \sqrt[5]{30}$, $x = 1.97$ (h) $x = -\sqrt[5]{30}$, $x = -1.97$

9 $x = \dfrac{1}{4}$

10 (a) $a = 0$ or $a = 1$ (b) $b = 1$ or $b = 2$

(c) $c = 0$ (d) $d = \dfrac{1}{2}$

11 (a) $5^{2x} + 7 \times 5^x + 10 = 0$ has no real solutions because, although it can be factorised the only possible solutions are $5^x = -5$ and $5^x = -2$, and powers with positive bases are positive for all values of the index.

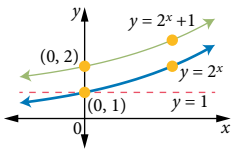
(b) $3^{2x} - 2 \times 3^x + 11 = 0$ has no real solutions because $a^2 - 2a + 11 = 0$ has no real solutions. Looking at the discriminant, $b^2 - 4ac$, the result is less than zero, so there are no solutions for 3^x and hence no solutions for x.

12 $x = 2\dfrac{1}{2}$ 13 $x = \dfrac{4}{5}$ 14 $x = -1, x = 3$

15 $x \geq 1$ 16 $x = 0, x = 1$

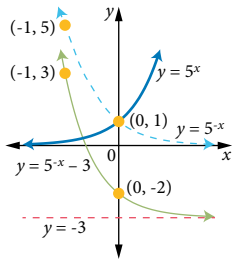
EXERCISE 7.2

1 (a)



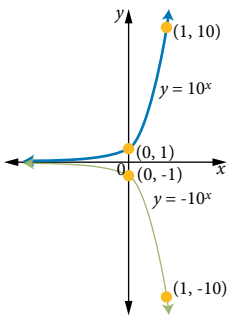
Translation: up 1

(b)



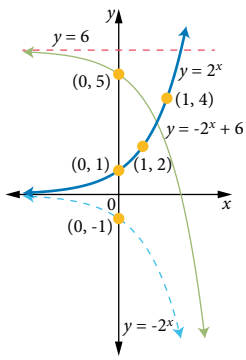
Reflection in the y-axis; translation: down 3

(c)



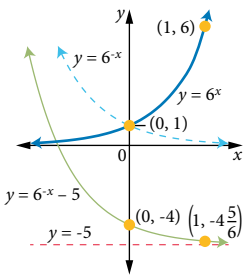
Reflection in the x-axis

(d)



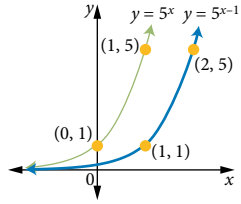
Reflection in x-axis; translation: up 6

(e)



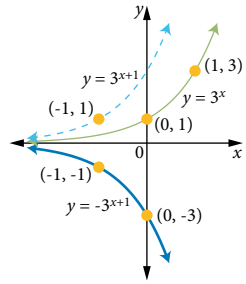
Reflection in y-axis; translation: down 5

2 (a)



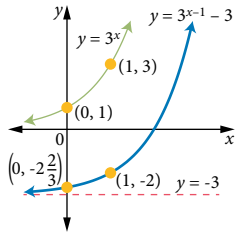
Translation: right 1

(b)



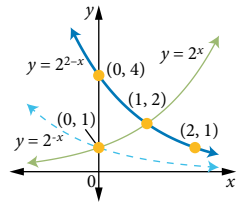
Translation: left 1; reflection in x-axis

(c)



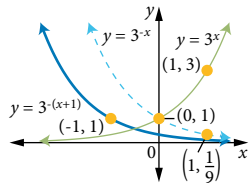
Translation: right 1 and down 3

(d)



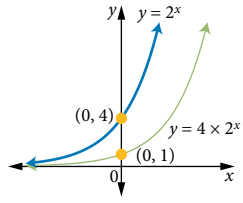
Reflection in y-axis; translation: right 2

(e)



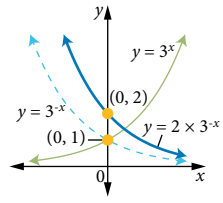
Reflection in y-axis; translation: left 1

3 (a)



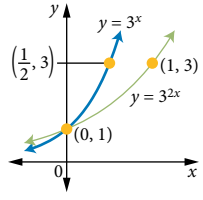
Dilation parallel to y-axis by a factor of 4

(b)



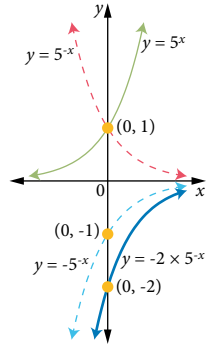
Reflection in y-axis; dilation parallel to the y-axis by a factor of 2

(c)



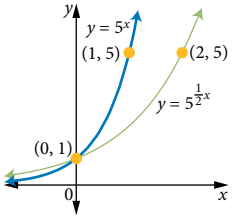
Dilation parallel to the x-axis by a factor of $\dfrac{1}{2}$

(d)



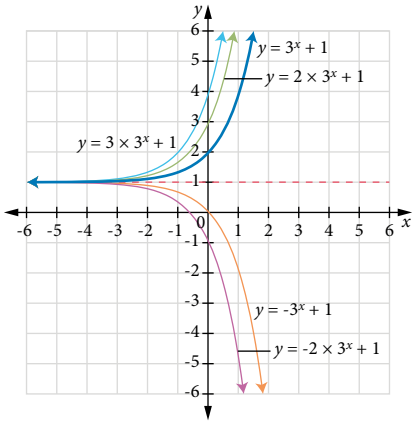
Reflection in both axes; dilation parallel to the y-axis by a factor of 2

(e)



Dilated parallel to the x-axis by a factor of 2

4



Rule	Asymptote	Size of dilation factor (parallel to the y-axis)	Reflection in x-axis	y-intercept
$y = 3^x + 1$	$y = 1$	1	no	(0, 2)
$y = 2 \times 3^x + 1$	$y = 1$	2	no	(0, 3)
$y = 3 \times 3^x + 1$	$y = 1$	3	no	(0, 4)
$y = -3^x + 1$	$y = 1$	1	yes	(0, 0)
$y = -2 \times 3^x + 1$	$y = 1$	2	yes	(0, -1)

5 $y = 3 - 2^{3x}$

6 D

7 (a) A

(b) The student may have forgotten to subtract the constant value of 3.

8 (a) $f(x) = 3 \times 4^x$

(b) $f(x) = \frac{1}{5} \times 4^x$

(c) $f(x) = -4^x$

(d) $f(x) = 4^{2x}$

(e) $f(x) = 4^{-x}$

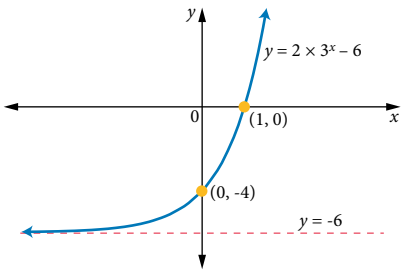
(f) $f(x) = 4^x - 7$

(g) $f(x) = 4^{x-3}$

(h) $f(x) = 4^{\frac{1}{3}x}$

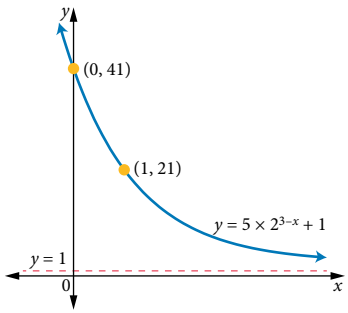
9 $f(-2) = 5$; $f(0) = -1$; constant term: -3

10 (a)



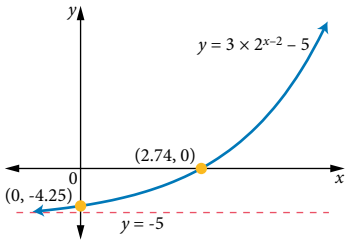
Domain: $\mathbb{R}$; Range: $(-6, \infty)$

(b)



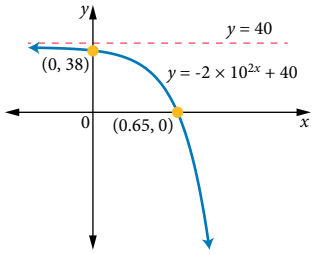
Domain: $\mathbb{R}$; Range: $(1, \infty)$

(c)



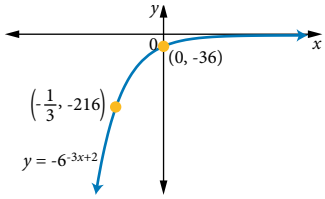
Domain: $\mathbb{R}$; Range: $(-5, \infty)$

(d)



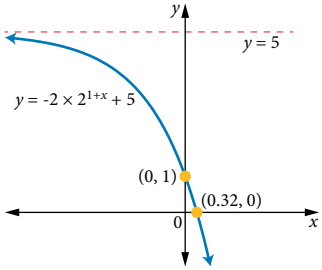
Domain: $\mathbb{R}$; Range: $(-\infty, 40)$

(e)



Domain: $\mathbb{R}$; Range: $(-\infty, 0)$

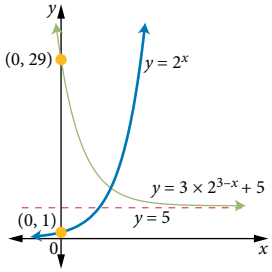
(f)



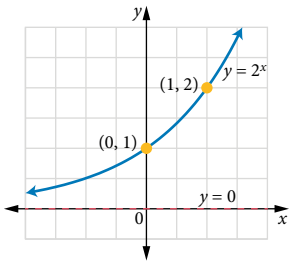
Domain: $\mathbb{R}$; Range: $(-\infty, 5)$

11 (a) $f(x) = 3 \times 2^{-(x-3)} + 5$ or $f(x) = 3 \times 2^{3-x} + 5$

(b) Graphs of $f(x) = 2^x$ and $f(x) = 3 \times 2^{3-x} + 5$:



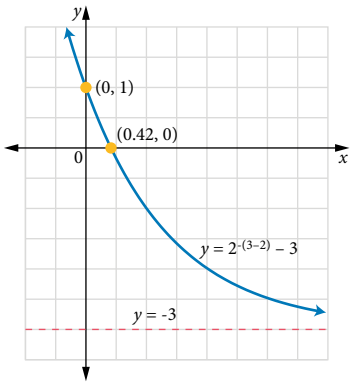
12 **(a)**



Domain: $x \in \mathbb{R}$; Range: $y \in \mathbb{R}^+$; Asymptote: $y = 0$; y-intercept: (0, 1)

(b) $g(x) = 2^{-(x-2)} - 3$ or $g(x) = 2^{2-x} - 3$

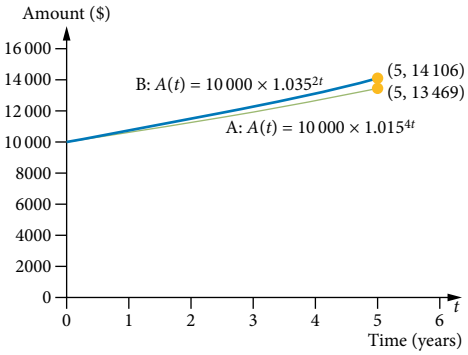
(c)



Domain: $x \in \mathbb{R}$; Range: $y > -3$; asymptote: $y = -3$; y-intercept: (0, 1); x-intercept; (0.42, 0)

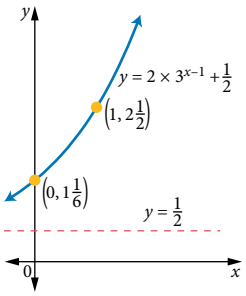
13 **(a)** A: $A(t) = 10\,000 \times 1.015^{4t}$; B: $A(t) = 10\,000 \times 1.035^{2t}$

(b)



(c) Bank B

14



Domain: $x \in \mathbb{R}$, Range: $y > \frac{1}{2}$

15 **(a)** $y = 2^{x+2} - 2$ **(b)** $y = \frac{20}{3}\left(2^{\frac{x}{25}} - 1\right)$

EXERCISE 7.3

1 **(a)** $\log_5(25) = 2$ **(b)** $\log_3(81) = 4$

(c) $\log_{10}(0.001) = -3$ **(d)** $\log_6(216) = 3$

(e) $\log_2(17) = x$ **(f)** $\log_3(23) = -x$

(g) $\log_8\left(\sqrt{2}\right) = \frac{1}{6}$ **(h)** $\log_{10}(63) = 2x + 3$

(i) $\log_{10}(2) = 0.301$ **(j)** $\log_{10}(50) = 1.699$

2 **(a)** C

(b) The student has mixed up the base and the index.

3 **(a)** 1 **(b)** 0 **(c)** 2 **(d)** 3

(e) 2 **(f)** 3

4 **(a)** $x = \log_5(37)$ **(b)** $x = 0$

(c) $x = -\log_2(12)$ or $x = \log_2\left(\frac{1}{12}\right)$

(d) $x = 4$ **(e)** $x = -\log_{10}(15)$

(f) $x = \log_5(8)$ **(g)** $b = \log_6(90)$

(h) $c = \frac{1}{5}\log_2\left(\frac{5}{2}\right)$ **(i)** $d = 7$

(j) $x = \log_3(2)$ **(k)** no solution

5 D

6 **(a)** 4.09 **(b)** 2.13 **(c)** 7.57 **(d)** 0.86

7 **(a)** $(\log_5(3), 3)$ **(b)** $(\log_6(2), -1)$

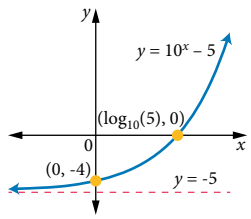
8 **(a)** $(0.79, 0.58)$ **(b)** $(-0.55, 0.55)$

(c) $(-0.39, 1.31)$

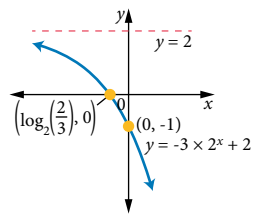
9 $x = \log_5(130)$, $x \approx 3.02$

10 **(a)** $x = -\frac{59}{2}$ **(b)** $x = 3$

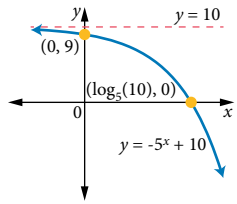
11 **(a)**



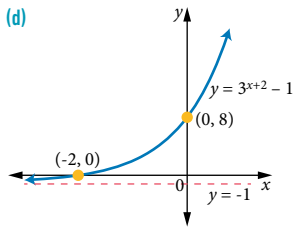
(b)



(c)



(d)



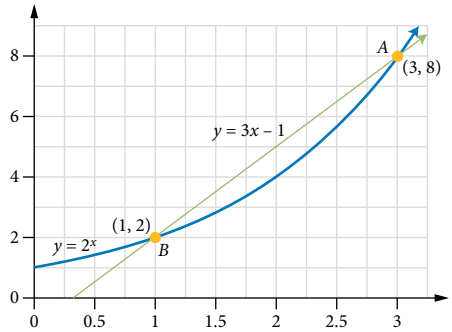
12 **(a)** $x = \log_3(5)$ **(b)** $a = 1$ or $a = \log_5(12)$

(c) $m = 1$ **(d)** $b = \log_7(3)$

(e) $n = 1$ or $n = \log_3(4)$ **(f)** no solutions

13 $x = \log_3\left(\frac{2}{3}\right)$ or $x \approx -0.37$

14 **(a)**



The solutions to $2^x = 3x - 1$ are $x = 1$ and $x = 3$.

(b) $x \le 1, x \ge 3$

15 $x = \dfrac{2y}{3}$

16 $2P = P\left(1 + \dfrac{0.08}{4}\right)^{4\times T}$

$2 = (1.02)^{4T}$

$T = \dfrac{1}{4}\log_{1.02}(2)$ years

After $8\dfrac{3}{4}$ years the principal has almost doubled.

After 9 years the principal has (more than) doubled for the first time.

EXERCISE 7.4

1 **(a)** $N_0 = 40.00$ **(b)** 35.78 mg

(c) $k = 0.050$ **(d)** $N(t) = 40 \times (0.8)^{0.050t}$

(e) 29 mg **(f)** 62 days

(g) 373

2 **(a)** $V(t) = 150\,000 \times (1.12)^t$ **(b)** \$114 351

(c) 9.69 years

3 **(a)** $V(t) = 378\,000 \times (1.07)^t$ **(b)** \$2 877 000

(c) \$1 195 200 **(d)** yes

4 **(a)** $V(t) = 400\,000(0.82)^t$ **(b)** \$72 000

(c) \$148 296 **(d)** \$26 214

(e) 7 years

5 **(a)** B

(b) $a^0 = 1$, so substituting $t = 0$ does not give the constant term in an exponential function.

6 D

7 **(a)** 11.73%

(b) $P(t) = 5897 \times (1.1173)^{0.5t}$

(c) 10 years

8 **(a)** 23°C **(b)** $A = 60$

(c) $z = 0.939$ **(d)** $T = 60(0.939)^t + 23$

(e) 32.1°C **(f)** 39 min 29 s

9 **(a)** $A = 3$, $a = 2$ and $k = 4$ **(b)** 2 hours 56 minutes

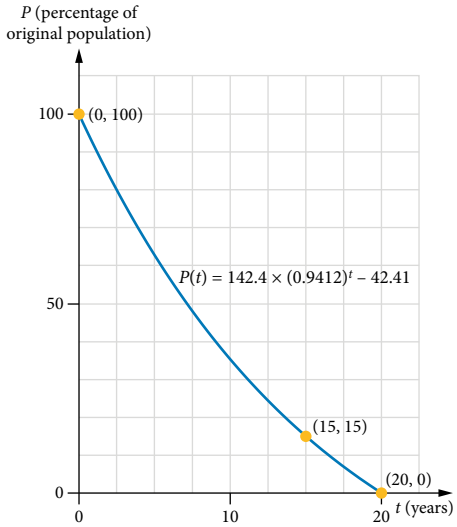
(c) 2.377×10^{29}

10 **(a)** $c = 3000$, $A = -1980$; $N(t) = 3000 - 1980(1.1)^{-t}$

(b) 1020 **(c)** 194%

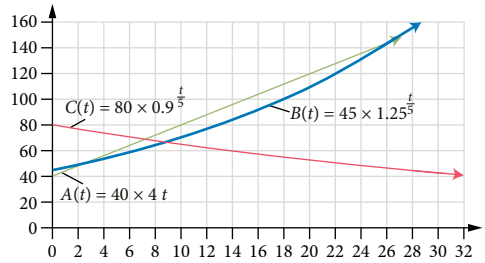
(d) The numbers were eventually close to triple so yes the campaign was successful.

11



Domain: $0 \le t \le 20$; Range: $0 \le P \le 100$

12 **(a)** 2018



(b) A little more than 7 years. **(c)** 2048

13 **(a)** 88 660

(b) **(i)** After 7 years he will have half the number of fish he started with, although the actual number of fish does increase and decrease.

(ii) The farmer will have sold all his fish in 11 years with only 15 331 fish to sell in this last year.

(iii) \$1 693 310

(c) End of the 7th year

14 **(a)** Possible models: $y = 2.5 \times (0.85)^x$; $y = 2.43 \times (0.857)^x$

(b) $y = 2.5 \times (0.85)^x$: $x = 5.5 \to y = 1.02$; $x = 15 \to y = 0.22$

$y = 2.43 \times (0.857)^x$: $x = 5.5 \to y = 1.04$; $x = 15 \to y = 0.24$

Values for $x = 5.5$ gave reliable values between the original data values for $x = 5$ and $x = 6$. Both gave values for $x = 15$ that are below the original data values for $x = 10$.

There is no original data near $x = 15$, so you cannot know whether the model is valid for this value, or whether the trend would continue.

CHAPTER REVIEW 7

1 **(a)** $x = 1$ **(b)** $x = \dfrac{5}{2}$ **(c)** $x = 3$

2 **(a)** $a = 1.51$ **(b)** $b = 0.73$ **(c)** $c = -1.40$

3 (a) $a = -2$ (b) $b = \pm 2$ (c) $c = 25$

4 (a) $x = \frac{1}{2}$ (b) $x = \frac{7}{2}$

5 $x = 3.82$

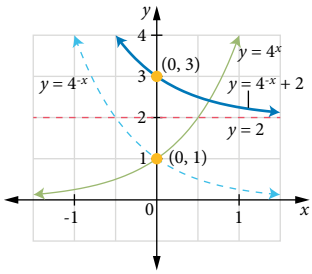
6 $g(x) = 5 \times 3^{-(x+3)} - 4$

7 (a) Asymptote: $y = 0$; domain: $(-\infty, \infty)$ or $\mathbb{R}$; range: $(0, \infty)$ or $\mathbb{R}^+$

(b) Asymptote: $y = 2$; domain: $(-\infty, \infty)$ or $\mathbb{R}$; range: $(-\infty, 2)$

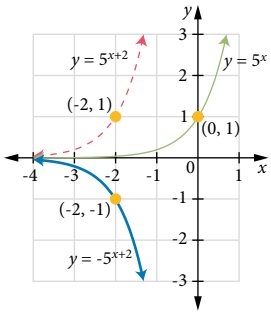
(c) Asymptote: $y = -7$; domain: $(-\infty, \infty)$ or $\mathbb{R}$; range: $(-7, \infty)$

8 (a)



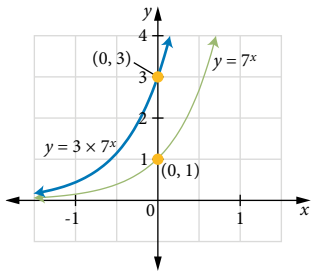
Reflection in the y-axis; translation: up 2

(b)



Translation: left 2; reflection in x-axis

(c)



Dilation parallel to y-axis by a factor of 3

9 Asymptote: $y = 2$; y-intercept: $(0, 5)$

10 C

11 $a = 2, b = 1$

12 (a) 0.86 (b) 0.89

13 (a) $\log_2(64) = 6$ (b) $\log_5(44) = 2 - x$

(c) $\log_8\left(\sqrt{46}\right) = 5x$

14 (a) $x = \log_3(42)$

(b) $x = \log_2(15) - 1$ or $x = \log_2(7.5)$

15 (a) $\log_3(42)$ (b) -1 (c) $\log_a(2)$

(d) $3\log_2(3)$ (e) $\frac{5}{2}$

16 C

17 $x = \frac{1}{3}\log_2(15)$ or $x = \log_2\left(\sqrt[3]{15}\right)$

18 (a) $a = 200$ (b) $n = -\frac{1}{4}$ (c) $y = 12.5$

19 (a) 50 (b) 40 minutes

(c) 5.37×10^{10} (d) 42.74 hours

20 (a) $x = 0$ (b) $x = 0$ or $x = 3$

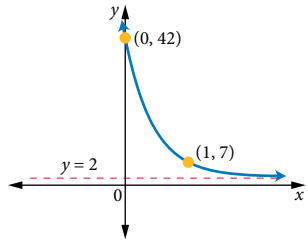
21 D

22 (a) Dilation of 5 parallel to the y-axis, reflection in the y-axis, dilation of $\frac{1}{3}$ parallel to the x-axis, horizontal translation of 1 unit to the right and vertical translation of 2 units up.

(b) $y = 2$ (c) Domain: $\mathbb{R}$; Range: $(2, \infty)$

(d) $g(0) = 42, g(1) = 7$

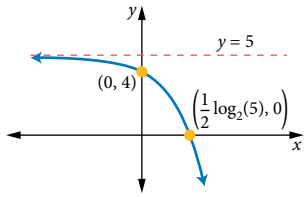
(e)



23 (a) $c = 3$ (b) Domain: $\mathbb{R}$; Range: $(-\infty, 3)$ (c) $a = 1.75$

(d) $n = 3$ (e) $(0.26, 0)$

24



25 (a) $x = 9$ (b) $a = 4, b = 2$

(c) LHS = RHS = $2 + 2^x + 2^{-x}$

26 D

27 (a) $V = P \times (1.0327)^t$ (b) \$910

(c) The average wage was \$9672 less than it would have been had it kept pace with inflation.

28 (a) $C_0 = 270$ (b) $k = 0.0732$ (c) 6 years

29 $N(t) = A \times (0.9)^t + 200$; 429

30 $1 < x < \log_3(6)$

31 $\frac{9}{2}$

32 (a) 9.9°C/minute (b) 14 minutes

(c) 8–13 minutes or 9–14 minutes

Chapter 8 Trigonometric functions

RECALL

1 (a) $\theta = 43°$ (b) $\theta = 43°$ (c) $\theta = 128°$

2 (a) $x = \frac{11}{2}$ (b) $x = -\frac{14}{5}$ (c) $x = -\frac{41}{12}$

3 (a) $x = 7$ or $x = -7$ (b) $x = 6$

(c) $x = 7$ or $x = -2$ (d) $x = -1$ or $x = -\frac{1}{3}$

4 (a) $\tan(47°) = \frac{x}{14}$ (b) $\sin(\theta°) = \frac{12}{20}$

5 (a) $x = 20.81$ cm (b) $x = 28.16$ cm

6 (a) 23.49 cm (b) 104.98 mm

7 (a) 415 cm^2 (b) 197 cm^2 (c) 62 cm^2

8 (a) Dilation by a factor of 2 parallel to the y-axis.

(b) Dilation by a factor of $\frac{3}{4}$ parallel to the y-axis and reflection in the x-axis.

(c) Translation up 2 (d) Translation left 1

EXERCISE 8.1

1 (a) $\frac{\pi}{6}$ (b) $\frac{4\pi}{15}$ (c) $\frac{5\pi}{6}$ (d) $\frac{7\pi}{6}$

(e) $\frac{5\pi}{4}$ (f) $\frac{3\pi}{4}$ (g) $\frac{5\pi}{3}$ (h) $\frac{11\pi}{6}$

2 (a) 3.46^c (b) 1.71^c (c) 6.20^c (d) 4.78^c

3 (a) 13° (b) 140° (c) 268° (d) 51°

4 (a) 45° (b) 40° (c) 108° (d) 105°

(e) 150° (f) 67.5° (g) 288° (h) 210°

5 (a) 10.47 cm (b) 12.57 cm (c) 97.39 mm (d) 14.81 m

(e) 122.68 cm (f) 67.86 m (g) 2.34 m

(h) 131.25 mm

6 (a) 577.27 cm^2 (b) 3534.29 cm^2

7 143° 8 4.43

9 (a) C

(b) The diameter was used instead of the radius.

10 (a) 73 cm (b) 3848 cm^2

11 (a) 382.5 cm^2 (b) 210 cm^2 (c) 3350 mm^2

12 A 13 $128(2\pi - 1)cm^2$

14 9.97 cm^2 15 $25\left(\sqrt{3} - \frac{\pi}{2}\right)m^2$

EXERCISE 8.2

1 (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) 1 (d) 1

(e) $\frac{\sqrt{3}}{2}$ (f) 1 (g) $\frac{1}{2}$ (h) undefined

2 (a) 1 (b) 0 (c) 0

(d) 0 (e) -1 (f) undefined

3 (a) -0.77 (b) -0.17 (c) -0.32

(d) 0.77 (e) 5.67 (f) 0.34

4 (a) $-\frac{1}{2}$ (b) $-\frac{\sqrt{2}}{2}$ (c) $-\frac{1}{2}$ (d) $-\frac{\sqrt{3}}{2}$

(e) $-\frac{\sqrt{3}}{2}$ (f) $-\frac{\sqrt{2}}{2}$ (g) $\frac{\sqrt{3}}{2}$ (h) $\frac{\sqrt{2}}{2}$

5 (a) $-\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{3}$ (c) $\frac{\sqrt{2}}{2}$

(d) $-\sqrt{3}$ (e) $\frac{\sqrt{2}}{2}$ (f) $-\frac{1}{2}$

6 (a) 0.8192 (b) 0.8192 (c) 0.8090 (d) 0.9093

(e) 0.6428 (f) 0.6428 (g) 0.6235 (h) 0.8912

7 C

8 (a) 60° anticlockwise (b) 30° anticlockwise

(c) 40° clockwise

9 (a) $\cos(\pi - \theta) = -0.669$, $\cos(\pi + \theta) = -0.669$, $\cos(4\pi - \theta) = 0.669$

(b) $\sin(\pi + \theta) = -0.324$, $\sin(2\pi + \theta) = 0.324$, $\sin(-\theta) = -0.324$

(c) $\tan(\pi - \theta) = -0.839$, $\tan(6\pi + \theta) = 0.839$, $\tan(5\pi + \theta) = 0.839$

10 (a) 0.8192 (b) 0.6428 (c) 0.9848

11 (a) $\tan(228°) = 1.1106$, $\tan(312°) = -1.1106$

(b) $\cos(36°) = 0.8090$, $\cos(144°) = -0.8090$

12 (a) $\cos(\theta)$ (b) $-\cos(\theta)$ (c) $-\sin(\theta)$ (d) $\sin(\theta)$

13 (a) $-\frac{\sqrt{2}}{2}$ (b) -1 (c) $\frac{\sqrt{3}}{2}$ (d) $-\frac{1}{2}$

14 (a) $\frac{\sqrt{3}}{3}$ (b) -1 (c) $-\sqrt{3}$ (d) $-\frac{\sqrt{3}}{3}$

15 (a) $-\frac{\sqrt{3}}{3}$ (b) $-\frac{\sqrt{2}}{2}$ (c) $-\frac{\sqrt{3}}{2}$

(d) $\sqrt{3}$ (e) $-\frac{\sqrt{2}}{2}$ (f) $-\frac{1}{2}$

(g) $\sqrt{3}$ (h) $\frac{1}{2}$ (i) $-\frac{\sqrt{2}}{2}$

16 (a) $\frac{1}{2}$ (b) -1 (c) $\frac{1}{2}$

(d) $-\frac{1}{2}$ (e) $\frac{\sqrt{3}}{2}$ (f) $-\frac{\sqrt{3}}{3}$

17 (a) D

(b) $\cos(-240°) = \cos(240°)$ not $-\cos(240°)$

18 (a) -0.84 (b) -0.84 (c) 0.84

19 (a) 7.0 cm or 70 mm (b) 60.8 cm or 608 mm

(c) 18.4 cm or 184 mm (d) 7 cm or 70 mm

EXERCISE 8.3

1 (a) Amplitude: 3, period: 2π (b) Amplitude: 1, period: $\frac{\pi}{2}$

(c) Amplitude: $\frac{5}{2}$, period: $\neq$ (d) Amplitude: $\frac{1}{2}$, period: 3π

(e) Amplitude: 2, period: $\frac{5\pi}{2}$ (f) Amplitude: $\frac{3}{2}$, period: $\frac{3\pi}{2}$

(g) Amplitude: 4, period: 1 (h) Amplitude: 3, period: 4

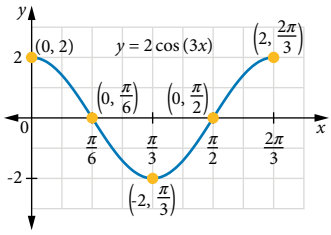
(i) Amplitude: $\frac{1}{3}$, period: 2

2 (a) π (b) 2π (c) π (d) $\frac{2\pi}{3}$

(e) 6π (f) 3π (g) $\frac{\pi}{3}$ (h) $\frac{6\pi}{5}$

(i) $\frac{4\pi}{5}$ (j) 1 (k) 4 (l) 5

3 (a)



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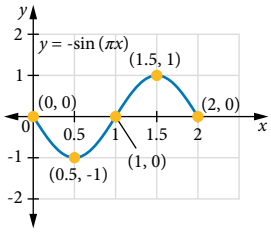

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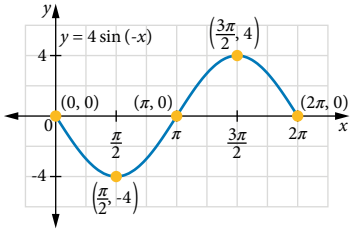

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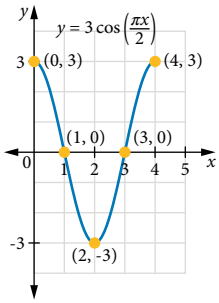

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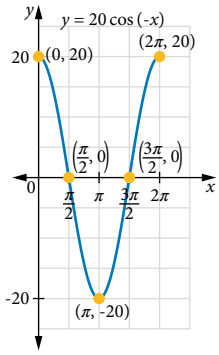

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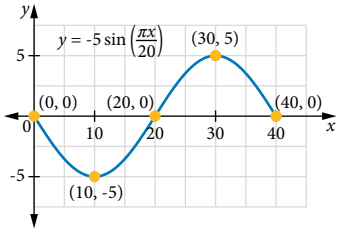


4 (a) Amplitude: 4, period: π; $y = 4\sin(2x)$, $x \in [-\pi, \pi]$

(b) Amplitude: 3, period: $\frac{4\pi}{3}$; $y = -3\cos\left(\frac{3x}{2}\right)$, $x \in [-2\pi, \pi]$

5 (a) $A = 3$, $B = \frac{10}{3}$ (b) $A = \frac{5}{2}$, $B = \frac{4}{15}$

6 A

7 (a) C

(b) A translation down by $\frac{\pi}{2}$ is not the same as a translation to the left by $\frac{\pi}{2}$.

8 (a) x-intercepts: $x = n\pi$, where n is any integer

Asymptotes: $x = \frac{\pi}{2} + n\pi$ or $x = \frac{(2n+1)\pi}{2}$, where n is any integer

(b) x-intercepts: $x = \frac{n\pi}{3}$, where n is any integer

Asymptotes: $x = \frac{\pi}{6} + \frac{n\pi}{3}$ or $x = \frac{(2n+1)\pi}{6}$, where n is any integer

(c) x-intercepts: $x = 3n\pi$, where n is any integer

Asymptotes: $x = \frac{3\pi}{2} + 3n\pi$ or $x = \frac{3(2n+1)\pi}{2}$, where n is any integer

(d) x-intercepts: $x = n$, where n is any integer $x = \frac{5n\pi}{3}$

Asymptotes: $x = \frac{1}{2} + n$ or $x = \frac{2n+1}{2}$, where n is any integer

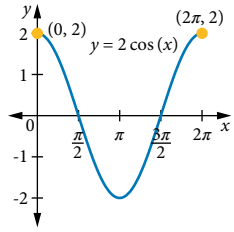
(e) x-intercepts: $x = 2n\pi$, where n is any integer

Asymptotes: $x = \pi + 2n\pi$ or $x = (2n+1)\pi$, where n is any integer

(f) x-intercepts: $x = \frac{5n\pi}{3}$, where n is any integer

Asymptotes: $x = \frac{5\pi}{6} + \frac{5n\pi}{3}$ or $x = \frac{5(2n+1)\pi}{6}$, where n is any integer

9 (a)



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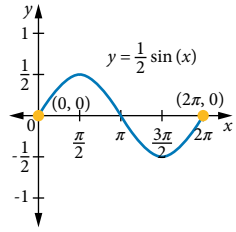

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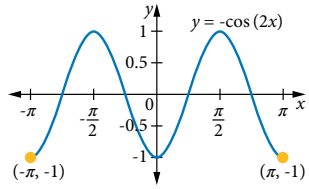

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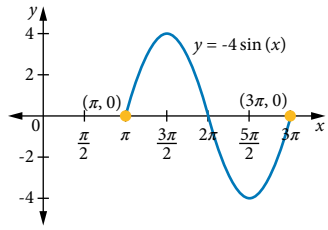

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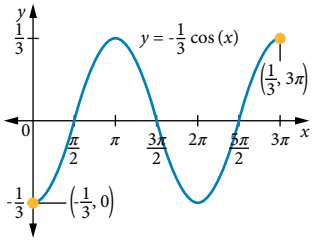

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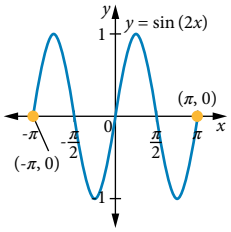

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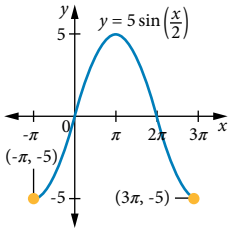
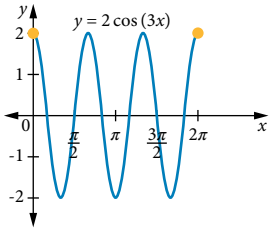


10 (a)



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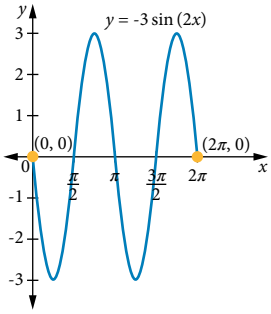

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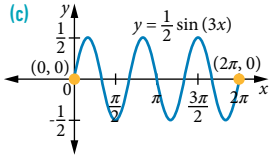

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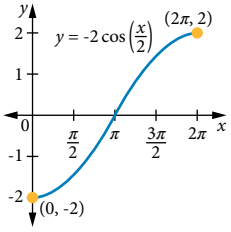

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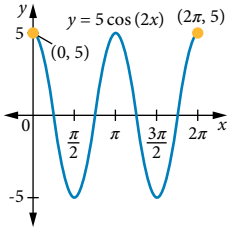

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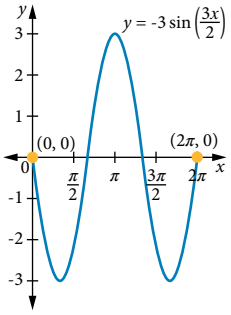

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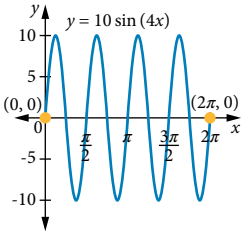

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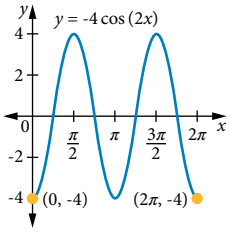

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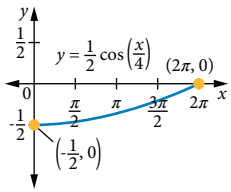
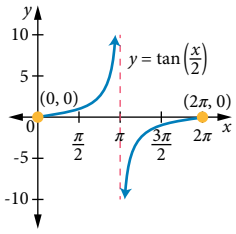


11 (a)



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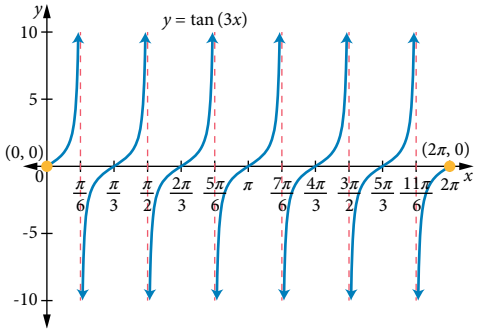
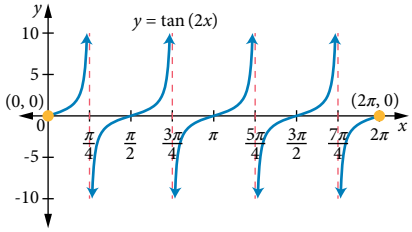
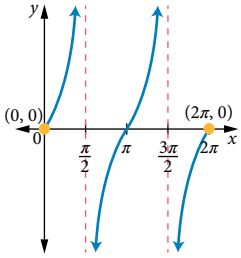


(c)



12 (a)



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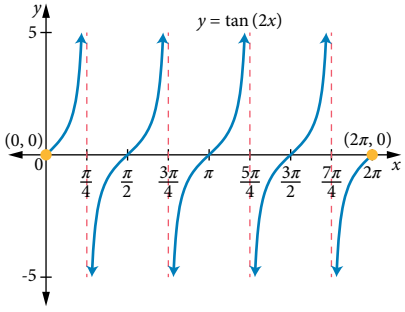
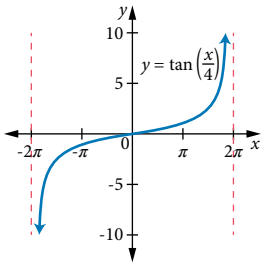
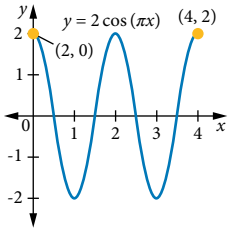


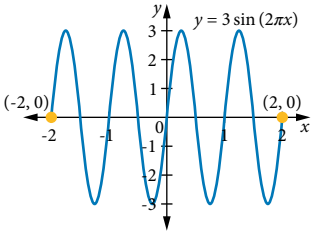
(c)



13 (a)



(b)



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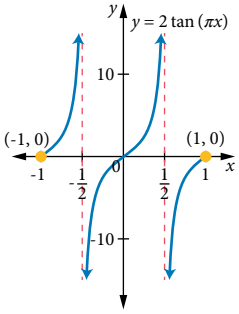
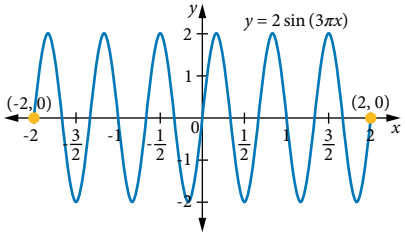
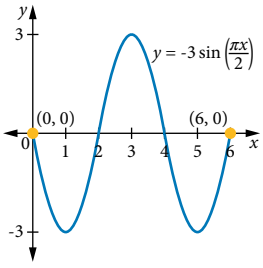


(d)



(e)



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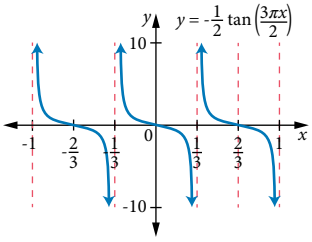

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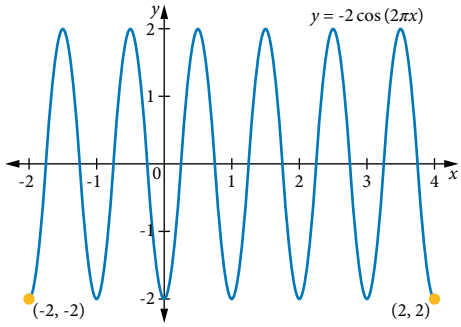
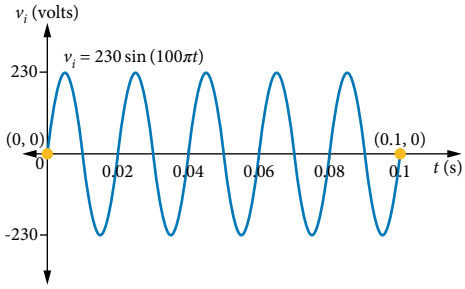
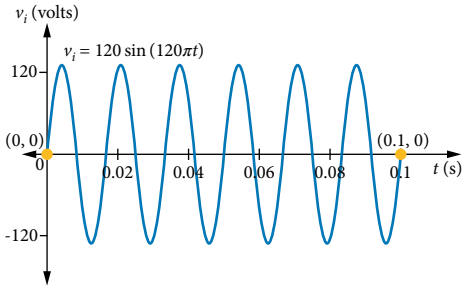


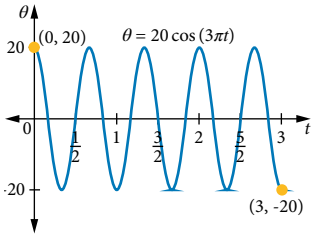
14 (a)



(b)



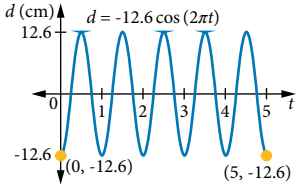
15 (a)



(b) 20°

(c) $\frac{2}{3}$ s

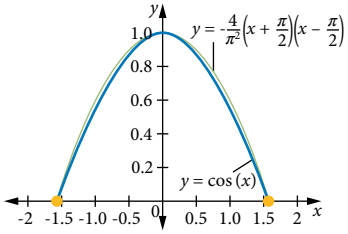
16 (a)



(b) 12.6 cm

(c) 1 s

17



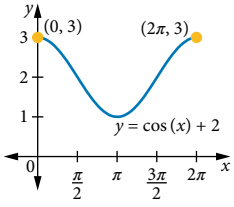
Sample answer: The graphs are both symmetrical about the y-axis and have similar shapes. The graph of the quadratic is slightly above the graph of the cosine function. They are very close around the intercepts, and coincide at the intercepts.

18 $y = \begin{cases} 3(x-1)^2 - 3, & 0 \le x \le 2 \\ -3(x-3)^2 + 3, & 2 < x < 4 \\ 3(x-5)^2 - 3, & 4 \le x \le 6 \end{cases}$

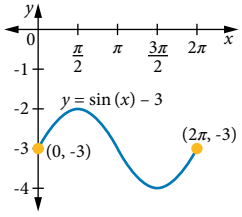
EXERCISE 8.4

1 (a) Median: 2, range: [1, 3]
(b) Median: -3, range: [-4, -2]
(c) Median: 3, range: $\mathbb{R}$
(d) Median: -1, range: [-6, 4]
(e) Median: 3, range: [-1, 7]
(f) Median: -7, range: $\mathbb{R}$

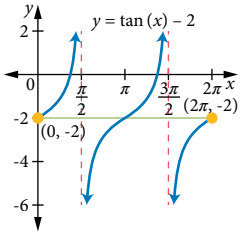
2 (a)



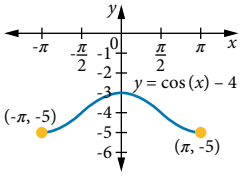
(b)



(c)

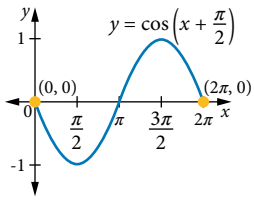


(d)

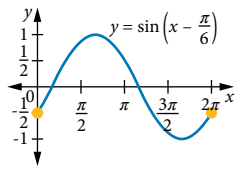


3 (a) Median: 0, range: [-1, 1], phase shift: $\frac{\pi}{2}$
(b) Median: 0, range: [-1, 1], phase shift: $-\frac{\pi}{4}$
(c) Median: 1, range: $\mathbb{R}$, phase shift: π
(d) Median: -1, range: [-2, 0], phase shift: $-\frac{\pi}{3}$
(e) Median: 2, range: [1, 3], phase shift: $-\pi$
(f) Median: 0, range: [-3, 3], phase shift: $\frac{\pi}{3}$
(g) Median: -5, range: $\mathbb{R}$, phase shift: $-\frac{\pi}{3}$
(h) Median: 2, range: [0, 4], phase shift: $-\frac{\pi}{5}$
(i) Median: -5, range: [-11, 1], phase shift: $-\frac{\pi}{8}$

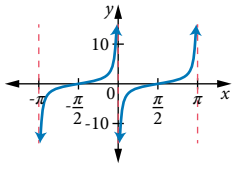
4 (a)



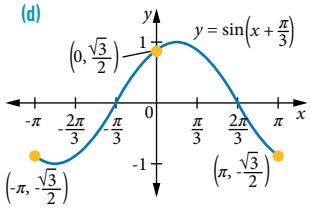
(b)



(c)



(d)



5 (a) $y = \cos(x)$ has experienced dilation parallel to the vertical axis of factor of 2, phase shift of $\frac{\pi}{2}$, translation of 1 unit up.
(b) $y = \sin(x)$ has experienced dilation parallel to the vertical axis of factor of $\frac{1}{2}$, dilation parallel to the horizontal axis of factor of $\frac{1}{2}$, translation of 3 units up.
(c) $y = \tan(x)$ has experienced reflection in the horizontal axis, dilation parallel to the vertical axis of factor 2, dilation parallel to the horizontal axis of factor 2, phase shift of $-\frac{\pi}{4}$.
(d) $y = \sin(x)$ has experienced reflection in the horizontal axis, dilation parallel to the vertical axis of factor 3, phase shift of $\frac{\pi}{4}$, vertical translation of 1 unit down.
(e) $y = \sin(x)$ has experienced reflection in the horizontal axis, dilation parallel to the vertical axis of factor $\frac{5}{2}$, dilation parallel to the horizontal axis of factor 2, phase shift of $-\frac{\pi}{2}$, vertical translation of 2 units up.
(f) $y = \cos(x)$ has experienced dilation parallel to the vertical axis of factor $\frac{1}{2}$, phase shift of $\frac{\pi}{3}$, vertical translation of 1 unit down.
(g) $y = \tan(x)$ has experienced reflection in the horizontal axis, dilation parallel to the vertical axis of factor $\frac{2}{5}$, dilation parallel to the horizontal axis of factor $\frac{1}{2}$, phase shift of $-\frac{\pi}{2}$, vertical translation of 1 unit down.
(h) $y = \cos(x)$ has experienced reflection in the horizontal axis, dilation parallel to the vertical axis of factor 2, dilation parallel to the horizontal axis of factor $\frac{1}{3}$, phase shift of $-\pi$, vertical translation of 3 units down.

6 (a) Period: 2π, domain: $[-\pi, \pi]$, $A = 3$, $C = -\frac{\pi}{2}$
(b) Period: 4π, domain: $[-2\pi, 2\pi]$, $A = -2$, $B = \frac{1}{2}$, $D = 1$

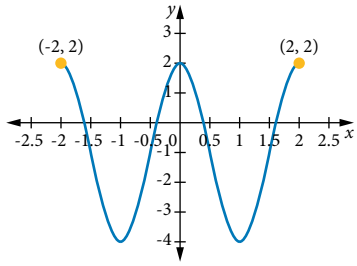
7 A

8 (a) A
(b) $y = \sin\left(x + \frac{\pi}{2}\right)$ has phase shift of $-\frac{\pi}{2}$

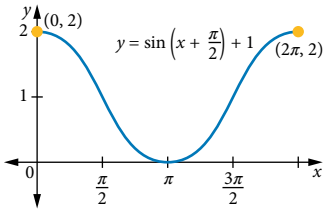
9 **(a)** Period: $\frac{2\pi}{b}$, amplitude: a, range: $[-a + c, a + c]$

(b) **(i)** Period: 2, amplitude: 3, range: [-4, 2]

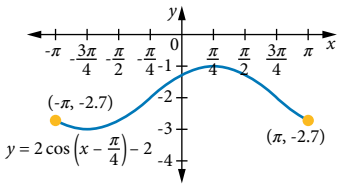
(ii) 4 solutions



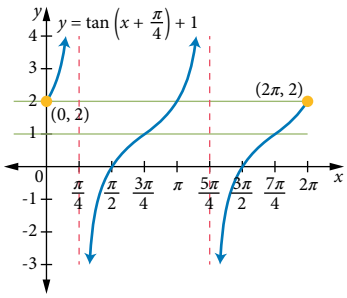
10 **(a)**



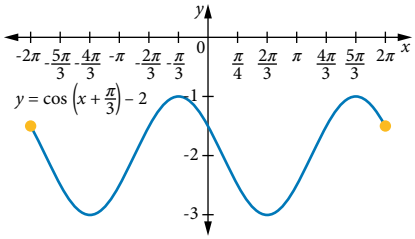
(b)



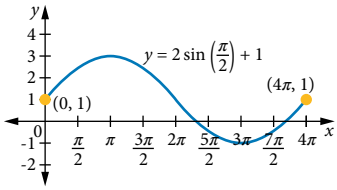
(c)



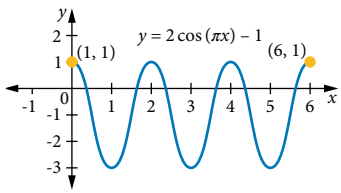
(d)



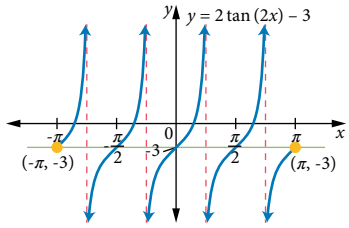
11 **(a)**



(b)



(c)



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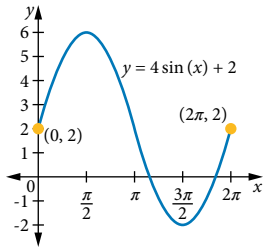

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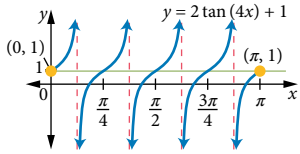

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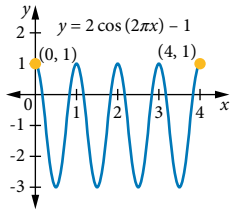
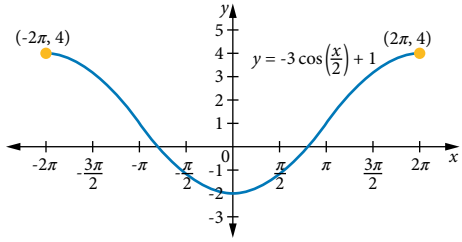


(g)



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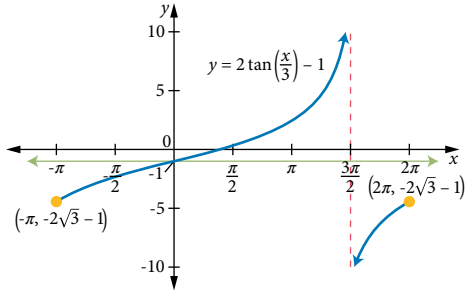
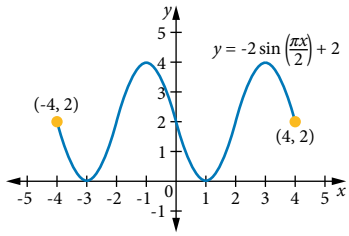
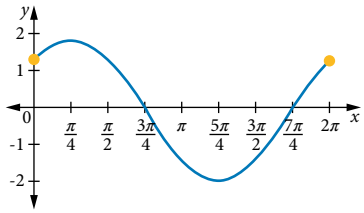


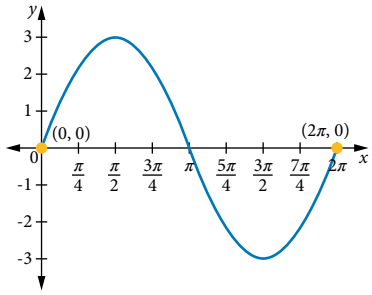
(i)



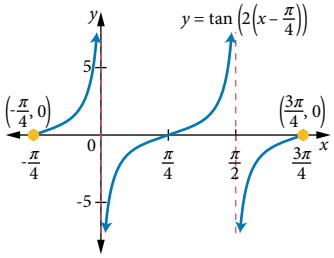
12 **(a)**



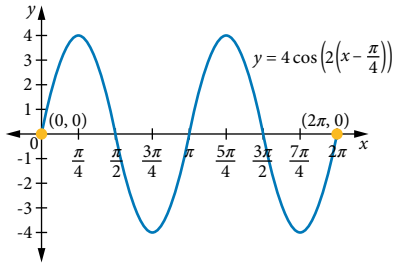
(b)



(c)



(d)



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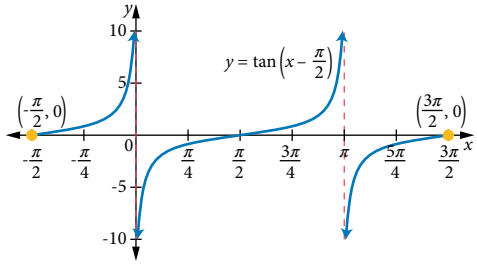
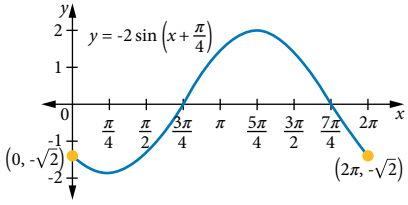
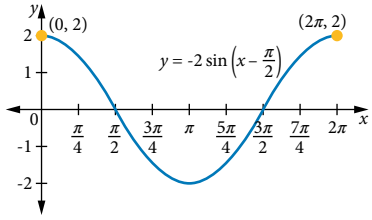


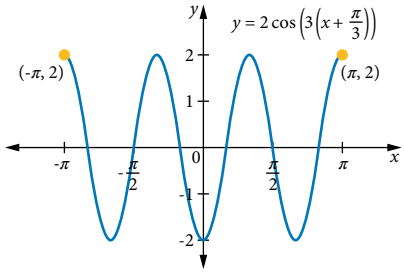
(f)



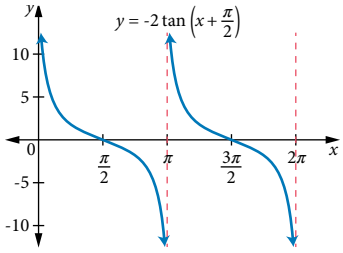
(g)



(h)



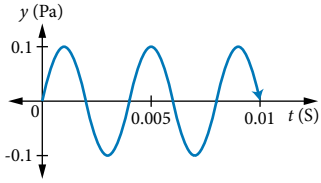
(i)



13 **(a)** $y = 5\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 1$

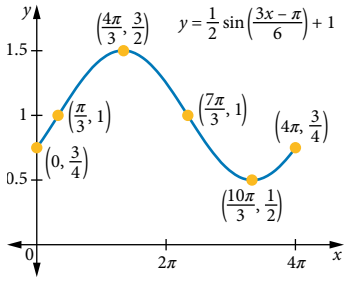
(b) $y = -2\cos\left(\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right) + 4$

14 **(a)**

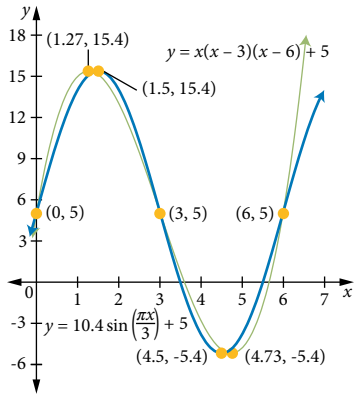


(b) 0.1 Pa **(c)** 0.004 s **(d)** 10π

15



16 $y = 10.4\sin\left(\frac{\pi x}{3}\right) + 5$ is a close match for [-0.1,6.1]. The median values are an exact match, but the turning points shifted towards the middle. Concavity is the same for [0,6], but the opposite to the left and right of this domain.



EXERCISE 8.5

1 (a) $x = \frac{\pi}{4}, x = \frac{7\pi}{4}$ (b) $x = \frac{\pi}{6}, x = \frac{5\pi}{6}$

(c) $x = \frac{5\pi}{6}, x = \frac{11\pi}{6}$ (d) $x = \frac{5\pi}{6}, x = \frac{7\pi}{6}$

(e) $x = \frac{\pi}{6}, x = \frac{11\pi}{6}$ (f) $x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$

(g) $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$ (h) $x = \frac{\pi}{3}, x = \frac{2\pi}{3}$

(i) $x = \frac{\pi}{3}, x = \frac{4\pi}{3}$

2 (a) B

(b) In the third quadrant, both x- and y-values are negative, hence $\tan(x)$ is positive.

3 C

4 (a) $x = 60°, x = 300°$ (b) $x = 60°, x = 120°$

(c) $x = 120°, x = 300°$ (d) $x = 135°, x = 225°$

(e) $x = -30°, x = 30°$ (f) $x = -120°, x = -60°$

(g) $x = -135°, x = 45°$ (h) $x = 30°, x = 150°$

5 (a) $x = 0.97, x = 5.31$ (b) $x = 0.13, x = 3.02$

(c) $x = 0.54, x = 3.68$ (d) $x = 1.72, x = 4.56$

(e) $x = 3.72, x = 5.70$ (f) $x = 1.91, x = 5.06$

(g) $x = 1.35, x = 4.94$ (h) $x = 3.37, x = 6.06$

6 (a) $x = 14°, x = 166°$ (b) $x = 117°, x = 297°$

(c) $x = 77°, x = 283°$ (d) $x = 29°, x = 209°$

(e) $x = 207°, x = 333°$ (f) $x = 93°$ or $x = 267°$

7 (a) $x = 0.11, 1.68, 3.25, 4.82$ (b) $x = 0.10, 0.23, 0.77, 0.90$

(c) $x = 2.87, 6.56$ (d) $x = 0.94, 1.99$

(e) $x = -1.29, -0.81, 0.81, 1.29$ (f) $x = -0.43, -0.07, 0.57, 0.93$

8 (a) $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$

(b) $x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{7\pi}{4} + 2n\pi$

(c) $x = \frac{\pi}{3} + n\pi$ (d) $x = \frac{5\pi}{4} + 2n\pi$ or $x = \frac{7\pi}{4} + 2n\pi$

(e) $x = \frac{5\pi}{6} + 2n\pi$ or $x = \frac{7\pi}{6} + 2n\pi$ (f) $x = \frac{3\pi}{4} + n\pi$

9 (a) $x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$ (b) $x = \frac{\pi}{12}, \frac{5\pi}{12}$

(c) $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

(d) $x = \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$

(e) $x = \frac{\pi}{2}, \frac{3\pi}{2}$ (f) $x = \frac{\pi}{6}$

10 (a) $x = 0, x = \frac{2\pi}{3}, x = \frac{4\pi}{3}, x = 2\pi$

(b) $x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x = \frac{3\pi}{2}$

(c) $x = 0, x = \pi, x = 2\pi$

(d) $x = \frac{\pi}{4}, x = \frac{3\pi}{4}, x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$

11 $x = -\frac{2\pi}{3}, x = -\frac{\pi}{3}, x = \frac{\pi}{3}, x = \frac{2\pi}{3}$

12 $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

13 $x = 1.11, x = 2.16, x = 4.25$

14 (a) $b = 6$, $5\sin(x) + 2x = 6$; $x = 0.96, x = 3.24, x = 5.20$

(b) $a = 1$, $5\sin(x) + 2x = 1$; $x = 0.14$

(c) $c = 10$, $5\sin(x) + 2x = 10$; $x = 5.91$

15 (a) $a = 0.04, b = -1$; $\cos(x) = 0.04x^2 - 1$; $x = -6.82, x = -4.53, x = -2.44, x = 2.44, x = 4.53, x = 6.82$

(b) $a = 0.01, b = 0.2$; $\cos(x) = 0.01x^2 + 0.2$; $x = -7.08, x = -5.20, x = -1.35, x = 1.35, x = 5.20, x = 7.08$

EXERCISE 8.6

1 (a) $t = 2$ (b) $t = 14$ (c) $t = 11$

(d) $t = 3.75$ (e) $t = 1.5$ (f) $t = 8\frac{2}{3} \approx 8.67$

2 (a) $t = 5$ (b) $t = 17$ (c) $t = 5.5$

(d) $t = 2.35$ (e) $t = 11.77$ (f) $t = 27.2$

3 (a) 11 pm (b) 7 am the next day

(c) 5 am the next day (d) 11.30 pm

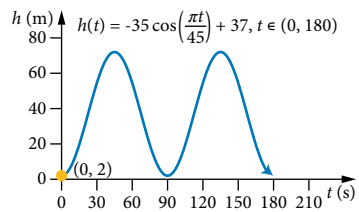
(e) 1.21 am the next day (f) 11.45 pm

4 (a) 31 August 2001 (b) 28 February 2002

(c) 30 June 2002 (d) 15 or 16 August 2001

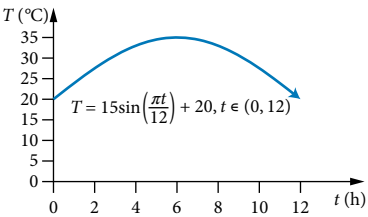
(e) 19 October 2001 (f) 15 or 16 May 2002

5 (a)



(b) 54.5 m (c) 23.73 s

6 (a)



(b) 24°C (c) 30.6°C

(d) 11:47 am and 6:13 pm (e) 54%

7 $T = 5\cos\left(\frac{\pi t}{6}\right) + 25$

8 (a) $A = 240, D = 0$ (b) $B = 100\pi$

(c) $V = 240\sin(100\pi t)$

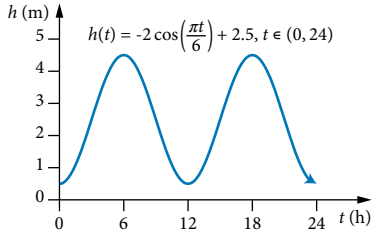
9 (a) B

(b) The reciprocal of B was not used to get $\frac{2\pi}{B}$.

10 $h(t) = -10\cos\left(\frac{\pi t}{15}\right) + 12$

11 (a) $h(t) = -2\cos\left(\frac{\pi t}{6}\right) + 2.5$ (b) 6.00 pm

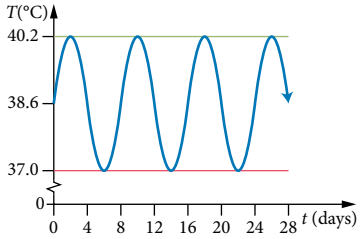
(c)



12 $h(t) = 5\cos\left(\frac{2\pi t}{5}\right) + 4$

13 (a) $T = 1.6\sin\left(\frac{\pi t}{4}\right) + 38.6$

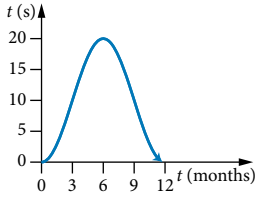
(b)



(c) 38.6°C (d) 38.6°C (e) 2 days (f) 37.0°C

14 D

15 (a)



(b) 12 months (c) \$20 000 in the middle of the year.

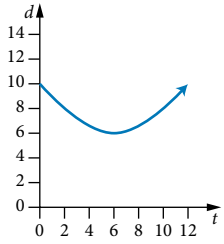
(d) \$0 at the beginning and end of the year.

16 (a) $T(t) = -7\sin\left(\frac{\pi}{12}(t+2)\right) + 15$ or (b) 4 pm

$T(t) = 7\sin\left(\frac{\pi}{12}(t-4)\right) + 15$

17 (a) 14 m, 6 m (b) 6 am

(c)

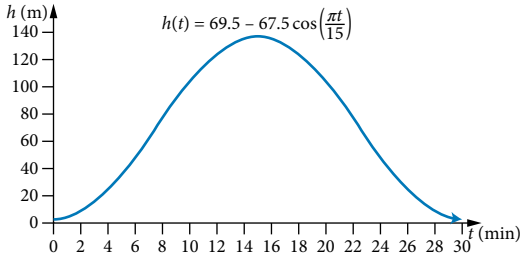


(d) 16 h

18 (a) $h(t) = 69.5 - 67.5\cos\left(\frac{\pi t}{15}\right)$

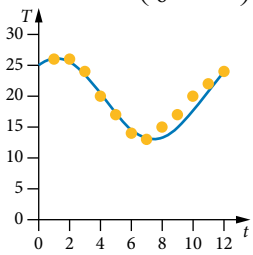
(b) 30 minutes (c) 69.5 m

(d)



(e) 10 min, 20 min

19 (a) $T_{max} = 6.5\sin\left(\frac{\pi}{6}(t+2)\right) + 19.5$



(b) $T_{min} = 4\sin\left(\frac{\pi}{6}(t+2)\right) + 10$

20 (a) 31 March: 800, 20 June: 702, 23 October: 838

(b) 1 March and 1 November

21 (a) 4 m (b) 15 m

CHAPTER REVIEW 8

1 (a) $\frac{4\pi}{15}$ (b) $\frac{25\pi}{18}$ (c) $\frac{7\pi}{4}$

2 (a) 60° (b) 540° (c) 50°

3 (a) 8π cm (b) $\frac{5\pi}{4}$ cm

4 (a) 6π cm^2 (b) 15π cm^2

5 D

6 (a) $\frac{1}{2}$ (b) $\sqrt{3}$ (c) $\frac{\sqrt{2}}{2}$ (d) $\frac{\sqrt{2}}{2}$

7 3 and 4

8 (a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{3}$ (d) $\frac{\sqrt{3}}{2}$

9 (a) 0.7234 (b) -0.7234 (c) 0.7234

(d) -0.7234 (e) -0.7234

10 (a) 2.658 (b) -2.658 (c) 2.658

(d) -2.658 (e) 2.658

11 A

12 Amplitude: 2, period: $\frac{2\pi}{3}$

13 2

14 $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

15 $x = \frac{7\pi}{6} + 2\pi n$ or $\frac{11\pi}{6} + 2\pi n$

16 **(a)** $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(b) $x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}$

17 $x = 0.38$ or 3.52

18 **(a)** $x = 0.12, 0.38, 1.12, 1.38$ **(b)** $x = 0.14, 0.92, 1.71, 2.49$

19 **(a)** $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2},$

$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

(b) **(i)** $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

(ii) $\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

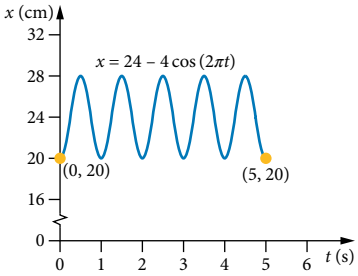
20 **(a)** $\cos(\theta)$ **(b)** $-\sin(\theta)$

21 D

22 $\sin(\theta) = \pm\frac{\sqrt{15}}{4}$

23 **(a)** $y = \frac{3}{2}\sin(4x) - 3.5$ **(b)** $y = -5\cos(x) - 1$

24 **(a)**

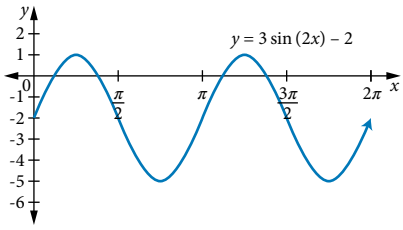


(b) **(i)** 24 cm **(ii)** 20 cm **(iii)** 28 cm

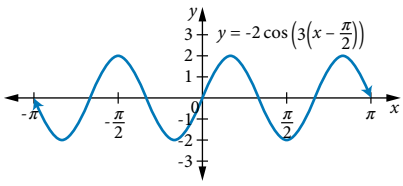
(c) 1 s

25 $y = \tan(3x) - 2$

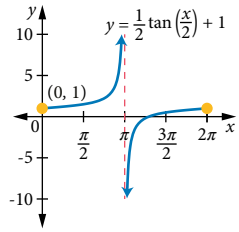
26 **(a)**



(b)

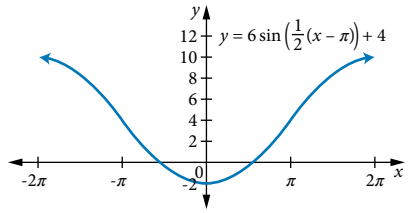


(c)



27 **(a)** $y = 6\sin\left[\frac{1}{2}(x - \pi)\right] + 4$

(b)



28 **(a)** $x = 0, \frac{2\pi}{3}, 2\pi$ **(b)** $x = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

(c) $x = \frac{27\pi}{20}$

29 $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

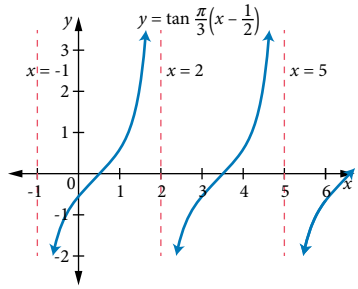
30 D

31 **(a)** 7.2 m **(b)** 0 m **(c)** 3.6 m

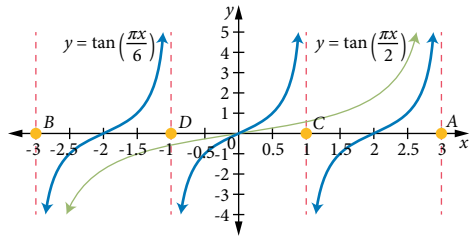
32 14 cm

33 $\cos\left(\frac{5\pi}{14}\right) = 0.43, \cos\left(\frac{9\pi}{14}\right) = -0.43$

34 **(a)** $x = 0.5, x = 3.5$



(b) $m = \frac{\pi}{6} + \frac{2n\pi}{3}$, where n is any positive integer.



35 $x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Chapter 9 Rates of change

RECALL

1 (a) Multiply by 1000 (b) Divide by 100 000

(c) Divide by 3600

2 (a) 2000 mL of oil (b) 5 L of petrol

3 (a) 0.25 (b) 0.05 (c) 1.56

4 (a) 5 (b) $\frac{7}{2}$ or $3\frac{1}{2}$ (c) 3

5 (a)

Time (min)	0	1	2	3	4
Volume (L)	50	55.3	60.6	65.9	71.2

(b)

Time (min)	0	10	20	30	40
Volume (L)	1500	1493	1486	1479	1472

6 (a) $m = 2$ (b) $m = -3$ (c) $m = 0.5$

7 (a) $y = 5x + 5$ (b) $y = -5x + 5$ (c) $y = 4x + 8$

8 (a) L3 (b) L4

9 (a) -1 (b) 7 (c) -5

EXERCISE 9.1

1 (a) 60 km/h (b) \$0.95/kg (c) 0.25 L/m^2

(d) 12 800 MB/h

2 (a) 86.73 km/h (b) 94.58 km/h (c) 91.82 km/h

3 \$282.50/month

4 19.2 mL 5 15 cm 6 32 hours and 40 minutes

7 (a) D

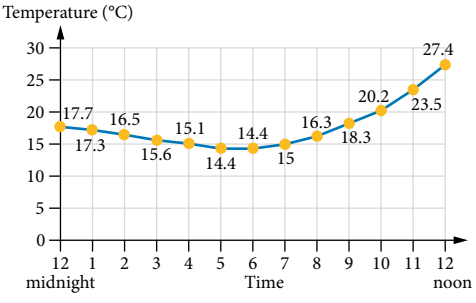
(b) $\frac{420}{20} = 21$kB/s The graph does not pass through the origin.

8 (a) B (b) D

9 -2°C/s 10 \$0/year

11 (a) 0.65 kg/week (b) 0.5 kg/week

12 (a)



(b) -0.66°C/h (c) 0°C/h (d) 2.17°C/h

13 (a) A (b) D (c) B (d) C

14 D

15 Find the gradient of the tangent at *C*.

16 (a) 2 km/min (b) 250 m/min (c) 8 km/h (d) 30 km/s

17 (a) After 1 day: 2 cm/day , after 2 days: 4 cm/day

(b) 6 cm/day

18 Median house price: \$19 150/year
Average wage: \$1795.80/year

House prices increased at a rate of almost 11 times the rate of wage increase.

19 (a) \$387.56 (b) \$180.73 (c) \$6.78

20 (a) (i) \$0 (ii) \$2156.50 (iii) \$14 160.50

(iv) \$48 785.60 (v) \$86 015.50

(b) (i) 0 cents in the \$ or 0%

(ii) 7.30 cents in the \$ or 7.30%

(iii) 20.35 cents in the \$ or 20.35%

(iv) 29.52 cents in the \$ or 29.52%

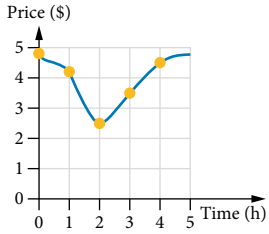
(v) 34.32 cents in the \$ or 34.32%

21 (a) (i) 0.5 L/s (ii) 1 L/s

(b) *BC*: 5 L/s (c) *CD*: 0.46 L/s

22 (a) 1987 to 1992: -3 days/year (b) 1997 to 2000: 5 days/year

23 (a)



(b) (i) -\$1.05 per hour (ii) -\$0.25 per hour

(iii) \$1.17 per hour

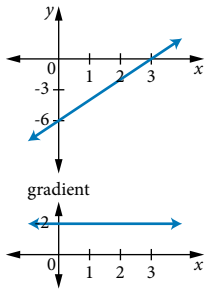
24 (a) 42%/year (b) 29%/year (c) 5.75%/year

25 (a) 97 700 (b) 78 300

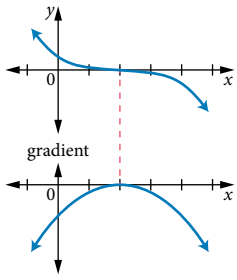
26 (a) -2.1 (b) ±2.6

EXERCISE 9.2

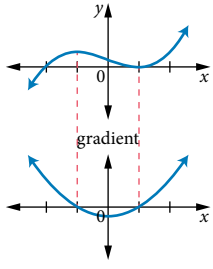
1 (a)



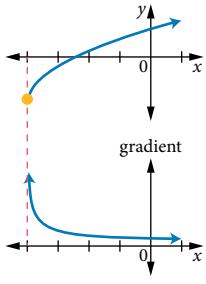
(b)



(c)



(d)



2 (a) D (b) C (c) B (d) A

3 (a) C

(b) The values for the numerator and denominator are not in corresponding order.

4 6 5 A

6 (a) A: (2, 2), B: (4, 8) (b) 3 (c) 3

7 (a) 18 (b) 2 (c) 9

8 (a) $f(1) = -1, f(4) = 8$ (b) 3

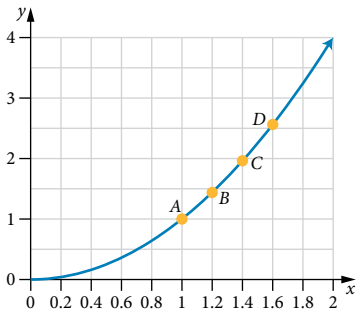
9 (a) (2, 1) (b) 1

10 (a) (-1.5, 1.75) (b) 3

11 (a) 2.4 (b) 2 (c) 1.6

(d) As the value of h gets smaller, the gradient of the secant approaches the value of the gradient of the tangent line at $x = 0.4$.

12 (a)



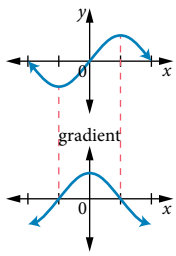
(b) (i) $m = 2.6$ (ii) $m = 2.4$ (iii) $m = 2.2$

(c) $m = 2$

(d) Points closer to point A give secant gradients closer to the gradient of the tangent at A.

13 (a) 0 m/s (b) 10 m/s (c) -32.4 m/s

14 (a)



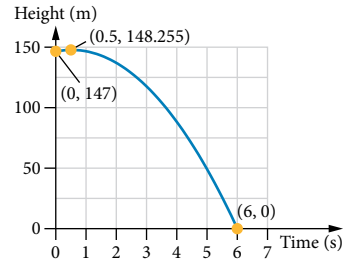
(b) (i) 0 (ii) 1 (iii) $\frac{1}{2}$

15 (a) 10 m^3 (b) 2 h (c) 18 m^3

(d) 3 h (e) 4 m^3/h (f) 6 m^3/h

(g) -2 m^3/h (h) 8 m^3/h (i) 12 m^3/h

16 (a)

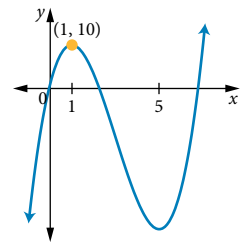


(b) 147 m (c) 0.5 s, 148.225 m

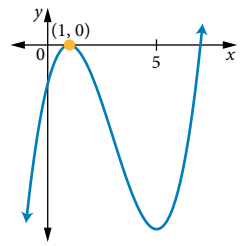
(d) 2.45 m/s (e) 4.9 m/s

(f) -26.95 m/s (g) -53.9 m/s

17 (a)



(b)



18 (a) For $A(-3, -15)$: AB: $m = 14.6$; AC: $m = 3.9$

(b) For $A(-3, -15)$: 34

(c) For $A(-3, -15)$, D is (4.33, 37.03)

(d) For $A(-3, -15)$: BD: $m = 3.9$; CD: $m = 14.5$

(e) Gradients of AB and CD are the same and gradients of AC and BD are the same due to the symmetry of the gradient function. This means that the cubic function will have rotational symmetry.

EXERCISE 9.3

1 (a) $f(x+h) = 4x + 4h - 4$, $f(x+h) - f(x) = 4h$

(b) $f(x+h) = x^2 + 2hx + h^2 - 2$, $f(x+h) - f(x) = 2hx + h^2$

(c) $f(x+h) = 5x^2 + 10xh + 5h^2 + 7$,

$f(x+h) - f(x) = 10xh + 5h^2$

(d) $f(x+h) = x + h - x^2 - 2hx - h^2$,

$f(x+h) - f(x) = h - 2hx - h^2$

(e) $f(x+h) = 5x + 5h - 6x^2 - 12xh - 6h^2$,

$f(x+h) - f(x) = 5h - 12xh - 6h^2$

(f) $f(x+h) = 7x^2 + 14hx + 7h^2 + 2x + 2h + 2$,

$f(x+h) - f(x) = 14hx + 7h^2 + 2$

2 (a) -1 (b) 27 (c) 2 (d) -9

(e) 7 (f) 10

3 (a) $f'(x) = 2$ (b) $f'(x) = -5$ (c) $f'(x) = -1$

(d) $f'(x) = 7$ (e) $f'(x) = 3$ (f) $f'(x) = -2$

4 D

5 (a) C

(b) The student has forgotten to change the sign in the second term when expanding brackets with a negative in front.

(c) In simplified form, every term in the numerator will have a factor of h.

6 (a) $2+h$ (b) 2

7 (a) $4+h$ (b) 4

8 (a) 4 (b) -1 (c) -3 (d) $\frac{1}{2}$

9 (a) $f'(x)=4x$ (b) $f'(x)=-2x$

(c) $f'(x)=2x+4$ (d) $f'(x)=-2x-7$

10 (a) $f'(x)=-3x^2$ (b) $f'(x)=3x^2$

(c) $g'(x)=3x^2-2$

11 (a) $f'(x)=2x-4$ (b) $g'(x)=3x^2-6$

12 (a) $f'(x)=4x^3$ (b) $f'(x)=5x^4$ (c) $f'(x)=6x^5$

13 (a) $f'(x)=-\frac{1}{x^2}$ (b) $f'(x)=\frac{2}{x^2}$

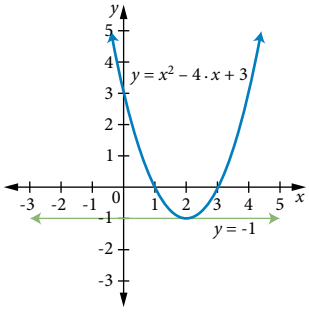
(c) $f'(x)=-\frac{2}{x^3}$ (d) $f'(x)=\frac{6}{x^3}$

14 (a) $f'(x)=-\frac{1}{(x+3)^2}$ (b) $g'(x)=-\frac{1}{x^2}-4$

15 (a) $h^2+6h+13$ (b) 13

16 (a) $f'(x)=2x-4$ (b) (2, -1) (c) 0

(d)



(e) $x>2$

17 (a) $f'(x)=-\frac{3}{x^4}$ (b) $f'(x)=-\frac{4}{x^5}$ (c) $f'(x)=-\frac{5}{x^6}$

18 (a) $g'(x)=\frac{4}{x^2}$ (b) (-2, 2) and (2, -2) (c) $4\sqrt{2}$ units

EXERCISE 9.4

1 (a) $f'(x)=2x$ (b) $f'(x)=3x^2$ (c) $f'(x)=5x^4$

(d) $f'(x)=4$ (e) $f'(x)=6x$ (f) $f'(x)=18x^5$

(g) $f'(x)=\frac{8}{3}x^3$ (h) $f'(x)=-15x^2$ (i) $f'(x)=-x^4$

2 B

3 (a) $g'(t)=5$ (b) $g'(t)=2t+6$

(c) $g'(t)=4t^3-8t$

4 (a) $\frac{dy}{dx}=x^2+\frac{1}{2}$ (b) $\frac{dy}{dx}=-2x^4+3x^3$

(c) $\frac{dy}{dx}=\frac{3}{5}x^2-\frac{4}{3}x$

5 (a) $\frac{dy}{dx}=0$ (b) $\frac{dy}{dx}=-9x^2+10x$

6 (a) B

(b) It is incorrect to multiply the derivatives of the individual factors.

7 $\frac{dy}{dx}=3x^2+8x+4$

8 (a) $f'(x)=3x^2-18x$ (b) $h'(x)=10(1-x)$

(c) $\frac{dy}{dx}=2ax+b$ (d) $\frac{dp}{dx}=3(x-1)^2$

(e) $f'(a)=a-2$ (f) $\frac{dh}{dt}=3t^2-4at+a^2$

9 (a) $f'(-1)=9, f'(2)=27$ (b) $f'(-1)=5, f'(2)=-1$

(c) $f'(-1)=3a-2b+c, f'(2)=12a+4b+c$

(d) $f'(-1)=-4, f'(2)=20$ (e) $f'(-1)=-1, f'(2)=5$

(f) $f'(-1)=-7, f'(2)=-1$

10 (a) $g'(x)=1$ (b) $g'(t)=6t+6$

11 (a) $\frac{dh}{dt}=-\frac{1}{t^2}$ (b) $\frac{dh}{dt}=-\frac{2}{t^3}$ (c) $\frac{dh}{dt}=-\frac{6}{5t^4}$

(d) $\frac{dh}{dt}=\frac{3}{4t^2}$ (e) $\frac{dh}{dt}=-\frac{1}{t^2}-\frac{4}{3t^3}$ (f) $\frac{dh}{dt}=\frac{3}{t^2}$

12 (a) $\frac{dy}{dx}=\frac{1}{2x^{\frac{1}{2}}}$ (b) $\frac{dy}{dx}=\frac{9}{2}x^{\frac{1}{2}}$ (c) $\frac{dy}{dx}=\frac{3}{\sqrt{x}}$

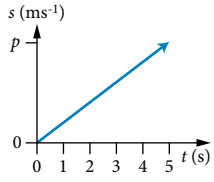
(d) $\frac{dy}{dx}=-\frac{1}{x\sqrt{x}}$ (e) $\frac{dy}{dx}=\frac{1}{3\sqrt[3]{x^2}}$ (f) $\frac{dy}{dx}=6\sqrt{x}$

13 (a) $f'(x)=\frac{4}{x^2}+5$ (b) $f'(x)=\frac{3}{2}x^2+12x+\frac{2}{x^3}$

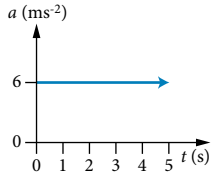
(c) $f'(x)=6-\frac{2}{\sqrt{x}}$

14 (a) 3 (b) $-\frac{15}{2}$ (c) $\frac{15}{2}$

15 (a) $d(t)=3t^2$ (b) $s(t)=6t$



(c) $a(t)=6$



16 (a) (i) $f'(x) = -\frac{1}{x^2}$ (ii) $f''(x) = \frac{2}{x^3}$

(iii) $f'''(x) = -\frac{6}{x^4}$ (iv) $f''''(x) = \frac{24}{x^5}$

(b) $\frac{(-1)^n n!}{x^{n+1}}$ (c) 0

17 (a) $-\frac{3}{(x-3)^2}$ (b) $-\frac{4}{(x-3)^2}$ (c) $-\frac{9}{(x-3)^2}$

CHAPTER REVIEW 9

1 (a) 90 km/h (b) 800 L/h (c) \$2.80 per kg

2 (a) \$66.60 (b) 2.8 m

3 (a) 12

(b) (i) 12 km/h (ii) 12 000 m/h (iii) 0.2 km/min

4 (a) 678 L/min (b) 2 days 13 hours

5 (a) 30.2 m/s (b) 20.4 m/s

6 (a) 36°C/min (b) -12°C/min

7 D 8 \$8 per year 9 $6\frac{1}{2}$ min

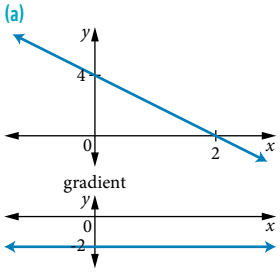
10 -30°C/h 11 120 km/h

12 (a) 200 km/h (b) 162.5 km/h

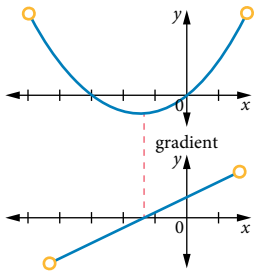
13 (a) 3 (b) 12 (c) 4

14 (a) 2, 18 (b) 8

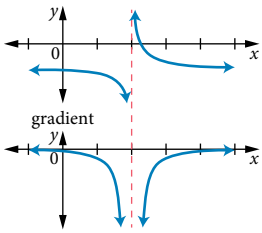
15 (a)



(b)



(c)

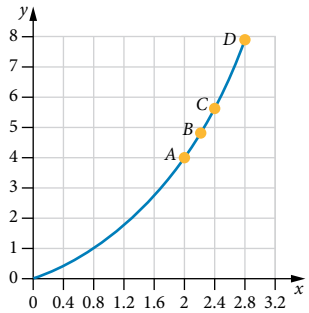


16 (a) C (b) A (c) D (d) B

17 $\Delta x = 1$

18 -2

19 (a)



(b) (i) $m = 4.8$ (ii) $m = 4.4$ (iii) $m = 4.2$

(c) The gradient is about 4.

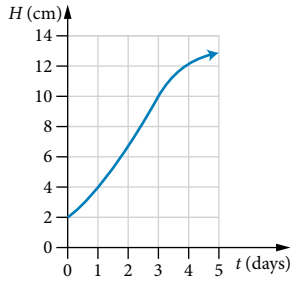
(d) As the points approach the point A on the curve, the gradient of the secants approach the gradient of the tangent at the point (2, 4).

20 (a) $f'(x) = 4x$ (b) $f'(x) = -12x^2 + 20x$

(c) $f'(x) = 8x - 12$ or $f'(x) = 4(2x - 3)$

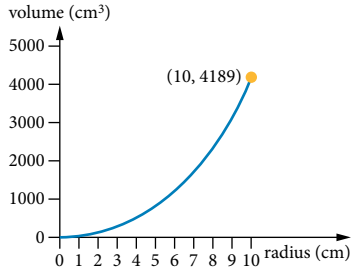
21 $4x - 8$

22 (a)



(b) Estimates may vary: $t = 1$, $2\frac{1}{2}$ cm/day; $t = 2$, 3 cm/day; $t = 3$, $2\frac{2}{3}$ cm/day

23 (a)



(b) 418.9 cm³/cm (c) 314.2 cm³/cm (d) 1256.5 cm³/cm

24 (a) $f'(x) = 10x$

(b) $f'(x) = -4x - 4$ or $f'(x) = -4(x + 1)$

25 (a) $h^2 + 6h + 15$ (b) 15

26 (a) $f'(x) = 12x^2 - 3$ or $f'(x) = 3(2x - 1)(2x + 1)$

(b) $f'(x) = 2x - \frac{18}{x^3}$ (c) $f'(x) = \frac{1}{2\sqrt{x}} - \frac{2}{x^2}$

27 (a) $f'(-2) = -17$ (b) $x > -1$

28 C 29 D

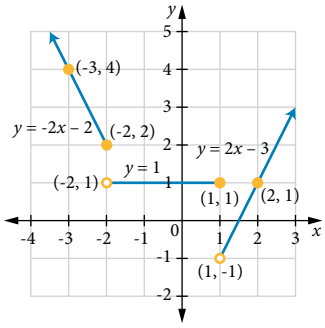
30 $y = -\frac{19}{3}x + 21\frac{26}{27}$ or $y = -\frac{19}{3}x + \frac{593}{27}$

31 $\dfrac{-4}{3(x-1)^3}$

32 $-\dfrac{x+1}{2x^2(x+2)^2}$

Mixed review: Chapters 1–9

1



2 (a) 1612, 1406, 1200, 994, 788 (b) 9th term: $t_9 = -36$

(c) 7128

3 \$4213.50

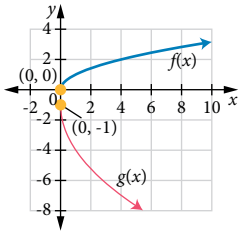
4 $y = (x+9)^2 - 2$

5 (a) $\Delta = 13$, two roots (b) $\Delta = 0$, one root

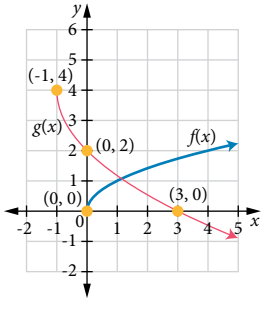
(c) $\Delta = 6$, two roots (d) $\Delta = -16$, no roots

6 $y = \dfrac{2}{3}x^2 + 3$

7 $g(x) = -3\sqrt{x} - 1$



8 Dilate $f(x)$ by factor 2 parallel to the y-axis, reflect in the x-axis, translate 1 unit left and translate 4 units up.



9 (a) $n = 13$ (b) $n = 26$ (c) $n = 17$

10 66

11 (a) $\dfrac{2}{x^3}$ (b) $4a^2b^4$ (c) $\dfrac{1}{x^4y^5}$

12 (a) 3.51×10^{-3} (b) 5.237×10^2

(c) 1.097×10^5 (4 s.f.)

13 (a) $x = -\dfrac{4}{3}$ (b) $x = \dfrac{4}{3}$ (c) $x = -3$

14 (a) 3.91 (b) 0.22 (c) -2.13

15 (a) 135° (b) 72.19° (c) $\dfrac{31\pi}{45}$ (d) 0.820

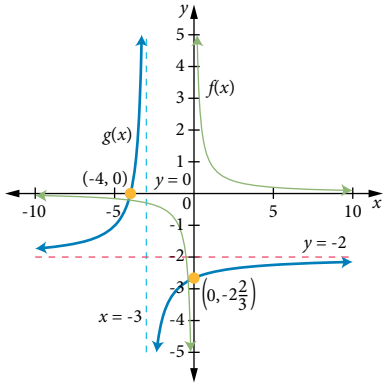
16 (a) $m_{PQ} = -4 - h$ (b) -4

17 $x = -3, y = 4$

18 (a) $(-\infty, 27]$ (b) $\left[-\dfrac{139}{8}, \infty\right)$ (c) $(-\infty, 16]$

19 (a) $x = -4, x = -3, x = 4$ (b) $x = -1, x = -5$

20 Reflection in the x-axis and dilation factor of 2 parallel to the y-axis. Translations of 3 units to the left and 2 units down.



21 (a) $5^n - 2$ (b) $\dfrac{8}{9}$

22 $126\,\text{cm}^2$

23 $y = 3\cos\left(\dfrac{4}{3}\left(x - \dfrac{\pi}{4}\right)\right) + 2$

24 $\dfrac{dy}{dx} = -\dfrac{1}{x^3} + \dfrac{9\sqrt{x}}{2}$

25 $y = \begin{cases} \sqrt{r^2 - (x+r)^2}, & x \le -r \\ r, & -r < x < r, \\ \sqrt{r^2 - (x-r)^2}, & x \ge r \end{cases}$

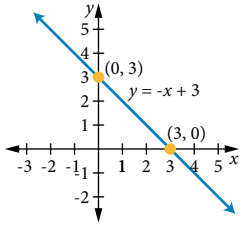
$y = \begin{cases} -\dfrac{1}{2}x - r, & -2r \le x < 0 \\ \dfrac{1}{2}x - r, & 0 \le x \le 2r \end{cases}$

26 (a) 3.6×10^{10} kg (b) 4.24×10^8 kg

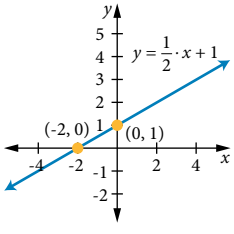
Chapter 10 Differential calculus

RECALL

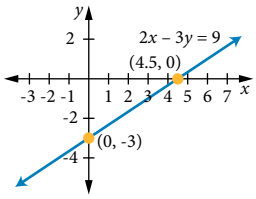
1 (a)



(b)



(c)



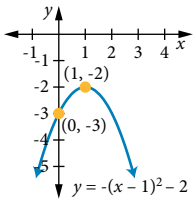
2 $y = -4x + 11$

3 (a) no solutions (b) two solutions (c) one solution

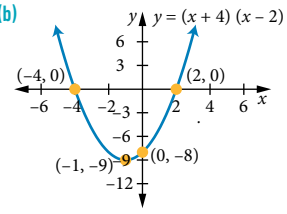
4 (a) $x = -4$ or $x = 4$ (b) $x = -2$ or $x = \frac{5}{2}$

(c) $x = \frac{2}{3}$ or $x = 2$ (d) $x = \frac{3-\sqrt{11}}{2}$ or $x = \frac{3+\sqrt{11}}{2}$

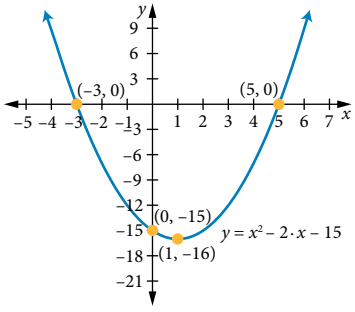
5 (a)



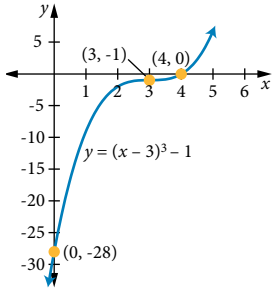
(b)



(c)



6



7 (a) $4x^2 - 12x + 9$ (b) $9x^2 - 16$

(c) $x^3 - 6x^2 + 12x - 8$

8 (a) $x = 3, y = 1$ (b) $x = \frac{3}{2}, y = \frac{1}{2}$

(c) $x = -\frac{11}{5}, y = -\frac{7}{5}$

9 (a) $x = 0, x = 1, x = 3$ (b) $x = -3, x = -2, x = -1$

(c) $x = -2, x = 2$

EXERCISE 10.1

1 (a) $g'(2) = 16$ and $g'(-1) = -8$

(b) $g'(2) = 1$ and $g'(-1) = 7$

(c) $g'(2) = -8$ and $g'(-1) = 1$

2 (a) D (b) $y = -4$, not $\frac{dy}{dx}$

3 (a) B (b) A

4 (a) $\frac{dy}{dx} = 3x^2 - 6x$ (b) 24

(c) (0, 0) and (2, -4) (d) $(-\infty, 0) \cup (2, \infty)$

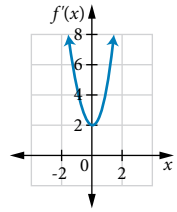
5 (-1, 6), (2, -21) 6 -7 7 $x > -4$

8 (a) 1

(b) (-4, 2), (4, 0)

9 (4, 4) 10 $y = -9$

11 $f'(x) = 3x^2 + 2$, you know this is a positive parabola with turning point at (0, 2), hence $f'(x) > 0$ for all values of x.



12 $\left(-\sqrt{3}, 1-\sqrt{3}\right), \left(\sqrt{3}, 1+\sqrt{3}\right)$ 13 $a = 3$ and $b = -2$

14 (a) (2, 0) and (4, 0) (b) (-2, -12) and (2, 0)

15 (a) $y = -8x - 15$ (b) $y = -15x - 15$

(c) $y = -48x + 24, y = 48x + 24$

16 $y = 2x + 10, y = 2x - 22$

17 (a) (i) $x = 0, x = 2$ (ii) $0 < x < 2$

(b) $y = -3x + 5$

18 (a) $x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$

(b) (i) $b^2 = 3ac$ or $b = \pm\sqrt{3ac}$ (ii) $b^2 > 3ac$

(iii) $b^2 < 3ac$

19 At (0, 0) the curves have gradients -2 and $\frac{1}{2}$; $-2 \times \frac{1}{2} = -1$

At (2, 0) the curves have gradients 2 and $-\frac{1}{2}$; $2 \times \left(-\frac{1}{2}\right) = -1$

20 $y = 2x^2 + 4x - 30$

21 (a) (3, -8)

(b) The x-value of the point of intersection is the average of the x-values of the intercepts.

22 $(6, 3)$

EXERCISE 10.2

1 (a) -3 (b) -3

2 (a) 8 (b) 11

3 (a) C

(b) The student has found the average rate over 3 s, instead of the instantaneous rate at 3 s.

4 (a) $\frac{28\pi}{3}$ cm³/cm (b) 4π cm³/cm (c) 28.27 cm³/cm

5 (a) 4°C/hour (b) 6°C/hour

6 (a) 27 L/min (b) 50 L/min

7 (a) $h'(-1) = 6$ (b) $h'(1) = 6$

(c) $h'(-1) = h'(1)$ (d) $h'(x) = h'(-x)$

8 (a) $\frac{31}{2}$ (b) $f'(2) = 12 - \frac{\sqrt{2}}{4}$

9 (a) 2 s, 20 m/s

(b) $\frac{dh}{dt} = -10$ m/s and so the body travels downwards at 10 m/s.

10 D

11 (a) 6 cm/min (b) $d'(5) = 2.4$ cm/min

12 (a) -7 (b) $-2h - 3$

(c) -3 (d) -11

13 (a) 160 L/min (b) $t = 0$ min

14 (a) $V = 15\pi r^2$ (b) 70π cm^3/cm (c) 60π cm^3/cm

(d) Yes, as r increases, the rate of increase in the volume is greater.

15 (a) (i) 0.0098 cm/cm^3 (ii) 0.0026 cm/cm^3

(iii) 0.0018 cm/cm^3

(b) Yes, the balloon's radius increases more quickly at the start.

(c) 0.0021 cm/cm^3

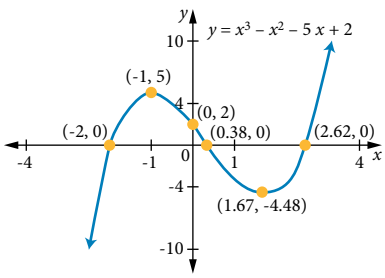
16 (a) $0 \le a \le 0.5$, $(6a - 10)$ m^3/h (b) -9.94 m^3/h

(c) Yes, the difference is only 0.06 m^3/h, so it is a good approximation.

(d) $a = 0.001$: -9.994 m^3/h. It is clear that the values for average rate of change approach -10 m^3/h as $a \to 0$.

EXERCISE 10.3

1



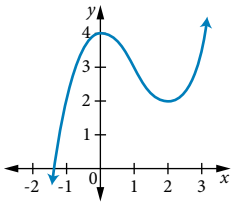
2 (a) D

(b) For parabolas, the turning points are found halfway between the x-intercepts. This is not true for cubics.

3 C

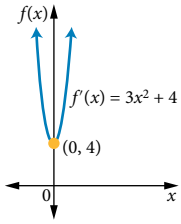
4 $c = -3$

5

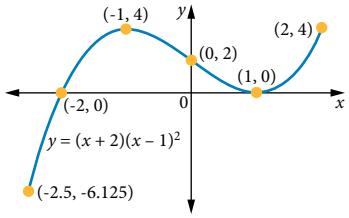


6 (a) $f'(x) = 3x^2 + 4$

(b) The derivative of $f(x)$ never equals zero, therefore there are no stationary points.



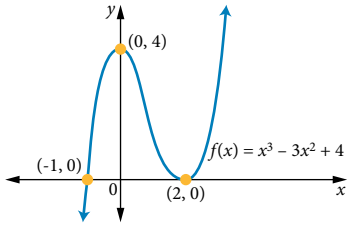
7 (a)



(b) $y_{min} = 0$, $y_{max} = 20$

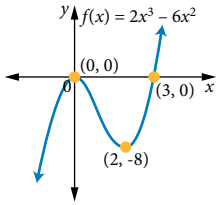
8 (a) (i) Local maximum: (0, 4)
Local minimum: (2, 0)

(ii)



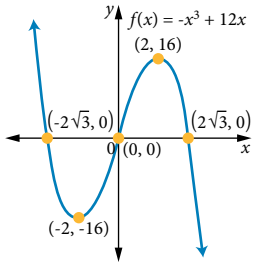
(b) (i) Local maximum: (0, 0)
Local minimum: (2, -8)

(ii)



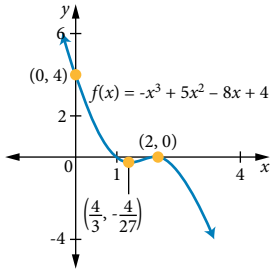
(c) (i) Local minimum: (-2, -16)
Local maximum: (2, 16)

(ii)



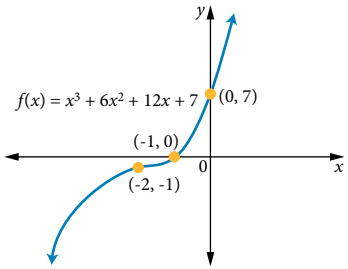
(d) (i) Local minimum: $\left(\frac{4}{3}, -\frac{4}{27}\right)$ or (1.33, -0.15)
Local maximum: (2, 0)

(ii)



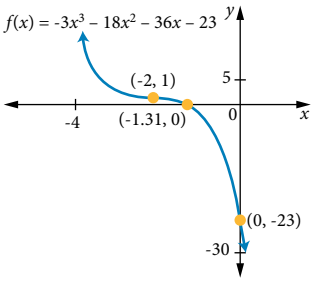
9 **(a)** **(i)** Stationary point of inflection: (-2, -1)

(ii)



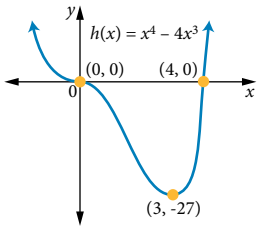
(b) **(i)** Stationary point of inflection: (-2, 1)

(ii)



(c) **(i)** Stationary point of inflection: (0, 0)
Local minimum: (3, -27)

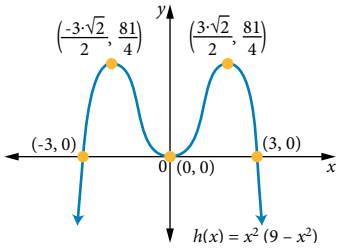
(ii)



(d) **(i)** Local maxima: $\left(-\frac{3\sqrt{2}}{2}, \frac{81}{4}\right)$ and $\left(\frac{3\sqrt{2}}{2}, \frac{81}{4}\right)$

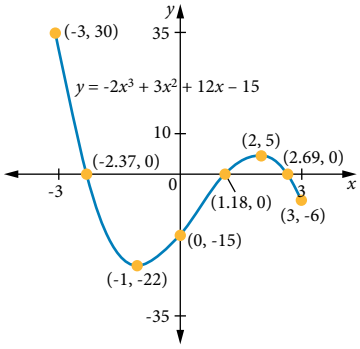
Local minimum: (0, 0)

(ii)



10 **(a)** Local minimum: (-1, -22)
Local maximum: (2, 5)

(b)



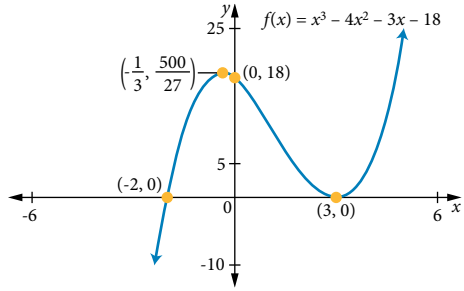
(c) Maximum: 30, minimum: -22

11 **(a)** $a = -4$ and $b = -3$

(b) Local maximum: $\left(-\frac{1}{3}, \frac{500}{27}\right)$ or $\left(-\frac{1}{3}, 18\frac{14}{27}\right)$

Local minimum: (3, 0)

(c)



12 **(a)** $f(-2) = 0$, hence $(x + 2)$ is a factor.

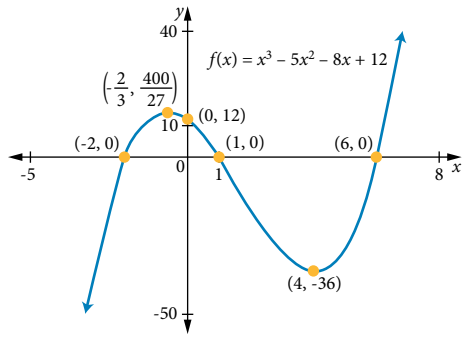
Others factors: $(x - 1)$ and $(x - 6)$

(b) x-intercepts: (-2, 0), (1, 0) and (6, 0); y-intercept: (0, 12)

(c) Local maximum: $\left(-\frac{2}{3}, \frac{400}{27}\right)$

Local minimum: (4, -36)

(d)



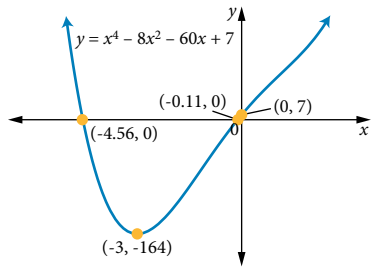
(e) $-\frac{2}{3} < x < 4$

13 **(a)** $\frac{dy}{dx} = 4x^3 - 16x + 60$

(b) $\frac{dy}{dx} = 0$ leads to $x^3 - 4x + 15 = 0$.

(c) $x^3 - 4x + 15 = (x + 3)(x^2 - 3x + 5)$ has only one root, $x = -3$ because the quadratic factor has no roots. Hence the curve has a single stationary point at $x = -3$.

(d)

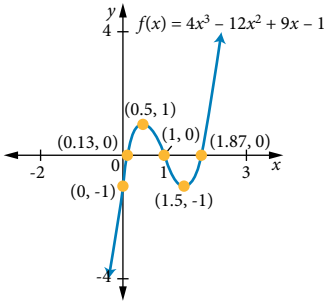


14 **(a)** $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ **(b)** $a > 0$

(c) $\frac{dy}{dx} = 0$ leads to the x-value of the turning point: $x = -\frac{b}{2a}$.
The quadratic formula gives the average of the x-intercepts as $x = -\frac{b}{2a}$.

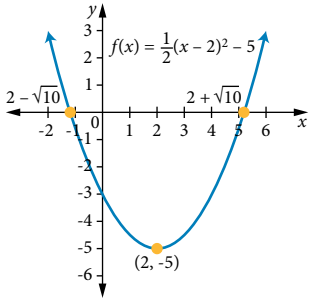
15 (a) $f(x) = 4x^3 - 12x^2 + 9x - 1$

(b)



(c) $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$

16



EXERCISE 10.4

1 (a) Maximum: 0.22, $x = 0.20$; Minimum: -125, $x = -2$

(b) Maximum: 47, $x = -8$; Minimum: -97, $x = 4$

(c) Maximum: 7.5, $x = 1$; Minimum: 4.00, $x = 0.04$

2 (a) Maximum: 18, $x = -1$; Minimum: -36, $x = 2$

(b) Maximum: 6, $x = 0$; Minimum: 2, $x = 2$

(c) Maximum: 10, $x = 0$; Minimum: -80, $x = 5$

3 (a) $A = l(15 - l)$ (b) $S = 2\pi r^2 + \frac{6000}{r}$

(c) $S = x^2 + (x + 40)^2$

4 (a) $y = 100 - 2x$ (b) $1250\,\text{m}^2$

(c) 25 m by 50 m with 50 m parallel to the riverbank.

5 288

6 (a) $y = 24 - x$

(b) $A = x(24 - x)$ or $A = 24x - x^2$

(c) $0 < x < 24$

(d) $x = 12$; A is a negative parabola, hence the stationary point will be a local (and absolute) maximum.

(e) $144\,\text{m}^2$; a 12 m by 12 m square paddock

7 (a) $I = 25x - 0.01x^2$ or $I = x(25 - 0.01x)$

(b) $P = -0.01x^2 + 20x - 5000$

(c) 1000

8 (a) B (b) D (c) C

9 (a) C

(b) The student has assumed that a minimum will occur when one of the values is zero. A second error here is that both numbers must be positive, so this solution would be rejected.

10 (a) $V(x) = x(4 - 2x)(2 - 2x)$, domain: $0 < x < 1$.

(b) $x = 0.42$ m, maximum volume: $1.54\,\text{m}^3$

11 (a) $x = 10 - \pi r$ (b) $A = 10r - \frac{\pi}{2}r^2$ (c) $r = \frac{10}{\pi}$ m

(d) $\frac{50}{\pi}\,\text{m}^2$, semicircular

12 $50\,\text{m}^2$

13 (a) $V = x(6 - 2x)^2$ or $V(x) = 4x^3 - 24x^2 + 36x$

(b) $0 < x < 3$

(c) Maximum volume: $16\,\text{m}^3$; dimensions: 1 m by 4 m by 4 m

14 (a) $h(x) = 18 - 2x$ (b) $V(x) = 18x^2 - 2x^3$

(c) $x = 6$ m

(d) Maximum volume: $216\,\text{m}^3$;
Dimensions: 6 m by 6 m by 6 m, cube

15 (a) The maximum value is 64.

(b) The minimum value is $\frac{256}{27}$.

16 (a) Maximum volume is when the side of square base is 2 m and height is 1 m; the model does not taken into account the overlapping edges to join the timber, the accuracy depends on the thickness of the timber used.

(b) $4\,\text{m}^3$

17 $\frac{1}{2}\sqrt{10}$

18 (a) $r = 4.30$ cm, $h = 8.60$ cm

(b) Range of values for the radius in cm is: $3.89 \le r \le 4.61$.

19 (a) $A = \frac{x^2}{4\pi}$ (b) $B = \left(\frac{5 - x}{4}\right)^2$

(c) $x = \frac{5\pi}{\pi + 4}$, circular piece: $\frac{20}{\pi + 4}$ m, square piece: $\frac{20}{\pi + 4}$ m

(d) The maximum area occurs when the wire is all used for the circle.

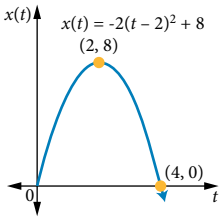
20 (a) The side length of the square cut out from each corner is one-sixth of the side length of the original square piece of cardboard.

(b) Volume: $\frac{2a^3}{27}\,\text{m}^3$; dimensions: $\frac{a}{6}$ m by $\frac{2a}{3}$ m by $\frac{2a}{3}$ m

(c) Base: $A = \frac{4a^2}{9}\,\text{m}^2$; side faces: $A = \frac{a^2}{9}\,\text{m}^2$ each

EXERCISE 10.5

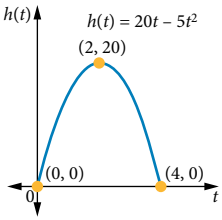
1 (a)



(b) 8 m/s (c) 16 m

(d) -4 m/s; travelling to the left at 4 m/s, towards the origin

2 (a)



(b) $t = 2$ s, $h = 20$ m (c) -10 m/s

3 (a) 8 m to the left of O. (b) -6 m/s

(c) $t = 1$ s

4 (a) D

(b) This answer gives the change of position, and ignores the fact that the particle travelled back for some distance.

5 1 second: 30 m, 2 seconds: 40 m, 3 seconds: 50 m

6 2 seconds: 12.8 m, 8 seconds: 48.80 m

7 (a) -7 m/s (b) 14.5 m

8 (a) $t = 1$ s and $t = 3$ s (b) $t = 3$ s, $v = 0$ m/s

9 (a) 35 m (b) 6.25 s (c) 113.125 m

(d) $t = 13.77$ s (e) 68 m

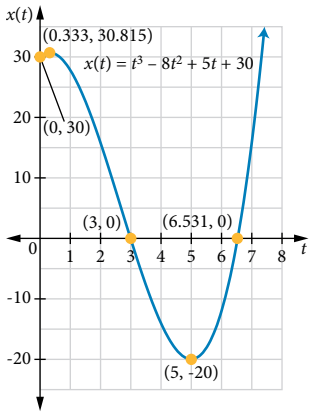
10 D

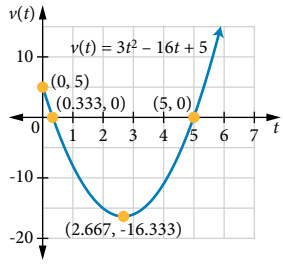
11 (a) $t = 1$ s

(b) 0 m/s

(c) $t = 0$ s

12 (a)





(b) $t = 0$: 30 m to the right of the origin, slowing down, 5 m/s away from the origin (to the right).

$t = 2$: 16 m to the right of the origin, speeding up, 15 m/s towards the origin (to the left).

$t = 3$: at the origin, slowing down, 16 m/s to the left.

$t = 6$: 12 m to the left of the origin, speeding up, 17 m/s towards the origin (to the right).

13 (a) 10 m to the right of the origin, slowing down, 10 m/s away from the origin (to the right).

(b) After 1.83 s, 22.17 m to the right of the origin.

(c) 28.39 m/s to the left

14 (a) After $\left(3-\sqrt{2}\right)$s, $\left(2\sqrt{2}-2\right)$m to the left of the origin; after $\left(3+\sqrt{2}\right)$s at $\left(2+2\sqrt{2}\right)$m to the right of the origin

(b) First: particle A is going further to the left at $\left(2\sqrt{2}-2\right)$m/s and particle B is going to the right (towards the origin) at a steady 2 m/s.

Second: particle A is speeding up and travelling to the right at $\left(2+2\sqrt{2}\right)$m/s and particle B is continuing to the right (away from the origin) at a steady 2 m/s.

15 (a) $x(t) = 4t^3 - 8t^2 + t + 7$

(b) After $\frac{2}{3}$ s the particle is $\frac{143}{27}$ m to the right of the origin and at an instantaneously constant speed of $\frac{13}{3}$ m/s as it travels towards the origin (to the left). The particle has slowed down and is about to speed up.

EXERCISE 10.6

1 (a) $f(g(x)) = (2x-3)^2$ (b) $g(f(x)) = 2x^2 - 3$

(c) $x = 3-\sqrt{3}$ or $x = 3+\sqrt{3}$

2 (a) $\frac{dy}{dx} = 18x\left(3x^2-1\right)^2$ (b) $\frac{dy}{dx} = 4(x-3)^3$

(c) $\frac{dy}{dx} = 6x^2\left(x^3-2\right)$ (d) $\frac{dy}{dx} = \frac{-12}{x^2}\left(1+\frac{4}{x}\right)^2$

(e) $\frac{dy}{dx} = -\frac{24x}{\left(3x^2-5\right)^5}$ (f) $\frac{dy}{dx} = -\frac{112x}{\left(2x^2-1\right)^5}$

(g) $\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}$ (h) $\frac{dy}{dx} = -\frac{3x^2}{\sqrt{x^3-1}}$

(i) $\frac{dy}{dx} = 4(2x-3)\left(x^2-3x+2\right)^3$

(j) $\frac{dy}{dx} = -\frac{2(2x-7)}{\left(x^2-7x+1\right)^2}$ (k) $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{(x-1)^2}}$

(l) $\frac{dy}{dx} = \frac{2}{5\sqrt[5]{(2x+4)^4}}$

3 $-\frac{\sqrt{3}}{3}$ 4 -6 5 B

6 (a) C

(b) $f(x)=\sqrt{x+1}$ is defined for $x \geq -1$, but $f'(x)$ has an expression in the denominator, so $x \neq -1$ for $f'(x)$.

7 $\dfrac{2x^3\left(f'\left(\sqrt{1+x^4}\right)\right)}{\sqrt{1+x^4}}$

8 The gradient at both (1, 0) and (2, 0) is 0.

9 $x > -1.65$

10 (a) Domain: $\mathbb{R}$; Range: $\left[\sqrt{5},\infty\right)$ (b) $x < \dfrac{\sqrt{3}}{3}$

11 (a) Domain: $[-1,\infty)$; Range: $[0,\infty)$

(b) Domain: $\mathbb{R}$; Range: $[0,\infty)$

(c) $f(g(x))=\sqrt{x^2+1}$

(d) $g(f(x))=x+1$

(e) The derivative of $f(g(0))$ at $x=0$ is 0.

(f) $-\dfrac{\sqrt{2}}{4} < x < \dfrac{\sqrt{2}}{4}$

12 (a) $k=1$ (b) Both have gradient $\dfrac{\sqrt{2}}{2}$.

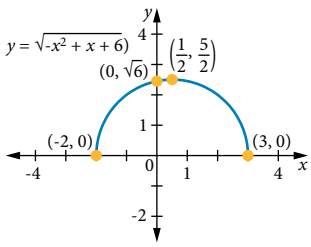
(c) $p=2$

13 (a) $\dfrac{dr}{dt}=2\,\text{cm/h}$, when $t=1\,\text{h}$, $\dfrac{dA}{dr}=10\,\text{cm}^2/\text{cm}$

(b) $\dfrac{dA}{dt}=20\,\text{cm}^2/\text{h}$ (c) $\dfrac{dA}{dt}=60\pi\,\text{cm}^2/\text{h}$

14 (a) $-2 \leq x \leq 3$

(b)



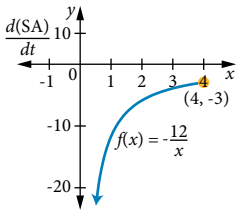
15 (a) $f(g(1))=1$, $g(f(1))=-4$ (b) $h'(x)>0$ for $x>\dfrac{7}{2}$

(c) $h(x)=x^2-7x+7$; The derivative of the composite function $h(x)$ is positive for $x>\dfrac{7}{2}$ as established in part (b).

16 (a) $\dfrac{dx}{dt}=-\dfrac{1}{x^2}\,\text{cm/min}$ (b) $6\,\text{cm}^2/\text{min}$

(c) $12\,\text{cm}^2/\text{min}$

(d)



As $x \to 0$ from the right, the surface area decreases at a rate approaching ∞.

EXERCISE 10.7

1 (a) $\dfrac{dy}{dx}=2x-2$ (b) $\dfrac{dy}{dx}=-3(x-3)(1-x)$

(c) $\dfrac{dy}{dx}=(3-x)^3(3-5x)$

2 (a) $f'(x)=3x^2+6x+1$ (b) $f'(x)=3x^2-10x+10$

3 8

4 (a) $f'(x)=3x^2+4x-1$

(b) The graph has zero gradient at $x=\dfrac{-2-\sqrt{7}}{3}$ and $x=\dfrac{-2+\sqrt{7}}{3}$.

5 $f'(-1)=-5$

6 $f'(1)=-2$

7 (a) $\dfrac{dy}{dx}=\dfrac{1}{(x+1)^2}$ (b) $\dfrac{dy}{dx}=\dfrac{x^2-2x-4}{(x-1)^2}$

8 (a) $-\dfrac{4}{5}$ (b) $-\dfrac{3}{4}$

9 (a) $f'(x)=\dfrac{1-x^2}{\left(1+x^2\right)^2}$

(b) The graph has zero gradient at $x=-1$ and $x=1$.

10 (a) $h'(-1)=20$ (b) $h'(-1)=-\dfrac{1}{4}$

11 $f'(1)=\dfrac{9}{4}$

12 (a) D

(b) $f'(4)g'(4)=4\times 6=24$

When $h(x)=f(x)g(x)$, $h'(x)\neq f'(x)g'(x)$

13 A

14 $\dfrac{dy}{dx}=\dfrac{3x+1}{2\sqrt{x}}$ 15 $f'(x)=\dfrac{x\left(3x^2+2\right)}{\sqrt{x^2+1}}$

16 (a) $g'(x)=\dfrac{3(2-x)}{2\sqrt{3-x}}, x<3$ (b) $x=2$

(c) $2\sqrt{2}<x<3$

17 (a) $\dfrac{dy}{dx}=\dfrac{3-2x}{2\sqrt{x}(2x+3)^2}$ (b) $\dfrac{dy}{dx}=-\dfrac{9}{2x^2\sqrt{9-x^2}}$

18 (a) $\dfrac{2\sqrt{3}}{9}$ (b) $\dfrac{1}{4}$

19 $-\dfrac{4}{9}$

20 (a) $f'(x)=\dfrac{\sqrt{x}+2}{2\left(\sqrt{x}+1\right)^2}$

(b) The gradient of the graph of $y=f(x)$ at $x=4$ is $\dfrac{2}{9}$

(c) 0

21 (a) $\dfrac{dy}{dx}=\dfrac{(x+1)\left(8-3x^2-x^3\right)}{\left(x^3-3x+4\right)^2}$

(b) (-1, -0.17), (1.36, 1.88)

22 (a) $\frac{dy}{dx} = (x+1)(3x-7)$

(b) (-1, 0): the gradient is 0; (4, 0): the gradient is 25

(c) (-1, 0), $\left(\frac{7}{3}, -\frac{500}{27}\right)$

(d) $x < \frac{4-\sqrt{106}}{6} \cup x > \frac{4+\sqrt{106}}{6}$

23 (a) $h_1(x) = (1-x)\sqrt{1+x}$; domain: $x \geq -1$; $h_1'(x) = -\frac{1+3x}{2\sqrt{1+x}}$

(b) $h_2(x) = \frac{1-x}{\sqrt{1+x}}$; domain: $x > -1$, $h_2'(x) = -\frac{3+x}{2(1+x)\sqrt{1+x}}$

(c) $h_1(x)$: $\left(-\frac{1}{3}, \frac{4\sqrt{6}}{9}\right)$; $h_2(x)$ has no stationary points.

(d) $x = \frac{-3+\sqrt{33}}{6}$

24 (a) $f'(x) = \frac{x^{p-1}\left((p-2)x^2+p\right)}{\left(x^2+1\right)^2}$

(b) $p = 1$; $\left(-1, -\frac{1}{2}\right), \left(1, \frac{1}{2}\right)$

(c) $p \geq 2$ or equivalently $p \in \mathbb{N} \setminus 1$; (0, 0)

25 (a) Range: $\mathbb{R}$

(b) $g'(t) = \frac{-3\left(t^2+1\right)}{\left(t^2-1\right)^2}$

(c) The gradient at (0, 0) is -3.

(d) The gradient of the graph is always negative.

(e) $k \in [-3, 0]$

CHAPTER REVIEW 10

1 (a) (3, -4) (b) $y = -2x + 1$

(c) (1, 0): gradient of -4; (5, 0): gradient of 4

2 (1, 2) 3 $-2 - h$ 4 $x > -3$

5 (a) 11 (b) 29

6 (a) 12π m²/m (b) 16π m²/m

7 (a) 1°C (b) 9°C at 12 noon

(c) $\frac{8}{9}$°C/hour (d) $\frac{4}{3}$°C/hour

(e) 4:30 pm (f) 2:15 pm

8 (a) $g'(x) = 3x^2 - 6x + 4$

(b) The graph $y = g(x)$ does not have stationary points.

9 B

10 Maximum turning point: (1, 0)

11 (a) 12 m/s (b) $t = 2$ s, $x = 16$ m.

12 (a) O (b) $t = \frac{2}{3}$ s and $t = 2$ s

13 D

14 $\frac{dy}{dx} = 3\left(1-3x+x^2\right)^2(2x-3)$

15 (a) $\frac{dy}{dx} = 2(x-3)\left(2x^2-3x+1\right)$

(b) $\frac{dy}{dx} = \frac{x^2-8x-2}{(x-4)^2}$

(c) $\frac{dy}{dx} = \frac{3(x-5)^3(2x-7)^4\left(15x^2-38x-137\right)}{(3x+1)^2}$

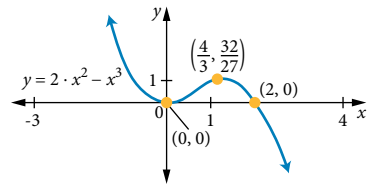
16 B 17 A

18 (a) 469 birds, after 5.75 months

(b) 3 months, 378 birds

19 (a) Local minimum: (0, 0); local maximum: $\left(\frac{4}{3}, \frac{32}{27}\right)$

(b)



20 12.5 m²

21 (a) $S = x^2 + \frac{4(25-x)^2}{\pi}$ or $S = x^2 + \frac{4(x-25)^2}{\pi}$

(b) 44 cm (circle), 56 cm (square)

(c) Do not cut the wire at all, but make a circle from the entire 100 cm.

22 (a) $a = -\frac{5}{64}$, $h = 20$ (b) $x = \frac{40}{3}$, maximum: 92.59

23 (a) Particle 1 : 2 m to the right of O, 3 m/s to the left; Particle 2: At O, 2 m/s to the right.

(b) 0.5 s and 2 s (c) 1.25 s

(d) At $t = \frac{1}{2}$, particle 1: left; particle 2: right.
At $t = 2$, particle 1: right: particle 2: left.

24 $\frac{dy}{dx} = \frac{-g(x)g'(x)}{\sqrt{1-(g(x))^2}}$

25 (a) $f'(x) = \frac{5x^2-6x}{\sqrt{2x-3}}$ (b) $x = 0$, $x = \frac{6}{5}$

26 $-\frac{3}{4}$ 27 $-4 < x < -\frac{2}{3}$

28 $\frac{f(x)g'(x)h(x) + f'(x)g(x)h(x) - f(x)g(x)h'(x)}{(h(x))^2}$

29 $y = -\frac{16}{3}x + 12$

30 (a) $\left(\frac{b}{3}, \frac{4b^3}{27}\right)$, $(b, 0)$ (b) Both have $x = \frac{2b}{3}$.

31 $x = 2.11$ m, $h = 4.59$ m

32 The particle starts at the origin and moves in a positive direction at an initial speed of 6 m/s. It is slows to stop at 9 m after 3 s. The particle then changes direction and returns to the origin, taking 1 s to speed up to 2 m/s and then keeping a steady pace for the final 4 s.

33 (a) 26 min (b) 4 h 7 min (c) $\sqrt{2}$ h

(d) 0.09% (e) $\sqrt{6}$ h (f) 0.08%

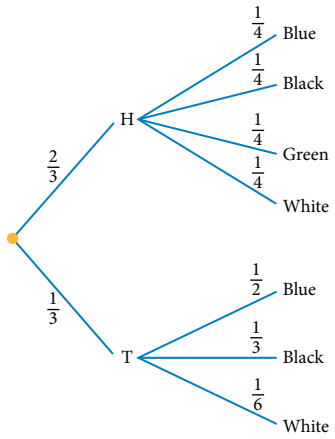
Chapter 11 Discrete random variables

RECALL

1 (a) (i) $\frac{3}{7}$ (ii) $\frac{11}{14}$ (b) $\frac{1}{243}$ (c) $\frac{7}{13}$

2 (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{3}{16}$

3 (a) $\frac{2}{3}$ (b)



(c) {(H, blue), (H, black), (H, green), (H, white), (T, blue), (T, black), (T, white)}

(d) $\frac{1}{3}$ (e) $\frac{1}{6}$

4 (a) $y = 3x + 29$ (b) $y = 65 - 5x$

5 (a) $x = 3, y = -2$ (b) $x = -4, y = -2$ (c) $x = 3, y = -2$

6 (a) 220 ways (b) 84 ways (c) 455 ways

EXERCISE 11.1

1 (a) $\frac{3}{8}$ (b) $\frac{7}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

2 (a) Yes, the sum is 1 and all values are between zero and one, inclusive.

(b) No, the sum is greater than 1.

(c) Yes, the sum is 1 and all values are between zero and one, inclusive.

(d) No, the sum is 1 but not all values are between zero and one, inclusive.

(e) Yes, the sum is 1 and all values are between zero and one, inclusive.

(f) Yes, the sum is 1 and all values are between zero and one, inclusive.

3 (a) $k = \frac{2}{9}$ (b) $k = \frac{1}{6}$

4 (a) discrete (b) continuous (c) discrete

(d) continuous (e) discrete (f) continuous

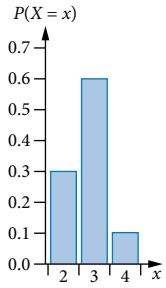
(g) discrete (h) discrete (i) discrete

5 C

6 (a)

x	2	3	4
$P(X = x)$	0.3	0.6	0.1

(b)



7 (a) lowest: 2, highest: 12

(b)

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

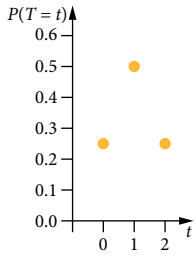
(c) $P(X \geq 6) = \frac{13}{18}$ (d) $P(X < 10) = \frac{5}{6}$

(e) $P(4 \leq X \leq 10) = \frac{5}{6}$

8 (a)

t	0	1	2
$P(T = t)$	0.25	0.5	0.25

(b)



9 (a) Smallest: 1, largest: 216

(b)

x	1	8	27	64	125	216
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(c) $P(X < 100) = \frac{2}{3}$

10 (a) $a = 48$

x	7	8	9	10	11
$P(X = x)$	0.12	0.24	0.36	0.24	0.04

(b) 0.6 (c) 0.0096

(d) (i)

x	7	8	9	10	11
$P(X = x)$	0.125	0.24	0.35	0.24	0.045

(ii) 0.0108

11 4 penalty kicks

12 (a) $k = \frac{2}{9}$ (b) $k = \frac{1}{12}$

13 (a) 0, 1, 2

(b)

x	0	1	2
$P(X = x)$	0.49	0.42	0.09

(c) 0.32 (d) 0.96

14 (a) D

(b) The student has omitted the known probability $P(X = 5) = \frac{1}{8}$ in the total.

15 $k = 0.15$

16 (a) $\frac{3}{8}$ (b) 0.244

(c) 0.060 (d) 8 attempts

17 (a)

x	1	2	3	4	5	6	9	16	25	36
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

(b) (i) $P(X \le 12) = \frac{3}{4}$ (ii) $P(X \ge 16) = \frac{1}{4}$

(iii) $P(5 < X < 9) = \frac{1}{12}$

18 (a)

x	0	1	2	3
$P(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

(b) (i) $\frac{54}{125}$ (ii) $\frac{98}{125}$ (iii) $\frac{81}{125}$ (iv) $\frac{44}{125}$

(c) The answers to parts (iii) and (iv) add to 1 because 'fewer than two' and 'at least two' are complementary events and probabilities of complementary events add to 1.

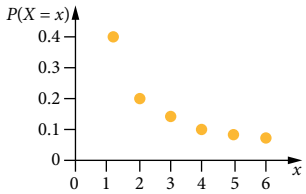
19 (a)

y	2	3	4	5	6	7	8	9	10	11	12
$P(Y = y)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{6}{81}$	$\frac{8}{81}$	$\frac{11}{81}$	$\frac{12}{81}$	$\frac{14}{81}$	$\frac{8}{81}$	$\frac{9}{81}$	$\frac{4}{81}$	$\frac{4}{81}$

(b) (i) $P(Y \ge 4) = \frac{76}{81}$ (ii) $P(Y < 7) = \frac{10}{27}$

(iii) $P(3 \le Y \le 9) = \frac{7}{9}$ (iv) $P(Y \ge 6 \mid Y \le 10) = \frac{54}{73}$

20



21 (a) (i) 3 shots (ii) 3 shots (iii) 3 shots

(b) 53%

EXERCISE 11.2

1 (a) $E(X) = 4.3$ (b) $E(X) = 0.5$

(c) $E(X) = 4\frac{17}{18}$ (d) $E(X) = 1\frac{1}{4}$

2 (a) $a = 0.1$ and $b = 0.3$ (b) $i = 0.1$ and $j = 0.25$

(c) $i = \frac{1}{20}$ and $j = \frac{1}{5}$ (d) $a = \frac{1}{5}$ and $b = \frac{1}{3}$

3 (a) $E(X) = 1.55$ (b) $E(2X - 3) = 0.1$

(c) $E(X^2 - 5) = -0.95$ (d) $E(X^2 + X - 4) = 1.6$

4 0.98

5 (a) $k = \frac{1}{24}$, $E(X) = 3\frac{5}{6}$ (b) $k = \frac{1}{20}$, $E(X) = -1\frac{1}{10}$

6 (a) $E(W) = 4\frac{7}{8}$ (b) $E(3W - 4) = 10\frac{5}{8}$

(c) $E(2W + 5) = 14\frac{3}{4}$ (d) $E(W^2 - 7) = 21\frac{3}{4}$

7 (a) C (b) $E(X^2 - 4) \ne (E(X))^2 - 4$

8 $E(X) = 2.9$, $E(Y) = 3.4$, $E(X + Y) = 6.3$

9 (a)

x	1	3	5
$P(X = x)$	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{1}{9}$

(b) $E(X) = 2\frac{1}{9}$ (c) \$2.11

10 (a) Expected value: 3.3, variance: 2.61

(b) Expected value: 6.9, variance: 1.79

11 1.4663 **12** D

13 First salary package, \$27.50 more per week.

14 (a) $3\frac{1}{2}$ (b) 0.73 to 6.27 (c) 100%

15 (a) $w = 10$ (b) $\text{Var}(T) = 1.6475$ (c) $\sigma = 1.28$

(d) $\text{Var}(2T - 6) = 6.59$ (e) $\text{Var}(5 - 3T) = 14.8275$

16 (a) $k = \frac{1}{20}$ (b) $E(G) = 1\frac{1}{2}$

(c) $\text{Var}(G) = 1\frac{3}{4}$ (d) $\sigma = 1.3229$

17 (a) $E(Y) = 3$ (b) $E(X) = 1$

(c) (i)

m	1	2	3	4	5	6	7	8
$P(M = m)$	0.06	0.13	0.21	0.3	0.1	0.09	0.07	0.04

(ii) $E(M) = 4$

(d) $E(M) = E(X) + E(Y)$

18 \$8 **19** 2.48 **20** \$1.35

EXERCISE 11.3

1 (a) $E(R) = 8\frac{1}{2}$ (b) $\text{Var}(R) = 21\frac{1}{4}$

2 (a) 5 (b) $6\frac{2}{3}$

3 (a) $E(F) = 2\frac{1}{2}$ (b) $\text{Var}(F) = 1\frac{1}{4}$

4 (a) C (b) D

5 (a) $E(X) = 3$ (b) $\text{Var}(X) = 2$

(c) (i) $E(Y) = 4$ (ii) $\text{Var}(Y) = 2$

(d) $Y = X + 1$, $E(Y) = E(X) + 1$

(e) $Y = X + 1$, $\text{Var}(Y) = \text{Var}(X)$

(f) (i) $E(Z) = 8$ (ii) $\text{Var}(Z) = 2$

6 (a) $E(T) = 29.5$

(b) $\text{Var}(T) = 299.92$, $\sigma(T) = 17.32$

7 (a) $13\frac{1}{2}$ (b) $2\frac{11}{12}$

8 (a) 25 (b) 125

9 (a) $Y = 9X + 19$, $n = 8$ (b) $Y = 3X - 13$, $n = 15$

(c) $Y = 0.1X + 3002$, $n = 10$

10 (a) C

(b) $\text{Var}(aX + b) = a^2(\text{Var}(X))$; the student forgot to square the value of the factor a.

11 (a) $E(Y) = 64$ (b) $\text{Var}(Y) = 242$

12 (a) $E(Y) = -23.5$ (b) $\text{Var}(Y) = 404.25$

13 (a) $E(Y) = 1.052$ (b) $\text{Var}(Y) = 0.000\,021$

14 Mean: 144.5, variance: 200

15 (a) The expected value and the variance for the double roll are each double the respective single roll values.

$E(X) = 3\frac{1}{2}$, $E(S) = 7$; $\text{Var}(X) = 2\frac{11}{12}$, $\text{Var}(S) = 5\frac{5}{6}$

(b) The expected value and the variance for the double spin are each double the respective single spin values.

$E(V) = 2\frac{1}{2}$, $E(T) = 5$; $\text{Var}(V) = 1\frac{1}{4}$, $\text{Var}(T) = 2\frac{1}{2}$

(c) The predictions that the expected value of the sum would be the total of the expected values of the individual distributions, and the variance for the sum would be the total of the individual variance values, were correct.

$E(V) = 2\frac{1}{2}$, $E(X) = 3\frac{1}{2}$, $\text{E(W)} = 6$; $\text{Var}(V) = 1\frac{1}{4}$,

$\text{Var}(X) = 2\frac{11}{12}$, $\text{Var}(W) = 4\frac{1}{6}$

EXERCISE 11.4

1 (a)

z	1	-2	-3	4	-5	6
$P(Z = z)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(b) $E(Z) = \frac{1}{6}$

(c) The game is not fair as it is biased in favour of the player.

2 (a) $k = 0.05$

(b)

x	1	2	3	4	5	6	7	8
$P(X = x)$	0.05	0.1	0.15	0.2	0.2	0.15	0.1	0.05

(c) $E(X) = 4.5$

(d) $\text{Var}(X) = 3.25$

3 (a) B (b) A

4 Tomino's probability distribution is incorrect because $P(X = x) = -\frac{1}{5}$ and probabilities cannot be negative.

5 (a) A

(b) $\text{Var}(Y)$ has been given instead of $\sigma(Y)$.

6 Expected value: 1.83, standard deviation: 1.86

7 Expected value: 17.33, standard deviation: 3.82

8 (a)

x	1	2	3	4
$P(X = x)$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{17}{216}$	$\frac{1}{216}$

(b) $E(X) = 1\frac{127}{216}$ (c) 0.427

9 (a)

y	0	1	2	3	4	5
$P(Y = y)$	0.310563	0.431337	0.209840	0.044177	0.003965	0.000119

(b) $E(Y) = 1.00$

10 (a)

z	1	2	3	4	5	6
$P(Z = z)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(b)

y	1	2	3	4	5	6
$P(Y = y)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(c) The two distributions are mirror images of each other.

(d) (i) $E(Z) = 4\frac{17}{36}$ (ii) $E(Y) = 2\frac{19}{36}$

(e) $1\frac{19}{36}$ (f) $1\frac{19}{36}$

(g) The two values are the same because one distribution is a mirror image of the other.

(h) $\text{Var}(Y) = \text{Var}(Z) = 1\frac{926}{1296}$

11 (a) $E(X) = 2$, $\text{Var}(X) = 0.5$

(b)

z	2	3	4	5	6
$P(X + Y = z)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

(c) $E(X + Y) = 4$, $\text{Var}(X + Y) = 1$

(d) $E(X + Y) = 2E(X) = 2E(Y)$

(e) $\text{Var}(X + Y) = 2\text{Var}(X) = 2\text{Var}(Y)$

12 (a) $E(X) = 2.6$

(b) $E(X) = 2.6$, $E(2X) = 5.2$; $E(2X) = 2E(X)$

(c) $\text{Var}(X) = 1.14$

(d) (i) $\text{Var}(2X) = 4.56$ (ii) $\text{Var}(3X) = 10.26$

(e) $\text{Var}(kX) = k^2\text{Var}(X)$

13 (a) $E(X) = 3.008$ (b) $E(Y) = 3.894$

(c) (i) $P(X = Y) = 0.207$ (ii) $P(X \neq Y) = 0.793$

(iii) $P(X + Y > 8) = 0.151$

14 (a) 0.8920 (b) 0.2777 (c) 0.3733

15 (a) $E(X) = n + 0.31$ (b) 0.207

(c) 0.745 (d) 0.003

CHAPTER REVIEW 11

1 (a) The value of k that makes the table a discrete probability distribution is $k = \frac{5}{16}$.

(b) The value of k that makes the table a discrete probability distribution is $k = \frac{1}{18}$.

(c) The value of k that makes the table a discrete probability distribution is $k = \frac{1}{15}$.

2 (a)

x	6	7	8	9	10	11	12	13	14	15	16
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

(b) $P(X \geq 11) = \frac{7}{12}$ (c) $P(X < 15) = \frac{11}{12}$

(d) $P(7 \leq X \leq 11) = \frac{5}{9}$

3 C 4 C

5

t	4	5	6	8	9	11
$P(T = t)$	0.1	0.1	0.1	0.3	0.1	0.3

6

b	0	2	3	4	8	10	11	12
$P(B = b)$	0.04	0.12	0.02	0.12	0.10	0.48	0.02	0.12

7 For the second set the probability when $X = 1$ is negative so this value is not valid.

8 (a) $P(X \geq 10) = \frac{1}{6}$ (b) $P(5 < X \leq 9) = \frac{5}{9}$

(c) $E(X) = 7$

9 (a) $E(X) = 0.25$ (b) $\text{Var}(X) = 1.8875$

(c) $E(3X - 2) = -1.25$ (d) $E(X^2 - 2X) = 1.45$

(e) $\text{Var}(3X - 2) = 16.9875$

10 \$1 11 $\sigma = 0.95$ 12 $\text{Var}(Y) = 1$

13 0.985; more than 95% of the values lie within two standard deviations of the expected value.

14 (a) $E(R) = 10\frac{1}{2}$ (b) $\text{Var}(R) = 33\frac{1}{4}$

15 $n = 15$

16 (a) $n = 52$ (b) $n = 35$ (c) $n = 201$

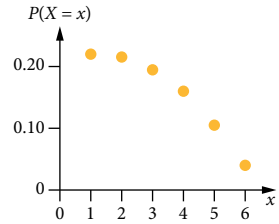
17 3 shots at goal

18 (a) $k = \frac{1}{125}$

(b)

x	1	2	3	4	5	6
$P(X = x)$	$\frac{6}{25}$	$\frac{29}{125}$	$\frac{26}{125}$	$\frac{21}{125}$	$\frac{14}{125}$	$\frac{1}{25}$

(c)



19 (a)

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

(b) (i) $E(X) = 5$ (ii) $\text{Var}(X) = 2\frac{1}{2}$

20 D 21 D

22 (a) Plot 1 and Plot 4

(b) Plot 1: $E(X) = 7$, $\text{Var}(X) = 8.2$; Plot 4: $E(X) = 23$, $\text{Var}(X) = 421$

23 (a) A uniform discrete distribution with consecutive data values.

(b)

x	1	2	3	4	5	6
$P(X = x)$	$\frac{17}{96}$	$\frac{16}{96}$	$\frac{17}{96}$	$\frac{12}{96}$	$\frac{14}{96}$	$\frac{20}{96}$

(c) The results suggest a uniform distribution.

(d) Sample: $E(X) = 3.52$, $\text{Var}(X) = 3.19$; theoretical: $E(X) = 3.5$, $\text{Var}(X) = 2.92$

The sample expected values are very close. The sample variation is about 9% higher than the theoretical value.

(e) Answers will vary.

(f) There are likely to be slight differences between each set of values.

(g) The values should be very close to the theoretical values.

24 (a)

y	0	1	2	3
$P(Y = y)$	$\frac{7}{24}$	$\frac{21}{40}$	$\frac{7}{40}$	$\frac{1}{120}$

(b) $E(Y) = \frac{9}{10}$, best guess 1 item

25 (a)

x	1	2	3
$P(X = x)$	$\frac{1}{3}$	$\frac{23}{36}$	$\frac{1}{36}$

(b) $E(X) = 1\frac{25}{36}$, a best guess would be 2. (c) 0.27

26 (a) $E(X) = 3\frac{1}{4}$, $\text{Var}(X) = 2\frac{41}{48}$

(b) $\frac{1}{8}$

27 (a) $E(Z) = 4\frac{1}{12}$ (b) $P(Z > 6) = \frac{1}{9}$

(c) $P(Z < 7) = \frac{8}{9}$ (d) $P(Z > 4 \mid Z \leq 8) = \frac{8}{17}$

28 (a)

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

(b) $E(X) = 1.9$, best guess is 2 rolls.

(c) This is just the same as the die question as the relevant probabilities are equal. So the expected number of flips is 1.9.

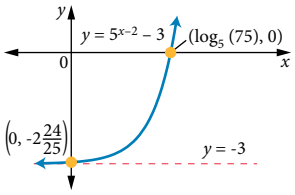
29 (a) $E(X) = 3.06$, about 3 packs per hour

(b) Profit of \$45.20 per hour.

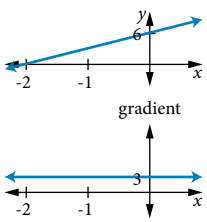
Unit 2 Exam review: Chapters 7–11

PAPER 1: TECHNOLOGY-FREE

1



2 (a)



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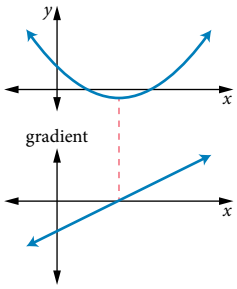

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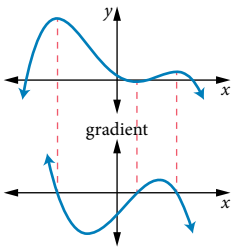
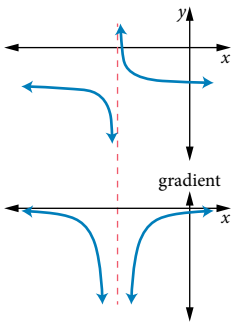


(d)



3 (a) $8\sqrt{2} + 1$ (b) $x > \frac{7}{8}$

4 $x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$ 5 $f'(x) = 2x - 4$

6 (a) $f'(x) = \frac{-x(5x-8)}{2\sqrt{2-x}}$ (b) $f'(x) = \frac{4 - 3x^2}{\sqrt{x}\left(x^2+4\right)^2}$

7 (a) $V(h) = \frac{1}{3}\pi h^2(2R - h)$ or $V = \frac{2\pi R}{3}h^2 - \frac{\pi}{3}h^3$

(b) $h = \frac{4}{3}$ m Maximum volume: $\frac{32}{81}$ m^3

PAPER 2: TECHNOLOGY-ACTIVE

1 (a) $x = 1.63$ (b) $x = 1.43$ (c) $x = 10.34$

2 (a) 0.803 (b) 2.182 (c) 5.341

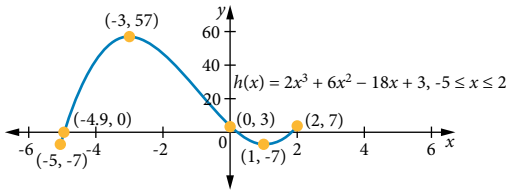
3 (a) 30.18 cm (b) 753.98 cm^2

4 1.5524

5 (a) $h(x) = 2x^3 + 6x^2 - 18x + 3$

(b) (-3, 57) local maximum
(1, -7) local minimum

(c) (-3, 57) global maximum
(-5, -7) and (1, -7) global minimums



6 (a) 22.5 (b) 17.64

7 (a) $N(t) = 35 \times (1.43)^t$; 3660 cockatoos in 2025

(b) $y = 500 - 465 \times 0.9^{1.33t}$, where y is the number of birds at any time t (years) after 2012.

The model predicts 425 cockatoos in 2025.

Units 1 & 2 Exam review: Chapters 1–11

PAPER 1: TECHNOLOGY-FREE

1 Gradient: $\frac{2}{5}$, y-intercept: $\left(0, \frac{3}{4}\right)$

2 $x = -2, y = -1$ 3 $y = -(x+1)^2 + 7$

4 (a) $a = 2, a = -\frac{1}{4}$ (b) $b = 16 \pm 4\sqrt{19}$ (c) $c = \pm\sqrt{3}$

5 (a) Dilation parallel to the m-axis of scale factor 2, reflection in the p-axis and translation up by 1 unit.

(b) Dilation parallel to the s-axis of factor 3, translation left by 5 units and translation down by 3 units.

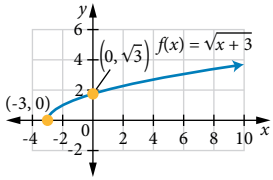
(c) Dilation parallel to the y-axis of factor $\frac{3}{4}$ and a reflection in the x-axis.

6 (a) $y = -\frac{1}{5}x^2$ (b) $y = 6x$ (c) $y = -\frac{1}{4}x^3$ (d) $y = 4x^4$

7 (a) $8x^3 - 12x^2y + 6xy^2 - y^3$ (b) $9m^2 + 12mn + 4n^2$

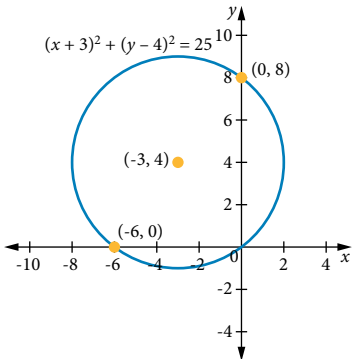
(c) $36x^2 - 24xy + 4y^2$ (d) $27x^3 + 27x^2 + 9x + 1$

8



Domain: $[-3, \infty)$; Range: $[0, \infty)$

9 Radius: 5, centre: (-3, 4)



10 (a) $P(\text{B or Q}) = \frac{7}{13}$ (b) $P(\text{R or N}) = \frac{11}{13}$

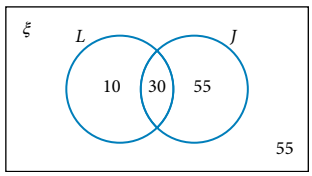
(c) $P(\text{S} \cup (\text{J or 5})) = \frac{19}{52}$

(d) $P(\text{C} \cup (\text{6 or Q or A})) = \frac{11}{26}$

(e) $P(\text{neither R nor Q}) = \frac{6}{13}$

(f) $\frac{6}{13}$

11 (a)



(b) (i) $P(J) = \frac{17}{30}$ (ii) $P(J \text{ and } L') = \frac{11}{30}$

(iii) $P(L \text{ and } J') = \frac{1}{15}$ (iv) $P(J \text{ or } L)' = \frac{11}{30}$

12 (a) $n^3 - 3n^2 + 2n$ (b) $n - 2$

(c) $n^2 - 11n + 30$ (d) $\frac{1}{n^2 - 5n + 6}$

13 (a) 3 (b) 8 (c) $\frac{1}{36}$

14 $f(x) = 3^x - 2$

15 (a) $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ (b) $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

16 13.3°C/min

17 (a) 100 km/h (b) $83\frac{1}{3}$ km/h

18 Graph 1: Container C Graph 5: Container D
Graph 2: Container F Graph 6: Container E
Graph 3: Container A Graph 7: Container B
Graph 4: Container G Graph 8: Container H

19 (a) $f'(x) = -8x$ (b) $f'(x) = 2x - 2$ or $f'(x) = 2(x - 1)$

20 58 L/s

21 (a) $\frac{dy}{dx} = \frac{5}{2\sqrt{5x-7}}$ (b) $\frac{dy}{dx} = 36x^2\left(4x^3 - 5\right)^2$

22 (a) $E(S) = 13$ (b) $\text{Var}(S) = 52$

23 (a) (i) (8, 0) (ii) (1, 0)

(b) (i) $y = -\sqrt{x+8} + 4$ (ii) $y = 2\sqrt{x+3} - 4$

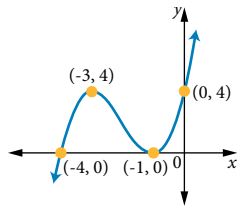
(c) (i) $\left(0, 4 - 2\sqrt{2}\right)$ (ii) $\left(0, 2\sqrt{3} - 4\right)$

24 1.50×10^8 km, 150 000 000 km

25 $a = 0$, $a = 2$

26 (a) (-3, 4): local maximum, (-1, 0): local minimum

(b)



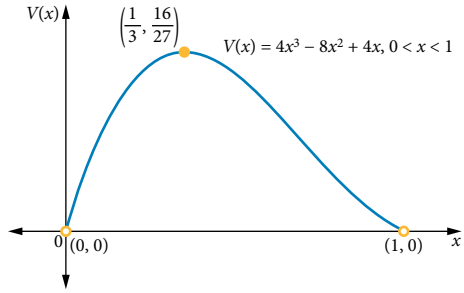
27 4.5 square units

28 (a) $V = 4x^3 - 8x^2 + 4x$ (b) $0 < x < 1$

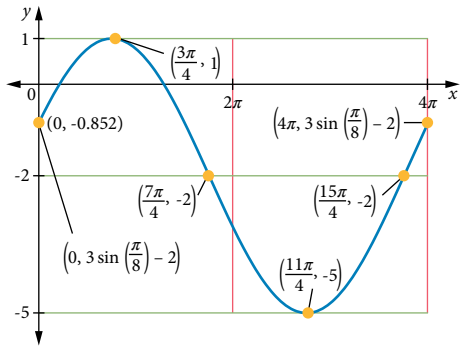
(c) $x = \frac{1}{3}$; maximum volume: $\frac{16}{27}$ m^3;

dimensions: $\frac{4}{3}$ m by $\frac{4}{3}$ m by $\frac{1}{3}$ m

(d)



29



PAPER 2: TECHNOLOGY-ACTIVE

1 (a) 960 ohms (b) 0.3 amps

2 (a) {16} (b) {2, 4, 6, 8, 10, 12, 14, 18, 20}

3 (a) 3.6×10^5 (b) 7.3×10^{-3}

4 $x = 0.8614$

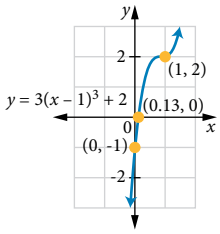
5 $x = 0.33, 1.37, 2.42, 3.47, 4.52, 5.56$

6 (a) -20, -11, -2, 7

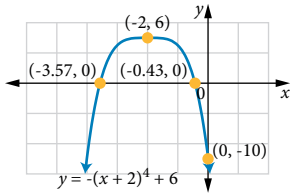
(b) Yes; there is a common difference.

(c) 3 (d) 37

7 **(a)**



(b)



8 **(a)** 22.2 L/h **(b)** 152.8 L/h **(c)** 375.0 L/h

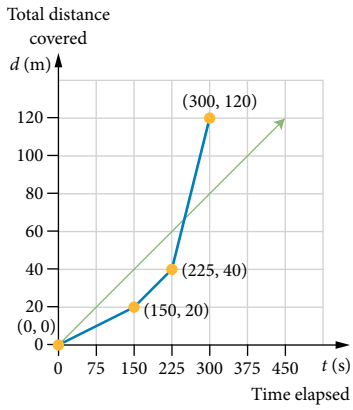
9 $a = \frac{1}{8}, b = \frac{1}{16}$

10 **(a)** 19 **(b)** 114

11 **(a)** $E(X) = \frac{1}{4}$ **(b)** $\text{Var}(X) = \frac{7}{32}$

12 **(a)** $d = \begin{cases} \frac{2}{15}t, 0 \le t \le 150 \\ 20 + \frac{4}{15}(t - 150), 150 < t \le 225 \\ 40 + \frac{16}{15}(t - 225), 225 < t \le 300 \end{cases}$

(b) $53\frac{1}{3}$ m



(c) $53\frac{1}{3}$ m

13 $A(x) = 100x + 4x^2$, 1.86 m

14 $\theta = 66°$ or $\theta = 1.15$ radians

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